COMP 540 HW 3

Peiguang Wang, Xinran Zhou

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1: MAP and MLE parameter estimation

1. Estimate for θ using MLE

   Solution. The maximum likelihood estimation of D given θ is that

   $$MLE = l(D|\theta) = p(x^{(i)}) = \prod_i^m \theta^{x^{(i)}}(1-\theta)^{1-x^{(i)}}$$

   take the NLL of MLE

   $$NLL = -\sum_i^m [x^{(i)} log\theta + (1-x^{(i)})log(1-\theta)]$$

   take the derivative of NLL and make it equal to zero

   $$\frac{\partial NLL}{\partial \theta} = -\sum_i^m [x^{(i)}\frac{1}{\theta} - \frac{1}{(1-\theta)}(1-x^{(i)})] = 0$$

   by computing this equation, we can get θ_{MLE}

   $$\theta_{MLE} = \frac{1}{m}\sum_i^m x^{(i)}$$

2. Compare the MAP and MLE estimates of θ

   Solution. If we add a conjugate prior and use both the D and this prior to make a estimation of θ, we can have this

   $$\begin{aligned} MAP &= l(D|\theta)Beta(D|a,b) \\ &= [\prod_i^m \theta^{x^{(i)}}(1-\theta)^{1-x^{(i)}}]\theta^{a-1}(1-\theta)^{b-1} \end{aligned}$$

   take the derivative of θ and make it equal to zero, we can get θ_{MAP}

   $$\theta_{MAP} = \frac{\sum_i^m x^{(i)} + a + 1}{m + a + b - 2}$$

   if $a = b = 1$ then

   $$\theta_{MAP} = \theta_{MLE} = \frac{1}{m}\sum_i^m x^{(i)}$$

2: Logistic regression and Gaussian Naive Bayes

1. For logistic regression, what is the posterior probability for each class, i.e., $P(y=1|x)$ and $P(y=0|x)$? Write the expression in terms of the parameter θ and the sigmoid function.

   Solution.

   $$P(y=1|x) = h_\theta(X) = \frac{1}{1+e^{-\theta^T X}}$$

   $$P(y=0|x) = 1 - h_\theta(X) = \frac{e^{-\theta^T X}}{1+e^{-\theta^T X}}$$

2. Derive the posterior probabilities for each class

   Solution. The Gaussian distribution and Bernoulli distribution that we assume

$$P(y=1)=\gamma$$

$$p(x_j|y=1)=N(\mu_j^1,\sigma_j^2)$$

$$P(x_j|;y=0)=N(\mu_j^0,\sigma_j^2)$$

   Naive Bayes model

$$p(x|y)=\prod_j^d P(x_j|y)$$

   Bayes rule

$$P(y=1|x)=\frac{P(y=1)P(x|y=1)}{\sum P(y=i)P(x|y=i)}$$

   so these are what we can use now, then we will derive the posterior probabilities

$$\begin{aligned}
p(y=1|x) &= \frac{P(y=1)P(x|y=1)}{P(y=0)P(x|y=0)+P(y=1)P(x|y=1)} \\
&= \frac{\gamma\prod_{j=1}^d N(\mu_j^1,\sigma_j^2)}{\gamma\prod_{j=1}^d N(\mu_j^1,\sigma_j^2)+(1-\gamma)\prod_{j=1}^d N(\mu_j^0,\sigma_j^2)} \\
&= \frac{1}{1+\frac{1-\gamma}{\gamma}\prod_{j=1}^d exp(\frac{(x_j-\mu_j^1)^2-(x_j-\mu_j^0)^2}{2\sigma_j^2})}
\end{aligned}$$

   the probability $P(y=0|x)$ can be derived using the same method or just subtract it from 1.

3. part 3

   Solution. Class 1 and class 0 are equally likely, that means $\gamma=\frac{1}{2}$ the probability equation can be written as

$$P(y=1|x)=\frac{1}{1+\frac{1-\gamma}{\gamma}\prod_{j=1}^d exp(\frac{(x_j-\mu_j^1)^2-(x_j-\mu_j^0)^2}{2\sigma_j^2})}$$

   if we set $\mu_j^0=-\mu_j^1$ then we have

$$P(y=1|x)=\frac{1}{1+\prod_{j=1}^d e^{(\frac{2x_j\mu_j^0}{\sigma_j^2})}}$$

   obviously it has the same form as logistic regression, if we see in this way

$$\theta=[\frac{2\mu_1^0}{\sigma_1^2},\frac{2\mu_2^0}{\sigma_2^2},...,\frac{2\mu_d^0}{\sigma_d^2}]^T$$

$$X=[x_1,x_2,...,x_d]^T$$

   the equation can be rewritten using θ and X

$$P(y=1|x)=\frac{1}{1+e^{-\theta^T X}}$$

3: Reject option in classifiers

1. part 1

   Solution. The loss of choosing a class j is

   $$loss = \lambda_s(1 - P(y = j|x))$$

   then

   $$\lambda_s(1 - P(y = j|x)) \leq \lambda_r$$

   so

   $$P(y = j|x) \geq 1 - \frac{\lambda_r}{\lambda_s}$$

4: Kernelizing k-nearest neighbors

5: Constructing kernels

1. $k(x, x^{'}) = Ck_1(x, x^{'})$

   Solution.

   $$Ck_1(x, x^{'}) = C\Phi_1(x)^T\Phi_1(x^{'}) = (\sqrt{C}\Phi_1(x)^T)(\sqrt{C}\Phi_1(x^{'})^T)$$

   if $C \geq 0$ then $k(x, x^{'})$ is a valid kernel.

2. $k(x, x^{'}) = f(x)k_1(x, x^{'})f(x^{'})$

   Solution.

   $$f(x)k_1(x, x^{'})f(x^{'}) =< f(x)\Phi_1(x), f(x^{'})\Phi_1(x^{'}) >$$

   since it satisfy the Mercer's theorem, $k(x, x^{'})$ is valid

3. $(x, x^{'}) = k_1(x, x^{'}) + k_1(x, x^{'})$

   Solution. Because $k_1(x, x^{'}$ and $k_2(x, x^{'}$ both are valid kernels, so by Mercer's theorem they both satisfy

   $$\int_d k(x, x^{'} f(x)f(x^{'}) \geq 0$$

   then