

# Math493: Honors Algebra I

Yanzhi Li

December 27, 2025

### **Abstract**

This course is **a basic introduction on finite group theory** and **representation theory**, containing my personal thoughts as well as lecture notes. My course instructor is [Mircea Mustață](#).

# Contents

<b>I</b>	<b>Finite Group Theory</b>	<b>3</b>
<b>1</b>	<b>Group Actions</b>	<b>4</b>
1.1	Introduction	4
1.2	Orbits and Orbits-Stablizer Theorem	6
<b>2</b>	<b>Sylow's Theorem</b>	<b>10</b>
2.1	Cauchy's Theorem	10
2.2	Sylow's First Theorem	11
2.3	Sylow's Second and Third Theorem	12
2.3.1	More on Group Action	12
2.3.2	Sylow's Second and Third Theorem	13
2.4	Simple Group	16
2.4.1	Special simple groups	16
2.4.2	Classification of Finite Simple Groups	19
2.5	Semidirect Product	19
2.5.1	External Semidirect Product	19
2.5.2	Internal Semidirect Product	20
<b>3</b>	<b>Classification of Finite Groups</b>	<b>22</b>
3.1	Composition Series and Jordan-Hölder Theorem	22
3.2	Solvable and Nilpotent Groups	25
3.2.1	Solvable Group	25
3.2.2	Nilpotent Group	28
3.3	Free groups	31
3.3.1	Construction of Free Groups	32
3.3.2	Presentations of Groups by Generators and Relations	34
<b>II</b>	<b>Representation Theory</b>	<b>37</b>
<b>4</b>	<b>Group Representations</b>	<b>38</b>
4.1	Basic Definitions	38
4.2	Category of $G$ -representations	39
4.2.1	Morphisms of $G$ -representations	39
4.2.2	Direct Sum of Representations	41
4.2.3	Hom between two Representations	43
4.2.4	Irreducible Representation	45
4.3	Group Representations on Multi-linear Algebra	47
4.3.1	Representation via Tensor Product	47
4.3.2	Representation via Exterior Product	48
4.3.3	Representation via Symmetric Product	50

---

<b>5</b>	<b>Character Theory</b>	<b>52</b>
5.1	Basic Definition . . . . .	52
5.2	Orthogonality of Characters . . . . .	54
5.3	Analysis on Decomposition . . . . .	57
5.3.1	Regular Repr. and Irred. Repr. . . . .	57
5.4	Characters and Conjugacy Classes . . . . .	59
5.5	Irreducible Representation of $S_4$ . . . . .	62

**Part I**

**Finite Group Theory**

# Chapter 1

## Group Actions

### 1.1 Introduction

We now lay our focus to group actions, group actions are useful because we can endowed the **symmetric structure** of a group into other mathematical objects through group actions, specifically:

- often groups acts on various mathematical structure, such as sets, topological spaces, manifolds, etc.
- It will be of great significance for us to consider the actions of a group on itself via **conjugation**.

**Definition 1.1.1 (Group Action).** Let's fix a group  $G$  and a set  $X$ , an **action** (say also a left action of )  $G$  on  $X$  is a map:

$$\begin{aligned} G \times X &\rightarrow X \\ (g, x) &\mapsto gx \end{aligned}$$

such that the following holds:

$$\begin{aligned} ex &= x \quad \forall x \in X \\ g(hx) &= (gh)x \quad \forall g, h \in G, x \in X \end{aligned}$$

We now introduce an **equivalent formulation** for group action:  
Recall that:

$$S_X = (\{\text{bijections } X \rightarrow X\}, \circ)$$

is a group.

**Definition 1.1.2 (Group Action Cont.).** Now suppose we have the action of  $G$  on  $X$  as above, we may define a map  $\varphi : G \rightarrow S_X$  as follows: for every  $g \in G$ ,  $\varphi(g)$  which written as  $\varphi_g$  is the map:

$$\varphi_g : X \rightarrow X, \varphi_g(x) = gx$$

It is easy to see that by inheritance of the existence of inverses in  $G$ ,  $\varphi_g$  is a bijection. In particular, one can see that it is actually a **group homomorphism**.

And the following conclusion is easy to deduce:

**Conclusion 1.1.1.**

$$\{\text{Actions of } G \text{ on } X\} \leftrightarrow \{\text{Group Homomorphism } G \rightarrow S_X\}$$

forms a **bijection**.

We then give some examples of group actions:

**Example 1.1.1.** Given any set  $X$ , we have the identity, **trivial** group action given by the group homomorphism:

$$S_X \xrightarrow{Id} S_X$$

which is equivalent to the action of  $S_X$  on  $X$  by:

$$S_X \times X \rightarrow X, (f, x) \mapsto f(x)$$

**Example 1.1.2.** If  $n > 3$  and  $P_n$  be the regular  $n$ -gon, we then have a group homomorphism:

$$D_{2n} \rightarrow S_{P_n}$$

which leads to an action of  $D_{2n}$  on  $P_n$

**Note.** See that in this case  $D_{2n}$  preserve the distance structure within the regular  $n$ -gon.

**Example 1.1.3.** The group  $GL_n(\mathbb{C})$  acts on  $\mathbb{C}^n$  via:

$$(A, u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}) \mapsto Au$$

which represent the matrix as **linear transformation**. Such corresponds to the group homomorphism:

$$GL_n(\mathbb{C}) \rightarrow S_{\mathbb{C}^n}$$

$A \mapsto \text{corresponds linear transformation on } \mathbb{C}^n$

**Example 1.1.4 (Cayley's Theorem).** Define an action of  $G$  on itself by:

$$G \times G \rightarrow G$$

$$(g, h) \mapsto g \cdot h$$

which acts by the natural left multiplication. Note such corresponds to a group homomorphism:

$$G \xrightarrow{\varphi} S_G$$

And we shall have:

**Proposition 1.1.1 (Cayley's Theorem).**  $\varphi$  is always injective

In particular, if  $G$  is finite,  $G$  is then **isomorphic** to a subgroup of  $S_n$ .

$$G \cong \text{Im}(\varphi) \subseteq S_G$$

The proof is immediate by showing  $\ker(\varphi) = \{e\}$  by cancellation.

**Example 1.1.5.** Suppose  $H \leq G$ , we have:

$$G \times (G/H)_I \rightarrow (G/H)_I$$

$$(g, ah) \mapsto gaH$$

easy to see such is a group action after checking well-definedness. Such action is induced by the action of group on itself, note  $H$  here is **not necessarily normal**.

**Example 1.1.6** (Group action by **Conjugation**). The following will be the most interesting example for us. First recall we have an **automorphism** given by  $g \in G$ :

$$\alpha_g : G \rightarrow G, \alpha_g(x) = gxg^{-1}$$

Moreover, observe  $\text{Aut}(G) \leq S_G$ , so we have a group homomorphism:

$$\begin{aligned} G &\rightarrow \text{Aut}(G) \leq S_G \\ g &\mapsto \alpha_g \end{aligned}$$

We can understand  $\text{Aut}(G)$  as those **permutation that preserve the group structure**. In particular, by our discussion, we get an action of  $G$  on itself:

$$(g, x) \mapsto gxg^{-1}$$

## 1.2 Orbits and Orbits-Stablizer Theorem

**Definition 1.2.1.** Write  $x \sim y$  for  $x, y \in X$ , if  $\exists g \in G$ , s.t.  $gx = y$ .

**Lemma 1.2.1.** Such gives us a equivalent relation, directly check by **reflexive, symmetric, transitive**.

**Conclusion 1.2.1.** We get a partition of  $X$  into equivalence classes, called **orbits**. If  $x \in X$ , then the corresponding equivalence classes is given by:

$$\{gx \mid g \in G\}$$

which is denoted by **Gx** or **O(x)**.

**Notation.**  $X/G$  denotes the sets of the orbits of  $X$ .

**Definition 1.2.2 (Transitive).** The action of  $G$  on  $X$  is transitive if  $X$  has only one orbits, which is:

$$\forall x, y \in X, \exists g \in G \text{ s.t. } gx = y$$

**Example 1.2.1.** The action given by the left multiplication of  $G$  on itself is **transitive**.

**Example 1.2.2.** Induced by above example, the action of  $G$  on the **set of left cosets** of  $H$  is also transitive.

**Definition 1.2.3.** For every  $x \in X$ , the stablizer of  $x \in G$  is given by:

$$\text{Stab}_G(x) = \{g \in G \mid gx = x\}$$

namely those elements in  $G$  that doesn't move the position of  $x$ .

**Lemma 1.2.2.**  $\text{Stab}_G(x) \leq G$  being a subgroup.

**Example 1.2.3.** Consider the action of  $G$  on itself by conjugation, the orbits of  $a \in G$  is called the



conjugate class of  $a$ . Two elements of  $G$  are conjugate of each other if they lie in the same conjugate class (**same orbit**).

What is the stablizer in this case?

$$\text{Stab}_G(x) = \{y \in G \mid yxy^{-1} = x\} =: C_G(x)$$

which is the centralizer of  $x$  in  $G$ .

**Note.**  $C_G(x) = G$  iff  $x \in Z(G)$

**Remark 1.2.1.** Consider the conjugacy classes of  $S_n$ , then  $\sigma, \tau \in S_n$  are conjugate of each other if and only if when they written as **product of disjoint cycles**, then  $\#$  of  $k$ -cycle for both of them is the **same** for all  $k$  (**they have same cycle type**).

We now introduce Orbit-Stablizer theorem.

**Theorem 1.2.1 (Orbit-Stablizer Theorem).** If  $G$  acts on  $X$ , then for every  $x \in X$ :

$$\#O(x) = (G : \text{Stab}_G(x))$$

In particular, if  $G$  is finite group, then

$$\#O(x) \mid |G|$$

**Proof.** Define a map:

$$\begin{aligned} f : (G/\text{Stab}_G(x))_I &\longrightarrow O(x) \\ f(g\text{Stab}_G(x)) &= gx \end{aligned}$$

- Well-defineness + injectivity:

$$\begin{aligned} g_1\text{Stab}_G(x) &= g_2\text{Stab}_G(x) \\ \Leftrightarrow g_2^{-1}g_1 &\in \text{Stab}_G(x) \\ \Leftrightarrow (g_2^{-1}g_1)x &= x \\ \Leftrightarrow g_2^{-1}(g_1x) &= x \\ \Leftrightarrow g_1x &= g_2x \end{aligned}$$

- Surjectivity:

$$\begin{aligned} y &\in O(x) \\ \Rightarrow y &= gx \\ \Rightarrow y &= f(g\text{Stab}_G(x)) \end{aligned}$$

So we see  $f$  is a bijection. The first isomorphism theorem direct yields the result. ■

**Note.** When the action is transitive, with  $X$  being finite, we have:

$$\#O(x) = (G : \text{Stab}_G(x)) = |X|$$

In particular, transitive means there is only one orbits.

**Proposition 1.2.1.** If  $G$  acts on  $X$ , then:

$$|X| = \sum_{i \in I} (G : \text{Stab}_G(x_i))$$

where  $x_i$  are a system of representative for the orbits of  $G$  in  $X$ .

**Proof.** We have a partition:

$$X = \bigsqcup_{i \in I} O(x_i) \Rightarrow \#X = \sum \#O(x_i)$$

with:

$$\#O(x_i) = (G : \text{Stab}_G(x_i))$$

■

**Example 1.2.4 (Class Equation).** Consider the action of  $G \times G \rightarrow G$  by **conjugation**, with  $G$  be finite group:

$$\begin{aligned} |G| &= \sum_{i \in I} (G : C_G(x_i)) \\ \Rightarrow |G| &= |Z(G)| + \sum_{i \in I'} (G : C_G(x_i)) \end{aligned}$$

where  $I'$  runs over indices such that  $(G : C_G(x_i)) > 1$ .

Such results direct yields from the fact that elements in  $Z(G)$  attains its centralizer (stablizer) to be the whole group.

**Note.** Such only works for actions by **conjugation**!

We now give an application for orbits-stablizer theorem, which is important when we study the construction of groups.

**Definition 1.2.4 (p-group).** If  $p$  be prime number, and  $G$  be group, if  $|G| = p^n$  for some  $n \geq 1$ , then we say  $G$  is a p-group.

**Proposition 1.2.2.** If  $G$  is a  $p$ -group, then:

$$Z(G) \neq \{e\}$$

**Proof.** Since  $p \mid |G|$  and  $(G : \text{Stab}_G(x)) \mid |G| = p^n$ , then:

$$p \mid (G : C_G(x_i))$$

whenever this is  $> 1$ , then class equation yields that:

$$p \mid |Z(G)|$$

■

**Corollary 1.2.1.** If  $p$  prime,  $|G| = p^2$ , then  $G$  is an abelian group.

To proof this we need first a lemma:

**Lemma 1.2.3.** For all group  $G$ , if  $G/Z(G)$  is cyclic, then  $G$  is abelian.

**Proof.** By cyclic property, first suppose that  $xZ(G)$  is a generator of  $G/Z(G)$ , then there exists

$i, j \in \mathbb{Z}$ , such that:

$$\begin{aligned}
 aZ(G) &= x^i Z(G) \\
 bZ(G) &= x^j Z(G) \\
 \Leftrightarrow a &= x^i a' \\
 b &= x^j b' \quad \text{for some } a', b' \in Z(G) \\
 \Rightarrow ab &= x^i a' x^j b' = x^{i+j} a' b' \\
 ba &= x^j b' x^i a' = x^{i+j} b' a' \\
 \Rightarrow ab &= ba \quad \text{since } a', b' \in Z(G)
 \end{aligned}$$

■

We now give proof to **Corollary 1.2.1**:

**Proof.** We know that:

$$Z(G) \neq \{e\}$$

Then by **Lagrange's theorem**, see that either  $Z(G) = G$  or  $|Z(G)| = p$ .

If  $Z(G) = G$ , we already done! Now suppose that  $|Z(G)| = p$ , now observe that every subgroup of  $Z(G)$  is a **normal subgroup** of  $G$  by definition.

In particular, we then consider the group  $G/Z(G)$ , this is of order  $p$ , so it is cyclic. Then by **Lemma 1.2.3**, see that  $G$  is abelian, contradict to the fact that  $Z(G) \neq G$ . ■

# Chapter 2

## Sylow's Theorem

**Sylow's Theorem** is important when we want to understand and classify the category of finite groups, simply by looking on the order of such finite group. Further, it lays foundation and gives tools for us when we want to understand the structure of the building blocks of finite group (simple group).

### 2.1 Cauchy's Theorem

Our motivation to study Cauchy's Theorem, as well as Sylow's Theorem, in the first place, is we want to give a partial inverse to **Lagrange's theorem**. Recall that Lagrange's theorem implies that:

**Proposition 2.1.1.** If  $G$  is a finite group, with order  $n$ , then  $\forall x \in G$ , it satisfies that:

$$|x| \mid n$$

In general, the converse of this statement is not true, in particular:

$$\exists q \mid |G|, \text{ but there exists no } g \in G, \text{ s.t. } |g| = q$$

for example we can take  $q = |G|$ , but  $G$  is **not a cyclic group**.

So we give partial converse of this statement by stating **Cauchy's Theorem** and later Sylow's theorem.

**Theorem 2.1.1 (Cauchy's Theorem).** If  $G$  is a finite group and  $p$  is a prime integer, s.t.  $p \mid |G|$ , then  $\exists g \in G, |g| = p$ .

We first proof for the case when  $G$  is **abelian group**, then one can easily derive the general cases by invoking **Z(G)** of any group.

**Proof.** We reason by deviding cases for abelian and non-abelian group:

- When  $G$  is **abelian**: We argue by contradiction, suppose that  $|x| \neq p, \forall x \in G$ , in this case see that  $p \nmid |x|, \forall x \in G$ , otherwise if  $|x| = m, p \mid m \Rightarrow |x^{\frac{m}{p}}| = p \nmid$ .

Let  $N = \text{lcm}\{|x| \mid x \in G\}$ , use prime factorization of  $N$  see that  $p \nmid N$ . Now suppose  $g_1, \dots, g_n$  are the elements of  $G$ , we define:

$$f : \underbrace{\mathbb{Z}/N\mathbb{Z} \times \dots \times \mathbb{Z}/N\mathbb{Z}}_{n \text{ times}} \longrightarrow G$$

$$f(a_1 + N\mathbb{Z}, \dots, a_n + N\mathbb{Z}) = g_1^{a_1} \dots g_n^{a_n}$$

one can easily check the well-definedness of such map. The point is  $G$  be an abelian group makes

this map a group homomorphism:

$$\begin{aligned}
 f((a_1 + N\mathbb{Z}, \dots, a_n + N\mathbb{Z}) + (b_1 + N\mathbb{Z}, \dots, b_n + N\mathbb{Z})) &= f(a_1 + b_1 + N\mathbb{Z}, \dots, a_n + b_n + N\mathbb{Z}) \\
 &= g_1^{a_1+b_1} \dots g_n^{a_n+b_n} \\
 &= (g_1^{a_1} \dots g_n^{a_n})(g_1^{b_1} \dots g_n^{b_n}) \quad \text{By } G \text{ is abelian} \\
 &= f((a_1 + N\mathbb{Z}, \dots, a_n + N\mathbb{Z})) \\
 &\quad + f((b_1 + N\mathbb{Z}, \dots, b_n + N\mathbb{Z}))
 \end{aligned}$$

so  $f$  indeed be a group homomorphism. One should then see that  $f$  is clearly surjective by taking:

$$g_i = f(0, \dots, \overset{i}{1}, \dots, 0)$$

By fundamental isomorphism theorem:

$$G \cong (\mathbb{Z}/N\mathbb{Z})^n / \ker(f) \Rightarrow |G| = \frac{|\mathbb{Z}/N\mathbb{Z}|^n}{|\ker(f)|}$$

And by Lagrange theorem:

$$|G| \mid |(\mathbb{Z}/N\mathbb{Z})^n| = N^n$$

however, since  $p \mid |G|$ , but  $p \nmid N$ , hence  $p \nmid N^n$ , leading to contradiction  $\nmid$ .

- When  $G$  is **non-abelian**: We argue by induction on  $|G|$ .
  - Base case: If  $|G| = p \Rightarrow G$  is cyclic, then we are done.
  - Inductive case: we assume we know the assertion for all  $G'$ , such that  $p \mid |G'|$ , with  $|G'| < |G|$ , and prove the assertion for  $G$ . Note that if there exists  $H \leq G$ , with  $H \neq G$ , but  $p \mid |H|$ , then by the induction hypothesis we are done. Hence, may assume:  $\forall H \leq G, H \neq G, p \nmid |H|$ . Then by Lagrange theorem:  $|G| = |H| \cdot (G : H) \Rightarrow p \mid (G : H)$ . By class equation:

$$|G| = |Z(G)| + \sum_{i=1}^d (G : C_G(x_i))$$

where  $x_i$  runs over a system of representatives of conjugacy classes with  $\geq 2$  elements. By our assumptions, we saw:

$$p \mid (G : C_G(x_i)), \quad p \mid |G| \Rightarrow p \mid |Z(G)|$$

Since among all subgroups of  $G$ , **only itself** can be divided by  $p$ , so this means that:

$$Z(G) = G$$

so we see  $G$  is abelian and yields back to the case we've proven before. ■

## 2.2 Sylow's First Theorem

With the help of Cauchy's theorem, we now give proof to Sylow's first theorem.

**Definition 2.2.1.** If  $G$  is finite group,  $p$  prime integer, s.t.  $p \mid |G|$  and  $p^m \mid |G|$ ,  $p^{m+1} \nmid |G|$ , then a subgroup of  $G$  with order  $p^m$  is called a  $p$ -Sylow subgroup of  $G$ .

**Theorem 2.2.1 (Sylow's First Theorem).** If  $G$  is a finite group and  $p$  is a prime integer, if  $p \mid |G| \Rightarrow G$  contains a  $p$ -Sylow subgroup.

**Note.** Sylow's first theorem actually implies Cauchy's theorem.

If  $H \leq G$  be a  $p$ -Sylow subgroup and  $g \in H \setminus \{e\} \Rightarrow |g| = p^r \Rightarrow |g^{p^{r-1}}| = p$ .

However, we will **use** Cauchy's theorem to prove Sylow's theorem.

**Proof.** We argue by induction on  $|G|$ .

- Base case:  $|G| = p$ , the assertion is clear,  $G$  is its own  $p$ -Sylow subgroup.
- Inductive case: Assume that we know the theorem for all groups  $G'$ , s.t.  $p \mid |G'|$ , and  $|G'| < |G|$ , we want to show the assertion for  $G$ .

