COMP 540 HW 1

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Part 0: Background refresher

1. Generate different distributions from uniform distribution:
   (a) Plot the histogram of a categorical distribution with probabilities [0.2,0.4,0.3,0.1].
   (b) Plot the univariate normal distribution with mean of 10 and standard deviation of 1.
   (c) Produce a scatter plot of the samples for a 2-D Gaussian with mean at [1,1] and a covariance matrix [[1,0.5],[0.5,1]]
   (d) Test your mixture sampling code by writing a function that implements an equal weighted mixture of four Gaussians in 2 dimensions, centered at $(\pm 1; \pm 1)$ and having covariance I. Estimate the probability that a sample from this distribution lies within the unit circle centered at (0.1, 0.2).

2. Prove that the sum of two independent Poisson random variables is also a Poisson random variable.

   Proof. The characteristic function of a Poisson random variable is

$$\Phi_1(u) = e^{\lambda_1(e^{iu}-1)}$$

   Let X_1 and X_2 denote two independent Poisson random variables. Let $X = X_1 + X_2$

   Let $\Phi_1(u)$ and $\Phi_2(u)$ denote the characteristic functions of X_1 and X_2:

$$\Phi_1(u) = e^{\lambda_1(e^{iu}-1)}$$

$$\Phi_2(u) = e^{\lambda_2(e^{iu}-1)}$$

   Let $\Phi(u)$ denote the characteristic functions of X. Since $X = X_1 + X_2$, we have:

$$\Phi(u) = \Phi_1(u)\Phi_2(u) = e^{\lambda_1(e^{iu}-1)}e^{\lambda_2(e^{iu}-1)}$$

   Simplify the equation above,

$$\Phi(u) = e^{(\lambda_1+\lambda_2)(\frac{\lambda_1}{\lambda_1+\lambda_2}e^{iu}+\frac{\lambda_2}{\lambda_1+\lambda_2}e^{iu})-1}.$$

   That is

$$\Phi(u) = e^{(\lambda_1+\lambda_2)(e^{iu}-1)}.$$

   Comparing with the characteristic function of Poisson distribution, we can see that X is also a Poisson random variable. □

3. Let X_0 and X_1 be continuous random variables. Show that if

$$P(X_0 = x_0) = \alpha_0 e^{-\frac{(x_0-\mu_0)^2}{2\sigma_0^2}}$$

$$P(X_1 = x_1|X_0 = x_0) = \alpha e^{-\frac{(x_1-x_0)^2}{2\sigma^2}}$$

   there exists α_1, μ_1 and σ_1 such that

$$P(X_1 = x_1) = \alpha_1 e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}}$$

   Write down expressions for these quantities in terms of α_0, α, μ_0, σ_0 and σ.

Solution. If X,Y are both Gaussian random variable, then

$$Y|X = x \sim N\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X), \quad \sigma_Y^2(1-\rho^2)\right)$$

where μ_X, μ_Y are mean of X and Y; σ_X^2, σ_Y^2 are variance of X and Y; ρ is the correlation coefficient between X and Y.

According to the problem, X_0, X_1 and $X_1|X_0$ are all Gaussian. So we have the following equations:

$$\begin{cases} \mu_1 + \rho\frac{\sigma_1}{\sigma_0}(x_0 - \mu_0) = x_0, for\ all\ x_0 \\ \sigma_1^2(1-\rho^2) = \sigma^2 \end{cases}$$

Solve the equation, then $\sigma_1^2 = \sigma^2 + \sigma_0^2$, $\mu_1 = -\mu_0$. And since $\alpha_1 = \frac{1}{\sqrt{2\pi}\sigma_1}$, we have

$$\alpha_1 = \sqrt{\frac{1}{(1/\alpha)^2 + (1/\alpha_0)^2}}$$

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4. Find the eigenvalues and eigenvectors of the following 2×2 matrix A.

$$A = \begin{pmatrix} 13 & 5 \\ 2 & 4 \end{pmatrix}$$

Solution. Let λ and $\boldsymbol{x}$ denote the eigenvalue and eigenvector of A. According to the definition of eigenvalue,

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Solve the equation to get eigenvalues

$$|A - \lambda I| = 0$$

That is,

$$\lambda^2 - 14\lambda + 42 = 0$$

A has two eigenvalues: $\lambda_1 = 14$, $\lambda_2 = 3$.

When $\lambda = 14$,

$$(A - \lambda I)\boldsymbol{x} = \begin{pmatrix} -1 & 5 \\ 2 & -10 \end{pmatrix}\boldsymbol{x} = 0$$

$$\boldsymbol{x} = \begin{pmatrix} 5 & 1 \end{pmatrix}^T$$

When $\lambda = 3$,

$$(A - \lambda I)\boldsymbol{x} = \begin{pmatrix} 10 & 5 \\ 2 & 1 \end{pmatrix}\boldsymbol{x} = 0$$

$$\boldsymbol{x} = \begin{pmatrix} 1 & -2 \end{pmatrix}^T$$

In summary, A has two eigenvalues, $\lambda_1 = 14$, $\lambda_2 = 3$. The corresponding eigenvectors are $\boldsymbol{x_1} = \begin{pmatrix} 5 & 1 \end{pmatrix}^T$ and $\boldsymbol{x_2} = \begin{pmatrix} 1 & -2 \end{pmatrix}^T$.

5. Provide one example for each of the following cases, where A and B are 2 2 matrices.

   (a) $(A+B)^2 \neq A^2 + 2AB + B^2$

   (b) $AB = 0, A \neq 0, B \neq 0$

Solution. (a) one example that satisfies (a) is:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Calculate left,

$$left = (A + B)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Calculate right,

$$right = A^2 + 2AB + B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \mathbf{0} + \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

And $left \neq right$

(b) one example that satisfies (b) is:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where $A \neq 0$, and $B \neq 0$. Calculate AB,

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$

6. Let u denote a real vector normalized to unit length. That is, $u^T u = 1$. Show that

$$A = I - 2uu^T$$

is orthogonal, i.e., $A^T A = 1$.

Proof. Derive from left,

$$A^T A = (I - 2uu^T)^T (I - 2uu^T) = (I - 2uu^T)(I - 2uu^T) = I - 2uu^T - 2uu^T + 4uu^T = I$$

So $left = right$. □

Part 1: Locally weighted linear regression

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