1 General Model

Timeline:

1. The bank observes the future loan return/performance r_L perfectly;
2. The bank sets the deposit rate r_D;
3. The loan return/performance r_L is realized and disclosed;
4. The deposit flow is given by $D(r_D, r_L)$, where $\frac{\partial D}{\partial r_D} > 0$, $\frac{\partial^2 D}{\partial r_D^2} \leq 0$ and $\frac{\partial D}{\partial r_L} > 0$ (depositors are more likely to invest in banks with stronger loan performance).

The bank solves the following optimization problem.

$$\max_{r_D | r_L} (r_L - r_D) D(r_D, r_L)$$

The first order condition is given by:

$$-D + (r_L - r_D^*) \frac{\partial D}{\partial r_D} = 0.$$

The second order condition is given by:

$$-\frac{2\partial D}{\partial r_D} + (r_L - r_D^*) \frac{\partial^2 D}{\partial r_D^2} = -\frac{2\partial D}{\partial r_D} + \frac{D}{\frac{\partial D}{\partial r_D}} \frac{\partial^2 D}{\partial r_D^2} = 0.$$

The second step uses the first order condition. All three terms are negative and hence there exists a unique optimal solution of r_D^*.

Using the implicit function theorem yields:

$$
\begin{aligned}
& -\frac{\partial D}{\partial r_D}\frac{\partial r_D^*}{\partial r_L}-\frac{\partial D}{\partial r_L}+\left(1-\frac{\partial r_D^*}{\partial r_L}\right)\frac{\partial D}{\partial r_D} \\
& +\left(r_L-r_D^*\right)\left(\frac{\partial^2 D}{\partial r_D^2}\frac{\partial r_D^*}{\partial r_L}+\frac{\partial^2 D}{\partial r_D \partial r_L}\right) \\
= \;& -\frac{\partial D}{\partial r_D}\frac{\partial r_D^*}{\partial r_L}-\frac{\partial D}{\partial r_L}+\left(1-\frac{\partial r_D^*}{\partial r_L}\right)\frac{\partial D}{\partial r_D} \\
& +\frac{D}{\frac{\partial D}{\partial r_D}}\left(\frac{\partial^2 D}{\partial r_D^2}\frac{\partial r_D^*}{\partial r_L}+\frac{\partial^2 D}{\partial r_D \partial r_L}\right) \\
= \;& 0.
\end{aligned}
$$

The second step uses the first order condition.

Hence

$$
\frac{\partial r_D^*}{\partial r_L}=-\frac{-\frac{\partial D}{\partial r_L}+\frac{D}{\frac{\partial D}{\partial r_D}}\frac{\partial^2 D}{\partial r_D \partial r_L}+\frac{\partial D}{\partial r_D}}{\frac{D}{\frac{\partial D}{\partial r_D}}\frac{\partial^2 D}{\partial r_D^2}-\frac{2\partial D}{\partial r_D}}.
$$

The denominator of $\frac{\partial r_D^*}{\partial r_L}$ is negative by the second order condition. The sign of $\frac{\partial r_D^*}{\partial r_L}$ is determined by the numerator. The first term is negative (as a higher r_L increases the deposit amount, it is more costly to pay a higher deposit rate). The sign of the second term is ambiguous and depends on the sign of $\frac{\partial^2 D}{\partial r_D \partial r_L}$. The last term is positive (as a higher r_L improves the return of the deposits, it is more profitable to pay a higher deposit rate in order to attract deposits).

For simplicity, assume that $\frac{\partial^2 D}{\partial r_D \partial r_L}=0$. Then the trade-off between the first and the last effects can make the sign of $\frac{\partial r_D^*}{\partial r_L}$ ambiguous. $\frac{\partial r_D^*}{\partial r_L}<0$ if and only if $\frac{\partial D}{\partial r_D}<\frac{\partial D}{\partial r_L}$.

