COMP 540 HW 1

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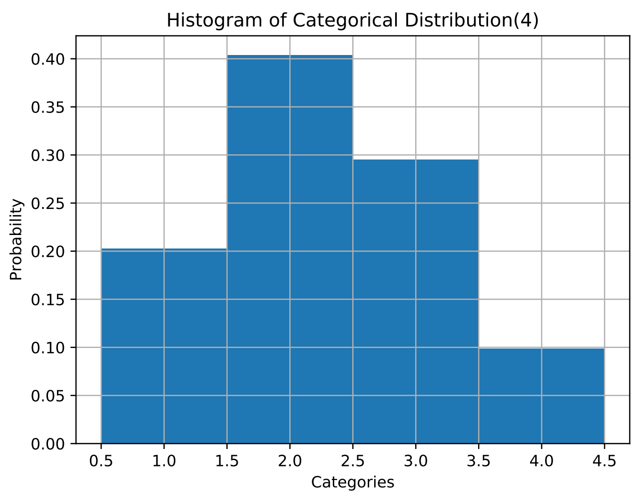
Due: 1/18/2018

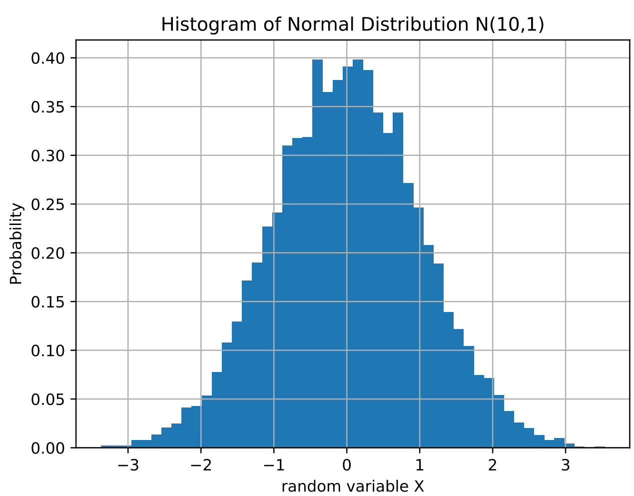
Part 0: Background refresher

1. Generate different distributions from uniform distribution:

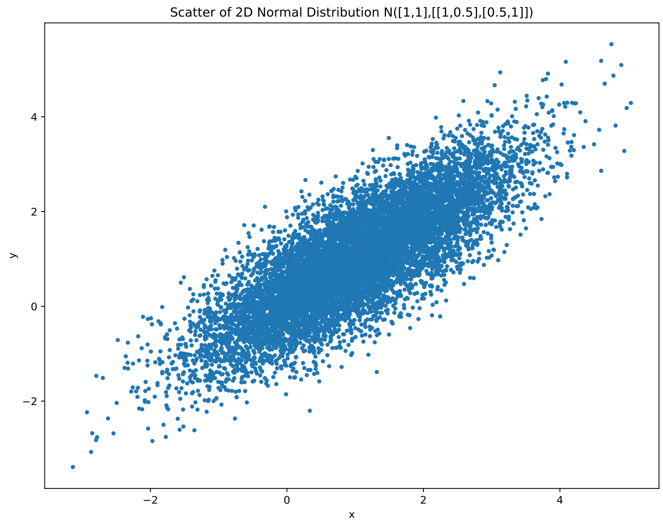
   (a) Plot the histogram of a categorical distribution with probabilities [0.2,0.4,0.3,0.1].



   (b) Plot the univariate normal distribution with mean of 10 and standard deviation of 1.



   (c) Produce a scatter plot of the samples for a 2-D Gaussian with mean at [1,1] and a covariance matrix [[1,0.5],[0.5,1]]



(d) Test your mixture sampling code by writing a function that implements an equal weighted mixture of four Gaussians in 2 dimensions, centered at $(\pm 1; \pm 1)$ and having covariance I. Estimate the probability that a sample from this distribution lies within the unit circle centered at (0.1, 0.2).
The Probability that falls in unit circle with center at (0.1,0.2) is 0.1815.

