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Subject: Discrete Mathematics.

Assignment On: Application of Inclusion, Exclusion.

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Submitted To: Dr. Muzaffar Rasool.

Inclusion and Exclusion:-

Principle of inclusion and exclusion is an approach which derives the method of finding the number of elements in the union of sets (finite sets) This is used for solving combinations and probability problems when it is necessary to find a counting method, which makes sure that an object is not counted twice.

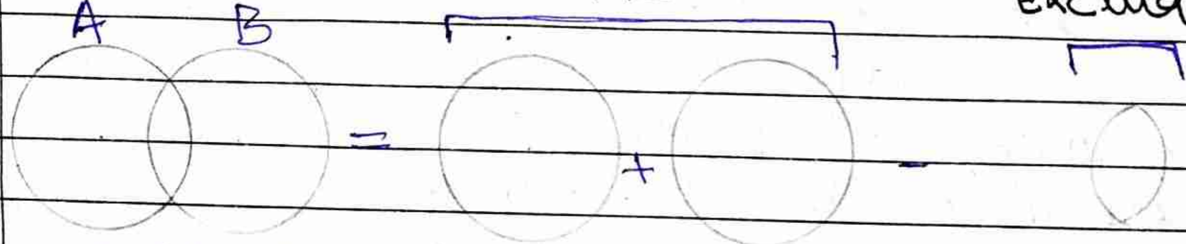
Consider two finite sets A and B. We can denote the principle of Inclusion and Exclusion formulla as follows,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Here $n(A)$ denotes the cardinality of set A, $n(B)$ denotes the cardinality of set B and $n(A \cap B)$ denotes the cardinality of $(A \cap B)$. We have included A and B and excluded their common elements.

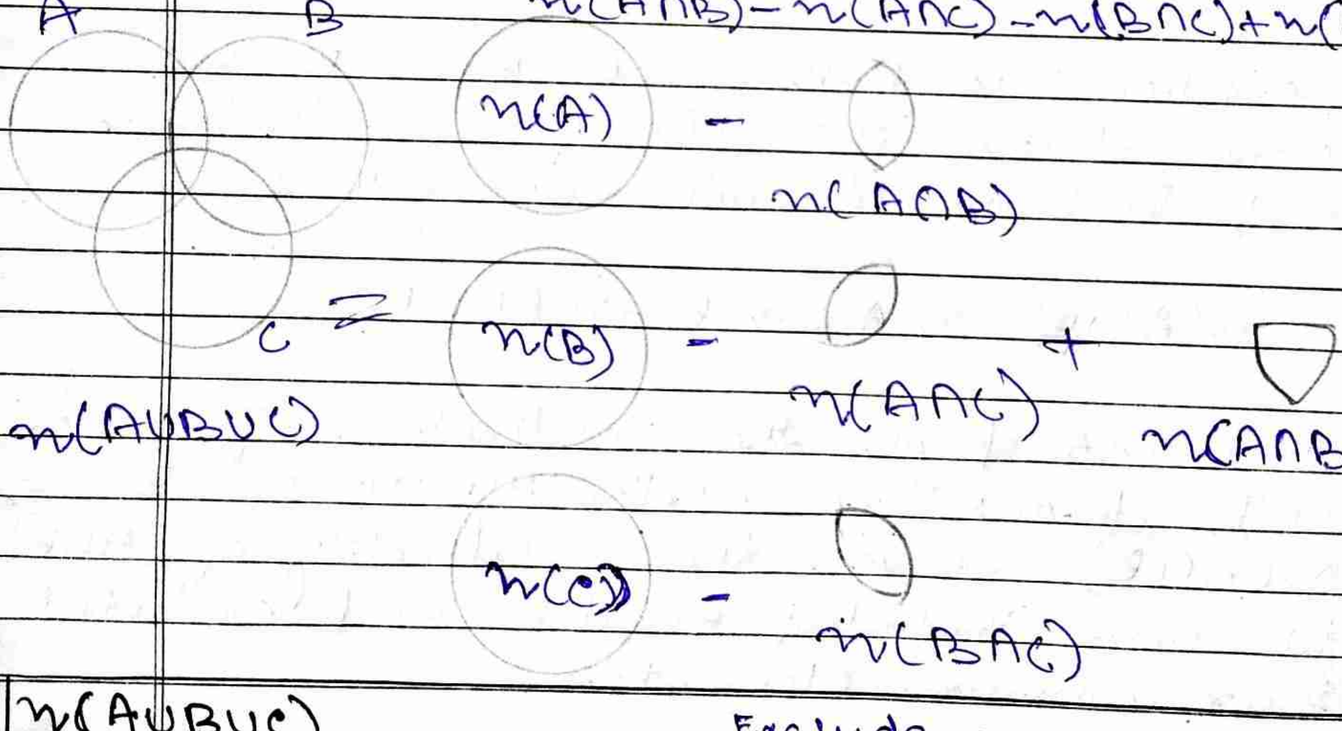
★ Inclusion simply means to 'include/add something & exclusion means to remove/substract something.