2 Imperfect information

Timeline:

1. The bank observes a noisy signal s about the future loan return/performance r_L, where the bank's belief about r_L is given by the distribution function $F\left(r_L|s\right)$, and a higher s makes r_L higher in the sense of first-order stochastic dominance;

2. The bank sets the deposit rate r_D;

3. The loan return/performance r_L is realized and disclosed;

4. The deposit flow is given by $D(r_D, r_L) = H(r_D) + G(r_L)$, where $H' > 0, H'' \leq 0$ and $G' > 0$.

The bank solves the following optimization problem.

$$\max_{r_D|s} E\left[(r_L - r_D) D(r_D, r_L) | s\right]$$

The first order condition is given by:

$$\begin{aligned} & E\left[(r_L - r_D^*) \frac{\partial D}{\partial r_D} - D|s\right] \\ = \; & E\left[(r_L - r_D^*) H'(r_D^*) - H(r_D^*) - G(r_L)|s\right] = 0. \end{aligned}$$

This becomes:

$$(E[r_L|s] - r_D^*) H'(r_D^*) - E[G(r_L)|s] = H(r_D^*).$$

The left-hand side is decreasing in r_D^* whereas the right-hand side is increasing in r_D^*. Hence, the solution to r_D^* is unique.

The left-hand side is increasing in s if and only if

$$\frac{\partial E[r_L|s]}{\partial s} H'(r_D^*) > \frac{\partial E[G(r_L)|s]}{\partial s}.$$

Assume that $G(r_L)$ is linear; hence

$$\frac{\partial E[r_L|s]}{\partial s} H'(r_D^*) > G'(r_L) \frac{\partial E[r_L|s]}{\partial s}.$$

Since $\frac{\partial E[r_L|s]}{\partial s} > 0$ by FOSD, we obtain that

$$H'(r_D^*) > G'(r_L).$$

Accordingly, we obtain the following empirical prediction:

Empirical Prediction: *The current deposit rate* r_D^* *is more likely to be positively associated with future loan charge-offs, if the loan performance has a stronger effect on the deposit flow (higher* $G'(r_L)$*) or the deposit flow is less responsive to the deposit rate (lower* $H'(r_D^*)$*), and vice versa.*

Note: we should look for shocks to expected loan performance, if feasible. In addition, we should verify in the data if the deposit flow responds positively to banks' reported loan performance and the deposit rate.

3 Bank's reporting discretion

Timeline:

1. The bank observes privately the future loan performance r_L.

2. The bank sets the deposit rate r_D and issues a potentially biased report about r_L:

$$R_L = r_L + b,$$

   where the cost of biasing the report is $\frac{c}{2}b^2$ and $c > 0$.

3. The deposit flow is given by $D(r_D, E[r_L|R_L]) = H(r_D) + \beta E[r_L|R_L]$, where $H' > 0$, $H'' \leq 0$ and $\beta > 0$.

The bank solves the following optimization problem.

$$\max_{r_D, b} (r_L - r_D) D(r_D, E[r_L|R_L]) - \frac{c}{2}b^2$$

The first order conditions are given by:

$$(r_L - r_D^*) \frac{\partial D}{\partial r_D} - D = (r_L - r_D^*) H'(r_D^*) - H(r_D^*) - \beta E[r_L|R_L] = 0,$$

where by rational expectations, $E[r_L|R_L] = R_L - b^*$, and

$$\left(r_L - r_D^*\right)\beta \frac{\partial E[r_L|R_L]}{\partial b} = \left(r_L - r_D^*\right)\beta = cb^*.$$

This yields

$$b^* = \frac{\left(r_L - r_D^*\right)\beta}{c}.$$

Using $E[r_L|R_L] = R_L - b^* = r_L + b^* - b^* = r_L$ yields:

$$\left(r_L - r_D^*\right) H'\left(r_D^*\right) - \beta r_L = H\left(r_D^*\right).$$

The bank's report is given by

$$R_L = r_L + b^* = r_L + \frac{\left(r_L - r_D^*\right)\beta}{c}.$$

Hence

$$cov\left(R_L, r_D^*\right) = cov\left(r_L + \frac{\left(r_L - r_D^*\right)\beta}{c}, r_D^*\right) = \left(1 + \frac{\beta}{c}\right) var\left(r_D^*\right)\left[\frac{cov\left(r_L, r_D^*\right)}{var\left(r_D^*\right)} - \frac{\beta}{c+\beta}\right].$$

The first term is positive if and only if $H'\left(r_D^*\right) > \beta$. The second term is always negative. Intuitively, a higher deposit rate reduces the manipulation incentive and leads to a lower report of loan performance. All else, equal, this term results in a negative association between R_L and r_D^* (i.e., a positive association between deposit rate and loan charge-offs).