2. Prove that the sum of two independent Poisson random variables is also a Poisson random variable.

Proof. The characteristic function of a Poisson random variable is

$$\Phi_1(u) = e^{\lambda_1 (e^{iu}-1)}$$

Let X_1 and X_2 denote two independent Poisson random variables. Let $X = X_1 + X_2$
Let $\Phi_1(u)$ and $\Phi_2(u)$ denote the characteristic functions of X_1 and X_2:

$$\Phi_1(u) = e^{\lambda_1 (e^{iu}-1)}$$

$$\Phi_2(u) = e^{\lambda_2 (e^{iu}-1)}$$

Let $\Phi(u)$ denote the characteristic functions of X. Since $X = X_1 + X_2$, we have:

$$\Phi(u) = \Phi_1(u)\Phi_2(u) = e^{\lambda_1 (e^{iu}-1)} e^{\lambda_2 (e^{iu}-1)}$$

Simplify the equation above,

$$\Phi(u) = e^{(\lambda_1+\lambda_2)(\frac{\lambda_1}{\lambda_1+\lambda_2}e^{iu}+\frac{\lambda_2}{\lambda_1+\lambda_2}e^{iu})-1}.$$

That is

$$\Phi(u) = e^{(\lambda_1+\lambda_2)(e^{iu}-1)}.$$

Comparing with the characteristic function of Poisson distribution, we can see that X is also a Poisson random variable. □

3. Let X_0 and X_1 be continuous random variables. Show that if

$$P(X_0 = x_0) = \alpha_0 e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}}$$

$$P(X_1 = x_1 | X_0 = x_0) = \alpha e^{-\frac{(x_1 - x_0)^2}{2\sigma^2}}$$

there exists α_1, μ_1 and σ_1 such that

$$P(X_1 = x_1) = \alpha_1 e^{-\frac{(x_1 - \mu_1)^2}{2\sigma_1^2}}$$

Write down expressions for these quantities in terms of α_0, α, μ_0, σ_0 and σ.

Solution. If X,Y are both Gaussian random variable, then

$$Y|X = x \sim N\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X), \quad \sigma_Y^2(1 - \rho^2)\right)$$

where μ_X, μ_Y are mean of X and Y; σ_X^2, σ_Y^2 are variance of X and Y; ρ is the correlation coefficient between X and Y.

According to the problem, X_0, X_1 and $X_1|X_0$ are all Gaussian. So we have the following equations:

$$\begin{cases} \mu_1 + \rho\frac{\sigma_1}{\sigma_0}(x_0 - \mu_0) = x_0, for\ all\ x_0 \\ \sigma_1^2(1 - \rho^2) = \sigma^2 \end{cases}$$

Solve the equation, then $\sigma_1^2 = \sigma^2 + \sigma_0^2$, $\mu_1 = -\mu_0$. And since $\alpha_1 = \frac{1}{\sqrt{2\pi}\sigma_1}$, we have

$$\alpha_1 = \sqrt{\frac{1}{(1/\alpha)^2 + (1/\alpha_0)^2}}$$

.

4. Find the eigenvalues and eigenvectors of the following 2×2 matrix A.

$$A = \begin{pmatrix} 13 & 5 \\ 2 & 4 \end{pmatrix}$$

Solution. Let λ and $\boldsymbol{x}$ denote the eigenvalue and eigenvector of A. According to the definition of eigenvalue,

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Solve the equation to get eigenvalues

$$|A - \lambda I| = 0$$

That is,

$$\lambda^2 - 14\lambda + 42 = 0$$

A has two eigenvalues: $\lambda_1 = 14$, $\lambda_2 = 3$.

