The Adoption of Blockchain-Based Decentralized Exchanges*

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Abstract

We show that pricing curves implemented by automated market makers allow arbitrageurs to extract profits from liquidity providers. This raises the cost of liquidity provision, and may lead to a liquidity freeze if token pairs are too volatile. A more convex pricing curve results in higher price impact, which inhibits arbitrage but also reduces investors' surplus. The convexity of the socially optimal pricing curve is higher for more volatile token pairs. Our empirical analysis reveals that gas price volatility is 60% lower for stable-coin pairs, and that deposit inflows are negatively correlated with volatility, and positively correlated with trading volume.

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1 Introduction

Since the emergence of Bitcoin in 2008, practitioners and academics have argued that financial innovations such as asset tokenization and decentralized ledgers, along with the backbone blockchain technology, will disrupt traditional financial services (see, e.g., Campbell (2016), Yermack (2017), Cong and He (2019), Chiu and Koeppl (2019), Gan, Tsoukalas, and Netessine (2021)).

A prominent class of financial services commonly referred to as decentralized finance (abbreviated through DeFi) has emerged in the mid of 2020. DeFi utilizes open-source smart contracts to provide services which typically rely on centralized financial intermediaries. Among DeFi services, decentralized exchanges (DEXes) account for nearly half of the entire DeFi market capitalization, and for more than 10% of the total spot crypto trading volume, as of December 2021 (see CoinGecko (2022)). Rather than utilizing order books and relying on centralized intermediaries, most DEXes make the market through an Automated Market Maker (AMM) smart contract.

AMMs replace competing market makers in a traditional exchange trading venue with collaborating liquidity providers. They create a new market structure based on liquidity pools, where any token holder can become a liquidity provider and own a share of the pool by depositing their tokens. Liquidity providers do not need to set prices, because pricing schedules are determined by pre-coded pricing curves, also referred to as "bonding curves". On the one hand, pooling simplifies the process of liquidity provision and allows AMMs to crowd-source liquidity from token holders. Traders do not need to be paired to complete a transaction, but they gain immediate access to liquidity by trading against the pooled deposits. On the other hand, relying on pre-determined curves to set the whole pricing schedule may appear rigid and expose liquidity providers to the risk of arbitrage. This leads to fundamental questions: under what circumstances would liquidity providers adopt the pools? How does the choice of pricing curves affect welfare, trading, and liquidity provision incentives?

In this paper, we argue that the current design of AMMs exposes liquidity providers to arbitrage losses, which raises the cost of liquidity provision. Liquidity providers are willing to supply liquidity only if the token exchange rate is not too volatile. We identify the convexity of the pricing curve as the key determinant of price impact, which in turn, affects trading and liquidity provision incentives. We design a socially optimal pricing curve that balances between price impact from trades and arbitrage losses incurred by liquidity providers.

Our model of liquidity provision consists of a DEX with a pre-specified pricing function, liquidity providers, investors, and arbitrageurs. The AMM supports the trading of two tokens, whose exchange rate is decided by a curve pinned down by the pricing function. The execution priority of transactions does not depend on their order of arrival, but rather on the "tip" amount, also referred to as the "gas fee", offered to validators of the underlying blockchain. Liquidity providers decide whether or not to deposit their tokens. Investors trade and extract benefits stemming from holding certain types of tokens. Because of exogenous shocks to fundamentals or price impact caused by investors' trades, the spot token exchange rate may deviate from its fundamental value. Arbitrageurs compete to exploit this price deviation, while liquidity providers submit exit orders to withdraw their tokens. Liquidity providers share the loss imposed by an arbitrage, while the arbitrageur whose order is executed first extracts the full profit from it. As a result, arbitrageurs have an incentive to bid a higher gas fee than liquidity providers. Then the arbitrage order will be executed prior to exit orders, effectively leading to a transfer of wealth from liquidity providers to arbitrageurs.

We show that if the token exchange rate is too volatile, the expected arbitrage loss suffered by liquidity providers is higher than the fee revenue collected from investors' trades. This leads to a "*liquidity freeze*" where liquidity providers do not deposit their tokens in the AMM, and no trade occurs.

As deposits in the AMM grow, the arbitrage becomes more profitable. This, in turn, intensifies gas fee bidding competition between arbitrageurs, and leads to an increase in both levels and volatility of gas fees. Such a surge imposes negative externalities on other applications operating on the same blockchain, because it creates high execution delays and prevents users from transacting efficiently.

Price impact increases with the convexity of the pricing curve. As a result, a higher convexity leads to smaller traded quantities, both for arbitrageurs and investors. This, in turn, reduces profits that arbitrageurs extract from the pool and the surplus captured by investors. Liquidity providers' incentives to deposit increase with the convexity of the curve if the convexity is not too high. If convexity exceeds a threshold, the benefit from reduced arbitrage losses is smaller than the cost of reduced fee revenues from investors' trades.

We construct a socially optimal pricing curve that maximizes aggregate welfare, which amounts to maximizing the aggregate benefit from investors' trades. Gains from trades depend on the depth of the liquidity pool, and on price impact. If the convexity is high, the price impact is high, so the investors trade small quantities and extract little benefits from trades. If the convexity is low, price impact is low, but the pool would be shallow because of the limited liquidity provision caused by the high expected losses from arbitrage. The convexity of the socially optimal pricing curve strikes the best tradeoff between these opposite economic forces. It is higher for more volatile token pairs, where the arbitrage problem is more severe.

We provide empirical support to the main implications of our model using transaction-level data from Uniswap V2 and Sushiswap AMMs, as well as token price data from Binance. We identify deposit, withdrawal, and swap orders from the raw transaction history of 12 AMMs offering the most actively traded token pairs, during the period April 16, 2021, through December 31, 2021. Consistently with our model implications, we find that token exchange rate volatility is negatively correlated with deposit inflow rate, while trading volume is positively correlated with it. An increase in weekly log spot rate volatility by 0.03 (around one-standard deviation) decreases the weekly deposit flow rate by 0.05 (around 30% of the standard deviation). Moreover, an increase in trading volume by one standard deviation increases the weekly deposit flow rate by around 25% of the standard deviation.

We exploit the segmentation of AMMs and divide token pairs into two groups: "stable pairs" and "non-stable pairs". "Stable pairs" consist of two stable coins, each pegged to one US dollar, and thus having low token exchange rate volatility compared to "unstable pairs". We find that gas fees for transactions of "stable pairs" are 8% lower than those of "non-stable pairs", and that the weekly volatility of gas fees for "stable pairs" is about 60% lower than for "non-stable pairs". Hence, consistently with our model implications, the (negative) externalities imposed by "stable pairs" on other applications using the Ethereum blockchain are smaller.

Literature Review. Our paper contributes to the rapidly growing literature on DeFi. We refer to Harvey, Ramachandran, and Santoro (2021) for an excellent and comprehensive survey.

There exist a few complementary studies to ours, which examine AMMs from different perspectives. Park (2021) shows that front-running arbitrage always exists regardless of the AMM pricing function. He then shows that uniform pricing limits the arbitrageurs' profit from frontrunning, but may incentivize order splitting. The frontrunning arbitrage in Park (2021) arises because of public observability of transactions in the blockchain mempool, and the victims are liquidity demanders whose transactions are pending. In our model instead, arbitrage arises due to deviations between the spot exchange rate and the fundamental rate, and imposes a loss on liquidity providers. Lehar and Parlour (2021) compare returns from liquidity provision at a centralized exchange with those at an AMM which utilizes a constant product function. They also document empirically that token prices at Uniswap closely track those at Binance where centralized limit order books are used. Aoyagi and Ito (2021) study how market prices and traders' behavior are affected by the coexistence of centralized and decentralized exchanges. They show that liquidity in DEXes complements that in CEXes, because informed and liquidity traders have different preferences for submission venues. We argue that arbitrage problems can arise at AMM even without any asymmetric information on fundamentals, because arbitrageurs can exploit liquidity providers even on public information. This stands in contrast with the studies of Lehar and Parlour (2021) and Aoyagi and Ito (2021), where asymmetric information is the main friction. Barbon and Ranaldo (2021) compare liquidity and price efficiency of CEXes and DEXes, and argue that AMMs are likely to become a competitive alternative to CEXes. A significant contribution of our work relative to all these studies is to highlight why the convexity of pricing curve is a key design characteristic of an AMM.

Our results also add to the existing literature on crypto trading. Griffins and Shams (2020), Cong et al. (2020), and Li, Shin, and Wang (2018) analyze trading activities and price manipulations at centralized crypto exchanges. We analyze trading patterns and liquidity provision at decentralized exchanges, whose trading volume has been growing steadily.

A related branch of literature studies blockchain as a platform for crypto transactions. Noticeable studies in this direction include Huberman, Leshno, and Moallemi (2021), Sockin and Xiong (2020), Easley, O'Hara, and Basu (2019), Pagnotta (2021), Schilling and Uhlig (2019), Athey et al. (2016), Cong, Li, and Wang (2020), and Irresberger et al. (2020). Unlike these works, which view blockchain as the technology underlying payment systems, we treat blockchain as an infrastructure for DEXes.

More broadly, our paper is related to market microstructure literature on "sniping risk" of stale orders in continuous limit-order-book exchanges (e.g., Foucault (1999), Budish, Cramton, and Shim (2015), Menkveld and Zoican (2017)). In Budish, Cramton, and Shim (2015), symmetrically observed public information creates arbitrage opportunities, as in our paper. However, in their limit-order-book setting, arbitrage opportunities arise because of the continuous-time serial processing, whereas in our setting they are a consequence of liquidity crowdsourcing and of the blockchain transaction execution mechanism.

The rest of the paper is organized as follows. Section 2 provides institutional details of DEXes. Section 3 develops the game theoretical model. Section 4 solves for the subgame perfect equilibrium of the game, and analyzes its economic implications. Section 5 studies the pricing function design of an AMM. Section 6 provides empirical support for our main model implications. Section 7 offers concluding remarks. Technical proofs are relegated to an Appendix.

2 Decentralized Exchanges

DEXes take in, manage, and execute orders through a blockchain-based platform (typically Ethereum). Rather than sending the orders directly to the exchange, users submit them to the blockchain and bid a transaction fee (the gas price and gas limit in the Ethereum network).

The validator that constructs the very next block will prioritize orders based on the bid gas fees, from highest to lowest. Since each block has a maximum size, the number of orders a validator can include in a single block is limited. Hence, an order with a too low gas fee may need to wait longer before being confirmed and executed.

Most DEXes are in the form of automated market makers (AMMs), which develop a new market structure called liquidity pool.[1] A liquidity pool allows for a direct exchange of two tokens, say A and B, instead of first selling tokens A for fiat currency and then purchasing tokens B using proceeds from the sale.[2] Pricing schedules are determined through an exogenously specified pricing curve, often referred to as the "bonding curve", in the sense that exchange rates are deterministic functions of the supply of tokens in the pool. This stands in contrast with centralized limit-order books, where the pricing schedule is endogenous and depends on limit orders strategically submitted by market makers.

We describe the mechanics of liquidity pools, which implement the "constant product function" (see Adams (2020) for a detailed treatment). This pricing function is implemented by the most prominent AMMs, including Uniswap V2, SushiSwap, and PancakeSwap. Assume a liquidity pool manages the exchange of two tokens, A and B, where each token A is worth p_A and each token B is worth p_B. Anyone who owns both tokens A and B can become a liquidity provider by depositing an equivalent value of each underlying token in the AMM and in return, receiving pool tokens which prove his share of the AMM. For example, if the current reserve consists of 10 tokens A and 5 tokens B, where the value of one A token is 1 dollar and that of one B token is 2 dollars, then the liquidity provider must deposit tokens A and B in the ratio 2:1. After depositing 10 tokens A and 5 tokens B, the liquidity provider can claim pool tokens that account for half of the total tokens in the liquidity pool. The provider can exit the liquidity pool by trading in his pool tokens, and receiving his share of the liquidity reserve in

[1] There exist DEXs that are not AMMs, such as SPEEDEX designed by Ramseyer, Goel, and Mazières (2021). AMMs account for at least 80% of total DEX trading volumes as of Dec 2021, according to the reports of CoinGecko (2022). Prominent AMMs are Uniswap, Sushiswap, Curve, and PancakeSwap.

[2] Despite liquidity pools with a single token pair being the most common, there exist pools that allow for multiple token pairs (e.g. Balancer AMMs). The trading mechanism of pools with more than two tokens is almost identical to that of pools with a single pair of tokens.

the AMM. For instance, if the liquidity pool contains 25 tokens A and 9 tokens B when the liquidity provider exits and no one deposits after him, then he receives 12.5 tokens A and 4.5 tokens B.

Suppose an investor arrives and wants to exchange tokens A for tokens B. To complete such a trade, the investor does not need to be matched with a counterparty who is willing to exchange tokens B for tokens A. Rather, she directly interacts with the AMM and gains access to tokens in the liquidity pool. By submitting a swap order, she deposits an amount Δ_A of tokens A and withdraws an amount Δ_B of tokens B from the pool. The exchange rate is determined by a constant product function: the quantities Δ_A and Δ_B have to satisfy the invariance rule $(20 + \Delta_A)(10 - \Delta_B) = 20 * 10 = 200$. Assume the initial liquidity reserve in the AMM consists of d_A tokens A and d_B tokens B. Then the trade needs to satisfy the condition $(d_A + \Delta_A)(d_B - \Delta_B) = d_A d_B$. In addition to the amount Δ_A of tokens A exchanged, the investor must pay a trading fee $f\Delta_A$. Most of this fee is added to the liquidity pool in the form of additional tokens A.[3] Hence, the trading fee increases both the total liquidity reserve of the AMM and the worth of the share of each liquidity provider. For Uniswap V2 and Sushiswap, this trading fee is currently set to 0.3% of the tokens that investors trade in.

The relationship $(d_A + \Delta_A)(d_B - \Delta_B) = d_A d_B$ can be generalized to $F(d_A + \Delta_A, d_B - \Delta_B) = F(d_A, d_B)$, where $F(x, y)$ is an arbitrary pricing curve referred to as the pricing function. The constant product function become a special case, where $F(x, y) = xy$. We refer to Xu et al. (2021) for an overview of pricing functions used by a wide range of AMMs.

3 Model

The timeline consists of 3 periods indexed by t, $t = 1, 2, 3$. There are three types of agents: liquidity providers, arbitrageurs, and investors. Each agent has access to two types of tokens, A and B, and discounts future utility at zero rate.

[3] In practice, a small portion of the trading fees may be collected by the underlying platform. Our model can be extended to incorporate this characteristic if the trading fee collected by the liquidity provider is multiplied by a constant term. Because this extension would not qualitatively change our results, we opted for leaving it out in the current model.

Token Fundamentals. The fundamental values of tokens A and B at the end of period t are denoted, respectively, as $p_A^{(t)}$ and $p_B^{(t)}$, and they are publicly observable information for all agents. The tokens can be costlessly liquidated at their fundamental values at the end of period 3 for a single consumption good which is used as a numeraire. Alternatively, the tokens can be used directly on the corresponding platforms[4] A and B. We will refer to the ratio of token prices, $\frac{p_A^{(t)}}{p_B^{(t)}}$, as the fundamental exchange rate.

The AMM Pricing Function. There exists a smart contract built on a blockchain that implements a twice continuously differentiable pricing function $F(x, y) : \mathbb{R}^2 \to \mathbb{R}$ to determine the exchange rate for any trade. If the liquidity reserve of the AMM consists of an amount d_A of tokens A and d_B of tokens B, any trade that exchanges Δ_A tokens A for Δ_B tokens B needs to satisfy the relation $F(d_A + \Delta_A, d_B - \Delta_B) = F(d_A, d_B),\ 0 \leq \Delta_B \leq d_B$. An additional amount $f\Delta_A$ of tokens A needs to be added to the AMM as a trading fee. The invariance relationship pins down the pricing curve. When the trading size is infinitesimal, the exchange rate is $\lim_{\Delta_A \to 0} \frac{\Delta_A}{\Delta_B} = \frac{F_x(d_A, d_B)}{F_y(d_A, d_B)}$ which we refer to as the spot exchange rate. We impose the following assumption on the pricing function.

Assumption 1. *The function $F(x, y) : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following properties:*

1. $F_x > 0, F_y > 0$.

2. $F_{xx} < 0, F_{yy} < 0, F_{xy} > 0$.

3. $\forall c \geq 0, c^l F(x, y) = F(cx, cy)$ *for some* $l > 0$.

4. $\lim_{x \to 0} \frac{F_x}{F_y} = \infty, \lim_{x \to \infty} \frac{F_x}{F_y} = 0,\ \lim_{y \to 0} \frac{F_x}{F_y} = 0, \lim_{y \to \infty} \frac{F_x}{F_y} = \infty$.

The first condition ensures that a positive amount of tokens A can be exchanged for a positive amount of tokens B. The second condition guarantees that the pricing curve $F(x, y) = C$, where C is a constant, is convex. This implies that if the demand for tokens A goes up, the exchange

[4]The platform that issues token B20 is Metapurse. On this platform, one can use token B20 to claim ownership of NFT collectibles. The Ethereum network is a platform, where one can use ETH tokens as a cryptocurrency to exchange for goods and other tokens, or to run applications.

rate used to convert from tokens B to tokens A correspondingly increases. Symmetrically, if the demand for tokens B increases, a higher amount of tokens A is needed to exchange for a single token B. The third condition guarantees that the spot exchange rate does not change if the ratio of deposited tokens stays the same. The last condition ensures that the AMM supports trading for any token exchange rate in the interval $[0, \infty)$. These four conditions can be verified to hold for the majority of existing AMMs (see, for instance, Xu et al. (2021)).

The Blockchain Execution Mechanism. All agents submit their transactions to the blockchain, and bid a gas fee[5] for each transaction. At the end of each period, all transactions submitted in that period are executed in decreasing order of gas fees. If two transactions bid the same gas fee, their processing order is random.

3.1 Liquidity Providers

There are $n > 1$ liquidity providers, indexed by $\mathcal{N} = \{1, ..., n\}$, each endowed with a positive amount of tokens A and B. We use $e_{A_i}^{(0)} > 0$ and $e_{B_i}^{(0)} > 0$ to denote the initial endowment of tokens A and B, respectively, for liquidity provider $i \in \mathcal{N}$. The aggregate initial endowments of tokens A and B for liquidity providers are, respectively, $\sum_{i=1}^{n} e_{A_i}^{(0)} = e_A^{(0)}$ and $\sum_{i=1}^{n} e_{B_i}^{(0)} = e_B^{(0)}$.

In period 1, liquidity providers decide the amount of tokens A and B to deposit in the pool. We denote the amount of tokens A and B deposited by liquidity provider i in period 1 by $d_{A_i}^{(1)}$ and $d_{B_i}^{(1)}$ respectively. The total amounts of tokens A and B at the end of period t are denoted respectively by $d_A^{(t)}$ and $d_B^{(t)}$. The AMM requires tokens to be deposited at the fundamental exchange rate[6]. After making his deposit in period 1, liquidity provider i claims a

[5]For Ethereum, the gas fee is equal to the gas price multiplied by the gas amount needed to execute the transaction. The gas price is defined as the amount of ETH paid per unit of gas used, and the gas amount measures the computational resources needed to execute a transaction on Ethereum. The unit of gas price is Gwei, that is, 10^{-9} ETH token. Transactions with higher gas prices are confirmed first, because validators prioritize them to maximize their gas fee revenue. Since transactions executed on the same AMM use similar gas amounts, transactions with higher gas prices have higher total gas fees. In our model, we assume that the agents directly submit a gas fee instead of a gas price.

[6]This is equivalent to $\left.\frac{F_x}{F_y}\right|_{(x,y)=(d_A^{(1)}, d_B^{(1)})} = \frac{p_A^{(1)}}{p_B^{(1)}}$. For example, an AMM which utilizes a constant product function requires the deposited tokens to have equal value, i.e., $d_A^{(2)} p_A^{(2)} = d_B^{(2)} p_B^{(2)}$.

share $w_i = \frac{d_{A_i}^{(1)}}{d_A^{(1)}} = \frac{d_{B_i}^{(1)}}{d_B^{(1)}}$ of the AMM. Liquidity provider i chooses the amount of deposited tokens to maximize the expected value of his portfolio at the end of period 3, i.e.,

$$w_i \mathbb{E}\left[p_A^{(3)} d_A^{(3)} + p_B^{(3)} d_B^{(3)}\right] + \mathbb{E}\left[p_A^{(3)}(e_{A_i}^{(0)} - d_{A_i}^{(1)}) + p_B^{(3)}(e_{B_i}^{(0)} - d_{B_i}^{(1)})\right] - \sum_{t\in\{1,3\}} g_{(lp,i)}^{(t)},$$

where $g_{(lp,i)}^{(t)}$ is the gas fee bid of liquidity provider i in period t. We also impose the following tie-breaking rule:

Assumption 2. *The liquidity providers deposit their tokens if they are indifferent whether or not to deposit.*

3.2 Investors' Arrival and Token Value Shocks

In period 2, one of the following three mutually exclusive and collectively exhaustive events occurs: "an investor arrives", "tokens are hit by price shocks", and "no shock occurs and no investor arrives".