Suppose that  $p^m \mid |G|$ , but  $p^{m+1} \nmid |G|$ , then there are two cases:

- If there is a **proper subgroup**  $H$  of  $G$ , s.t.  $p^m \mid |H|$ , then by induction hypothesis,  $H$  contains a  $p$ -Sylow subgroup, and this is also a  $p$ -Sylow subgroup of  $G$ .
- If **for all proper subgroup**  $H$  of  $G$ ,  $p^m \nmid |H|$ , then  $p^m$  has to **divide the index of such subgroup**. Then by class equation:

$$|G| = |Z(G)| + \sum_{i=1}^d \underbrace{(G : C_G(x_i))}_{\geq 2 \text{ hence divided by } p}.$$

And see that:

$$p \mid |G| \Rightarrow p \mid |Z(G)|$$

By Cauchy's theorem for the abelian group  $Z(G)$ , there exists  $x \in Z(G)$ , s.t.  $|x| = p$ . Since  $x \in Z(G) \Rightarrow \langle x \rangle$  is a normal subgroup of  $G$ . Now consider:  $G' := G / \langle x \rangle$

- \* If  $m = 1$ , then we are done! Since  $\langle x \rangle$  is a  $p$ -Sylow subgroup of  $G$ .
- \* If  $m \geq 2$ , see that  $|G'| = \frac{|G|}{p} < |G|$  is divisible by  $p$ , so we apply induction on  $G'$ , see that  $G'$  contains a  $p$ -Sylow subgroup, which is of the form:  $H/\langle x \rangle$  for some subgroup  $H$  of  $G$ . Notice that:

$$|H / \langle x \rangle| = p^{m-1} \Rightarrow |H| = p^{m-1} \cdot p = p^m$$

by prime factorization,  $G = p^m \cdot (\dots)$ , so see  $|G'| = \frac{|G|}{p} = p^{m-1} \cdot (\dots)$

## 2.3 Sylow's Second and Third Theorem

Sylow's Second theorem basically tells us how the  $p$ -Sylow subgroup of  $G$  relate to each other, and Sylow's third theorem tells us the possible number of each  $p$ -Sylow subgroup can take. Before diving into the proof, we need to dig more intuition on what group actions can do, i.e. how it **permute the endian sets**.

### 2.3.1 More on Group Action

**Example 2.3.1.** Suppose  $G$  is a group acting on a set  $X$ :

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \end{aligned}$$

If  $\mathcal{P}(X)$  is the set of all subsets of  $X$ , we then get an **induced group action**:

$$\begin{aligned} G \times \mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ (g, A) &\longmapsto gA = \{ga \mid a \in A\} \end{aligned}$$

and it's clear that such satisfy the group action properties, we get a group action of  $G$  acting on  $\mathcal{P}(X)$ .

**Important example:** If in this case we consider the action on  $G$  acting on itself by conjugation, we can get a correspond action of  $G$  on  $\mathcal{P}(G)$ :

$$(g, A) \longmapsto gAg^{-1} = \{gag^{-1} \mid a \in A\}$$

**Note.** If  $A$  is a subgroup of  $G$ , then  $gAg^{-1}$  is also a subgroup of  $G$ .

the conjugation map is a group automorphism

**Remark 2.3.1.** If  $H$  is a  $p$ -Sylow subgroup of  $G$  and  $\sigma : G \rightarrow G$  be an automorphism, then  $\sigma(H)$  will also be a  $p$ -Sylow subgroup, in particular, **every conjugation** of  $H$  which is  $p$ -Sylow subgroup will also be a  $p$ -Sylow subgroup.

## 2.3.2 Sylow's Second and Third Theorem

**Theorem 2.3.1 (Sylow's Second Theorem).** If  $H$  is a  $p$ -Sylow subgroup of  $G$ ,  $K$  is **any**  $p$ -subgroup of  $G$ , then there exists  $a \in G$ , s.t.  $K \subseteq aHa^{-1}$ .

In particular: if  $K$  is a  $p$ -Sylow subgroup, too. Then  $H, K$  are conjugate of each other:  $K = aHa^{-1}$  for some  $a \in G$ .

**Theorem 2.3.2 (Sylow's Third Theorem).** If we denote:

$$n_p = \#p\text{-Sylow subgroups of } G$$

Then we have:

$$\begin{cases} n_p \equiv 1 \pmod{p} \\ n_p = (G : N_G(H)) \Rightarrow n_p \mid (G : H) \end{cases}$$

with:

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\}$$

where  $H$  is a  $p$ -Sylow subgroup of  $G$ .

**Note.**  $N_G(H)$  will be the largest subgroup of  $G$  in which  $H$  is a normal subgroup.

**Note.** The assertion:

$$n_p = (G : N_G(H)) \Rightarrow n_p \mid (G : H)$$

is derived from Lagrange's theorem:

$$\begin{aligned} H &\leq N_G(H) \leq G \\ \Rightarrow (G : H) &= \frac{|G|}{|H|} = \frac{|G|}{|N_G(H)|} \cdot (N_G(H) : H) \\ \text{And } (G : N_G(H)) &= \frac{|G|}{|N_G(H)|} \end{aligned}$$

We shall prove both of the theorem at the same time.

**Proof.** We've seen in **Example 2.3.1**,  $G$  has an action on  $\mathcal{P}(G) = \{T \mid T \leq G\}$  by conjugation:  $(g, T) \rightarrow gTg^{-1}$ . We now define:

$$\mathcal{S} = \{H = H_1, H_2, \dots, H_r\}$$

which is the collection of the orbits of  $H$  with respect to such action. By **Orbit-Stabilizer Theorem 1.2.1**:

$$\begin{aligned} r &= (G : \text{Stab}_G(H)) \\ \text{Stab}_G(H) &= \{g \in G \mid gHg^{-1} = H\} = N_G(H) \\ \Rightarrow r &= (G : N_G(H)) \end{aligned}$$

Now suppose we have a  $p$ -subgroup  $K$  of  $G$ , the action of  $G$  on  $\mathcal{P}(G)$  then induces an action of  $K$  on  $\mathcal{P}(G)$ , and we shall have the canonical injective homomorphism:

$$K \cap N_G(H) / K \cap H \hookrightarrow N_G(H) / H$$

For the right hand side, see that:

$$|N_G(H) / H| = \frac{|N_G(H)|}{|H|} \frac{|G|}{|H|} = \frac{|G|}{p^m}$$

Note that  $\frac{|G|}{p^m}$  is relatively primed to  $p$ .

For the left hand side, it attains a non-negative order (can be 0), since  $K$  is a  $p$ -group, it should attain its order to be **power of  $p$**  by **Lagrange's Theorem**.

Now since it is a injective homomorphism, it attains a trivial kernel, so by **Lagrange's theorem + first isomorphism theorem**:

$$|K \cap N_G(H) / K \cap H| \mid |N_G(H) / H|$$

But notice that the right hand side is coprime with  $p$  but left hand side divides  $p$ , this leads to:

$$|K \cap N_G(H) / K \cap H| = 1 \Rightarrow K \cap N_G(H) = K \cap H \quad (2.1)$$

Now suppose  $I \subseteq \{1, \dots, r\}$  are those such that  $\{H_i \mid i \in I\}$  which gives a system of representatives for the orbits of the  $K$  action on  $\mathcal{S}$ , then by class equation:

$$r = |\mathcal{S}| = \sum_{i \in I} (K : \underbrace{\text{Stab}_K(H_i)}_{=K \cap \text{Stab}_G(H_i) = K \cap H_i \text{ by 2.1}}) \quad (2.2)$$

Now we want to prove:

$$(K : K \cap H_i) \Leftrightarrow K \subseteq H_i \quad (2.3)$$

If it is not the case, then  $(K : K \cap H_i)$  is divisible by  $p$  since  $K$  is a  $p$ -group. First take  $K = H$ , since  $|H| = |H_i|$ ,  $\forall i$ , we have  $H \leq H_i \Leftrightarrow i = 1$ . Hence by **Equation 2.2**,  $r \equiv 1 \pmod{p}$ .

Suppose that  $K$  is an arbitrary  $p$ -group, if  $K \not\subseteq H_i$ ,  $\forall i \in I \Rightarrow p \mid r$  by **Equation 2.2**, which contradicts to  $r \equiv 1 \pmod{p}$ . Hence there exists  $i$ , s.t.  $K \subseteq H_i = aHa^{-1}$  for some  $a \in G$ , such yields **Sylow's Second Theorem**.

In particular, we see  $\mathcal{S} = \{p\text{-Sylow subgroups of } G\} \Rightarrow r = n_p$ :

$$\begin{aligned} \Rightarrow n_p &\equiv 1 \pmod{p} \\ n_p &= r = (G : N_G(H)) \end{aligned}$$

which yields **Sylow's Third Theorem**. ■

**Note.** All  $p$ -Sylow subgroups are conjugate to each other.



**Remark 2.3.2.**

$$n_p = 1 \Leftrightarrow H \trianglelefteq G$$

We state a small proposition that is helpful when we analyze the group structure along with Sylow's Theorem, it is also the midterm problem of this course.

**Proposition 2.3.1.** Let  $H, K \trianglelefteq G$ , and  $H \cap K = \{e\}$ , then:

$$\begin{aligned} H \times K &\longrightarrow HK \\ (h, k) &\longmapsto hk \end{aligned}$$

is a group isomorphism.

We now give two applications of Sylow's Theorem, and give proof to the second one.

**Proposition 2.3.2.** Let  $G$  be a group such that:  $|G| = pq$  where  $p, q$  are primes, let  $P_p$  and  $P_q$  be two the  $p, q$ -Sylow subgroups of  $G$  respectively, satisfying  $P_p, P_q \trianglelefteq G$ , and if  $p < q$ ,  $q \not\equiv 1 \pmod{p}$ , we have:

$$G \cong \mathbb{Z} / pq\mathbb{Z}$$

and thus  $G$  is **cyclic** and thus abelian.

**Proposition 2.3.3.** Suppose  $G$  be a group with order  $30 = 2 \cdot 3 \cdot 5$ , then:

1. there is a subgroup  $H \leq G$  of order 15.
2.  $n_5(G) = 1$ ,  $n_3(G) = 1$ .

**Proof.** Let  $H$  be a 5-Sylow subgroup of  $G$  and  $K$  be a 3-Sylow subgroup of  $G$ . See that  $|H \cap K| = 1$  since it has to divide both 3 and 5.

If  $H \trianglelefteq G$ , then by **second isomorphism theorem**, see that  $HK \leq G$  and:

$$\begin{aligned} HK / H &\cong K / H \cap K \cong K \\ \Rightarrow |HK| &= |H| \cdot |K| = 15 \end{aligned}$$

Similarly we will get a subgroup of order 15 if  $K$  is normal.

Then suppose that both  $H, K$  are not normal subgroups of  $G$ . By **Sylow's Third Theorem**:

$$\left. \begin{aligned} n_3(G) \mid \frac{|G|}{5} = 6 \\ n_3(G) \equiv 1 \pmod{5} \end{aligned} \right\} \Rightarrow n_5(G) = 6 \quad (n_5(G) \neq 1 \Leftrightarrow H \not\trianglelefteq G)$$

And similarly we have  $n_3(G) = 10$ .

Notice that the intersection of any 2 distinct 5-Sylow subgroups is the identity, since any of these subgroups is generated by elements that are not the identity. We then have  $6 \times 4 = 24$  elements of order 5 in  $G$ , and similarly we get  $10 \times 2 = 20$  elements of order 3. Since  $24 + 20 > 30$ , such leads to contradiction  $\nexists$ .

Thus we reach to the conclusion that there exists subgroup  $G'$  of  $G$  of order 15, and since  $(G : G') = 2 \Rightarrow G' \trianglelefteq G$ . Now since  $|G'| = 3 \times 5$  and  $5 \not\equiv 1 \pmod{3}$ , by **Proposition 2.3.2**,  $G'$  is cyclic thus abelian.

By Sylow's second theorem,  $n_5(G') = n_3(G') = 1$ , and it should deduce that  $n_3(G) = n_5(G) = 1$ :

Say  $A$  is a 5-Sylow subgroup of  $G'$ , it should also be a 5-Sylow subgroup of  $G$ . Now if  $A'$  is another 5-Sylow subgroup of  $G$ , by Sylow's Second theorem, there exists  $g \in G$ ,  $A' = gAg^{-1}$ . Since  $G'$  is abelian,  $A \trianglelefteq G' \trianglelefteq G \Rightarrow A' = gAg^{-1} \subseteq gG'g^{-1} = G' \Rightarrow gAg^{-1}$  is also a 5-Sylow subgroup of  $G'$  and hence equal to  $A$ . ■

## 2.4 Simple Group

“Simple groups are the basic **building blocks** of groups”. Given a group  $H \trianglelefteq G$ , if we understand  $H$ ,  $G/H$ , we can then hope to get some information of  $G$ , and thus to decompose such idea, we obtain the concept of **simple group**.

**Definition 2.4.1 (Simple Group).** A group  $G$ , (**not necessarily finite**), is simple, if the following:

1.  $G \neq \{e\}$
2. whenever  $H \trianglelefteq G$ , we have either  $H = \{e\}$  or  $H = G$ .

In particular, there is **no** interesting normal subgroup of  $G$ .

We shall give an overview of classifying some good types of groups, but now we already seen that cyclic group and abelian groups give us good enough property. We may ask what will happen to an abelian group if it is also simple.

**Proposition 2.4.1.** If  $G$  is abelian, then  $G$  is simple **if and only if**  $G \cong \mathbb{Z}/p\mathbb{Z}$ .

**Proof.** May assume that  $G \neq \{e\}$ , since  $G$  is abelian, it means that every subgroup of  $G$  is normal subgroup. Hence  $G$  is simple if and only if  $G$  has no non-trivial subgroups.

Equivalently:  $\forall x \in G, x \neq e$ , we have  $G = \langle x \rangle$ . This is ok if  $G \cong \mathbb{Z}/p\mathbb{Z}$ .

Conversely: suppose  $G$  satisfying  $G = \langle x \rangle$ , in particular, see that  $G$  is cyclic, so  $G \cong \mathbb{Z}$  or  $G \cong \mathbb{Z}/n\mathbb{Z}$ ,  $n \geq 2$ . Clearly  $\mathbb{Z}$  does not satisfy generated by any element of it: e.g.  $\langle 2 \rangle \neq \mathbb{Z}$ . So  $G \cong \mathbb{Z}/n\mathbb{Z}$ ,  $n \geq 2$ . If  $p$  prime, and  $p \mid n$ , this implies:

$$\left. \begin{array}{l} \left| \frac{n}{p} \right| = p \\ \langle \frac{n}{p} \rangle = G \text{ by assumption} \end{array} \right\} \Rightarrow G \cong \mathbb{Z}/p\mathbb{Z}$$

■

We shall give some examples of it.

**Example 2.4.1.**  $n \geq 3$ , Dihedral groups, see that:

$$D_{2n} \neq \text{simple groups}$$

since

$$\langle \sigma \rangle \trianglelefteq D_{2n}$$

because it has index 2.

**Example 2.4.2.**  $n \geq 3$ ,  $S_n$  is not simple as  $A_n \trianglelefteq S_n$ .

**Example 2.4.3.** If  $p, q$  prime integers,  $|G| = p^2q \Rightarrow G$  is not simple.

**Proof. Sketch of Proof:** Consider whether  $Z(G) = G$ , if  $Z(G) = G$ , then by **Proposition 2.4.1**, it is not simple. If they are not equal, use Sylow's theorem to deduce that it cannot happen that  $n_q \neq 1 \neq n_p$ . ■

### 2.4.1 Special simple groups

We now give an important theorem, which gives us a big type of simple finite groups.

**Theorem 2.4.1.** For every  $n \geq 5$ ,  $A_n$  is simple.

**Proof.** We will proceed by induction on  $n \geq 5$ .

- **Base case ( $n = 5$ ):**  $|A_5| = 2^2 \cdot 3 \cdot 5 = 60$ . Suppose  $H$  be the non-trivial **normal** subgroup of  $A_5$ , we want to deduce a contradiction.

- If  $5 \nmid |H|$ : notice that  $n_5(H) = n_5(G)$  by similar reasoning as we've done in **Proposition 2.3.3**: (Sylow's Second theorem +  $H \trianglelefteq G$ ).

See that:

$$\left. \begin{array}{l} n_5(G) \equiv 1 \pmod{5} \\ n_5(G) \mid \frac{|G|}{5} = 12 \end{array} \right\} \Rightarrow n_5(G) = 1 \text{ or } 6$$

- \* If  $n_5(G) = n_5(H) = 6$ : then  $H$  contains more than 24 elements of order 5. And by  $|H| \mid |G|$ ,  $|H| = 30$ . By **Proposition 2.3.3**, see that  $n_5(H) = 1$ , leads to contradiction  $\nexists$ .
- \* If  $n_5(G) = n_5(H) = 1$ : then  $G$  only has 4 elements of order 5 in  $G$ , but  $G = A_5$ , these are precisely the 5-cycles (They have sign to be 1). But we have  $4 \times 3 \times 2 \times 1$  such cycles, in  $A_5 \nexists$ .

So we reach the conclusion that we cannot have  $5 \nmid |H|$ .

- Since  $|H| \mid 60$  and  $\gcd(5, |H|) = 1 \Rightarrow |H| \mid 12 \Rightarrow |H| \in \{2, 3, 4, 6, 12\}$ . The main idea is to try to **contract to the acse of 2,3,4**.

- \* If  $|H| = 6$ , then:

$$\left. \begin{array}{l} n_3(H) \equiv 1 \pmod{3} \\ n_3(H) \mid \frac{|H|}{3} = 2 \end{array} \right\} \Rightarrow n_3(H) = 1$$

Again by sylow's second theorem +  $H \trianglelefteq G \Rightarrow n_3(H) = n_3(G) = 1 \Rightarrow G$  has a 3-Sylow subgroup that is normal.

- \* If  $|H| = 12$ , then:

$$\left. \begin{array}{l} n_3(H) = n_3(G) \equiv 1 \pmod{3} \\ n_3(H) = n_3(G) \mid \frac{12}{3} = 4 \end{array} \right\} \Rightarrow n_3(H) = n_3(G) = 1 \text{ or } 4$$

- If  $n_3(G) = 1$ , then replacing  $H$  by a 3-Sylow subgroup get a normal subgroup of  $G$  or order 3.
- If  $n_3(G) = n_3(H) = 4$ , we have  $4 \times 2 = 8$  elements of order 3 in  $H$ , thus we get 3 elements of order different from 1 and 3 in  $H$ .  
Since  $|H| = 12$ , it has a subgroup  $P$  with 4 elements (a 2-Sylow subgroup), thus  $P$  is the unique 2-Sylow subgroup of  $H$ , and thus be a unique 2-Sylow subgroup of  $G$ . So we repalce  $H$  by  $P$  and get a normal subgroup of  $G$  with 4 elements.
- \* Hence assume  $|H| \in \{2, 3, 4\}$ , thus  $|G/H| \in \{30, 20, 15\}$ .

**Claim.**  $G/H$  contains a normal subrgoup with 5 elements, then this has to be of the form  $K/H$  where  $K \trianglelefteq G$ ,  $H \subseteq K \Rightarrow 5 \mid |K|$ , which contradicts to **the initial case!**

Let  $\overline{G} = G/H$ :

- If  $|\overline{G}| = 30$ , then by **Proposition 2.3.3**,  $n_5(\overline{G}) = 1$ , and we get a normal subgroup with 5 elements.
- If  $|\overline{G}| = 20$ , then  $n_5(\overline{G}) \equiv 1 \pmod{5}$  and  $n_5(\overline{G}) \mid \frac{20}{5} = 4 \Rightarrow n_5(\overline{G}) = 1$ , conclusion the same.

· If  $|\overline{G}| = 15$ , then similarly as the case just proved,  $n_5(\overline{G}) = 1$ , conclusion the same.  
So the claim holds and the base case is done.

- **Inductive case:** Suppose that  $n \geq 6$  and we know that  $A_{n-1}$  is simple.

Suppose that  $H$  is non-trivial normal subgroup of  $A_n = G$ . For  $i \in [n]$ , let:

$$G_i = \{\sigma \in G \mid \sigma(i) = i\} \cong A_{n-1} \quad (2.4)$$

Note that if  $\alpha \in S_n$ , then:

$$\alpha G_i \alpha^{-1} = G_{\alpha(i)} \quad (2.5)$$

Since **Formula 2.4** is clear, if  $i = 1 \Rightarrow$  it follows for all  $i$  by taking some  $\alpha$ , s.t.  $\alpha(i) = 1$ .