When $\lambda = 14$,

$$(A - \lambda I)\boldsymbol{x} = \begin{pmatrix} -1 & 5 \\ 2 & -10 \end{pmatrix} \boldsymbol{x} = 0$$

$$\boldsymbol{x} = \begin{pmatrix} 5 & 1 \end{pmatrix}^T$$

When $\lambda = 3$,

$$(A - \lambda I)\boldsymbol{x} = \begin{pmatrix} 10 & 5 \\ 2 & 1 \end{pmatrix} \boldsymbol{x} = 0$$

$$\boldsymbol{x} = \begin{pmatrix} 1 & -2 \end{pmatrix}^T$$

In summary, A has two eigenvalues, $\lambda_1 = 14$, $\lambda_2 = 3$. The corresponding eigenvectors are $\boldsymbol{x_1} = \begin{pmatrix} 5 & 1 \end{pmatrix}^T$ and $\boldsymbol{x_2} = \begin{pmatrix} 1 & -2 \end{pmatrix}^T$.

5. Provide one example for each of the following cases, where A and B are 2 2 matrices.

   (a) $(A+B)^2 \neq A^2 + 2AB + B^2$

   (b) $AB = 0, A \neq 0, B \neq 0$

   Solution. (a) one example that satisfies (a) is:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Calculate left,

$$left = (A+B)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Calculate right,

$$right = A^2 + 2AB + B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathbf{0} + \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

And $left \neq right$

   (b) one example that satisfies (b) is:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where $A \neq 0$, and $B \neq 0$. Calculate AB,

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$

6. Let u denote a real vector normalized to unit length. That is, $u^T u = 1$. Show that

$$A = I - 2uu^T$$

is orthogonal, i.e., $A^T A = 1$.

Proof. Derive from left,

$$A^T A = (I - 2uu^T)^T (I - 2uu^T) = (I - 2uu^T)(I - 2uu^T) = I - 2uu^T - 2uu^T + 4uu^T = I$$

So $left = right$. □

Part 1: Locally weighted linear regression

1. Show that $J(\theta)$ can be written in the form

$$J(\theta) = (X\theta - y)^T W (X\theta - y)$$

for an appropriate diagonal matrix W, where X is the $m \times d$ input matrix and y is a $m \times 1$ vector denoting the associated outputs. State clearly what W is.

Proof. We know that $J(\theta)$ can also be written as

$$J(\theta) = \frac{1}{2} \sum_{i=1}^{m} w^{(i)} (\theta^T x^{(i)} - y^{(i)})^2$$

where $x^{(i)}$ is $d \times 1$ vector and θ is a $d \times 1$ vector. We consider each row of the matrix X as as a $1 \times d$vector x^i, so we can write $X = [x^1, x^2, ..., x^m]^T$. So

$$J(\theta) = (X\theta - y)^T W (X\theta - y) = [x^1\theta - y^1, x^2\theta - y^2, ..., x^m\theta - y^m] W [x^1\theta - y^1, x^2\theta - y^2, ..., x^m\theta - y^m]^T$$

So W is a $m \times m$ diagonal matrix

$$W = \begin{pmatrix} 2w^{(1)} & 0 & 0 & 0 \\ 0 & 2w^{(2)} & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 2w^{(m)} \end{pmatrix}$$

□

2. If all the $w^{(i)}$'s are equal to 1, the normal equation to solve for the parameter θ is:

$$X^T X\theta = X^T y$$

and the value of θ that minimizes $J(\theta)$ is $(X^T X)^{-1} X^T y$. By computing the derivative of the weighted $J(\theta)$ and setting it equal to zero, generalized the normal equation to the weighted setting and solve for θ in closed form in terms of W, X and y.

Proof.

$$J(\theta) = (X\theta - y)^T W(X\theta - y) = \theta^T X^T W X\theta^T - \theta^T X^T W y - y^T W X\theta + y^T W y$$

Compute the derivative of $J(\theta)$

$$\frac{\partial J(\theta)}{\partial(\theta)} = 2X^T W X\theta - X^T W y - X^T W^T y$$

Since W is a diagonal matirx $W = W^T$, the equation can be written as

$$\frac{\partial J(\theta)}{\partial(\theta)} = 2X^T W X\theta - 2X^T W y$$

By setting it equal to zero, we can find the value of θ that minimizes $J(\theta)$, the equation is :

$$X^T W X\theta = X^T W y$$

So the value of θ in form in terms of W, X and y is $(X^T W X)^{-1} X^T W y$. □

3. To predict the target value for an input vector x, one choice for the weighting function $w^{(i)}$ is:

$$w^{(i)} = \exp(-\frac{(x - x^{(i)})^T (x - x^{(i)})}{2\tau^2})$$

Points near x are weighted more heavily than points far away from x. The parameter τ is a band width defining the sphere of influence around x. Note how the weights are defined by the input x. Write down an algorithm for calculating θ by batch gradient descent for locally weighted linear regression. Is locally weighted linear regression a parametric or a non-parametric method?