Accordingly, we obtain the following empirical prediction:

Empirical Prediction: *The current deposit rate r_D^* is more likely to be positively associated with future loan charge-offs, if the loan performance has a stronger effect on the deposit flow (higher β), the deposit flow is less responsive to the deposit rate (lower $H'\left(r_D^*\right)$), and/or the manipulation incentive is strong (lower c).*

4 Imperfect information II

Timeline:

1. The bank observes a noisy signal s about the future loan return/performance r_L,

$$s = r_L + \varepsilon,$$

   where $r_L \sim N\left(\bar{r}, \frac{1}{h_L}\right)$ and $\varepsilon \sim N\left(0, \frac{1}{h_\varepsilon}\right)$; given the information structure,

$$E\left[r_L|s\right] = \frac{h_L \bar{r} + h_\varepsilon s}{h_L + h_\varepsilon};$$

2. The bank sets the deposit rate r_D;

3. The bank reports the expected loan performance $E\left[r_L|s\right]$ to depositors;

4. The deposit flow is given by $D\left(r_D, E\left[r_L|s\right]\right) = H\left(r_D\right) + G\left(E\left[r_L|s\right]\right)$, where $H' > 0, H'' \leq 0$ and $G' > 0$.

The bank solves the following optimization problem.

$$\max_{r_D|s} E\left[\left(r_L - r_D\right) D\left(r_D, r_L\right)|s\right]$$

The first order condition is given by:

$$\begin{aligned} & E\left[\left(r_L - r_D^*\right)\frac{\partial D}{\partial r_D} - D|s\right] \\ = \quad & E\left[\left(r_L - r_D^*\right) H'\left(r_D^*\right) - H\left(r_D^*\right) - G\left(E\left[r_L|s\right]\right)|s\right] = 0. \end{aligned}$$

This becomes:

$$\left(E\left[r_L|s\right] - r_D^*\right) H'\left(r_D^*\right) - G\left(E\left[r_L|s\right]\right) = H\left(r_D^*\right).$$

The left-hand side is decreasing in r_D^* whereas the right-hand side is increasing in r_D^*. Hence, the solution to r_D^* is unique.

The left-hand side is increasing in the expected loan performance $E[r_L|s]$ if and only if

$$H'(r_D^*) > G'(E[r_L|s]).$$

Accordingly, we obtain the following empirical prediction:

Empirical Prediction: *The current deposit rate r_D^* is more likely to be positively associated with future loan charge-offs, if the loan performance has a stronger effect on the deposit flow (higher $G'(E[r_L|s])$) or the deposit flow is less responsive to the deposit rate (lower $H'(r_D^*)$), and vice versa.*

Next, consider the bank's discretionary reporting of its loan performance. For simplicity, assume that the bank observes its loan performance perfectly, i.e., $r_L = s$. In addition, assume that $G(E[r_L|s]) = \beta E[r_L|s]$, where $\beta > 0$.

Timeline:

1. The bank observes privately the future loan performance r_L.

2. The bank sets the deposit rate r_D and issues a potentially biased report about r_L:

$$R_L = r_L + b,$$

   where the cost of biasing the report is $\frac{c}{2}b^2$ and $c > 0$.

3. The deposit flow is given by $D(r_D, E[r_L|R_L]) = H(r_D) + \beta E[r_L|R_L]$, where $H' > 0$, $H'' \leq 0$ and $\beta > 0$.

The bank solves the following optimization problem.

$$\max_{r_D, b} (r_L - r_D) D(r_D, E[r_L|R_L]) - \frac{c}{2}b^2$$

The first order conditions are given by:

$$(r_L - r_D^*) \frac{\partial D}{\partial r_D} - D = (r_L - r_D^*) H'(r_D^*) - H(r_D^*) - \beta E[r_L|R_L] = 0,$$

where by rational expectations, $E[r_L|R_L] = R_L - b^*$, and

$$(r_L - r_D^*)\,\beta \frac{\partial E[r_L|R_L]}{\partial b} = (r_L - r_D^*)\,\beta = cb^*.$$