We now consider  $H \cap G_i$  be normal subgroup of  $G_i$ , with  $G_i$  being simple group by **IH**, there is only two possibilities:

1.  $H \cap G_i = \{e\}$
2.  $H \cap G_i = G_i \Rightarrow G_i \subseteq H$ 
  - If there exists  $i$ , s.t.  $G_i \subseteq H$ , then for every  $j$ , if  $\sigma \in G$  is s.t.  $\sigma(i) = j$ , then by **Formula 2.5** and  $H \trianglelefteq G$ ,  $G_j \subseteq H \Rightarrow \langle G_1, \dots, G_n \rangle \subseteq H$ .  
But  $\langle G_1, \dots, G_n \rangle = A_n$  by the fact that every  $\sigma \in A_n$  are product of even number of transposition of the form  $(ij)(kl)$ , and if  $q \neq i, j, k, l \Rightarrow (ij)(kl) \in G_q$ .
  - If  $G_i \cap H = \{e\}$ ,  $\forall i$ : then if  $\sigma, \sigma' \in H$ , s.t.  $\sigma(i) = \sigma'(i)$  for some  $i$ , then  $\sigma(\sigma')^{-1} \in H \cap G_i \Rightarrow \sigma = \sigma'$ .

Suppose  $\sigma \in H, \sigma \neq e$

- \* In the decomposition of  $\sigma$  into disjoint cycles, there is a cycle of order  $\geq 3$ :

$$\sigma = (a_1 a_2 a_3 \dots) \dots$$

Let  $\alpha \in A_n$ , s.t.  $\alpha(a_1) = a_1, \alpha(a_2) = a_2, \alpha(a_3) \neq a_3$ , there must exist such  $\alpha$ , since  $n \geq 6$ . then:

$$\sigma' = \alpha \sigma \alpha^{-1} = (a_1 a_2 \alpha(a_3) \dots) \dots \in H \text{ by normality of } H$$

See that  $\sigma(a_1) = \sigma'(a_1)$  but  $\sigma(a_2) \neq \sigma'(a_2)$ , leading to contradiction  $\frac{1}{2}$ .

- \* The decomposition of  $\sigma$  into disjoint cycles only has transpositions, by Induction hypothesis, it can't fix any elements as  $G_i \cap H = \{e\}$ ,  $\sigma \in H$ .

$$\sigma = (a_1 a_2)(a_3 a_4)(a_5 a_6) \dots$$

Take  $\alpha \in A_n$ , s.t.  $\alpha = (a_1 a_2)(a_3 a_5)$ , and let  $\sigma' = \alpha \sigma \alpha^{-1}$ , see that  $\sigma'(a_1) = a_2 = \sigma(a_1)$ ,  $\sigma'(a_3) = a_6 \neq \sigma(a_3) \Rightarrow \sigma \neq \sigma'$ , leading to contradiction  $\frac{1}{2}$ .

■

**Remark 2.4.1.**  $A_4$  is **not** simple group!

$$H = \{e, (12)(34), (13)(24), (14)(23)\} \trianglelefteq A_4$$

Note that  $H$  is isomorphic to the **Klein group**. See that  $H$  is a normal subgroup since **conjugation preserves the cycle type decomposition** and the element of  $H$  that are  $\neq e$  are **precisely the one** that decomposition as product of 2 disjoint cycles.

**Remark 2.4.2.** Elementary to check: if  $|G| < 60$ ,  $G$  is simple  $\Rightarrow G \cong \mathbb{Z} / p\mathbb{Z}$ ,  $p$  prime.

## 2.4.2 Classification of Finite Simple Groups

There exists an **explicit classification** of finite simple groups!

1.  $\mathbb{Z} / p\mathbb{Z}$ ,  $p$  primes.
2.  $A_n$ ,  $n \geq 5$
3. 12 series of “finite groups of Lie types”.

For example:  $\text{PSL}_n(\mathbb{Z} / p\mathbb{Z})$  or more generally  $\text{PSL}_n(\mathbb{F}_q)$ .

$$\begin{aligned}\text{SL}_n(\mathbb{Z} / p\mathbb{Z}) &= \{A \in M_n(\mathbb{Z} / p\mathbb{Z}) \mid \det A = 1\} \\ \text{PSL}_n(\mathbb{Z} / p\mathbb{Z}) &= \text{SL}_n(\mathbb{Z} / p\mathbb{Z}) / \{A = \lambda I_n \mid \lambda \in \mathbb{Z} / p\mathbb{Z}, \lambda^n = 1\}\end{aligned}$$

4. 26 “sporaolic examples”.

The largest one: the Monster

$$\begin{aligned}\text{has order } &2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \\ &\cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \\ &\sim 8 \cdot 10^{53}\end{aligned}$$

This can be embedded as a subgroup of  $GL_n(\mathbb{C})$ ,  $n = 196,883$

## 2.5 Semidirect Product

**General Motivation:** Suppose  $G$  be a group,  $N \trianglelefteq G$ , s.t. we understand  $N$  and  $G / N$ , what can we say about  $G$ ? In general, not much. However, in particular cases when we have  $G \xrightarrow{\pi} G / N$  and there is a section of  $\pi$  to be a group homomorphism:  $s : G / N \rightarrow G$ , s.t.  $\pi \circ s = Id$ , then we can completely describe  $G$  if we know one more piece of data.

The external semidirect product is built by two separate independent group, while the internal semidirect product is to decompose an existing group at hand.

### 2.5.1 External Semidirect Product

**Definition 2.5.1 (External Semidirect Product).** Suppose  $N, H$  be two groups, and we have a group homomorphism

$$\varphi : H \rightarrow \text{Aut}(N)$$

We define an operation  $\star$  on  $\mathbf{N} \times \mathbf{H}$  by:

$$(n_1, h_1) \star (n_2, h_2) = (n_1 \varphi_{h_1}(n_2), h_1 h_2)$$

Such defines a group, denoted as  $N \rtimes H$ , which is called the **external semidirect product** of  $N$  and  $H$  with respect to  $\varphi$ .

Left for the reader to check that it is indeed a group, checking associativity, existence of identity and existence of inverses are not hard.

**Note.** • If  $\varphi : H \rightarrow \text{Aut}(N)$ ,  $h \mapsto Id$ , we recover the usual group structure on  $N \times H$ , namely the direct product.

- We have a map  $H \rightarrow N \rtimes_{\varphi} H$ ,  $h \mapsto (e_N, h)$ , this is a group homomorphism, it is clearly injective, with image to be:

$$H' = \{(n, h) \in N \rtimes_{\varphi} H \mid n = e_N\}$$

Similarly, we have  $N \rightarrow N \rtimes_{\varphi} H$ ,  $n \mapsto (n, e_H)$ , which is also a group homomorphism, injective,

with image to be:

$$N' = \{(n, h) \in N \rtimes_{\varphi} H \mid h = e_H\}$$

In fact this is a normal subgroup of  $N \rtimes_{\varphi} H$ , can be verified by proving:

$$(n', h')^{-1} \star (n, e) \star (n', h') \in N'$$

- We can also prove the following claim by checking:

$$(e_N, h) \star (n, e_H) \star (e_N, h)^{-1} = (\varphi_h(n), e_H)$$

**Claim.** via the isomorphism  $N \cong N'$ ,  $H \cong H'$ , the morphism:

$$\begin{aligned} H' &\rightarrow \text{Aut}(N') \\ g &\mapsto (n \mapsto gng^{-1}) \end{aligned}$$

is given by  $\varphi$ .

- We have a group homomorphism:

$$\begin{aligned} N \rtimes H &\rightarrow H \\ (n, h) &\mapsto h \end{aligned}$$

This is surjective group homomorphism, with the kernel to be  $N' \cong N \Rightarrow N \rtimes_{\varphi} H / N' \cong H$ .

- We have a section given by:

$$H \rightarrow N \rtimes_{\varphi} H$$

**Definition 2.5.2 (Section of a Group Extension).** Let explicit the short exact sequence:

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow 1$$

A homomorphism  $s : H \rightarrow G$  is called a **section** (or a splitting) of the extension if it satisfies:

$$\pi \circ s = \text{id}_H$$

where  $\text{id}_H$  represents the identity map on  $H$ .

If such a section exists, we say the extension *splits*, and  $G$  is isomorphic to the semidirect product  $N \rtimes H$ .

## 2.5.2 Internal Semidirect Product

**Definition 2.5.3 (Internal Semidirect Product).** Suppose  $G$  be any group, and  $H, N$  be subgroups of  $G$  with  $N \trianglelefteq G$  normal. And define:

$$\begin{aligned} \varphi : H &\rightarrow \text{Aut}(N) \\ h &\mapsto \varphi_h \\ \varphi_h(n) &= hnh^{-1} \quad \text{ok since } N \trianglelefteq G \end{aligned}$$

We get a map  $N \rtimes_{\varphi} H \xrightarrow{\alpha} G$ ,  $(n, h) \mapsto nh$ . And we claim it is a **group homomorphism**, with the operation defined as:

$$(n_1, h_1) \star (n_2, h_2) = (n_1 \varphi_{h_1}(n_2), h_1 h_2)$$

One should easily verify that.

We define  $G$  to be the **internal semidirect product** of its subgroups  $N, H$  if  $\alpha$  is an isomorphism.

We then give a more straightforward equivalence of this definition.

---

**Proposition 2.5.1.** Let  $H, N, G$  be as in the definition,  $G$  is the internal semidirect product of  $N, H$  if and only if:

1.  $N \trianglelefteq G$
2.  $G = N \cdot H$
3.  $H \cap N = \{e\}$

**Proof.**  $\alpha$  is isomorphism if and only if it is injective and surjective. It is surjective if and only if  $G = N \cdot H$  (This is subgroup since  $N \trianglelefteq G$ ). It is injective if and only if:

$$\begin{aligned}\ker(\alpha) &= \{(n, h) \mid n \in N, h \in H, nh = e\} \\ &= \{(n, n^{-1}) \mid n \in H \cap N\} \\ &= \{e\}\end{aligned}$$

■

# Chapter 3

## Classification of Finite Groups

Finite group theory is already a well-studied field, people are already been able to classify different types of groups in finite order with good or relatively good property, and decompose the finite group into the building blocks, namely the simple groups. We've studied simple groups before, so in this chapter we shall introduce how we will decompose the finite groups into simple factors through **Jordan-Hölder theorem**. We will then introduce some special groups with relatively good properties, which are built by abelian groups. A small glimpse of it will be:

$$\text{cyclic} \subseteq \text{abelian} \subseteq \text{nilpotent} \subseteq \text{solvable}$$

and we will also try to dig some more property of  $p$ -group.

### 3.1 Composition Series and Jordan-Hölder Theorem

**Definition 3.1.1 (Composition Series).** Given a group  $G$ , a composition series of  $G$  is given by a sequence of subgroups:

$$\{e\} \leq N_r \leq N_{r-1} \leq \dots \leq N_0 = G$$

such that  $\forall i \in [r-1]$ :

1.  $N_{i+1} \trianglelefteq N_i$
2.  $N_i / N_{i+1}$  are simple groups

We call  $r$  to be the length of the series, and the  $N_i / N_{i+1}$  to be the simple factors in the series

**Note.** Some trivial case worth noting:

- $r = 0 \Leftrightarrow G = \{e\}$
- $r = 1 \Leftrightarrow G$  is simple

**Definition 3.1.2.** If  $G$  has a composition series, then  $G$  has finite length.

**Proposition 3.1.1.** Every finite group has finite length

**Proof. Sketch of proof:** proof by induction on  $|G|$ , and **glue the composition series** of  $N$  and  $G / N$  if there exists  $N \trianglelefteq G$ . ■

**Theorem 3.1.1 (Jordan-Hölder Theorem).** Given two composition series for  $G$ , denoted as:

$$\begin{aligned} \{e\} = N_r &\leq \dots \leq N_1 \leq N_0 = G \\ \{e\} = N'_s &\leq \dots \leq N'_1 \leq N'_0 = G \end{aligned}$$



Then:

- $r = s$
- and

$$\begin{aligned} & N_{r-1}/N_r, \dots, N_0/N_1 \\ & N'_{s-1}/N'_s, \dots, N'_0/N'_1 \end{aligned}$$

are pairwise isomorphism, after possibly reordering.

**Note.** This is basically stating that the **simple factors** are the **building blocks**.

We need a lemma to proof the original theorem.

**Lemma 3.1.1.** If  $G$  has a composition series,  $N \trianglelefteq G$ , then  $N$  has a composition series.

**Proof. (Proof of Lemma 3.1.1)** Suppose we start with a composition series of  $G$ , and consider for  $N$ :

$$\{e\} = N \cap G_r \leq \dots \leq N \cap G_1 \leq N \cap G_0 = N$$

Now what can we say about the following injective map?

$$N \cap G_i / N \cap G_{i+1} \hookrightarrow G_i / G_{i+1}$$

By normality of  $N$  in  $G$ , see:

$$N \cap G_i / N \cap G_{i+1} \subseteq G_i / G_{i+1}$$

With the image of the map also normal in  $G_i / G_{i+1}$ , and since the simple factor is simple:

- either  $N \cap G_i / N \cap G_{i+1} = \{e\}$
- or  $N \cap G_i / N \cap G_{i+1} = G_i / G_{i+1}$

Then **After removing repeated factors, i.e. Those  $\{e\}$** , we get a composition series of  $N$ . ■

**Proof. (Proof of Jordan-Hölder Theorem)** We shall proceed the proof by induction on the shortest length of a composition series of  $G$ , in particular, our aim is to see that if  $G$  has a composition series with length  $r$ , then all composition series of  $G$  should have length  $r$  and the pairwise isomorphism statement holds.

- **Base case:** If this is 0:  $G = \{e\}$ , which is trivial, if this is 1, then  $G$  is simple, every composition series should have length to be 1, also trivial.
- **Inductive case:** Suppose  $r \leq s$ , and we assume by Induction we know the theorem for those groups that admit a composition series with length  $r - 1$ , then there are two cases:

–  $N_1 = N'_1$ : we then get a composition series for those two subgroups, and we can apply induction hypothesis see that  $r - 1 = s - 1$  and the pairwise isomorphism statement also holds and call a day.

–  $N_1 \neq N'_1$ : We then look at what we can say about  $N_1 N'_1$ ?

First note that  $N_1 \not\subseteq N'_1$ , otherwise:

$$\{e\} \neq N'_1 / N_1 \trianglelefteq G / N_1 \text{ (Simple)} \quad \nexists$$

Similarly,  $N'_1 \not\subseteq N_1$ .

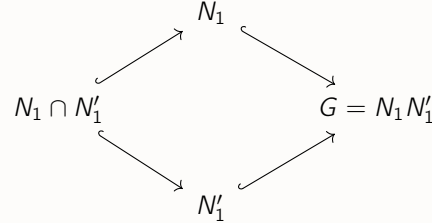
Then  $N_1 \subseteq N_1 N'_1 \trianglelefteq G$ , since both  $N_1, N'_1$  are normal subgroups.

Since  $G/N$  is simple, then either  $N = N_1 N'_1 \Rightarrow N'_1 \subseteq N_1 \not\subseteq$  or  $N_1 N'_1 = G$ , clearly the latter case.

Hence  $G = N_1 N'_1$ , then by the second isomorphism theorem:

$$G/N_1 = N_1 N'_1/N_1 \cong N'_1/N_1 \cap N'_1 \Rightarrow G/N'_1 \cong N_1/N_1 \cap N'_1$$

We then want to use a composition series of  $N_1 \cap N'_1$  to apply induction.



and by the second isomorphism theorem:

$$\begin{aligned} N_1/N_1 \cap N'_1 &\cong G/N'_1 \\ N'_1/N_1 \cap N'_1 &\cong G/N_1 \end{aligned} \tag{3.1}$$

And in particular, groups on the **LHS** are **simple**.

Since  $N_1 \cap N'_1 \subseteq G$  and  $G$  has a composition series, then by **Lemma 3.1.1**,  $N_1 \cap N'_1$  also has a composition series, and in particular  $N_1 \cap N'_1 \trianglelefteq G$ .

We now choose one such composition series:

$$\{e\} = N''_t \leq \dots \leq N''_1 \leq N''_0 = N_1 \cap N'_1$$

Then we obtain four kinds of composition series for  $G$ :

1.  $\{e\} = N_r \leq \dots \leq N_2 \leq N_1 \leq G$
2.  $\{e\} = N''_t \leq \dots \leq N''_1 \leq N_1 \cap N'_1 \leq N_1 \leq G$
3.  $\{e\} = N''_t \leq \dots \leq N''_1 \leq N_1 \cap N'_1 \leq N'_1 \leq G$
4.  $\{e\} = N'_s \leq \dots \leq N'_2 \leq N'_1 \leq G$

Since  $N_1$  has a composition series of length  $r-1$ , by **IH**, the **first and second** series satisfy the condition of the theorem.

By **Formula 3.1**, the **second and third** series satisfy the condition in the theorem.

By  $N'_1$  has a composition series of length  $r-1$ , by **IH**, the **third and fourth** series satisfy the condition in the theorem.

Thus the **first and fourth** series satisfy the condition of the theorem. ■

**Remark 3.1.1.** If  $G$  being finite, and:

$$\{e\} = G_r \leq \dots \leq G_1 \leq G = G_0 \text{ is a composition series.}$$

Then induction on  $i$  + Lagrange theorem gives us:

$$\begin{aligned} |G_i| &= \prod_{j=i}^{r-1} |G_j / G_{j+1}| \\ \text{by } |G_i| &= |G_{i+1}| \cdot |G_i / G_{i+1}| \\ \Rightarrow |G| &= \prod_{i=0}^{r-1} |G_i / G_{i+1}| \end{aligned}$$

**Remark 3.1.2.** If  $G$  is abelian and finite, then since any of its subgroup will be normal, then given a composition series denoted as above, we have:

$$|G_i / G_{i+1}| = \text{prime integer}$$

In fact we will see later this is an **equivalent** definition for a group to be **nilpotent**.

**Proof. Sketch of Proof:** This directly yields by doing prime factorization on  $|G|$  and matching the size of the simple factor on the prime number and combined with **Proposition 2.3.2**. ■

**Example 3.1.1.** Both  $\mathbb{Z}/4\mathbb{Z}$  and  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  have simple factors  $\mathbb{Z}/2\mathbb{Z}$  twice, in particular this tells us simple factors are not unique properties to a group, it is only related to  $|G|$ , **just like the prime numbers to a particular integers**.

## 3.2 Solvable and Nilpotent Groups

As we stated before, those two kinds of groups are built out of abelian groups, and both of them are classes of groups that characterized by such fact. Without making confusion we state the definition of characterized group here.

**Definition 3.2.1.** A group  $H \leq G$  is called characterized if  $\forall \varphi \in \text{Aut}(G), \varphi(H) = H$ . Denoted as  $H\text{char}G$ .

### 3.2.1 Solvable Group

**Definition 3.2.2 (Solvable Group).** A group  $G$  is solvable if there exists a **finite** sequence of subgroups

$$\{e\} = G_r \leq \dots \leq G_1 \leq G_0 = G$$

such that:

- $G_{i+1} \trianglelefteq G_i$  for all  $i \in \llbracket 0, r-1 \rrbracket$ .
- $G_{i+1} / G_i$  is abelian for all  $i \in \llbracket 0, r-1 \rrbracket$ .

**Remark 3.2.1.** This notation is important in both representation theory and Galois theory.

**Proposition 3.2.1.** Let  $G$  be a group,  $N \leq G$ , then:

1. If  $G$  is solvable  $\Rightarrow N$  is solvable.
2. If  $N \trianglelefteq G$ ,  $G$  is solvable  $\Rightarrow G / N$  is solvable.

3. If  $N \trianglelefteq G$ ,  $N, G/N$  is solvable  $\Rightarrow G$  is solvable.

**Proof. (First statement):** Suppose we have a sequence of subgroups as stated in the definition of solvable group. If  $N \leq G$ , we get a corresponding sequences of subgroups in  $N$ :

$$\{e\} = N \cap G_r \leq \dots \leq N \cap G_1 \leq N = N \cap G_0$$

Easy to see that  $N \cap G_{i+1} \trianglelefteq N \cap G_i$ . Moreover, we have an injective group homomorphism:

$$\begin{aligned} N \cap G_i / N \cap G_{i+1} &\rightarrow G_i / G_{i+1} \\ x(N \cap G_{i+1}) &\mapsto xG_{i+1} \end{aligned}$$

In particular, if  $G$  is solvable, then have such sequence with all  $G_i / G_{i+1}$  abelian, and thus have  $N \cap G_i / N \cap G_{i+1}$  abelian for all  $i$ , thus  $N$  is solvable, which yields the first statement.

**Note.** Abelian property is **carried through** by the injective group homomorphism.

**(Second Statement):** Suppose we have a sequence of subgroups as stated in the definition of solvable group for  $G$ . Consider  $\forall i$ , that  $N \subseteq G_i N \leq G$  (It is a subgroup by the normality of  $N$ ).