Part 2: Properties of the linear regression estimator

1. Show that $E[\theta] = \theta^*$ for the least squares estimator.

Proof. In part 1 problem 2, we can get thee value of θ given the normal equation $X^T X\theta = X^T y$ is

$$\theta = (X^T X)^{-1} X^T y$$

The date comes from the linear model:

$$y^{(i)} = \theta^T x^{(i)} + \epsilon^{(i)}$$

the expectation of θ is

$$\begin{aligned}
E[\theta] &= E[(X^TX)^{-1}X^Ty] \\
&= E[(X^TX)^{-1}X^T(X\theta^* + \epsilon)] \\
&= E[(X^TX)^{-1}(X^TX\theta^* + X^T\epsilon)] \\
&= E[(X^TX)^{-1}X^TX\theta^* + (X^TX)^{-1}X^T\epsilon] \\
&= E[\theta^*] + E[(X^TX)^{-1}X^T\epsilon]
\end{aligned}$$

since each $\epsilon^{(i)}$is an independent random variable drawn from a normal distribution with zero mean and variance σ^2 and θ^* is a true parameter that has certain value. Then $E[\theta]$ can be written as

$$E[\theta] = E[\theta^*] + 0 = \theta^*$$

□

2. Show that the variance of the least squares estimator is $Var(\theta) = (X^TX)^{-1}\sigma^2$.

Proof.

$$Var(\theta) = E[\theta^2] - (E[\theta])^2$$

since we already knew that $E[\theta] = \theta^*$. So in order to get $Var(\theta)$, all we need to do is to compute $E[\theta^2]$.

$$\begin{aligned}
E[\theta^2] &= E[((X^TX)^{-1}X^Ty)((X^TX)^{-1}X^Ty)^T] \\
&= E[(\theta^* + (X^TX)^{-1}X^T\Sigma)(\theta^* + (X^TX)^{-1}X^T\Sigma)^T] \\
&= E[\theta^*\theta^{*T} + (X^TX)^{-1}X^T\Sigma\theta^{*T} + \theta^*\Sigma^TX(X^TX)^{-1} + (X^TX)^{-1}X^T\Sigma\Sigma^TX(X^TX)^{-1}]
\end{aligned}$$

Σ is the covariance matrix generated by ϵ and each $\epsilon^{(i)}$is an independent random variable drawn from a normal distribution with zero mean and variance σ^2. Therefore the expectation of Σ is zero. Σ is also independent to X and θ^*, $\Sigma = \sigma^2 I$, where I is the identity matrix. Therefore

$$E[\theta^2] = (\theta^*)^2 + \sigma^2 I(X^TX)^{-1}$$

Then we have

$$\begin{aligned}
Var(\theta) &= E[\theta^2] - (E[\theta])^2 \\
&= (\theta^*)^2 + \sigma^2 I(X^TX)^{-1} - (\theta^*)^2 \\
&= (X^TX)^{-1}\sigma^2
\end{aligned}$$

