A Theory of Name Resolution

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Abstract. We describe a language-independent theory for name binding and resolution, suitable for programming languages with complex scoping rules including both lexical scoping and modules. We formulate name resolution as a two stage problem. First a language-independent scope graph is constructed using language-specific rules from an abstract syntax tree. Then references in the scope graph are resolved to corresponding declarations using a language-independent resolution process. We introduce a resolution calculus as a concise, declarative, and language-independent specification of name resolution. We develop a resolution algorithm that is sound and complete with respect to the calculus. Based on the resolution calculus we develop language-independent definitions of α-equivalence and rename refactoring. We illustrate the approach using a small example language with modules. In addition, we show how our approach provides a model for a range of name binding patterns in existing languages.

1 Introduction

Naming is a pervasive concern in the design and implementation of programming languages. Names identify *declarations* of program entities (variables, functions, types, modules, etc.) and allow these entities to be *referenced* from other parts of the program. Name *resolution* associates each reference to its intended declaration(s), according to the semantics of the language. Name resolution underlies most operations on languages and programs, including static checking, translation, mechanized description of semantics, and provision of editor services in IDEs. Resolution is often complicated, because it cuts across the local inductive structure of programs (as described by an abstract syntax tree). For example, the name introduced by a **let** node in an ML AST may be referenced by an arbitrarily distant child node. Languages with explicit name spaces introduce further complexity; for example, resolving a qualified reference in Java requires first resolving the class or package name to a context, and then resolving the member name within that context. But despite this diversity, it is intuitively clear that the basic concepts of resolution reappear in similar form across a broad range of lexically-scoped languages.

In practice, the name resolution rules of real programming languages are usually described using *ad hoc* and informal mechanisms. Even when a language

is formalized, its resolution rules are typically encoded as part of static and dynamic judgments tailored to the particular language, rather than being presented separately using a uniform mechanism. This lack of modularity in language description is mirrored in the implementation of language tools, where the resolution rules are often encoded multiple times to serve different purposes, e.g., as the manipulation of a symbol table in a compiler, a use-to-definition display in an IDE, or a substitution function in a mechanized soundness proof. This repetition results in duplication of effort and risks inconsistencies. To see how much better this situation might be, we need only contrast it with the the realm of syntax definition, where context-free grammars provide a well-established declarative formalism that underpins a wide variety of useful tools.

Formalizing resolution. This paper describes a formalism that we believe can help play a similar role for name resolution. It consists of a *scope graph*, which represents the naming structure of a program, and a *resolution calculus*, which describes how to resolve references to declarations within a scope graph. The scope graph abstracts away from the details of a program AST, leaving just the information relevant to name resolution. Its nodes include name references, declarations, and "scopes," which (in a slight abuse of conventional terminology) we use to mean minimal program regions that behave uniformly with respect to name resolution. Edges in the scope graph associate references to scopes, declarations to scopes, or scopes to "parent" scopes (corresponding to lexical nesting in the original program AST). The resolution calculus specifies how to construct a path through the graph from a reference to a declaration, which corresponds to a possible resolution of the reference. Hiding of one definition by another "closer" definition is modeled by providing an ordering on resolution paths. Ambiguous references correspond naturally to multiple resolution paths starting from the same reference node; unresolved references correspond to the absence of resolution paths. To describe programs involving explicit name spaces, the scope graph also supports giving names to scopes, and can include "import" edges to make the contents of a named scope visible inside another scope. The calculus supports complex import patterns including transitive and cyclic import of scopes.

This language-independent formalism gives us clear, abstract definitions for concepts such as scope, resolution, hiding, and import. We build on these concepts to define generic notions of α-equivalence and valid renaming. We also give a practical algorithm for computing conventional static environments mapping bound identifiers to the AST locations of the corresponding declarations, which can be used to implement a deterministic, terminating resolution function that is consistent with the calculus. We expect that the formalism can be used as the basis for other language-independent tools. In particular, any tool that relies on use-def information, such as an IDE offering code completion for identifiers, or a live variable analysis in a compiler, should be specifiable using scope graphs.

On the other hand, the construction of a scope graph from a given program is a language-*dependent* process. For any given language, the construction can be specified by a conventional syntax-directed definition over the language gram-

mar. We would also like a more generic *binding specification language* which could be used to describe how to construct the scope graph for an arbitrary object language. We do not present such a language in this paper. However, the work described here was inspired in part by our previous work on NaBL [17], a DSL that provides high-level, non-algorithmic descriptions of name binding and scoping rules suitable for use by a (relatively) naive language designer. The NaBL implementation integrated into the Spoofax Language Workbench [15] automatically generates an incremental name resolution algorithm that supports services such as code completion and static analysis. However, the NaBL language itself is defined largely by example and lacks a high-level semantic description; one might say that it works well in practice, but not in theory. Because they are language-independent, scope graphs can be used to give a formal semantics for NaBL specifications, although we defer detailed exploration of this connection to further work.

Relationship to Related Work. The study of name binding has received a great deal of attention, focused in particular on two topics. The first is how to represent (already resolved) programs in a way that makes the binding structure explicit and supports convenient program manipulation "modulo α-equivalence" [7, 20, 3, 10, 4]. Compared to this work, our system is novel in several significant respects. (i) Our representation of program binding structure is *independent* of the underlying language grammar and program AST, with the benefits described above. (ii) We support representation of ill-formed programs, in particular, programs with ambiguous or undefined references; such programs are the normal case in IDEs and other front-end tools. (iii) We support description of binding in languages with explicit name spaces, such as modules or OO classes, which are common in practice.

A second well-studied topic is binding specification languages, which are usually enriched grammar descriptions that permit simultaneous specification of language syntax and binding structure [22, 8, 14, 23, 25]. This work is essentially complementary to the design we present here.

Specific contributions.

- *Scope Graph and Resolution Calculus*: We introduce a language-independent framework to capture the relations among *references*, *declarations*, *scopes*, and *imports* in a program. A scope graph can be obtained via a straightforward mapping from an abstract syntax tree. We give a declarative specification of the resolution of references to declarations by means of a calculus that defines resolution paths in a scope graph (Section 2).
- *Variants*: We illustrate the modularity of our core framework design by describing several variants that support more complex binding schemes (Section 2.5).
- *Coverage*: We show that the framework covers interesting name binding patterns in existing languages, including various flavors of let bindings, qualified names, and inheritance in Java (Section 3).

- *Resolution algorithm*: We define a deterministic and terminating resolution algorithm based on the construction of binding environments, and prove that it is sound and complete with respect to the calculus (Section 4).
- *α-equivalence and renaming*: We define a language-independent characterization of α-equivalence of programs, and use it to define a notion of valid renaming (Section 5).

2 Scope Graphs and Resolution Paths

Defining name resolution directly in terms of the abstract syntax tree leads to complex scoping patterns. In unary lexical binding patterns, such as lambda abstraction, the scope of the bound variable is the subtree dominated by the binding construct. However, in name binding patterns such as the sequential `let` in ML, or the variable declarations in a block in Java, the set of abstract syntax tree locations where the bindings are visible does not necessarily form a contiguous region. Similarly, the list of declarations of formal parameters of a function is contained in a subtree of the function definition that does not dominate their use positions. Informally, we can understand these name binding patterns by a conceptual mapping from the abstract syntax tree to an underlying pattern of *scopes*. However, this mapping is not made explicit in conventional descriptions of programming languages.

We introduce the language-independent concept of a *scope graph* to capture the scoping patterns in programs. A scope graph is obtained by a language-specific mapping from the abstract syntax tree of a program. The mapping collapses all abstract syntax tree nodes that behave uniformly with respect to name resolution into a single 'scope' node in the scope graph. In this paper, we do not discuss how to specify such mappings for arbitrary languages, which is the task of a binding specification language, but we show how it can be done for specific examples in a toy language. We assume that it should be possible to build a scope graph in a single traversal of the abstract syntax tree. Furthermore, the mapping should be *syntactic*; *no name resolution* should be necessary to construct the mapping. In previous work [17] we have developed a generic name binding language, NaBL, that has these properties. The scope graph model developed in this paper is a generalization and formal definition of the resolution model underlying NaBL.

Figures 1 to 3 define the full theory. Fig. 1 defines the structure of scope graphs. Fig. 2 defines the structure of *resolution paths*, a subset of resolution paths that are *well-formed*, and a *specificity ordering* on resolution paths. Finally, Fig. 3 defines the *resolution calculus*, which consists of the definition of *edges* between scopes in the scope graph and their transitive closure, the definition of *reachable* and *visible* declarations in a scope, and the *resolution* of references to declarations. In the rest of this section we motivate and explain this theory.

2.1 Example Language

To illustrate the scope graph framework we use the toy language LM, defined in Fig. 4, which contains a rather eclectic combination of features chosen to exhibit

References and declarations

- d_x^i:S: declaration with name x at position i and optional associated named scope S
- r_x^i: reference with name x at position i

Scope graph

- $\mathcal{G} = \{S \mid S \text{ is a scope}\}$: scope graph
- $\mathcal{D}(S)$: declarations d_x^i:S' in S
- $\mathcal{R}(S)$: references r_x^i in S
- $\mathcal{I}(S)$: imports r_x^i in S
- $\mathcal{P}(S)$: parent scope of S

Well-formedness properties

- $\mathcal{P}(S)$ is partial function
- The parent relation is well-founded
- Each r_x^i and d_x^i appears in exactly one scope S

Fig. 1. Scope graphs

Resolution paths

$$s := \mathbf{D}(d_x^i) \mid \mathbf{I}(r_x, d_x^i{:}S) \mid \mathbf{P}$$
$$p := [] \mid s \mid p \cdot p$$
(inductively generated)
$$[] \cdot p = p \cdot [] = p$$
$$(p_1 \cdot p_2) \cdot p_3 = p_1 \cdot (p_2 \cdot p_3)$$

Well-formed paths

$$WF(p) \Leftrightarrow p \in \mathbf{P}^* \cdot \mathbf{I}(\_, \_)^*$$

Specificity ordering on paths

$$\frac{}{\mathbf{D}(\_) < \mathbf{I}(\_, \_)} \quad (DI)$$

$$\frac{}{\mathbf{I}(\_, \_) < \mathbf{P}} \quad (IP)$$

$$\frac{}{\mathbf{D}(\_) < \mathbf{P}} \quad (DP)$$

$$\frac{s_1 < s_2}{s_1 \cdot p_1 < s_2 \cdot p_2} \quad (Lex1)$$

$$\frac{p_1 < p_2}{s \cdot p_1 < s \cdot p_2} \quad (Lex2)$$

Fig. 2. Resolution paths, well-formedness predicate, and specificity ordering.

Edges in scope graph

$$\frac{\mathcal{P}(S_1) = S_2}{\mathbb{I} \vdash \mathbf{P} : S_1 \longrightarrow S_2} \quad (P)$$

$$\frac{r_y \in \mathcal{I}(S_1) \quad r_y \notin \mathbb{I} \quad \mathbb{I} \vdash p : r_y \longmapsto d_y{:}S_2}{\mathbb{I} \vdash \mathbf{I}(r_y, d_y{:}S_2) : S_1 \longrightarrow S_2} \quad (I)$$

Transitive closure

$$\frac{}{\mathbb{I} \vdash [] : A \twoheadrightarrow A} \quad (N)$$

$$\frac{\mathbb{I} \vdash s : A \longrightarrow B \quad \mathbb{I} \vdash p : B \twoheadrightarrow C}{\mathbb{I} \vdash s \cdot p : A \twoheadrightarrow C} \quad (T)$$

Reachable declarations

$$\frac{d_x \in \mathcal{D}(S') \quad \mathbb{I} \vdash p : S \twoheadrightarrow S' \quad WF(p)}{\mathbb{I} \vdash p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x} \quad (R)$$

Visible declarations

$$\frac{\mathbb{I} \vdash p : S \rightarrowtail d_x \qquad \forall d_x', p' (\mathbb{I} \vdash p' : S \rightarrowtail d_x' \Rightarrow \neg(p' < p))}{\mathbb{I} \vdash p : S \longmapsto d_x} \quad (V)$$

Reference resolution

$$\frac{r_x \in \mathcal{R}(S) \quad \{r_x\} \cup \mathbb{I} \vdash p : S \longmapsto d_x}{\mathbb{I} \vdash p : r_x \longmapsto d_x} \quad (X)$$

Fig. 3. Resolution calculus

$$
\begin{aligned}
\mathit{program} &= \mathit{decl}^* \\
\mathit{decl} &= \textbf{module}\ \mathit{id}\ \{\ \mathit{decl}^*\ \} \mid \textbf{import}\ \mathit{id} \mid \textbf{def}\ \mathit{id} = \mathit{exp} \\
\mathit{exp} &= \mathit{int} \mid \mathit{exp} \oplus \mathit{exp} \mid (\ \mathit{exp}\ ) \mid \textbf{ifz}\ \mathit{exp}\ \textbf{then}\ \mathit{exp}\ \textbf{else}\ \mathit{exp} \\
&\quad \mid \mathit{id} \mid \mathit{exp}\ .\ \mathit{id} \mid \textbf{fun}\ \mathit{id}\ \{\ \mathit{exp}\ \} \mid \textbf{fix}\ \mathit{id}\ \{\ \mathit{exp}\ \} \mid \mathit{exp}\ \mathit{exp} \\
&\quad \mid \textbf{let}\ \mathit{bind}^*\ \textbf{in}\ \mathit{exp} \mid \textbf{letrec}\ \mathit{bind}^*\ \textbf{in}\ \mathit{exp} \mid \textbf{letpar}\ \mathit{bind}^*\ \textbf{in}\ \mathit{exp} \\
\mathit{bind} &= \mathit{id} = \mathit{exp}
\end{aligned}
$$

Fig. 4. Syntax of LM, a small language with Lets and Modules used to illustrate the resolution theory in this paper.

both simple and challenging name binding patterns. LM supports the following constructs for binding variables:

- Lambda and Mu: The functional abstractions **`fun`** and **`fix`** represent lambda and mu terms, respectively; both have basic unary lexically scoped bindings.
- Let: The various flavors of let bindings (sequential **`let`**, **`letrec`**, and **`letpar`**) challenge the unary lexical binding model.
- Definition: A definition (**`def`**) declares a variable and binds to it the value of an initializing expression. The definitions in a module are not ordered (no requirement for 'def-before-use'), giving rise to mutually recursive definitions.