Investors' Arrival. With probability κ_I, an investor arrives at the AMM. The investor is characterized by an intrinsic type, that is "type A" or "type B". A "type A" investor extracts a benefit of $(1+\alpha)p_A^{(1)}$ from using one token A on its corresponding platform in period 2. A "type A" investor does not extract a benefit from platform B, and thus she only receives $p_B^{(1)}$ for each token B. Symmetrically, a "type B" investor receives $(1+\alpha)p_B^{(1)}$ for each token B and $p_A^{(1)}$ for each token A.[7] The arriving investor is of "type A" or of "type B" with equal probability. The investor chooses the traded quantities of tokens A and B, $(\Delta Q_A^{(2)}, \Delta Q_B^{(2)})$, and the gas fee bid $g_{(inv)}^{(2)}$ to maximize her total surplus from the transaction, that is, $S_A = (1+\alpha)p_A^{(1)}\Delta Q_A^{(2)} + p_B^{(1)}\Delta Q_B^{(2)} - g_{(inv)}^{(2)}$ for "type A" investors, and $S_B = (1+\alpha)p_B^{(1)}\Delta Q_B^{(2)} + p_A^{(1)}\Delta Q_A^{(2)} - g_{(inv)}^{(2)}$ for "type B" investors.

[7] If an investor prefers to gain exposure to NFT collectibles rather than stable currencies, then she prefers token B20 over stable coins such as USDT and USDC. If an investor needs to run applications on the Ethereum blockchain, then she extracts a benefit from holding ETH tokens. If an investor prefers tokens with low volatility and high liquidity, then she would extract benefit from holding stable coins.

Token Value Shock. In period 2, the fundamental values of tokens A and B may be hit by exogenous shocks ζ_A, ζ_B:

$$\begin{aligned} &\zeta_A \sim Bern(\kappa_A), \zeta_B \sim Bern(\kappa_B), \zeta_A \perp \zeta_B, \kappa_A \geq \kappa_B, \\ &p_i^{(2)} = (1 + \beta\zeta_i)p_i^{(1)}, i = A, B. \end{aligned} \tag{1}$$

That is, with probability $0 < \kappa_A < 1$, the shock ζ_A occurs and the value of tokens A increases by β; with probability $0 < \kappa_B < 1$, the shock ζ_B occurs and the value of tokens B increases by β. It can be easily seen that the standard deviation of the fundamental exchange rate increases in β.

3.3 Arbitrage and Token Withdrawal

An investor's arrival or a token value shock in period 2 make the spot exchange rate $\frac{F_x(d_A^{(2)}, d_B^{(2)})}{F_y(d_A^{(2)}, d_B^{(2)})}$ deviate from its fundamental value $\frac{p_A^{(2)}}{p_B^{(2)}}$. This deviation presents a profitable opportunity for arbitrageurs. In period 3, arbitrageurs submit orders, while liquidity providers submit withdrawal orders.

Stale Price Arbitrage Opportunity. A token value shock presents an opportunity for arbitrageurs. To see this, assume a shock hits either token A or token B, and the fundamental exchange rate $\frac{p_A^{(2)}}{p_B^{(2)}}$ changes. The spot exchange rate at the AMM no longer reflects the fundamentals and becomes "stale". For instance, assume that the shock hits token B, and its fundamental value rises from $p_B^{(1)}$ to $p_B^{(2)} = (1+\beta)p_B^{(1)}$. Arbitrageurs may then find it profitable to exchange tokens A for B. We refer to this trading opportunity as the stale price arbitrage.

Reverse Trade Arbitrage Opportunity. Deviation of AMM prices from fundamentals may also be caused by the price impact of an investor's trade, which leads to a reverse trade arbitrage opportunity. Suppose an investor trades in period 2, the spot exchange rate at the AMM changes due to price impact while the fundamental exchange rate remains the same. To see this, assume a "type A" investor arrives at $t = 2$. After the investor swaps tokens B for tokens A, the amount of tokens A in the AMM decreases while the amount of tokens B increases. As a result, the spot rate at which tokens A are exchanged for tokens B increases and becomes higher than the fundamental exchange rate:

$$\left.\frac{F_x}{F_y}\right|_{(x,y)=(d_A^{(2)},d_B^{(2)})} > \left.\frac{F_x}{F_y}\right|_{(x,y)=(d_A^{(1)},d_B^{(1)})} = \frac{p_A^{(1)}}{p_B^{(1)}}.$$

If the deviation is large, arbitrageurs may find it profitable to trade in the opposite direction of the investor. Hence, the price impact imposed by the initial trade of the investor is only transient, because the price change caused by her trade is reversed by the opposite trade executed by the arbitrageur.[8] In this way, investors subsidize not only liquidity providers, but also arbitrageurs who benefit from the price impact caused by investors' trades.

Arbitrageurs. There are two arbitrageurs indexed by $\mathcal{M} = \{1, 2\}$. They do not use tokens on either platform, and only exchange tokens for consumption good. In period 3, arbitrageurs take advantage of price deviations between fundamental and AMM exchange rates, and trade to maximize profits. If a price deviation occurs, arbitrageur $j, j = 1, 2$, submits its arbitrage order $(\Delta q_{A_j}^{(3)}, \Delta q_{B_j}^{(3)})$ and bid a gas fee $g_{(arb,j)}^{(3)}$.

Definition 1. *An order $(\Delta q_{A_j}^{(3)}, \Delta q_{B_j}^{(3)})$ is an arbitrage order if*

1. *it satisfies the invariance relationship, $F(d_A^{(2)}, d_B^{(2)}) = F(d_A^{(2)} - \Delta q_A^{(3)}, d_B^{(2)} - \Delta q_B^{(3)})$;*

2. *it yields a positive profit when swapping a token A for a token B, i.e., $p_A^{(2)}(1+f)\Delta q_A^{(3)} + p_B^{(2)}\Delta q_B^{(3)} > 0$, or vice-versa when swapping a token B for a token A, i.e., $p_A^{(2)}\Delta q_A^{(3)} + p_B^{(2)}(1+f)\Delta q_B^{(3)} > 0$.*

An arbitrage order is optimal if the profit is maximized.

[8] The exchange rate needs not to be entirely reversed to its fair value, i.e., to the rate before investor's trade. This is because the arbitrageur who "backruns" the investor's trade needs to pay trading fees.

If the order of arbitrageur j is confirmed before that of the other arbitrageur and of the liquidity providers, its payoff is

$$p_B^{(2)} \Delta q_{B_j}^{(3)} + p_A^{(2)} \Delta q_{A_j}^{(3)} - g_{(arb,j)}^{(3)}.$$

Without loss of generality, we assume that if the gas fees for two arbitrage orders are the same, each will be executed first with the same probability. Arbitrageurs can deploy a smart contract to avoid any downside risk[9] from execution. Thus, the payoff of an arbitrageur is zero if the other arbitrager has already exploited the opportunity or the liquidity provider has withdrawn the tokens, because the smart contract would terminate if the arbitrage opportunity no longer exists. In this case, the transaction would be deemed as failed, and the corresponding gas fee is close to zero because the total gas used is negligible.

Token Withdrawal. In period 3, liquidity provider i withdraws his tokens from the AMM by submitting an order and bidding a non-negative gas fee $g_{(lp,i)}^{(3)}$ with it. If the withdrawal order is executed, liquidity provider i pays the gas fee bid and receives an amount $d'_A w_i$ of tokens A and an amount $d'_B w_i$ of tokens B, where d'_A, d'_B are the total reserves in the AMM before any withdrawal. We recall that w_i is the share of reserves of liquidity provider i. Liquidity providers liquidate the tokens at their fundamental values after the withdrawal.

We visualize the timeline of the game in Figure 1.

3.4 Equilibrium

A strategy profile includes the deposit and gas fee bidding strategies of liquidity providers, the trading and gas fee bidding strategies of investors, and the trading and gas fee bidding strategies of arbitrageurs. A strategy profile is an equilibrium if all agents attain their maximum expected payoff and each agent is not strictly better off deviating from it. Our equilibrium concept is

[9] Alternatively, arbitrageurs can submit their transactions privately to validators (e.g. using Flashbots). In this case, the bidding for arbitrage execution would take place off-chain and follow a seal-bid first price auction where only the winning arbitrageur pays gas fees. If an arbitrageur submits through Flashbots and fails to execute before the opportunity is gone, its gas cost would be exactly zero. (See, Flashbots (2021)).

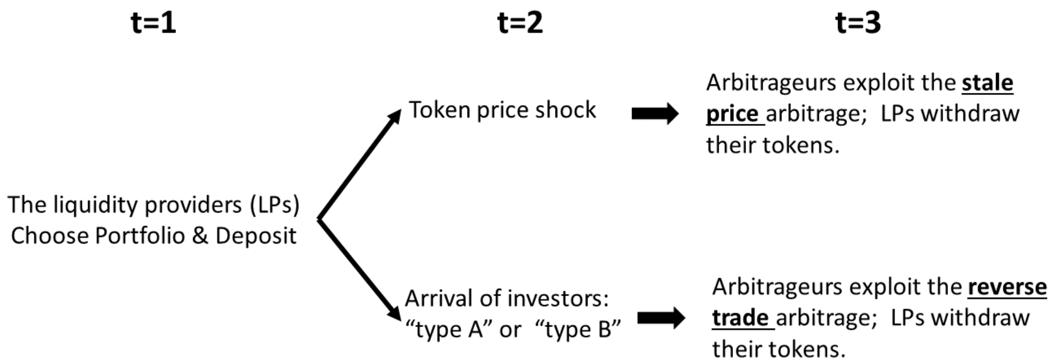

Figure 1: Timeline of The Model.

that of a *pure-strategy subgame perfect Nash equilibrium*.

4 When do Liquidity Providers Deposit?

We provide the conditions under which liquidity providers find it incentive-compatible to deposit tokens. In Section 4.1, we show that arbitrageurs can extract profits from liquidity providers. In Section 4.2, we analyze the subgame perfect equilibrium of the game and characterize the parameter region under which no token is deposited.

4.1 Identification of Arbitrage Opportunities

Arbitrage opportunities arise in period 3 if spot exchange rates deviate enough from fundamentals. As stated in the following lemma, a stale price arbitrage opportunity arises if one of the tokens is hit by a large enough price shock, and a reverse trade arbitrage opportunity arises if an investor trades a high enough quantity of tokens in the pool.

Lemma 1. *1. There exists a stale price arbitrage opportunity in period 3 if and only if a shock hits only one token and the value change of that token exceeds the trading fee, i.e., $\beta > f$. The optimal stale price arbitrage order $(\Delta q_A^{(3)*}, \Delta q_B^{(3)*})$ is unique, and if it executed, it generates a profit $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}) > 0$ that is increasing in β and decreasing in f.*

2. *There exists a reverse trade arbitrage opportunity in period 3 if and only if an investor arrives and trades enough to impose a sufficiently large price impact. The optimal reverse arbitrage order* $(\Delta q_A^{(3)*}, \Delta q_B^{(3)*})$ *is unique, and if its executed, it generates a profit* $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}) > 0$ *that is decreasing in* f.

Whether or not arbitrageurs are able to exploit the opportunity depends on their incentives to bid a higher gas fee than liquidity providers, and thus frontrun their withdrawal orders. The next proposition provides an affermative answer to this question.

Proposition 1. *Suppose there exists an arbitrage opportunity in period 3. Both arbitrageurs submit an optimal order and bid a gas fee in the amount equal to the profit from the optimal arbitrage order* $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. *One of the arbitrage orders is executed before all withdrawal orders of liquidity providers in period 3, and it yields a loss* $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ *for liquidity provider* $i, i = 1, 2, ..., n$.

If an optimal arbitrage order is executed, the revenue of this arbitrageur equals the aggregate loss in token value incurred by liquidity providers. Liquidity provider i incurs a token value loss in proportion to his share of the AMM, i.e., he loses the amount $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. To avoid the loss, the withdrawal order of this liquidity provider must be executed before the orders of both arbitrageurs. This means that the liquidity provider must pay a gas fee higher than the fees bid by the two arbitrageurs. Because of competition, the gas fee bid by either arbitrageur equals its gain $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ from the optimal arbitrage. Observe that it is not incentive-compatible for liquidity provider i to bid a gas fee higher than $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, because he would otherwise incur a cost higher than the loss from arbitrage. Hence, arbitrageurs always extract profits from liquidity providers when an opportunity arises.

Remark 1. *It is worth remarking that arbitrage losses suffered from liquidity providers in AMMs differ from those incurred by market makers in continuous limit-order book markets. Market makers are subject to a "sniping problem" for not being fast enough in the serial-process limit order book market, which leads to an arms race for speed (see, for instance, Budish,* *Cramton, and Shim (2015)).*[10] *In AMMs, the speed of information processing is less concerning for liquidity providers because transactions submitted to the blockchain are processed in discrete time. Rather, liquidity providers face an arbitrage problem because of two core properties of the AMM market design: liquidity pooling and the order-processing mechanism that prioritizes transactions with high gas fees.*

The arbitrage problem is not alleviated if an AMM pools multiple tokens together. This is in spite of anecdotal evidence suggesting that pooling multiple tokens may reduce arbitrage losses because of the higher diversification benefits. In Appendix A, we analyze how pooling three tokens increases the stale price arbitrage loss of liquidity providers.

4.2 The Adoption of Automated Market Makers

In this section, we examine the conditions under which it is incentive compatible for liquidity providers to deposit their tokens into the AMM. If none of the liquidity providers deposits their tokens in period 1, no trading activities can occur in that period. We refer to this market scenario as a "liquidity freeze".

Theorem 2. *For any $\alpha, \beta, f, \kappa_I, \kappa_A, \kappa_B$, there exists a unique subgame perfect equilibrium. A "liquidity freeze" occurs in equilibrium if and only if the fundamental exchange rate volatility is sufficiently high, i.e., $\beta > \overline{\beta_{frz}}$, where $\overline{\beta_{frz}} \in [0, +\infty)$. Moreover, the threshold $\overline{\beta_{frz}}$ is increasing in α and κ_I.*

If the fundamental token exchange rate is sufficiently volatile (β is sufficiently large), the liquidity providers do not get enough revenue from investors' trading fees to cover the expected arbitrage losses. This in turn leads to a "liquidity freeze", because liquidity providers are better off not depositing. The comparative static results are intuitive. If investors' benefits of holding tokens go up (α increases) or the arrival probability of an investor goes up (κ_I increases), the expected trading volume increases and thus liquidity providers collect a higher trading

[10] At DEXs, arbitrageurs are typically bots which scan token prices and submit arbitrage orders. Liquidity providers are token holders seeking for higher yields. At limit order books, both market makers and arbitrageurs who "snipe" stale orders are high-frequency firms.

fee. Hence, their incentives to deposit tokens in the AMM are stronger, and the threshold for "liquidity freeze" becomes higher.

The above result suggests that AMMs will be adopted for pairs whose token values are not too volatile, such as stable coins. Moreover, adoption is higher for tokens with large liquidity demand. These tokens attract investors, who in turn generate high trading volumes. Liquidity providers can then earn large trading fees and be compensated for the arbitrage problem they face.

Proposition 3. *Suppose liquidity providers deposit $d_A^{(1)}$ and $d_B^{(1)}$ amount of tokens A and B in period 1. In the subgame originating from period 2, the expectation and variance of gas fees submitted by the arbitrageurs, $\mathbb{E}[g^{(3)}_{(arb,j)}]$ and $Var[g^{(3)}_{(arb,j)}]$, are both increasing in β and in the amount of tokens $d_A^{(1)}$ and $d_B^{(1)}$ deposited by liquidity providers.*

The above proposition highlights an undesirable consequence of AMM adoption. If the fundamental exchange rate is too volatile or a large amount of tokens is deposited in the pool, we expect arbitrageurs to extract higher profit from each arbitrage opportunity. Hence, they are willing to bid higher gas fees to prioritize their execution. A surge in gas fees imposes negative externalities on other users of the same blockchain, who may experience large delays in the execution of their orders because of the high gas fee bidding transactions.

5 Socially Optimal Pricing Curve

We construct the pricing function which maximizes aggregate welfare. As we demonstrate in this section, this requires to pin down the convexity of the pricing curve $F(x, y) = C$, which we show to be tightly linked to the price impact from trades. Price impact affects the economic incentives of market participants, because it determines both the severity of the arbitrage problem and the trading volume of investors.

Recall that any trade executed in period $t, t = 2, 3$, satisfies the relation:

$$F(d_A^{(t-1)} + \Delta_A, d_B^{(t-1)} + \Delta_B) = F(d_A^{(t-1)}, d_B^{(t-1)}), \Delta_B \geq -d_B^{(t-1)}, \Delta_A \geq -d_A^{(t-1)},$$

where Δ_A, Δ_B are, respectively, the amounts of tokens A and B added to (or withdrawn from, if the sign is negative) the AMM. The pricing curve defined by the above equation pins down the relationship between Δ_A and Δ_B, which can be written as $\Delta_B = g(\Delta_A)$[11]. The slope of the curve $g'(\Delta_A) = -\frac{F_x(d_A^{(t-1)}+\Delta_A, d_B^{(t-1)}+\Delta_B)}{F_y(d_A^{(t-1)}+\Delta_A, d_B^{(t-1)}+\Delta_B)}$, is the negative of the marginal exchange rate. Hence, the pricing curve determines the pricing schedule. The second-order derivative of the curve at each point, $g'' \geq 0$, captures the rate of change of the marginal exchange rate. The price impact of an order which exchanges an amount Δ_A of tokens A for tokens B can be written as:

$$|g'(\Delta_A) - g'(0)| = \int_0^{\Delta_A} g''(x)dx. \tag{2}$$

Hence, if the pricing curve has higher convexity, g'' is larger, and the price impact is higher.

Example: We present an example to further illustrate the relation between the convexity of the curve and price impact. Suppose the liquidity pool consists of 2 tokens A and 2 tokens B, whose fundamental prices are $p_A^{(1)} = p_B^{(1)} = 1$. Consider the case of a linear pricing function, i.e., $F_0(x,y) = p_A^{(1)}x + p_B^{(1)}y$. Then the pricing curve is $x + y = 4$. Observe from Figure 2 that the slope of this pricing curve is -1 and its second-order derivative is 0. This means that the marginal exchange rate is fixed at 1, and a swap trade will not change the marginal exchange rate and thus has no price impact. As it can be seen from Figure 2, an investor can use 2 tokens A to exchange for 2 tokens B , and the marginal exchange rates before and after the trade are both 1. Next, consider the case where the AMM deploys the constant product pricing function used by Uniswap V2 and Sushiswap, i.e., $F_1(x,y) = xy$, and the resulting pricing curve is then $xy = 4$. The slope of the curve is $-\frac{y}{x}$, and the pricing curve is strictly convex. This implies that the marginal cost of acquiring an additional A or B token increases with the trading size, and any trade has an impact on the price. To see this, observe from Figure 2 that 2 tokens A can only exchange for 1 B token, the pre-trade marginal exchange rate of this trade is 1, and the post-trade marginal exchange rate is only 0.25. Compared with the linear pricing curve, the

[11] The existence and the convexity of such a function g is guaranteed by the Implicit Function Theorem and Assumption 1.

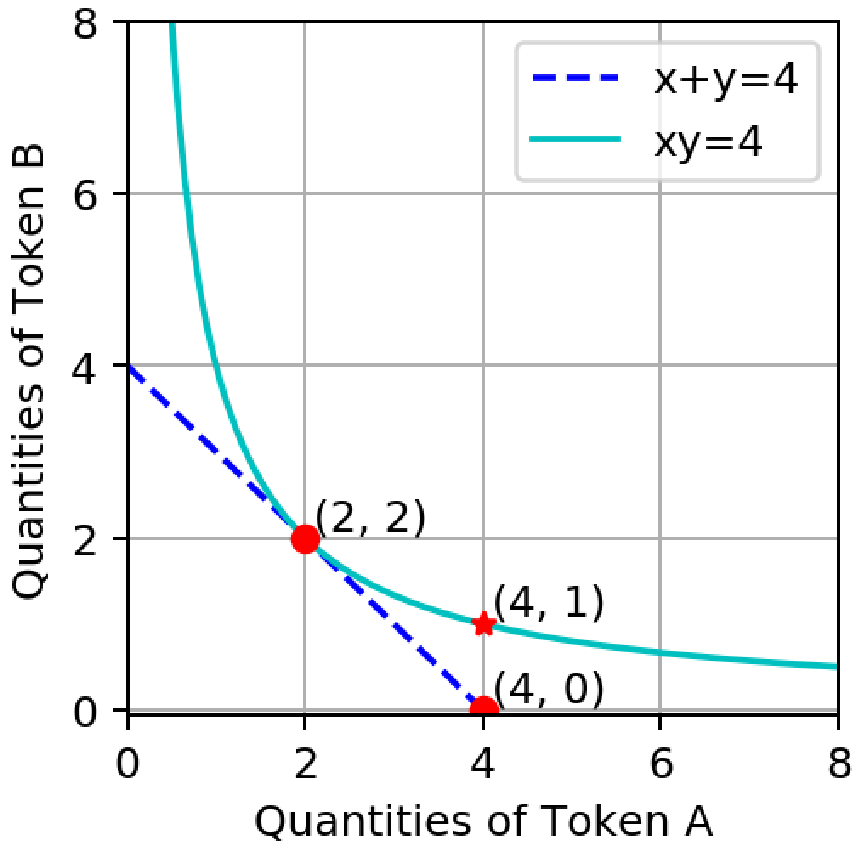

Figure 2: The plot illustrates how the price impact changes with the convexity of the pricing curve. Suppose there are 2 tokens A and 2 tokens B. If the pricing curve is linear, trades have zero price impact, the exchange rates are fixed at 1, and 2 tokens A can be exchanged for 2 tokens B. If the pricing curve is the constant product curve, 2 tokens A can be used to exchange for only 1 token B. This trade has a price impact: the marginal exchange rate decreases from 1 to 0.25 after the trade.

constant product pricing curve has a larger convexity and thus leads to a higher price impact, which explains why 2 tokens A are exchanged for less than 2 tokens B .