□

Part 3: Implementing Linear Regression

Problem 3.1.A1 Implementing gradient descent

Include Figure 1, Figure 2 and Figure 3

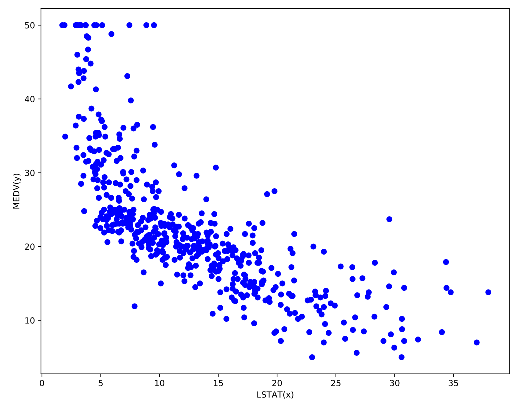


Figure 1: Scatter plot of training data

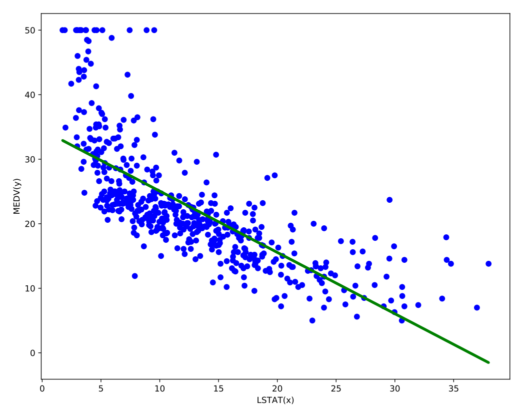


Figure 2: Fitting a linear model to the data in Figure 1

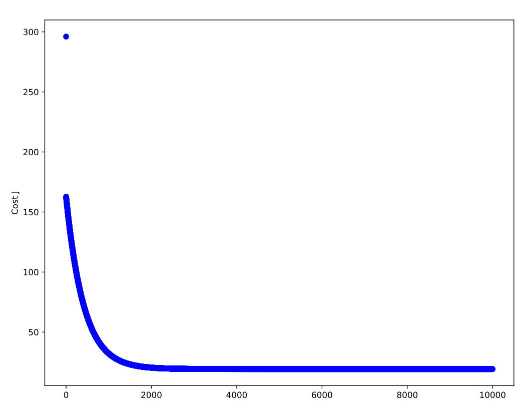


Figure 3: Convergence of gradient descent to fit the linear model in Figure 2

1. Qualitative analysis of the linear fit. What can you say about the quality of the linear fit for this data? In your assignment writeup.pdf, explain how you expect the model to perform at the low and high ends of values for LSTAT? How could we improve the quality of the fit?
   As we can see in the Figure 2, the linear fit for this data is not that good, especially at the high and low ends. The regression should have some curve at the low and high ends of values for LSTAT, which We can replace x with some non-linear function to model non-linear relationship. Using polynomial regression.

Problem 3.1.A3 Predicting on unseen data

1. For lower status percentage = 5, we predict a median home value of 298034.4941220727
   For lower status percentage = 50, we predict a median home value of -129482.12889798547

2. Comparing with sklearn's linear regression model.
   The coefficients computed by sklearn: 34.5538408794 and -0.950049353758

Problem 3.1.B2 Loss function and gradient descent

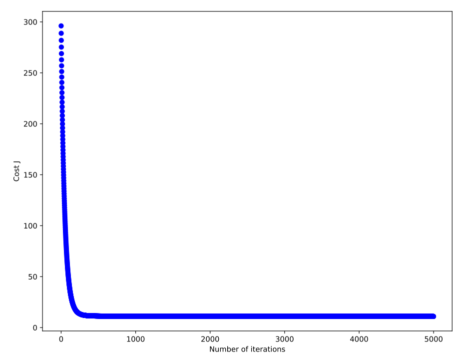


Figure 4: Number of iteration

Problem 3.1.B3 Predicting on unseen data

For average home in Boston suburbs, we predict a median home value of 225328.063241

Problem 3.1.B4: Normal equations (5 points)

For average home in Boston suburbs, we predict a median home value of 225328.06324113606. The prediction matches.