Most programming languages have some notion of *module* to divide a program into separate units and a notion of *imports* that make elements of a module available in another. Modules change the standard lexical scoping model, since names can be declared either in the lexical parent or in an imported module. The modules of LM support the following features:

- Qualified names: Elements of modules can be addressed by means of a qualified name using conventional dot notation.
- Imports: All declarations in an imported module are made visible without the need for qualification.
- Transitive imports: The definitions imported into an imported module are themselves visible in the importing module.
- Cyclic imports: Modules can (indirectly) mutually import each other, leading to cyclic import chains.
- Nested modules: Modules may have sub-modules, which can be accessed using dot notation or by importing the containing module.

In the remainder of this section, we use LM examples to illustrate the basic features of our framework. In Section 3 and Appendix A we explore the full power of the framework by applying it to a wide range of name binding patterns from both LM and real languages.

2.2 Declarations, References, and Scopes

We now introduce and motivate the various elements of the name binding framework, gradually building up to the full system described in Figures 1 to 3. The

central concepts in the framework are *declarations*, *references*, and *scopes*. A *declaration* (also known as *binding occurrence*) *introduces* a name. For example, the **def** x = e and **module** m { .. } constructs in LM introduce names of variables and modules, respectively. (A declaration may or may not also *define* the name; this distinction is unimportant for name resolution—except in the case where the declaration defines a module, as discussed in detail later.) A *reference* (also known as *applied occurrence*) is the *use* of a name that refers to a declaration with the same name. In LM, the variables in expressions and the names in import statements (e.g. the x in **import** x) are references. Each reference and declaration is unique and is distinguished not just by its name, but also by its position in the program's AST. Formally, we write r_x^i for a reference with name x at position i and d_x^i for a declaration with name x at position i. When positions are not interesting in the current context, we drop the superscript.

A *scope* is an abstraction over a group of nodes in the abstract syntax tree that behave uniformly with respect to name resolution. Each program has a *scope graph* $\mathcal{G}$, whose nodes are a finite set of scopes $\mathcal{S}(\mathcal{G})$. Every program has at least one scope, the global or *root* scope. Each scope S has an associated finite set $\mathcal{D}(S)$ of declarations and finite set $\mathcal{R}(S)$ of references (at particular program positions), and each declaration and reference in a program belongs to a unique scope. A scope is the atomic grouping for name resolution: roughly speaking, each reference r_x in a scope resolves to a declaration of the same variable d_x in the scope, if one exists. Intuitively, a single scope corresponds to a group of mutually recursive definitions, e.g., a **letrec** block, the declarations in a module, or the set of top-level bindings in a program. Below we will see that edges between nodes in a scope graph determine visibility of declarations in one scope from references in another scope.

Name resolution. We write $\mathcal{R}(\mathcal{G})$ and $\mathcal{D}(\mathcal{G})$ for the (finite) sets of all references and all declarations, respectively, in the program with scope graph $\mathcal{G}$. Name resolution is specified by a relation $\longmapsto \subseteq \mathcal{R}(\mathcal{G}) \times \mathcal{D}(\mathcal{G})$ between references and corresponding declarations in $\mathcal{G}$. In the absence of edges, this relation is very simple:

$$\frac{r_x \in \mathcal{R}(S) \quad d_x \in \mathcal{D}(S)}{r_x \longmapsto d_x} \qquad (X_0)$$

That is, a reference r_x resolves to a declaration d_x, if the scope S in which r_x is contained also contains d_x. We say that there is a *resolution path* from r_x to d_x. We will see soon that paths will grow beyond the one step relation defined by the rule above.

Scope graph diagrams. It can be illuminating to depict a scope graph graphically. In a scope graph diagram, a scope is depicted as a circle, a reference as a box with an arrow pointing *into* the scope that contains it, and a declaration as a box with an arrow *from* the scope that contains it. Fig. 5 shows an LM program consisting of a set of mutually-recursive global definitions, its scope graph, and the resolution paths for variables a,b, and c. In concrete example programs and scope diagrams we write both r_x^i and d_x^i as x_i, relying on context to distinguish

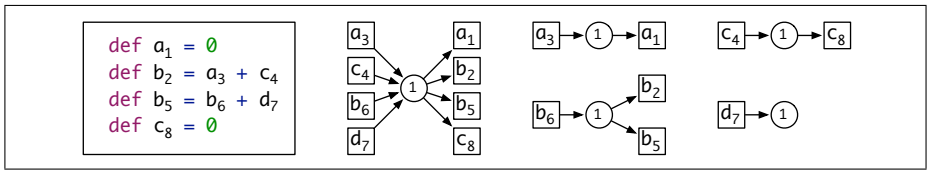

Fig. 5. Declarations and references in global scope.

references and declarations. For example, in Fig. 5, all occurrences $\texttt{b}_i$ denote the *same name* `b` at *different positions*. In scope diagrams, the numbers in scope circles are arbitrarily chosen, and are just used to identify different scopes so that we can talk about them.

Duplicate declarations. It is possible for a scope to contain multiple references and/or declarations with the same name. For example, scope 1 in Fig. 5 has two declarations of the variable `b`. While the existence of multiple references is normal, multiple declarations may give rise to multiple resolutions. For example, the $\texttt{b}_3$ reference in Fig. 5 resolves to *each* of the two declarations $\texttt{b}_2$ and $\texttt{b}_5$.

Typically, correct programs will not declare the same identifier at two different locations in the same scope, although some languages have constructs (e.g. or-patterns in OCAML [18]) that are most naturally modeled this way. But even when the existence of multiple resolutions implies an erroneous program, we want the resolution calculus to identify *all* these resolutions, since IDEs and other front-end tools need to be able to represent erroneous programs. For example, a rename refactoring should support consistent renaming of identifiers, even in the presence of ambiguities (see Section 5). The ability of our calculus to describe ambiguous resolutions distinguishes it from systems, such as [4], that inherently require unambiguous resolution of references.

2.3 Lexical Scope

We model lexical scope by means of the *parent* relation on scopes. In a well-formed scope graph, each scope has at most one parent and the parent relation is well-founded. Formally, the partial function $\mathcal{P}(\_)$ maps a scope S to its *parent* scope $\mathcal{P}(S)$. Given a scope graph with parent relation we can define the notion of *reachable* and *visible* declarations in a scope.

Fig. 6 illustrates how the parent relation is used to model common lexical scope patterns. Lexical scoping is typically presented through nested regions in the abstract syntax tree, as illustrated by the nested boxes in Fig. 6. Expressions in inner boxes may refer to declarations in surrounding boxes, but not vice versa. Each of the scopes in the program is mapped to a scope (circle) in the scope graph. The three scopes correspond to the global scope, the scope for **`fix`** $\texttt{f}_2$, and the scope for **`fun`** $\texttt{n}_3$. The edges from scopes to scopes correspond to the parent relation. The resolution paths on the right of Fig. 6 illustrate the consequences of the encoding. From reference $\texttt{f}_6$ both declarations $\texttt{f}_1$ and $\texttt{f}_2$ are *reachable*, but from reference $\texttt{f}_9$ only declaration $\texttt{f}_1$ is reachable. The duplicate declaration of variable `f` does not indicate a program error in this situation because only $\texttt{f}_2$ is *visible* from the scope of $\texttt{f}_6$.

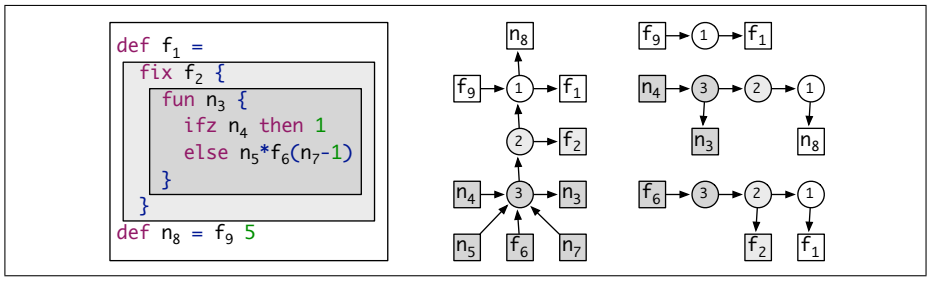

Fig. 6. Lexical scoping modeled by edges between scopes in the scope graph with example program, scope graph, and reachability paths for references.

Reachability. The first step towards a full resolution calculus is to take into account reachability. We redefine rule (X_0) as follows:

$$\frac{r_x \in \mathcal{R}(S_1) \quad p : S_1 \twoheadrightarrow S_2 \quad d_x \in \mathcal{D}(S_2)}{p : r_x \longmapsto d_x} \qquad (X_1)$$

That is, r_x in scope S_1 can be resolved to d_x in scope S_2, if S_2 is *reachable* from S_1, i.e. if $S_1 \twoheadrightarrow S_2$. Reachability is defined in terms of the parent relation as follows:

$$\frac{\mathcal{P}(S_1) = S_2}{\mathbf{P} : S_1 \longrightarrow S_2} \qquad \frac{}{[] : A \twoheadrightarrow A} \qquad \frac{s : A \longrightarrow B \quad p : B \twoheadrightarrow C}{s \cdot p : A \twoheadrightarrow C}$$

The parent relation between scopes gives rise to a direct edge $S_1 \longrightarrow S_2$ between child and parent scope, and $A \twoheadrightarrow B$ is the transitive closure of the direct edge relation. In order to reason about the different ways in which a reference can be resolved, we record the resolution path p. For example, in Fig. 6 reference $\texttt{f}_6$ can be resolved with path $\mathbf{P}$ to declaration $\texttt{f}_2$ and with path $\mathbf{P} \cdot \mathbf{P}$ to $\texttt{f}_1$.

Visibility. Under lexical scoping, multiple possible resolutions are not problematic, as long as the declarations reached are not declared in the same scope. A declaration is *visible* unless it is shadowed by a declaration that is 'closer by'. To formalize visibility, we first extend reachability of scopes to *reachability of declarations*:

$$\frac{d_x \in \mathcal{D}(S') \quad p : S \twoheadrightarrow S'}{p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x} \qquad (R_2)$$

That is, a declaration d_x in S' is reachable from scope S ($S \rightarrowtail d_x$), if scope S' is reachable from S.

Given multiple reachable declarations, which one should we prefer? A reachable declaration d_x is *visible* in scope S ($S \longmapsto d_x$) if there is no other declaration for the same name that is reachable through a *more specific* path:

$$\frac{p : S \rightarrowtail d_x \qquad \forall d'_x, p' (p' : S \rightarrowtail d'_x \Rightarrow \neg(p' < p))}{p : S \longmapsto d_x} \qquad (V_2)$$

where the *specificity ordering* $p' < p$ on paths is defined as

$$\frac{}{\mathbf{D}(\_) < \mathbf{P}} \qquad \frac{s_1 < s_2}{s_1 \cdot p_1 < s_2 \cdot p_2} \qquad \frac{p_1 < p_2}{s \cdot p_1 < s \cdot p_2}$$

That is, a path with fewer parent transitions is more specific than a path with more parent transitions. This formalizes the notion that a declaration in a "nearer" scope shadows a declaration in a "farther" scope.

Finally, a reference resolves to a declaration, if that declaration is visible in the scope of the reference.

$$\frac{r_x \in \mathcal{R}(S) \quad p : S \longmapsto d_x}{p : r_x \longmapsto d_x} \qquad (X_2)$$

Example. In Fig. 6 the scope (labeled 3) containing reference $\texttt{f}_6$ can reach two declarations: $\mathbf{P} \cdot \mathbf{D}(d^2_{\texttt{f}}) : S_3 \rightarrowtail d^2_{\texttt{f}}$ and $\mathbf{P} \cdot \mathbf{P} \cdot \mathbf{D}(d^1_{\texttt{f}}) : S_3 \rightarrowtail d^1_{\texttt{f}}$. Since the first path is more specific than the second path, only $\texttt{f}_2$ is visible, i.e. $\mathbf{P} \cdot \mathbf{D}(d^2_{\texttt{f}}) : S_3 \longmapsto d^2_{\texttt{f}}$. Therefore $\texttt{f}_6$ resolves to $\texttt{f}_2$, i.e. $\mathbf{P} \cdot \mathbf{D}(d^2_{\texttt{f}}) : r^6_{\texttt{f}} \longmapsto d^2_{\texttt{f}}$.

Scopes, revisited. Now that we have defined the notions of reachability and visibility, we can give a more precise description of the sense in which scopes "behave uniformly" with respect to resolution. For every scope S:

- Each declaration in the program is either visible at every reference in $\mathcal{R}(S)$ or not visible at any reference in $\mathcal{R}(S)$.
- For each reference in the program, either every declaration in $\mathcal{D}(S)$ is reachable from that reference, or no declaration in $\mathcal{D}(S)$ is reachable from that reference.
- Every declaration in $\mathcal{D}(S)$ is visible at every reference in $\mathcal{R}(S)$.

2.4 Imports

Introducing modules and imports complicates the name binding picture. Declarations are no longer visible only through the lexical context, but may be visible through an import as well. Furthermore, resolving a reference may require first resolving one or more imports, which may in turn require resolving further imports, and so on.

We model an *import* by means of a reference r_x in the set of imports $\mathcal{I}(S)$ of a scope S. (Imports are also always references and included in some $\mathcal{R}(S')$, but not necessarily in the same scope in which they are imports.) We model a *module* by associating a scope S with a declaration $d_x{:}S$. This associated *named scope* (i.e., named by x) represents the declarations introduced by, and encapsulated in, the module. (We write the $:S$ only in rules where it is required; where it is missing, the associated scope is optional.) Thus, *importing* entails resolving the import reference to a declaration and making the declarations in the scope associated with that declaration available in the importing scope.

Note that 'module' is not a built-in concept in our framework. A module is any construct that (1) is named, (2) has an associated scope that encapsulates

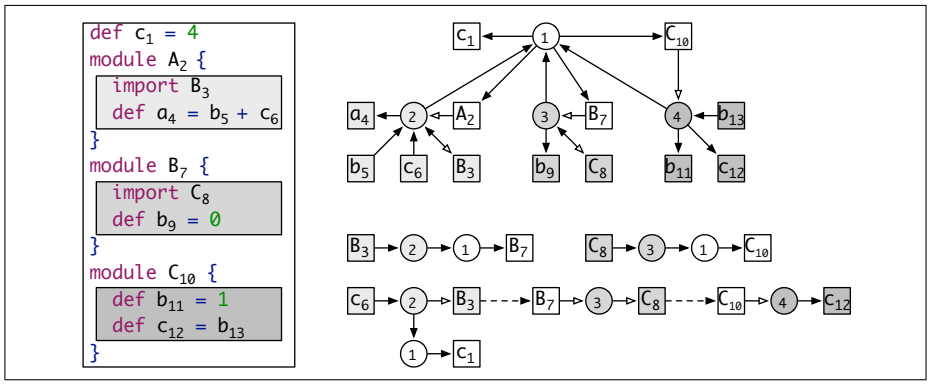

Fig. 7. Modules and imports with example program, scope graph, and reachability paths for references.

declarations, and (3) can be imported into another scope. Of course, this can be used to model the module systems of languages such as ML. But it can be applied to constructs that are not modules at first glance. For example, a class in Java encapsulates class variables and methods, which are imported into its subclasses through the 'extends' clause. Thus, a class plays the role of module and the extends clause that of import. In the next section we discuss further applications.

