

Math493: Honors Algebra I

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Abstract

This course is **a basic introduction on finite group theory** and **representation theory**, containing my personal thoughts as well as lecture notes. My course instructor is [Mircea Mustață](#).

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Chapter 1

Group Actions

1.1 Introduction

We now lay our focus to group actions, group actions are useful because we can endowed the **symmetric structure** of a group into other mathematical objects through group actions, specifically:

- often groups acts on various mathematical structure, such as sets, topological spaces, manifolds, etc.
- It will be of great significance for us to consider the actions of a group on itself via **conjugation**.

Definition 1.1.1. Let's fix a group G and a set X , an **action** (say also a left action of G) on X is a map:

$$G \times X \rightarrow X$$

$$(g, x) \mapsto gx$$

such that the following holds:

$$ex = x \quad \forall x \in X$$

$$g(hx) = (gh)x \quad \forall g, h \in G, x \in X$$

We now introduce an **equivalent formulation** for group action:

Recall that:

$$S_X = (\{\text{bijections } X \rightarrow X\}, \circ)$$

is a group.

Definition 1.1.2. Now suppose we have the action of G on X as above, we may define a map $\varphi : G \rightarrow S_X$ as follows: for every $g \in G$, $\varphi(g)$ which written as φ_g is the map:

$$\varphi_g : X \rightarrow X, \varphi_g(x) = gx$$

It is easy to see that by inheritance of the existence of inverses in G , φ_g is a bijection. In particular, one can see that it is actually a **group homomorphism**.

And the following conclusion is easy to deduce:

Conclusion 1.1.1.

$$\{\text{Actions of } G \text{ on } X\} \leftrightarrow \{\text{Group Homomorphism } G \rightarrow S_X\}$$

forms a **bijection**.

We then give some examples of group actions:

Example. Given any set X , we have the identity, **trivial** group action given by the group homomor-

phism:

$$S_X \xrightarrow{Id} S_X$$

which is equivalent to the action of S_X on X by:

$$S_X \times X \rightarrow X, (f, x) \mapsto f(x)$$

Example. If $n > 3$ and P_n be the regular n -gon, we then have a group homomorphism:

$$D_{2n} \rightarrow S_{P_n}$$

which leads to an action of D_{2n} on P_n

Note. See that in this case D_{2n} preserve the distance structure within the regular n -gon.

Example. The group $GL_n(\mathbb{C})$ acts on \mathbb{C}^n via:

$$(A, u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}) \mapsto Au$$

which represent the matrix as **linear transformation**. Such corresponds to the group homomorphism:

$$\begin{aligned} GL_n(\mathbb{C}) &\rightarrow S_{\mathbb{C}^n} \\ A &\mapsto \text{corresponds linear transformation on } \mathbb{C}^n \end{aligned}$$

Example. (Cayley's Theorem) Define an action of G on itself by:

$$\begin{aligned} G \times G &\rightarrow G \\ (g, h) &\mapsto g \cdot h \end{aligned}$$

which acts by the natural left multiplication. Note such corresponds to a group homomorphism:

$$G \xrightarrow{\varphi} S_G$$

And we shall have:

Proposition 1.1.1. (Cayley) φ is always injective

In particular, if G is finite, G is then **isomorphic** to a subgroup of S_n .

$$G \cong \text{Im}(\varphi) \subseteq S_G$$

The proof is immediate by showing $\ker(\varphi) = \{e\}$ by cancellation.

Example. Suppose $H \leq G$, we have:

$$\begin{aligned} G \times (G/H)_I &\rightarrow (G/H)_I \\ (g, ah) &\mapsto gaH \end{aligned}$$

easy to see such is a group action after checking well-definedness. Such action is induced by the action of group on itself, note H here is **not necessarily normal**.

Example. (Group action by **Conjugation**) The following will be the most interesting example for us. First recall we have an **automorphism** given by $g \in G$:

$$\alpha_g : G \rightarrow G, \alpha_g(x) = gxg^{-1}$$

Moreover, observe $\text{Aut}(G) \leq S_G$, so we have a group homomorphism:

$$\begin{aligned} G &\rightarrow \text{Aut}(G) \leq S_G \\ g &\mapsto \alpha_g \end{aligned}$$

We can understand $\text{Aut}(G)$ as those **permutation that preserve the group structure**. In particular, by our discussion, we get an action of G on itself:

$$(g, x) \mapsto gxg^{-1}$$

1.2 Orbits and Orbits-Stabilizer Theorem

Definition 1.2.1. Write $x \sim y$ for $x, y \in X$, if $\exists g \in G$, s.t. $gx = y$.

