

Math493: Honors Algebra I

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Abstract

This course is **a basic introduction on finite group theory** and **representation theory**, containing my personal thoughts as well as lecture notes. My course instructor is [Mircea Mustață](#).

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Chapter 1

Group Actions

1.1 Introduction

We now lay our focus to group actions, group actions are useful because we can endowed the **symmetric structure** of a group into other mathematical objects through group actions, specifically:

- often groups acts on various mathematical structure, such as sets, topological spaces, manifolds, etc.
- It will be of great significance for us to consider the actions of a group on itself via **conjugation**.

Definition 1.1.1. Let's fix a group G and a set X , an **action** (say also a left action of G) on X is a map:

$$G \times X \rightarrow X$$

$$(g, x) \mapsto gx$$

such that the following holds:

$$ex = x \quad \forall x \in X$$

$$g(hx) = (gh)x \quad \forall g, h \in G, x \in X$$

We now introduce an **equivalent formulation** for group action:

Recall that:

$$S_X = (\{\text{bijections } X \rightarrow X\}, \circ)$$

is a group.

Definition 1.1.2. Now suppose we have the action of G on X as above, we may define a map $\varphi : G \rightarrow S_X$ as follows: for every $g \in G$, $\varphi(g)$ which written as φ_g is the map:

$$\varphi_g : X \rightarrow X, \varphi_g(x) = gx$$

It is easy to see that by inheritance of the existence of inverses in G , φ_g is a bijection. In particular, one can see that it is actually a **group homomorphism**.

And the following conclusion is easy to deduce:

Conclusion 1.1.1.

$$\{\text{Actions of } G \text{ on } X\} \leftrightarrow \{\text{Group Homomorphism } G \rightarrow S_X\}$$

forms a **bijection**.

We then give some examples of group actions:

Example. Given any set X , we have the identity, **trivial** group action given by the group homomor-

phism:

$$S_X \xrightarrow{Id} S_X$$

which is equivalent to the action of S_X on X by:

$$S_X \times X \rightarrow X, (f, x) \mapsto f(x)$$

Example. If $n > 3$ and P_n be the regular n -gon, we then have a group homomorphism:

$$D_{2n} \rightarrow S_{P_n}$$

which leads to an action of D_{2n} on P_n

Note. See that in this case D_{2n} preserve the distance structure within the regular n -gon.

Example. The group $GL_n(\mathbb{C})$ acts on \mathbb{C}^n via:

$$(A, u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}) \mapsto Au$$

which represent the matrix as **linear transformation**. Such corresponds to the group homomorphism:

$$\begin{aligned} GL_n(\mathbb{C}) &\rightarrow S_{\mathbb{C}^n} \\ A &\mapsto \text{corresponds linear transformation on } \mathbb{C}^n \end{aligned}$$

Example. (Cayley's Theorem) Define an action of G on itself by:

$$\begin{aligned} G \times G &\rightarrow G \\ (g, h) &\mapsto g \cdot h \end{aligned}$$

which acts by the natural left multiplication. Note such corresponds to a group homomorphism:

$$G \xrightarrow{\varphi} S_G$$

And we shall have:

Proposition 1.1.1. (Cayley) φ is always injective

In particular, if G is finite, G is then **isomorphic** to a subgroup of S_n .

$$G \cong \text{Im}(\varphi) \subseteq S_G$$

The proof is immediate by showing $\ker(\varphi) = \{e\}$ by cancellation.

Example. Suppose $H \leq G$, we have:

$$\begin{aligned} G \times (G/H)_I &\rightarrow (G/H)_I \\ (g, ah) &\mapsto gaH \end{aligned}$$

easy to see such is a group action after checking well-definedness. Such action is induced by the action of group on itself, note H here is **not necessarily normal**.

Example. (Group action by **Conjugation**) The following will be the most interesting example for us. First recall we have an **automorphism** given by $g \in G$:

$$\alpha_g : G \rightarrow G, \quad \alpha_g(x) = gxg^{-1}$$

Moreover, observe $\text{Aut}(G) \leq S_G$, so we have a group homomorphism:

$$\begin{aligned} G &\rightarrow \text{Aut}(G) \leq S_G \\ g &\mapsto \alpha_g \end{aligned}$$

We can understand $\text{Aut}(G)$ as those **permutation that preserve the group structure**. In particular, by our discussion, we get an action of G on itself:

$$(g, x) \mapsto gxg^{-1}$$

Appendix