Reachability. To define name resolution in the presence of imports, we first extend the definition of reachability. We saw above that the parent relation on scopes induces an edge $S_1 \longrightarrow S_2$ between a scope S_1 and its parent scope S_2 in the scope graph. Similarly, an import induces an edge $S_1 \longrightarrow S_2$ between a scope S_1 and the scope S_2 associated with a declaration imported into S_1:

$$\frac{r_y \in \mathcal{I}(S_1) \quad p : r_y \longmapsto d_y{:}S_2}{\mathbf{I}(r_y, d_y{:}S_2) : S_1 \longrightarrow S_2} \qquad (I_3)$$

Note the recursive invocation of the resolution relation on the name of the imported scope.

Figure 7 illustrates extensions to scope graphs and paths to describe imports. Association of a name to a scope is indicated by an open-headed arrow from the name declaration box to the scope circle. (For example, scope 2 is associated to declaration $\texttt{A}_2$.) An import into a scope is indicated by an open-headed arrow from the scope circle to the import name reference box. (For example, scope 2 imports the contents of the scope associated to the resolution of reference $\texttt{B}_3$; note that since $\texttt{B}_3$ is also a reference within scope 2, there is also an ordinary arrow in the opposite direction, leading to a double-headed arrow in the scope graph.) Edges representing the resolution of imported scope names to their definitions are drawn dashed. (For example, reference $\texttt{B}_3$ resolves to declaration $\texttt{B}_7$, which has associated scope 3.) The paths at the bottom right of the figure illustrate that the scope (labeled 2) containing reference $\texttt{c}_6$ can reach two declarations: $\mathbf{P} \cdot \mathbf{D}(d^1_{\mathrm{C}}) : S_2 \rightarrowtail d^1_{\mathrm{C}}$ and $\mathbf{I}(r^3_{\mathrm{B}}, d^7_{\mathrm{B}}{:}S_3) \cdot \mathbf{I}(r^8_{\mathrm{C}}, d^{10}_{\mathrm{C}}{:}S_4) \cdot \mathbf{D}(d^{12}_{\mathrm{C}}) : S_2 \rightarrowtail d^{12}_{\mathrm{C}}$, making use of the subsidiary resolutions $r^3_{\mathrm{B}} \longmapsto d^7_{\mathrm{B}}$ and $r^8_{\mathrm{C}} \longmapsto d^{10}_{\mathrm{C}}$.

Visibility. Imports cause new kinds of ambiguities in resolution paths, which require extension of the visibility policy.

The first issue is illustrated by Fig. 8. In the scope of reference $\mathtt{b}_{10}$ we can reach declaration $\mathtt{b}_7$ with path $\mathbf{D}(d^7_{\mathtt{b}})$ and declaration $\mathtt{b}_4$ with path $\mathbf{I}(r^6_A, d^2_A{:}S_A){\cdot}\mathbf{D}(d^4_{\mathtt{b}})$ (where S_A is the scope named by declaration A_2). We resolve this conflict by extending the specificity order with the rule $\mathbf{D}(\_) < \mathbf{I}(\_, \_)$. That is, local declarations override imported declarations. Similarly, in the scope of reference $\mathtt{a}_8$ we can reach declaration $\mathtt{a}_1$ with path $\mathbf{P}\cdot\mathbf{D}(d^1_{\mathtt{a}})$ and declaration $\mathtt{a}_3$ with path $\mathbf{I}(r^6_A, d^2_A{:}S_A)\cdot\mathbf{D}(d^3_{\mathtt{a}})$. We resolve this conflict by extending the specificity order with the rule $\mathbf{I}(\_, \_) < \mathbf{P}$. That is, resolution through imports is preferred over resolution through parents. In other words, declarations in imported modules override declarations in lexical parents.

```
def a1 = ...
module A2 {
  def a3 = ...
  def b4 = ...
}
module C5 {
  import A6
  def b7 = a8
  def c9 = b10
}
```

Fig. 8. Parent vs Import

```
def a1 = ...
module B2 {
}
module C3 {
  def a4 = ...
  module D5 {
    import B6
    def e7 = a8
  }
}
```

Fig. 9. Parent of import

The next issue is illustrated in Fig. 9. In the scope of reference $\mathtt{a}_8$ we can reach declaration $\mathtt{a}_4$ with path $\mathbf{P}\cdot\mathbf{D}(d^4_{\mathtt{a}})$ and declaration $\mathtt{a}_1$ with path $\mathbf{P}\cdot\mathbf{P}\cdot\mathbf{D}(d^1_{\mathtt{a}})$. The specificity ordering guarantees that only the first of these is visible, giving the resolution we expect. However, with the rules as stated so far, there is another way to reach $\mathtt{a}_1$, via the path $\mathbf{I}(r^6_{\mathtt{B}}, d^2_{\mathtt{B}}{:}S_B){\cdot}\mathbf{P}{\cdot}\mathbf{D}(d^1_{\mathtt{a}})$. That is, we first import module `B`, and then go to its lexical parent, where we find the declaration. In other words, when importing a module, we import not just its declarations, but all declarations in its lexical context. This behavior seems undesirable; to our knowledge, no real languages exhibit it. To rule out such resolutions, we define a well-formedness predicate $WF(p)$ that requires paths p to be of the form $\mathbf{P}^*\cdot\mathbf{I}(\_, \_)^*$, i.e. forbidding the use of parent steps after one or more import steps. We use this predicate to restrict the reachable declarations relation by only considering scopes reachable through a well-formed path:

$$\frac{d_x \in \mathcal{D}(S') \quad p : S \twoheadrightarrow S' \quad WF(p)}{p\cdot\mathbf{D}(d_x) : S \rightarrowtail d_x} \qquad (R_3)$$

The complete definition of well-formed paths and specificity order on paths is given in Fig. 2. In Section 2.5 we discuss how alternative visibility policies can be defined by just changing the well-formedness predicate and specificity order.

Seen imports. Consider the example in Fig. 11. Is declaration $\mathtt{a}_3$ reachable in the scope of reference $\mathtt{a}_6$? This reduces to the question whether the import of $\mathtt{A}_4$ can resolve to module $\mathtt{A}_2$. Surprisingly, it can, in the calculus as discussed so far, as shown by the derivation in Fig. 10 (which takes a few shortcuts). The conclusion of the derivation is that $r^4_A \longmapsto d^2_A{:}S_{A_2}$. This conclusion is obtained by *using the import at* $\mathtt{A}_4$ to conclude at step (*) that $S_{root} \longrightarrow S_{A_1}$, i.e. that the

```
module A1 {
  module A2 {
    def a3 = ...
  }
}
import A4
def b5 = a6
```

Fig. 11. Self import

$$\dfrac{\dfrac{\dfrac{\dfrac{r_A^4 \in \mathcal{I}(S_{root}) \qquad \dfrac{r_A^4 \in \mathcal{R}(S_{root}) \qquad d_A^1{:}S_{A_1} \in \mathcal{D}(S_{root})}{r_A^4 \longmapsto d_A^1{:}S_{A_1}}}{S_{root} \longrightarrow S_{A_1} \quad (*)} \qquad d_A^2{:}S_{A_2} \in \mathcal{D}(S_{A_1})}{S_{root} \rightarrowtail d_A^2{:}S_{A_2}}}{r_A^4 \in \mathcal{R}(S_{root}) \quad S_{root} \longmapsto d_A^2{:}S_{A_2}}}{r_A^4 \longmapsto d_A^2{:}S_{A_2}}$$

Fig. 10. Derivation for $r_A^4 \longmapsto d_A^2{:}S_{A_2}$.

body of module $\texttt{A}_1$ is reachable! In other words, the import of $\texttt{A}_4$ is used in its own resolution. Intuitively, this is nonsensical.

To rule out this kind of behavior we extend the calculus to keep track of the set of *seen imports* $\mathbb{I}$ using judgements of the form $\mathbb{I} \vdash p : r_x \longmapsto d_x$. We need to extend all rules to pass the set $\mathbb{I}$, but only the rules for resolution and import are truly affected:

$$\frac{r_x \in \mathcal{R}(S) \quad \{r_x\} \cup \mathbb{I} \vdash p : S \longmapsto d_x}{\mathbb{I} \vdash p : r_x \longmapsto d_x} \qquad (X)$$

$$\frac{r_y \in \mathcal{I}(S_1) \quad r_y \notin \mathbb{I} \quad \mathbb{I} \vdash p : r_y \longmapsto d_y{:}S_2}{\mathbb{I} \vdash \mathbf{I}(r_y, d_y{:}S_2) : S_1 \longrightarrow S_2} \qquad (I)$$

With this final ingredient, we reach the full calculus in Fig. 3. It is not hard to see that the resolution relation is well-founded. The only recursive invocation (via the I rule) uses a strictly larger set $\mathbb{I}$ of seen imports (via the X rule); since the set $\mathcal{R}(G)$ is finite, $\mathbb{I}$ cannot grow indefinitely.

Anomalies. Although the calculus produces the desired resolutions for a wide variety of real language constructs, its behavior can be surprising on corner cases. Even with the "seen imports" mechanism, it is still possible for a single derivation to resolve a given import in two different ways, leading to unintuitive results. For example, in the program in Fig. 12, $\texttt{x}_{11}$ resolves to $\texttt{x}_3$ and $\texttt{y}_{12}$ resolves to $\texttt{y}_6$. (Derivations left as an exercise to the curious reader!) In our experience, phenomena like this occur only in the presence of mutually-recursive imports; to our knowledge, no real language has these (perhaps for good reason). We defer deeper exploration of these anomalies to future work.

```
module A1 {
  module B2 {
    def x3 = 1
  }
}
module B4 {
  module A5 {
    def y6 = 2
  }
}
module C7 {
  import A8
  import B9
  def z10 = x11
          + y12
}
```

Fig. 12. Anomalous resolution

2.5 Variants

The resolution calculus presented so far reflects a number of binding policy decisions. For example, we enforce imports to be transitive and local declarations to be preferred over imports. However, not every language behaves like this. We now present how other common behaviors can easily be represented with slight modifications of the calculus. Indeed, the modifications do not have to be done

on the calculus itself (the $\longrightarrow$, $\twoheadrightarrow$, $\rightarrowtail$ and $\longmapsto$ relations) but can simply be encoded in the *WF* predicate and the $<$ ordering on paths depending on the kind of policy we want to introduce.

Reachability policy. Reachability policies defines how a reference can access a particular definition, i.e. what rules can be used during the resolution. We can change our reachability policy by modifying the *WF* predicate. For example, if we want to rule out transitive imports, we can change *WF* to be

$$WF(p) \Leftrightarrow p \in \mathbf{P}^* \cdot \mathbf{I}(\_, \_)?$$

where ? denotes the *at most one* operation on regular expressions. Therefore, an import can only be used once at the end of the chain of scopes.

For a language that supports both transitive and non-transitive imports, we can add a label on references corresponding to imports. If r_x is a reference representing a non-transitive import and $\bar{r}_x$ a reference corresponding to a transitive import, then the *WF* predicate simply becomes:

$$WF(p) \Leftrightarrow p \in \mathbf{P}^* \cdot \mathbf{I}(\bar{r}\_, \_)^* \cdot \mathbf{I}(r\_, \_)?$$

Now no import can occur after the use of a non-transitive one.

Similarly, we can modify the rule to handle the *Export* declaration in Coq, which forces transitivity (a resolution can always use an exported module even after importing from a non-transitive one). Assume r_x is a reference representing a non-transitive import and $\bar{r}_x$ a reference corresponding to an export; then we can use the following predicate:

$$WF(p) \Leftrightarrow p \in \mathbf{P}^* \cdot \mathbf{I}(r\_, \_)? \cdot \mathbf{I}(\bar{r}\_, \_)^*$$

Visibility policy. We can modify the visibility policy, i.e. how resolutions shadow each other, by changing the definition of the specificity ordering. For example, we might want imports to act like textual inclusion, so the declarations in the included module have the same precedence as local declarations. This is similar to Standard ML's `include` mechanism. In the program in Fig. 13, the reference x_7 should be treated as having duplicate resolutions, to either x_5 or x_2; the former should not hide the latter. To handle this situation, we can drop the rule $\mathbf{D}(\_) < \mathbf{I}(\_, \_)$ so that definitions and references will get the same precedence, and a definition will not shadow an imported definition. To handle both `include` and ordinary imports, we can once again differentiate the references, and define different order rules depending on the reference used in the import step.

```
module A1 {
  def x2 = 3
}
module B3 {
  include A4;
  def x5 = 6;
  def z6 = x7
}
```

Fig. 13. Include

3 Coverage

In the previous section we introduced the scope graph framework and illustrated it on basic name binding patterns in the LM language. To what extent does this framework cover name binding systems that live in the world of real programming languages? It is not possible to *prove* complete coverage of the framework,

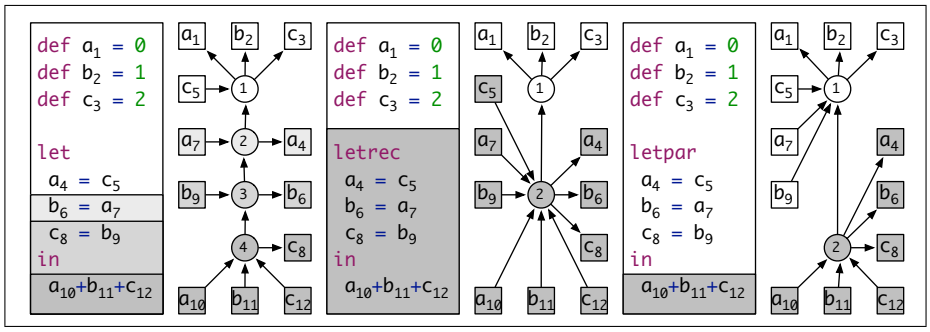

Fig. 14. Example LM programs with sequential, recursive, and parallel lets, and their encodings as scope graphs.

in the sense of being able to encode all possible name binding systems that exist or may be designed in the future. Indeed, given that name binding systems are typically implemented in compilers with algorithms in Turing-complete programming languages, the approach is likely *not* to be complete. However, we believe that the approach covers the rules underlying a broad range of lexically-scoped languages. The design of the framework was informed by an investigation of a wide range of name binding patterns in existing languages, their (attempted) formalization in the NaBL name binding language [17], and their encoding in scope graphs. In this section, we discuss three such examples: let bindings, qualified names, and inheritance in Java. This should provide the reader with a good sense of how name binding patterns can be expressed using scope graphs. Appendix A provides further examples of definition-before-use, compilation units and packages in Java, and namespaces and partial classes in C#. These should satisfy the curious reader, but are not crucial for understanding of the framework.