Since  $G_{i+1} \trianglelefteq G_i \Rightarrow G_{i+1} N \trianglelefteq G_i N$ , we then get a sequence of subgroups of  $G/N$  given by:

$$\{e\} = G_r N / N \subseteq G_{r-1} N / N \subseteq \dots \subseteq G_0 N / N = G / N$$

with:

$$G_{i+1} N / N \trianglelefteq G_i N / N$$

By the third isomorphism theorem:

$$\begin{aligned} G_i N / N / G_{i+1} N / N &\cong G_i N / G_{i+1} N \leftarrow G_i / G_{i+1} \quad \text{gp. hom. by abelian group} \\ &\quad \underbrace{g \cdot G_{i+1} N \leftarrow g \cdot G_{i+1}}_{\text{This is surjective} \Rightarrow G_{i+1} N / G_i N \text{ is abelian.}} \end{aligned}$$

which yields the second statement.

**(Third Statement):** Suppose we have two sequences of subgroups as in the definition of solvable groups:

$$\begin{aligned} N_0 = \{e\} &\subseteq N_1 \subseteq \dots \subseteq N_r = N \\ \{e\} = N_r / N &\subseteq N_{r+1} / N \subseteq \dots \subseteq N_{r+s} / N = G / N \end{aligned}$$

The third isomorphism theorem tells us the sequence of subgroups

$$N_0 \subseteq \dots \subseteq N_r \subseteq N_{r+1} \subseteq \dots \subseteq N_{r+s} = G$$

satisfy the condition to make  $G$  solvable. ■

We shall give some examples to commonly seen solvable groups.

**Example 3.2.1.** Every abelian group  $G$  is solvable.

**Example 3.2.2.**  $D_{2n}$  is solvable since  $\exists H \leq D_{2n}$  with  $H$  abelian, take  $H = \langle \sigma \rangle$ , see that  $H$  is normal and  $D_{2n} / H \cong \mathbb{Z} / 2\mathbb{Z}$ .

**Example 3.2.3.** If  $G$  is simple, but non-abelian, then  $G$  is not solvable. For example  $A_n$ ,  $n \geq 5$  is not solvable.

**Example 3.2.4.** By **Proposition 3.2.1**,  $S_n$ ,  $n \geq 5$  is not solvable.

**Example 3.2.5.**  $S_4$  is solvable, take  $K$  to be the Klein group, and we have the sequences as:

$$\{e\} \leq K \leq A_4 \leq S_4$$

where:

$$\begin{aligned} K &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ A_4/K &\cong \mathbb{Z}/3\mathbb{Z} \\ S_4/A_4 &\cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

**Definition 3.2.3 (Commutator).** Given  $G$  be a group, define:

$$[G, G] = \langle [g, h] \mid g, h \in G, [g, h] = ghg^{-1}h^{-1} \rangle$$

to be the commutator of  $G$ . See that  $[G, G] \trianglelefteq G$  and

$$G^{ab} := G/[G, G] \text{ is abelian.}$$

We can then construct recursively a sequence of subgroups of  $G$ :

- $G^{(0)} = G$ ,  $G^{(1)} = [G, G]$ .
- If  $G^{(n)}$  is constructed, then  $G^{(n+1)} = [G^{(n)}, G^{(n)}] \trianglelefteq G^{(n)}$

Thus defined a series for  $G$ .

We now give an equivalent definition for solvable groups.

**Proposition 3.2.2.**  $G$  is solvable if and only if there exists  $n$ , s.t.  $G^{(n)} = \{e\}$ .

**Proof.** The “( $\Leftarrow$ )” part is clear by definition, so we only care about “( $\Rightarrow$ )” part.

Suppose we have:

$$\{e\} = N_r \leq \dots \leq N_1 \leq N_0 = G$$

s.t.

- $N_{i+1} \trianglelefteq N_i$ ,  $\forall i \in \llbracket 0, r-1 \rrbracket$ .
- $N_i / N_{i+1}$  abelian.

It is enough to show that for  $i \in \llbracket 0, r-1 \rrbracket$ , we have  $G^{(i)} \subseteq N_i$ , then when  $i = r \Rightarrow G^{(r)} = \{e\}$ .

We argue by induction on  $i \geq 0$ :

- **Base case:** ok when  $i = 0$ .
- **Inductive case:** Suppose  $i \leq r-1$  and we know the assertion for  $i$ :

$$G^{(i+1)} = \underbrace{[G^{(i)}, G^{(i)}]}_{\text{By induction, } G^{(i)} \subseteq N_i} \subseteq [N_i, N_i] \subseteq N_{i+1}$$

The last inclusion needs further explanation:

$$\begin{aligned} N_i / N_{i+1} &\text{ is abelian} \\ N_i / [N_i, N_i] &\text{ is the largest abelian group for the quotient} \end{aligned}$$

### 3.2.2 Nilpotent Group

**Definition 3.2.4 (Nilpotent Group).** Let  $G$  be any group, we define a sequence of subgroups  $\{Z_i(G)\}_{i \geq 0}$  as follows:

- $Z_0(G) = \{e\}$
- $Z_1(G) = Z(G)$
- If  $Z_i(G) \leq G$  is constructed, consider:

$$Z\left(G / Z_i(G)\right) = Z_{i+1}(G) / Z_i(G)$$

for some  $Z_{i+1}(G) \trianglelefteq G$  and containing  $Z_i(G)$ .

The group  $G$  is nilpotent if there exists  $n$ , s.t.  $Z_n(G) = G$ .

We then give some examples for nilpotent groups.

**Example 3.2.6.** nilpotent group is solvable group, since

$$Z\left(G / Z_i(G)\right) = Z_{i+1}(G) / Z_i(G) \text{ is abelian.}$$

**Example 3.2.7.** abelian groups are nilpotent:  $Z(G) = G = Z_1(G)$ .

**Example 3.2.8.**  $S_3 \cong D_6$  is solvable but not nilpotent, since  $Z(S_3) = \{e\}$  but  $S_3 \neq \{e\}$ .  
In particular,  $S_3, S_4$  are solvable, but not nilpotent.

**Proposition 3.2.3.** A group  $G$  is nilpotent if and only if there exists a sequence of subgroups:

$$\{e\} = G_0 \subseteq G_1 \subseteq \dots \subseteq G_r = G$$

s.t.

1.  $G_i \trianglelefteq G, \forall i \in \llbracket 0, r-1 \rrbracket$ .
2.  $G_i / G_{i+1} \subseteq Z(G / G_{i-1}), \forall i \in \llbracket 1, r \rrbracket$ .

**Proof. Sketch of Proof:** One direction is trivial. The other direction can be treated by prove  $G_i \subseteq Z_i(G), \forall i \in \llbracket 0, r \rrbracket$  by induction on  $i$ . ■

**Proposition 3.2.4.** Let  $N \leq G$ , then:

1.  $G$  nilpotent  $\Rightarrow N$  is nilpotent.
2.  $N \trianglelefteq G, G$  is nilpotent  $\Rightarrow G / N$  is nilpotent.
3. If  $N \leq \mathbf{Z}(G)$  and  $G / N$  is nilpotent  $\Rightarrow G$  is nilpotent.

**Note.** If we just know  $N \trianglelefteq G$  nilpotent and  $G / N$  nilpotent  $\nRightarrow G$  nilpotent.

**Proof.** Almost the same as what we have done for solvable group in **Propositpion 3.2.1**. Using the characterization via sequences of subgroups by taking:

$$\begin{aligned} & \{G_i \cap N\}_{i \in I} \text{ for } N \\ & \left\{ G_i N / N \right\}_{i \in J} \text{ for } G / N \end{aligned}$$

We give more assertion for the **third statement**:

If  $\{e\} = G_0 / N \leq G_1 / N \leq \dots \leq G_r / N = G / N$  as characterization of  $G / N$  being nilpotent, then

$$\{e\} \leq \underbrace{N}_{\text{Since } N \subseteq Z(G)} = G_0 \leq G_1 \leq \dots \leq G$$

satisfies the characterization of  $G$  being nilpotent group. ■

We now look at what  $p$ -group has.

**Proposition 3.2.5.** If  $G$  is a  $p$ -group, then  $G$  is nilpotent.

**Proof.** We proceed by induction on  $|G|$ .

- **Base case:** If  $|G| = p \Rightarrow G \cong \mathbb{Z} / p\mathbb{Z} \Rightarrow$  nilpotent.
- **Inductive case:** We've proven  $Z(G) \neq \{e\}$  before in **Proposition 1.2.2**, it follows that:

$$\left| G / Z(G) \right| < |G|$$

and

$$\left| G / Z(G) \right| \mid |G| \Rightarrow G / Z(G) = \{e\} \text{ or a } p\text{-group.}$$

If  $G = Z(G)$  then  $G$  is abelian thus nilpotent. Otherwise we apply induction hypothesis on  $G / Z(G)$  and learn that it is nilpotent. By **Proposition 3.2.4**,  $G$  is nilpotent. ■

We now give several equivalent definition to nilpotent groups to strengthen the intuition. It is actually strict for some groups to be nilpotent or even solvable.

**Theorem 3.2.1.** Let  $G$  be a finite group,  $p_1, p_2, \dots, p_r$  are the primes in the prime factorization of  $|G|$ . Let  $P_i$  be the  $p_i$ -Sylow subgroup of  $G$ , then the following are equivalent:

1.  $G$  is nilpotent.
2. For every  $H \leq G$ , we have  $H \leq N_G(H)$ .
3.  $P_i \trianglelefteq G, \forall i$ .
4.  $G \cong P_1 \times \dots \times P_r$ .

**Lemma 3.2.1 (Frattini's Argument).** If  $G$  is a finite group with normal subgroup  $H$ , and if  $P$  is a Sylow  $p$ -subgroup of  $H$ , then

$$G = N_G(P)H$$

in particular, by applying it to  $N_G(N_G(P))$ , one can show that

$$N_G(N_G(P)) = N_G(P)$$

whenever  $G$  is a fnite group and  $P$  is a Sylow  $p$ -subgroup of  $G$ .

**Proof.** The main idea is to show:  $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$ .

- **(1  $\Rightarrow$  2):** Proceed by induction on  $|G|$ . If  $|G| = 1$ , things are clear. For the inductive step, If  $Z(G) \not\subseteq H \Rightarrow \exists a \in Z(G) \setminus H$ , and so  $a \in N_G(H) \setminus H$ .

If  $Z(G) \subseteq H$ , take  $\bar{G} = G / Z(G) \supsetneq \bar{H} = H / Z(G)$ . Since  $G$  is nilpotent, then  $|Z(G)| > 1 \Rightarrow |\bar{G}| < |G|$ . Note that  $\bar{G}$  is nilpotent by **Proposition 3.2.4**. We now apply induction for  $\bar{H}$  and obtain:  $\bar{H} \subsetneq N_{\bar{G}}(\bar{H}) \Rightarrow \exists aZ(G) \in H / Z(G)$ ,  $a \notin H$ , s.t.  $aZ(G)\bar{H}a^{-1}Z(G) = \bar{H} \Rightarrow aHa^{-1} = H$ , such completes the induction step.

- **(2  $\Rightarrow$  3):** By **Frattini's Argument 3.2.1**,  $N_G(N_G(P_1)) = N_G(P_1)$ ,  $\forall i$ , by condition in 2,  $N_G(P_i) = G \Leftrightarrow P_i \trianglelefteq G$ .
- **(3  $\Rightarrow$  4):** One shall utilize **Proposition 1.2.2** for this. We show by induction on  $n$ ,  $n \in \llbracket 1, r \rrbracket$ , that:

$$P_1 \times \cdots \times P_n \rightarrow P_1 \cdots P_n \text{ is an isomorphism.} \quad (3.2)$$

Note that  $P_1 \cdots P_n \trianglelefteq G$  since  $P_i \trianglelefteq G \forall i$ .

If this is ok, take  $n = r$ , since

$$|P_1 \times \cdots \times P_r| = |G| \Rightarrow P_1 \cdots P_r = G$$

and yields the result. We now try to proceed the inductive step:

We have  $P_1 \cdots P_n \trianglelefteq G$  and  $P_{n+1} \trianglelefteq G$  and  $P_1 \cdots P_n \cap P_{n+1} = \{e\}$ . By

$$\left| P_1 \cdots P_n \cap P_{n+1} \right| \left| \begin{array}{l} |P_1 \cdots P_n| = \prod_{i \leq n} |P_i| \\ |P_{n+1}| \end{array} \right\} \text{relatively prime}$$

By lagrange theorem and our previous proposition, such holds:

$$(P_1 \cdots P_n) \times P_{n+1} \rightarrow P_1 \cdots P_n P_{n+1} \text{ is an isomorphism.} \\ (a, b) \mapsto ab$$

And thus:

$$(P_1 \times \cdots \times P_n) \times P_{n+1} \xrightarrow[\text{By IH}]{\sim} (P_1 \cdots P_n) \times P_{n+1} \rightarrow P_1 \cdots P_n P_{n+1} \text{ is an isomorphism.}$$

The composition is also an isomorphism.

- **(4  $\Rightarrow$  1):** Since we know that each  $P_i$  is nilpotent by being a  $p_i$ -group, it is enough to show the following lemma:

**Lemma 3.2.2 (Structrual theorem).** If  $G, H$  are nilpotent group, then  $G \times H$  is still nilpotent.



**Proof.** Check that:

$$Z(G \times H) = Z(G) \times Z(H)$$

then:

$$G \times H / Z(G \times H) \cong (G / Z(G)) \times (H / Z(H))$$

And apply induction on  $i$ :

$$Z_i(G \times H) = Z_i(G) \times Z_i(H)$$

and if  $i \gg 0$ :

$$\begin{aligned} Z_i(G) &= G \\ Z_i(H) &= H \\ \Rightarrow Z_i(G \times H) &= G \times H \end{aligned}$$

■

■

### 3.3 Free groups

In this section, we will give some introduction on the free groups and more generally we hope to grab some idea of **free objects** in Category Theory.

We shall start with the **forget functor**. Suppose that  $\mathcal{C}$  is a category consisting of “sets with extra structure”, then the forget functor is given by:

$$G_{\text{forget}} : \mathcal{C} \rightarrow \underline{\text{Sets}}$$

Its easy to understand that the forget functor is to strip off the extra structure on the sets and leaving only its basic set structure, for example, we can view a vector space as a bunch of elements forming a set.

The main idea is to see the **free functor** is the **left adjoint** of the **forget functor**.

Given a set  $S$ , the **free object corresponding** to  $S$  is an object  $F(S) \in \text{Ob}(\mathcal{C})$ , such that there is a **functorial bijection** for all  $X \in \text{Ob}(\mathcal{C})$ :

$$\text{Hom}_{\mathcal{C}}(F(S), X) \cong \text{Hom}_{\underline{\text{Sets}}}(S, G_{\text{forget}}(X))$$

Or say the following diagram is commutative:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(F(S), X) & \xrightarrow{\sim} & \text{Hom}_{\underline{\text{Sets}}}(S, G_{\text{forget}}(X)) \\ \downarrow f_* = (u \mapsto f \circ u) & & \downarrow \\ \text{Hom}_{\mathcal{C}}(F(S), X') & \xrightarrow{\sim} & \text{Hom}_{\underline{\text{Sets}}}(S, G_{\text{forget}}(X')) \end{array}$$

where  $f$  is any **morphism** from  $X$  to  $X'$  here. Or  $f \in \text{Hom}_{\mathcal{C}}(X, X')$ .

It maybe helpful to understand it with an example in vector spaces and corresponding bases:

**Example 3.3.1.** Let  $\mathcal{C} = \text{Vect} / \mathbb{R}$  with  $S \in \text{Ob}(\text{Sets}) \rightarrow F(S) =$  vector spaces with a basis indexed by  $S$

In this case, see that every object in  $\text{Vect}$  is isomorphic to  $F(S)$ , for some  $S$ , i.e. **every object in the category of vector spaces is free**.

**Note.** This probably give us some sense that why Linear Algebra is easier to study than group theory. Objects in vector space category are all free, so we just count things in terms of basis to classify all of them. But in groups, not every groups are free, so in general it is more complex to study.

**Intuition.** In short,  $F(S)$  are free objects **generated** by  $S$  under some rule.

### 3.3.1 Construction of Free Groups

We introduce the construction of free groups in this section.

Fix a set  $S$ , let  $S^{-1}$  be a set in **bijection** with  $S$ , written as:

$$\begin{aligned} S &\rightarrow S^{-1} \\ g &\mapsto g^{-1} \end{aligned}$$

**Notation.** For  $x \in S^{-1}$ , s.t.  $x = a^{-1}$ , write  $x^{-1} = a$ .

Consider a **separate** element, denoted as 1, by notation,  $1^{-1} = 1$ . We will work with:

$$T = S \sqcup S^{-1} \sqcup \{1\}$$

**Definition 3.3.1.** A word in  $T$  is a sequence  $(x_1, x_2, \dots)$ , such that:

$$\begin{cases} x_n = 1, \text{ for } n \gg 0 \\ x_n \in T \forall n \geq 1 \end{cases}$$

**Notation.** Use the notation  $1 = (1, 1, \dots)$ , to refer the empty word.

A reduced word  $(x_1, x_2, \dots)$  is a word, s.t.:

1. For every  $n \geq 1$ ,  $x_{n+1} \neq x_n^{-1}$ , unless  $x_n = x_{n+1} = 1$ .
2. If  $x_n = 1$ , then  $x_p = 1$ ,  $\forall p \geq n$ .

**Definition 3.3.2 (Free Group).** Define  $F(S) = \{\text{reduced word on } T\}$ .

Define an operation  $\star$  on  $F(S)$  as follow "concatenation", given:

$$\begin{aligned} x &= (x_1, x_2, \dots, x_m \neq 1, 1, 1, \dots) \\ y &= (y_1, y_2, \dots, y_n \neq 1, 1, 1, \dots) \end{aligned}$$

consider

$$z = (x_1, \dots, \underbrace{x_m, y_1}_{\text{inverse?}}, \dots, y_n, 1, 1, \dots)$$

This is reduced unless  $x_m = y_1^{-1}$ , and if this is the case, simply remove both of them and repeat and after finite times later, we get **a reduced words**, which is  $\underline{x \star y}$ .

**Note.** Note that:

$$\begin{aligned} 1 \star y &= y \text{ if } m = 0 \\ x \star 1 &= x \text{ if } n = 0 \end{aligned}$$

It is straightforward but tedious to prove associativity for it. It has identity 1 and every element have inverses:

$$\begin{aligned} x &= (x_1, \dots, x_m \neq 1, 1, \dots) \\ x^{-1} &= (x_m^{-1}, \dots, x_1^{-1}, 1, \dots) \end{aligned}$$

So  $(F(S), \star)$  is a group and is called the free group on the set  $S$ .

**Note.** If we defined with **word**, the condition existence of inverses will failed, but we can define monoid with it. The reduced word give us equivalence relation so we can guarantee the uniqueness of inverse elements.

**Notation.** We will then follow the followin notation later:

1.  $(x_1, x_2, \dots, x_m, 1, \dots) = x_1 x_2 \cdots x_m$
2.  $x_1 x_1 x_2 x_3^{-1} x_3^{-1} x_3^{-1} = x_1^2 x_2 x_3^{-3}$
3.  $x_i \in S$  or  $S^{-1}$

**Note.** Every  $x \in F(S)$  is (finite) product of element in  $S$  or inverses of such elements, so  $F(S)$  is generated by  $S$ .

**Proposition 3.3.1 (Universal Property).** For every group  $G$ , the map:

$$\{\text{gp. hom. } F(S) \rightarrow G\} \rightarrow \{\text{functions } S \rightarrow G\} \text{ is bijective}$$

$$f \mapsto f|_S$$

Explicitly:  $\forall$  map  $\varphi : S \rightarrow G$ , there exists a **unique** group homomorphism  $f : F(S) \rightarrow G$ , s.t.  $f(s) = \varphi(s)$ ,  $\forall s \in S$ .

**Proof.** Prove existence and uniqueness separately.

- Uniqueness follows by the fact that  $S$  generates  $F(S)$ . Specifically, let  $H$  be a group and  $A \subseteq H$  such that  $H = \langle A \rangle$ . Let  $f, g : H \rightarrow G$  be group homomorphism such that  $f|_A = g|_A \Rightarrow f = g$ .  
**If they map the same on the generator, then they map the same on the whole group.**
- Existence: Define:

$$f : F(S) \rightarrow G$$

$$f(x_1, x_2, \dots, x_m \neq 1, 1, \dots) = u(x_1) \cdots u(x_m)$$

where

$$u(x_i) = \begin{cases} \varphi(x_i), & x_i \in S \\ \varphi(x_i^{-1})^{-1}, & x_i^{-1} \in S \end{cases}$$

The idea of define  $u$  here is to maintain the group structure after the mapping of generator. Check that  $f$  is a group homomorphism and  $f|_S = \varphi$ .

