Abstract

In recent years many leading companies have explored the possibility of combining a truck with drones to make deliveries more efficient. Finding different delivery approaches and efficient solution methods are crucial to reduce costs and speed up the delivery.
As many drone-assisted delivery approaches can be interpreted as a natural extension of the well-known traveling salesman problem (TSP), the NP-hard nature of this problem makes it very challenging to compute an optimal solution.
This work summarizes standard optimization approaches for drone-assisted delivery, like the traveling salesman problem with drone (TSP-D), flying sidekick TSP (FSTSP) and the parallel drone scheduling TSP (PDSTSP). Furthermore, it explores and connects these approaches to different elements of the TSP.
One result of this master thesis is the refinement of the problem formulation, such as the notions and the mathematical framework used in the TSP-D to fit the standard notations in the TSP. Furthermore, we study specific combinatorial structures and their properties and analyze how this information translates to the TSP-D. Finally, we introduce an optimization approach inspired by the PDSTSP to present an approximation result based on the optimal solution of the TSP and a lower bound result based on the minimum spanning tree.
This work emphasizes the mathematical the connection between elements of the well-known TSP to the optimization approaches of drone-assisted delivery.

6 Parallel TSP with drone (PTSP-D)

This section aims to formulate a simplified version of the PDSTSP introduced in [22] to find similar approximation results as for the TSP-D in [6]. Therefore, we aim to reformulate the PDSTSP to fit the notation given in section 4.
As we take a similar approach to this problem as Agatz et al. in [6], we call the simplified PDSTSP the parallel traveling salesman problem with drone (PTSP-D). The main difference to the PDSTSP is that we only work with exactly one drone instead of a fleet of drones and each node has to be visited at least once. Further, we assume that the drone can visit every node $v \in V$, i.e., there are no endurance and range limitations. The aim of this is to prove an approximation and lower bound result in subsections 6.2 and 6.3. Note that most proof approaches in this section were inspired by [6].

6.1 Problem definition and notation

Like the TSP-D, the PTSP-D is an optimization approach using a single truck and a drone to serve a set of customers. However, both vehicles work independently from one another. The truck starts at the depot, serves a set of customers during its delivery and finishes the tour at the depot. Meanwhile, the drone flies independently from the depot to a customer, delivering the parcel and then flying back to the depot where it is prepared for the next delivery.
In this model, we assume that every customer and depot is connected. Further, we want the truck and the drone to finish their delivery at the depot.
We have the following conditions for a solution of the PTSP-D:

- truck and drone start and end their delivery at the depot
- each customer has to be visited at least once by either the drone or the truck
- the drone can only serve one customer at a time and has to return to the depot before its next delivery.

In this solution approach, we use most of the notation we introduced in section 4. Analog to the TSP-D, the PTSP-D is defined in a strongly connected directed graph $G = (V, E)$. Unless stated otherwise, the notation we use in this section is identical to that in section 4.

The main reason for introducing the notion of an operation in the TSP-D is to formalize the interplay of truck and drone during the delivery. As both truck and drone work independently in the PTSP-D, we do not have the notion of an operation in the sense of

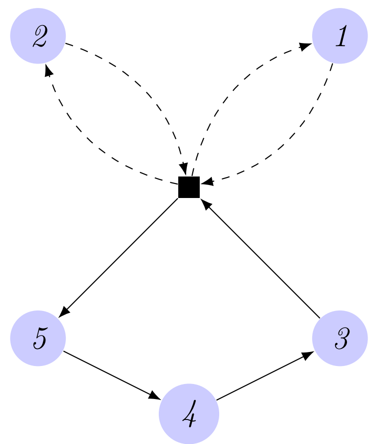
Definition 4.2. Thus, we define the notion of a PTSP-D tour with the notation introduced before without using operations:

Definition 6.1 (PTSP-D tour)**.** *Let $G = (V, E)$ be a strongly connected directed graph. A (feasible) PTSP-D tour T is a tuple of paths $(\mathcal{R}, \mathcal{D})$ with the following conditions:*

- *the start and end node the paths $\mathcal{R}$ and $\mathcal{D}$ are both the depot*
- *every $v \in V$ is part of at least one of the paths $\mathcal{R}$ and $\mathcal{D}$*
- *every $d_i \in V$ with even index $0 \leq i \leq p$, which is part of the path $\mathcal{D} = (d_0, d_1, \dots, d_p)$ is the depot v_0.*

We call the paths $\mathcal{R}$ and $\mathcal{D}$ the truck and drone tours.
Since the truck and the drone do not work in tandem, we do not have the set of combined nodes V_C, which are jointly visited by both vehicles in the TSP-D.