We next demonstrate that increasing the convexity of a pricing curve results in reduced arbitrage losses for liquidity providers but also decreases investors' trades. To see why this is the case, consider the following family of pricing functions:

$$F_k(x,y) = (1-k)\ A\ F_0(x,y) + k\ F_1(x,y),$$

where $k \in [0,1]$, and $A = \left(\frac{d_A^{(1)} d_B^{(1)}}{p_A^{(1)} p_B^{(1)}}\right)^{1/2}$ is a scaling coefficient. The convexity of the pricing curve

$F_k(x,y) = C$ is increasing in k. If $k = 0$, the pricing curve is a straight line whose second-order derivative is zero; if $k = 1$, the pricing curve is the constant product function.

Lemma 2. *Suppose the liquidity providers deposit a positive amount of tokens A and B, respectively, in period* 1*. Then, in the subgame starting in period 2:*

1. *the expected arbitrage loss of liquidity providers is decreasing in k.*

2. *the equilibrium expected investors' surplus is decreasing in k.*

As k increases, so does the convexity of the pricing curve. This in turn leads to a higher price impact, and thus is it more expensive to purchase tokens. Both arbitrageurs and investors reduce trading quantities. As a result, arbitrageurs extract smaller profits from liquidity providers, and investors earn less surplus.

The liquidity providers' willingness to deposit depends on the convexity of the pricing curve, as stated in the following proposition.

Proposition 4. *Suppose that $\alpha > \beta$. Then there exists a critical threshold $k^* \in (0,1)$ such that*

1. *The expected payoff of liquidity providers in equilibrium is increasing in k, for $k \in [0, k^*]$, and decreasing in k, for $k \in [k^*, 1]$.*

2. *A pricing function with $k = k^*$ minimizes the occurrence of a "liquidity freeze", i.e., if a liquidity freeze occurs when $k = k^*$, then it occurs for any other $k \in [0,1]$.*

As the convexity increases, there are two opposite economic forces at play: (1) a reduced arbitrage loss for liquidity providers, and (2) a decrease in fee revenues. If k is small, the former force dominates the latter, and thus increasing convexity makes liquidity providers have stronger incentive to deposit. Conversely, if k is large, the loss from smaller fee revenues exceeds the gains from reduced arbitrage, which in turn reduces liquidity providers incentive to deposit.

Despite F_{k^*} being the optimal pricing function for liquidity providers, its convexity may still be too high from a social welfare perspective. We characterize the socially optimal pricing curve $F_{k^{opt}}$, which maximizes aggregate welfare, defined as the ex-ante expected payoff of liquidity providers, arbitrageurs, and investors plus the gas fees earned by validators.

Theorem 5. *Suppose that $\alpha > \beta$. Then there exists $k^{opt} \leq k^*$ such that a pricing function with $k = k^{opt}$ maximizes aggregate welfare. The aggregate welfare is increasing in k, for $k \in [0, k^{opt}]$, and decreasing in k, for $k \in [k^{opt}, 1]$. Moreover, k^{opt} increases in β.*

All transactions are transfers of wealth in the system except for investors' trades. Welfare thus is only contributed by the benefits that investors extract from purchasing tokens. There are two sources of welfare loss. First, a higher price impact reduces the quantities of tokens exchanged by investors and gains from trade. Second, the arbitrage problem reduces liquidity providers' incentives to deposit tokens. This in turn diminishes the traded quantities of investors and may even lead to a liquidity freeze. If the convexity is too high, that is, $k > k^*$, further increasing k not only leads to a higher price impact but also weakens liquidity providers' incentives to deposit, as shown in Proposition 4. The smaller depth of the pool combined with the higher trading costs reduces investors' traded quantities as well as their benefits from trades. Hence, the convexity of the socially optimal pricing curve is such that $k^{opt} \in [0, k^*]$. A higher value of k in this interval reduces arbitrage loss for liquidity providers and thus improves their incentives to supply liquidity. However, a higher convexity means larger trading costs, less quantities traded, and thus smaller investors' gains from trading tokens. The socially optimal pricing curve balance between these two economic forces, and also depends on the token characteristics. As the token exchange rate becomes more volatile, the convexity of the curve needs to be higher to counteract the increased severity of the arbitrage problem and incentivize liquidity providers to deposit.

6 Empirical Analysis

We provide empirical support to the main testable implications of our model. We state the tested implications in Section 6.1, and describe our dataset in Section 6.2. In Section 6.3, we specify how we construct the variables used for our regressions. We discuss the regression results in Section 6.4.

6.1 Testable Implications

Our model generates the following implications:

(1) Exchange rate volatility is negatively correlated with the amount of deposited tokens. Theorem 2 implies that if the volatility of the token exchange rate increases, liquidity providers have weaker incentives to deposit their tokens.

(2) Trading volumes are positively correlated with the amount of deposited tokens. As shown in Theorem 2, liquidity providers have stronger incentives to deposit if the expected trading gains of investors are higher ($\alpha, \kappa_I \uparrow$). Higher α, κ_I also imply a higher expected trading volume.

(3) Levels and volatility of gas fee bids are positively correlated with the volatility of token pairs. Proposition 3 states that as the volatility of the token exchange rate increases ($\beta \uparrow$), so does the expectation and variance of gas fees.

6.2 Data

The dataset contains histories of all trades, deposits, and withdrawals for a sample of 12 AMMs with actively traded pairs. Among the 12 AMMs, 6 of them are from Uniswap V2, and the rest are from Sushiswap. Two pairs consist of only stable coins pegged to one US dollar, and they are denoted as "stable pairs". The dataset also contains the price of each pair from the largest centralized exchange, Binance, at 5-minute intervals. The time series cover the 37-week period from April 16, 2021 to December 31, 2021.

For each AMM, the transaction-level data include the time stamp, the address of the investor, the gas price bid, as well as the name attribute and amount of tokens that the investor trades in or takes out of the AMM. If an investor trades in (takes out) both tokens in a transaction, then we identify the transaction as a deposit (withdrawal); if instead, the investor trades in one token and takes out the other token, we identify the transaction as a swap.

6.3 Definitions of Variables

We next list and describe the main variables in our empirical analysis.

Token Exchange Rate Volatility. We measure the weekly volatility of a token pair using the standard deviation of the log exchange rate between the two tokens during that week. This measurement is invariant with respect to the choice of base currency[12] of the pair and to scalar multiplication of token values. It is worth remarking that some tokens may be more valuable than others. A normalized measure ensures that the volatility of the token exchange rate is comparable across pairs. We use Binance rather than DEX data to estimate the exchange rate volatility. We make this choice to shut down the channel that the amount of deposits may impact the exchange rate volatility in the AMM. Specifically, a trade of a given size has a larger price impact in a liquidity pool with smaller depth, leading to higher price volatility.

Deposit Inflow and Outflow. Denote the pair of tokens in AMM j by A_j and B_j. We measure the change of deposits for AMM j during week t as

$$Depositflow_{jt} = \operatorname{sgn}(DepositA_{jt}) \times \left(\frac{DepositA_{jt}}{TokenA_{jt}} \times \frac{DepositB_{jt}}{TokenB_{jt}} \right)^{1/2}, \tag{3}$$

where $DepositA_{jt}$ and $DepositB_{jt}$ are the total amount of A_j and B_j tokens deposited (if positive), or withdrawn (if negative) by liquidity providers of AMM j, with deposit or withdrawal orders submitted during week t. $TokenA_{jt}$ and $TokenB_{jt}$ are, respectively, the total liquidity reserves of A_j and B_j tokens in AMM j at the beginning of week t. $\operatorname{sgn}(DepositA_{jt})$ is positive if the net deposit is larger than the net withdrawal, and negative otherwise.

Trading Volume. We measure the total trading volume in AMM j during the week t as

$$Volume_{jt} = \left(\frac{TradeA_{jt}}{TokenA_{jt}} \times \frac{TradeB_{jt}}{TokenB_{jt}} \right)^{1/2}, \tag{4}$$

[12] In the foreign exchange market, the first listed currency of a pair is denoted as the base currency, and the second currency is referred to as the quote currency. We follow the same convention, and refer to the first token in the AMM pair as the base token, and to the second token as the quote token.

where $TradeA_{jt}$ and $TradeB_{jt}$ are, respectively, the total amount of A_j and B_j tokens traded by investors in AMM j during week t using swap orders, and $TokenA_{jt}, TokenB_{jt}$ are, respectively, the total liquidity reserves of A_j and B_j tokens in AMM j at the beginning of week t. The above measure captures the total trading volume of investors relative to the total reserve in AMM j during week t. Observe that our measurement of trading volume is normalized by the total amount of deposits in the AMM, thus a large liquidity pool does not necessarily have a higher trading volume.

Gas Price Volatility. We measure the gas volatility in AMM j during week t using the standard deviation of gas price bids associated with all transactions executed on AMM j during week t. We consider AMMs which are all built on the same Ethereum blockchain, and thus levels and volatility of gas prices are comparable across pairs.

Table 1 presents summary statistics of the data. Most of the variables have large in-sample variations. Since "stable pairs" are pairs of stable coins pegged to one US dollar, the log spot exchange rates are very close to 0, and their volatility is much lower than the volatility of "unstable pairs".

6.4 Empirical Results

In this section, we test the main model implications listed in Section 6.1.

6.4.1 Exchange Rate Volatility, Trading Volume, and Deposit

We estimate the following panel regressions to measure the impact of token exchange rate volatility and trading volume on deposit flow rates:

$$Depositflow_{jt} = \gamma_t + \rho_1 Volatilitd_{jt} + \epsilon_{jt} \tag{5}$$

$$Depositflow_{jt} = \gamma_t + \delta_1 Volume_{jt} + \epsilon_{jt} \tag{6}$$

$$Depositflow_{jt} = \gamma_t + \rho_2 Volatilitd_{jt} + \delta_2 Volume_{jt} + \epsilon_{jt}, \tag{7}$$

Table 1: Summary statistics of the data set. It covers the 37-week period from April 16, 2021 to Dec 31, 2021. The values of weekly-level variables are given in Panel A, and transaction-level data are reported in Panel B.

	N	Mean	SD	10th	50th	90th
Panel A: Weekly-level Data						
Log Rate Volatility , All	444	0.0356	0.0289	0.0003	0.0314	0.0657
Log Rate Volatility , Stable	74	0.0003	0.0001	0.0002	0.0003	0.0004
Log Rate Volatility , Unstable	370	0.0427	0.0265	0.0183	0.0372	0.0723
Token Inflow Rate, All	444	-0.0139	0.1597	-0.1575	-0.0049	0.1176
Token Inflow Rate, Stable	74	0.0194	0.2666	-0.2036	-0.0002	0.2320
Token Inflow Rate, Unstable	370	-0.0206	0.1269	-0.1479	-0.0054	0.0849
Trading Volume, All	444	1.3856	1.6452	0.2579	0.8112	2.9688
Trading Volume, Stable	74	0.5568	0.5868	0.1733	0.3871	1.0155
Trading Volume, Unstable	370	1.5513	1.7362	0.3068	1.0042	3.2147
Gas Price Volatility, All	444	161.0302	231.0983	29.9368	89.5011	320.3129
Gas Price Volatility, Stable	74	68.2301	77.8994	16.8105	41.9542	141.7429
Gas Price Volatility, Unstable	370	179.5902	246.5912	38.6047	102.9704	362.1330
Panel B: Transaction-level Data						
Gas Price (Gwei), All	2,965,754	89.986	295.789	16.500	64.000	166.287
Gas Price (Gwei), Stable	146,523	86.541	127.123	19.000	67.017	161.000
Gas Price (Gwei), Nonstable	2,819,231	90.165	301.989	16.400	64.000	166.833
Absolute Value of Log Spot Rate, All	2,965,754	6.897	2.215	2.709	7.685	8.222
Absolute Value of Log Spot Rate, Stable	146,523	0.007	0.006	0.001	0.006	0.013
Absolute Value of Log Spot Rate, Nonstable	2,819,231	7.256	1.601	4.429	7.702	8.232

where j indexes the AMM, t indexes time, $Depositflow_{jt}$ is the deposit flow rate (inflow if positive and outflow if negative), γ_t are the time fixed effects, $Volatilitd_{jt}$ is the volatility of the token exchange rate of AMM j in week t, $Volume_{jt}$ is the trading volume at AMM j in week t, and ϵ_{jt} is an error term. We cluster our standard errors at the AMM level. The coefficients ρ_1, ρ_2 quantify the sensitivity of deposit flow on token volatility, and the coefficients δ_1, δ_2 give

Table 2: Results from regressing weekly deposit flow rates of AMMs on token exchange rate volatility and trading volumes. The data set covers 12 AMMs for a 37-week period from April 16, 2021 to Dec 31, 2021. The dependent variable is the weekly deposit flow rate of AMMs. The independent variables are the weekly spot exchange rate volatility and the weekly trading volume of each AMM. Week fixed effects are included for all regressions. Standard errors are clustered at the AMM level. Asterisks denote significance levels (***=1%, **=5%, *=10%).

	Dependent variable: Deposit Inflow Rate		
	(a)	(b)	(c)
Intercept	0.033	-0.030	0.002
	(0.046)	(0.073)	(0.041)
Exchange Rate Volatility	-0.797^{***}		-1.478^{***}
	(0.306)		(0.578)
Trading Volume		0.011^{*}	0.026^{***}
		(0.006)	(0.013)
Week fixed effects?	yes	yes	yes
Observations	444	444	444
R^2	0.13	0.14	0.17
Note:			*p<0.1; **p<0.05; ***p<0.01

the sensitivity of deposit flow to trading volume.

Table 2 shows a negative, statistically significant relationship between the token exchange rate volatility and the deposit flow rate ($\rho_1, \rho_2 < 0$), which is consistent with our model predictions. After controlling for trading volume, a one-standard-deviation increase in weekly spot rate volatility (which is equal to 0.03) decreases the deposit inflow rate by 30% standard deviations of that variable. Columns (b) and (c) show that there exists a positive, statistically significant relationship between trading volume and deposit flow rate ($\delta_1, \delta_2 > 0$), which confirms our model predictions. After controlling for exchange rate volatility, a one-standard-deviation increase in trading volume increases the deposit flow rate by 25% standard deviations of that variable. In summary, Table 2 confirms our model implications that the amount of deposited tokens decreases with exchange rate volatility, and increases with trading volume. Moreover, the regression estimates show that these effects are economically significant.

6.4.2 Token Exchange Rate Volatility and Gas Price

We group all token pairs into two categories, "stable pairs" and "unstable pairs", and examine whether transactions in "stable pairs" are associated with lower average levels and volatility of gas fees. "Stable pairs" consist of two stable coins pegged to one US dollar, and have a lower price volatility than "unstable pairs" where at least one token is not a stable coin.

Specifically, we estimate the following two linear models:

$$GasVolatility_{jt} = \gamma_t + \kappa_A \mathbb{1}_{StablePair} + \epsilon_{jt} \tag{8}$$

$$Gas_{js} = \gamma_t + \kappa_B \mathbb{1}_{StablePair} + \epsilon_{js}, \tag{9}$$

where j indexes AMMs, t indexes time, and s indexes transactions. We run regressions at weekly frequency[13] for the model in equation (8) and at daily frequency for the model in equation (9). $GasVolatility_{jt}$ is the volatility of gas price during week t in AMM j, Gas_{js} is the gas price of transaction s in AMM j, γ_t are time fixed effects, $\mathbb{1}_{StablePair}$ is the dummy variable for "stable pairs", which consist of two stable coins pegged to one US dollar, and ϵ_{jt}, ϵ_{js} are error terms. We cluster our standard errors at the AMM level. The coefficients κ_A and κ_B quantify the differences in levels and volatility of gas fees, respectively, between "stable pairs" and "unstable pairs".

Table 3 indicates that the weekly gas price volatility of "stable pairs" is about 65% lower than that of "non-stable pairs". Additionally, the gas price level for transactions of "stable pairs" is around 8% lower than that of "non-stable pairs". Altogether, Table 3 confirms our model implications that levels and volatility of gas fees are higher for pairs with a larger exchange rate volatility, and additionally illustrates that the identified relationships are economically significant.

[13] We choose weekly, instead of daily frequency, to have enough observations for estimating the gas price volatility.

Table 3: Results from regressing a binary variable indicating whether or not the AMM contains a "stable pair" on gas price levels and weekly gas price volatility. The data set covers 12 AMMs for a 37-week period from April 16, 2021 to Dec 31, 2021. The dependent variable in column (a) is the weekly gas price volatility, and the dependent variable in column (b) is the gas price of transactions. The independent variable is a dummy equal to one if the AMM contains a pair of stable coins pegged to one US dollar and zero otherwise. Time fixed effects are included for all regressions. Standard errors are clustered at the AMM level. Asterisks denote significance levels (***=1%, **=5%, *=10%).

	Dependent variables:	
	Gas Price Volatility	Gas Price
	(a)	(b)
Intercept	179.59***	90.327***
	(9.65)	(1.19)
Stable	-111.36***	-6.913***
	(14.67)	(1.78)
Week fixed effects?	yes	no
Day fixed effects?	no	yes
Observations	444	2,965,727
R^2	0.28	0.04
Note:		*p<0.1; **p<0.05; ***p<0.01

7 Concluding Remarks

Decentralized exchanges allow for digital tokens to be deposited into a liquidity pool. A smart contract manages the exchange of pool tokens, whose pricing schedule is determined by a pre-coded pricing curve. On the one hand, liquidity provision becomes easier because liquidity providers no longer need to set prices and compete, but rather only need to decide whether or not to deposit their tokens. This allows AMMs to crowd-source liquidity from token holders. On the other hand, pre-determined pricing schedules may be too rigid and suboptimal for liquidity providers. They risk of being exploited by arbitrageurs as soon as the spot exchange rate at the AMM deviates from its fundamental value. These considerations make it crucial to choose a suitable pricing function.

We have demonstrated that that the key characteristic of a pricing function is its convexity, and provided guidance on how to design pricing functions which maximize aggregate welfare.

Many innovations of AMMs target on the convexity of pricing curves. For instance, AMMs which manage trading of stable coins, such as Curve, often implement pricing curves with very small convexity. Our findings suggest that stable pairs, which are typically subject to rare and small shocks to fundamentals, do not require pricing curves to have a high convexity to limit arbitrage problems for liquidity providers. Rather, these AMMs aim at boosting investors' trading through the implementation of pricing curves with very small convexity. Our model can be used to calibrate parameters of different liquidity pools, and to specify the socially optimal pricing functions for each of these pools. Socially optimal pricing curves help increase the adoption of AMMs and improve allocative efficiencies.

A major protocol innovation in DEXes has been Uniswap V3, launched in May 2021. Uniswap V3 allows liquidity providers to choose the convexity of pricing curves by selecting the maximum allowed price impact. For stable pairs, liquidity providers typically choose pricing curves with smaller price impact. For instance, for the stable pair USDT-DAI, the maximum possible price impact is chosen to be around 1-2bps. The pricing curve has small convexity and is almost equivalent to a linear pricing curve. By contrast, pricing curves with large convexity are often chosen for unstable pairs (e.g. USDC-ETH), and the maximum price impact allowed often exceeds 10,000 bps. This market design requires liquidity providers to be more sophisticated and able to calculate their returns from depositing under different pricing curves. However, liquidity providers are often naive depositors, and according to Loesch et al. (2021) most of them end up making negative returns at Uniswap V3. Our model can illuminate liquidity providers on the choice of the pricing curve, F_{k^*}, that maximizes their payoffs. We remark that liquidity providers' incentives are not aligned with social incentives, so the convexity of the pricing curve, if chosen by liquidity providers only, may be too high from a social welfare perspective.

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A Pooling Multiple Tokens Does Not Reduce the Arbitrage Problem

We demonstrate our statement by considering three token types: A, B, and C. Consider two pools, one which manages tokens A and B only, and the other which manages all three token types. Both pools utilize an AMM with constant product function, that is, $F_{AB}(x,y) = xy$ for the first AMM, and $F_{ABC}(x,y,z) = xyz$ for the second AMM. We denote the fundamental value of a single token C at time t by $p_C^{(t)}$.