The following figure gives us more idea on this "Include" "Exclude"



$n(A\cup B)$ $n(A)$ $n(B)$ $n(A\cap B)$

If we have 3 sets A, B, and C, then according to the principle of inclusion and Exclusion, $n(A\cup B\cup C) = n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)-n(B\cap C)+n(A\cap B\cap C)$



$$n(A\cup B\cup C) = \underbrace{n(A)+n(B)+n(C)}_{\text{Include}}-\overbrace{n(A\cap B)-n(A\cap C)-n(B\cap C)}^{\text{Exclude}}+\underbrace{n(A\cap B\cap C)}_{\text{Include}}$$

In general, $n(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)$

$$= \sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} n(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n)$$

neither A_1, A_2, A_3 or A_n.

$$n(A_1' \cap A_2' \cap A_3' \cap \cdots \cap A_n') = \underbrace{n(U)}_{\text{Total}} - \underbrace{n(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n)}_{\text{at least}}.$$

APPLICATION:

DERANGEMENTS:

The number of rearrangements, if "n" things are arranged in a row, such that none of them will occupy their original positions are called Derangements.

In simple words it is the dearranging the things, number of ways of dearranging such that nothing goes into the right place.

The number of Derangements of n distinct things can be denoted by D_n. (i.e. D_n is derangement of n objects in a row)

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \cdot \frac{1}{n!}\right) \text{ where } n \geq 2$$

or

$$D_n = n!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!}\right)$$

Proof:-

let D_n denotes the number of derangements of n elements.

$$D_n = \left\{ \begin{array}{l} \text{Total no. of Permutations } (n!) \\ \text{Number of permutations in which} \\ \text{atleast one element is in it original} \\ \text{Position } (k) \end{array} \right\}$$

$D_n = n! - k \ldots (i)$

let A_i be the set of permutations in which i^{th} element is in its original position.

$\therefore k = n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)$

Inclusion

$= \underbrace{\sum_i n(A_i) - \sum_i n(A_i \cap A_j)}_{\text{Exclusion}} + \sum_i n(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} n(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)$

$\boxed{\text{k is at least one element in its original position.}}$

$= {}^nC_1 (n-1)! - {}^nC_2 (n-2)! + {}^nC_3 (n-3)! - \ldots + (-1)^{n-1} \cdot 1$

$= n (n-1)! - \frac{n(n-1)}{2!} (n-2)! + \frac{n(n-1)(n-2)}{3!} (n-3)! - \ldots + (-1)^{n-1} \cdot 1$

$= n! - \frac{n!}{2!} + \frac{n!}{3!} + \ldots + (-1)^{n-1} \cdot 1.$

$= n! \left(1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{n-1} \frac{1}{n!}\right) \rightarrow$ value of k.

Now put this value of k in eq — (i).

$D_n = n! - k$

$= n! - n! \left(1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{n-1} \frac{1}{n!}\right).$

$= n! - n! + \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n \cdot 1$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)$$

Hence this is our formulla.

$$\boxed{D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}\right) \text{ where } n \geq 2.}$$

~~[illegible]~~

(2). 2nd Application.

Counting Integers.

As a simple example of the use of the principle of inclusion-exclusion, consider the question.