■

**Example 3.3.2.**  $\#S = 1 \Rightarrow F(S) \cong \mathbb{Z}$ . To see it, use the universal property.

$$\{\text{gp. hom. } \mathbb{Z} \rightarrow G\} \rightarrow G \cong \{\text{functions } S \rightarrow G\} \text{ is bijection.}$$

$$f \mapsto f(1)$$

$$f(n) = (f(1))^n$$

**Remark 3.3.1.** Given two sets  $S, S'$  and a map  $\varphi : S \rightarrow S'$ , the **universal property** of the free group  $F(S)$  implies:

$$\exists! \text{ group homomorphism } F(\varphi) : F(S) \rightarrow F(S')$$

such that the following diagram is **commutative** (think: sending basis elements to basis elements):

$$\begin{array}{ccc}
S & \xrightarrow{\varphi} & S' \\
\downarrow \iota & & \downarrow \iota' \\
F(S) & \xrightarrow{F(\varphi)} & F(S')
\end{array}$$

This construction gives a **functor**  $F$  from the category of Sets to the category of Groups:

$$\begin{aligned}
F : \underline{\text{Sets}} &\rightarrow \underline{\text{Gps}} \\
S &\rightsquigarrow F(S) \\
(\varphi : S \rightarrow S') &\rightsquigarrow (F(\varphi) : F(S) \rightarrow F(S'))
\end{aligned}$$

To verify  $F$  is a functor, we need to check two conditions:

1. **Identity:**  $F(\text{Id}_S) = \text{Id}_{F(S)}$  for all sets  $S$ .
2. **Composition:** For  $\psi : S' \rightarrow S''$  and  $\varphi : S \rightarrow S'$ ,

$$F(\psi) \circ F(\varphi) = F(\psi \circ \varphi)$$

Consider the following diagram consisting of two adjacent squares:

$$\begin{array}{ccccc}
S & \xrightarrow{\varphi} & S' & \xrightarrow{\psi} & S'' \\
\downarrow \iota & & \downarrow \iota' & & \downarrow \iota'' \\
F(S) & \xrightarrow{F(\varphi)} & F(S') & \xrightarrow{F(\psi)} & F(S'')
\end{array}$$

- By the definition of  $F(\varphi)$  and  $F(\psi)$ , both the **left square** and the **right square** are commutative. This implies the **outer rectangle** is also commutative.
- By the uniqueness part of the universal property for  $F(S)$ , the composition  $F(\psi) \circ F(\varphi)$  must be the unique homomorphism that satisfies the defining property of  $F(\psi \circ \varphi)$ .
- Therefore,  $F(\psi) \circ F(\varphi) = F(\psi \circ \varphi)$ .

**Remark 3.3.2.** In particular, since  $F$  is a functor, if  $\varphi$  is a bijection, then  $F(\varphi)$  is a group isomorphism, this states that  $F(S)$  **only depends on the cardinality of  $S$  up to isomorphism**.

**Notation.** So we denote  $F_n = F(S)$ , when  $\#S = n$ .

**Theorem 3.3.1 (Schreier).** Every subgroup of a free group is again free.

### 3.3.2 Presentations of Groups by Generators and Relations

The question is raised naturally from the universal property of free groups.

Given a group  $G$  and a subset  $S \subseteq G$ . By the universal property of  $F(S)$ , there exists a unique group homomorphism

$$\begin{aligned}
f : F(S) &\rightarrow G \\
f|_S &= \text{the inclusion map}
\end{aligned}$$

Since  $\text{Im}(f) = \langle S \rangle$  since  $F(S)$  is generated by  $S$ , in particular, if  $S$  generates  $G$ ,  $f$  is surjective.

The question is: **How to describe the kernel of it?** If we can describe its kernel, we can construct isomorphism through isomorphism theorem.

**Definition 3.3.3 (Normal Closure).** Given a group  $G$  and a subset  $A$ , the normal closure of  $A$  is the **smallest normal subgroup** of  $G$  containing  $A$ .

$$\bigcap_{A \subseteq H, H \leq G} H$$

Also one can show that the normal closure of  $A$  is

$$\langle \{gag^{-1} \mid g \in G, a \in A\} \rangle$$

Given  $A$ , if  $N$  is the normal closure of  $A$ , may consider  $G / N$  is the “largest” quotient of  $G$  in which the elements of  $A$  is **identity**.

**Definition 3.3.4 (Presentation).** A presentation of a group  $G$  is a pair  $(S, \mathfrak{R})$  where  $S$  is a set,  $\mathfrak{R} \subseteq F(S)$ , such that if  $N$  is the normal closure of  $\mathfrak{R}$  in  $F(S)$ , then:

$$F(S) / N \cong G$$

**Intuition.**  $S$  as generator and  $\mathfrak{R}$  as the relation, so there is no redundant relation by the normal closure, thus we have an isomorphism.

**Example 3.3.3.** If  $n \geq 3$ , then  $D_{2n}$  has a presentation with

$$\begin{aligned} S &= \{\sigma, \tau\} \\ \mathfrak{R} &= \{\sigma^n, \tau^2, \tau\sigma^{-(n-1)}\tau\sigma\} \end{aligned}$$

usually written as:

$$D_{2n} = \{\sigma, \tau \mid \sigma^n = e, \tau^2 = e, \tau\sigma = \sigma^{n-1}\tau\}$$

**Proof.** Let  $N$  be the normal closure of  $\mathfrak{R} \subseteq F(\{\tau, \sigma\})$ , it is clear:

$$\begin{aligned} \text{if } F(\{\tau, \sigma\}) &\xrightarrow{\varphi} D_{2n} \\ \text{and } \mathfrak{R} &\subseteq \ker(\varphi) \\ \Rightarrow N &\subseteq \ker(\varphi) \end{aligned}$$

We get a group homomorphism which is also surjective:

$$F(\{\tau, \sigma\}) / N \xrightarrow{\bar{\varphi}} D_{2n}$$

By the relation  $\tau\sigma^{-(n-1)}\tau\sigma$ , every element in  $F(\{\tau, \sigma\}) / N$  will be in the form  $\sigma^i\tau^j$ . By the relation  $\sigma^n, \tau^2$ , may assume  $i \in \llbracket 0, n-1 \rrbracket, j \in \llbracket 0, 1 \rrbracket$ , then we have  $\leq 2n$  elements in  $F(\{\tau, \sigma\}) / N$ , then it has to be injection and thus bijection, thus be an isomorphism. ■

**Definition 3.3.5.** A group  $G$  is finitely presented if there exists presentation  $(S, \mathfrak{R})$  of  $G$  with both  $\mathfrak{R}, S$  finite.

**Note.**

1. In general, finite generated  $\not\cong$  finite presented.
2. Know that given  $(S_1, \mathfrak{R}_1), (S_2, \mathfrak{R}_2)$ , figuring out whether:

$$F(S_1) / N_1 \cong F(S_2) / N_2$$

is **undecidable**. (very hard, no algorithm to do)

**Theorem 3.3.2.** Every finite group is finitely presented.

**Proof.** Let  $G$  be finite group, take  $S = G$  and  $\mathfrak{R} = \{g_1 g_2 g_3^{-1} \in F(G) \mid g_3 = g_1 g_2, g_1, g_2, g_3 \in G\}$

**Note.** Note that  $g_1 g_2 g_3^{-1}$  is in  $F(G)$  not  $G$ , so there is really no inverse cancellation. The relation here is intended to **go through all multiplication combination**, and for each pair of  $(g_i, g_j)$ , we have  $g_k = g_i \cdot g_j$  in  $G$ , and thus we construct a word by such three **alphabet** as  $g_i g_j g_k^{-1}$ .

May consider the map:

$$\begin{aligned} \varphi : F(G) &\rightarrow G \\ \varphi|_G &= Id \ (g_i \mapsto g_i) \end{aligned}$$

such is clearly surjective, with the left hand side is the alphabet and the right hand side is the corresponding group element.

By construction, clear that  $\mathfrak{R} \subseteq \ker(\varphi)$ , let  $N$  be normal closure of  $\mathfrak{R}$  in  $F(G)$ , then  $N \subseteq \ker(\varphi)$ , so

$$\exists! \bar{\varphi} : F(G) / N \rightarrow G$$

such that:

$$\begin{array}{ccc} F(S) & \xrightarrow{\pi} & F(S)/N \\ \varphi \downarrow & \swarrow \bar{\varphi} & \\ G & & \end{array}$$

by the universal property of quotient.

Now let  $A = \{\pi(g) \mid g \in G\}$ , see that:

$$\left. \begin{array}{l} \langle A \rangle = F(G) / N \text{ since } \langle g \mid g \in G \rangle = F(G) \\ A \text{ is **closed** under multiplication and inverses} \end{array} \right\} \Rightarrow A = F(G) / N$$

$A$  itself is a group, so it equals to its generator

thus:

$$\left. \begin{array}{l} |F(G) / N| \leq |G| \\ \bar{\varphi} \text{ surjective} \end{array} \right\} \Rightarrow \bar{\varphi} \text{ is isomorphism.}$$

■

# **Part II**

## **Representation Theory**

## Chapter 4

# Group Representations

Group representation theory is intended to understand group by how it maps to matrices. In general its more kinds of flavor of Category Theory, basically study group by study the morphism between group and general linear transformation.

### 4.1 Basic Definitions

General idea: group representation are linearized version of group actions on sets.

Sets  $\rightsquigarrow$  Vector Spaces

Actions by permutations  $\rightsquigarrow$  Actions by linear isomorphisms

**Definition 4.1.1 (Group Representation).** Let  $G$  be a group, a representation of  $G$  is an action of  $G$  on a **complex** vector space  $V$ , such that for all  $g \in G$ , the map:

$$\begin{aligned} V &\rightarrow V \\ v &\mapsto gv \end{aligned}$$

is linear.

**Remark 4.1.1.** Recall that giving an action of  $G$  on  $V$  is equivalent to giving a group homomorphism:  $G \xrightarrow{\rho} S_V, g \mapsto (v \mapsto gv)$ .

Let  $GL(V) = \{f : V \rightarrow V \mid f = \text{linear isomorphism}\} \subseteq S_V$  be a subgroup.

Hence giving a group representation is equivalent to  $\text{Im}(\rho) \subseteq GL(V) \Leftrightarrow$  giving a group homomorphism:  $\rho : G \rightarrow GL(V)$ .

#### Notation.

1. We sometimes denote a representation by  $V$ , with the  $G$ -action being understood or by  $\rho$ .
2. From now on, we will always assume  $\dim_{\mathbb{C}}(V) < \infty$ , this dimension is sometimes called the **degree of the representation**.
3. We shall always assume  $G$  to be **finite** group later.

**Definition 4.1.2.** A representation  $\rho : G \rightarrow GL(V)$  is faithful if  $\ker(\rho) = \{e\}$ .



**Remark 4.1.2.** If  $\dim_{\mathbb{C}}(V) = n$  and  $e_1, \dots, e_n$  is a basis of  $V$ , then:

$$\begin{aligned} \mathrm{GL}_n(V) &\cong \mathrm{GL}_n(\mathbb{C}) \\ (\varphi : V \rightarrow V) &\rightarrow A = (a_{ij})_{i,j \in \llbracket 1, n \rrbracket} \\ \text{s.t. } \varphi(e_j) &= \sum_{i=1}^n a_{ij} e_i \end{aligned}$$

We then give some examples for finite group representation.

**Example 4.1.1 (Trivial Representation).** A trivial representation is the representation  $V$ , s.t.

$$gv = v, \forall g \in G, v \in V \Leftrightarrow G \rightarrow \mathrm{GL}(V), g \mapsto \mathrm{Id}_V$$

**Example 4.1.2 (Permutation Representation).** If  $G$  acts on a finite set  $X$ , take  $V$  vector spaces with basis  $\{e_x \mid x \in X\}$ , the map is determined by its value on the basis, we put:

$$g \cdot e_x = e_{gx}$$

Since we have an action of  $G$  on  $X$ , so the above is a representation, called **permutation representation**.

**Example 4.1.3 (Regular Representation).** Given  $G$  be finite group, consider the action of  $G \times G \rightarrow G$ ,  $(g, h) \mapsto gh$  given by left multiplication, we get a representation:

$$V = \bigoplus_{g \in G} \mathbb{C}e_g (\cong \mathbb{C}^{|G|}), \text{ s.t. } h \cdot \left( \sum_{g \in G} a_g e_g \right) = \sum_{g \in G} a_g e_{hg}$$

where:

$$e_{hg} := h \cdot e_g$$

which gives us **regular representation** of  $G$ .

**Example 4.1.4.** If  $V$  is a 1-dimensional vector space over  $\mathbb{C}$ , then:

$$G \xrightarrow{\rho} \mathrm{GL}(V) \cong \mathbb{C}^\times$$

Since  $\mathbb{C}^\times$  is abelian, then for every  $\rho$  above, there exists a **unique** map satisfying the universal property of **quotients**:

$$\bar{\rho} : G^{ab} = G / [G, G] \rightarrow \mathrm{GL}(V)$$

Hence  $G$  and  $G^{ab}$  have the same 1-dimensional representation, since the **images** of  $\rho$  and  $\bar{\rho}$  are the same.

## 4.2 Category of $G$ -representations

### 4.2.1 Morphisms of $G$ -representations

**Definition 4.2.1 (Morphism of Group Representations).** Fix  $G$  a group, Given two  $G$ -representation  $V, W$  of  $G$ , a morphism of representations is a **linear map**:

$$\begin{aligned} f : V &\rightarrow W \\ \text{s.t. } f(gv) &= gf(v), \forall g \in G, v \in V \end{aligned}$$

To check well-definedness, we need to check the property of category and verify the definition of general

try with  
commu-  
tative dia-  
gram

morphism.

It is easy to see:

1. If  $V_1 \xrightarrow{f} V_2$  and  $V_2 \xrightarrow{g} V_3$  are morphism of  $G$ -representation, then  $g \circ f$  is a morphism of  $G$ -representation.
2.  $\text{Id}_V$  is clearly a morphism of  $G$ -representation.

**Definition 4.2.2 (Isomorphism of  $G$ -repr).** We get a category  $\underline{G\text{-repr}}$  with objects be  $G$ -representations and morph be morphism of  $G$ -representation.

In particular, we get a notion of isomorphism of  $G$ -repr as a morphism  $f : V \rightarrow W$  of  $G$ -representation, s.t.  $\exists g : W \rightarrow V$  being morphism of  $G$ -representation, s.t.  $g \circ f = 1_V$ ,  $f \circ g = 1_W$ .

**Lemma 4.2.1.** A morphism of  $G$ -repr  $f : V \rightarrow W$  is an isomorphism of  $G$ -repr if and only if  $f$  is bijective.

The proof is straightforward.

**Remark 4.2.1.** Given a representation  $V$  of  $G$  if we choose a basis  $v_1, \dots, v_n$  of  $V$ , get an **isomorphism of vector spaces**

$$\begin{aligned} V &\xrightarrow[\varphi]{\sim} \mathbb{C}^n \\ v_i &\leftrightarrow e_i \quad i \in \llbracket 1, n \rrbracket \end{aligned}$$

Can use  $\varphi$  to define a structure of  $G$ -repr on  $\mathbb{C}^n$  such that we get an **isomorphism of  $G$ -repr**: if  $u \in \mathbb{C}^n$ ,  $gu = \varphi(g\varphi^{-1}(u))$ . Specifically, it is defined as:

$$\begin{aligned} \psi : (G \rightarrow \text{GL}_n(\mathbb{C})) &\xrightarrow{\sim} (G \rightarrow \text{GL}(V)) \\ \psi(g)(u) &= \varphi(g \underbrace{\varphi^{-1}(u)}_{\text{this vector in } V}) \end{aligned} \quad (4.1)$$

Conclusion, up to isomorphism, may always assume  $V = \mathbb{C}^n$  for some  $n$ . To give a structure of  $G$ -repr on  $\mathbb{C}^n$ :

$$G \xrightarrow{\rho} \text{GL}_n(\mathbb{C})$$

**Question 4.2.1.** Given two  $G$ -repr  $\rho_1, \rho_2 : G \rightarrow \text{GL}_n(\mathbb{C})$ , when are they **isomorphism**?

By definition, this means: there exists linear isomorphism  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ , s.t. the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{\varphi} & \mathbb{C}^n \\ \rho_1(g) \downarrow & & \downarrow \rho_2(g) \\ \mathbb{C}^n & \xrightarrow{\varphi} & \mathbb{C}^n \end{array}$$

which is also means, by what we have done in **Formula 4.1**:

$$\begin{aligned} \psi : (G \rightarrow \underbrace{\text{GL}(\underbrace{\mathbb{C}^n}_{=V_1})}_{=\rho_1}) &\xrightarrow{\sim} (G \rightarrow \underbrace{\text{GL}(\underbrace{\mathbb{C}^n}_{=V_2})}_{=\rho_2}) \\ \psi(g)(u) &= \varphi(g \underbrace{\varphi^{-1}(u)}_{\text{this vector in } V_1}) \end{aligned}$$

The diagram is **commutative**, meaning:

$$\varphi \circ \rho_1(g) = \rho_2(g) \circ \varphi$$

Hence  $\rho_1 \cong \rho_2$  if and only if there exists  $A \in GL_n(\mathbb{C})$ , s.t.

$$\rho_1(g) = A^{-1} \rho_2(g) A \quad \forall g \in G$$

**Example 4.2.1.** Suppose that  $n = 1$ ,  $\rho_1, \rho_2 : G \rightarrow \mathbb{C}^\times$ , then  $\rho_1 \cong \rho_2$  iff  $\rho_1 = \rho_2$ .  
It is straightforward since  $\mathbb{C}^\times$  is an abelian group on multiplication.

**Example 4.2.2.**  $G \cong \mathbb{Z} / n\mathbb{Z}$ . The isomorphism classes of 1-dim repr of  $G$ : They are in bijection with group isomorphism

$$\mathbb{Z} / n\mathbb{Z} \longrightarrow \mathbb{C}^\times$$

This is easy to see since it is 1-dimensional representation.

We look at the generator of these two groups and realize the set of this group homomorphism is in bijection with

$$\{\lambda \in \mathbb{C} \mid \lambda^n = 1\}$$

Since the group homomorphism is determined by:

$$i + n\mathbb{Z} \longrightarrow \lambda^i$$

We now look at the subobject of a representation of a given group.

**Definition 4.2.3 (Subrepresentation).** If  $V$  is a  $G$ -repr, a subrepresentation of  $V$  is a linear subspace  $W \subseteq V$ , s.t.  $gv \in W$ ,  $\forall v \in W$ .

In this case,  $W$  has a structure of  $G$ -repr, s.t.

$$\begin{aligned} \mathbf{i} : W &\rightarrow V \\ i(x) &= x \end{aligned}$$

is a **morphism** of  $G$ -repr.

**Proposition 4.2.1.** If  $f : V \rightarrow W$  is a morphism of  $G$ -repr, then  $\ker(f) \subseteq V$  and  $\text{Im}(f) \subseteq W$  are subrepresentation.

## 4.2.2 Direct Sum of Representations

**Definition 4.2.4 (External Direct Sum).** Let  $G$  be group and  $V_1, \dots, V_n$  are  $G$ -representations, the (external) direct sum:

$$V_1 \oplus \dots \oplus V_n$$

is the vector spaces  $V_1 \oplus \dots \oplus V_n = V_1 \times \dots \times V_n$  with the  $G$ -action given by:

$$g \cdot (v_1, \dots, v_n) = (gv_1, \dots, gv_n)$$

**Proposition 4.2.2 (Universal Property of Direct Sum).** According to our construction above, we have the universal property:

For  $j \in \llbracket 1, n \rrbracket$ , we have a morphism of  $G$ -repr:

$$\begin{aligned} \alpha_j : V_j &\longrightarrow V_1 \oplus \dots \oplus V_n \\ v &\longmapsto (0, \dots, \overset{j}{v}, \dots, 0) \end{aligned}$$

For every  $G$ -repr  $W$  and morphism of  $G$ -repr  $V_j \xrightarrow{\varphi_j} W$ , there exists a **unique** morphism of  $G$ -repr  $\varphi : V_1 \oplus \dots \oplus V_n \rightarrow W$ , s.t.  $\varphi \circ \alpha_j = \varphi_j$ ,  $\forall j \in \llbracket 1, n \rrbracket$ .

One can check that  $\varphi(v_1, \dots, v_n) = \sum_{j=1}^n \varphi_j(v_j)$ .

**Definition 4.2.5 (Internal Direct Sum).** If  $V$  is a  $G$ -repr,  $V_1, \dots, V_n \subseteq V$  are subrepr, we say that  $V$  is the (internal) direct sum of  $V_1, \dots, V_n$ , written as:

$$V = V_1 \oplus \dots \oplus V_n$$

if and only if the morphism of  $G$ -repr:

$$\begin{aligned} \varphi : V_1 \oplus \dots \oplus V_n &\rightarrow V \\ \varphi(v_1, \dots, v_n) &= v_1 + \dots + v_n \end{aligned}$$

is an **isomorphism**.

