exercise1 (Score: 14.0 / 14.0)

1. Test cell (Score: 2.0 / 2.0)
2. Test cell (Score: 1.0 / 1.0)
3. Test cell (Score: 1.0 / 1.0)
4. Test cell (Score: 1.0 / 1.0)
5. Test cell (Score: 1.0 / 1.0)
6. Test cell (Score: 3.0 / 3.0)
7. Test cell (Score: 2.0 / 2.0)
8. Task (Score: 3.0 / 3.0)

Lab 5

1. 提交作業之前，建議可以先點選上方工具列的**Kernel**，再選擇**Restart & Run All**，檢查一下是否程式跑起來都沒有問題，最後記得儲存。
2. 請先填上下方的姓名(name)及學號(stduent_id)再開始作答，例如：

```
name = "我的名字"
student_id= "B06201000"
```

3. 演算法的實作可以參考lab-5 (https://yuanyuyuan.github.io/itcm/lab-5.html), 有任何問題歡迎找助教詢問。
4. **Deadline: 12/11(Wed.)**

In [1]:

```
name = "林以翎"
student_id = "B06201024"
```

Exercise 1

An $m \times m$ *Hilbert matrix* H_m has entries $h_{ij} = 1/(i+j-1)$ for $1 \leq i,j \leq m$, and so it has the form

$$\left [

$$\begin{array}{cccc} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

\right].$$

In [2]:

```
import numpy as np
from numpy import linalg as LA
import matplotlib.pyplot as plt
```

Part 1

Generate the Hilbert matrix of order m, for $m = 2, 3, \ldots, 12$.

For each m, compute the condition number of H_m, *ie* , in p-norm for $p = 1$ and 2, and make a plot of the results.

Part 1.1

Define the function of Hilbert matrix

In [3]:

(Top)

```
def hilbert_matrix(m):
    '''
    Return:
        2D np.array, the Hildert Matrix of order m
    '''
    # ===== 請實做程式 =====
    return np.array([[1/(i + j + 1) for j in range(m)] for i in range(m)])
    # ====================
```

Test your function.

In [4]:

hilbert_matrix (Top)

```
print('H_2:\n', hilbert_matrix(2))
### BEGIN HIDDEN TESTS
assert np.mean(np.array(hilbert_matrix(3)) - np.array([[1, 1/2, 1/3], [1/2, 1/3, 1/4], [1/3, 1/4, 1/5]]))
< 1e-7
### END HIDDEN TESTS
```

```
H_2:
 [[1.         0.5       ]
 [0.5        0.33333333]]
```

Part 1.2

Collect all Hilbert matrices into the list `H_m` for m = 2, 3, ..., 12.

In [5]:

(Top)

```
H_m = []
# ===== 請實做程式 =====
H_m = [hilbert_matrix(m) for m in range(2,13)]
# ====================
```

Check your Hilbert matrix list.

In [6]:

hilbert_matrices (Top)

```
for i in range(len(H_m)):
    print('H_%d:' % (i+2))
    print(H_m[i])
    print()
### BEGIN HIDDEN TESTS
error = 0
for m in range(2, 13):
    error += LA.norm(hilbert_matrix(m) - np.array([[1/(i + j + 1) for j in range(m)] for i in range(m)]))
assert error < 1e-16
### END HIDDEN TESTS
```

```
H_2:
```

```
[[1.         0.5       ]
 [0.5        0.33333333]]

H_3:
[[1.         0.5        0.33333333]
 [0.5        0.33333333 0.25      ]
 [0.33333333 0.25       0.2       ]]

H_4:
[[1.         0.5        0.33333333 0.25      ]
 [0.5        0.33333333 0.25       0.2       ]
 [0.33333333 0.25       0.2        0.16666667]
 [0.25       0.2        0.16666667 0.14285714]]

H_5:
[[1.         0.5        0.33333333 0.25       0.2       ]
 [0.5        0.33333333 0.25       0.2        0.16666667]
 [0.33333333 0.25       0.2        0.16666667 0.14285714]
 [0.25       0.2        0.16666667 0.14285714 0.125     ]
 [0.2        0.16666667 0.14285714 0.125      0.11111111]]

H_6:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125     ]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1       ]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909]]

H_7:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125     ]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1       ]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308]]

H_8:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125     ]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1       ]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667]]

H_9:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1       ]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625    ]
```

```
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353]]

H_10:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111 0.1       ]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1        0.09090909]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909 0.08333333]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333 0.07692308]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308 0.07142857]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857 0.06666667]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667 0.0625    ]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625     0.05882353]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353 0.05555556]
 [0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667
  0.0625     0.05882353 0.05555556 0.05263158]]

H_11:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111 0.1        0.09090909]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1        0.09090909 0.08333333]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909 0.08333333 0.07692308]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333 0.07692308 0.07142857]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308 0.07142857 0.06666667]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857 0.06666667 0.0625    ]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667 0.0625     0.05882353]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625     0.05882353 0.05555556]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353 0.05555556 0.05263158]
 [0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667
  0.0625     0.05882353 0.05555556 0.05263158 0.05      ]
 [0.09090909 0.08333333 0.07692308 0.07142857 0.06666667 0.0625
  0.05882353 0.05555556 0.05263158 0.05       0.04761905]]

H_12:
[[1.         0.5        0.33333333 0.25       0.2        0.16666667
  0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333]
 [0.5        0.33333333 0.25       0.2        0.16666667 0.14285714
  0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308]
 [0.33333333 0.25       0.2        0.16666667 0.14285714 0.125
  0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857]
 [0.25       0.2        0.16666667 0.14285714 0.125      0.11111111
  0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667]
 [0.2        0.16666667 0.14285714 0.125      0.11111111 0.1
  0.09090909 0.08333333 0.07692308 0.07142857 0.06666667 0.0625    ]
 [0.16666667 0.14285714 0.125      0.11111111 0.1        0.09090909
  0.08333333 0.07692308 0.07142857 0.06666667 0.0625     0.05882353]
 [0.14285714 0.125      0.11111111 0.1        0.09090909 0.08333333
  0.07692308 0.07142857 0.06666667 0.0625     0.05882353 0.05555556]
 [0.125      0.11111111 0.1        0.09090909 0.08333333 0.07692308
  0.07142857 0.06666667 0.0625     0.05882353 0.05555556 0.05263158]
 [0.11111111 0.1        0.09090909 0.08333333 0.07692308 0.07142857
  0.06666667 0.0625     0.05882353 0.05555556 0.05263158 0.05      ]
 [0.1        0.09090909 0.08333333 0.07692308 0.07142857 0.06666667
  0.0625     0.05882353 0.05555556 0.05263158 0.05       0.04761905]
 [0.09090909 0.08333333 0.07692308 0.07142857 0.06666667 0.0625
  0.05882353 0.05555556 0.05263158 0.05       0.04761905 0.04545455]
 [0.08333333 0.07692308 0.07142857 0.06666667 0.0625     0.05882353
  0.05555556 0.05263158 0.05       0.04761905 0.04545455 0.04347826]]
```

Part 1.3

Plot the condition number of H_m for $m = 2, 3, \ldots, 12$

Collect all condition numbers in 1-norm of `H_m` into a list `one_norm`

In [7]:

(Top)

```
one_norm = []
# ===== 請實做程式 =====
for H in H_m:
    one_norm.append(LA.norm(H, 1) * LA.norm(LA.inv(H),1))
# ====================
```

In [8]:

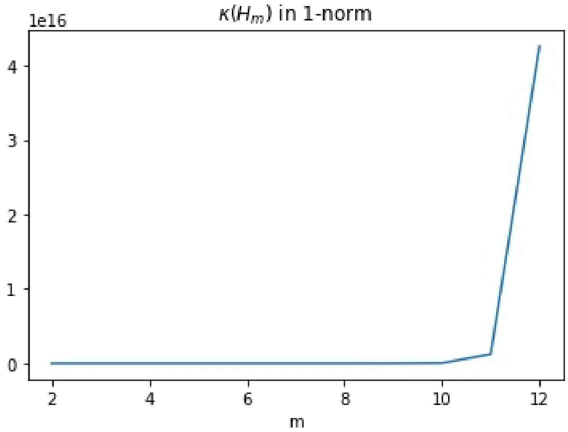
kappa_one_norm (Top)

```
print('one_norm:\n', one_norm)
### BEGIN HIDDEN TESTS
assert len(one_norm) == 11
### END HIDDEN TESTS
```

```
one_norm:
 [27.000000000000001, 748.0000000000027, 28375.00000000183, 943656.0000063396, 29070279.003768
865, 985194889.5765331, 33872792384.49069, 1099651993126.3992, 35356843615851.68, 12345355201
09080.0, 4.2559090171614424e+16]
```

In [9]:

```
plt.plot(range(2,13), one_norm)
plt.xlabel('m')
plt.title(r'$\kappa(H_m)$ in 1-norm')
plt.show()
```



Collect all condition numbers in 2-norm of `H_m` into a list `two_norm`

In [10]:

(Top)

```
two_norm = []
# ===== 請實做程式 =====
for H in H_m:
    two_norm.append(LA.norm(H, 2) * LA.norm(LA.inv(H),2))
# ====================
```

In [11]:

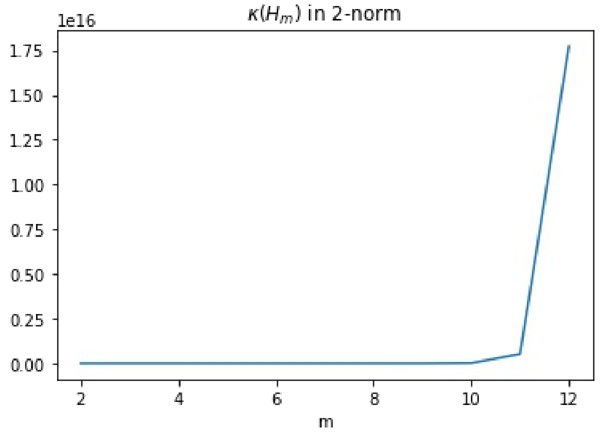
kappa_two_norm (Top)

```python
print('two_norm:\n', two_norm)
### BEGIN HIDDEN TESTS
assert len(two_norm) == 11
### END HIDDEN TESTS
```

```
two_norm:
 [19.281470067903975, 524.0567775860627, 15513.738738933602, 476607.2502457645, 14951058.6420
7495, 475367356.4677417, 15257576321.957926, 493153786012.41656, 16026019477413.041, 52342065
6648738.0, 1.7715806936210974e+16]
```

In [12]:

```python
plt.plot(range(2,13), two_norm)
plt.xlabel('m')
plt.title(r'$\kappa(H_m)$ in 2-norm')
plt.show()
```



Part 2

Now generate the m-vector $b_m = H_m x$ also, where x is the m-vector with all of its components equal to 1.

Use Gaussian elimination to solve the resulting linear system $H_m x = b_m$ with H_m and b given above, obtaining an approximate solution $\tilde{x}$.

Part 2.1

Construct the m-vector b_m for $m = 2, 3, \ldots, 12$. Store all 1D `np.array` b_m into the list `b_m`.

In [13]:

(Top)

```python
'''
Hint:
    b_m = ?
'''
# ===== 請實做程式 =====
b_m = [H @ np.ones(len(H)) for H in H_m]
# ====================
```

In [14]:

```
H_m[0]
```

Out[14]:

```
array([[1.        , 0.5       ],
       [0.5       , 0.33333333]])
```