How many integers in $\{1, \dots, 100\}$ are not divisible by 2, 3 or 5?

let $S = \{1, \dots, 100\}$ and P_1 the property that an integer is divisible by 2, P_2 the property that an integer is divisible by 3 and P_3 the property that an integer is divisible by 5 letting A_i be the subset of S whose elements have property P_i we have by elementary

Counting: $(A_1) = 50$, $(A_2) = 33$, & $(A_3) = 20$. There are 16 of these integers divisible by 6, 10 divisible by 10, and 6 divisible by 15, Finally there are just 3 integers divisible by 30, so the number of integers not divisible by any of 2, 3 or 5 is given by

$$100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

③ 3rd Applications

Counting Intersections

The principle of inclusion-exclusion, combined with De Morgan's law, can be used to count the cardinality of the intersection of sets as well.
Let $\overline{A_k}$ represents the complement of A_k with respect to some universal set A such that $A_k \subseteq A$ for each k. The we have

$$\bigcap_{i=1}^{n} A_i = \overline{\bigcup_{i=1}^{n} \overline{A_i}}$$

thereby turning the problem of finding an intersection into the problem of finding a union.

④ <u>Graph coloring:</u>

The inclusion exclusion principle forms the basis of algorithms for a number of NP-hard graph partitioning problems, such as graph coloring.

A well known application of the principle is the construction of the chromatic polynomial of a graph.

⑤ <u>Bipartite graph perfect matchings:</u>

The number of perfect matchings of a bipartite graph can be calculated using this principle.

⑥ <u>Number of onto functions.</u>

Given finite sets A & B, how many surjective functions (onto functions) are there from A to B? without loss of generality we may take $A = \{1, \ldots, k\}$ & $B = \{1, \ldots, n\}$, since only the cardinalities of the sets matter. By using S as the set of all functions from A to B, and defining for i in B, the property P_i as "the function misses the element i in B" (i is not in the image of the function), the principle of inclusion-exclusion gives the number of onto functions between A and B as:

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^k.$$

⑦ <u>Permutations with forbidden positions:</u>

A permutation of the set $S=\{1,\dots,n\}$ each element of S is restricted to not being in certain positions (here the permutation is considered as an ordering of the elements of S) is called a permutation with forbidden positions. For example, with $S=\{1,2,3,4\}$, the permutations with the restriction that the element 1 can not be in positions 1 or 3, and the element 2 can not be in position 4 are: 2134, 3124, 4123, 2341, 2431, 3241, 3421, 4231 and 4321. By letting A_i be the set of positions that the element i is not allowed to be in and the property P_i to be the property that a permutation puts element i into a position in A_i, the principle of inclusion-exclusion can be used to count the number of permutations which satisfy all the restrictions.

In the given example, there are $12 = 2(3!)$ permutations with property P_1, $6 = 3!$ permutation with property P_2 and no permutations have properties P_3 only as there are no restrictions for these two elements. The number of permutations satisfying the restrictions is thus:

$$4! - (12+6+0+0) + (4) = 24 - 18 + 4 = 10$$

The final 4 in this computation is the number of permutations having both properties P_1 and P_2. There are no other non-zero contributions to the formula.

(8) Stirling numbers of the second kind:

The stirling numbers of the second kind $S(n,k)$ count the number of partition of a set of n elements into k non-empty subsets (indistinguishable boxes).

(9) Rook Polynomials:

A rook polynomial is the generating function of the number of ways to place non-attacking rooks on a board B that looks like a subset of the squares of a checkerboard.

that is, no two rooks may be in the same row or column.

⑧ **Euler's Phi function:-**

Euler's totient or phi function, $\phi(n)$ is an arithmetic function that counts the number of positive integers less than or equal to "n" that are relatively prime to n. That is, if n is a positive integer, then $\phi(n)$ is the number of integers k in the range $1 < k \leq n$ which have no common factor with "n" other than 1. The principle of inclusion-exclusion is used to obtain a formula for $\phi(n)$. Let S be the set $\{1, \ldots, n\}$ & define the property P_i to be that a number in S is divisible by the prime number P_i, for $1 < i \leq r$, where the prime factorization of

$$n = P_1^{a_1} P_2^{a_2} \ldots P_r^{a_r}.$$

Then,

$$\boxed{\phi(n) = n - \sum_{i=1}^{r} \frac{n}{P_i} + \sum_{1 \leq i < j \leq r} \frac{n}{P_i P_j} - \ldots = n \prod_{i=1}^{r} \left(1 - \frac{1}{P_i}\right)}$$