Let bindings. The several flavors of let bindings in languages such as ML, Haskell, and Scheme do not follow the unary lexical binding pattern in which the binding construct dominates the abstract syntax tree that makes up its scope. The LM language from Fig. 4 has three flavors of let bindings: sequential, recursive, and parallel let, each with a list of bindings and a body expression. Fig. 14 shows the encoding into scope graphs for each of the constructs and makes precise how the bindings are interpreted in each flavour. In the recursive **`letrec`**, the bindings are visible in all initializing expressions, so a single scope suffices for the whole construct. In the sequential **`let`**, each binding is visible in the *subsequent* bindings, but not in its own initializing expression. This requires the introduction of a new scope for each binding. In the parallel **`letpar`**, the variables being bound are not visible in any of the initializing expressions, but only in the body. This is expressed by means of a single scope (2) in which the bindings are declared; any references in the initializing expressions are associated to the parent scope (1).

Qualified names. Qualified names refer to declarations in named scopes outside the lexical scoping. They can be either used as simple references or as imports.

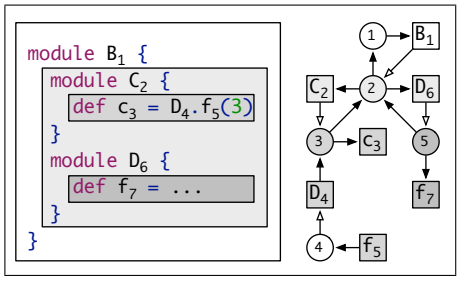

Fig. 15. Example LM program with partially-qualified name.

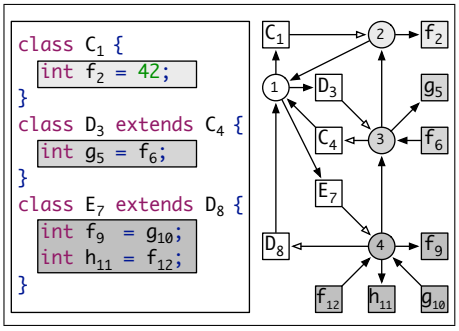

Fig. 16. Class inheritance in Java modeled by import edges.

For example, fully-qualified names of Java classes can be used to refer to (or import) classes from other packages. While fully-qualified names allow to navigate named scopes from the root scope, partially-qualified names give access to lexical subscopes, which are otherwise hidden from lexical parent scopes.

The LM program in Fig. 15 uses a partially-qualified name `D.f` to access function `f` in submodule `D`. We can model this pattern using an anonymous scope (4), which is not linked to the lexical context. The relative name ($\texttt{f}_5$) is a reference in the anonymous scope. We add the qualifying scope name ($\texttt{D}_4$) as an import in the anonymous scope.

Inheritance in Java. As we alluded to in Section 2.4, the concepts of named scopes and imports in the scope graph framework can be used to model more than just language constructs called 'module'. Inheritance in object-oriented languages is another language construct that can be modeled with named scopes and imports. For example, Fig. 16 shows a hierarchy of three Java classes. Class `C` declares a field `f`. Class `D` **`extends`** `C` and inherits its field `f`. Class `E` **`extends`** `D`, inheriting the fields of `C` and `D`. Each class name is declared in the same package scope (1), and associated with the lexical scope of its class body. The name of the super class is a reference in the package scope. We import declarations from the corresponding named scope transitively into class bodies. Thereby, local declarations hide imported declarations. In the example, $\texttt{f}_{12}$ refers to the local declaration $\texttt{f}_9$, which hides the transitively imported $\texttt{f}_2$.

4 Resolution Algorithm

The calculus of Section 2 gives a precise definition of resolution. In principle, we can search for derivations in the calculus to answer questions such as "Does this variable reference resolve to this declaration?" or "Which variable declarations does this reference resolve to?" Automating this search process is not trivial, because of the need for back-tracking and because derivations can have cycles, and hence can grow arbitrarily long.

$$
\begin{aligned}
\mathit{Res}[\mathbb{I}](r_x) &:= \{d_x^i \mid \exists S \ s.t. \ r_x \in \mathcal{R}(S) \wedge d_x^i \in \mathit{Env}_V[\{r_x\} \cup \mathbb{I}, \emptyset](S)\} \\
\mathit{Env}_V[\mathbb{I}, \mathbb{S}](S) &:= \mathit{Env}_L[\mathbb{I}, \mathbb{S}](S) \triangleleft \mathit{Env}_P[\mathbb{I}, \mathbb{S}](S) \\
\mathit{Env}_L[\mathbb{I}, \mathbb{S}](S) &:= \mathit{Env}_D[\mathbb{I}, \mathbb{S}](S) \triangleleft \mathit{Env}_I[\mathbb{I}, \mathbb{S}](S) \\
\mathit{Env}_D[\mathbb{I}, \mathbb{S}](S) &:= \begin{cases} \emptyset \text{ if } S \in \mathbb{S} \\ \mathcal{D}(S) \end{cases} \\
\mathit{Env}_I[\mathbb{I}, \mathbb{S}](S) &:= \begin{cases} \emptyset \text{ if } S \in \mathbb{S} \\ \bigcup \{\mathit{Env}_L[\mathbb{I}, \{S\} \cup \mathbb{S}](S_y) \mid r_y \in \mathcal{I}(S) \setminus \mathbb{I} \wedge d_y{:}S_y \in \mathit{Res}[\mathbb{I}](r_y)\} \end{cases} \\
\mathit{Env}_P[\mathbb{I}, \mathbb{S}](S) &:= \begin{cases} \emptyset \text{ if } S \in \mathbb{S} \\ \mathit{Env}_V[\mathbb{I}, \{S\} \cup \mathbb{S}](\mathcal{P}(S)) \end{cases}
\end{aligned}
$$

Fig. 17. Resolution algorithm

In this section we describe a deterministic and terminating *algorithm* for computing resolutions, which provides a practical basis for implementing tools based on scope graphs, and prove that it is sound and complete with respect to the calculus. This algorithm also connects the calculus, which talks about resolution of a single variable at a time, to more conventional descriptions of binding which use "environments" or "contexts" to describe *all* the visible or reachable declarations accessible from a program location.

For us, an *environment* is just a set of declarations d_x^i. This can be thought of as a function from identifiers to (possible empty) sets of declaration positions. (In this paper, we leave the representation of environments abstract; in practice, one would use a hash table or other dictionary data structure.) We construct an atomic environment corresponding to the declarations in each scope, and then combine atomic environments to describe the sets of reachable and visible declarations resulting from the parent and import relations. The key operator for combining environments is *shadowing*, which returns the union of the declarations in two environments restricted so that if a variable x has any declarations in the first environment, no declarations of x are included from the second environment. More formally:

Definition 1 (Shadowing). *For any environments E_1, E_2, we write:*

$$E_1 \triangleleft E_2 := E_1 \cup \{d_x \in E_2 \mid \nexists d'_x \in E_1\}$$

Figure 17 specifies an algorithm $\mathit{Res}[\mathbb{I}](r_x)$ for resolving a reference r_x to a set of corresponding declarations d_x^i. Like the calculus, the algorithm avoids trying to use an import to resolve itself by maintaining a set $\mathbb{I}$ of "already seen" imports. The algorithm works by computing the full environment $\mathit{Env}_V[\mathbb{I}, \mathbb{S}](S)$ of declarations that are visible in the scope S containing r_x, and then extracting just the declarations for x. The full environment, in turn, is built from the more basic environments Env_D of immediate declarations, Env_I of imported declarations, and Env_P of lexically enclosing declarations, using the shadowing operator. The order of construction matches both the *WF* restriction from the calculus that prevents the use of parent after an import and the path ordering $<$ that prefers immediate declarations over imports, and imports over declarations from the

parent scope. The algorithm is naturally implemented by constructing the various environments lazily. A key difference from the calculus is that the shadowing operator is applied at each stage in environment construction, rather than applying the visibility criterion just once at the "top level" as in calculus rule V. This difference is a natural consequence of the fact that the algorithm computes sets of declarations rather than full derivations paths, so it does not maintain enough information to delay the visibility computation.

Termination The algorithm is terminating using the well-founded lexicographic measure $(|\mathcal{R}(\mathcal{G}) \setminus \mathbb{I}|, |\mathcal{S}(\mathcal{G}) \setminus \mathbb{S}|)$. Termination is straightforward by unfolding the calls to Res in Env_I and then inlining the definitions of Env_V and Env_L: this gives an equivalent algorithm (presented in Appendix B.1) in which the measure strictly decreases at every recursive call.

4.1 Correctness of Resolution Algorithm

The resolution algorithm is sound and complete with respect to the calculus.

Theorem 1. $\forall\ \mathbb{I}, r_x, d_x, (d_x \in Res[\mathbb{I}](r_x)) \iff (\exists p\ s.t.\ \mathbb{I} \vdash p : r_x \longmapsto d_x)$.

We sketch the proof of this theorem here; details of the supporting lemmas and proofs are in Appendix B. To begin with, we must deal with the fact that the calculus can generate derivations with cycles but the algorithm does not follow cycles. In fact, derivations with cycles do not contribute to the set of possible resolutions. We have:

Definition 2 (Cycle). *A path p contains a cycle if there exist p_1, p_2, p_3 such that $p = p_1 \cdot p_2 \cdot p_3$, $p_2 \neq []$, and there exist A, B and S such that $\mathbb{I} \vdash p_1 : A \twoheadrightarrow S$, $\mathbb{I} \vdash p_2 : S \twoheadrightarrow S$, and $\mathbb{I} \vdash p_3 : S \twoheadrightarrow B$.*

Lemma 1 (Cycle-free Path Existence). *If there is a path for a resolution then there is a cycle-free path for this resolution.*

$\forall \mathbb{I}, r_x, d_x, (\exists p\ s.t.\ \mathbb{I} \vdash p : r_x \longmapsto d_x) \Longrightarrow (\exists p, p\ is\ cycle\text{-}free \wedge \mathbb{I} \vdash p : r_x \longmapsto d_x)$

We therefore begin our attack on the proof of Theorem 1 by defining an alternative version of the calculus that prevents construction of cyclic paths. This alternative calculus consists of the original rules $(P), (I)$ from Figure 3 together with the new rules $(N'), (T'), (R'), (V'), (X')$ from Figure 18. The new rules describe transitions that include a "seen scopes" component $\mathbb{S}$ which is used to enforce acyclicity of paths. By inspection, this is the only difference between the "primed" system and original one. Thus, by Lemma 1, we have

Lemma 2. $\forall \mathbb{I}, \mathbb{S}, d_x, (\exists p\ s.t.\ \mathbb{I} \vdash p : S \longmapsto d_x) \iff (\exists p\ s.t.\ \mathbb{I}, \emptyset \vdash p : S \longmapsto d_x)$.

Hereinafter, we can work with the primed system.

To relate the calculus to the algorithm, we also need a way to describe shadowing and visibility in terms of paths.

Definition 3. *For any path p, $\delta(p) := d_x$ iff $\exists p'\ d_x\ s.t.\ p = p' \cdot \mathbf{D}(d_x)$.*

Transitive closure

$$\frac{}{\mathbb{I}, \mathbb{S} \vdash [] : A \twoheadrightarrow A} \quad (N')$$

$$\frac{\mathbb{I} \vdash s : A \longrightarrow B \quad B \notin \mathbb{S} \quad \mathbb{I}, \{B\} \cup \mathbb{S} \vdash p : B \twoheadrightarrow C}{\mathbb{I}, \mathbb{S} \vdash s \cdot p : A \twoheadrightarrow C} \quad (T')$$

Reachable variables

$$\frac{d_x \in \mathcal{D}(S') \quad S \notin \mathbb{S} \quad \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p : S \twoheadrightarrow S' \quad \mathit{WF}(p)}{\mathbb{I}, \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x} \quad (R')$$

Visible variables

$$\frac{\mathbb{I}, \mathbb{S} \vdash p : S \rightarrowtail d_x \qquad \forall d'_x, p' (\mathbb{I}, \mathbb{S} \vdash p' : S \rightarrowtail d'_x \Rightarrow \neg(p' < p))}{\mathbb{I}, \mathbb{S} \vdash p : S \longmapsto d_x} \quad (V')$$

Reference resolution

$$\frac{r_x \in \mathcal{R}(S) \quad \{r_x\} \cup \mathbb{I}, \emptyset \vdash p : S \longmapsto d_x}{\mathbb{I} \vdash p : r_x \longmapsto d_x} \quad (X')$$

Fig. 18. Resolution calculus with "seen scopes" component

Definition 4. *For any set of paths P, $\Delta(P) := \{\delta(p) \mid p \in P\}$.*

Definition 5 (Visible). *For any set of paths P, the subset of visible (i.e., non-shadowed, or most-specific) paths is*

$$\mathit{visible}(P) := \{p \in P \mid \nexists p' \in P \ s.t. \ \delta(p) = d_x \wedge \delta(p') = d'_x \wedge p' < p\}.$$

The relationship between the hiding operator $\triangleleft$ and the *visible* operator is given by the following lemma.

Lemma 3 (Shadowing and visible). *If $\forall p \in P, \forall q \in Q, p < q$ then*

$$\Delta(\mathit{visible}(P \cup Q)) = \Delta(\mathit{visible}(P)) \ \triangleleft \ \Delta(\mathit{visible}(Q))$$

Next we define a family of sets $\mathbb{P}$ of derivable paths in the (primed) calculus.