**Example 4.2.3.** If  $n = 2$ , then  $V = V_1 \oplus V_2$  iff  $V_1 \cap V_2 = \{0\}$  and  $V_1 + V_2 = V$ .

**Proposition 4.2.3.** If  $\rho : G \rightarrow \text{GL}(V)$  is any 1-dimensional subrepr of  $V$ , then it is of the form  $\mathbb{C} \cdot v$ , where  $v$  is a common eigenvector for each  $\rho(g)$ .

**Example 4.2.4.**  $G = S_3$ ,  $G$  has a standard action on  $\{1, 2, 3\}$  by permutation and consider the corresponding  $G$ -repr on  $\mathbb{C}^3$  with the basis  $(e_1, e_2, e_3)$ .

$$\sigma \cdot e_i = e_{\sigma(i)}$$

**Question 4.2.2.** What is the 1-dim subrepr of such  $G$ -repr?

A 1-dim subrepr is  $\mathbb{C} \cdot (e_1 + e_2 + e_3) = V_1$ .

A 2-dim subrepr is  $V_2 = \{a_1 e_1 + a_2 e_2 + a_3 e_3 \mid a_1 + a_2 + a_3 = 0\}$ , and see that  $V = V_1 \oplus V_2$ .

There is also a 1-dimensional **sign** representation for it. But now we want to explicitly express  $V_2$ : Given  $G \rightarrow \text{GL}(V_2)$ , a basis of  $V_2$  is given by:

$$\begin{cases} u_1 = e_1 - e_2 \\ u_2 = e_2 - e_3 \end{cases}$$

Let  $\tau(12), \sigma = (123)$  as two examples;

$$\rho(\tau) : u_1 \mapsto e_2 - e_1 = -u_1$$

$$u_2 \mapsto e_3 - e_1 = u_1 - u_2 \Rightarrow \rho(\tau) \text{ correspond to } \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\rho(\sigma) : u_1 \mapsto e_2 - e_3 = u_2$$

$$u_2 \mapsto e_3 - e_1 = -u_1 - u_2 \Rightarrow \rho(\sigma) \text{ correspond to } \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

**Theorem 4.2.1 (Complement of Subrepresentation).**  $G$  be a finite group,  $V$  is a  $G$ -repr,  $W \subseteq V$  is a subrepr. Then there exists  $W' \subseteq V$  be a subrepr, such that:

$$V = W \oplus W'$$

**Remark 4.2.2.** The same assertion holds for all finite dimensional **vector spaces**. We use similar project method idea to deal with this problem. Main idea is that in vector spaces, we use linear map to project, but now in  $G$ -representation, we use **morphism** to project. The way to transform the project map is similar as we transform the **isomorphism between vector spaces** to **isomorphism between  $G$ -representations** + we use a **averaging procedure**.

**Proof.** The **issue** in linear algebra proof is that for the linear map  $p$ , we don't yet know  $\ker(p)$  will be a **subrepr** of  $V$ .

Define the projection as the linear map  $p : V \rightarrow W$ , s.t.  $p(x) = x \ \forall x \in W$ . Now define:

$$p' : V \rightarrow W$$

$$p'(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p(gv) \in W \text{ Since } W \text{ is a subrepr.}$$

It is clear that  $p'$  is a linear map, we want to see it is a morphism as well as a projection.

1. **Projection:** If  $v \in W \Rightarrow gv \in W \Rightarrow p(gv) = gv$ , then:

$$p'(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot (gv) = v \text{ indeed a projection.}$$

2. **Morphism:**  $p'$  is a morphism of  $G$ -repr, for  $h \in G$ :

$$p'(hv) = \frac{1}{|G|} \sum_{g \in G} \underbrace{g^{-1} p(ghv)}_{=h(gh)^{-1}p(ghv)} = \frac{1}{|G|} h \sum_{g' \in G} (g')^{-1} p(g'v) = hp'(v)$$

Indeed a morphism.

Thus  $\ker(p')$  will be a subrepr of  $V$ . Then follow similar procedure of proof in linear algebra:

$$V = W \oplus \ker(p')$$

where

$$W' := \ker(p')$$

■

### 4.2.3 Hom between two Representations

We already have a sense that we can obtain the category of  $G$ -repr, and clearly the objects are those  $G$ -repr, but what about the morphisms? namely the "Hom"?

Recall that if  $V, W$  are vector spaces over  $\mathbb{C}$ , then  $\text{Hom}_{\mathbb{C}}(V, W)$  is again a vector space, with:

$$\begin{cases} (f + g)(v) = f(v) + g(v) \\ (\lambda f)(v) = \lambda(f(v)) \end{cases}$$

If  $e_1, \dots, e_n$  is basis of  $V$  and  $f_1, \dots, f_m$  is basis of  $W$ :

$$\text{Hom}_{\mathbb{C}}(V, W) \xrightarrow{\sim} M_{m,n}(\mathbb{C})$$

$$f \mapsto (a_{i,j})_{i \in \llbracket 1, m \rrbracket, j \in \llbracket 1, n \rrbracket} \text{ s.t. } f(e_j) = \sum a_{ij} e_i$$

and in particular:

$$\dim(\text{Hom}_{\mathbb{C}}(V, W)) = m \cdot n$$

Now what if  $V, W$  are  $G$ -reprs? Abuse the notation to let  $\text{Hom}_{\mathbb{C}}(V, W)$  become a  $G$ -repr, with the group action defined as: for  $\varphi \in \text{Hom}_{G\text{-repr}}(V, W)$ ,  $\varphi_0 \in \text{Hom}_{\mathbb{C}}(V, W)$

$$\begin{aligned} (g\varphi)(v) &= g \cdot \varphi_0(g^{-1}v) \\ \underbrace{\left( \underbrace{g\varphi}_{g \text{ acting on } \varphi} \right)}_{\text{G-action in } W}(v) &= g \cdot \underbrace{\varphi_0 \left( \underbrace{g^{-1}v}_{\text{G-action in } V} \right)}_{\text{map vector to } W} \quad \forall v \in V \end{aligned}$$

easy to see that  $g\varphi$  is linear, remains to see it is a group action, check:

$$\begin{aligned}
 (h(g\varphi))(v) &= h \cdot ((g\varphi) \cdot (h^{-1}v)) \\
 &= h \cdot (g \cdot \varphi(g^{-1}h^{-1}v)) \\
 &= (h \cdot g) \cdot \varphi((hg)^{-1}v) \\
 &= ((hg)\varphi)(v) \\
 \Rightarrow h(g\varphi) &= (hg)\varphi
 \end{aligned}$$

**Example 4.2.5.** There are some special cases for the  $G$ -repr  $\text{Hom}_{\mathbb{C}}(V, W)$ :

1. Let  $V = \mathbb{C}^\times$  be the trivial  $G$ -action:  $gv = v$ ,  $\forall g \in G$ , then

$$\begin{aligned}
 \text{Hom}_{\mathbb{C}}(\mathbb{C}, V) &\xrightarrow{\sim} V \\
 \varphi &\mapsto \varphi(1) \text{ is an isomorphism of } G\text{-action.}
 \end{aligned}$$

2.  $W = \mathbb{C}$  with trivial  $G$ -action, we obtain the dual representation of  $V$ :

$$V^* := \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

For example,  $G = \mathbb{Z} / n\mathbb{Z}$  and  $V$  be the 1-dim repr given by:

$$\begin{aligned}
 \mathbb{Z} / n\mathbb{Z} &\rightarrow \text{GL}(V) = \mathbb{C}^\times \\
 1 &\mapsto \lambda \in \mathbb{C}^\times, \lambda^n = 1
 \end{aligned}$$

then  $V^*$  correspond to  $\lambda^{-1}$  in this case.

More generally, we state the definition of dual representation here:

**Definition 4.2.6 (Dual Representation).** Let  $(\rho, V)$  be a representation of a group  $G$ . The dual representation is the vector space  $V^* = \text{Hom}(V, \mathbb{C})$  equipped with the  $G$ -action defined by:

$$(g \cdot f)(v) = f(g^{-1}v)$$

for all  $g \in G$ ,  $f \in V^*$ , and  $v \in V$ , and  $\mathbb{C}$  here is the 1-dimensional **trivial representation**.

*Note:* Alternatively, denoted using  $\rho^*$ :

$$\rho^*(g) = \rho(g^{-1})^T$$

The inverse  $g^{-1}$  is necessary to ensure that it satisfies the group homomorphism property (i.e., to make it a *left* action).

**Question 4.2.3.** Can we use the *sign representation* for the codomain  $\mathbb{C}$  instead of the *trivial representation*?

No, strictly speaking.

- The standard **Dual Representation**  $V^*$  is defined as  $\text{Hom}(V, \mathbb{C}_{\text{triv}})$ . This ensures the natural pairing is  $G$ -invariant:

$$\langle g\phi, gv \rangle = \langle \phi, v \rangle$$

- If we use the sign representation (denoted  $\epsilon$ ), we obtain a different object, the **Twisted Dual**:

$$\text{Hom}(V, \mathbb{C}_\epsilon) \cong V^* \otimes \epsilon$$

In this case, the action involves the sign of the permutation:

$$(g \cdot \phi)(v) = \text{sgn}(g)\phi(g^{-1}v)$$

In the context of dual representations, this “inner product” denotes the **Natural Pairing** (or Canonical Pairing), not necessarily a Hermitian inner product.

- **Natural Pairing** (Used here):

$$\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{C}, \quad \langle \phi, v \rangle := \phi(v)$$

It is **bilinear** (linear in both arguments). No complex conjugation involved.

- **Hermitian Inner Product** (Used in Unitary Reps):

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$$

It is **sesquilinear** (linear in one, conjugate-linear in the other).

## 4.2.4 Irreducible Representation

We now introduce the irreducible representation, which is the building blocks of any representation, it is similar to what simple groups means to finite groups.

**Definition 4.2.7 (Irreducible Representation).** A  $G$ -repr  $V$  is irreducible if

- $V \neq \{0\}$ .
- For every subrepr  $W \subseteq V$ , either  $W = \{0\}$  or  $W = V$ .

**Example 4.2.6.**

1. Every 1-dimensional repr is irreducible.
2. If  $\dim V = 2 \Rightarrow V$  is an irreducible repr iff there exists no common eigenvector for all  $\rho(g)$ ,  $g \in \text{groups}$ .

**Theorem 4.2.2.** If  $V$  is a repr of the finite group  $G$ , then there exists irreducible subrepr  $W_1, \dots, W_r$  s.t.

$$V = W_1 \oplus \dots \oplus W_r$$

**Proof. (Sketch of Proof):** Use induction on dimension of  $V$ , and utilize **Theorem 4.2.1**. ■

The following lemma is very important in terms of **classifying irreducible representation**, thus we can classify different representation using irreducible representation.

**Lemma 4.2.2 (Schur's Lemma).** Let  $f : V \rightarrow W$  be a **morphism** between **irreducible**  $G$ -repr, with  $G$  be a finite group. Then

1. If  $V \not\cong W \Rightarrow f = 0$ .
2. If  $V = W \Rightarrow f = \lambda \text{Id}_V$  for some  $\lambda \in \mathbb{C}$ .

**Proof.** Suppose  $f : V \rightarrow W$  is non-zero, then  $\ker(f) \subseteq V$  is a subrepr, and  $f \neq 0 \Rightarrow \ker(f) \neq V$ . Since  $V$  is irreducible, then  $\ker(f) = \{0\} \Rightarrow f$  is injective.

Let's take  $\text{Im}(f) \subseteq W$  to be subrepr, similarly we see  $\text{Im}(f) = W \Rightarrow f$  is surjective. Thus conclude that  $f$  is isomorphism.

This implies: Suppose now  $V = W$ , let  $\lambda$  be an eigenvalue for  $f$ , it is an automorphism, so  $\ker(f - \lambda \text{Id}_V) \neq \{0\}$ . And clearly  $f - \lambda \text{Id}_V : V \rightarrow V$  is morphism of  $G$ -repr.

Since  $V$  is irreducible, since  $\ker(f - \lambda \text{Id}_V) \neq \{0\} \Rightarrow f - \lambda \text{Id}_V = 0$ . ■

We now focus more on what a trivial representation can derive.

**Definition 4.2.8.** Given ant  $G$ -repr  $V$ , define:

$$V^G := \{v \in V \mid gv = v\} \subseteq V \text{ is a subrepr.}$$

**Example 4.2.7.** Suppose  $V, W$  are  $G$ -repr, we have:

$$\text{Hom}_{\mathbb{C}}(V, W)^G = \{\varphi \in \text{Hom}_{\mathbb{C}}(V, W) \mid g(\varphi(g^{-1}v)) = \varphi(g^{-1}v) \forall g \in G\}$$

By definition of morphisms of  $G$ -repr, obtain:

$$\text{Hom}_{\mathbb{C}}(V, W)^G = \{f : V \rightarrow W \mid f \text{ is morphism of } G\text{-repr.}\}$$

**Parenthesis.** If  $V = V_1 \oplus \dots \oplus V_r$  where  $V_1, \dots, V_r$  are  $G$ -repr,  $W$  is a  $G$ -repr, then:

$$\begin{aligned} \text{Hom}_{\mathbb{C}}(V, W) &\xrightarrow{\sim} \text{Hom}_{\mathbb{C}}(V_1, W) \oplus \dots \oplus \text{Hom}_{\mathbb{C}}(V_r, W) \\ \varphi &\rightarrow (\varphi|_{V_1}, \dots, \varphi|_{V_r}) \end{aligned}$$

which is an isomorphism by the universal properties of direct sums of  $G$ -repr, thus be a unique decomposition.

And we give an application to this example:

**Proposition 4.2.4.** Suppose  $V = V_1 \oplus \dots \oplus V_r$  where  $V_i$  are irreducible  $G$ -repr,  $W$  be arbitrary irreducible repr:

$$\#\{i \mid V_i \cong W\} = \dim_{\mathbb{C}}(\text{Hom}_{\mathbb{C}}(V, W)^G)$$



In particular, the LHS is **independent** of the decomposition of  $V$ .

**Proof.** See that:

$$\begin{aligned}\operatorname{Hom}_{\mathbb{C}}(V, W) &\cong \operatorname{Hom}_{\mathbb{C}}(V_1, W) \oplus \cdots \oplus \operatorname{Hom}_{\mathbb{C}}(V_r, W) \\ \Rightarrow \operatorname{Hom}_{\mathbb{C}}(V, W)^G &\cong \operatorname{Hom}_{\mathbb{C}}(V_1, W)^G \oplus \cdots \oplus \operatorname{Hom}_{\mathbb{C}}(V_r, W)^G\end{aligned}$$

Thus we obtain a decomposition of **morphisms**, so we can apply **Schur's Lemma 4.2.2**,

$$\operatorname{Hom}_{\mathbb{C}}(V_i, W)^G \cong \begin{cases} 0, & \text{if } V_i \not\cong W \text{ (Contribute no dim.)} \\ \mathbb{C}, & \text{if } V_i \cong W \text{ (Add 1 dim.)} \end{cases}$$

■

## 4.3 Group Representations on Multi-linear Algebra

In this section we focus on how to get representation via tensor product, exterior product as well as symmetric product. They are special way to derive special vector spaces in linear algebra, and it will be natural to explore its usage in representation theory.

### 4.3.1 Representation via Tensor Product

Let's first consider the definition of tensor product.

**Definition 4.3.1 (Tensor Product of Vector Spaces).** If  $V, W$  are finite dimensional vector spaces over  $\mathbb{C}$ , then the tensor product of these two vector spaces  $V \otimes_{\mathbb{C}} W$  is another vector space with a bilinear map:

$$\begin{aligned}g : V \times W &\rightarrow V \otimes_{\mathbb{C}} W \\ (v, w) &\mapsto v \otimes w\end{aligned}$$

which has a **universal property** that if  $f : V \times W \rightarrow U$  is bilinear, then there exists a **unique linear map  $h$** , such that the following diagram is commutative:

$$\begin{array}{ccc} V \times W & \xrightarrow{g} & V \otimes_{\mathbb{C}} W \\ & \searrow f & \downarrow h \\ & & U \end{array}$$

**Remark 4.3.1.**

- The universal property of tensor product implies that if

$$\begin{aligned}V_1 &\xrightarrow{f_1} W_1 \\ V_2 &\xrightarrow{f_2} W_2\end{aligned}$$

are both linear maps, then we get a new linear map

$$\begin{aligned}V_1 \otimes V_2 &\xrightarrow{f_1 \otimes f_2} W_1 \otimes W_2 \\ v_1 \otimes v_2 &\mapsto f_1(v_1) \otimes f_2(v_2)\end{aligned}$$

In particular, such map behaves well with respect to composition

$$(g_1 \circ f_1) \otimes (g_2 \circ f_2) = (g_1 \otimes g_2) \circ (f_1 \otimes f_2)$$

- If  $e_1, \dots, e_n$  is basis for  $V$  and  $f_1, \dots, f_m$  is basis of  $W$ , then a basis for  $V \otimes W$  is given by:

$$\{e_i \otimes f_j \mid i \in \llbracket 1, n \rrbracket, j \in \llbracket 1, m \rrbracket\}$$

Then we step into tensor product of representations.

**Definition 4.3.2 (Tensor Product of Group Representations).** Let  $G$  be finite group, and have 2 representations:

$$\rho_1 : G \rightarrow \text{GL}(V_1)$$

$$\rho_2 : G \rightarrow \text{GL}(V_2)$$

We get a new representation  $\rho_1 \otimes \rho_2$ , which is the tensor product of 2 representations, such that:

$$G \rightarrow \text{GL}(V_1 \otimes V_2)$$

$$\rho_1 \otimes \rho_2(g) = \rho_1(g) \otimes \rho_2(g)$$

One can check that such map results is invertible and itself yields a group homomorphism, thus it is indeed a well-defined group representation.

One can also try an example of calculating  $\rho \otimes \rho((12))$  and  $\rho \otimes \rho((123))$  with the same setup in **Example 4.2.4**. The point is to write the basis of tensor product as  $f_1 \otimes f_1, f_1 \otimes f_2, f_2 \otimes f_1, f_2 \otimes f_2$  and take advantage of the property of **bilinearity**.

**Proposition 4.3.1.** If  $\rho_1, \rho_2$  are 2  $G$ -repr, then:

$$\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1}(g) \cdot \chi_{\rho_2}(g) \quad \forall g \in G$$

**Proof.** Suppose:

$$\rho_1 : G \rightarrow \text{GL}(V) \quad e_1, \dots, e_n \text{ basis of } V$$

$$\rho_2 : G \rightarrow \text{GL}(W) \quad f_1, \dots, f_m \text{ basis of } W$$

then:

$$\rho_1(g) : e_j \mapsto \sum_{i=1}^n a_{ij} e_i \quad \chi_{\rho_1}(g) = \sum_{i=1}^n a_{ii}$$

$$\rho_2(g) : f_l \mapsto \sum_{k=1}^m b_{kl} f_k \quad \chi_{\rho_2}(g) = \sum_{k=1}^m b_{kk}$$

thus:

$$\rho_1 \otimes \rho_2(g) : e_j \otimes f_l \mapsto \left( \sum_{i=1}^n a_{ij} e_i \right) \otimes \left( \sum_{k=1}^m b_{kl} f_k \right) = \sum_{i=1}^n \sum_{k=1}^m a_{ij} b_{kl} (e_i \otimes f_k)$$

For  $i = j, k = l$ , thus:

$$\begin{aligned} \text{Tr}(\rho_1 \otimes \rho_2(g)) &= \sum_{i,k} a_{ii} b_{kk} \\ &= \left( \sum a_{ii} \right) \cdot \left( \sum b_{kk} \right) \\ &= \chi_{\rho_1}(g) \cdot \chi_{\rho_2}(g) \end{aligned}$$

■

**Remark 4.3.2.**

$$\chi_{\rho \otimes \rho}(g) = (\chi_{\rho}(g))^2$$

### 4.3.2 Representation via Exterior Product

Let's first state the definition for exterior product for vector spaces.

