Math493: Honors Algebra I

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Abstract

This course is **a basic introduction on finite group theory** and **representation theory**, containing my personal thoughts as well as lecture notes. My course instructor is Mircea Mustaţă.

Contents

Chapter 1

Group Actions

1.1 Introduction

We now lay our focus to group actions, group actions are useful because we can endowed the **symmetric structure** of a group into other mathematical objects through group actions, specifically:

- often groups acts on various mathematical structure, such as sets, topological spaces, manifolds, etc.
- It will be of great significance for us to consider the actions of a group on itself via **conjugation**.

Definition 1.1.1. Let's fix a group G and a set X, an **action** (say also a left action of) G on X is a map:

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx \end{aligned}$$

such that the following holds:

$$ex = x \quad \forall\, x \in X$$

$$g(hx) = (gh)x \quad \forall\, g, h \in G,\ x \in X$$

We now introduce an **equivalent formulation** for group action:
Recall that:

$$S_X = (\{\text{bijections } X \to X\}, \circ)$$

is a group.

Definition 1.1.2. Now suppose we have the action of G on X as above, we may define a map $\varphi : G \to S_X$ as follows: for every $g \in G$, $\varphi(g)$ which written as φ_g is the map:

$$\varphi_g : X \to X, \varphi_g(x) = gx$$

It is easy to see that by inheritance of the existance of inverses in G, φ_g is a bijection. In particular, one can see that it is actually a **group homomorphism**.

And the following conclusion is easy to deduce:

Conclusion 1.1.1.

$$\{\textbf{Actions of } G \textbf{ on } X\} \leftrightarrow \{\textbf{Group Homomorphism } G \to S_X\}$$

forms a **bijection**.

We then give some examples of group actions:

Example. Given any set X, we have the identity, **trivial** group action given by the group homomorphism:

$$S_X \xrightarrow{Id} S_X$$

which is equivalent to the action of S_X on X by:

$$S_X \times X \to X \ , \ (f, x) \mapsto f(x)$$

Example. If $n > 3$ and P_n be the regular n-gon, we then have a group homomorphism:

$$D_{2n} \to S_{P_n}$$

which leads to an action of D_{2n} on P_n

Note. See that in this case D_{2n} preserve the distance structure within the regular n-gon.

Example. The group $GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$ via:

$$(A, u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}) \mapsto Au$$

which represent the matrix as **linear transformation**. Such corresponds to the group homomorphism:

$$GL_n(\mathbb{C}) \to S_{\mathbb{C}^n}$$
$$A \mapsto \text{corresponds linear transformation on } \mathbb{C}^n$$

Example. (Cayley's Theorem) Define an action of G on itself by:

$$G \times G \to G$$
$$(g, h) \mapsto g \cdot h$$

which acts by the natural left multiplication. Note such corresponds to a group homomorphism:

$$G \xrightarrow{\varphi} S_G$$

And we shall have:

Proposition 1.1.1. (Cayley) φ is always injective

In particular, if G is finite, G is then **isomorphic** to a subgroup of S_n.

$$G \cong \text{Im}(\varphi) \subseteq S_G$$

The proof is immediate by showing $\ker(\varphi) = \{e\}$ by cancellation.

Example. Suppose $H \leq G$, we have:

$$G \times (G/H)_l \to (G/H)_l$$
$$(g, ah) \mapsto gaH$$

easy to see such is a group action after checking well-definedness. Such action is induced by the action of group on itself, note H here is **not necessarily normal**.

Example. (Group action by **Conjugation**) The following will be the most interesting example for us. First recall we have an **automorphism** given by $g \in G$:

$$\alpha_g : G \to G \ , \ \alpha_g(x) = gxg^{-1}$$

Moreover, observe $Aut(G) \leq S_G$, so we have a group homomorphism:

$$\begin{aligned} G &\to Aut(G) \leq S_G \\ g &\mapsto \alpha_g \end{aligned}$$

We can understand $Aut(G)$ as those **permutation that preserve the group structure**. In particular, by our discussion, we get an action of G on itself:

$$(g, x) \mapsto gxg^{-1}$$

1.2 Orbits and Orbits-Stablizer Theorem

Definition 1.2.1. Write $x \sim y$ for $x, y \in X$, if $\exists\, g \in G$, s.t. $gx = y$.

Lemma 1.2.1. Such gives us a equivalent relation, directly check by **reflexive, symmetric, transitive**.

Conclusion 1.2.1. We get a partition of X into equivalence classes, called **orbits**. If $x \in X$, then the corresponding equivalence classes is given by:

$$\{gx \mid g \in G\}$$

which is denoted by **Gx** or **O(x)**.

Notation. X/G denotes the sets of the orbits of X.

Definition 1.2.2. The action of G on X is transitive if X has only one orbits, which is:

$$\forall\, x, y \in X, \ \exists\, g \in G \ s.t. \ gx = y$$

Example. The action given by the left multiplication of G on itself is **transitive**.

Example. Induced by above example, the action of G on the **set of left cosets** of H is also transitive.

Definition 1.2.3. For every $x \in X$, the stablizer of $x \in G$ is given by:

$$Stab_G(x) = \{g \in G \mid gx = x\}$$

namely those elements in G that doesn't move the position of x.

Lemma 1.2.2. $Stab_G(x) \leq G$ being a subgroup.

Example. Consider the action of G on itself by conjugation, the orbits of $a \in G$ is called the conjugate class of a. Two elements of G are conjugate of each other if they lie in the same conjugate class (**same orbit**).

What is the stablizer in this case?

$$Stab_G(x) = \{y \in G \mid yxy^{-1} = x\} =: C_G(x)$$

which is the centralizer of x in G.

Note. $C_G(x) = G$ iff $x \in Z(G)$

Remark. Consider the conjugacy classes of S_n, then $\sigma, \tau \in S_n$ are conjugate of each other if and only if when they written as **product of disjoint cycles**, then $\#$ of k-cycle for both of them is the **same** for all k (**they have same cycle type**).

Appendix