HW5_Writting_assignment

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1. $f(x) = \dfrac{1}{\sqrt{(2\pi)^k |\Sigma|}} \cdot e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$, $\Sigma \succ 0$ ($\Sigma = LL^T$, L is invertable)

let $y = \Sigma^{-\frac{1}{2}}(x-\mu) \Rightarrow x = \mu + \Sigma^{\frac{1}{2}}y$, $dx = |\det L|\, dy = |\Sigma|^{\frac{1}{2}}\, dy$

$\int_{\mathbb{R}^k} f(x)\, dx = \dfrac{1}{(2\pi)^{\frac{k}{2}} \cdot |\Sigma|^{\frac{1}{2}}} \int_{\mathbb{R}^k} e^{(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))}\, dx$

substitute $x = \mu + Ly$

$= \dfrac{1}{(2\pi)^{\frac{k}{2}} |\Sigma|^{\frac{1}{2}}} \cdot \int_{\mathbb{R}^k} e^{-\frac{1}{2}y^T y} \cdot |\Sigma|^{\frac{1}{2}}\, dy = \dfrac{1}{(2\pi)^{\frac{k}{2}}} \int_{\mathbb{R}^k} e^{-\frac{\|y\|^2}{2}}\, dy$

$= \dfrac{1}{(2\pi)^{\frac{k}{2}}} \cdot (2\pi)^{\frac{k}{2}} = 1$ ※

2. (a) $-d\, \text{trace}(AB) = \text{trace}(dA\, B) = \text{trace}((B)^T\, dA)$

By def. $d\, \text{trace}(AB) = \text{trace}((\partial \text{trace}(AB)/\partial A)^T\, dA)$

$\Rightarrow (\dfrac{\partial\, \text{trace}(AB)}{\partial A})^T = B \Rightarrow \dfrac{\partial\, \text{trace}(AB)}{\partial A} = B^T$ ※

(b) $\because \text{trace}(u\, v^T) = v^T u$

$\therefore x^T A x = \text{trace}(x^T A x) = \text{trace}(A x x^T) = \text{trace}(x x^T A)$ ※

(c) Given $x_1 \cdots x_n \in \mathbb{R}^k$, $\ell(\mu, \Sigma) = -\dfrac{n}{2}\log|\Sigma| - \dfrac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^T \Sigma^{-1}(x_i - \mu)$

$\dfrac{d\ell}{d\mu} = \Sigma^{-1}\sum_{i=1}^{n}(x_i - \mu) = \Sigma^{-1} n(\bar{x} - \mu) = 0 \Rightarrow \hat{\mu} = \bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$

$\dfrac{\partial}{\partial \Sigma}$: $d\log|\Sigma| = \text{trace}(\Sigma^{-1} d\Sigma)$ $d\,\text{trace}(\Sigma^{-1}S) = -\text{trace}(\Sigma^{-1}S\Sigma^{-1}d\Sigma)$

let $S = \sum_{i=1}^{n}(x_i - \hat{\mu})(x_i - \hat{\mu})^T$, $d\ell = -\dfrac{n}{2}\text{trace}(\Sigma^{-1}d\Sigma) - \dfrac{1}{2}(-\text{trace}(\Sigma^{-1}S\Sigma^{-1}d\Sigma))$

let coefficient $= 0$.

$-\dfrac{n}{2}\Sigma^{-1} + \dfrac{1}{2}\Sigma^{-1}S\Sigma^{-1} = 0 \Rightarrow S = n\Sigma \Rightarrow \hat{\Sigma} = \dfrac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})(x_i - \hat{\mu})^T$ ※

3. Unanswered question.

Q. $\log|\Sigma|$ 項在 MLE 中扮演什麼角色?