Problem 3.1.B5: Exploring convergence of gradient descent
Figure 5

1. Exploring convergence of gradient descent. What are good learning rates and number of iterations for this problem?

Problem 3.2 Visualizing the dataset

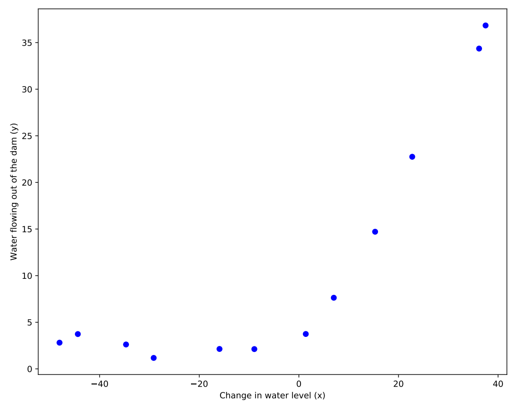


Figure 5: Figure 6: The training data for regularized linear regression

Problem 3.2.A2 Regularized linear regression cost function

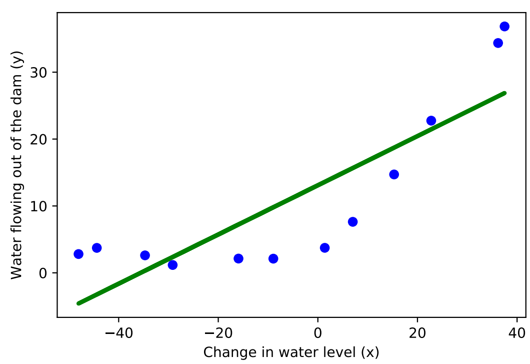


Figure 6: The best fit line for the training data

Problem 3.2.A3 Learning curves

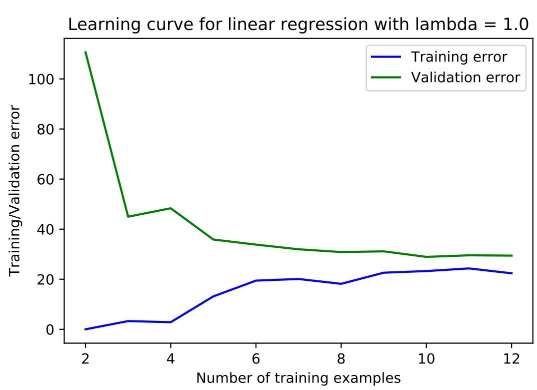


Figure 7: Learning curves

Problem 3.2 Learning polynomial regression models

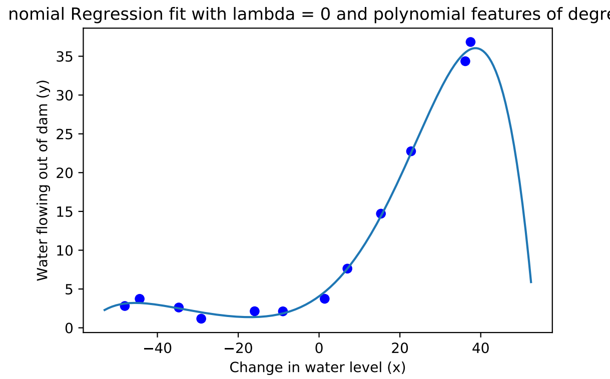


Figure 8: Polynomial fit for lambda = 0 with a p=6 order model.

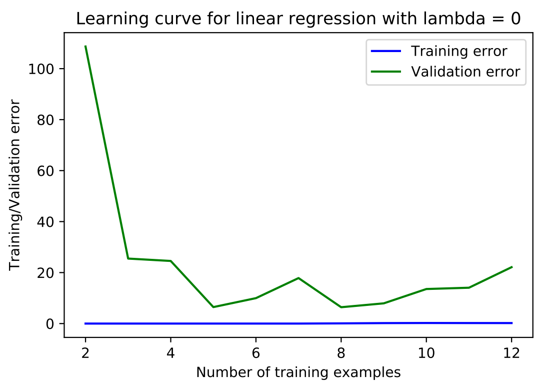


Figure 9: Learning curve for lambda = 0.