Lemma 1.2.1. Such gives us a equivalent relation, directly check by **reflexive, symmetric, transitive**.

Conclusion 1.2.1. We get a partition of X into equivalence classes, called **orbits**. If $x \in X$, then the corresponding equivalence classes is given by:

$$\{gx \mid g \in G\}$$

which is denoted by **Gx** or **O(x)**.

Notation. X/G denotes the sets of the orbits of X .

Definition 1.2.2. The action of G on X is transitive if X has only one orbits, which is:

$$\forall x, y \in X, \exists g \in G \text{ s.t. } gx = y$$

Example. The action given by the left multiplication of G on itself is **transitive**.

Example. Induced by above example, the action of G on the **set of left cosets** of H is also transitive.

Definition 1.2.3. For every $x \in X$, the stabilizer of $x \in G$ is given by:

$$\text{Stab}_G(x) = \{g \in G \mid gx = x\}$$

namely those elements in G that doesn't move the position of x .

Lemma 1.2.2. $\text{Stab}_G(x) \leq G$ being a subgroup.

Example. Consider the action of G on itself by conjugation, the orbits of $a \in G$ is called the

conjugate class of a . Two elements of G are conjugate of each other if they lie in the same conjugate class (**same orbit**).

What is the stabilizer in this case?

$$Stab_G(x) = \{y \in G \mid yxy^{-1} = x\} =: C_G(x)$$

which is the centralizer of x in G .

Note. $C_G(x) = G$ iff $x \in Z(G)$

Remark. Consider the conjugacy classes of S_n , then $\sigma, \tau \in S_n$ are conjugate of each other if and only if when they written as **product of disjoint cycles**, then # of k -cycle for both of them is the **same** for all k (**they have same cycle type**).

We now introduce Orbit-Stabilizer theorem.

Theorem 1.2.1. If G acts on X , then for every $x \in X$:

$$\#O(x) = (G : Stab_G(x))$$

In particular, if G is finite group, then

$$\#O(x) \mid |G|$$

Proof. Define a map:

$$\begin{aligned} f : (G/Stab_G(x))_I &\longrightarrow O(x) \\ f(gStab_G(x)) &= gx \end{aligned}$$

- Well-defineness + injectivity:

$$\begin{aligned} g_1Stab_G(x) &= g_2Stab_G(x) \\ \Leftrightarrow g_2^{-1}g_1 &\in Stab_G(x) \\ \Leftrightarrow (g_2^{-1}g_1)x &= x \\ \Leftrightarrow g_2^{-1}(g_1x) &= x \\ \Leftrightarrow g_1x &= g_2x \end{aligned}$$

- Surjectivity:

$$\begin{aligned} y &\in O(x) \\ \Rightarrow y &= gx \\ \Rightarrow y &= f(gStab_G(x)) \end{aligned}$$

So we see f is a bijection. The first isomorphism theorem direct yields the result. ■

Note. When the action is transitive, with X being finite, we have:

$$\#O(x) = (G : Stab_G(x)) = |X|$$

In particular, transitive means there is only one orbits.

Proposition 1.2.1. If G acts on X , then:

$$|X| = \sum_{i \in I} (G : Stab_G(x_i))$$

where x_i are a system of representative for the orbits of G in X .

Proof. We have a partition:

$$X = \bigsqcup_{i \in I} O(x) \Rightarrow \#X = \sum \#O(x_i)$$

with:

$$\#O(x_i) = (G : Stab_G(x_i))$$

■

Example. (Class Equation: A important special case) Consider the action of $G \times G \rightarrow G$ by **conjugation**, with G be finite group:

$$\begin{aligned} |G| &= \sum_{i \in I} (G : C_G(x_i)) \\ \Rightarrow |G| &= |Z(G)| + \sum_{i \in I'} (G : C_G(x_i)) \end{aligned}$$

where I' runs over indices such that $(G : C_G(x_i)) > 1$.

Such results direct yields from the fact that elements in $Z(G)$ attains its centralizer (stabilizer) to be the whole group.

Note. Such only works for actions by **conjugation!**

We now give an application for orbits-stabilizer theorem, which is important when we study the construction of groups.

Definition 1.2.4. If p be prime number, and G be group, if $|G| = p^n$ for some $n \geq 1$, then we say G is a p -group.

Proposition 1.2.2. If G is a p -group, then:

$$Z(G) \neq \{e\}$$

Proof. Since $p \mid |G|$ and $(G : Stab_G(x)) \mid |G| = p^n$, then:

$$p \mid (G : C_G(x_i))$$

whenever this is > 1 , then class equation yields that:

$$p \mid |Z(G)|$$

■

Appendix