Definition 6 (Path Sets).

$$\begin{aligned}
\mathbb{P}_D[\mathbb{I}, \mathbb{S}](S) &:= \{p \mid \exists d_x \ s.t. \ p = \mathbf{D}(d_x) \wedge \mathbb{I}, \mathbb{S} \vdash p : S \rightarrowtail d_x\} \\
\mathbb{P}_P[\mathbb{I}, \mathbb{S}](S) &:= \{p \mid \exists \ p' \ d_x \ s.t. \ p = \mathbf{P} \cdot p' \wedge \\
&\qquad \mathbb{I}, \mathbb{S} \vdash p : S \rightarrowtail d_x \wedge \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p' : \mathcal{P}(S) \longmapsto d_x\} \\
\mathbb{P}_I[\mathbb{I}, \mathbb{S}](S) &:= \{p \mid \exists d_x \ r_y \ d_y{:}S' \ s.t. \ p = \mathbf{I}(r_y, d_y{:}S') \cdot p' \wedge \\
&\qquad \mathbb{I}, \mathbb{S} \vdash p : S \rightarrowtail d_x \wedge \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p' : S' \longmapsto d_x\} \\
\mathbb{P}_L[\mathbb{I}, \mathbb{S}](S) &:= \mathit{visible}(\mathbb{P}_D[\mathbb{I}, \mathbb{S}](S) \cup \mathbb{P}_I[\mathbb{I}, \mathbb{S}](S)) \\
\mathbb{P}_V[\mathbb{I}, \mathbb{S}](S) &:= \mathit{visible}(\mathbb{P}_L[\mathbb{I}, \mathbb{S}](S) \cup \mathbb{P}_P[\mathbb{I}, \mathbb{S}](S))
\end{aligned}$$

These sets are designed to correspond to the various classes of environments Env_C. $\mathbb{P}_D$, $\mathbb{P}_P$, and $\mathbb{P}_I$ contain all reachability derivations starting with a $\mathbf{D}(\_)$, $\mathbf{P}$, or $\mathbf{I}(\_, \_)$ respectively, with the further condition that the *tail* of each derivation is minimal among all reachability derivations (i.e. is a visibility derivation). $\mathbb{P}_V$ describes the union of all these derivation paths after applying hiding. ($\mathbb{P}_L$ is similar, but omits paths including $\mathbf{P}$s, because well-formedness prevents using these steps after an import step.)

We have two key lemmas connecting the sets $\mathbb{P}$ with the calculus on the one hand and the algorithm on the other. First, $\mathbb{P}_V$ consists of exactly the resolution paths described by the calculus:

Lemma 4 (Resolution and $\mathbb{P}_V$ equivalence).

$$\forall\ \mathbb{I}\ \mathbb{S}\ S\ p\ d_x, p \cdot \mathbf{D}(d_x) \in \mathbb{P}_V[\mathbb{I}, \mathbb{S}](S) \iff \mathbb{I}, \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \longmapsto d_x$$

Proof. We first prove two auxiliary lemmas about reachability and visibility after one scope transition:

$$\forall\ \mathbb{I}\ \mathbb{S}\ s\ p\ S\ d\ s.t.\ \mathbb{I}, \mathbb{S} \vdash s \cdot p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x \Longrightarrow$$
$$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash s : S \longrightarrow S' \Longrightarrow \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S' \rightarrowtail d_x \quad (\Diamond)$$

$$\forall\ \mathbb{I}\ \mathbb{S}\ s\ p\ S\ d\ s.t.\ \mathbb{I}, \mathbb{S} \vdash s \cdot p : S \longmapsto d_x \Longrightarrow$$
$$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash s : S \longrightarrow S' \Longrightarrow \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p : S' \longmapsto d_x \quad (\blacklozenge)$$

Then we proceed by case analysis on the first step of $p = s \cdot p'$.

($\Rightarrow$) The first step s tells which set among $\mathbb{P}_D$,$\mathbb{P}_P$,$\mathbb{P}_I$ p comes from. Thus we have $\mathbb{I}, \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x$ and must prove that p is minimal. Assume $\bar{s} \cdot \bar{p} < s \cdot p'$. Then using definition of *visible* and $\Diamond$ we can prove that $\bar{p}$ is a reachable path smaller than p', contradicting the minimality of p'.

($\Leftarrow$) The lemma $\blacklozenge$ provides the resolution of the tail which allows us to place p in the corresponding set $\mathbb{P}_D$,$\mathbb{P}_P$,$\mathbb{P}_I$. Then knowing that p is minimal we lift it through the *visible* calls to $\mathbb{P}_V$. Details are in Appendix B.

Secondly, $\mathbb{P}$ sets correspond to *Env* sets. For compactness, we state this result uniformly over all classes of sets:

Lemma 5 (Algorithm and path sets). *For each class $C \in \{V, L, D, I, P\}$:*

$$\forall\ \mathbb{I}\ \mathbb{S}\ S, \mathit{Env}_c[\mathbb{I}, \mathbb{S}](S) = \Delta(\mathbb{P}_C[\mathbb{I}, \mathbb{S}](S))$$

Proof. By two nested inductions, the outer one on $\mathbb{I}$ (or, more strictly, on $|\mathcal{R}(\mathcal{G}) \setminus \mathbb{I}|$, the number of references *not* in $\mathbb{I}$) and the inner one on $\mathbb{S}$ (more strictly, on $|\mathcal{S}(\mathcal{G}) \setminus \mathbb{S}|$, the number of scopes *not* in $\mathbb{S}$). We then proceed by cases on the class C, using Lemmas 3 and 4 and a number of other technical results. Details are in Appendix B.

With these lemmas in hand we proceed to prove Theorem 1.

Proof. Fix $\mathbb{I}$,r_x, and d_x. Given S, the (unique) scope such that $r_x \in \mathcal{R}(S)$:

$$d_x \in \mathit{Res}[r_x](\mathbb{I}) \Leftrightarrow d_x \in \mathit{Env}_V[\{r_x\} \cup \mathbb{I}, \emptyset](S)$$

By the V case of Lemma 5, this is equivalent to

$$\exists p\ s.t.\ p \cdot \mathbf{D}(d_x) \in \mathbb{P}_S[\{r_x\} \cup \mathbb{I}, \emptyset](S)$$

By Lemma 4, this is equivalent to:

$$\exists p\ s.t.\ \{r_x\} \cup \mathbb{I}, \emptyset \vdash p : S \longmapsto d_x$$

which, by Lemma 2 and rule X, is equivalent to $\exists p\ s.t.\ \mathbb{I} \vdash p : r_x \longmapsto d_x$. □

5 α-equivalence and Renaming

The choice of a particular name for a bound identifier should not affect the meaning of a program. This notion of name irrelevance is usually referred to as α-equivalence, but definitions of α-equivalence exist only for some languages and are language-specific. In this section we show how the scope graph and resolution calculus can be used to specify α-equivalence in a language-independent way.

Free variables. A free variable is a reference that does not resolve to any declaration (r_x is free if $\forall\ d_x\ p, \neg \vdash p : r_x \longmapsto d_x$); a bound variable has at least one declaration. For uniformity, we introduce a program-independent artificial declaration $d_x^{\bar{x}}$ to represent the free variable x, with an artificial position $\bar{x}$. These declarations do not belong to any scope but are reachable through a particular well-formed path $\top$, which is less specific than any other path, according to the following rules:

$$\frac{}{\mathbb{I} \vdash \top : S \rightarrowtail d_x^{\bar{x}}} \qquad \frac{p \neq \top}{p < \top}$$

This path representing the resolution of a free reference is shadowed by any existing path leading to a concrete declaration; therefore the resolution of bound variables is unchanged.

5.1 α-Equivalence

We now define α-equivalence using scope graphs. Except for the leaves representing identifiers, two α-equivalent programs must have the same abstract syntax tree. We write `P` $\simeq$ `P'` (pronounced "`P` and `P'` are similar") when the ASTs of `P` and `P'` are equal up to identifiers. To compare two programs we first compare their AST structures; if these are similar then we compare how identifiers behave in these programs. Since two potentially α-equivalent programs are similar, the identifiers occur at the same positions. In order to compare the identifiers' behavior, we define equivalence classes of positions of identifiers in a program: positions in the same equivalence class are declarations of, or references to, the same entity. The abstract position $\bar{x}$ identifies the equivalence class corresponding to the free variable x.

Given a program `P`, we write $\mathbb{P}$ for the set of positions corresponding to references and declarations and $\mathbb{PX}$ for $\mathbb{P}$ extended with the artificial positions (e.g. $\bar{x}$). We define the $\overset{\texttt{P}}{\sim}$ equivalence relation between elements of $\mathbb{PX}$ as the reflexive symmetric and transitive closure of the resolution relation.

Definition 7 (Position equivalence).

$$\frac{\vdash p : r_x^{\,i} \longmapsto d_x^{\,i'}}{i \overset{\texttt{P}}{\sim} i'} \qquad \frac{i' \overset{\texttt{P}}{\sim} i}{i \overset{\texttt{P}}{\sim} i'} \qquad \frac{i \overset{\texttt{P}}{\sim} i' \quad i' \overset{\texttt{P}}{\sim} i''}{i \overset{\texttt{P}}{\sim} i''} \qquad \frac{}{i \overset{\texttt{P}}{\sim} i}$$

In this equivalence relation, the class containing the abstract free variable declaration can not contain any other declaration. So the references in a particular class are either all free or all bound.

Lemma 6 (Free variable class). *The equivalence class of a free variable does not contain any other declaration, i.e.* $\forall\ d_x^i$ *s.t.* $i \overset{\mathtt{P}}{\sim} \bar{x} \implies i = \bar{x}$

Proof. Detailed proof is in appendix B.5, we first prove:
$\forall\ r_x^i,\ (\vdash \top : r_x^i \longmapsto d_x^{\bar{x}}) \implies \forall\ p\ d_x^{i'},\ p \vdash r_x^i \longmapsto d_x^{i'} \implies i' = \bar{x} \wedge p = \top$
and then proceed by induction on the equivalence relation.

The equivalence classes defined by this relation contain references to or declarations of the same entity. Given this relation, we can state that two program are α-equivalent if the identifiers at identical positions refer to the same entity, that belong to the same equivalence class:

Definition 8 (α-equivalence). *Two programs* `P1` *and* `P2` *are α-equivalent (denoted* $\mathtt{P1} \overset{\alpha}{\approx} \mathtt{P2}$*) when they are similar and have the same $\sim$-equivalence classes:*

$$\mathtt{P1} \overset{\alpha}{\approx} \mathtt{P2} \ \triangleq\ \mathtt{P1} \simeq \mathtt{P2} \wedge \forall\ e\ e',\ e \overset{\mathtt{P1}}{\sim} e' \Leftrightarrow e \overset{\mathtt{P2}}{\sim} e'$$

Remark 1. $\overset{\alpha}{\approx}$ is an equivalence relation since $\simeq$ and $\Leftrightarrow$ are equivalence relations.

Free variables. The $\overset{\mathtt{P}}{\sim}$ equivalence classes corresponding to free variables x also contain the artificial position $\bar{x}$. Since the equivalence classes of two equivalent programs `P1` and `P2` have to be exactly the same, every element equivalent to $\bar{x}$ (i.e. a free reference) in `P1` is also equivalent to $\bar{x}$ in `P2`. Therefore the free references of α-equivalent programs have to be identical.

Duplicate declarations. The definition allows us to also capture α-equivalence of programs with duplicate declarations. Assume that a reference $r_x^{i_1}$ resolves to two definitions $d_x^{i_2}$ and $d_x^{i_3}$; then i_1, i_2 and i_3 belong to the same equivalence class. Thus all α-equivalent programs will have the same ambiguities.

5.2 Renaming

Renaming is the substitution of a bound variable by a new variable throughout the program. It has several practical applications such as rename refactoring in an IDE, transformation to a program with unique identifiers, or as an intermediate transformation when implementing capture-avoiding substitution.

A valid renaming should respect α-equivalence classes. To formalize this idea we first define a generic transformation scheme on programs that also depends on the position of the sub-term to rewrite:

Definition 9 (Position dependent rewrite rule). *Given a program* `P`*, we denote $(t_i \to t' \mid F)$ the transformation that replaces the occurrences of the sub-term t at positions i by t' if the condition F is true. (T)*`P` *denotes the application of the transformation T to the program* `P`*.*

Given this definition we can now define the renaming transformation that replaces the identifier corresponding to an entire equivalence class:

Definition 10 (Renaming). *Given a program* P *and a position* i *corresponding to a declaration or a reference for the name* x*, we denote* $[x_i{:=}y]\mathtt{P}$ *the program* P' *corresponding to* P *where all the identifiers* x *at positions* $\overset{\mathtt{P}}{\sim}$*-equivalent to* i *are replaced by* y*:*

$$[x_i := y]\mathtt{P} \triangleq (x_{i'} \to y \mid i' \overset{\mathtt{P}}{\sim} i)\mathtt{P}$$

However, not every renaming is acceptable: a renaming might provoke variable captures and completely change the meaning of a program. We consider that a renaming is *valid* if and only if it produces an α-equivalent program.

Definition 11 (Valid renamings). *Given a program* P*, renaming* $[x_i := y]$ *is valid only if it produces an* α*-equivalent program, i.e.* $[x_i := y]\mathtt{P} \overset{\alpha}{\approx} \mathtt{P}$

Remark 2. This definition prevents the renaming of free variables since α-equivalent program have exactly the same free variables.

Intuitively, valid renamings are those that do not accidentally "capture" variables, but the precise characterization of capture in our general setting is complex, and we leave it for future work.

6 Related Work

Binding-sensitive program representations. There has been a great deal of work on representing program syntax in ways that take explicit note of binding structure, usually with the goal of supporting program transformation or mechanized reasoning tools that respect α-equivalence by construction. Notable techniques include de Bruijn indexing [7], Higher-Order Abstract Syntax (HOAS) [20], locally nameless representations [3], and nominal sets [10]. (Aydemir, et al. [2] give a survey in the context of mechanized reasoning.) However, most of this work has concentrated on simple lexical binding structures, such as single-argument λ-terms. Cheney [4] gives a catalog of more interesting binding patterns and suggests how nominal logic can be used to describe many of them. However, he leaves treatment of module imports as future work.

Binding specification languages. The *Ott* system [22] allows definition of syntax, name binding and semantics. This tool generates language definitions for theorem provers along with a notion of α-equivalence and functions such as capture-avoiding substitution that can be proven correct in the chosen proof assistant modulo with respect to the α-equivalence. Avoiding capture is also the basis of hygienic macros in Scheme. Dybvig [8] gives an algorithmic description of what hygiene means. Herman and Wand [14, 13] introduce static binding specifications to formalize a notion of α-equivalence that does not depend on macro expansion. Stansifer and Wand's Romeo system [23] extends these specifications to somewhat more elaborate binding forms, such as sequential `let`. *Unbound* [25] is another recent domain specific language for describing bindings that supports moderately complex binding forms. Again, none of these systems treat modules or imports.

