Math493: Honors Algebra I

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Abstract

This course is **a basic introduction on finite group theory** and **representation theory**, containing my personal thoughts as well as lecture notes. My course instructor is Mircea Mustaţă.

Contents

Chapter 1

Group Actions

1.1 Introduction

We now lay our focus to group actions, group actions are useful because we can endowed the **symmetric structure** of a group into other mathematical objects through group actions, specifically:

- often groups acts on various mathematical structure, such as sets, topological spaces, manifolds, etc.
- It will be of great significance for us to consider the actions of a group on itself via **conjugation**.

Definition 1.1.1. Let's fix a group G and a set X, an **action** (say also a left action of) G on X is a map:

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx \end{aligned}$$

such that the following holds:

$$ex = x \quad \forall\, x \in X$$

$$g(hx) = (gh)x \quad \forall\, g, h \in G,\ x \in X$$

We now introduce an **equivalent formulation** for group action:
Recall that:

$$S_X = (\{\text{bijections } X \to X\}, \circ)$$

is a group.

Definition 1.1.2. Now suppose we have the action of G on X as above, we may define a map $\varphi : G \to S_X$ as follows: for every $g \in G$, $\varphi(g)$ which written as φ_g is the map:

$$\varphi_g : X \to X, \varphi_g(x) = gx$$

It is easy to see that by inheritance of the existance of inverses in G, φ_g is a bijection. In particular, one can see that it is actually a **group homomorphism**.

And the following conclusion is easy to deduce:

Conclusion 1.1.1.

$$\{\textbf{Actions of } G \textbf{ on } X\} \leftrightarrow \{\textbf{Group Homomorphism } G \to S_X\}$$

forms a **bijection**.

Appendix