Example 6.2.



The above example consists of the depot v_0 and five customers. The node set is given by $V = \{v_0, \dots, v_5\}$ and the customers are $v_1, \dots, v_4$. Consider the truck tour given by $\mathcal{R} = (v_0, v_5, v_4, v_3, v_0)$ and the drone tour given by $\mathcal{D} = (v_0, v_1, v_0, v_2, v_0)$. Together the tuple $(\mathcal{R}, \mathcal{D})$ defines a PTSP-D tour. $(\mathcal{R}, \mathcal{D})$ is a feasible TSP-D tour, since:

- *$\mathcal{R}$ and $\mathcal{D}$ start and end at the depot v_0*
- *every $v \in V$ is part of either the path $\mathcal{R}$ or $\mathcal{D}$*
- *every second location of the drone tour $\mathcal{D}$ is the depot v_0.*

Since the order of the drone nodes does not change the travel time of the drone, $\mathcal{D}' = (v_0, v_2, v_0, v_1, v_0)$ is also a valid drone tour, which means $(\mathcal{R}, \mathcal{D}')$ is a feasible PTSP-D tour with the same travel time.

With our two cost functions $c^t, c^d : E \longrightarrow \mathbb{R}^+$ defined on the edges of the graph $G = (V, E)$, we can compute the duration time of a PTSP-D tour.

Definition 6.3. *Let $(\mathcal{R}, \mathcal{D})$ be a feasible PTSP-D tour. Let $\mathcal{R} = (r_0, \dots, r_m)$ and $\mathcal{D} = (d_0, \dots, d_p)$ with $r_i, d_i \in V$ be the truck and drone routes and $c^t, c^d : E \longrightarrow \mathbb{R}^+$ two cost functions. We define:*

- $c^t(\mathcal{R}) = \sum_{i=0}^{m-1} c^t(r_i, r_{i+1})$ *as the duration time of* $\mathcal{R}$
- $c^d(\mathcal{D}) = \sum_{i=0}^{p-1} c^d(d_i, d_{i+1})$ *as the duration time of* $\mathcal{D}$.

The overall delivery time of the PTSP-D route $(\mathcal{R}, \mathcal{D})$ can then be computed as follows:

$$c(\mathcal{R}, \mathcal{D}) = \max\{c^t(\mathcal{R}), c^d(\mathcal{D})\}.$$

Now, we are ready to formulate the problem statement of the PTSP-D:

Definition 6.4 (Problem statement PTSP-D)**.** *Let $G = (V, E)$ be a strongly connected directed graph and let $c^t, c^d : E \longrightarrow \mathbb{R}^+$ be two cost functions defined on the edges E. The PTSP-D aims to find a feasible PTSP-D tour on G with minimal cost. This means the PTSP-D aims to solve the following minimization problem:*

$$\min\Big\{c(\mathcal{R}, \mathcal{D}) = \max\{c^t(\mathcal{R}), c^d(\mathcal{D})\} : (\mathcal{R}, \mathcal{D}) \text{ is a feasible PTSP-D tour in } G\Big\}. \tag{56}$$

We say $(\mathcal{R}, \mathcal{D})$ is the optimal TSP-D tour if it is the minimal argument of the set (56).

6.2 Approximation of PTSP-D by TSP

This section wants to prove a similar approximation result for PTSP-D by the TSP as Agatz proved for the TSP-D in [6]. Therefore, our approach for this proof is similar to [6].

First, we define a fixed constant $\alpha \geq 1$ to describe following relation between the cost functions c^t and c^d :

$$c^t(v, v_0) = \alpha c^d(v, v_0) \text{ for all } v \in V.$$

In the coming subsections, we work with symmetric graphs. In this case, we have the following proposition:

Proposition 6.5. *Let $G = (V, E)$ be a strongly connected symmetric directed graph and $(\mathcal{R}, \mathcal{D})$ be a PTSP-D tour. Then we can compute the duration time of the drone tour $\mathcal{D} = (d_0, \dots, d_p)$ as follows:*

$$c^d(\mathcal{D}) = 2\sum_{i=0}^{p} c^d(v_0, d_i)$$

Proof. Since $(\mathcal{R}, \mathcal{D})$ is a PTSP-D tour, for every even index $0 \leq i \leq p$, d_i is the depot v_0. Since G is a symmetric graph, we have:

$$c^d(\mathcal{D}) = \sum_{j=0}^{p-1} c^d(d_j, d_{j+1}) = 2c^d(v_0, d_1) + 2c^d(v_0, d_3) + \cdots + c^d(v_0, d_p) = 2\sum_{i=0}^{p} c^d(v_0, d_i).$$