Language engineering. In language engineering approaches, name bindings are often realized using a random-access symbol table such that multiple analysis and transformation stages can reuse the results of a single name resolution pass [1]. Another approach is to represent the result of name resolution by means of *reference attributes*, direct pointers from the uses of a name to its definition [12]. Erdweg, et al. [9] describe a system for defining capture-free transformations that requires name resolution algorithms for the source and target languages.

Semantics engineering. Semantics engineering approaches to name binding vary from first-order representation with substitution [16], to explicit or implicit environment propagation [21, 19, 6], to HOAS [5]. Identifier bindings represented with environments are passed along in derivation rules, rediscovering bindings for each operation. This approach is inconvenient for more complex patterns such as mutually recursive definitions.

7 Conclusion and Future Work

We have introduced a generic, language-independent framework for describing name binding in programming languages. Its theoretical basis is the notion of a scope graph, which abstracts away from syntax, together with a calculus for deriving resolution paths in the graph. Scope graphs are expressive enough to describe a wide range of binding patterns found in real languages, in particular those involving modules or classes. We have presented a practical resolution algorithm, which is provably correct with respect to the resolution calculus. We can use the framework to define generic notions of α-equivalence and renaming.

As future work, we plan to explore and extend the theory of scope graphs, in particular to find ways to rule out anomalous examples and to give precise characterizations of variable capture and substitution. On the practical side, we will use our formalism to give a precise semantics to the NaBL DSL, and verify (using proof and/or testing) the current NaBL implementation conforms to this semantics.

Our broader vision is that of a complete language designer's workbench that includes NaBL as the domain-specific language for name binding specification and also includes languages for type systems and dynamic semantics specifications. In this setting, we also plan to study the interaction of name resolution and types, including issues of dependent types and name disambiguation based on types. Eventually we aim at deriving a complete mechanized meta-theory for the languages defined in this workbench and prove the correspondence between static name binding and name binding in dynamics semantics as outlined in [24].

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A More Name Binding Patterns

In this appendix we discuss more name binding patterns and their encoding using scope graphs.

A.1 Definition-before-use

Our scopes are sets of mutually recursive declarations. This makes it easy to model a construct such as **`letrec`** as we have seen in Section 3. However, the scope constructs in many languages have a 'definition before use' policy in which bindings are only visible to references appearing 'later' in the program. For example, the variable declarations in a block in C or Java are only visible to the subsequent statements in the block. While colloquially such a block is thought of as a single scope, in fact each variable declaration opens a new scope, much like the bindings in the sequential let construct. Fig. 19 shows how the policy is encoded for blocks in Java by placing each declaration in a new scope that dominates the references in subsequent statements.

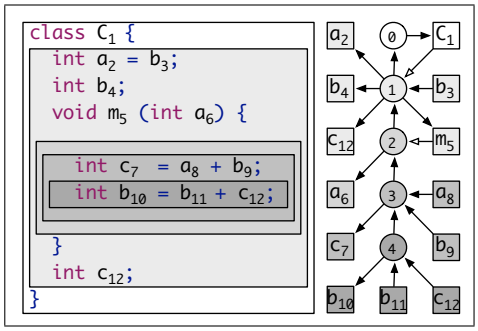

Fig. 19. Subsequent scoping of Java variables modeled by nested scopes.

A.2 Compilation Units and Packages in Java

Scopes do not always correspond to single, continuous program fragments. For example, in Java [11], the scope of a top-level type name is all type declarations in the package in which it is declared. Those type declarations can be spread over multiple, discontinuous program fragments, since a package can consist of a number of compilation units.

A compilation unit can be modeled as a scope, which imports other scopes associated with the same package name. For example, Fig. 20 shows two compilation units contributing to the same package `p`. Each compilation unit declares package `p` in the root scope and associates the declaration with a scope for top-level type declarations. Types `C` and `D` are declared in separate scopes, but are reachable in both scopes via imports from scopes associated with `p`. This approach follows the Java terminology, where each compilation unit declares its package.

Import Declarations. Fig. 21 shows a compilation unit with two import declarations. Java supports two kinds of type-import declarations in compilation units. A *single-type-import declaration* imports a single type. A *type-import-on-demand declaration* allows all accessible types of a package or type to be imported as needed. Thereby, types imported by single-import declarations shadow types of

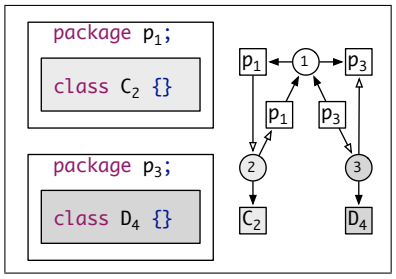

Fig. 20. Java compilation units modeled by import edges.

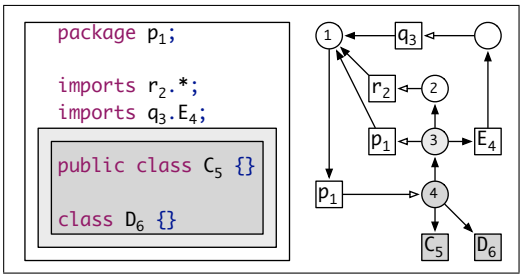

Fig. 21. Java import declarations modeled by nested lexical scopes.

the same name imported by import-on-demand declarations, as well as top-level types of the same name declared in other compilation units of the same package. The scope of an imported type name is all type declarations in the compilation unit in which the import declaration appears.

We model visibility of imported declarations by three lexical scopes per compilation unit. The first scope handles all import-on-demand declarations. The second scope imports type declarations from other compilation units of the same package, and declares aliases for single-type-import declarations. Those aliases act as a declaration and as a reference to the imported name. According to the default visibility policy, the alias declarations shadow imported declarations from the second and first scopes. The third scope is associated with the package name and declares the types of the compilation unit. Those local declarations shadow declarations from other compilation units or packages.

Example. The scope graph diagram in Fig. 21 shows a root scope for package names and three lexical scopes for the compilation unit. Scope 2 imports from a package `r` and scope 3 imports from other compilation units from package `p`. The single-import declaration `q.E` is modeled as a qualified name resolution in the global scope and as a declaration in scope 3. Scope 4 declares the top-level types `C` and `D`.

Accessibility. In Java, the reachability of type declarations can be influenced by access modifiers. In the example, `C` and `D` can be accessed in other compilation units of `p`, but `D` cannot be accessed in other packages. To model accessibility, we need to distinguish **`public`** from *package-private* classes, and imports of compilation units of the same package from regular imports in the scope graph. We can then define a Java-specific well-formedness predicate which rejects paths with regular imports of package-private classes.

A.3 Namespaces and Partial Classes in C#

While Java packages can be declared in multiple compilation units, C# namespaces are open-ended, allowing also for multiple declarations of a namespace in

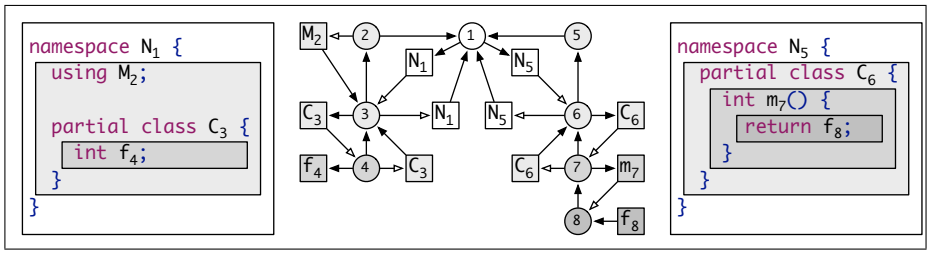

Fig. 22. C# namespaces and partial classes modeled by import edges.

the same compilation unit. In a similar way, C# supports open-ended class declarations. Fig. 22 shows two declarations of namespace `N`, both partially declaring a class `C`. The left part declares a field `f`, while the right part is referring to this field. Similar to Java import declarations, the using directive in the left part only affects the left namespace declaration.

We can model those partial scopes similarly to packages in Java. Each namespace declaration corresponds to two lexical scopes, where local declarations in the second scope hide imported declarations in the first scope. The second scope also imports from other namespace declarations for the same namespace. The same pattern can be applied for partial class declarations. However, C# supports proper nesting of namespaces and partially qualified names. Thus, we can no longer resolve to other partial declarations of the same name in global scope. With the default reachability policy, we might find other partial declarations for the same name in lexical scope. Instead, we need to make sure to resolve only to parts in the same surrounding namespace (which might again be declared in several parts). This can be achieved by distinguishing imports of parts from regular imports, and a reachability policy for the former, which rejects parent paths.

B Proofs

B.1 Termination of resolution algorithm

Figure 23 presents the inlined resolution algorithm that only use strictly decreasing recursive calls.

$$
\begin{aligned}
Res[\mathbb{I}](r_x) &:= \left\{ d_x^i \;\middle|\; \exists S \text{ s.t. } r_x \in \mathcal{R}(S) \wedge d_x^i \in \begin{pmatrix} Env_D[\{r_x\} \cup \mathbb{I}, \emptyset](S) \\ \triangleleft\ Env_I[\{r_x\} \cup \mathbb{I}, \emptyset](S) \\ \triangleleft\ Env_P[\{r_x\} \cup \mathbb{I}, \emptyset](S) \end{pmatrix} \right\} \\
Env_D[\mathbb{I}, \mathbb{S}](S) &:= \begin{cases} \emptyset \text{ if } S \in \mathbb{S} \\ \mathcal{D}(S) \end{cases} \\
Env_I[\mathbb{I}, \mathbb{S}](S) &:= \begin{cases} \emptyset \text{ if } S \in \mathbb{S} \\ \bigcup \left\{ \begin{array}{l} Env_D[\mathbb{I}, \{S\} \cup \mathbb{S}](S_y) \\ \triangleleft\ Env_I[\mathbb{I}, \{S\} \cup \mathbb{S}](S_y) \end{array} \;\middle|\; \begin{array}{l} r_y \in \mathcal{I}(S) \setminus \mathbb{I} \wedge \\ \exists S_y \text{ s.t. } r_y \in \mathcal{R}(S_y) \wedge \\ d_y{:}S_y \in \begin{pmatrix} Env_D[\{r_x\} \cup \mathbb{I}, \emptyset](S_y) \\ \triangleleft\ Env_I[\{r_x\} \cup \mathbb{I}, \emptyset](S_y) \\ \triangleleft\ Env_P[\{r_x\} \cup \mathbb{I}, \emptyset](S_y) \end{pmatrix} \end{array} \right\} \end{cases} \\
Env_P[\mathbb{I}, \mathbb{S}](S) &:= \begin{cases} \emptyset \text{ if } S \in \mathbb{S} \\ \begin{pmatrix} Env_D[\mathbb{I}, \{S\} \cup \mathbb{S}](\mathcal{P}(S)) \\ \triangleleft\ Env_I[\mathbb{I}, \{S\} \cup \mathbb{S}](\mathcal{P}(S)) \\ \triangleleft\ Env_P[\mathbb{I}, \{S\} \cup \mathbb{S}](\mathcal{P}(S)) \end{pmatrix} \end{cases}
\end{aligned}
$$

Fig. 23. Inlined resolution algorithm

B.2 Cycles in Resolution Paths

To enforce the termination of the resolution algorithm, we only look for cycle free paths in this algorithm. Looking for cycle free paths is sufficient since if there is a cyclic path from a particular reference to a declaration there is also a cycle free one:

Lemma 7 (Lemma 1: Cycle-free Path Existence). *If there is a path for a resolution then there is a cycle-free path for this resolution.*

$$
\begin{aligned}
\forall\ \mathbb{I}, r_x, d_x, (\exists p \text{ s.t. } \mathbb{I} \vdash p : r_x \longmapsto d_x) \Longrightarrow \\
(\exists\ p \text{ s.t. } p \text{ is cycle-free } \wedge\ \mathbb{I} \vdash p : r_x \longmapsto d_x)
\end{aligned}
$$

Proof. The set of all resolution paths $\{p \mid \mathbb{I} \vdash p \cdot \mathbf{D}(d_x) : r_x \longmapsto d_x\}$ must contain (at least one) shortest path p (we use *shorter* when referring to the length of the path and *smaller* when referring to the path ordering $<$). We claim p contains no cycle. For suppose it does, with p_1, p_2, p_3 as in Definition 2 and consider the *shrunken* path $p' = p_1 \cdot p_3$. Then:

$$r_x \in S_r \wedge d_x \in S_d \wedge \mathbb{I} \vdash p' : S_r \twoheadrightarrow S_d$$

Moreover, by considering all possible patterns of steps for p_2, we can see that p' is well-formed. Hence

$$\mathbb{I} \vdash p' : S_r \rightarrowtail d_x$$

We claim p' is also a minimal (most-specific) path resolving r_x under $\mathbb{I}$. For suppose not. Then there is a q such that:

$$\mathbb{I} \vdash q : r_x \rightarrowtail d_x \wedge q < p_1 \cdot p_3$$

Then we have two cases:

- $q = q_1 \cdot q_2$ and $q_1 < p_1$; and therefore $q < p_1 \cdot p_2 \cdot p_3 = p$
- $q = p_1 \cdot q_2$ and $q_2 < p_3$; in this case $p_1 \cdot p_2 \cdot q_2$ is also a well-formed path more specific than $p_1 \cdot p_2 \cdot p_3 = p$,

In either case, we have exhibited a well-formed path more specific than p, contradicting the assumed minimality of p. Therefore p' is indeed minimal, so we have:

$$\mathbb{I} \vdash p' : r_x \longmapsto d_x$$

But p' is strictly shorter than p, which is a contradiction. So p must not contain a cycle. □

B.3 Shadowing and Resolution

We present here a lemma that allows to exhibit a minimal path for the resolution of a reference as soon as one know that some declaration is reachable from the reference. In proofs of minimality of p, it will allow us to not only suppose (by contradiction) that p is not minimal (therefore there is smaller path p' such that $\mathbb{I} \vdash p' : S \rightarrowtail d_x$) but that there is a smaller minimal path p' such that $\mathbb{I} \vdash p' : S \longmapsto d_x$.

We first define a measure on the well-formed paths and prove it is compatible with the usual ordering on paths.