Print b_m

In [15]:

```
        b_m                                                                    (Top)

for i in range(len(b_m)):
    print('b_%d:' % (i+2))
    print(b_m[i])
    print()
### BEGIN HIDDEN TESTS
error = 0
for m in range(2,13):
    error += LA.norm(b_m[m-2] - np.array([[1/(i + j + 1) for j in range(m)] for i in range(m)])@np.ones(m
))
assert error < 1e-16
### END HIDDEN TESTS
```

```
b_2:
[1.5        0.83333333]

b_3:
[1.83333333 1.08333333 0.78333333]

b_4:
[2.08333333 1.28333333 0.95       0.75952381]

b_5:
[2.28333333 1.45       1.09285714 0.88452381 0.74563492]

b_6:
[2.45       1.59285714 1.21785714 0.99563492 0.84563492 0.73654401]

b_7:
[2.59285714 1.71785714 1.32896825 1.09563492 0.93654401 0.81987734
 0.73013376]

b_8:
[2.71785714 1.82896825 1.42896825 1.18654401 1.01987734 0.89680042
 0.80156233 0.72537185]

b_9:
[2.82896825 1.92896825 1.51987734 1.26987734 1.09680042 0.96822899
 0.86822899 0.78787185 0.72169538]

b_10:
[2.92896825 2.01987734 1.60321068 1.34680042 1.16822899 1.03489566
 0.93072899 0.84669538 0.77725094 0.7187714 ]

b_11:
[3.01987734 2.10321068 1.68013376 1.41822899 1.23489566 1.09739566
 0.98955252 0.90225094 0.82988251 0.7687714  0.71639045]

b_12:
[3.10321068 2.18013376 1.75156233 1.48489566 1.29739566 1.15621919
 1.04510808 0.95488251 0.87988251 0.81639045 0.761845   0.71441417]
```

In [16]:

```
H_m[0]
```

Out[16]:

```
array([[1.        , 0.5       ],
       [0.5       , 0.33333333]])
```

Part 2.2

Implement the function of **Gaussian elimination**.

(Note that you need to implement it by hand, simply using some package functions is not allowed.)

In [17]:

(Top)

```
def gaussian_elimination(
    A,
    b
):
    '''
    Arguments:
        A : 2D np.array
        b : 1D np.array

    Return:
        x : 1D np.array, solution to Ax=b
    '''

    # ===== 請實做程式 =====
    U = np.copy(A)
    m = len(A)
    x = np.zeros(m)
    new_b = np.copy(b)

    for k in range(m-1):
        L = np.diag(np.ones(m))
        for j in range(k + 1, m):
            L[j, k] = -U[j, k] / U[k, k]
            U[j, k:] = U[j, k:] + L[j, k]*U[k, k:]
            new_b[j] = new_b[j] + L[j, k]*new_b[k]

    for i in range(m):
        if i == 0:
            x[-1] = new_b[-1] / U[-1, -1]
        else:
            x[-1-i] = (new_b[-1-i] - U[-1-i, -i:]@x[-i:]) / U[-1-i, -1-i]

    return x
    # ====================
```

Store all approximate solutions $\tilde{x}$ of H_m into a list `x_m` for $m = 2, 3, \ldots, 12$

In [18]:

```
x_m = []
for i in range(len(H_m)):
    x = gaussian_elimination(H_m[i], b_m[i])
    x_m.append(x)
```

Part 3

Investigate the error behavior of the computed solution $\tilde{x}$.

(i) Compute the ∞-norm of the residual $r = b - H_m\tilde{x}$.

(ii) Compute the error $\delta x = \tilde{x} - x$, where x is the vector of all ones.

(iii) How large can you take m before there is no significant digits in the solution ?

Part 3.1

Compute the ∞-norm of the residual $r_m = b_m - H_m\tilde{x}$ for $m = 2, 3, \ldots, 12$. And store the values into the list `r_m` .