As in the main model of the paper, the values of tokens A, B, and C may change due to exogenous idiosyncratic shocks $\zeta_A, \zeta_B, \zeta_C$, respectively:

$$\begin{aligned} &\zeta_i \sim Bern(\kappa), \zeta_A \perp \zeta_B, \perp \zeta_B \perp \zeta_C, \perp \zeta_A \perp \zeta_C, \\ &p_i^{(2)} = (1+\beta\zeta_i)p_i^{(1)}, i = A, B, C. \end{aligned} \tag{A.1}$$

Apart from this adjustment, the model remains the same as the one presented in the main body. The following theorem compares the equilibrium arbitrage losses of the two AMMs, and its proof is reported in Appendix B.

Theorem A.1. *Suppose liquidity providers deposit the same value of tokens in both AMMs. Then their expected loss from stale price arbitrage is larger in the AMM which pools three tokens.*

The above theorem states that arbitrageurs can extract a larger profit from the pool with three tokens. The reason is twofold. First, an arbitrage opportunity is more likely to emerge if the number of tokens increases. Second, for each realized opportunity, the arbitrageur can extract a larger portion of the shocked token from the deposits in the AMM with three tokens. If the price of token A increases due to a shock, arbitrageurs can only use token B to exchange for it in the AMM pooling two tokens. By contrast, in the AMM which pools three tokens, arbitrageurs can use both tokens B and C to exchange for token A.

The logic described above extends to AMMs which pool $M > 3$ tokens, and suggests that if the number of token types increases, the losses imposed on liquidity providers by arbitrageurs are higher.

B Technical Results and Proofs

Proof of Lemma 1. Suppose that arbitrageurs trade in tokens A for tokens B .[14] Arbitrageurs aim for the optimal arbitrage, i.e., they choose the buy order, $(\Delta q_A^{(3)}, \Delta q_B^{(3)})$, which solves the following optimization problem:

$$\begin{aligned} \max_{\Delta q_A^{(3)}, \Delta q_B^{(3)}} \quad & p_A^{(2)}(1+f)\Delta q_A^{(3)} + p_B^{(2)}\Delta q_B^{(3)} \\ \text{s.t.} \quad & F(d_A^{(2)}, d_B^{(2)}) = F(d_A^{(2)} - \Delta q_A^{(3)}, d_B^{(2)} - \Delta q_B^{(3)}) \\ & \Delta q_A^{(3)} \leq 0, d_B^{(2)} \geq \Delta q_B^{(3)} \geq 0, \end{aligned} \tag{B.2}$$

where $p_A^{(2)}(1+f)\Delta q_A^{(3)}$ is the trading cost, that is, the value of tokens A traded in plus the trading fee paid to the liquidity providers, and $p_B^{(2)}\Delta q_B^{(3)}$ is the value of tokens B received by the arbitrageur from the order. There exists an arbitrage opportunity if and only if the solution of (B.2), $(\Delta q_A^{(3)*}, \Delta q_B^{(3)*})$, yields a positive profit, i.e., $p_A^{(2)}(1+f)\Delta q_A^{(3)*} + p_B^{(2)}\Delta q_B^{(3)*} > 0$.

(1) **Stale price arbitrage**. Without loss of generality, we assume the shock hits token B. That is, $p_A^{(2)} = p_A^{(1)}$ and $p_B^{(2)} = p_B^{(1)}(1+\beta)$. The case of a shock leading to an appreciation of token A can be handled symmetrically.

We begin by showing that the two-variable constrained optimization problem (B.2) can be reduced to a single-variable unconstrained optimization problem which is easier to analyze. To obtain this equivalence result, we use the pricing function F to pin down the price schedule, i.e., to determine the amount $\Delta q_B^{(3)}$ of tokens B received by the arbitrageur upon depositing $-\Delta q_A^{(3)}$ tokens A. Recall that F is twice continuously differentiable, and $F_x > 0, F_y > 0$. By the Implicit Function Theorem, we can then rewrite the constraint of the arbitrageur's optimization

[14] The case where arbitrageurs swap tokens B for tokens A is symmetric.

problem stated in (B.2), i.e., the constraint imposed by the pricing curve, as

$$\Delta q_B^{(3)} = g^{(3)}(\Delta q_A^{(3)}), -\infty < \Delta q_A^{(3)} \leq 0, \tag{B.3}$$

where $g^{(3)}$ is a twice differentiable function. Hence, we write the amount of tokens B the arbitrageur can withdraw from the AMM, $\Delta q_B^{(3)}$, as a function of the amount of tokens A deposited into the AMM by the arbitrageur, before paying the fee. The first order derivative of the above function, $\frac{d(g^{(3)})}{d(\Delta q_A^{(3)})}$, is the negative of the marginal exchange rate. Again, by the Implicit Function Theorem, we obtain:

$$\frac{d(g^{(3)})}{d(\Delta q_A^{(3)})} = -\frac{F_x}{F_y}\bigg|_{(x,y)=(d_A^{(2)}-\Delta q_A^{(3)}, d_B^{(2)}-\Delta q_B^{(3)})} \leq 0.$$

Differentiating the above equation on both sides with respect to $\Delta q_A^{(3)}$, we obtain

$$\frac{d^2(g^{(3)})}{d(\Delta q_A^{(3)})^2} = -\frac{-F_y^2 F_{xx} + 2F_y F_x F_{xy} - F_x^2 F_{yy}}{F_y^3}\bigg|_{(x,y)=(d_A^{(2)}-\Delta q_A^{(3)}, d_B^{(2)}-\Delta q_B^{(3)})}.$$

By the assumed properties of the pricing function (see part 1 and 2 of Assumption 1), we obtain that the above expression is negative, which means that $g^{(3)}$ is concave.

Using the relations $p_A^{(2)} = p_A^{(1)}$ and $p_B^{(2)} = p_B^{(1)}(1+\beta)$ as well as (B.3), we obtain a single variable optimization problem equivalent to the arbitrageur's optimization problem stated in (B.2):

$$\max_{-\infty < \Delta q_A^{(3)} \leq 0} \quad p_A^{(1)}(1+f)\Delta q_A^{(3)} + p_B^{(1)}(1+\beta) g^{(3)}(\Delta q_A^{(3)}), \tag{B.4}$$

where the first term is the cost of the arbitrage order, and the second term is the value of tokens B received by the arbitrageur.

We can now solve the arbitrageur's optimization problem, using the monotonicity and convexity properties derived above. The first-order derivative of the objective function in (B.4) is:

$$p_A^{(1)}(1+f) - \left(-p_B^{(1)}(1+\beta)\frac{dg^{(3)}}{d(\Delta q_A^{(3)})}\right) \tag{B.5}$$

The first term is the arbitrageur's marginal cost of exchanging tokens A for tokens B , while the second term is the marginal benefit of this exchange. If the above expression is positive, then the marginal cost of trading exceeds the marginal benefit, which means that the arbitrageur should decrease the amount of swapped tokens A. By the concavity of $g^{(3)}$, we can conclude that the above expression is decreasing in $\Delta q_A^{(3)}$.

The arbitrageur would use tokens A to exchange for tokens B as long as the marginal benefit is larger than the marginal cost. We first calculate the marginal exchange rate if $\Delta q_A^{(3)} = 0$, and solve for the condition under which the marginal benefit exceeds the marginal cost in this case. The marginal exchange rate $\frac{dg^{(3)}}{d(\Delta q_A^{(3)})}\Big|_{\Delta q_A^{(3)}=0} = -\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(1)},d_B^{(1)})} = -\frac{p_A^{(1)}}{p_B^{(1)}}$. This is because in period 1, the liquidity providers deposit their tokens in the AMM at the fundamental exchange rate. Hence, the first-order derivative in (B.5) is equal to $(f-\beta)p_A^{(1)}$ at $\Delta q_A^{(3)} = 0$, and it is positive if and only if $f > \beta$.

If $f > \beta$, then the marginal cost exceeds the marginal benefit when $\Delta q_A^{(3)} = 0$. Recall that the above first order derivative is decreasing. Thus, for any $\Delta q_A^{(3)} < 0$, the marginal cost is larger than the marginal benefit from exchanging. This means that the any trade will not be profitable for the arbitrageur. Thus, the optimal amount of tokens A exchanged for tokens B is $\Delta q_A^{(3)*} = 0$. If $\Delta q_A^{(3)} = 0$, the value of the objective function is 0 because no arbitrage occurs. Hence, there is no stale price arbitrage opportunity.

If $\beta > f$, the optimal exchange amount is attained when the marginal cost equals the marginal benefit of exchanging:

$$p_A^{(1)}(1+f) = (-p_B^{(1)}(1+\beta)\frac{dg^{(3)}}{d(\Delta q_A^{(3)})}). \tag{B.6}$$

We denote the solution of the above equation by $\Delta q_A^{(3)*}$. We first show that it exists and is unique. By concavity of $g^{(3)}$, the first order derivative in (B.5) is decreasing in the interval $(-\infty, 0]$; it is also negative at 0 because $\beta > f$. By part 4 of Assumption 1, the first order derivative is positive as $\Delta q_A^{(3)} \to -\infty$. Thus, by continuity and monotonicity of the first-order derivative, there exists a unique $\Delta q_A^{(3)*}$ at which the derivative stated in (B.5) is 0, and (B.6) is satisfied.

We then prove that arbitrageurs can earn strictly positive profit from this optimal arbitrage. Recall that when $\beta > f$, $p_A^{(1)}(1+f) < (-p_B^{(1)}(1+\beta)\frac{d(g^{(3)})}{d(\Delta q_A^{(3)})})$ at $\Delta q_A^{(3)} = 0$. As a result, (B.6) is not satisfied at $\Delta q_A^{(3)} = 0$. This implies that $\Delta q_A^{(3)*} \neq 0$. Because the arbitrageur's objective function attains a strictly higher value at $\Delta q_A^{(3)*} \neq 0$ than at $\Delta q_A^{(3)} = 0$, and the objective function has value 0 at $\Delta q_A^{(3)} = 0$, we obtain $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}) > 0$ if $\beta > f$.

We next prove that $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, $|\Delta q_A^{(3)*}|$, and $|\Delta q_B^{(3)*}|$ are increasing in β and decreasing in f. If $\beta \leq f$, the arbitrageur does not trade, and thus $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}) = 0$, $|\Delta q_A^{(3)*}|= 0$,$|\Delta q_B^{(3)*}|= 0$, and thus they are all independent of f and β.

If $\beta > f$, we apply the Envelope Theorem and obtain $\frac{\partial \pi}{\partial f} = p_A^{(1)} \Delta q_A^{(3)*} < 0$, and $\frac{\partial \pi}{\partial \beta} = p_B^{(1)} \Delta q_B^{(3)*} > 0$. Therefore, $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ increases in β and decreases in f.

$\Delta q_A^{(3)*}$ is defined by the condition stated in (B.6). By differentiating both sides of (B.6) with respect to f, we have

$$p_A^{(1)} = -p_B^{(1)}(1+\beta)\frac{d^2 g^{(3)}}{d(\Delta q_A^{(3)})^2}\frac{d\Delta q_A^{(3)*}}{df}.$$

Recall that $g^{(3)}$ is concave, which means that $\frac{d^2 g^{(3)}}{d(\Delta q_A^{(3)})^2} < 0$. This, along with the equality above, implies that $\frac{d\Delta q_A^{(3)*}}{df} > 0$, i.e., $\Delta q_A^{(3)*}$ increases in f. Following the same procedure as above, we can show that $\Delta q_A^{(3)*}$ decreases in β. Since $\Delta q_A^{(3)*} < 0$, $|\Delta q_A^{(3)*}|$ decreases in f, and increases in β.

Since $\Delta q_B^{(3)*} = g^{(3)}(\Delta q_A^{(3)*})$ and the derivative of $g^{(3)}$ is negative, we have that $\Delta q_B^{(3)*}$ decreases in $\Delta q_A^{(3)*} < 0$ and increases in $|\Delta q_A^{(3)*}|= -\Delta q_A^{(3)*}$. Because $|\Delta q_A^{(3)*}|$ decreases in f, and increases in β, we have $\Delta q_B^{(3)*}$ decreases in f, and increases in β.

(2) **Reverse trade arbitrage**. Without loss of generality, we assume that the investor who arrives is of "type A".

After the investor trades, the amount of tokens A and B remaining in the AMM are, respectively, $d_A^{(2)}$ and $d_B^{(2)}$. The arbitrageur again solves the problem stated in (B.2). We solve the optimal arbitrage by following the same procedure as in the proof of part 1 of Lemma 1. We obtain that there exists a reverse trade arbitrage opportunity if and only if $\frac{1}{1+f}\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(2)},d_B^{(2)})} > \frac{p_A^{(1)}}{p_B^{(1)}}$, that is, the price impact of investor's trade is sufficiently large. This is because the marginal benefit of trading for an arbitrageur is larger than the marginal cost at $\Delta q_A^{(3)} = 0$ if and only if $\frac{1}{1+f}\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(2)},d_B^{(2)})} > \frac{p_A^{(1)}}{p_B^{(1)}}$. If this condition is not satisfied, then the arbitrageur does not trade, because the marginal cost is always larger than the marginal benefit.

If $\frac{1}{1+f}\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(2)},d_B^{(2)})} > \frac{p_A^{(1)}}{p_B^{(1)}}$, then the arbitrageur chooses the optimal quantities $\Delta q_A^{(3)*}, \Delta q_B^{(3)*}$ such that the marginal benefit breaks even with the marginal cost:

$$\frac{1}{1+f}\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(2)}-\Delta q_A^{(3)*},d_B^{(2)}-\Delta q_B^{(3)*})} = \frac{p_A^{(1)}}{p_B^{(1)}}. \tag{B.7}$$

The optimal arbitrage order $(\Delta q_A^{(3)*}, \Delta q_B^{(3)*})$ is unique as in part 1 of Lemma 1, and the reason is also identical. Applying the Implicit Function Theorem to $F(d_A^{(2)} - \Delta q_A^{(3)*}, d_B^{(2)} - \Delta q_B^{(3)*}) = F(d_A^{(2)}, d_B^{(2)})$ allows us to parameterize $\Delta q_B^{(3)}$ as a concave function of $\Delta q_A^{(3)}$, i.e., $\Delta q_B^{(3)} = g^{(3)}(\Delta q_A^{(3)})$. The derivative function $\frac{d(g^{(3)})}{d(\Delta q_A^{(3)})} = \frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(2)}-\Delta q_A^{(3)*},d_B^{(2)}-\Delta q_B^{(3)*})}$ is monotone in $\Delta q_A^{(3)}$ due to the concavity of the function. If $\Delta q_A^{(3)} = 0$, $\frac{1}{1+f}\frac{d(g^{(3)})}{d(\Delta q_A^{(3)})} = \frac{1}{1+f}\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(2)},d_B^{(2)})} > \frac{p_A^{(1)}}{p_B^{(1)}}$. If $\Delta q_A^{(3)} \to -\infty$, by part 4 of Assumption 1, $\frac{1}{1+f}\frac{d(g^{(3)})}{d(\Delta q_A^{(3)})} \to 0$. By the intermediate value theorem and the monotonicity of $\frac{d(g^{(3)})}{d(\Delta q_A^{(3)})}$, we obtain that $(\Delta q_A^{(3)*}, \Delta q_B^{(3)*})$ exists and is unique. Following the same procedure as in the proof of part 1 of Lemma 1, we deduce that $(\Delta q_A^{(3)*}, \Delta q_B^{(3)*}) \neq (0,0)$, and the profit from the optimal arbitrage is strictly positive. We then apply the Envelope Theorem and obtain that $\frac{\partial \pi}{\partial f} = p_A^{(1)}\Delta q_A^{(3)*} \leq 0$.

$\square$

Lemma B.1. *If $f < \alpha$, then an arriving investor trades and earns a positive surplus from the transaction, $s(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)}) > 0$. Moreover, the surplus, $s(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})$, and the optimal trading quantities of an arriving investor, $|Q_A^{(2)*}|, |Q_B^{(2)*}|$, are strictly positive, increasing in α, and decreasing in f. The arriving investor does not trade if $f \geq \alpha$, .*

Proof. Without loss of generality, we assume that the investor who arrives is of "type A". The case where the investor who arrives is of "type B" can be handled using symmetric arguments. The investor decides the amount of A token to swap for, and she faces the following optimization problem:

$$\begin{aligned} \max_{\Delta Q_A^{(2)}, \Delta Q_B^{(2)}} \quad & (1+\alpha) p_A^{(1)} \Delta Q_A^{(2)} + (1+f) p_B^{(1)} \Delta Q_B^{(2)} \\ \text{s.t.} \quad & F(d_A^{(1)}, d_B^{(1)}) = F(d_A^{(1)} - \Delta Q_A^{(2)}, d_B^{(1)} - \Delta Q_B^{(2)}) \\ & d_A^{(1)} \geq \Delta Q_A^{(2)} \geq 0, \Delta Q_B^{(2)} \leq 0, \end{aligned} \tag{B.8}$$

where $(1+\alpha)p_A^{(1)} \Delta Q_A^{(2)}$ is the "type A" investor's benefit from a trade , and $-(1+f)p_B^{(1)} \Delta Q_B^{(2)}$ is the total value of tokens B paid by the investor.

Recall that F is twice continuously differentiable, and $F_x > 0, F_y > 0$. We first pin down the pricing schedules. By the Implicit Function Theorem, we can rewrite the constraint of the investor's optimization problem stated in (B.8) as

$$\Delta Q_A^{(2)} = g^{(2)}(\Delta Q_B^{(2)}), -\infty < \Delta Q_B^{(2)} \leq 0, \tag{B.9}$$

where $g^{(2)}$ is a twice differentiable function. In this way, we can write the amount of tokens A the investors can withdraw from the AMM as a function of the negative of the amount of tokens B she has to deposit into the AMM, excluding the fee. The first order derivative of the above function is the negative of the marginal exchange rate:

$$\frac{dg^{(2)}}{d(\Delta Q_B^{(2)})} = -\frac{F_y}{F_x}\bigg|_{(x,y)=(d_A^{(1)} - \Delta Q_A^{(2)}, d_B^{(1)} - \Delta Q_B^{(2)})} < 0.$$

It follows from Assumption 1 that the function $g^{(2)}$ is concave, that is, $\frac{d^2 g^{(2)}}{d(\Delta Q_A^{(2)})^2} \leq 0$.

Plugging (B.9) into the two-variable optimization problem stated in (B.8), we obtain the following equivalent, single-variable, optimization problem:

$$\max_{-\infty<\Delta Q_B^{(2)}\leq 0} \quad (1+\alpha)p_A^{(1)}g^{(2)}(\Delta Q_B^{(2)})+(1+f)p_B^{(1)}\Delta Q_B^{(2)}, \tag{B.10}$$

where the first term is the total benefits from trades of investors, and the second term is the trading cost.

The first-order derivative of the objective function in (B.10) is

$$(1+f)p_B^{(1)}-\left(-(1+\alpha)p_A^{(1)}\frac{dg^{(2)}}{d(\Delta Q_B^{(2)})}\right) \tag{B.11}$$

The first term is the marginal cost of exchanging a B token for an A token, and the second term is the marginal benefit of investors. By concavity of the function $g^{(2)}$, the above expression decreases in $\Delta Q_B^{(2)}$. If $\Delta Q_B^{(2)}=0$, then $\frac{dg^{(2)}}{d(\Delta Q_B^{(2)})}=-\frac{p_B^{(1)}}{p_A^{(1)}}$ because in period 1 the liquidity providers deposit their tokens in the AMM at the fair rate. Hence, the first-order derivative of the objective function stated in (B.11) is equal to $(f-\alpha)p_B^{(1)}$ at $\Delta Q_B^{(2)}=0$, which is positive if and only if $f>\alpha$.

If $f\geq\alpha$, then the marginal cost exceeds the marginal benefit at $\Delta Q_B^{(2)}=0$. Since (B.11) decreases in $\Delta Q_B^{(2)}$, then (B.11) is positive at any $\Delta Q_B^{(2)}\leq 0$. In this way, the marginal cost exceeds the marginal benefit at any $\Delta Q_B^{(2)}\leq 0$. Hence, the optimal trading size is zero, i.e., the investor does not trade when the fee is larger than the private benefit.

If $\alpha>f$, the optimal exchange amount is obtained by equating the marginal cost and the marginal benefit of exchanging:

$$(1+f)p_B^{(1)}=\left(-(1+\alpha)p_A^{(1)}\frac{dg^{(2)}}{d(\Delta Q_B^{(2)})}\right). \tag{B.12}$$

We denote the solution of the equation above as $\Delta Q_B^{(2)*}$. We first prove its existence and uniqueness. Since $\frac{dg^{(2)}}{d(\Delta Q_B^{(2)})}$ is monotone, (B.12) admits at most one solution. Hence, if a solution $\Delta Q_B^{(2)*}$ exists, it is unique. The existence of a solution $\Delta Q_B^{(2)*}$ follows from the intermediate value theorem: the derivative (B.11) is continuous, negative if $\Delta Q_B^{(2)}=0$, and positive if $\Delta Q_B^{(2)}\to-\infty$. Moreover, $\Delta Q_B^{(2)*}\neq 0$ because, if $\alpha>f$, $(1+f)p_B^{(1)}<(-(1+\alpha)p_A^{(1)}\frac{dg^{(2)}}{d(\Delta Q_B^{(2)})})$

when $\Delta Q_B^{(2)} = 0$, which means that (B.12) is not satisfied at $\Delta Q_B^{(2)} = 0$. Therefore, the investor's objective function attains a higher value at $\Delta Q_B^{(2)*}$ than at 0, and thus the ("type A") investor's maximum surplus $s_A(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})$ is strictly positive if $\alpha > f$.