Lemma 8 (Well-formed path measure). *For all well-formed paths p, there exist i and j such that $p = \mathbf{P}^i \cdot \mathbf{I}(\_,\_)^j$. We write $m(p)$ for the corresponding pair (i, j) and write $<$ for the lexicographic ordering on these pairs.*

Lemma 9 (WF path measure and specificity ordering are compatible). *For all well-formed p, p',*

$$p < p' \Longrightarrow m(p) < m(p')$$

Proof. Assume $p < p'$ and $m(p) = (i_p, j_p) \wedge m(p) = (i_{p'}, j_{p'})$. Then in the first step where they differ (first application of $Lex1$ rule), we have 3 possibilities (we denote s_k the k-th step of p):

- $s_k = \mathbf{D}(\_) \wedge s'_k = \mathbf{P}\_$: then $i_p < k \leq i_{p'}$ and then $m(p) < m(p')$
- $s_k = \mathbf{D}(\_) \wedge s'_k = \mathbf{I}(\_,\_)$: then $i_p = i_{p'} \wedge j_p < j_{p'}$ and then $m(p) < m(p')$

– $s_k = \mathbf{I}(\_, \_) \wedge s'_k = \mathbf{P}$: then $i_p < k \leq i_{p'}$ and then $m(p) < m(p')$ □

Using this measure we can directly prove the existence of a minimal well-formed resolution path when a well-formed resolution path exists. This means that as soon as a declaration is reachable from a reference, it can only be shadowed by other definitions and that this shadowing process stops somewhere..

Lemma 10 (Shadowing preserves resolution). *If a reference has a reachable declaration, then it has a visible declaration:*

$$\forall\ \mathbb{I}, r_x,\ (\exists\ p\ d_x\ s.t.\ \mathbb{I} \vdash p : r_x \rightarrowtail d_x) \Longrightarrow (\exists\ p'\ d'_x\ s.t.\ \mathbb{I} \vdash p' : r_x \longmapsto d'_x \wedge p' \leq p)$$

Proof. Fix $\mathbb{I}$, r_x, and p. Let $W := \{p' \mid \exists d'_x\ s.t.\ \mathbb{I} \vdash p' : r_x \rightarrowtail d'_x \wedge p' \leq p\}$ be the set of well-formed resolution paths for r_x that are at least as specific as p. We need to show that W contains a minimal element, i.e., an element q such that: $\forall d'_x\ p', \mathbb{I} \vdash p' : r_x \rightarrowtail d'_x \Rightarrow \neg(p' < q)$. W is not empty (it contains p) and the set of associated measures of these well formed paths, i.e.

$$m(W) = \{m(p') \mid \exists\ d'_x\ s.t.\ \mathbb{I} \vdash\ p' : r_x \rightarrowtail d'_x \wedge p' \leq p\}$$

is a set of pairs of natural numbers, on which the lexicographic ordering is well-founded. Thus it has at least one smallest element $m(q)$ and using lemma 9, q is minimal for $<$ in W. Assume q is not minimal among all the reachable paths, then there is a path $q' < q$ such that

$$\mathbb{I} \vdash\ q' : r_x \rightarrowtail d'_x \wedge q' < q$$

But since q is in W then $q < p$ and $<$ is transitive so $q' < p$ and then $q' \in W$. Therefore q is not minimal in W which gives a contradiction. Therefore we have an element q such that:

$$\mathbb{I} \vdash\ q : r_x \longmapsto d'_x \wedge q \leq p$$

□

B.4 Correctness of resolution algorithm

We now proceed to prove the correctness of the resolution algorithm. We first address some auxiliary lemmas that we will need in the proof. First, the path uniquely encode the sequence of scopes in the resolution:

Lemma 11 (Transition unique).

$$\mathbb{I}, \mathbb{S} \vdash s : S \longrightarrow S_1 \Longrightarrow \mathbb{I}, \mathbb{S} \vdash s : S \longrightarrow S_2 \Longrightarrow S_1 = S_2$$

Proof. Trivial by case analysis on s, imported scope S_y is stored in the import step and the parent is unique. □

Then we state that we can cut a resolution path for the reachability relation, that is if a declaration is reachable from a scope S then it is also reachable from any scope appearing in the resolution.

Lemma 12 (Reachable tail). *If a definition is reachable from a scope S through scope S' then it is also reachable from S':*

$$\begin{aligned}\forall\ \mathbb{I}\ \mathbb{S}\ s\ p\ S\ d\ s.t.\ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash s : S \longrightarrow S' \Longrightarrow \\ \mathbb{I}, \mathbb{S} \vdash s \cdot p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x \Longrightarrow \\ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S' \rightarrowtail d_x\end{aligned}$$

Proof. If $s \cdot p$ is well formed then p is also well formed. By inversion of:

$$\mathbb{I}, \mathbb{S} \vdash s \cdot p\mathbf{D}(d_x) : S \rightarrowtail d$$

And using lemma 11, we get:

$i)$ $\mathbb{I}, \{S, S'\} \cup \mathbb{S} \vdash p : S' \twoheadrightarrow S_d$
$ii)$ $S \notin \mathbb{S}$
$iii)$ $S' \notin \mathbb{S}$
$iv)$ $d_x \in \mathcal{D}(S_d)$

Therefore, using $i)$, $iii)$ and $iv)$, we get by rule (R'):

$$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S' \rightarrowtail d_x$$

□

And we can prove that we have the property for visibility, if a declaration is visible from a reference then it is visible all along the path.

Lemma 13 (Visible tail). *If a definition is visible from a scope S through scope S' then it is also visible from S':*

$$\begin{aligned}\forall\ \mathbb{I}\ \mathbb{S}\ s\ p\ S\ d\ s.t.\ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash s : S \longrightarrow S' \Longrightarrow & \quad (i) \\ \mathbb{I}, \mathbb{S} \vdash s \cdot p : S \longmapsto d_x \Longrightarrow & \quad (ii) \\ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p : S' \longmapsto d_x & \end{aligned}$$

Proof. By inversion on (ii) we get:

$(*)$ $\mathbb{I}, \mathbb{S} \vdash s \cdot p : S \rightarrowtail d_x$
$(**)$ $\forall\ p'\ d'_x\ s.t.\ \mathbb{I}, \mathbb{S} \vdash p' : S \rightarrowtail d'_x \Rightarrow \neg p' < s \cdot p$

By lemma 12 and $(*)$ we have:

$(\Diamond)$ $\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p : S' \rightarrowtail d_x$

Assume p is not minimal, then there is a p' such that:

(Hr) $\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p' : S' \rightarrowtail d'_x$
$(<)$ $p' < p$

We have that $s \cdot p'$ is well formed: if p' starts with a $\mathbf{P}$ then $(<)$, p also starts with a $\mathbf{P}$ and we know that $s \cdot p$ is well formed so $s = \mathbf{P}$ else $p \in \mathbf{I}(\_, \_) \cdot \mathbf{D}()$, thus $s \cdot p'$ is well formed. Therefore, by inversion of (Hr) and using (i) we have:

$$\mathbb{I}, \mathbb{S} \vdash s \cdot p' : S \rightarrowtail d'_x$$

which contradicts $(**)$. Therefore p is minimal and using $(\Diamond)$ we have:

$$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p : S' \longmapsto d_x$$

□

We now prove a lemma about the different paths sets from Definition 6. To shorten notation, we write $\mathbb{P}_C$ for $\mathbb{P}_C[\mathbb{I}, \mathbb{S}](S)$ for each clause class C. Next lemma states that the elements in the $\mathbb{P}_C$ sets are not comparable. Therefore $\mathit{visible}(\mathbb{P}_C) = \mathbb{P}_C$.

Lemma 14 (Path set minima).

$$\forall\ p\ p'\ s.t. p \in \mathbb{P}_C \Rightarrow p' \in \mathbb{P}_C \Rightarrow \neg p < p'$$

Proof. Assume $p < p' (*)$, by case analysis on C, trivial for L, V and D:

- Case I: by definition of $\mathbb{P}_I$:
  - $i)$ $p = \mathbf{I}(r_y, d_y{:}S_y, \cdot)\bar{p}$
  - $ii)$ $p' = \mathbf{I}(r_z, d_z{:}S_z, \cdot)\bar{p}'$
  - $iii)$ $\mathbb{I}, \{S\} \cup \mathbb{S} \vdash \bar{p} : S_y \longmapsto d_x$
  - $iv)$ $\mathbb{I}, \{S\} \cup \mathbb{S} \vdash \bar{p}' : S_z \longmapsto d'_x$

  And by inversion of $(*)$, $y = z$ and thus $S_y = S_z$, so we can not have $\bar{p} < \bar{p}'$ since both $\bar{p}$ and $\bar{p}'$ are minimal.
- Case P: Proof is the same for case P, both $\bar{p}$ and $\bar{p}'$ are resolutions from $\mathcal{P}(S)$ □

Let us now prove an equivalent definition of $\mathbb{P}_V[\mathbb{I}, \mathbb{S}](S)$ stating that the paths in $\mathbb{P}_V[\mathbb{I}, \mathbb{S}](S)$ are exactly the resolution paths from S:

Lemma 15 (Lemma 4).

$$\forall\ \mathbb{I}\ \mathbb{S}\ S\ p\ d_x, p \cdot \mathbf{D}(d_x) \in \mathbb{P}_V[\mathbb{I}, \mathbb{S}](S) \iff \mathbb{I}, \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \longmapsto d_x$$

Proof: We fix $\mathbb{I}$, $\mathbb{S}$ and proceed by case analysis on the first step of p.

$(\Rightarrow)$ Suppose $p \cdot \mathbf{D}(d_x) \in \mathbb{P}_V$. By definition of $\mathbb{P}_V$ and $\mathbb{P}_L$ sets we have

$$p \cdot \mathbf{D}(d_x) \in \mathit{visible}(\mathit{visible}(\mathbb{P}_D \cup \mathbb{P}_I) \cup \mathbb{P}_P)$$

Depending on the first step of p, we know the set $\mathbb{P}_D$, $\mathbb{P}_I$ or $\mathbb{P}_P$ which p comes from:

Case $p = []$:
since $\mathbf{D}(d_x) \in \mathbb{P}_D$, it is trivial by definition of $\mathbb{P}_D$ since $\mathbf{D}(\_)$ is always minimal.

Case $p \cdot \mathbf{D}(d_x) = \mathbf{I}(r_y, d_y{:}S') \cdot p'$:
$p \in \mathbb{P}_I$ and $\mathbb{P}_D$ does not contain a resolution for x (if not $p \cdot \mathbf{D}(d_x)$ is not in the *visible*). By definition of $\mathbb{P}_I$ we have:

$$(*)\ \ \mathbb{I}, \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x$$
$$(**)\ \ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p' : S \longmapsto d_x$$

Given $(*)$, to get the conclusion we need to prove that $p \cdot \mathbf{D}(d_x)$ is minimal. Assume there is a path $\bar{p}$ such that:

$$(\diamond)\ \ \mathbb{I}, \mathbb{S} \vdash \bar{p} : S \rightarrowtail d'$$

smaller than $\mathbf{I}(r_y, d_y{:}S') \cdot p'$ then we have 2 cases:

- $\bar{p} = \mathbf{D}(d'_x)$:
  then by definition of $\mathbb{P}_D$ it contains a resolution for x: contradiction
- $\bar{p} = \mathbf{I}(r_y, d_y{:}S') \cdot \bar{p}' \wedge \bar{p}' < p'$:
  then by $(\diamond)$ and lemma 12, we have that:
  $\mathbb{I}, \{S\} \cup \mathbb{S} \vdash \bar{p}' : S' \rightarrowtail d'_x$
  which contradicts $(**)$ since $\bar{p}' < p'$.

Case $p \cdot \mathbf{D}(d_x) = \mathbf{P} \cdot p'$:
$p \in \mathbb{P}_P$ and $\mathbb{P}_D \cup \mathbb{P}_I$ does not contain a resolution for x (if not p is not in the *visible*). By definition of $\mathbb{P}_P$ we have:

$$(*)\ \ \mathbb{I}, \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x$$
$$(**)\ \ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p' : \mathcal{P}(S) \longmapsto d_x$$

Given $(*)$, to get the conclusion we need to prove that $p \cdot \mathbf{D}(d_x)$ is minimal. Assume there is a path $\bar{p}$ such that:

$$(\diamond)\ \ \mathbb{I}, \mathbb{S} \vdash \bar{p} : S \rightarrowtail d'_x$$

smaller than $p \cdot \mathbf{D}(d_x)$ then we have 3 cases:

- $\bar{p} = \mathbf{D}(d'_x)$:
  then by definition of $\mathbb{P}_D$ it contains a resolution for x: contradiction
- $\bar{p} = \mathbf{I}(r_y, d_y{:}S') \cdot \bar{p}'$:
  then by $(\diamond)$ and lemma 12, we have that:
  $\mathbb{I}, \{S\} \cup \mathbb{S} \vdash \bar{p}' : S' \rightarrowtail d'_x$
  Using lemma 10, we get a path $\bar{p}''$ and a definition d''_x such that:
  $(i)\ \ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash \bar{p}'' : S' \longmapsto d''_x$
  $(ii)\ \ \bar{p}'' < \bar{p}'$
  Since $\bar{p}$ and $\bar{p}''$ are well-formed and using (ii) then we have that:
  $\bar{p}'' \in \mathbf{I}(\_, \_)^* \cdot \mathbf{D}()$
  and thus:
  $WF(\mathbf{I}(r_y, d_y{:}S') \cdot \bar{p}'')$
  Therefore using (i) and $(\diamond)$ we get:

(iii) $\mathbb{I}, \mathbb{S} \vdash \mathbf{I}(r_y, d_y{:}S') \cdot \bar{p}'' : S \rightarrowtail d''_x$

Then by (i) and (iii), $\bar{p}''$ is a resolution for x in $\mathbb{P}_I$ which is a contradiction.

– $\bar{p} = \mathbf{P} \cdot \bar{p}' \wedge \bar{p}' < p'$:
then by $(\diamond)$ and lemma 12, we have that:
$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash \bar{p}' : \mathcal{P}(S) \rightarrowtail d'_x$ which contradicts $(**)$ since $\bar{p}' < p'$

$(\Leftarrow)$ Suppose that:

$(*)$ $\mathbb{I}, \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \longmapsto d_x$

By rule (V') we have that:

(i) $\mathbb{I}, \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x$
(ii) $p \cdot \mathbf{D}(d_x)$ is a minimal path for x under $\mathbb{I}, \mathbb{S}$.

By case analysis on the first step of p:

Case $p = []$:
then $p \cdot \mathbf{D}(d_x)$ is trivially in $\mathbb{P}_D$ and thus in the *visible*.