Problem 3.2.A4: Adjusting the regularization parameter

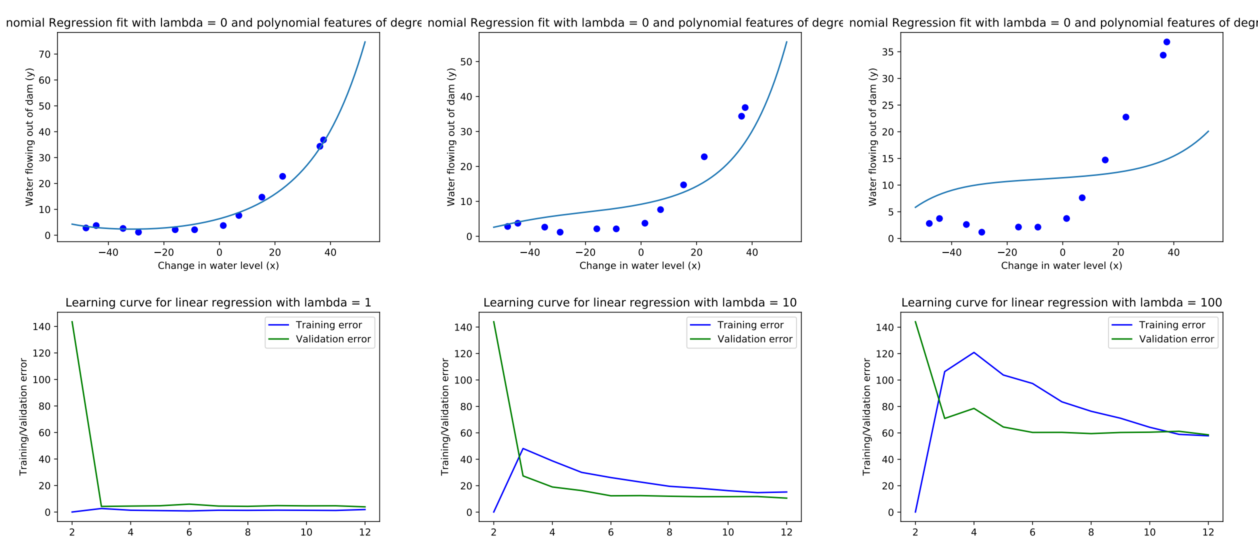


Figure 10: Adjusting the regularization parameter

Increasing lambda results in less overfitting but also greater bias. The training error and testing error increase as long as the lambda increases.

Problem 3.2.A5: Selecting using a validation set

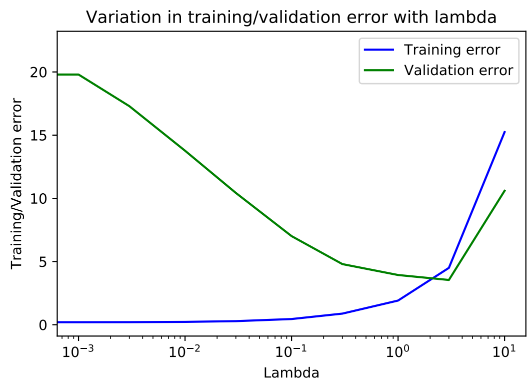


Figure 11: training and validation error on different lambda

The best model is lambda = 3. When lambda = 3, the validation error is the smallest.

Problem 3.2.A6: Computing test set error on the best model

When lambda = 0.3, the model has the smallest validation error. The test error is: 5.857077821089781

Problem 3.2.A7: Plotting learning curves with randomly selected examples

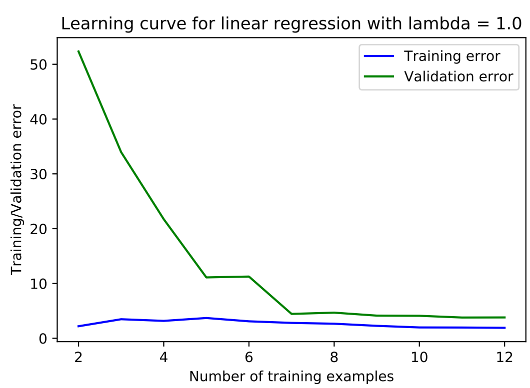


Figure 12: Averaged Learning curve for lambda = 1