We next prove that $s_A(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})$, $|\Delta Q_A^{(2)*}|$, and $|\Delta Q_B^{(2)*}|$ are increasing in α and decreasing in f. For the case $\alpha \leq f$, the investor does not trade, and thus $s_A(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}) = 0$, $|\Delta Q_A^{(2)*}| = 0$, $|\Delta Q_B^{(2)*}| = 0$. As a result, $\frac{\partial s_A}{\partial f} = \frac{\partial s_A}{\partial \alpha} = 0, \frac{\partial |\Delta Q_A^{(2)*}|}{\partial f} = \frac{\partial |\Delta Q_A^{(2)*}|}{\partial \alpha} = \frac{\partial |\Delta Q_B^{(2)*}|}{\partial f} = \frac{\partial |\Delta Q_B^{(2)*}|}{\partial \alpha} = 0$.

For the case $\alpha > f$, we apply the Envelope Theorem and obtain $\frac{\partial s_A}{\partial f} = p_B^{(1)} \Delta Q_B^{(2)*} < 0$, and $\frac{\partial \pi}{\partial \alpha} = p_A^{(1)} \Delta Q_A^{(2)*} > 0$. Therefore, $s_A(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})$ increases in α and decreases in f.

$\Delta Q_B^{(2)*}$ is defined by the condition in (B.12). Differentiating both sides of (B.12) with respect to f, we obtain

$$p_B^{(1)} = -(1+\alpha) p_A^{(1)} \frac{d^2 g^{(2)}}{d(\Delta Q_B^{(2)})^2} \frac{d\Delta Q_B^{(2)*}}{df}.$$

Recall that $g^{(2)}$ is concave, thus $\frac{d^2 g^{(2)}}{d(\Delta Q_B^{(2)})^2} < 0$. It then follows that $\frac{d\Delta Q_B^{(2)*}}{df} > 0$ because $p_B^{(1)} > 0, p_A^{(1)} > 0, (1+\alpha) > 0$. Using the same argument, we can show $\frac{d\Delta Q_B^{(2)*}}{d\alpha} < 0$. Since $\Delta Q_B^{(2)*} < 0$, we have that $|\Delta Q_B^{(2)*}|$ decreases in f, and increases in α. Since $\Delta Q_A^{(2)*} = g^{(2)}(\Delta Q_B^{(2)*})$, and the derivative of $g^{(2)}$ is negative, we have $\frac{\partial \Delta Q_A^{(2)*}}{\partial f} = \frac{dg^{(2)}}{d\Delta Q_B^{(2)*}} \frac{\partial \Delta Q_B^{(2)*}}{\partial f} < 0$, and $\frac{\partial \Delta Q_A^{(2)*}}{\partial \alpha} = \frac{dg^{(2)}}{d\Delta Q_B^{(2)*}} \frac{\partial \Delta Q_B^{(2)*}}{\partial \alpha} > 0$. Because $\Delta Q_A^{(2)*} > 0$, we deduce that $|\Delta Q_A^{(2)*}|$ decreases in f and increases in α. $\square$

Proof of Proposition 1. Suppose that there exists a stale price or reverse trade arbitrage opportunity. The profit from the optimal arbitrage, $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}) > 0$, is obtained by solving the arbitrageur's optimization problem stated in (B.2). We then prove that in the subgame of period 3, there exists a unique Nash equilibrium where arbitrageurs submit an optimal order and bid a gas fee equal to their profit $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, and all liquidity providers bid zero gas fee together with their exit orders.

The arbitrageurs will never deviate from the unique optimal arbitrage order, $(\Delta q_A^{(3)*}, \Delta q_B^{(3)*})$, because any other trade results in a profit strictly smaller than than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. We next show that for both arbitrageurs, bidding a gas fee other than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ will not result in a strictly higher payoff. First, if an arbitrageur were to bid a gas fee higher than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, its arbitrage order would be executed first, because the bid gas is the highest. However, this arbitrageur would attain a negative payoff because the profit from the executed order would be smaller than the gas fee paid. Second, if an arbitrageur deviates to bid a gas fee strictly smaller than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, its order will not be executed because the other arbitrageur bids a higher fee. The payoff of this arbitrageur is then zero, and thus smaller than the payoff achieved with the optimal order.

We next show that liquidity providers have no incentive to bid a non-zero gas fee for their withdrawal orders. Liquidity provider i owns $w_i \leq 1$ proportion of the tokens in the AMM. As a result, if the arbitrage order is executed before its withdrawal order, then he will incur a loss $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. If liquidity provider i bids a gas fee strictly smaller than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, then his order will be executed after one of the arbitrageurs, and he will incur a loss of $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ from the arbitrage order. Hence, bidding a gas fee larger than zero and smaller than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ will not reduce the loss of liquidity provider i. We then consider the case where the liquidity provider i bids a gas fee larger than or equal to $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. The withdrawal order of liquidity provider i may then be executed before any arbitrageur's order, and thus he would avoid the loss $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. However, the gas fee bid by liquidity provider i would be larger than or equal to $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, i.e., larger than the avoided loss $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. Hence, the liquidity provider i has no incentive to bid a gas fee higher than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$.

We then show that there does not exist any other Nash equilibrium. Liquidity provider i never bids more than $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ in equilibrium. Otherwise, the gas fee paid exceeds the avoided loss, which means that deviating to a zero gas fee bid would be profitable. Moreover, both arbitrageurs bid $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ in equilibrium. If any arbitrageur were to bid higher than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, then the arbitrageur who bids higher, or arbitrageur 1 if both bid equally, would achieve a negative expected payoff, which means that bidding zero fee is a profitable deviation. If only one arbitrageur bids higher than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, then this arbitrageur can bid zero and have a strictly higher payoff. Hence, both arbitrageurs would not bid more than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ in equilibrium. If at least on arbitrageur bids less than $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, then the arbitrageur with the lower bid has a profitable deviation by bidding $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. This means that both arbitrageurs bid $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ in equilibrium. Next, we show that liquidity provider i bid zero in equilibrium. Recall that liquidity provider i never bids more than $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, and both arbitrageurs bid $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. This means that liquidity provider i will be surely exploited by the arbitrageurs and lose $w_i\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$. Bidding above zero will only lead to a lower payoff.

$\square$

Lemma B.2. *For any constant $c > 0$, the following relations hold:*

1. $\Delta q_B^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)}) = c\Delta q_B^{(3)*}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$,
2. $\Delta q_A^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)}) = c\Delta q_A^{(3)*}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$,
3. $\pi(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)}) = c\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$.

Proof of Lemma B.2. Recall that the constraint of the arbitrageur's optimization problem defined in (B.2) is:

$$F(d_A^{(2)}, d_B^{(2)}) = F(d_A^{(2)} - \Delta q_A^{(3)}, d_B^{(2)} - \Delta q_B^{(3)}).$$

Scaling deposits of A and B tokens by a factor c, i.e., $(cd_A^{(2)}, cd_B^{(2)})$, the constraint may be rewritten as

$$\begin{aligned} F(cd_A^{(2)}, cd_B^{(2)}) &= F(cd_A^{(2)} - \Delta q_A^{(3)}, cd_B^{(2)} - \Delta q_B^{(3)}) \\ \iff c^l F(d_A^{(2)}, d_B^{(2)}) &= c^l F(d_A^{(2)} - \frac{\Delta q_A^{(3)}}{c}, d_B^{(2)} - \frac{\Delta q_B^{(3)}}{c}) \\ \iff F(d_A^{(2)}, d_B^{(2)}) &= F(d_A^{(2)} - \frac{\Delta q_A^{(3)}}{c}, d_B^{(2)} - \frac{\Delta q_B^{(3)}}{c}), \end{aligned}$$

where the equivalence result above comes from the third property of the pricing function stated in Assumption 1. The arbitrageur's optimization problem given deposits $cd_A^{(2)}, cd_B^{(2)}$ can be written as:

$$\begin{aligned} \max_{\Delta q_A^{(3)}, \Delta q_B^{(3)}} \quad & p_A^{(2)}(1+f)\Delta q_A^{(3)} + p_B^{(2)}\Delta q_B^{(3)} \\ \text{s.t.} \quad & F(d_A^{(2)}, d_B^{(2)}) = F(d_A^{(2)} - \frac{\Delta q_A^{(3)}}{c}, d_B^{(2)} - \frac{\Delta q_B^{(3)}}{c}) \\ & \Delta q_A^{(3)} \leq 0, \Delta q_B^{(3)} \geq 0. \end{aligned} \tag{B.13}$$

We use $(q_A^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)}), q_B^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)}))$ to denote the solution of (B.13).

With the following change of variables, $\Delta q_A^{(3)'} = \Delta q_A^{(3)}/c, \Delta q_B^{(3)'} = \Delta q_B^{(3)}/c$, the arbitrageur's optimization problem stated in (B.13) may be rewritten as

$$\begin{aligned} \max_{\Delta q_A^{(3)'}, \Delta q_B^{(3)'}} \quad & cp_A^{(2)}(1+f)\Delta q_A^{(3)'} + cp_B^{(2)}\Delta q_B^{(3)'} \\ \text{s.t.} \quad & F(d_A^{(2)}, d_B^{(2)}) = F(d_A^{(2)} - \Delta q_A^{(3)'}, d_B^{(2)} - \Delta q_B^{(3)'}) \\ & \Delta q_A^{(3)'} \leq 0, \Delta q_B^{(3)'} \geq 0, \end{aligned} \tag{B.14}$$

which corresponds to the optimization problem stated in (B.2) except that the objective function is now multiplied by a constant c. Therefore, the maximum value for the optimization problem in (B.14) is c times the maximum of the optimization problem stated in (B.2). Moreover, the optimization problem (B.14) admits the same solution as the optimization problem stated in (B.2), and given by $(\Delta q_A^{(3)*}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}), \Delta q_B^{(3)*}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}))$. Recall the change of variables to obtain (B.14). The solution of (B.14) is then obtained by the solution of (B.13) divided by c, $(\frac{\Delta q_A^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)})}{c}, \frac{\Delta q_B^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)})}{c})$. To recover $\Delta q_A^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ and $\Delta q_B^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)})$, we multiply $(\Delta q_A^{(3)*}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}), \Delta q_B^{(3)*}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}))$ by c, and we obtain

$$\begin{aligned} \Delta q_A^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)}) &= c\Delta q_A^{(3)*}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}), \\ \Delta q_B^{(3)*}(cd_A^{(2)}, cd_B^{(2)}, p_B^{(2)}, p_A^{(2)}) &= c\Delta q_B^{(3)*}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}). \end{aligned}$$

$\square$

Lemma B.3. *For any constant $c > 0$, the following relations hold:*

1. $\Delta Q_B^{(2)*}(cd_A^{(1)}, cd_B^{(1)}, p_B^{(1)}, p_A^{(1)}) = c\Delta Q_B^{(2)*}(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})$,

2. $\Delta Q_A^{(2)*}(cd_A^{(1)}, cd_B^{(1)}, p_B^{(1)}, p_A^{(1)}) = c\Delta Q_A^{(2)*}(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})$,

3. $s_A(cd_A^{(1)}, cd_B^{(1)}, p_B^{(1)}, p_A^{(1)}) = cs_A(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})$,

4. $s_B(cd_A^{(1)}, cd_B^{(1)}, p_B^{(1)}, p_A^{(1)}) = cs_B(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})$.

Proof of Lemma B.3. The proof is analogous to that of Lemma B.2, except that we now solve the investor's optimization problem (B.8) instead of the arbitrageur's optimization problem (B.2). If the amount of deposits are multiplied by c in (B.2), we can apply the same change of variable as in proof of Lemma B.3 by introducing a new variable which is $\frac{1}{c}$ of the original variables. The investor's optimization problem when deposits are multiplied by c can then be rewritten as in (B.8). We then multiply the solution by c to recover $(Q_A^{(2)*}(cd_A^{(1)}, cd_B^{(1)}, p_B^{(1)}, p_A^{(1)})$ and $Q_B^{(2)*}(cd_A^{(1)}, cd_B^{(1)}, p_B^{(1)}, p_A^{(1)}))$. We omit the proof details. $\square$

Lemma B.4. *For any $p_A, p_B, d_A, d_B > 0$, there exists a unique solution (a^*, b^*) of the optimization problem*

$$\begin{aligned} \min_{a,b} \quad & p_A a + p_B b \\ s.t. \quad & F(d_A, d_B) = F(d_A + a, d_B + b). \end{aligned} \tag{B.15}$$

This solution satisfies the condition $\left.\frac{F_x}{F_y}\right|_{(x,y)=(d_A+a^*,d_B+b^*)} = \frac{p_A}{p_B}$, i.e., for any point on the pricing curve, the value of deposits is the lowest when the spot exchange rate is equal to the fundamental rate.

Proof of Lemma B.4. Recall that F is twice continuously differentiable, and $F_x > 0, F_y > 0$ by the first property of Assumption 1. By the Implicit Function Theorem, we can rewrite the constraint of the optimization problem (B.15) as

$$b = m(a), a \in \mathbb{R}, \tag{B.16}$$

where $m(a)$ is a twice differentiable function, whose first order derivative is

$$m'(a) = -\frac{F_x}{F_y}\bigg|_{(x,y)=(d_A+a,d_B+b)} < 0.$$

By Assumption 1, $m(a)$ is convex, that is, $m''(a) > 0$. We then plug $b = m(a)$ into the optimization problem (B.15), and obtain an equivalent single variable unconstrained optimization problem

$$\min_{a} \quad p_A a + p_B m(a), \tag{B.17}$$

where the objective function is strictly convex. Hence, the optimization problem admits a unique solution a^* which satisfies the first-order condition:

$$p_A + p_B m'(a^*) = 0.$$

We then have:

$$-m'(a^*) = \frac{F_x}{F_y}\bigg|_{(x,y)=(d_A+a^*,d_B+b^*)} = \frac{p_A}{p_B}.$$

□

Proof of Theorem 2. We follow the proof strategy of Zermelo's theorem. To establish existence and uniquenss of a subgame perfect equilibrium, we identify a unique strategy profile through backward induction.

We start from the last period $t = 3$. If the liquidity providers do not deposit in period 1, then nothing happens. We then consider the case where liquidity providers deposit. The arguments used in the proof of Proposition 1 show that the optimal gas fees that liquidity providers bid with their exit orders are zero. If there exists an arbitrage opportunity at period 3, then by Lemma 1, the optimal trading order is unique. Both arbitrageurs bid a gas fee equal to the profit from the optimal order, and both arbitrageurs must choose the optimal trade order which solves the arbitrageur's optimization problem stated in (B.2).

In period 2, by Lemma B.1, if an investor arrives then she chooses the unique optimal trading order which solves the optimization problem (B.8). It is incentive compatible for the investor to bid the lowest gas fee, i.e., zero.

In period 1, each liquidity provider maximizes its payoff. Since the AMM requires tokens to be deposited at the current fundamental exchange rate $\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(1)},d_B^{(1)})} = \frac{p_A^{(1)}}{p_B^{(1)}}$, the ratio of tokens A and B deposited, $\frac{d_B^{(1)}}{d_A^{(1)}}$, is already pinned down by the fundamental exchange rate $\frac{p_A^{(1)}}{p_B^{(1)}}$ in period 1. Hence, the liquidity provider only chooses the amount of tokens A to deposit. Suppose liquidity provider i deposits an amount $d_{A_i}^{(1)} > 0$ and $d_{B_i}^{(1)} > 0$, respectively, of tokens A and B . Liquidity provider i owns a share $w_i = \frac{d_{A_i}^{(1)}}{d_A^{(1)}}$ of the pool. The expected payoff of liquidity provider i is then:

$$w_i\mathbb{E}\left[p_A^{(3)}d_A^{(3)} + p_B^{(3)}d_B^{(3)}\right] + \mathbb{E}\left[p_A^{(3)}(e_{A_i}^{(0)} - d_{A_i}^{(1)}) + p_B^{(3)}(e_{B_i}^{(0)} - d_{B_i}^{(1)})\right]. \tag{B.18}$$

In the above expression, the first term is the expected payoff from the tokens deposited in the AMM, and the second term is the expected payoff from non-deposited tokens. We may rewrite (B.18) as follows:

$$w_i(p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)})\mathbb{E}\Big[1+R_D\Big] + \mathbb{E}\Big[1+R_A\Big]p_A^{(1)}(e_{A_i}^{(0)} - w_i d_A^{(1)}) + \mathbb{E}\Big[1+R_B\Big]p_B^{(1)}(e_{B_i}^{(0)} - w_i d_B^{(1)}).$$

The expected return from deposits made in period 1 is:

$$\mathbb{E}\Big[R_D\Big] := \frac{d_A^{(1)}}{(p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)})}\mathbb{E}\left[p_A^{(3)}\frac{d_A^{(3)}}{d_A^{(1)}} + p_B^{(3)}\frac{d_B^{(3)}}{d_A^{(1)}}\right] - 1,$$

and the expected returns from investing in tokens A and B are:

$$\mathbb{E}\Big[R_A\Big] := \mathbb{E}\left[\frac{p_A^{(2)}}{p_A^{(1)}}\right] - 1, \mathbb{E}\Big[R_B\Big] := \mathbb{E}\left[\frac{p_B^{(2)}}{p_B^{(1)}}\right] - 1.$$

We first show that $\mathbb{E}\Big[R_D\Big], \mathbb{E}\Big[R_A\Big], \mathbb{E}\Big[R_B\Big]$ do not depend on the decision variable $d_{A_i}^{(1)} = w_i d_A^{(1)}$ of liquidity provider i. First, the value of $\frac{d_B^{(1)}}{d_A^{(1)}}$ does not depend on the amount of deposits,

$d_A^{(1)}$ and on liquidity provider i's share, w_i. This is because $\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(1)},d_B^{(1)})} = \frac{p_A^{(1)}}{p_B^{(1)}}$, and this pins down the ratio of tokens A and B deposited. In other words, no matter how many tokens each liquidity provider deposits, the ratio is already decided by the fundamental exchange rate in period 1. In period 2, we have $\frac{d_A^{(2)}}{d_A^{(1)}} = 1$ if the investor does not arrive; $d_A^{(2)} = d_A^{(1)} - \Delta Q_A^{(2)*}$, $\frac{d_A^{(2)}}{d_A^{(1)}} = 1 - \frac{\Delta Q_A^{(2)*}}{d_A^{(1)}}$ if a "type A" investor arrives; $d_A^{(2)} = d_A^{(1)} - (1+f)\Delta Q_A^{(2)*}$, $\frac{d_A^{(2)}}{d_A^{(1)}} = 1 - \frac{(1+f)\Delta Q_A^{(2)*}}{d_A^{(1)}}$ if a "type B" investor arrives. We know from Lemma B.3 that $\Delta Q_A^{(2)*}$ is proportional to $d_A^{(1)}$, hence $\frac{d_A^{(2)}}{d_A^{(1)}}$ does not depend on the choice of w_i and $d_A^{(1)}$ by liquidity providers. Using symmetric arguments, we obtain that the realization of $\frac{d_B^{(2)}}{d_B^{(1)}}$ for any event does not depend on w_i and $d_A^{(1)}$. Since we have also shown that $\frac{d_B^{(1)}}{d_A^{(1)}}$ is independent of $w_i, d_A^{(1)}$, we conclude that $\frac{d_B^{(2)}}{d_A^{(1)}} = \frac{d_B^{(1)}}{d_A^{(1)}}\frac{d_B^{(2)}}{d_B^{(1)}}$ is independent of w_i and $d_A^{(1)}$. Using Lemma B.2 and the same arguments above, we deduce that $\frac{d_A^{(3)}}{d_A^{(2)}}$ and $\frac{d_B^{(3)}}{d_A^{(2)}}$ are independent of w_i and $d_A^{(1)}$. As a result, $\frac{d_A^{(3)}}{d_A^{(1)}} = \frac{d_A^{(3)}}{d_A^{(2)}}\frac{d_A^{(2)}}{d_A^{(1)}}$, and $\frac{d_B^{(3)}}{d_B^{(1)}} = \frac{d_B^{(3)}}{d_B^{(2)}}\frac{d_B^{(2)}}{d_B^{(1)}}$ also do not depend on w_i and $d_A^{(1)}$. Therefore, the expected return from deposits $\mathbb{E}\Big[R_D\Big]$ does not depend on the amount of tokens deposited by liquidity provider i, $w_i d_A^{(1)}$, and only depends on exogenous parameters. Also, the expected returns of holding tokens A and B , $\mathbb{E}\Big[R_A\Big]$, $\mathbb{E}\Big[R_B\Big]$, are obviously constant.