Case $p \cdot \mathbf{D}(d_x) = \mathbf{I}(r_y, d_y{:}S') \cdot p'$: by lemma 13 and $(*)$ we have:
$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p' : S' \longmapsto d_x$
So, using (i), we have $p \cdot \mathbf{D}(d_x) \in \mathbb{P}_I$ and by lemma 14 $p \cdot \mathbf{D}(d_x) \in \mathit{visible}(\mathbb{P}_I)$.
If $\mathbb{P}_D$ contains a path for x then it is trivially smaller than $p \cdot \mathbf{D}(d_x)$ and thus $p \cdot \mathbf{D}(d_x)$ can not be minimal which contradicts (ii). So $\mathbb{P}_D$ does not contain a path for x and therefore:
$p \cdot \mathbf{D}(d_x) \in \mathit{visible}(\mathbb{P}_D \cup \mathbb{P}_I)$.
Therefore, using lemma 3 for $\mathbb{P}_P$, $p \cdot \mathbf{D}(d_x) \in \mathbb{P}_V$.

Case $p \cdot \mathbf{D}(d_x) = \mathbf{P} \cdot p'$:
by lemma 13 and $(*)$ we have:
$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p' : \mathcal{P}(S) \longmapsto d_x$
So, using (i), we have $p \cdot \mathbf{D}(d_x) \in \mathbb{P}_P$ and by lemma 14 $p \cdot \mathbf{D}(d_x) \in \mathit{visible}(\mathbb{P}_P)$.
If $\mathbb{P}_D$ or $\mathbb{P}_I$ contains a path for x then it is trivially smaller than $p \cdot \mathbf{D}(d_x)$ and thus $p \cdot \mathbf{D}(d_x)$ can not be minimal which contradicts (ii). Therefore, $\mathbb{P}_P$, $p \cdot \mathbf{D}(d_x) \in \mathbb{P}_V$). □

Lemma 16. *Similar to lemma 4 for $\mathbb{P}_L$:*
$\forall\, \mathbb{I}\ \mathbb{S}\ S\ p\ d_x, p{\cdot}\mathbf{D}(d_x) \in \mathbb{P}_L[\mathbb{I}, \mathbb{S}](S) \iff \mathbb{I}, \mathbb{S} \vdash p{\cdot}\mathbf{D}(d_x) : S \longmapsto d_x \wedge p \in \mathbf{I}(\_, \_)^*$

Proof. Fix $\mathbb{I}$, $\mathbb{S}$, S, p and d_x
$(\Rightarrow)$ If $p \cdot \mathbf{D}(d_x) \in \mathbb{P}_L[\mathbb{I}, \mathbb{S}](S)$ then $p \in \mathbb{P}_V[\mathbb{I}, \mathbb{S}](S)$ since for all p' in $\mathbb{P}_P[\mathbb{I}, \mathbb{S}](S)$ we have $p \cdot \mathbf{D}(d_x) < p'$. Then by lemma 4, we have that:
$\mathbb{I}, \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \longmapsto d_x$
We now need to prove that $p \in \mathbf{I}(\_, \_)^*$, we have 2 cases:

– if $p \cdot \mathbf{D}(d_x) \in \mathbb{P}_D[\mathbb{I}, \mathbb{S}](S)$ then $p = []$ and thus $p \in \mathbf{I}(\_, \_)^*$

– if $p \cdot \mathbf{D}(d_x) \in \mathbb{P}_I[\mathbb{I},\mathbb{S}](S)$ then:
$\exists\ r_y\ d_y\ p'\ s.t.\ p = \mathbf{I}(r_y, d_y) \cdot p'$
and by inversion on: $\mathbb{I},\mathbb{S} \vdash \mathbf{I}(r_y, d_y) \cdot p' \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x$ we have $WF(\mathbf{I}(r_y, d_y) \cdot p')$ and thus $\mathbf{I}(r_y, d_y) \cdot p'(= p) \in \mathbf{I}(\_, \_)^*$.

($\Leftarrow$) Assume: $\mathbb{I},\mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \longmapsto d_x \wedge p \in \mathbf{I}(\_, \_)^*$ then by lemma 4, $p \cdot \mathbf{D}(d_x) \in \mathbb{P}_V[\mathbb{I},\mathbb{S}](S)$. Since p in $\mathbf{I}(\_, \_)^*$ then p can not be in $\mathbb{P}_P[\mathbb{I},\mathbb{S}](S)$ so p has to be in $\mathbb{P}_L[\mathbb{I},\mathbb{S}](S)$.

We now proceed to prove the different functions of the algorithm compute definitions corresponding to paths in the $\mathbb{P}_C$ sets.

Lemma 17 (Lemma 5). *For each class $C \in \{V, L, D, I, P\}$:*

$$\forall\ \mathbb{I}\ \mathbb{S}\ S, \mathit{Env}_c[\mathbb{I},\mathbb{S}](S) = \Delta(\mathbb{P}_C[\mathbb{I},\mathbb{S}](S))$$

Proof. The proof is by two nested inductions, the outer one on $\mathbb{I}$ (or, more strictly, on the size of $\bar{\mathbb{I}}$, the references *not* in $\mathbb{I}$) and the inner one on $\mathbb{S}$ (more strictly, the size of $\bar{\mathbb{S}}$, the scopes *not* in $\mathbb{S}$). We then pick a d_x and proceed by cases on the class C.
Case D:
$d_x \in \mathit{Env}_D[\mathbb{I},\mathbb{S}](S) \iff$ (by definition of Env_D)
$d_x \in \mathcal{D}(S) \wedge S \notin \mathbb{S} \iff$ (There is no p such that $p < \mathbf{D}(d_x)$)
$d_x \in \mathcal{D}(S) \wedge S \notin \mathbb{S} \wedge \forall\ p'\ d'_x\ s.t.\ \mathbb{I},\mathbb{S} \vdash p' : S \longmapsto d'_x \Rightarrow \neg p' < \mathbf{D}(d_x)$
$\iff$ (by R')
$\mathbb{I},\mathbb{S} \vdash \mathbf{D}(d_x) : S \longmapsto d_x \iff$ (by definition of $\mathbb{P}_D$)
$\mathbb{I},\mathbb{S} \vdash \mathbf{D}(d_x) : S \rightarrowtail d_x \iff$ (by $\mathbf{D}(\_)$ always minimal)
$\mathbf{D}(d_x) \in \mathbb{P}_D[\mathbb{I},\mathbb{S}](S) \iff$ (by definition of $\Delta(\_)$)
$d_x \in \Delta(\mathbb{P}_D[\mathbb{I},\mathbb{S}](S))$

Case P:
$d_x \in \mathit{Env}_P[\mathbb{I},\mathbb{S}](S) \iff$ (by definition of Env_P)
$d_x \in \mathit{Env}_S[\mathbb{I}, \{S\} \cup \mathbb{S}](\mathcal{P}(S)) \wedge S \notin \mathbb{S} \iff$ (by inner induction)
$d_x \in \Delta(\mathbb{P}_V[\mathbb{I}, \{S\} \cup \mathbb{S}](\mathcal{P}(S))) \wedge S \notin \mathbb{S} \iff$ (by Lemma 4)
$\exists p\ s.t.\ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : \mathcal{P}(S) \longmapsto d_x \wedge S \notin \mathbb{S}$
$\iff$ (by rule (P) and inversion of resolution $\mathcal{P}(S) \longmapsto d_x$)
$\exists p\ s.t.\ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : \mathcal{P}(S) \longmapsto d_x \wedge S \notin \mathbb{S} \wedge$
$\mathbb{I},\mathbb{S} \vdash \mathbf{P} : S \longrightarrow \mathcal{P}(S) \wedge \mathcal{P}(S) \notin \{S\} \cup \mathbb{S}$
$\iff$ (by rule R' and T')
$\exists p\ s.t.\ \mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : \mathcal{P}(S) \longmapsto d_x \wedge \mathbb{I},\mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x$
$\iff$ (by definition of $\mathbb{P}_P$)
$d_x \in \Delta(\mathbb{P}_P[\mathbb{I},\mathbb{S}](S))$

Case I:
$d_x \in \mathit{Env}_I[\mathbb{I},\mathbb{S}](S) \iff$ (definition of Env_I)
$d_x \in \bigcup \{\mathit{Env}_L[\mathbb{I}, \{S\} \cup \mathbb{S}](S_y) \mid r_y \in \mathcal{I}(S) \setminus \mathbb{I} \wedge d_y{:}S_y \in \mathit{Res}[\mathbb{I}](r_y)\} \wedge S \notin \mathbb{S}$
$\iff$ (by unfolding of $\mathit{Res}[\mathbb{I}](r_y)$ and outer induction)

$$d_x \in \bigcup \left\{ Env_L[\mathbb{I}, \{S\} \cup \mathbb{S}](S_y) \;\middle|\; \begin{array}{l} r_y \in \mathcal{I}(S) \setminus \mathbb{I} \wedge \\ \exists S', r_y \in \mathcal{R}(S) \wedge \\ \quad d_y{:}S_y \in \Delta(\mathbb{P}_V[\{r_y\} \cup \mathbb{I}, \emptyset](S')) \end{array} \right\} \wedge S \notin \mathbb{S}$$

$\iff$ (by Lemma 4 and rule (X'))

$$d_x \in \bigcup \left\{ Env_L[\mathbb{I}, \{S\} \cup \mathbb{S}](S_y) \;\middle|\; \begin{array}{l} r_y \in \mathcal{I}(S) \setminus \mathbb{I} \wedge \\ \exists q \ s.t. \ \mathbb{I} \vdash q : r_y \longmapsto d_y{:}S_y \end{array} \right\} \wedge S \notin \mathbb{S}$$

$\iff$ (by L case of inner induction)

$\exists r_y \ q \ d_y{:}S_y \ p \ s.t.$
$r_y \in \mathcal{I}(S) \setminus \mathbb{I} \wedge \mathbb{I} \vdash q : r_y \longmapsto d_y{:}S_y \wedge$
$p \cdot D(d_x) \in \mathbb{P}_L[\mathbb{I}, \{S\} \cup \mathbb{S}](S_y) \wedge S \notin \mathbb{S}$

$\iff$ (by lemma 16)

$\exists r_y \ q \ d_y{:}S_y \ p \ s.t. \ r_y \in \mathcal{I}(S) \setminus \mathbb{I} \wedge$
$\mathbb{I} \vdash q : r_y \longmapsto d_y{:}S_y \wedge$
$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot D(d_x) : S_y \longmapsto d_x \wedge$
$p \in \mathbf{I}(\_, \_)^* \wedge$
$S \notin \mathbb{S}$

$\iff$ (by Import rule)

$\exists r_y \ d_y{:}S_y,$
$\mathbb{I}, \mathbb{S} \vdash \mathbf{I}(r_y, d_y{:}S_y) : S \longrightarrow S_y$
$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S_y \longmapsto d_x \wedge$
$p \in \mathbf{I}(\_, \_)^* \wedge$
$S \notin \mathbb{S}$

$\iff$ (by (V') and (R') we have)

$\exists \ S_x \ r_y \ d_y{:}S_y,$
$\mathbb{I}, \mathbb{S} \vdash \mathbf{I}(r_y, d_y{:}S_y) : S \longrightarrow S_y$
$S_y \notin \{S\} \cup \mathbb{S}$
$\mathbb{I}, \{S, S_y\} \cup \mathbb{S} \vdash p : S_y \twoheadrightarrow S_x$
$d_x \in S_x \wedge$
$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S_y \longmapsto d_x \wedge$
$p \in \mathbf{I}(\_, \_)^* \wedge$
$S \notin \mathbb{S}$

$\iff$ (by (T') and WF definition)

$\exists \ S_x \ r_y \ d_y{:}S_y,$
$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash \mathbf{I}(r_y, d_y{:}S_y) \cdot p : S \twoheadrightarrow S_x$
$d_x \in S_x \wedge$
$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S_y \longmapsto d_x \wedge$
$WF(\mathbf{I}(r_y, d_y{:}S_y) \cdot p \cdot \mathbf{D}(d_x)) \wedge$
$S \notin \mathbb{S}$

$\iff$ (by (R'))

$\exists \ r_y \ d_y{:}S_y,$
$\mathbb{I}, \mathbb{S} \vdash \mathbf{I}(r_y, d_y{:}S_y) \cdot p \cdot \mathbf{D}(d_x) : S \rightarrowtail d_x \wedge$
$\mathbb{I}, \{S\} \cup \mathbb{S} \vdash p \cdot \mathbf{D}(d_x) : S_y \longmapsto d_x$

$\iff$ (by definition of $\mathbb{P}_I$)

$\mathbf{I}(r_y, d_y{:}S_y) \cdot p \cdot \mathbf{D}(d_x) \in \mathbb{P}_I[\mathbb{I}, \mathbb{S}](S)$

$\iff$ (by definition of Δ)

$d_x \in \Delta(\mathbb{P}_I[\mathbb{I}, \mathbb{S}](S))$

Cases L,S: Direct using Lemma 3. □

B.5 α-equivalence

We prove of Lemma 6 about the absence of declaration in the equivalence class of a free variable:

Lemma 18 (Lemma 6). *The equivalence class of a free variable can not contain any other declaration:*

$$\forall d_x^i, i \overset{\mathsf{P}}{\sim} \bar{x} \Longrightarrow i = \bar{x}$$

Proof. First, since $\top$ is the only path to the free variable definition we have:
$\forall\ r_x^i\ p,\ (\vdash p : r_x^i \longmapsto d_x^{\bar{x}}) \Longrightarrow p = \top$
Then since $\top$ is bigger than any other path and using the (M) rule we have:
$\forall\ r_x^i,\ (\vdash \top : r_x^i \longmapsto d_x^{\bar{x}}) \Longrightarrow \forall\ p\ d_x^{i'},\ p \vdash r_x^i \longmapsto d_x^{i'} \Longrightarrow i' = \bar{x}$
So if there is resolution to the free variable declaration, then it is the only possible resolution for the reference. Finally, we prove by induction on the equivalence relation $i \overset{\mathsf{P}}{\sim} i'$ that any term equivalent to $\bar{x}$ resolves to $\bar{x}$ or is $\bar{x}$ itself:

$$\begin{aligned}\forall\ i\ j\ s.t.\ i \overset{\mathsf{P}}{\sim} j \Longrightarrow \\ &((j = \bar{x} \vee\ \vdash \top : r_x^j \longmapsto d_x^{\bar{x}}) \Rightarrow (i = \bar{x} \vee\ \vdash \top : r_x^i \longmapsto d_x^{\bar{x}})) \wedge \\ &((i = \bar{x} \vee\ \vdash \top : r_x^i \longmapsto d_x^{\bar{x}}) \Rightarrow (j = \bar{x} \vee\ \vdash \top : r_x^j \longmapsto d_x^{\bar{x}}))\end{aligned}$$

Which terminates the proof by instantiating j with $\bar{x}$.