**Definition 4.3.3 (Exterior Product for Vector Spaces).** Suppose  $V$  be any finite dimensional vector space over  $\mathbb{C}$ ,  $p \geq 1$ , then the  $p^{\text{th}}$  exterior product is defined:

$$\bigwedge^p V := V^{\otimes p} / \text{span}\{x_1 \otimes \cdots \otimes x_p \mid \exists i < j, x_i = x_j\}$$

along with the **multi-linear and alternating map**:

$$\begin{aligned} g : \underbrace{V \times \cdots \times V}_{p \text{ times}} &\rightarrow \bigwedge^p V \\ (v_1, \dots, v_p) &\mapsto v_1 \wedge \cdots \wedge v_p \end{aligned}$$

which has the **universal property** that if  $f : V^{\times p} \rightarrow U$  is multi-linear and alternating, then there exists a **unique linear map  $h$** , such that the following diagram is commutative:

$$\begin{array}{ccc} \underbrace{V \times \cdots \times V}_{p \text{ times}} & \xrightarrow{g} & \bigwedge^p V \\ & \searrow f & \downarrow h \\ & & U \end{array}$$

In fact, this gives us a functor, if  $f : V \rightarrow W$  is a linear map, then the following diagram is commutative:

$$\begin{array}{ccc} \underbrace{V \times \cdots \times V}_{p \text{ times}} & \xrightarrow{f \times \cdots \times f} & \underbrace{W \times \cdots \times W}_{p \text{ times}} \\ \downarrow \varphi_V & & \downarrow \varphi_W \\ \bigwedge^p V & \xrightarrow{\bigwedge^p f} & \bigwedge^p W \end{array}$$

The universal property of  $\bigwedge^p V$  implies that there exists a unique linear map:

$$\begin{aligned} \bigwedge^p f : \bigwedge^p V &\rightarrow \bigwedge^p W \\ v_1 \wedge \cdots \wedge v_p &\rightarrow f(v_1) \wedge \cdots \wedge f(v_p) \end{aligned}$$

The upshot:

$$\bigwedge^p : \text{Vect}/_{\mathbb{C}} \rightarrow \text{Vect}/_{\mathbb{C}} \text{ is a functor.}$$

**Question 4.3.1.** What is  $\dim \bigwedge^p V$ ?

**Theorem 4.3.1 (Dimension of Exterior Product).** If  $e_1, \dots, e_n$  be basis of  $V$ , then a basis of  $\bigwedge^p V$  is given by:

$$\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n\}$$

In particular,  $\bigwedge^p V = 0$  if  $p > n$ , and  $\dim \bigwedge^n V = 1$ .

**Proof.** Since  $e_{i_1} \otimes \cdots \otimes e_{i_p}$  for  $1 \leq i_1, \dots, i_p \leq n$  form a basis of  $\underbrace{V \otimes \cdots \otimes V}_{p \text{ times}}$ . Then  $e_{i_1} \wedge \cdots \wedge e_{i_p}$  generates  $\bigwedge^p V$ . But  $e_{i_1} \wedge \cdots \wedge e_{i_p} = 0$  if  $i_k = i_l$  for some  $k < l$  and

$$e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(p)}} = \epsilon(\sigma)(e_{i_1} \wedge \cdots \wedge e_{i_p}) \quad \forall \sigma \in S_p$$

Hence  $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n\}$  generates  $\bigwedge^p V$ . Let's then check these elements are linearly independent.

We denote if  $I \subseteq \{1, \dots, n\}$ ,  $|I| = p$ , then put  $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$  where  $I = i_1 < \cdots < i_p$ .

Suppose  $\sum_{I \subseteq [n], |I|=p} a_I e_I = 0$  in  $\bigwedge^p V$ , want to see:  $a_I = 0 \forall I$ , we want to construct a linear map:

$$\begin{aligned} \alpha_I : \bigwedge^p V &\rightarrow \mathbb{C} \\ \alpha_I(e_I) &= \pm 1 \\ \alpha_I(e_J) &= 0 \quad \forall J \neq I \end{aligned}$$

This would imply what we need.

To get this, let  $I$  be fixed, and  $\bar{I} := \{1, 2, \dots, n\} - I$  with elements being  $j_1, \dots, j_{n-p}$ , then define:

$$\begin{aligned} \tilde{\alpha} : \underbrace{V \times \cdots \times V}_{p \text{ times}} &\rightarrow \mathbb{C} \\ \alpha(v_1, \dots, v_p) &= \det \left( \begin{array}{c|c|c|c|c|c} \begin{array}{c} | \\ v_1 \\ | \end{array} & \begin{array}{c} | \\ v_2 \\ | \end{array} & \cdots & \begin{array}{c} | \\ v_p \\ | \end{array} & \begin{array}{c} | \\ e_{j_1} \\ | \end{array} & \cdots & \begin{array}{c} | \\ e_{j_{n-p}} \\ | \end{array} \end{array} \right) \end{aligned}$$

where the matrix on the right hand side is written as column vectors w.r.t.  $e_1, \dots, e_n$ . By properties of determinant, this is a multi-linear alternating map. By the **universal property** of  $\bigwedge^p V$ , this gives us a linear map:

$$\begin{aligned} \bigwedge^p &\rightarrow \mathbb{C} \\ v_1 \wedge \cdots \wedge v_p &\mapsto \det \left( \begin{array}{c|c|c|c|c|c} \begin{array}{c} | \\ v_1 \\ | \end{array} & \begin{array}{c} | \\ v_2 \\ | \end{array} & \cdots & \begin{array}{c} | \\ v_p \\ | \end{array} & \begin{array}{c} | \\ e_{j_1} \\ | \end{array} & \cdots & \begin{array}{c} | \\ e_{j_{n-p}} \\ | \end{array} \end{array} \right) \end{aligned}$$

and its clear that this maps satisfy that  $e_J \rightarrow 0$  if  $J \neq I$  and  $e_I \rightarrow \pm 1$ . ■

### 4.3.3 Representation via Symmetric Product

This section will be basically almost the same as what we introduced for the exterior product, but the universal property we get is w.r.t. multi-linear symmetric map.

**Definition 4.3.4 (Symmetric Product for Vector Spaces).** Suppose  $V$  be any finite dimensional vector spaces over  $\mathbb{C}$ , then the  **$p^{\text{th}}$  symmetric product** is defined:

$$S^p V := V^{\otimes p} / \text{span}\{v_1 \otimes \cdots \otimes v_p - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)} \mid v_1, \dots, v_p \in V, \sigma \in S_p\}$$

along with the **multi-linear symmetric map**:

$$\begin{aligned} g : \underbrace{V \times \cdots \times V}_{p \text{ times}} &\rightarrow S^p V \\ (v_1, \dots, v_p) &\mapsto v_1 \cdots v_p \end{aligned}$$

which has the universal property that if  $f : V^{\times p} \rightarrow U$  is multi-linear and symmetric, then there exists a **unique linear map  $h$** , such that the following diagram is commutative:

$$\begin{array}{ccc} \underbrace{V \times \cdots \times V}_{p \text{ times}} & \xrightarrow{g} & S^p V \\ & \searrow f & \downarrow h \\ & & U \end{array}$$

which defines a functor  $S^p : \text{Vect} / \mathbb{C} \rightarrow \text{Vect} / \mathbb{C}$ .

**Theorem 4.3.2.** If  $e_1, \dots, e_n$  is basis of  $V$ , then a basis of  $S^p V$  is given by:

$$\{e_{i_1} \cdots e_{i_p} \mid 1 \leq i_1 \leq \cdots \leq i_p \leq n\}$$

Analyzing the property of representation induced by exterior product and symmetric product is similar to what we have done for tensor product: write out the basis and use the multi-linearity and symmetric/alternating property. Here is an interesting statement:

**Proposition 4.3.2.**

$$\rho \otimes \rho \cong (S^2 \rho) \oplus \left( \bigwedge^2 \rho \right)$$

# Chapter 5

## Character Theory

At this point we are able to have some sense that irreducible representation are the building blocks of representations. We have also developed ways to classify irreducible representation, namely the **Schur's Lemma 4.2.2**. However we still lack the way to classify general representations in a convenient way. In this chapter we will introduce **Character Theory** on group representation, which give us a good sense on how to classify them.

### 5.1 Basic Definition

**Definition 5.1.1 (Character of Group Representations).** If  $V$  is a  $G$ -repr, then the character of  $V$  is a function:

$$\begin{aligned}\chi_V : G &\rightarrow \mathbb{C} \\ \chi_V(g) &= \text{Tr}(\rho(g))\end{aligned}$$

where  $\rho : G \rightarrow \text{GL}(V)$  gives the  $G$ -action on  $V$ , and  $\text{Tr}$  is the sum of the diagonal elements of the matrix.

A typical lemma from Linear Algebra, we state here without proof

**Lemma 5.1.1.**

$$\text{Tr}(AB) = \text{Tr}(BA)$$

Hence:

$$\text{Tr}(A) = \text{Tr}(BAB^{-1})$$

**Remark 5.1.1.** If  $V \cong W$  are isomorphism representations, then

$$\chi_V = \chi_W$$

To see it we can choose basis  $e_1, \dots, e_n$  for  $V$  and basis  $\varphi(e_1), \dots, \varphi(e_n)$  for  $W$ , and get the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\sim} & W \\ \rho_V(g) \downarrow & & \downarrow \rho_W(g) \\ V & \xrightarrow{\sim} & W \end{array}$$

Thus the matrix for  $\rho_V(g)$  and  $\rho_W(g)$  is the same thing for all  $g \in G$ .

**Remark 5.1.2.** If  $V$  is a  $G$ -repr, then  $\chi_V$  is a class function. i.e. it takes some value on the elements in each **conjugacy class**.

$$\chi_V(g) = \chi_V(hgh^{-1}) \quad \forall g, h \in G$$

The proof is straightforward by considering  $\rho$  is a group homomorphism as well as the Lemma we just

stated above.

**Example 5.1.1.** Let  $V$  be any repr, then  $\chi_V(e) = \dim V$ .

**Example 5.1.2.** If  $V$  is a 1-dimensional repr, then:

$$G \xrightarrow{\rho} \text{GL}(V) \cong \mathbb{C}^\times \rightsquigarrow \chi_V(g) = \rho(g)$$

**Example 5.1.3.** Consider the permutation representation on  $S_3$  corresponding to the obvious action of  $S_3$  on  $\{1, 2, 3\}$ , see that:

$$V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$$

and given  $W$  as the standard representation:

$$W = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1 + a_2 + a_3 = 0\} \subseteq V \text{ is a subrepr.}$$

By similar manner in **Example 4.2.4**, we can compute out:

$$\chi_W(e) = 2 \quad \chi_W(\tau) = 0 \quad \chi_W(\sigma) = -1$$

Note that since  $\chi_W$  is class function, it is enough to calculate the above to obtain all the information.

We state two more important property of character.

**Proposition 5.1.1.** If  $V \cong V_1 \oplus V_2$  on  $G$ -repr, then:

$$\chi_V = \chi_{V_1} + \chi_{V_2}$$

The proof of it is straightforward once written out the basis and write the matrix of  $\rho_{V_1 \oplus V_2}(g)$  as **Block Diagonal Matrix**.

**Proposition 5.1.2.** If  $V$  is a  $G$ -repr, then:

$$\chi_V(g^{-1}) = \overline{\chi_V(g)} \quad \forall g \in G$$

We need a important lemma from linear algebra to prove it.

**Definition 5.1.2 (Minimal Polynomial).** Given  $A \in M_n(\mathbb{C})$ , the minimal polynomial of  $A$ , denoted as  $P_0$  is the lowest degree monic polynomial that make  $P(A) = 0$ .

**Theorem 5.1.1.** If polynomial  $Q$  s.t.  $Q(A) = 0$ , then  $P_0 \mid Q$ .

$A \in M_n(\mathbb{C})$  is diagonalizable if and only if the minimal polynomial of  $A$  has simple roots/distinct roots.

**Proof of Proposition 5.1.2.**

**Claim.**  $\rho_V(g)$  is diagonalizable for all  $g \in G$ .

**Proof of the Claim.** If  $N = |G| \Rightarrow g^N = \text{Id} \quad \forall g \in G \Rightarrow (\rho_V(g))^N = \text{Id}$ . So by **Theorem 5.1.1**, the minimal polynomial of  $\rho_V(g)$  divides  $X^N - 1$ . Since  $X^N - 1$  has no multiple roots (no numerical multiplicity bigger than 1), it follows  $\rho_V(g)$  is diagonalizable. ■

So far, see that there exists a basis of  $e_1, \dots, e_n$  of  $V$ , s.t. the matrix corresponds to  $\rho_V(g)$  is

diagonal matrix, with eigenvalues being  $\lambda_1, \dots, \lambda_n$ , and  $\lambda_i^N = 1 \forall i$ , then:

$$|\lambda_i| = 1 \Rightarrow \lambda_i^{-1} = \overline{\lambda_i}$$

Thus the matrix for  $\rho_V(g^{-1}) = \rho_V(g)^{-1}$ , so

$$\begin{aligned} \text{Tr}(\rho_V(g^{-1})) &= \sum_{i=1}^n \lambda_i^{-1} \\ &= \sum_{i=1}^n \overline{\lambda_i} \\ &= \overline{\sum_{i=1}^n \lambda_i} \\ &= \overline{\text{Tr}(\rho_V(g))} \end{aligned}$$

Thus yields the result. ■

## 5.2 Orthogonality of Characters

We would gradually define the Hermitian product of characters in this section, notably, there is orthonormality within.

Consider  $\text{Fun}(G) = \{f : G \rightarrow \mathbb{C}\}$  be arbitrary functions, this is a vector space over  $\mathbb{C}$ :

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \forall x \in G \\ (\lambda f)(x) &= \lambda(f(x)) \forall x \in G \end{aligned}$$

A basis of  $\text{Fun}(G)$  is given by  $\{f_x \mid x \in G\}$ , with:

$$f_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

**Note.**  $h \in \text{Fun}(G)$ ,  $h = \sum_{x \in G} h(x) \cdot f_x$ .

Thus we see

$$\dim(\text{Fun}(G)) = |G|$$

**Definition 5.2.1 (Hermitian Product over  $\text{Fun}(G)$ ).** We have on  $\text{Fun}(G)$ , the Hermitian product  $\langle -, - \rangle$  defined by:

$$\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)} \in \mathbb{C}$$

such that:

1.  $\langle -, \psi \rangle : \text{Fun}(G) \rightarrow \mathbb{C}$  is linear.
2.  $\langle \varphi, \psi \rangle = \overline{\langle \psi, \varphi \rangle}$ .
3.  $\langle \varphi, \varphi \rangle > 0$ , if  $\varphi \neq 0$ , it is  $\frac{1}{|G|} \sum_{g \in G} |\varphi(g)|^2$ .

### Theorem 5.2.1.

1. If  $\rho_1, \rho_2$  are **non-isomorphic irreducible** repr of  $G$ , then

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = 0$$



2. If  $\rho_1 \cong \rho_2$  are **irreducible**, then

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = 1$$

**Proof.**

**Idea.** Starting with any linear map  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^m$  use the “averaging procedure” from the proof of **Theorem 4.2.1** to get from  $\varphi$  a morphism of  $G$ -repr  $\psi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ , then by **Schur’s Lemma 4.2.2**,  $\psi = 0$  in case 1 and  $\psi = \lambda \text{Id}$  in case 2. Choose  $\varphi$  suitably and taking the trace shall yield the results.

May assume:

$$\begin{aligned}\rho_1 : G &\rightarrow \text{GL}_n(\mathbb{C}) \\ \rho_2 : G &\rightarrow \text{GL}_n(\mathbb{C})\end{aligned}$$

In the setting of 2, may assume  $\rho_1 = \rho_2$  since by **Remark 5.1.1**, isomorphic repr have the same character. Given any linear map  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ , take  $\psi$  given by

$$\psi(v) = \sum_{g \in G} g^{-1} \varphi(gv)$$

the **key point** is: this is a morphism of  $G$ -repr:

$$\psi(hv) = \sum_{g \in G} \underbrace{g^{-1}}_{h(gh)^{-1}} \varphi(ghv) = h \sum_{g \in G} (g^{-1} \varphi(gv)) = h\varphi(v)$$

**Note.**

1.  $gv$  here is  $\rho_1(g)v$ , the group action for  $v \in \mathbb{C}^n$ .
2.  $g^{-1}$  here is  $\rho_2(g^{-1})$ , the group action for  $\text{Im}(\varphi) \in \mathbb{C}^m$ .

If  $\varphi$  described w.r.t. the standard basis of  $\mathbb{C}^n, \mathbb{C}^m$  by the matrix  $(\gamma_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$  and

$$\begin{aligned}\rho_2(g^{-1}) &\longleftrightarrow (b_{pq}(g))_{1 \leq p, q \leq m} \\ \rho_1(g) &\longleftrightarrow (a_{kl}(g))_{1 \leq k, l \leq n}\end{aligned}$$

then  $\psi$  correspond to th matrix

$$\sum_{g \in G} (b_{pq}(g)) \cdot (\gamma_{ij}) \cdot (a_{kl}(g)) \tag{5.1}$$

- Suppose we are in **Case 1**: By Schur’s Lemma,  $\psi = 0$ . Now fix  $i \leq m, j \leq n$  and take

$$\gamma_{i'j'} = \begin{cases} 1, & \text{if } i' = i, j' = j \\ 0, & \text{otherwise} \end{cases}$$

Now **Formula 5.1** is  $0 \Rightarrow \forall p \leq m, \forall l \leq n :$

$$\sum_{g \in G} b_{pi}(g) a_{jl}(g) = 0$$

Take  $p = i, l = j \Rightarrow \sum_{g \in G} b_{ii}(g) a_{jj}(g) = 0$ , take  $\sum_{i,j}$  on both sides:

$$\sum_{g \in G} \underbrace{\text{Tr}(\rho_2(g^{-1}))}_{\chi_{\rho_2}(g^{-1})} \underbrace{\text{Tr}(\rho_1(g))}_{\chi_{\rho_1}(g)} = 0 \Rightarrow \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = 0$$

- Suppose we are in **Case 2**: Then  $\rho_1 = \rho_2$  are irreducible repr, by **Schur's Lemma 4.2.2**,  $\forall \varphi = (\gamma_{ij})$ , we have  $\psi = \lambda(\varphi) \text{Id}$  for some  $\lambda(\varphi)$ .

For every fixed  $i, j \leq n$ , let

$$\gamma_{i'j'} = \begin{cases} 1, & \text{if } i = i', j = j' \\ 0, & \text{otherwise} \end{cases}$$

For all  $p, l \leq n$ , we have

$$\sum_{g \in G} b_{pi}(g) a_{jl}(g) = \lambda_{i,j} \delta_{p,l} \quad \text{for some } \lambda_{i,j} \in \mathbb{C}$$

Take  $p = i, l = j \Rightarrow$

$$\sum_{g \in G} b_{ii}(g) a_{jj}(g) = \lambda_{i,j} \delta_{i,j} \quad (5.2)$$

Take sum  $\sum_{i,j}$  on both sides, obtain:

$$\underbrace{\sum_{g \in G} \text{Tr}(\rho_2(g^{-1})) \text{Tr}(\rho_1(g))}_{|G| \cdot \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle} = \sum_{i=1}^n \lambda_{i,i}$$

Recall that  $\rho_1 = \rho_2 \Rightarrow b_{ij} = (a_{ji})^{-1}$ , take  $i = j$  in **Formula 5.2** and take the sum  $\sum_{i=1}^n$ , we have:

$$\sum_{g \in G} \delta_{p,l} = \sum_{i=1}^n \lambda_{i,i} \delta_{p,l}$$

$$\begin{aligned} \text{Let } p = l &\Rightarrow \sum \lambda_{i,i} = |G| \\ &\Rightarrow |G| \cdot \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = |G| \\ &\Rightarrow \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = 1 \end{aligned}$$

■

Thus we see the **irreducible representations** yields a **orthonormal system**.

**Corollary 5.2.1.** There are only finitely many classes of  $G$ -repr if  $G$  be finite group.

**Proof.** If  $\rho_1, \dots, \rho_n$  are pairwise non-isomorphic irreducible representation, the theorem implies:

$$\langle \chi_{\rho_i}, \chi_{\rho_j} \rangle = \delta_{i,j}$$

This implies  $\chi_{\rho_1}, \dots, \chi_{\rho_n}$  are linearly independent, and  $n$  will have an upperbound, which is

$$\dim(\text{Fun}(G)) = |G|$$

■

**Corollary 5.2.2.** A  $G$ -repr  $\rho$  is irreducible if and only if  $\chi_\rho, \chi_\rho = 1$ .

**Proof.** Consider a decomposition into irreducible representation

$$\rho \cong \bigoplus_{i=1}^r V_i^{\oplus a_i}, \quad V_i \not\cong V_j \text{ if } i \neq j$$

then

$$\chi_\rho = \sum_{i=1}^r a_i \chi_{V_i}$$

The theorem tells us  $\langle \chi_\rho, \chi_\rho \rangle = \sum_{i=1}^r a_i^2$ , hence  $\langle \chi_\rho, \chi_\rho \rangle = 1$  iff  $r = 1$  and  $a = 1$ . ■

## 5.3 Analysis on Decomposition

We shall see further application on how to take advantages to the orthogonality of the characters of the irreducible representations.