The expected return of not depositing is

$$\mathbb{E}\Big[R_{ND}\Big] := \mathbb{E}\Big[1+R_A\Big]\frac{p_A^{(1)}d_A^{(1)}}{(p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)})} + \mathbb{E}\Big[1+R_B\Big]\frac{p_B^{(1)}d_B^{(1)}}{(p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)})} - 1.$$

Following the same argument, we obtain that the return of not depositing is independent of the decision variables of liquidity providers, but only depends on exogenous parameters.

If $\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_{ND}\Big] = \mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_A\Big]\frac{p_A^{(1)}d_A^{(1)}}{(p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)})} - \mathbb{E}\Big[R_B\Big]\frac{p_B^{(1)}d_B^{(1)}}{(p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)})} > 0$, then the unique optimal action of each liquidity provider is to deposit all his tokens. If instead $\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_{ND}\Big] < 0$, the unique optimal action of each liquidity provider is to not deposit. If $\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_{ND}\Big] = 0$, then the liquidity providers are indifferent between depositing in the AMMs or holding tokens. By our tie-breaking rule given in Assumption 2, the unique optimal action of each liquidity provider is then to deposit as many as possible. If a liquidity provider decides to deposit, he always pays zero gas fee in period 1.

We now derive the threshold for a liquidity freeze. To do so, we need to determine the conditions under which $\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_{ND}\Big] < 0$. In period 2, with probability $\frac{\kappa_I}{2}$, a " type A" investor arrives, and with probability $\frac{\kappa_I}{2}$, a " type B" investor arrives. With probability $\kappa_A(1-\kappa_B)$, the price of an A token increases and the price of a B token stays unchanged, which leads to an arbitrage opportunity for the arbitrageur. Symmetrically, with probability $\kappa_B(1-\kappa_A)$, the price of a token B increases and the price of a token A stays unchanged, which again leads to an arbitrage opportunity.

The return from deposits conditional on the occurrence of a shock. If both tokens A and B are hit by a shock, their prices co-move and the return from depositing is β. If neither token A nor token B is hit by a shock, the return is zero.

If it is only token A to be hit by a shock, there may exist a stale price arbitrage opportunity. In this case, the return from deposits is

$$R^{(2)}_{arb_A} := \beta \frac{p^{(1)}_A d^{(1)}_A}{p^{(1)}_A d^{(1)}_A + p^{(1)}_B d^{(1)}_B} - \frac{\pi(d^{(1)}_A, d^{(1)}_B, p^{(1)}_B, (1+\beta)p^{(1)}_A)}{p^{(1)}_A d^{(1)}_A + p^{(1)}_B d^{(1)}_B}, \tag{B.19}$$

where the first term is the return from token A, and the second term is the arbitrage ratio, i.e., the token value loss divided by the initial value of deposits. Alternatively, if it is only token B to be hit by a shock, then the return from deposits is

$$R^{(2)}_{arb_B} := \beta \frac{p^{(1)}_B d^{(1)}_B}{p^{(1)}_A d^{(1)}_A + p^{(1)}_B d^{(1)}_B} - \frac{\pi(d^{(1)}_A, d^{(1)}_B, (1+\beta)p^{(1)}_B, p^{(1)}_A)}{p^{(1)}_A d^{(1)}_A + p^{(1)}_B d^{(1)}_B}. \tag{B.20}$$

We know from Lemma 1 that $\frac{\partial \pi(d^{(1)}_A, d^{(1)}_B, (1+\beta)p^{(1)}_B, p^{(1)}_A)}{\partial \beta} = p^{(1)}_B \Delta q^{(3)*}_B > 0$, and $\frac{\partial \Delta q^{(3)*}_B}{\partial \beta} > 0$. This implies that $\frac{\pi(d^{(1)}_A, d^{(1)}_B, (1+\beta)p^{(1)}_B, p^{(1)}_A)}{p^{(1)}_A d^{(1)}_A + p^{(1)}_B d^{(1)}_B}$ is increasing in β. Moreover, for any M such that $\beta > M > f$, we have the following inequality:

$$\frac{\partial \pi(d_A^{(1)}, d_B^{(1)}, (1+\beta)p_B^{(1)}, p_A^{(1)})}{\partial \beta} = p_B^{(1)} \Delta q_B^{(3)*}(d_A^{(1)}, d_B^{(1)}, (1+\beta)p_B^{(1)}, p_A^{(1)})$$
$$\geq p_B^{(1)} \Delta q_B^{(3)*}(d_A^{(1)}, d_B^{(1)}, (1+M)p_B^{(1)}, p_A^{(1)}) > 0, \tag{B.21}$$

where we have used that $\frac{\Delta q_B^{(3)*}}{\partial \beta} > 0$. Applying the inequality above, we obtain

$$\begin{aligned}\pi(d_A^{(1)}, d_B^{(1)}, (1+\beta)p_B^{(1)}, p_A^{(1)}) &= \int_f^\beta \frac{\partial \pi(d_A^{(1)}, d_B^{(1)}, (1+x)p_B^{(1)}, p_A^{(1)})}{\partial x} dx \\ &\geq \int_M^\beta \frac{\partial \pi(d_A^{(1)}, d_B^{(1)}, (1+x)p_B^{(1)}, p_A^{(1)})}{\partial x} dx \\ &\geq (\beta - M) p_B^{(1)} \Delta q_B^{(3)*}(d_A^{(1)}, d_B^{(1)}, (1+M)p_B^{(1)}, p_A^{(1)}).\end{aligned}$$

Recall from the proof of Lemma 1 that the optimal exchange amount $\Delta q_B^{(3)*}(d_A^{(1)}, d_B^{(1)}, (1+M)p_B^{(1)}, p_A^{(1)})$ is positive if the shock size $M > f$. This means that $\frac{\pi(d_A^{(1)}, d_B^{(1)}, (1+\beta)p_B^{(1)}, p_A^{(1)})}{p_A^{(1)} d_A^{(1)} + p_B^{(1)} d_B^{(1)}}$ converges to ∞ as $\beta \to \infty$. Following the same procedure, we can also show that $\frac{\pi(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, (1+\beta)p_A^{(1)})}{p_A^{(1)} d_A^{(1)} + p_B^{(1)} d_B^{(1)}}$ is increasing in β and converges to ∞ as $\beta \to \infty$.

The return from deposits conditional on the investor's arrival. We only consider the case where a "type A" investor arrives to the AMM. The case where a "type B" investor arrives can be obtained following the same procedure. By Lemma B.1, the investor chooses her optimal trading sizes, $\Delta Q_A^{(2)*}, \Delta Q_B^{(2)*}$. If the trading sizes are nonzero, then after the investor trades, the spot rate at which tokens A are exchanged for tokens B is higher than the fundamental exchange rate:

$$\left.\frac{F_x}{F_y}\right|_{(x,y)=(d_A^{(2)}, d_B^{(2)})} > \frac{p_A^{(1)}}{p_B^{(1)}} = \frac{p_A^{(2)}}{p_B^{(2)}}.$$

If $\left.\frac{1}{1+f} \frac{F_x}{F_y}\right|_{(x,y)=(d_A^{(2)}, d_B^{(2)})} < \frac{p_A^{(1)}}{p_B^{(1)}}$, then there is no reverse trade arbitrage opportunity after the investor trades. This is because for arbitrageurs, the marginal benefit of trading is lower than the marginal trading cost. The return from the deposit in this case is

$$\begin{aligned}\frac{p_A^{(3)}d_A^{(3)}+p_B^{(3)}d_B^{(3)}}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}-1&=\frac{p_A^{(1)}(d_A^{(1)}-\Delta Q_A^{(2)*})+p_B^{(1)}(d_B^{(1)}-(1+f)\Delta Q_B^{(2)*})}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}-1\\&=-\frac{p_A^{(1)}\Delta Q_A^{(2)*}+p_B^{(1)}(1+f)\Delta Q_B^{(2)*}}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}\\&=\frac{\int_{\Delta Q_B^{(2)*}}^{0}p_A^{(1)}(g^{(2)}(x))'+p_B^{(1)}(1+f)dx}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}},\end{aligned}\tag{B.22}$$

where we have used the relationship $\Delta Q_A^{(2)*}=g^{(2)}(\Delta Q_B^{(2)*})$ for $\Delta Q_B^{(2)*}\leq 0$, which has been derived in (B.9). Because $g^{(2)}$ is concave, the marginal exchange rate $-(g^{(2)}(x))'<-(g^{(2)}(0))'=\frac{p_B^{(1)}}{p_A^{(1)}}$ for any $x<0$. We next show that the expression in (B.22) increases in α. The integrand $p_A^{(1)}(g^{(2)}(x)')+p_B^{(1)}((1+f))>0$ for any $x<0$, so the expression in (B.22) increases in the limit of integration $|\Delta Q_B^{(2)*}|$. The integrand is invariant to α, and $|\Delta Q_B^{(2)*}|$ increases in α by Lemma B.1. Therefore, the return from deposits given in (B.22) increases in α.

If $\frac{1}{1+f}\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(2)},d_B^{(2)})}>\frac{p_A^{(1)}}{p_B^{(1)}}$, then there exists a reverse trade arbitrage opportunity because the marginal benefit of trading is larger than the marginal cost. The arbitrageur chooses the trading sizes $\Delta q_A^{(3)*},\Delta q_B^{(3)*}$ such that the marginal benefit breaks even with the marginal cost:

$$\frac{1}{1+f}\frac{F_x}{F_y}\Big|_{(x,y)=(d_A^{(2)}-\Delta q_A^{(3)*},d_B^{(2)}-\Delta q_B^{(3)*})}=\frac{p_A^{(1)}}{p_B^{(1)}},\tag{B.23}$$

where $\Delta q_A^{(3)*}$ and $\Delta q_B^{(3)*}$ satisfy the constraint, $F(d_A^{(2)}-\Delta q_A^{(3)*},d_B^{(2)}-\Delta q_B^{(3)*})=F(d_A^{(2)},d_B^{(2)})$, and (B.23) pins down the ratio of tokens A to tokens B, $\frac{d_A^{(2)}-\Delta q_A^{(3)*}}{d_B^{(2)}-\Delta q_B^{(3)*}}$.

Before calculating the return from deposits we analyze the sensitivity of $F(d_A^{(2)}-\Delta q_A^{(3)*},d_B^{(2)}-$

$\Delta q_B^{(3)*}$) to α:

$$
\begin{aligned}
\frac{dF(d_A^{(2)} - \Delta q_A^{(3)*}, d_B^{(2)} - \Delta q_B^{(3)*})}{d\alpha} &= \frac{dF(d_A^{(2)}, d_B^{(2)})}{d\alpha} \\
&= \frac{dF(d_A^{(1)} - \Delta Q_A^{(2)*}, d_B^{(1)} - (1+f)\Delta Q_B^{(2)*})}{d\alpha} \\
&= -F_x \frac{d\Delta Q_A^{(2)*}}{d\alpha} - (1+f)F_y \frac{d\Delta Q_B^{(2)*}}{d\alpha} \\
&= -fF_y \frac{d\Delta Q_B^{(2)*}}{d\alpha} > 0,
\end{aligned}
$$

where the first equality follows the invariance property of the pricing function, and to deduce the third equality we have used the condition $-F_x \frac{d\Delta Q_A^{(2)*}}{d\alpha} - F_y \frac{d\Delta Q_B^{(2)*}}{d\alpha} = 0$, which is achieved by differentiating the equation $F(d_A^{(1)} - \Delta Q_A^{(2)*}, d_B^{(1)} - \Delta Q_B^{(2)*}) = F(d_A^{(1)}, d_B^{(1)})$ on both sides with respect to α, and the inequality follows from Lemma B.1. Hence, we have shown that $F(d_A^{(2)} - \Delta q_A^{(3)*}, d_B^{(2)} - \Delta q_B^{(3)*})$ is increasing in α. Thus, by the third property of the pricing function F stated in Assumption 1, and the fact that the ratio of tokens A and B, $\frac{d_A^{(2)} - \Delta q_A^{(3)*}}{d_B^{(2)} - \Delta q_B^{(3)*}}$ is already determined by (B.23), we conclude that both $d_A^{(2)} - \Delta q_A^{(3)*}$ and $d_B^{(2)} - \Delta q_B^{(3)*}$ are increasing in α.

We next show that the reverse trade amount $-\Delta q_A^{(3)*}$ increases in α. The intuition behind this monotonicity result is that the larger α, the higher the price impact of an investor's trade, and the larger the trade size of the reverse trade by the arbitrageur. The quantity $-\Delta q_A^{(3)*}$ is pinned down by (B.23) and the constraint $F(d_A^{(2)} - \Delta q_A^{(3)*}, d_B^{(2)} - \Delta q_B^{(3)*}) = F(d_A^{(2)}, d_B^{(2)})$. Differentiating (B.23) with respect to α, we have:

$$
\begin{aligned}
&\frac{1}{1+f} F_{xx} \left(\frac{d}{d\alpha} d_A^{(2)} + \frac{d(-\Delta q_A^{(3)*})}{d\alpha} \right) + \frac{1}{1+f} F_{xy} \left(\frac{d}{d\alpha} d_B^{(2)} + \frac{d(-\Delta q_B^{(3)*})}{d\alpha} \right) \\
&- \frac{p_A^{(1)}}{p_B^{(1)}} \left(F_{yy} \left(\frac{d}{d\alpha} d_B^{(2)} + \frac{d(-\Delta q_B^{(3)*})}{d\alpha} \right) - F_{xy} \left(\frac{d}{d\alpha} d_A^{(2)} + \frac{d(-\Delta q_A^{(3)*})}{d\alpha} \right) \right) = 0.
\end{aligned}
$$

Recall that $d_A^{(2)} = (d_A^{(1)} - \Delta Q_A^{(2)*})$, $d_B^{(2)} = d_B^{(1)} - (1+f)\Delta Q_B^{(2)*}$. We can then rewrite the equation above as

$$\left(\frac{1}{1+f}F_{xx}-\frac{p_A^{(1)}}{p_B^{(1)}}F_{xy}\right)\frac{d(d_A^{(1)}-\Delta Q_A^{(2)*})}{d\alpha}+\left(\frac{1}{1+f}F_{xy}-\frac{p_A^{(1)}}{p_B^{(1)}}F_{yy}\right)\frac{d(d_B^{(1)}-(1+f)\Delta Q_B^{(2)*})}{d\alpha}$$
$$+\left(\frac{1}{1+f}F_{xx}-\frac{p_A^{(1)}}{p_B^{(1)}}F_{xy}\right)\frac{d(-\Delta q_A^{(3)*})}{d\alpha}+\left(\frac{1}{1+f}F_{xy}-\frac{p_A^{(1)}}{p_B^{(1)}}F_{yy}\right)\frac{d(-\Delta q_B^{(3)*})}{d(-\Delta q_A^{(3)*})}\frac{d(-\Delta q_A^{(3)*})}{d\alpha}=0.$$

By Assumption 1, $F_{xx}<0, F_{xy}>0, F_{yy}<0$; by Lemma B.1, we have $\frac{d(d_A^{(1)}-\Delta Q_A^{(2)*})}{d\alpha}>0$, and $\frac{d(d_B^{(1)}-(1+f)\Delta Q_B^{(2)*})}{d\alpha}<0$. By the constraint $F(d_A^{(2)}-\Delta q_A^{(3)*}, d_B^{(2)}-\Delta q_B^{(3)*})=F(d_A^{(2)},d_B^{(2)})$, we have $\frac{d(-\Delta q_B^{(3)*})}{d(-\Delta q_A^{(3)*})}<0$. Combining those conditions, we conclude that $\frac{d(-\Delta q_A^{(3)*})}{d\alpha}>0$.

We are now ready to compute the return from deposits. If a reverse trade arbitrage occurs after the investor trades, the return from deposits is:

$$\begin{aligned}
&\frac{p_A^{(3)}d_A^{(3)}+p_B^{(3)}d_B^{(3)}}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}-1\\
=&\frac{p_A^{(1)}(d_A^{(1)}-\Delta Q_A^{(2)*}-(1+f)\Delta q_A^{(3)*})+p_B^{(1)}(d_B^{(1)}-(1+f)\Delta Q_B^{(2)*}-\Delta q_B^{(3)*})}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}-1\\
=&\frac{p_A^{(1)}(d_A^{(1)}-\Delta Q_A^{(2)*}-\Delta q_A^{(3)*})+p_B^{(1)}(d_B^{(1)}-(1+f)\Delta Q_B^{(2)*}-\Delta q_B^{(3)*})}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}\\
+&\frac{p_A^{(1)}(-f\Delta q_A^{(3)*})}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}-1\\
=&\frac{p_A^{(1)}(d_A^{(2)}-\Delta q_A^{(3)*})+p_B^{(1)}(d_B^{(2)}-\Delta q_B^{(3)*})}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}+\frac{p_A^{(1)}(-f\Delta q_A^{(3)*})}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}-1,
\end{aligned}\tag{B.24}$$

where the first equality follows from the conditions $d_A^{(3)}=d_A^{(1)}-\Delta Q_A^{(2)*}-(1+f)\Delta q_A^{(3)*}$ and $d_B^{(3)}=d_B^{(1)}-\Delta Q_B^{(2)*}-(1+f)\Delta q_B^{(3)*}$. Those two conditions reflect the fact that the deposit in the AMM is only altered by the investor's trade and the arbitrageur's trade. Recall we have shown that both the first and the second term of (B.24) is increasing in α, and thus the return from deposits after an investor's trade is increasing in α.

We next show that the first term in (B.24), given by $\frac{p_A^{(1)}(d_A^{(2)}-\Delta q_A^{(3)*})+p_B^{(1)}(d_B^{(2)}-\Delta q_B^{(3)*})}{p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)}}$, is greater than 1. Since the AMM requires the liquidity providers to deposit at the fundamental exchange rate, the condition $\left.\frac{F_x}{F_y}\right|_{(x,y)=(d_A^{(1)},d_B^{(1)})}=\frac{p_A^{(1)}}{p_B^{(1)}}$ must hold. We also have $\left.\frac{F_x}{F_y}\right|_{(x,y)=(d_A^{(2)}-\Delta q_A^{(3)*},d_B^{(2)}-\Delta q_B^{(3)*})}\neq$

$\frac{p_A^{(1)}}{p_B^{(1)}}$ by (B.23). Moreover, we have the following inequality:

$$
\begin{aligned}
F(d_A^{(2)} - \Delta q_A^{(3)*}, d_B^{(2)} - \Delta q_B^{(3)*}) = F(d_A^{(2)}, d_B^{(2)}) &= F(d_A^{(1)} - \Delta Q_A^{(2)*}, d_B^{(1)} - (1+f)\Delta Q_B^{(2)*}) \\
&> F(d_A^{(1)} - \Delta Q_A^{(2)*}, d_B^{(1)} - \Delta Q_B^{(2)*}) \\
&= F(d_A^{(1)}, d_B^{(1)}),
\end{aligned}
$$

where the inequality follows from the fact that $F_y > 0$ and $\Delta Q_B^{(2)*} < 0$,. The equalities follows from the fact that the value of the pricing function, without accounting for fees, must stay unchanged after each transaction. Hence, the value of pricing function increases after the investors trade.

By Assumption 1, we can write $F(d_A^{(2)} - \Delta q_A^{(3)*}, d_B^{(2)} - \Delta q_B^{(3)*})$ as $c^l F(\frac{d_A^{(2)} - \Delta q_A^{(3)*}}{c}, \frac{d_B^{(2)} - \Delta q_B^{(3)*}}{c})$ where $c > 1, l > 0$, and $F(\frac{d_A^{(2)} - \Delta q_A^{(3)*}}{c}, \frac{d_B^{(2)} - \Delta q_B^{(3)*}}{c}) = F(d_A^{(1)}, d_B^{(1)})$. It follows from Lemma B.4 that

$$
\begin{aligned}
p_A^{(1)} d_A^{(1)} + p_B^{(1)} d_B^{(1)} &\leq p_A^{(1)} \left(\frac{d_A^{(2)} - \Delta q_A^{(3)*}}{c} \right) + p_B^{(1)} \left(\frac{d_B^{(2)} - \Delta q_B^{(3)*}}{c} \right) \\
&< p_A^{(1)} (d_A^{(2)} - \Delta q_A^{(3)*}) + p_B^{(1)} (d_B^{(2)} - \Delta q_B^{(3)*}),
\end{aligned}
$$

which means that the first term in (B.24) is greater than 1. Moreover, since $\Delta q_A^{(3)*} < 0$ we know that the second term $\frac{p_A^{(1)}(-f\Delta q_A^{(3)*})}{p_A^{(1)} d_A^{(1)} + p_B^{(1)} d_B^{(1)}}$ is positive. It then follows that the return from deposits in (B.24) is positive.

We denote the return from deposit after a trade of a "type A" and "type B" investor as $R_{inv_A}^{(2)}$ and $R_{inv_B}^{(2)}$, respectively. Using similar arguments as above, we can conclude that these returns are both increasing in α. By Lemma B.3, they are independent of the amount of deposits and only depend of the token price ratio $\frac{p_A^{(1)}}{p_B^{(1)}}$.