This yields

$$b^* = \frac{(r_L - r_D^*)\,\beta}{c}.$$

Using $E[r_L|R_L] = R_L - b^* = r_L + b^* - b^* = r_L$ yields:

$$(r_L - r_D^*)\,H'(r_D^*) - \beta r_L = H(r_D^*)\,.$$

The bank's report is given by

$$R_L = r_L + b^* = r_L + \frac{(r_L - r_D^*)\,\beta}{c}.$$

Hence

$$cov\,(R_L, r_D^*) = cov\left(r_L + \frac{(r_L - r_D^*)\,\beta}{c}, r_D^*\right) = \left(1 + \frac{\beta}{c}\right) var\,(r_D^*)\left[\frac{cov\,(r_L, r_D^*)}{var\left(r_D^*\right)} - \frac{\beta}{c+\beta}\right].$$

The first term is positive if and only if $H'(r_D^*) > \beta$. The second term is always negative. Intuitively, a higher deposit rate reduces the manipulation incentive and leads to a lower report of loan performance. All else, equal, this term results in a negative association between R_L and r_D^* (i.e., a positive association between deposit rate and loan charge-offs).

Accordingly, we obtain the following empirical prediction:

Empirical Prediction: *The current deposit rate r_D^* is more likely to be positively associated with future loan charge-offs, if the loan performance has a stronger effect on the deposit flow (higher β), the deposit flow is less responsive to the deposit rate (lower $H'(r_D^*)$), and/or the manipulation incentive is strong (lower c).*

5 Bank run model

Timeline:

1. The bank is endowed with a project (a loan). The project can either succeed and return $V > 0$ for each unit of investment or fail and return 0. The success of the project depends on its quality/performance θ and disruptions caused by withdrawal of deposits, l. Assume that θ follows an improper prior over the real line for simplicity. The project succeeds if $\theta > \delta l$, where $\delta > 0$ represents the cost of disruption from deposits withdrawals.

2. The bank observes a noisy signal y (e.g., reports of loan loss provisioning) about θ:

$$y = \theta + \eta,$$

where $\eta \sim N\left(0, \frac{1}{\alpha}\right)$;

3. The bank sets the deposit rate r_D;

4. The bank attracts deposits from a group of depositors $i \in [0, 1]$;

5. The bank reports the signal y to the group of depositors and each depositor also observes a private signal about θ:

$$x_i = \theta + \varepsilon_i,$$

where $\varepsilon_i \sim N\left(0, \frac{1}{\beta}\right)$;

6. Each depositor chooses whether to withdraw the deposit at the face value. If a depositor withdraws, she receives 1 and if the depositor stays, the payoff is r_D if the project succeeds and 0 if the project fails.

We first solve the equilibrium in depositors' withdrawal decisions. When depositor i observes the signal x_i and the bank's report y: her posterior distribution of θ is normal with mean

$$\xi_i \equiv \frac{\alpha y + \beta x_i}{\alpha + \beta},$$

and precision $\alpha+\beta$. Suppose that each depositor adopts a switching strategy such that she stays if her posterior $\xi_i \geq \xi$. This reduces into:

$$x_i \geq x(\xi, y) = \frac{\alpha+\beta}{\beta}\xi - \frac{\alpha}{\beta}y.$$

In addition, define ψ such that at $\theta = \psi$, the project is on the margin of success and failure, i.e., $\psi = \delta l$. Given the depositor's switching strategy, at $\theta = \psi$, the fraction of depositors withdrawing is

$$l = \Pr(x_i < x) = \Pr\left(\sqrt{\beta}(x_i - \psi) < \sqrt{\beta}(x - \psi)\right) = \Phi\left(\sqrt{\beta}(x - \psi)\right).$$

This gives

$$\delta\Phi\left(\sqrt{\beta}(x - \psi)\right) = \psi,$$

which can be simplified as:

$$\delta\Phi\left(\frac{\alpha}{\sqrt{\beta}}(\xi - y) + \sqrt{\beta}(\xi - \psi)\right) = \psi,$$

Finally, at $\xi_i = \xi$, the depositor should be indifferent between withdrawing and staying. The project succeeds whenever $\theta \geq \psi$. Therefore,

$$1 = r_D\left(1 - \Phi\left(\sqrt{\alpha+\beta}(\psi - \xi)\right)\right).$$

This gives that

$$\psi - \xi = -\frac{\Phi^{-1}(1/r_D)}{\sqrt{\alpha+\beta}}.$$

Combining the two equations gives that:

$$\begin{aligned}
\psi &= \delta\Phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\psi - y + \frac{\sqrt{\alpha+\beta}}{\alpha}\Phi^{-1}(1/r_D)\right)\right), \\
\xi &= \psi + \frac{\Phi^{-1}(1/r_D)}{\sqrt{\alpha+\beta}}.
\end{aligned}$$