□

Now we can prove our approximation result:

Theorem 6.6. *Let $G = (V, E)$ be a strongly connected symmetric directed graph with $V^t = \emptyset$. An optimal solution to the TSP is a $(1 + \alpha)$-approximation to the PTSP-D.*

Proof. Let $(\mathcal{R}^*, \mathcal{D}^*)$ be an optimal solution to the PTSP-D and $\mathcal{R}_{TSP}$ an optimal solution to the TSP. Define $V_{\mathcal{D}^*}$ and $V_{\mathcal{R}^*}$ as the drone and truck nodes of the sequences $\mathcal{D}^* = (d_0, \dots, d_p)$ and $\mathcal{R}^*$.
We now construct a TSP tour $\mathcal{R}$ from the optimal truck route $\mathcal{R}^*$ in the PTSP-D as follows:

1. Start with $\mathcal{R}^*$.

2. For all $d \in V_{\mathcal{D}^*}$ in the drone tour $\mathcal{D}$, we pick the closest node w in the node set $V_{\mathcal{R}^*}$ of the truck sequence $\mathcal{R}^*$ with respect to the cost function $c^t : E \longrightarrow \mathbb{R}^+$, i.e.,

$$\min_{w \in V_{\mathcal{R}^*}} c^t(d, w).$$

3. Add the edges (d, w) and (w, d) for all d in the drone tour $\mathcal{D}$ to the tour $\mathcal{R}$.

This process gives us the TSP tour $\mathcal{R}$. Note that every TSP tour $\mathcal{R}$ can naturally be interpreted as the TSP-D tour $(\mathcal{R}, \emptyset)$.
So, we have with $c(\mathcal{R}_{TSP}, \emptyset) = c^t(\mathcal{R}_{TSP})$:

$$\begin{aligned} c^t(\mathcal{R}_{TSP}) \leq c^t(\mathcal{R}) &= c^t(\mathcal{R}^*) + 2\sum_{i=0}^{p} \min_{w \in V_{\mathcal{R}^*}} c^t(d_i, w) \\ &\leq c^t(\mathcal{R}^*) + 2\sum_{i=0}^{p} c^t(d_i, v_0) \\ &\leq c^t(\mathcal{R}^*) + c^t(\mathcal{D}^*) \end{aligned} \tag{57}$$

Further we have:

$$c(\mathcal{R}^*, \mathcal{D}^*) \geq \max\{c^t(\mathcal{R}^*), c^d(\mathcal{D}^*)\} \geq \max\{c^t(\mathcal{R}^*), \frac{1}{\alpha}c^t(\mathcal{D}^*)\}$$

Using (57) we get:

$$c(\mathcal{R}^*, \mathcal{D}^*) \geq \max\{c^t(\mathcal{R}_{TSP}) - c^t(\mathcal{D}^*), \frac{1}{\alpha}c^t(\mathcal{D}^*)\} \tag{58}$$

The inequality is minimal if:

$$c^t(\mathcal{R}_{TSP}) - c^t(\mathcal{D}^*) = \frac{1}{\alpha}c^t(\mathcal{D}^*) \Leftrightarrow \frac{1}{\alpha+1}c^t(\mathcal{R}_{TSP}) = \frac{1}{\alpha}c^t(\mathcal{D}^*). \tag{59}$$

Using (59) in (58) we get:

$$c(\mathcal{R}^*, \mathcal{D}^*) \geq \max\{c^t(\mathcal{R}_{TSP}) - c^t(\mathcal{D}^*), \frac{1}{\alpha}c^t(\mathcal{D}^*)\} \geq \frac{1}{1+\alpha}c^t(\mathcal{R}_{TSP}).$$

So we have

$$c^t(\mathcal{R}_{TSP}) \leq (1+\alpha)c(\mathcal{R}^*, \mathcal{D}^*).$$

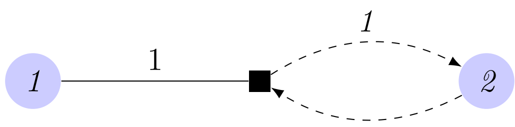
This is analog to

$$c(\mathcal{R}_{TSP}, \emptyset) \leq (1+\alpha)c(\mathcal{R}^*, \mathcal{D}^*),$$

which proves our theorem. □

With this approximation result of the PTSP-D, one can ask if a saving of $(1+\alpha)$ can be obtained using a drone.