**Proposition 5.3.1.** Given a representation  $V$  and an irreducible representation  $W$ , the # of times  $W$  shows up in a decomposition of  $V$  into irreducible repr is

$$\langle \chi_W, \chi_V \rangle$$

**Note.** This actually gives the proof for the fact that the number is independent on the way of decomposition.

**Proof.** Let  $V \cong V_1 \oplus \dots \oplus V_r$ , where  $V_i$  irreducible, then

$$\chi_V = \sum_{i=1}^r \chi_{V_i}$$

thus

$$\langle \chi_W, \chi_V \rangle = \sum_{i=1}^r \langle \chi_W, \chi_{V_i} \rangle \stackrel{\text{Thm. 5.2.1}}{=} \#\{i \mid W \cong V_i\}$$

■

**Proposition 5.3.2.** Two repr  $V_1, V_2$  are isomorphic iff  $\chi_{V_1} = \chi_{V_2}$ .

**Proof.** The forward direction is cleared as we've seen in **Remark 5.1.1**. Conversely, followed from **Proposition 5.3.1** as above: Suppose  $W_1, \dots, W_r$  are pairwise non-isomorphic irreducible repr of  $G$ , s.t.

$$\left. \begin{array}{l} V_1 \cong \bigoplus_{i=1}^r W_i^{\oplus a_i}, \quad V_2 \cong \bigoplus_{i=1}^r W_i^{\oplus b_i} \\ \text{Prop. 5.3.1} \Rightarrow a_i = \langle \chi_{W_i}, \chi_{V_1} \rangle = \langle \chi_{W_i}, \chi_{V_2} \rangle = b_i \quad \forall i \end{array} \right\} \Rightarrow V_1 \cong V_2$$

■

This tells us: If we understand the **characters of a repr.**, then we understand the repr.

### 5.3.1 Regular Repr. and Irred. Repr.

In this section, we shall use the regular representation  $V^{\text{reg}}$  of  $G$  to deduce information about the irreducible representation of  $G$ .

Recall that the regular representation  $V^{\text{reg}}$  has a basis  $\{e_g \mid g \in G\}$  with  $G$  acting via  $h \cdot e_g = e_{hg}$ ,  $\forall g, h \in G$ . Since  $hg = g$  iff  $h = e$ , so

$$\chi_{V^{\text{reg}}} = \begin{cases} 0, & \text{if } h \neq e \\ |G|, & \text{if } h = e \end{cases}$$

Suppose  $V_1, \dots, V_r$  are **all** the pairwise non-isomorphic irred. repr. of  $G$ , assume:

$$V^{\text{reg}} \cong V_1^{\oplus a_1} \oplus \dots \oplus V_r^{\oplus a_r}$$

We saw:

$$\begin{aligned} a_i &= \langle \chi_{V_i}, \chi_{V^{\text{reg}}} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{V_i}(g) \overline{\chi_{V^{\text{reg}}}(g)} \\ &= \frac{1}{|G|} \cdot \dim V_i \cdot |G| \\ &= \dim V_i \end{aligned}$$

Thus we obtain the following proposition.

**Proposition 5.3.3.** If  $W$  is an irred. repr. of  $G$ , then the  $\#$  of times this appears in a decomposition of  $V^{\text{reg}}$  in irred. repr. is  $\dim W$ .

In particular, if  $V_1, \dots, V_r$  are the irred.  $G$ -repr, then:

$$V^{\text{reg}} \cong V_1^{\dim V_1} \oplus \dots \oplus V_r^{\dim V_r}$$

$$\Rightarrow |G| = \sum_{i=1}^r (\dim V_i)^2$$

**Note.** This is extremely useful for **determining all the irred. repr.** of a specific **finite** group.

**Remark 5.3.1 (Decomposition of Characters of Regular Representation).**

Let  $V_1, \dots, V_r$  be all the irreducible  $G$ -representations.

$$\chi_{V^{\text{reg}}}(g) = \sum_{i=1}^r \dim V_i \cdot \chi_{V_i}(g)$$

$$\Rightarrow \sum_{i=1}^r \dim V_i \cdot \chi_{V_i}(g) = \begin{cases} 0, & g \neq e \\ |G|, & g = e \end{cases}$$

**Example 5.3.1.**  $G = S_3$ , it will have the following irred. repr.:

- the trivial 1-dim repr.
- the repr on  $\{(a_1, a_2, a_3) \mid \sum a_i = 0\}$ , which is 2-dim.
- the 1-dim sign repr.

Since  $1^2 + 1^2 + 2^2 = 6 = |S_3|$ , by the proposition, these are **all** the irreducible repr of  $S_3$ .

**Example 5.3.2.**  $G = S_4$ , we have known it will have the following repr:

- the trivial 1-dim repr.
- the 1-dim sign repr.
- seen in homework that there are no other 1-dim irred. repr.

By the **Proposition 5.3.3**, there will be remain one 2-dim and two 3-dim irred. repr.

**Proposition 5.3.4.** For every **abelian group**  $G$ , all irreducible  $G$ -repr are 1-dimensional.

**Fact 5.3.1.** Given a finite set of endomorphisms  $V \rightarrow V$  which are commute and diagonalizable, then there exists a basis of  $V$ , s.t. they're all given by diagonal matrices.

Such statement can be conclude by induction on the size of such sets, and analyzing on the mapping of eigenvectors.

**Proof.** Suppose  $\rho : G \rightarrow \text{GL}(V)$  be  $G$ -repr, we've seen  $\rho(g)$  is diagonalizable for all  $g \in G$ . One can apply the fact above for  $\{\rho(g) \mid g \in G\}$  to get a basis  $e_1, \dots, e_n$ , s.t. each  $\rho(g)$  is given by a diagonal matrix, then each  $\mathbb{C}e_i$  is a  $G$ -subrepr of  $V$ . In particular, if  $V$  is irreducible, it means  $\dim V = 1$ . And by **Proposition 5.3.3**, we will have  $|G|$  number of irreducible  $G$ -repr. ■

## 5.4 Characters and Conjugacy Classes

In this section, our general goal is to see how the number of irreducible representation of a given group is connected to the number of conjugacy classes of the group. In particular, we will see that they are in fact equal. In general there is no algorithm to derived them from each other, but we have such quantative relationship, which is magical.

We first introduce the setup of this section: Given  $G$  be a finite group, define the **class function** as:

$$\mathcal{C}(G) := \{f : G \rightarrow \mathbb{C} \mid f(g) = f(hgh^{-1}), \forall g, h \in G\} \subseteq \text{Fun}(G)$$

which is clearly a linear subspace. It has a basis  $(f_C)_C$  be conjugacy class of  $G$ , such that:

$$f_C(g) = \begin{cases} 1, & \text{if } g \in C \\ 0, & \text{otherwise} \end{cases}$$

In particular,  $\dim \mathcal{C}(G) = \#$  conjugacy classes of  $G$ . We've seen: if  $\rho$  be representation of  $G$ , then  $\chi_\rho \in \mathcal{C}(G)$  given by the trace property.

### Theorem 5.4.1.

If  $\rho_1, \dots, \rho_r$  are all the irreducible  $G$ -repr, then  $\chi_{\rho_1}, \dots, \chi_{\rho_r}$  is a basis of  $\mathcal{C}(G)$ .  
In particular,  $r = \#$  conjugacy classes of  $G$ .

**Remark 5.4.1.** We know,  $\chi_{\rho_1}, \dots, \chi_{\rho_r} \in \mathcal{C}(G)$  and

$$\langle \chi_{\rho_i}, \chi_{\rho_j} \rangle = \delta_{i,j} \quad \forall i, j$$

i.e. they are linearly independent. It is left to proof:

$$\chi_{\rho_1}, \dots, \chi_{\rho_r} \text{ span } \mathcal{C}(G)$$

We need some preparation to give the proof.

**Lemma 5.4.1.** Suppose  $f \in \mathcal{C}(G)$  and  $\rho : G \rightarrow \text{GL}(V)$  be a  $G$ -repr. Define:

$$\begin{aligned} \rho_f &\in \text{End}_{\mathbb{C}}(V) \\ \rho_f &= \sum_{g \in G} f(g) \rho(g) \end{aligned}$$

1.  $\rho_f : V \rightarrow V$  is a morphism of  $G$ -repr. See that:

$$\begin{aligned} \rho_f(hv) &= \sum_{g \in G} f(g) \underbrace{ghv}_{gv = \rho(g)v} = h \cdot \left( \sum_{g \in G} f(g) \cdot h^{-1}ghv \right) \\ &= h \cdot \left( \sum_{g \in G} (h^{-1}gh) \cdot (h^{-1}gh)v \right) \\ &= h \cdot \left( \sum_{g' \in G} f(g') g'v \right) \\ &= h \cdot \left( \sum_{g' \in G} f(g') \rho(g')(v) \right) \\ &= f \cdot \rho_f(v) \end{aligned}$$

2. If  $\rho$  is irreducible, then by **Schur's Lemma 4.2.2** and that  $\rho_f$  gives a morphism from  $V \rightarrow V$ , see that  $\rho_f = \lambda \text{Id}_V$  for some  $\lambda \in \mathbb{C}$ .

Take trace on both sides:

$$\begin{aligned}\sum_{g \in G} f(g) \chi_\rho(g) &= \lambda \cdot \dim(V) \\ &= |G| \langle \chi_\rho, \bar{f} \rangle \\ \Rightarrow \lambda &= \langle \chi_\rho, \bar{f} \rangle \frac{|G|}{\dim V} \\ \text{where } \bar{f} : G &\rightarrow \mathbb{C}, \bar{f}(g) = \overline{f(g)}\end{aligned}$$

3. If  $\rho = \rho_1 \oplus \rho_2$ , then by  $V = V_1 \oplus V_2$  and  $\rho(g) = (\rho_1(g), \rho_2(g)) \Rightarrow \rho_f = (\rho_{f_1}, \rho_{f_2})$ .

**Proof of Thm. 5.4.1.** By **Remark 5.4.1**, it is enough to show:

$$\text{span}\{\chi_{\rho_1}, \dots, \chi_{\rho_r}\} = \mathcal{C}(G)$$

**Fact 5.4.1.** Since  $\langle -, - \rangle$  is a Hermitian product on  $\mathcal{C}(G)$ , it is enough to show:

$$\{f \in \mathcal{C}(G) \mid \langle \chi_{\rho_i}, f \rangle = 0 \forall i\} = \{0\}$$

Reckon this in terms of orthogonal projection shall yield the results.

**Parenthesis (From Linear Algebra).** If  $W \subseteq \mathcal{C}(G)$  be linear subspace, then  $W \oplus W^\perp = \mathcal{C}(G)$ , where:

$$W^\perp = \{\varphi \mid \langle w, \varphi \rangle = 0, \forall w \in W\}$$

By definition of dual, we have such isomorphism:

$$\begin{aligned}\mathcal{C}(G) &\xrightarrow{\sim} \mathcal{C}(G)^* \\ \varphi &\mapsto \langle -, \bar{\varphi} \rangle\end{aligned}$$

It is injective:  $\varphi \neq 0 \Rightarrow \langle \bar{\varphi}, \bar{\varphi} \rangle \neq 0$ . They have same dimension, thus bijection and hence isomorphism.

If  $W \hookrightarrow \mathcal{C}(G)$  be subspace, then:

$$\mathcal{C}(G) \xrightarrow{\sim} \mathcal{C}(G)^* \rightarrow W^* \text{ is surjective.}$$

with kernel  $\{\psi \mid \langle \psi, \bar{\varphi} \rangle = 0, \forall \varphi \in W\}$

$$\Rightarrow \underbrace{\dim W^*}_{=\dim W} + \underbrace{\dim \overline{W}^\perp}_{=\dim W^\perp} = \dim \mathcal{C}(G)^* = \dim \mathcal{C}(G)$$

by the rank theorem. Since  $W \cap W^\perp = \{0\}$  since  $\langle \varphi, \varphi \rangle \neq 0 \forall \varphi$

$$\mathcal{C}(G) = W \oplus W^\perp$$

Hence we want to show: if  $f \in \mathcal{C}(G)$ , s.t.

$$\langle \chi_{\rho_i}, f \rangle = 0, \forall i \Rightarrow f = 0$$

By **Lemma 5.4.1 2**,  $(\rho_i)_{\bar{f}} = 0 \forall i$ , since:

$$(\rho_i)_{\bar{f}} = \underbrace{\langle \chi_{\rho_i}, f \rangle}_{=0} \frac{|G|}{\dim(\rho_i)}$$

By **Lemma 5.4.1 3**, since every repr. is a direct sum of irred. repr., then  $\rho_f = 0 \forall \rho$  as representation.

Apply for  $\rho = \rho^{\text{reg}}$  which is the regular representation:

$$V = \bigoplus_{g \in G} \mathbb{C}e_g \Rightarrow \sum_{g \in G} \bar{f}(g)\rho^{\text{reg}}(g) = 0$$

this is a endormorphism, so we can apply it to  $e_1 \in V$  who is the **identity element** of  $V$ , and get:

$$\sum_{g \in G} \bar{f}(g)e_g = 0 \quad (\text{since } \rho^{\text{reg}}(g)(e_1) = g \cdot e_1 = e_{g \cdot 1} = e_g)$$

The  $e_g$  are **linearly independent** by definition of regular representation, thus  $\bar{f}(g) = 0 \forall g \in G \Rightarrow \bar{f} = 0 \Rightarrow f = 0$ . ■

**Note.** There is **no natural canonical way to associate conjugacy classes and irreducible representation!**

However, it is recommended to make a table for a group  $G$  with x-axis being irreducible characters and y-axis being the conjugacy classes.

**Proposition 5.4.1 (Second Orthogonal Property).**

1. If  $g, h \in G$  are not conjugate to each other, and  $\rho_1, \dots, \rho_r$  are irred.  $G$ -repr, then:

$$\sum_{i=1}^r \chi_{\rho_i}(g)\overline{\chi_{\rho_i}(h)} = 0$$

2. we have:

$$\sum_{i=1}^r \chi_{\rho_i}(g)\overline{\chi_{\rho_i}(g)} = \frac{|G|}{c(g)}$$

where  $c(g) = \#\{\text{conjugacy classes of } g\}$

**Proof. (Sketch of Proof):** Choose  $f \in \mathcal{C}(G)$ , s.t.

$$f(x) = \begin{cases} 1, & \text{if } x \in \text{conjugacy classes of } h \\ 0, & \text{otherwise} \end{cases}$$

We know we can write  $f = \sum_{i=1}^r a_i \chi_{\rho_i}$ , with :

$$a_i = \langle f, \chi_{\rho_i} \rangle = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\chi_{\rho_i}(g)} = \frac{c(h)\overline{\chi_{\rho_i}(h)}}{|G|} \quad (5.3)$$

By construction: if  $g \notin \text{conjugacy class of } h \Rightarrow f(g) = 0$ :

$$\Rightarrow \sum_{i=1}^r \langle f, \chi_{\rho_i} \rangle \chi_{\rho_i}(g) = 0$$

if  $g \in \text{conjugacy class of } h$ :

$$\Rightarrow \sum_{i=1}^r \langle f, \chi_{\rho_i} \rangle \chi_{\rho_i}(g) = 0$$

now plugin **Equation 5.3** yields the results. ■

**Remark 5.4.2.** We knew that for an abelian group,  $\#$  of irred. repr. is  $|G|$ , thus the  $\#$  of conjugacy classes is also  $|G|$ .

## 5.5 Irreducible Representation of $S_4$

In this section we will give a comprehensive analysis of the irreducible representation of  $S_4$ , as an example, to reveal some technique of finding them. In general, we will introduce induced representation later, which also gives us a powerful way to discover the irreducible representation of a give group by deriving representation from its subgroup. Such will be given in an example of  $D_{2n}$ .

Recall that  $S_4$  have two 1-dim irred. repr, which are the **trivial** and **sign** representation, and no other 1-dim irred. repr. Also recall that there are two 3-dim and one 2-dim irred. repr., we denote them as  $V_1, V_2, V_3$ , so in this section, our goal is to determine  $V_1, V_2, V_3$ .

1.  **$V_1$** : We start with the representation of  $S_4$  on  $\mathbb{C}^4$  with the action of  $S_4$  permuting the entries of a vector space:

$$V_1 = \{(a_1, a_2, a_3, a_4) \mid a_1 + a_2 + a_3 + a_4 = 0\} \subseteq V \text{ is a subrepr.}$$

We want to see if its irreducible. So we look into its character, as it's class function we only look at conjugacy classes of  $S_4$ , in particular its cycle type and count the number of elements in the orbits.

- $\chi_{V_1}(e) = 3$  as its a 3-dim repr. May use basis  $f_1 = e_1 - e_2, f_2 = e_2 - e_3, f_3 = e_3 - e_4$ .
- $\chi_{V_1}((12))$  (**transposition**):

$$f_1 \mapsto -f_1$$

$$f_2 \mapsto f_1 + f_2 \Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \chi_{V_1}((12)) = 1$$

$$f_3 \mapsto f_3$$

There are 6 of such elements.

- $\chi_{V_1}((12)(34))$  **prod. of disjoint cycles**:

$$f_1 \mapsto -f_1$$

$$f_2 \mapsto f_1 + f_2 + f_3 \Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \chi_{V_1}((12)(34)) = -1$$

$$f_3 \mapsto -f_3$$

There are 3 of such elements.

- $\chi_{V_1}((123))$  **3-cycles**:

$$f_1 \mapsto f_2$$

$$f_2 \mapsto -f_1 - f_2 \Rightarrow \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \chi_{V_1}((123)) = 0$$

$$f_3 \mapsto f_1 + f_2 + f_3$$

There are 8 of such elements.

- $\chi_{V_1}((1234))$  **4-cycles**:

$$f_1 \mapsto f_2$$

$$f_2 \mapsto f_3 \Rightarrow \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \chi_{V_1}((1234)) = -1$$

$$f_3 \mapsto -(f_1 + f_2 + f_3)$$

There are 6 of such elements.



$$\langle \chi_{V_1}, \chi_{V_1} \rangle = \frac{1}{24}(3^2 + 6 \cdot 1^2 + 3 \cdot (-1)^2 + 8 \cdot 0^2 + 6 \cdot (-1)^2) = 1$$

Indeed irreducible.

2.  $\mathbf{V}_2$ : Now  $\rho_1 : S_4 \rightarrow GL(V_1)$ , let  $\rho_2 = \rho_1 \otimes \rho_{\text{sign}}$  whose corresponding vector space  $V_2 = V_1$ , with  $\rho_2(\sigma) = \epsilon(\sigma)\rho_1(\sigma)$ . Thus:

$$\chi_{\rho_2}(\sigma) = \epsilon(\sigma)\chi_{\rho_1}(\sigma) \quad \forall \sigma \in S_4$$

Take  $\sigma$  to be an odd permutation, so  $\chi_{\rho_1}(\sigma) \neq 0 \Rightarrow \chi_{\rho_2}(\sigma) \neq \chi_{\rho_1}(\sigma) \Rightarrow \rho_1 \not\cong \rho_2$ .

$$\langle \chi_{\rho_2}, \chi_{\rho_2} \rangle = \langle \chi_{\rho_1}, \chi_{\rho_1} \rangle = 1 \Rightarrow \rho_2 \text{ is also irreducible.}$$

3.  $\mathbf{V}_3$ : To get the 2-dim irred. repr. Look into the Klein subgroup:

$$K = \{e, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$$

We've seen its normality before. See that  $|S_4 / K| = 6$ , with  $a = (12) \in S_4$ ,  $b = (123) \in S_4$  and their images  $\bar{a}, \bar{b} \in S_4 / K$ . Since  $a^2 = b^3 = e \Rightarrow \bar{a}^2 = \bar{b}^3 = e \cdot K$ ,  $a, b \notin K \Rightarrow |\bar{a}| = 2$ ,  $|\bar{b}| = 3$ .

See that  $ab = (12)(123) = (23) \Rightarrow abab = e \Rightarrow \bar{a}\bar{b}\bar{a}\bar{b} = e \cdot K \Rightarrow \bar{b}\bar{a} = \bar{a}\bar{b}^2 \Rightarrow S_4 / K$  is not abelian, thus  $S_4 / K \cong S_3$ .

We know we have a 2-dim irred. repr. for  $S_3$ :

$$\rho' : S_3 \rightarrow GL(V_3)$$

Let  $\rho_3$  be the corresponding composition:

$$\begin{array}{ccccc} S_4 & \xrightarrow{\text{surj}} & S_3 & \xrightarrow{\rho'} & GL(V_3) \\ \downarrow \pi & & \nearrow \cong & & \\ S_4 / K & & & & \end{array}$$

$\rho_3$

We claim  $\rho_3$  is irreducible since  $\rho'$  is irreducible. This is given by the fact that  $S_4 \rightarrow S_3$  is surjective, so the  $S_3$ -subrepr of  $V_3$  are the same as  $S_4$ -subrepr, namely their images the same. The fact that the surjection is given by  $\pi$  is **important** as it **averaging over the whole group** so that  $\langle \chi_{\rho_3}, \chi_{\rho_3} \rangle = \langle \chi_{\rho'}, \chi_{\rho'} \rangle = 1$ .

Conclusion: We've find **all** irreducible representation of  $S_4$ .

# Appendix