Liquidity freeze threshold. By the law of total expectation, we have the following chain of equalities:

$$
\begin{aligned}
&\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_{ND}\Big] \\
=&\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_A\Big]\frac{p_A^{(1)}d_A^{(1)}}{(p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)})} - \mathbb{E}\Big[R_B\Big]\frac{p_B^{(1)}d_B^{(1)}}{(p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)})} \\
=&\frac{\kappa_I}{2}R_{inv_A}^{(2)} + \frac{\kappa_I}{2}R_{inv_B}^{(2)} + \kappa_A(1-\kappa_B)(R_{arb_A}^{(2)} - \beta\frac{p_A^{(1)}d_A^{(1)}}{(p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)})}) \\
+&\kappa_B(1-\kappa_A)(R_{arb_B}^{(2)} - \beta\frac{p_A^{(1)}d_A^{(1)}}{(p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)})}) \\
=&\frac{\kappa_I}{2}R_{inv_A}^{(2)} + \frac{\kappa_I}{2}R_{inv_B}^{(2)} - \Bigg(\kappa_A(1-\kappa_B)\frac{\pi(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, (1+\beta)p_A^{(1)})}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}} \\
+&\kappa_B(1-\kappa_A)\frac{\pi(d_A^{(1)}, d_B^{(1)}, (1+\beta)p_B^{(1)}, p_A^{(1)})}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}}\Bigg)
\end{aligned}
\tag{B.25}
$$

where, in the last expression, we have used the expressions of $R_{arb_A}^{(2)}$ and $R_{arb_B}^{(2)}$ given, respectively in equations (B.19) and (B.20). The first two terms in the last equality of Eq. (B.25) are the expected returns from fees, and the third term is the expected token value loss due to arbitrage. The first two terms are non-negative, and the remaining terms are non-positive.

If $\frac{\kappa_I}{2}R_{inv_A}^{(2)} + \frac{\kappa_I}{2}R_{inv_B}^{(2)} = 0$, then (B.25) is non-positive, and thus there exists a liquidity freeze for any $\beta > f$.

Next, consider the case $\frac{\kappa_I}{2}R_{inv_A}^{(2)} + \frac{\kappa_I}{2}R_{inv_B}^{(2)} > 0$. Recall that the arbitrage loss ratios $\frac{\pi(d_A^{(1)}, d_B^{(1)}, (1+\beta)p_B^{(1)}, p_A^{(1)})}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}}$ and $\frac{\pi(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, (1+\beta)p_A^{(1)})}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}}$ are increasing in β, equal to zero if $\beta \leq f$, and converge to infinity as $\beta \to \infty$. Thus, (B.25) is positive when $\beta \to 0$, and negative when $\beta \to \infty$. Since the first two terms are independent of β, while the last term is decreasing in β, the expression in (B.25) is monotonically decreasing in β. It is also continuous in β because all terms are differentiable respect to β. Therefore, there exists a threshold $\overline{\beta_{frz}}$ such that (B.25) is negative if and only if $\beta > \overline{\beta_{frz}}$.

Hence, we can conclude that there exists a critical threshold $\overline{\beta_{frz}} \in \mathbb{R}$ such that a "liquidity freeze" occurs if and only if $\beta > \overline{\beta_{frz}}$.

Comparative Statics. We show that $\overline{\beta_{frz}}$ is increasing in α, κ_I.

For the case $\frac{\kappa_I}{2} R^{(2)}_{inv_A} + \frac{\kappa_I}{2} R^{(2)}_{inv_B} = 0$, we have that $\overline{\beta_{frz}} = f$.

We next consider the case $\frac{\kappa_I}{2} R^{(2)}_{inv_A} + \frac{\kappa_I}{2} R^{(2)}_{inv_B} > 0$. The threshold $\overline{\beta_{frz}}$ is defined by the equation: $\mathbb{E}\left[R_D - R_{ND}\right] = 0$. Using the relation (B.25), we take the partial derivative of $\mathbb{E}\left[R_D - R_{ND}\right]$ with respect to α, κ_I, and obtain

$$\frac{\partial \mathbb{E}\left[R_D - R_{ND}\right]}{\partial \alpha} = \frac{\kappa_I}{2} \left(\frac{\partial R^{(2)}_{inv_A}}{\partial \alpha} + \frac{\partial R^{(2)}_{inv_B}}{\partial \alpha} \right) > 0, \tag{B.26}$$

$$\frac{\partial \mathbb{E}\left[R_D - R_{ND}\right]}{\partial \kappa_I} = \frac{1}{2} R^{(2)}_{inv_A} + \frac{1}{2} R^{(2)}_{inv_B} > 0, \tag{B.27}$$

An application of the Implicit Function Theorem to the curve $\mathbb{E}\left[R_D - R_{ND}\right] = 0$ yields $\frac{\partial \overline{\beta_{frz}}}{\partial \alpha} > 0$, and $\frac{\partial \overline{\beta_{frz}}}{\partial \kappa_I} > 0$.

□

Proof of Proposition 3. Recall that the gas fee bid with the arbitrage order is equal to the largest possible profit from the arbitrage, given by $\pi(d^{(2)}_A, d^{(2)}_B, p^{(2)}_B, p^{(2)}_A)$. By Lemma B.2,

$$\pi(d^{(2)}_A, d^{(2)}_B, p^{(2)}_B, p^{(2)}_A) = d^{(1)}_A \pi \left(\frac{d^{(2)}_A}{d^{(1)}_A}, \frac{d^{(2)}_B}{d^{(1)}_A}, p^{(2)}_B, p^{(2)}_A \right).$$

Recall that the realization of the random variable $\frac{d^{(2)}_A}{d^{(1)}_A}, \frac{d^{(2)}_B}{d^{(1)}_A}$ does not depend on $d^{(1)}_A$. The profit $\pi \left(\frac{d^{(2)}_A}{d^{(1)}_A}, \frac{d^{(2)}_B}{d^{(1)}_A}, p^{(2)}_B, p^{(2)}_A \right)$ is a random variable whose value does not depend on the amount of tokens A deposited in the AMM, $d_A^{(1)}$. We have that

$$\begin{aligned}\mathbb{E}[g_{(arb,j)}^{(3)}] &= \mathbb{E}[\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})] \\ &= \mathbb{E}\left[d_A^{(1)}\pi\left(\frac{d_A^{(2)}}{d_A^{(1)}}, \frac{d_B^{(2)}}{d_A^{(1)}}, p_B^{(2)}, p_A^{(2)}\right)\right] \\ &= d_A^{(1)}\mathbb{E}\left[\pi\left(\frac{d_A^{(2)}}{d_A^{(1)}}, \frac{d_B^{(2)}}{d_A^{(1)}}, p_B^{(2)}, p_A^{(2)})\right)\right],\end{aligned}$$

which is increasing in $d_A^{(1)}$ because $\pi\left(\frac{d_A^{(2)}}{d_A^{(1)}}, \frac{d_B^{(2)}}{d_A^{(1)}}, p_B^{(2)}, p_A^{(2)}\right)$ is a non-negative random variable, and thus $\mathbb{E}\left[\pi\left(\frac{d_A^{(2)}}{d_A^{(1)}}, \frac{d_B^{(2)}}{d_A^{(1)}}, p_B^{(2)}, p_A^{(2)}\right)\right] \geq 0$. The required ratio of deposits $\frac{d_A^{(1)}}{d_B^{(1)}}$ is also uniquely pinned down by the token price ratio before period 1, which means that $d_B^{(1)}$ is a constant multiple of $d_A^{(1)}$. Hence, the quantity $\mathbb{E}[g_{(arb,j)}^{(3)}]$ is also increasing in $d_B^{(1)}$ because $d_B^{(1)}$ is just $d_A^{(1)}$ multiplied by a constant. Using a similar argument, we find that $Var[g_{(arb,j)}^{(3)}] = (d_A^{(1)})^2 Var\left[\pi\left(\frac{d_A^{(2)}}{d_A^{(1)}}, \frac{d_B^{(2)}}{d_A^{(1)}}, p_B^{(2)}, p_A^{(2)}\right)\right]$, which implies that the variance of gas fee increases in $d_A^{(1)}$ and $d_B^{(1)}$.

We next show that $\mathbb{E}[g_{(arb,j)}^{(3)}] = \mathbb{E}[\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})]$ is increasing in β. It suffices to show that any possible realization of $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ is increasing in β. If no event occurs or an investor arrives, $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ is independent of β. If a token price shock occurs, by Lemma 1, we deduce that $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})$ is increasing in β. Hence, we obtain that $\mathbb{E}[g_{(arb,j)}^{(3)}] = \mathbb{E}[\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})]$ is increasing in β. By a similar argument, we can show that $Var[g_{(arb,j)}^{(3)}]$ is increasing in β.

□

Proof of Lemma 2. We first show that the expected investor surplus is decreasing in k. We then show that arbitrageurs' profits from stale price arbitrage and reverse trade arbitrage is decreasing in k. The calculations for the investors' surplus and arbitrage losses follow the same procedure used for the proof of Lemma 1, except that we now have an explicit expression of the pricing function F_k and thus can obtain explicit expressions for the trading quantities of investors and arbitrageurs. This allows us to examine the sensitivities of their payoffs to the parameter k.

Recall that liquidity providers must deposit their tokens at the spot price $\left.\frac{F_x}{F_y}\right|_{(x,y)=(d_A^{(1)},d_B^{(1)})} = \frac{p_A^{(1)}}{p_B^{(1)}}$. This condition guarantees that a liquidity provider deposits an equivalent value of tokens A and B, i.e., $\frac{p_A^{(1)}}{p_B^{(1)}} = \frac{d_B^{(1)}}{d_A^{(1)}}$ in period 1.

We calculate the expected investors' surplus for "type A" investors and "type B" investors, respectively given by $\mathbb{E}\left[s_A(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})\right]$, and $\mathbb{E}\left[s_B(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})\right]$. We consider the case where a "type B" investor arrives. The case where a "type A" investor arrives can be easily handled with symmetric arguments. We plug the expression of the pricing function $F_k(x,y) = (1-k)\ A\ F_0(x,y) + k\ F_1(x,y) = (1-k)\ A\ (p_A^{(1)}x + p_B^{(1)}y) + k\ xy$ into the investor's optimization problem (B.1). After algebraic manipulations and using the condition $\frac{p_A^{(1)}}{p_B^{(1)}} = \frac{d_B^{(1)}}{d_A^{(1)}}$, we obtain $\Delta Q_B^{(2)} = \frac{\Delta Q_A^{(2)} d_B^{(1)}}{k\Delta Q_A^{(2)} - d_A^{(1)}}$ and the following single-variable optimization problem equivalent to (B.1):

$$\max_{\frac{d_A^{(1)}}{-1+k} \leq \Delta Q_A^{(2)} \leq 0} \quad p_A^{(1)}(1+f)\Delta Q_A^{(2)} + (1+\alpha)p_B^{(1)} \frac{\Delta Q_A^{(2)} d_B^{(1)}}{k\Delta Q_A^{(2)} - d_A^{(1)}} \tag{B.28}$$

As in the more general case considered in the proof of Lemma 1, the optimal trading amount is $\Delta Q_A^{(2)*} = 0$ if $\alpha \leq f$. If $\frac{1+f}{(1-k)^2} - 1 > \alpha > f$, then the optimal trading amount is attained if the marginal benefit equals the marginal cost, i.e.,

$$\Delta Q_A^{(2)*} = \frac{d_A^{(1)}}{k}\left(1 - \sqrt{\frac{1+\alpha}{1+f}}\right). \tag{B.29}$$

If $1+\alpha > \frac{1+f}{(1-k)^2}$, then the optimal trading amount is achieved if all tokens B are purchased by the investor, i.e.,

$$\Delta Q_A^{(2)*} = \frac{d_A^{(1)}}{-1+k}. \tag{B.30}$$

If $\alpha \leq f$, we have $s_B(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)}) = 0$. If $\frac{1+f}{(1-k)^2} - 1 > \alpha > f$, then we plug (B.29) into (B.28) and obtain that the ("type B") investor's maximum surplus is given by

$$s_B(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)}) = \frac{(\sqrt{1+\alpha} - \sqrt{1+f})^2}{k} p_A^{(1)} d_A^{(1)},$$

which is decreasing in k.

If $\alpha \geq \frac{1+f}{(1-k)^2} - 1$, then we plug (B.30) into (B.28) and obtain that the investor's maximum surplus is given by

$$s_B(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)}) = \left((1+\alpha) - \frac{1+f}{1-k}\right) p_A^{(1)} d_A^{(1)},$$

which is decreasing in k.

Recall that the probability "type A" and "type B" investors' arrival are constant and equal to $\frac{\kappa_I}{2}$. Therefore, by the law of total probability, $\mathbb{E}\left[\sum_{i=A,B} s_i(d_A^{(1)}, d_B^{(1)}, p_B^{(1)}, p_A^{(1)})\right]$ is decreasing in k.

We now show that profits from a stale price arbitrage is decreasing in k. Suppose a shock only hits token B. The case where a shocks hits token A can be proven via symmetric arguments. We plug the specification of the pricing curve, $F_k(x,y) = (1-k)\ A\ F_0(x,y) + k\ F_1(x,y) = (1-k)\ A\ (p_A^{(1)} x + p_B^{(1)} y) + k\ xy$, into the arbitrageur's optimization problem (B.2). After straightforward algebraic manipulations and using the condition $\frac{p_A^{(1)}}{p_B^{(1)}} = \frac{d_B^{(1)}}{d_A^{(1)}} = \frac{d_B^{(2)}}{d_A^{(2)}}$, we obtain the following arbitrageur's optimization problem in period 3:

$$\begin{aligned} \max_{\Delta q_A^{(3)}, \Delta q_B^{(3)}} \quad & p_A^{(2)}(1+f)\Delta q_A^{(3)} + p_B^{(2)} \Delta q_B^{(3)} \\ \text{s.t.} \quad & q_B^{(3)} = \frac{q_A^{(3)} d_B^{(2)}}{k q_A^{(3)} - d_A^{(2)}}, \\ & \Delta q_A^{(3)} \leq 0, d_B^{(2)} \geq \Delta q_B^{(3)} \geq 0. \end{aligned} \tag{B.31}$$

Plugging $q_B^{(3)} = \frac{q_A^{(3)} d_B^{(2)}}{k q_A^{(3)} - d_A^{(2)}}$ into $d_B^{(2)} \geq \Delta q_B^{(3)} \geq 0$, we obtain $\frac{d_A^{(2)}}{-1+k} \leq \Delta q_A^{(3)} \leq 0$. We then use $\frac{q_A^{(3)} d_B^{(2)}}{k q_A^{(3)} - d_A^{(2)}}$ to replace $q_B^{(3)}$ in (B.31), which leads to the following equivalent single variable and unconstrained optimization problem:

$$\max_{\frac{d_A^{(2)}}{-1+k} \leq \Delta q_A^{(3)} \leq 0} \quad p_A^{(1)}(1+f)\Delta q_A^{(3)} + (1+\beta) p_B^{(1)} \frac{\Delta q_A^{(3)} d_B^{(2)}}{k \Delta q_A^{(3)} - d_A^{(2)}} \tag{B.32}$$

As in the more general case considered in the proof of Lemma 1, the optimal trading amount

$\Delta q_A^{(3)} = 0$ if $\beta \le f$. If $\beta > f$, then the optimal trading amount is achieved either when the marginal benefit is equal to the marginal cost, i.e.

$$p_A^{(1)}(1+f) - (1+\beta)p_B^{(1)} \frac{d_A^{(2)} d_B^{(2)}}{(k\Delta q_A^{(3)*} - d_A^{(2)})^2} = 0,$$

which leads to

$$\Delta q_A^{(3)*} = \frac{d_A^{(2)}}{k}\left(1 - \sqrt{\frac{1+\beta}{1+f}}\right) \tag{B.33}$$

or when arbitrageur acquires all tokens B in the AMM, i.e.,

$$p_A^{(1)}(1+f) - (1+\beta)p_B^{(1)} \frac{d_A^{(2)} d_B^{(2)}}{(k\Delta q_A^{(3)*} - d_A^{(2)})^2} < 0, \frac{\Delta q_A^{(3)*} d_B^{(2)}}{k\Delta q_A^{(3)*} - d_A^{(2)}} = d_B^{(2)},$$

which leads to the following solution

$$\Delta q_A^{(3)*} = \frac{d_A^{(2)}}{-1+k}. \tag{B.34}$$

The latter case occurs if $1+\beta > \frac{1+f}{(1-k)^2}$, i.e., if the convexity of the pricing function is sufficiently small.

We next plug the optimal trading amount into (B.32). If $\beta \le f$, we have $\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(1)}) = 0$. If $\frac{1+f}{(1-k)^2} - 1 > \beta > f$, plugging (B.33) into the objective function given by (B.32), we obtain

$$\begin{aligned} \pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}) &= \frac{(\sqrt{\frac{1+\beta}{1+f}} - 1)(\sqrt{(1+\beta)(1+f)}p_B^{(1)} d_B^{(2)} - p_A^{(1)}(1+f)d_A^{(2)})}{k} \\ &= \frac{(\sqrt{1+\beta} - \sqrt{1+f})^2 p_A^{(1)} d_A^{(1)}}{k}, \end{aligned} \tag{B.35}$$

where the second equality has been obtained using the identity $\frac{p_A^{(1)}}{p_B^{(1)}} = \frac{d_B^{(1)}}{d_A^{(1)}}$, and the fact that $d_A^{(2)} = d_A^{(1)}, d_B^{(2)} = d_B^{(1)}$ if only a token price shock arrives in period 2. The above expression is decreasing in k.

If $\beta \ge \frac{1+f}{(1-k)^2} - 1$, we plug (B.34) into the objective function given by (B.32). Following the same procedure above, we obtain

$$\pi(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}) = (1+\beta)p_B^{(1)} d_B^{(2)} - (1+f)p_A^{(1)} \frac{d_A^{(2)}}{1-k}$$
$$= (1+\beta)p_B^{(1)} d_B^{(1)} - (1+f)p_A^{(1)} \frac{d_A^{(1)}}{1-k}, \tag{B.36}$$

where the last equality is obtained using the identity $\frac{p_A^{(1)}}{p_B^{(1)}} = \frac{d_B^{(1)}}{d_A^{(1)}}$, and the fact that $d_A^{(2)} = d_A^{(1)}, d_B^{(2)} = d_B^{(1)}$ if a token price shock occurs in period 2. Clearly, the above expression is decreasing in k.

We then prove that profits from reverse trade arbitrage are decreasing in k. Assume there exists a reverse trade arbitrage opportunity after a "type B" investor trades. The case where a reverse trade occurs after a "type A" investor trades can be handled symmetrically. We plug the expression of the pricing curve $F_k(x,y) = (1-k)\ A\ F_0(x,y) + k\ F_1(x,y) = (1-k)\ A\ (p_A^{(1)}x + p_B^{(1)}y) + k\ xy$, into the arbitrageur's optimization problem (B.2). After straightforward algebraic manipulations and using that $p_B^{(1)} = p_B^{(2)}, p_A^{(1)} = p_A^{(2)}$ if no price shock occurs, we obtain the following equivalent single variable and unconstrained optimization problem:

$$\max_{\Delta q_B^{(3)} \leq 0} \quad p_B^{(1)}(1+f)\Delta q_B^{(3)} + p_A^{(1)}\Delta q_B^{(3)} \frac{A(1-k)p_B^{(1)} + k d_A^{(2)}}{A(-1+k)p_A^{(1)} + k(\Delta q_B^{(3)} - d_B^{(2)})}. \tag{B.37}$$

Using the relations $d_B^{(2)} = d_B^{(1)} - \Delta Q_B^{(2)*}$, $d_A^{(2)} = d_A^{(1)} - \Delta Q_A^{(2)*}$, $p_B^{(1)} d_B^{(1)} = p_A^{(1)} d_A^{(1)}$, $A = (\frac{p_A^{(1)} p_B^{(1)}}{d_B^{(1)} d_A^{(1)}})^{1/2}$, $\Delta Q_B^{(2)} = \frac{\Delta Q_A^{(2)} d_B^{(1)}}{k\Delta Q_A^{(2)} - d_A^{(1)}}$ and (B.29) to simplify (B.37), we can rewrite (B.37) as

$$\max_{\Delta q_B^{(3)} \leq 0} \quad p_B^{(1)}(1+f)\Delta q_B^{(3)} + p_A^{(1)}\Delta q_B^{(3)} \frac{d_A^{(2)}(1+\alpha - f\frac{\sqrt{1+\alpha}}{\sqrt{1+f}})}{-d_B^{(2)} + k\frac{\sqrt{1+\alpha}}{\sqrt{1+f}}\Delta q_B^{(3)}}. \tag{B.38}$$

As in the more general case considered in the proof of Lemma 1, the optimal trading amount $\Delta q_B^{(3)*}$ exists and is unique. An application of the Envelope theorem yields

$$\frac{\partial \pi}{\partial k} = -p_A^{(1)}(\Delta q_B^{(3)})^2(1+\alpha)\frac{\sqrt{(1+f)(1+a)} - f}{(1-f)\left(-d_B^{(2)} + k\frac{\sqrt{1+\alpha}}{\sqrt{1+f}}\Delta q_B^{(3)}\right)^2} < 0,$$

where $\sqrt{(1+f)(1+a)} - f$ is negative because $\alpha > f$. If $\alpha \leq f$, investors would not trade in period 2, so there would not be any reverse trade arbitrage. Hence, liquidity providers' losses from a reverse trade arbitrage are decreasing in k.