In [19]:

(Top)

```python
r_m = []
# ===== 請實做程式 =====
for i in range(len(H_m)):
    r_m.append(LA.norm(b_m[i] - H_m[i]@x_m[i], np.inf))
# ====================
```

In [20]:

infty_norm (Top)

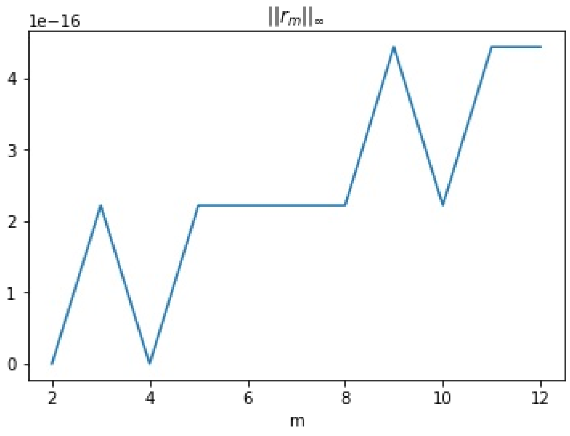
```python
print('r_m:\n', r_m)
### BEGIN HIDDEN TESTS
assert np.sum(r_m) < 1e-12
### END HIDDEN TESTS
```

```
r_m:
 [0.0, 2.220446049250313e-16, 0.0, 2.220446049250313e-16, 2.220446049250313e-16, 2.2204460492
50313e-16, 2.220446049250313e-16, 4.440892098500626e-16, 2.220446049250313e-16, 4.44089209850
0626e-16, 4.440892098500626e-16]
```

Plot the figure of the ∞-norm of the residual for $m = 2, 3, \ldots, 12$

In [21]:

```python
plt.plot(range(2,13), r_m)
plt.xlabel('m')
plt.title(r'$||r_m||_\infty$')
plt.show()
```



Part 3.2

Compute the error $\delta x = \tilde{x} - x$, where x is the vector of all ones. And store the values into the list `delta_x` .

In [22]:

(Top)

```python
delta_x = []
# ===== 請實做程式 =====
for x in x_m:
    delta_x.append(x - np.ones(len(x)))
# ====================
```

Collect all errors δx in 2-norm into the list `delta_x_two_norm` for $m = 2, 3, \ldots, 12$

In [23]:

(Top)

```
delta_x_two_norm = []
# ===== 請實做程式 =====
for delta in delta_x:
    delta_x_two_norm.append(LA.norm(delta, 2))
# ====================
```

In [24]:

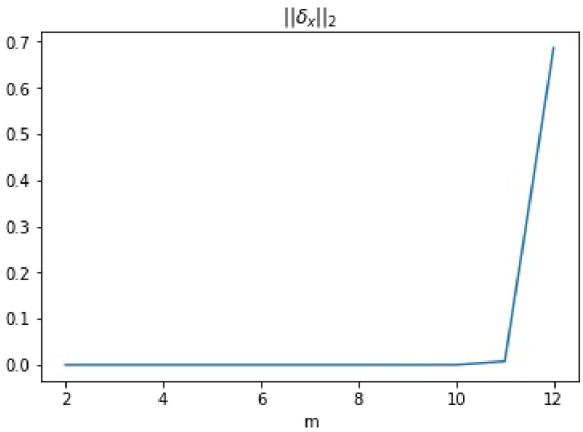
delta_x_two_norm (Top)

```
print('delta_x_two_norm =', delta_x_two_norm)
### BEGIN HIDDEN TESTS
assert (len(delta_x_two_norm) == 11) and (np.mean(delta_x_two_norm) <= 0.1)
### END HIDDEN TESTS
```

```
delta_x_two_norm = [8.005932084973442e-16, 1.762179615616027e-14, 1.531093756529614e-13, 3.47
26989976779822e-12, 3.380198688368993e-10, 9.687978241052163e-09, 4.244691764807721e-07, 1.73
93602379023114e-07, 0.0001452735113877698, 0.007757231295806999, 0.6867691718060317]
```

In [25]:

```
plt.plot(range(2,13), delta_x_two_norm)
plt.xlabel('m')
plt.title(r'$||\delta_x||_2$')
plt.show()
```



(Top)

Part 3.3

How large can you take m before there is no significant digits in the solution ?

By the graph, for $m \leq 11$, there is no significant digits in the solution.