Example 6.7. *In Example 4.9, we saw that a saving of* $(1+\alpha)$ *is achieved using the TSP-D.*
If we regard the same graph $G = (V, E)$ *in the PTSP-D setting we get the following optimal route:*



In this example, the optimal TSP-D solution and PTSP-D solution are the same. Note that this is generally not the case.
The only drone node v_2 *is served independently from the truck straight from the depot. After the delivery, the drone heads to the depot again, so* $(\mathcal{R}, \mathcal{D})$ *with* $\mathcal{R} = (v_0, v_1, v_0)$ *and* $\mathcal{D} = (v_0, v_2, v_0)$ *is a feasible PTSP-D tour.*
In example 4.9, we already saw that this route assignment saves $(1+\alpha)$ *compared to the TSP.*

Since the TSP is a well-studied problem compared to PTSP-D, this connection can be used analogously to [6] to find lower bounds using approximation algorithms from the (metric) TSP. This can be seen as follows:

Corollary 6.8. *Let* A *be a* γ*- approximation algorithm of the (metric) TSP. Then* A *is a* $\gamma(1+\alpha)$*-approximation for PTSP-D.*

Proof. Let $(\mathcal{R}^*, \mathcal{D}^*)$ be an optimal PTSP-D solution. Since A is a γ- approximation algorithm of the TSP we have

$$c(A, \emptyset) \leq \gamma c(\mathcal{R}_{TSP}, \emptyset).$$

With theorem 6.6, we get the final result

$$c(A, \emptyset) \leq \gamma c^t(\mathcal{R}_{TSP}) \leq \gamma(1+\alpha) c(\mathcal{R}^*, \mathcal{D}^*).$$

□

Example 6.9. *The well-known Christofides approximation algorithm for the TSP based on minimal spanning tree algorithm provides a worst case ratio of* $\gamma = \frac{3}{2}$ *(e.g., [24]).*
With a fixed α *and Corollary 6.8 we get that the Christriofides algorithm provides a* $(\frac{3}{2} + \frac{3}{2}\alpha)$ *approximation to the PTSP-D.*

6.3 Lower bound for the PTSP-D

In this subsection, we want to connect the minimum spanning tree of the graph to the PTSP-D.
Therefore, we prove following lower bound result:

Theorem 6.10. *Let $G = (V, E)$ be a strongly connected directed symmetric graph. Further let $(\mathcal{R}^*, \mathcal{D}^*)$ be an optimal PTSP-D solution and $T = (V_T, E_T)$ a minimum spanning tree in G. Then we have*

$$c(\mathcal{R}^*, \mathcal{D}^*) \geq \frac{2}{2+\alpha} c^t(T).$$

Proof. Let $V_{\mathcal{R}^*}$ be the truck node set of $\mathcal{R}^*$ and $V_{\mathcal{D}^*}$ the drone set of $\mathcal{D}^*$. We want to construct a spanning tree T' based on $\mathcal{R}^*$ for G as follows:

1. Start with the truck tour $\mathcal{R}^*$
2. Remove one edge of $\mathcal{R}^*$ to construct a spanning tree $T_{\mathcal{R}^*}$ of $V_{\mathcal{R}^*}$.
3. Connect all the drone nodes $d \in V_{\mathcal{D}^*} \backslash \{v_0\}$ to depot v_0. This gives us a spanning tree T' of graph G.

With this procedure, we get

$$c^t(T') = c^t(T_{\mathcal{R}^*}) + \sum_{i=0}^{p} c^t(v_0, d_i).$$

Together, we have:

$$\begin{aligned}
c(\mathcal{R}^*, \mathcal{D}^*) &\geq \max\{c^t(\mathcal{R}^*), c^d(\mathcal{D}^*)\} \\
&\geq \max\{c^t(\mathcal{R}^*), \frac{1}{\alpha} c^t(\mathcal{D}^*)\} \\
&\geq \max\{c^t(T_{\mathcal{R}^*}), \frac{1}{\alpha} c^t(\mathcal{D}^*)\} \\
&\geq \max\{c^t(T_{\mathcal{R}^*}), \frac{2}{\alpha} \sum_{i=0}^{p} c^t(v_0, d_i)\} \\
&= \max\{c^t(T_{\mathcal{R}^*}), \frac{2}{\alpha}(c^t(T') - c^t(T_{\mathcal{R}*}))\}.
\end{aligned}$$

This is minimal if

$$c^t(T_{\mathcal{R}^*}) = \frac{2}{\alpha}(c^t(T') - c^t(T_{\mathcal{R}^*})) \Leftrightarrow c^t(T_{\mathcal{R}^*}) = \frac{2}{\alpha+2} c^t(T').$$

Together, this shows

$$c(\mathcal{R}^*, \mathcal{D}^*) \geq \frac{2}{2+\alpha} c^t(T') \geq \frac{2}{2+\alpha} c^t(T)$$

which proves our statement. □

10 References

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