We have shown that liquidity providers' losses from a stale price arbitrage and reverse trade arbitrage are both decreasing in k. Moreover, the probability that a stale price arbitrage or a reverse trade arbitrage occurs are independent of k. By the law of total probability, the expected arbitrage loss of liquidity providers is decreasing in k.

□

Proof of Proposition 4. We follow the proof strategy of Theorem 2. Let $k_1 := 1 - \sqrt{\frac{1+f}{1+\alpha}}$, which yields the condition $\alpha = \frac{1+f}{(1-k_1)^2} - 1$. Let $k_2 := 1 - \sqrt{\frac{1+f}{1+\beta}}$, which yields the condition $\beta = \frac{1+f}{(1-k_2)^2} - 1$. It follows from the assumption $\alpha \geq \beta$ that $k_1 \geq k_2$.

We distinguish between two cases: (1) there does not exist a reverse trade arbitrage opportunity after the investor arrives and trades in period 2 , (2) there exists a reverse trade arbitrage opportunity after the investor arrives and trades. We only consider the case (1), because case (2) follows from similar arguments and using a slightly different expression for the return. Case (1) occurs if $\alpha - f(1 + \frac{\sqrt{1+\alpha}}{\sqrt{1+f}}) \leq 0$.

As in the more general case considered in the proof of Proposition 4, the liquidity provider will deposit if and only if $\mathbb{E}\Big[R_D\Big] > \mathbb{E}\Big[R_{ND}\Big]$. The expected return from not depositing tokens is $\mathbb{E}\Big[R_{ND}\Big] := \mathbb{E}\Big[1 + R_A\Big]\frac{p_A^{(1)}d_A^{(1)}}{(p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)})} + \mathbb{E}\Big[1 + R_B\Big]\frac{p_B^{(1)}d_B^{(1)}}{(p_A^{(1)}d_A^{(1)}+p_B^{(1)}d_B^{(1)})} - 1$. It remains to calculate the expected return from depositing $\mathbb{E}\Big[R_D\Big]$.

We first consider the case where a "type B" investor arrives to the AMM, and calculate the return from depositing. Recall from (B.22) that

$$R_{inv_B}^{(2)} = \frac{(1+f)p_A^{(1)}(-\Delta Q_A^{(2)*}) + p_B^{(1)}(-\Delta Q_B^{(2)*})}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}}. \tag{B.39}$$

If $k > k_1$, plugging $\Delta Q_A^{(2)*} = \frac{d_A^{(1)}}{k}\left(1 - \sqrt{\frac{1+\alpha}{1+f}}\right)$ from (B.29) into (B.39) and using the relation

$Q_B^{(2)*} = \frac{\Delta Q_A^{(2)*} d_B^{(2)}}{k\Delta Q_A^{(2)*} - d_A^{(2)}}$, we obtain

$$
\begin{aligned}
R^{(2)}_{inv_B} &= \frac{(1+f)p_A^{(1)}\left(\frac{d_A^{(1)}}{k}\left(\sqrt{\frac{1+\alpha}{1+f}}-1\right)\right) - p_B^{(1)}\frac{d_B^{(1)}}{k\sqrt{\frac{1+\alpha}{1+f}}}\left(\sqrt{\frac{1+\alpha}{1+f}}-1\right)}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}} \\
&= \frac{(1+f-\sqrt{\frac{1+f}{1+\alpha}})(\sqrt{\frac{1+\alpha}{1+f}}-1)}{2k}. \qquad \text{(B.40)}
\end{aligned}
$$

If $k < k_1$, plugging $\Delta Q_A^{(2)*} = \frac{d_A^{(1)}}{-1+k}$ from (B.30) into (B.39) and using the relationship $Q_B^{(2)*} = \frac{\Delta Q_A^{(2)*} d_B^{(2)}}{k\Delta Q_A^{(2)*} - d_A^{(2)}}$, we have

$$
\begin{aligned}
R^{(2)}_{inv_B} &= \frac{(1+f)p_A^{(1)}\frac{d_A^{(1)}}{1-k} - p_B^{(1)}d_B^{(1)}}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}} \\
&= \frac{1}{2}\left(\frac{1+f}{1-k} - 1\right). \qquad \text{(B.41)}
\end{aligned}
$$

Following the same procedure above, we obtain $R^{(2)}_{inv_A} = R^{(2)}_{inv_B}$.

We next consider the event that token B is hit by a shock. Then there exists an arbitrage opportunity. Recall that in this case, the return from depositing is

$$
R^{(2)}_{arb_B} = \beta\frac{p_B^{(1)}d_B^{(1)}}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}} - \frac{\pi(d_A^{(1)}, d_B^{(1)}, (1+\beta)p_B^{(1)}, p_A^{(1)})}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}}
$$

If $k > k_2$, using (B.35), the identity $\frac{p_A^{(1)}}{p_B^{(1)}} = \frac{d_B^{(1)}}{d_A^{(1)}}$, and the fact that $d_A^{(2)} = d_A^{(1)}, d_B^{(2)} = d_B^{(1)}$ if no investor arrives in period 2, we obtain

$$
R^{(2)}_{arb_B} = \frac{\beta}{2} - \frac{(\sqrt{1+\beta} - \sqrt{1+f})^2}{2k}. \qquad \text{(B.42)}
$$

Similarly, if $k < k_2$, using (B.36) and the above identities, we obtain

$$
R^{(2)}_{arb_B} = \frac{\beta}{2} - \frac{(1+\beta-\frac{1+f}{1-k})}{2}. \qquad \text{(B.43)}
$$

Following the same procedure above, we obtain $R^{(2)}_{arb_B} = R^{(2)}_{arb_A}$. We next compare the expectation of the return from depositing, $\mathbb{E}\Big[R_D\Big]$, with the expectation of return from not depositing $\mathbb{E}\Big[R_{ND}\Big]$. Recall from (B.25) that

$$\begin{aligned}
&\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_{ND}\Big] \\
=&\frac{\kappa_I}{2}R^{(2)}_{inv_A} + \frac{\kappa_I}{2}R^{(2)}_{inv_B} - \Bigg(\kappa_A(1-\kappa_B)\frac{\pi(d^{(1)}_A, d^{(1)}_B, p^{(1)}_B, (1+\beta)p^{(1)}_A)}{p^{(1)}_A d^{(1)}_A + p^{(1)}_B d^{(1)}_B} \\
&+\kappa_B(1-\kappa_A)\frac{\pi(d^{(1)}_A, d^{(1)}_B, (1+\beta)p^{(1)}_B, p^{(1)}_A)}{p^{(1)}_A d^{(1)}_A + p^{(1)}_B d^{(1)}_B}\Bigg).
\end{aligned} \tag{B.44}$$

If $k \geq k_1 \geq k_2$, plugging (B.40) and (B.42) into (B.44), and using the identities $R^{(2)}_{arb_B} = R^{(2)}_{arb_A}$ and $R^{(2)}_{inv_A} = R^{(2)}_{inv_B}$, we obtain

$$\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_{ND}\Big] = \frac{1}{k}\tau_1, \tag{B.45}$$

where

$$\tau_1 = \Bigg(\kappa_I\frac{(1+f-\sqrt{\frac{1+f}{1+\alpha}})(\sqrt{\frac{1+\alpha}{1+f}}-1)}{2} - \frac{(\sqrt{1+\beta}-\sqrt{1+f})^2}{2}(\kappa_A(1-\kappa_B)+\kappa_B(1-\kappa_A))\Bigg).$$

If $\tau_1 < 0$, then all terms in (B.45) are negative, and we have $\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_{ND}\Big] < 0$, that is, the expected return from depositing is smaller than the expected return from holding A tokens. As a result, there is a liquidity freeze. The liquidity providers' expected return is then $\mathbb{E}\Big[R_{ND}\Big]$, which is independent of k.

If $\tau_1 > 0$, then $\mathbb{E}\Big[R_D\Big] - \mathbb{E}\Big[R_{ND}\Big]$ is decreasing in k. Since $\mathbb{E}\Big[R_{ND}\Big]$ is independent of k, we have that $\mathbb{E}\Big[R_D\Big]$ is decreasing in k. Hence, the liquidity providers' expected return, $\max\Big\{\mathbb{E}\Big[R_D\Big], \mathbb{E}\Big[R_{ND}\Big]\Big\}$ is decreasing in k.

If $k_1 \geq k \geq k_2$, plugging (B.41) and (B.42) into (B.44), and using the identities $R^{(2)}_{arb_B} =$

$R^{(2)}_{arb_A}$ and $R^{(2)}_{inv_A} = R^{(2)}_{inv_B}$, we obtain

$$\mathbb{E}\left[R_D\right] - \mathbb{E}\left[R_{ND}\right] = \tau_2, \tag{B.46}$$

where

$$\tau_2 = \frac{\kappa_I}{2}\left(\frac{1+f}{1-k} - 1\right) - \frac{1}{k}\frac{\sqrt{1+\beta} - \sqrt{1+f})^2}{2}(\kappa_A(1-\kappa_B) + \kappa_B(1-\kappa_A)),$$

which is increasing in k because all terms in τ_2 are increasing in k.

If $k_2 \geq k \geq 0$, plugging (B.40) and (B.43) into (B.44), and using the identities $R^{(2)}_{arb_B} = R^{(2)}_{arb_A}$ and $R^{(2)}_{inv_A} = R^{(2)}_{inv_B}$, we have:

$$\mathbb{E}\left[R_D\right] - \mathbb{E}\left[R_{ND}\right] = \tau_3, \tag{B.47}$$

where

$$\tau_3 = \left(\frac{\kappa_I}{2}\left(\frac{1+f}{1-k} - 1\right) - \frac{(1+\beta - \frac{1+f}{1-k})}{2}(\kappa_A(1-\kappa_B) + \kappa_B(1-\kappa_A))\right),$$

which is increasing in k because all terms in τ_3 are increasing in k.

Therefore, $\mathbb{E}\left[R_D\right] - \mathbb{E}\left[R_{ND}\right]$ is increasing in k if $k < k_1$. Since $\mathbb{E}\left[R_{ND}\right]$ is independent of k, we have the expected return from depositing, $\mathbb{E}\left[R_D\right]$ is increasing in k. Moreover, we also have the liquidity providers' expected return $\max\left\{\mathbb{E}\left[R_D\right], \mathbb{E}\left[R_{ND}\right]\right\}$ is increasing in k if $k < k_1$.

The liquidity providers' expected return $\max\left\{\mathbb{E}\left[R_D\right], \mathbb{E}\left[R_{ND}\right]\right\}$ is maximized at $k^* = k_1$, because it increases in k on the interval $[0, k_1]$ and decreases in k on the interval $[k_1, 1]$. Since $\mathbb{E}\left[R_D\right] - \mathbb{E}\left[R_{ND}\right]$ is also maximized at $k^* = k_1$, if a liquidity freeze occurs at $k = k^*$, i.e, $\mathbb{E}\left[R_D\right] - \mathbb{E}\left[R_{ND}\right] < 0$, then $\mathbb{E}\left[R_D\right] - \mathbb{E}\left[R_{ND}\right] < 0$ for any other $k \in [0, 1]$, which means that a liquidity freeze also occurs for other values of $k \in [0, 1]$.

$\square$

Proof of Theorem 5. All transactions, except for investors' trades, are transfers of wealth.

Hence, the aggregate welfare can be written as the total expected benefits earned by investors from trading:

$$\frac{\kappa_I}{2}\alpha p_B^{(1)}\Delta Q_B^{(2)*} + \frac{\kappa_I}{2}\alpha p_A^{(1)}\Delta Q_A^{(2)*}. \tag{B.48}$$

Recall from the proof of Proposition 4 that the payoff of liquidity providers from depositing increases in $[0, k^*]$ and decreases in $[k^*, 1]$.

Consider the trivial case where liquidity providers' return from depositing is lower than not depositing if $k = k^*$. This means that liquidity providers will not deposit for any value of $k \in [0, 1]$. Then welfare is zero for any $k \in [0, 1]$. Then, we can set $k^{opt} = k^*$.

Consider the second trivial case $\alpha < f$. Recall from the proof of Lemma 1 that investors would not trade, and welfare would be zero for any $k \in [0, 1]$. We can then set $k^{opt} = k^*$.

Consider the case where liquidity providers' return from depositing is higher than from not depositing if $k = k^*$. This means that liquidity providers deposit tokens when $k = k^*$. Recall from the proof of Proposition 4 that liquidity providers' return from depositing increases in $[0, k^*]$ and decreases in $[k^*, 1]$, and the payoff is a continuous in k. By monotonicity and continuity, we know that there exist k_l, k_h, where $0 \leq k_l \leq k^* \leq k_h \leq 1$, such that liquidity providers' return from depositing is higher than or equal to the corresponding return from not depositing if and only if $k \in [k_l, k_h]$. Hence, liquidity providers all deposit if and only if $k \in [k_l, k_h]$. First, $k^{opt} \in [k_l, k_h]$ because liquidity providers would not deposit if $k \notin [k_l, k_h]$, and there would be no trade. In such case, the aggregate welfare would be zero, and this would lead to a welfare lower than at k^*. Second, we show $k^{opt} = k_l$. Recall from the proof of Lemma (2) that if a liquidity freeze does not occur, quantities traded by investors $\Delta Q_B^{(2)*}, \Delta Q_A^{(2)*}$ are decreasing in k. This means that the aggregate welfare in (B.48) is decreasing in k in the interval $[k_l, k_h]$, and thus $k^{opt} = k_l$. Third, we show that $\frac{\partial k^{opt}}{\partial \beta} > 0$. The quantity k_l is defined by the relationship $\mathbb{E}[R_D] - \mathbb{E}[R_{ND}] = 0$. Observe from (B.45), (B.46), and (B.47) that $\frac{\partial(\mathbb{E}[R_D] - \mathbb{E}[R_{ND}])}{\partial \beta} < 0$. We apply the implicit function theorem and obtain $\frac{\partial k^{opt}}{\partial \beta} > 0$.

□

Proof of Theorem A.1. If $\beta \leq f$, then no arbitrage occurs, and the stale price arbitrage loss for liquidity providers in both AMMs is zero.

Next, consider the case $\beta > f$. Consider the AMM which manages tokens A and B only. We are considering an AMM utilizing a constant product function, that is, a special case of F_k where we set $k = 1$. Plugging $k = 1$ into (B.35), we have that the realized token value loss when a stale price arbitrage opportunity occurs in period 3, is given by

$$\pi_{AB}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)}) = (\sqrt{1+\beta} - \sqrt{1+f})^2 p_A^{(1)} d_A^{(1)}.$$

The probability that a stale price arbitrage opportunity occurs is $2(1-\kappa)\kappa$. In period t, the expected arbitrage ratio of the AMM that pools two tokens is

$$\mathbb{E}\left[\frac{\pi_{AB}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})}{p_A^{(1)} d_A^{(1)} + p_B^{(1)} d_B^{(1)}}\right] = ((1-\kappa)\kappa(\sqrt{1+\beta} - \sqrt{1+f})^2,$$

where we have used that liquidity providers must deposit at the spot rate, and thus the ratio of deposits is uniquely pinned down by the relations $p_A^{(1)} d_A^{(1)} = p_B^{(1)} d_B^{(1)}$.

We then calculate the arbitrage ratio of the AMM that pools three tokens. An arbitrage occurs if the token price shock hits only one or two tokens. The probability that one token only is hit by a shock is $3(1-\kappa)^2\kappa$, and the probability that two tokens are hit by (idiosyncratic) shocks is $3(1-\kappa)\kappa^2$. Suppose that only B token is hit by a price shock, then the arbitrageur solves the following optimization problem:

$$\begin{aligned} \max_{\Delta q_A^{(3)}, \Delta q_B^{(3)}, \Delta q_C^{(3)}} \quad & p_A^{(1)}(1+f)\Delta q_A^{(3)} + (1+\beta) p_B^{(1)} \Delta q_B^{(3)} + (1+f) p_C^{(1)} \Delta q_C^{(3)} \\ \text{s.t.} \quad & d_A^{(2)} d_B^{(2)} d_C^{(2)} = (d_A^{(2)} - \Delta q_A^{(3)})(d_B^{(2)} - \Delta q_B^{(3)})(d_C^{(2)} - \Delta q_C^{(3)}) \\ & \Delta q_A^{(3)} \leq 0, \Delta q_B^{(3)} \geq 0, \Delta q_C^{(3)} \leq 0. \end{aligned} \tag{B.49}$$

The arbitrage profit is maximized if the following first-order conditions are satisfied:

$$\frac{(1+\beta)p_B^{(1)}}{(1+f)p_A^{(1)}} = \frac{(d_A^{(2)} - \Delta q_A^{(3)})}{(d_B^{(2)} - \Delta q_B^{(3)})}, \frac{(1+\beta)p_B^{(1)}}{(1+f)p_C^{(1)}} = \frac{(d_C^{(2)} - \Delta q_C^{(3)})}{(d_B^{(2)} - \Delta q_B^{(3)})}.$$

The relations above hold if the marginal benefit of exchanging tokens is equal to the marginal cost of the trade.

Using the condition above, the identities $p_A^{(1)}d_A^{(1)} = p_B^{(1)}d_B^{(1)} = p_C^{(1)}d_C^{(1)}, d_A^{(1)} = d_A^{(2)}, d_B^{(1)} = d_B^{(2)}, d_C^{(1)} = d_C^{(2)}$ and the constraint $d_A^{(2)}d_B^{(2)}d_C^{(2)} = (d_A^{(2)} - \Delta q_A^{(3)})(d_B^{(2)} - \Delta q_B^{(3)})(d_C^{(2)} - \Delta q_C^{(3)})$, we obtain

$$\begin{aligned} &p_A^{(1)}(d_A^{(2)} - \Delta q_A^{(3)}) = p_C^{(1)}(d_C^{(2)} - \Delta q_C^{(3)}) = \left(\frac{1+\beta}{1+f}\right)^{\frac{1}{3}} p_A^{(1)}d_A^{(1)}, \\ &p_B^{(1)}(d_B^{(2)} - \Delta q_B^{(3)}) = \left(\frac{1+\beta}{1+f}\right)^{-\frac{2}{3}} p_A^{(1)}d_A^{(1)} \end{aligned} \tag{B.50}$$

Plugging (B.50) into the objective function given in (B.49), we obtain that the realized arbitrage loss if token B only is hit by a shock is

$$\begin{aligned} \pi_{ABC}(d_A^{(2)}, d_B^{(2)}, d_C^{(2)}, p_C^{(2)}, p_B^{(2)}, p_A^{(2)}) &= p_A^{(1)}d_A^{(1)}(3 + \beta + 2f - 3(1+f)^{\frac{2}{3}}(1+\beta)^{\frac{1}{3}}) \\ &\geq p_A^{(1)}d_A^{(1)}(2 + \beta + f - 2(1+f)^{\frac{1}{2}}(1+\beta)^{\frac{1}{2}}) \\ &= p_A^{(1)}d_A^{(1)}(\sqrt{1+\beta} - \sqrt{1+f})^2, \end{aligned}$$

where we have used the geometric inequality $1 + f + 2(1+f)^{\frac{1}{2}}(1+\beta)^{\frac{1}{2}}) \geq 3(1+f)^{\frac{2}{3}}(1+\beta)^{\frac{1}{3}}$.

Following an analogous procedure, we can verify that the following inequality holds if two tokens are hit by price shocks:

$$\pi_{ABC}(d_A^{(2)}, d_B^{(2)}, d_C^{(2)}, p_C^{(2)}, p_B^{(2)}, p_A^{(2)}) \geq p_A^{(1)}d_A^{(1)}(\sqrt{1+\beta} - \sqrt{1+f})^2. \tag{B.51}$$

Recall that the probability that only one token is hit by a shock is $3(1-\kappa)^2\kappa$, and the probability that two tokens are hit by shocks is $3(1-\kappa)\kappa^2$. Applying the inequality in (B.51) and the identity $p_A^{(1)}d_A^{(1)} = p_B^{(1)}d_B^{(1)} = p_C^{(1)}d_C^{(1)}$, we deduce

$$\begin{aligned}
&\mathbb{E}\left[\frac{\pi_{ABC}(d_A^{(2)}, d_B^{(2)}, d_C^{(2)}, p_C^{(2)}, p_B^{(2)}, p_A^{(2)})}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)} + p_C^{(1)}d_C^{(1)}}\right] \\
&\geq ((1-\kappa)^2\kappa + (1-\kappa)\kappa^2)(\sqrt{1+\beta} - \sqrt{1+f})^2 \\
&= (1-\kappa)\kappa(\sqrt{1+\beta} - \sqrt{1+f})^2 \\
&= \mathbb{E}\left[\frac{\pi_{AB}(d_A^{(2)}, d_B^{(2)}, p_B^{(2)}, p_A^{(2)})}{p_A^{(1)}d_A^{(1)} + p_B^{(1)}d_B^{(1)}}\right]
\end{aligned}$$

This means that if liquidity providers deposit the same value of tokens in both exchanges, the expected loss from stale price arbitrage is higher in the pool with three tokens. □