To ensure a unique equilibrium, we need

$$\frac{\delta}{\sqrt{2\pi}}\frac{\alpha}{\sqrt{\beta}} < 1.$$

Conditional on θ, the bank's payoff is given by

$$1_{\theta\geq\psi}\left(V-r_D\right)l\left(\theta\right) = 1_{\theta\geq\psi}\left(V-r_D\right)\left[1-\Phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\xi-y\right)+\sqrt{\beta}\left(\xi-\theta\right)\right)\right].$$

Given that θ is normally distributed with y and precision α, the bank's expected payoff is given by:

$$\begin{aligned} & \left(V-r_D\right)\int_{\psi}^{\infty}\left[1-\Phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\xi-y\right)+\sqrt{\beta}\left(\xi-\theta\right)\right)\right]f\left(\theta;y,1/\alpha\right)d\theta \\ = & \left(V-r_D\right)\int_{\psi}^{\infty}\left[1-\Phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\psi-y+\Phi^{-1}\left(1/r_D\right)\frac{\sqrt{\alpha+\beta}}{\alpha}\right)+\sqrt{\beta}\left(\psi-\theta\right)\right)\right]f\left(\theta;y,1/\alpha\right)d\theta, \end{aligned}$$

where

$$\psi = \delta\Phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\psi-y+\frac{\sqrt{\alpha+\beta}}{\alpha}\Phi^{-1}\left(1/r_D\right)\right)\right).$$

The deposit flow is increasing in y (the loan performance), and increasing in r_D (the deposit rate). The loan success probability is increasing in y and the deposit rate.

Taking the first-order condition with respect to r_D yields:

$$\begin{aligned} & -\int_{\psi}^{\infty}\left[1-\Phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\psi-y+\Phi^{-1}\left(1/r_D\right)\frac{\sqrt{\alpha+\beta}}{\alpha}\right)+\sqrt{\beta}\left(\psi-\theta\right)\right)\right]f\left(\theta;y,1/\alpha\right)d\theta \\ & -\left(V-r_D\right)\frac{\partial\psi}{\partial r_D}\left[1-\Phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\psi-y+\Phi^{-1}\left(1/r_D\right)\frac{\sqrt{\alpha+\beta}}{\alpha}\right)\right)\right]f\left(\psi;y,1/\alpha\right) \\ & +\left(V-r_D\right)\left\{\int_{\psi}^{\infty}\left[\phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\psi-y+\Phi^{-1}\left(1/r_D\right)\frac{\sqrt{\alpha+\beta}}{\alpha}\right)+\sqrt{\beta}\left(\psi-\theta\right)\right)\right]f\left(\theta;y,1/\alpha\right)d\theta\right\} \\ & \frac{\alpha}{\sqrt{\beta}}\left(\frac{1}{\phi\left(1/r_D\right)}\frac{1}{r_D^2}\frac{\sqrt{\alpha+\beta}}{\alpha}\right). \end{aligned}$$

6 Numerical problem

$$
\begin{aligned}
r_D^* \quad &\in \\
\arg\max_{r_D} \qquad & (V - r_D) \int_{\psi}^{\infty} \left[1 - \Phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\psi - y + \Phi^{-1}(1/r_D)\frac{\sqrt{\alpha+\beta}}{\alpha}\right) + \sqrt{\beta}(\psi - \theta)\right)\right] f(\theta; y, 1/\alpha)\, d\theta, \\
s.t., \qquad & \psi = \delta\Phi\left(\frac{\alpha}{\sqrt{\beta}}\left(\psi - y + \frac{\sqrt{\alpha+\beta}}{\alpha}\Phi^{-1}(1/r_D)\right)\right), \\
& 1 \leq r_D \leq V,
\end{aligned}
$$

where $\Phi(\cdot)$ is the CDF of standard normal distribution, and $f(\cdot)$ is the density of θ, which is normally distributed with mean y and variance $1/\alpha$.

Without loss of generality, set $V = 2$. Assume that $\frac{\delta}{\sqrt{2\pi}}\frac{\alpha}{\sqrt{\beta}} < 1$, $\delta > 0$, $\alpha > 0$ and $\beta > 0$. Analyze how the optimal level of r_D^* varies with y, depending the values of α, β and δ.