

# Math494: Honors Algebra II

Yanzhi Li

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### **Abstract**

This is the note containing my personal thoughts as well as lecture notes. My course instructor is Prof. **Mircea Immanuel Mustață**.

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# Chapter 1

## Ring Theory

We've learnt about group theories which represents the symmetry for objects, which is kind of abstract. Rings are groups with extra structures, it is naturally more complicated, however it is closer to our intuition due to the same reason.

### 1.1 Ring and Ring Homomorphism

**Definition 1.1.1 (Ring).** A Ring is a tuple  $(R, +, \cdot)$  being a set  $R$  endowed with 2 binary operations  $(+)$  and  $(\cdot)$ , s.t.:

1.  $(R, +)$  is an **abelian** group, with identity element  $0_R$  or  $0$ .
2.  $(\cdot)$  is associative, and has an identity element  $1_R$  or  $1$  (any element multiply with it will be itself).
3. It satisfy distributivity:
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c), \forall a, b, c \in R.$
  - $(b + c) \cdot a = (b \cdot a) + (c \cdot a), \forall a, b, c \in R.$

#### Notation.

1. Usually write  $ab$  for  $a \cdot b$ .
2. If we don't use parentheses, the order of operations is First  $(\cdot)$  then  $(+)$ .
3. If  $(+), (\cdot)$  are understood, simply denote the ring by  $R$ .
4. Write  $na$  for  $a \in R$  and  $n \in \mathbb{Z}$  for addition for multiple times.
5. Write  $a^n$  for  $a \in R$  and  $n \in \mathbb{Z}_{\geq 0}$  for multiplication for multiple times.

#### Remark 1.1.1.

1. As always, with identity elements  $0_R, 1_R$  are unique.
2. For every  $a \in R$ , we have a unique inverse w.r.t  $(+)$ , denoted by  $-a$ .
3. In general, don't require  $xy = yx \forall x, y \in R$ , if this is the case, then  $R$  is a commutative ring.
4. Sometimes the definition of a ring does not require existence of  $1_R$ , then when there is an identity it is called as unitary ring.

**Example 1.1.1.**

1.  $\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}$  are rings w.r.t.  $(+), (\cdot)$ .
2. If  $n \in \mathbb{Z}_{>0}$ , then  $\mathbb{Z} / n\mathbb{Z}$  carries two operations:

$$\begin{aligned}[a] + [b] &:= [a + b] \\ [a] \cdot [b] &:= [ab]\end{aligned}$$

where  $[a] := a + n\mathbb{Z}$ , this is well-defined since operations holds regardless of the choice of representatives This is a ring with  $0_{\mathbb{Z}/n\mathbb{Z}} = [0]$  and  $1_{\mathbb{Z}/n\mathbb{Z}} = [1]$ .

3. Let  $R$  be any ring, then:

$$M_n(R) := \{A = (a_{ij})_{1 \leq i, j \leq n} \mid a_{ij} \in R \forall i, j\}$$

with “usual” addition and mult. for matrices:

$$\begin{aligned}(a_{ij}) + (b_{ij}) &:= (a_{ij} + b_{ij}) \\ (a_{ij}) \cdot (b_{ij}) &:= (c_{ij}) \rightsquigarrow c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}\end{aligned}$$

then  $(M_n(R), +, \cdot)$  is a ring with w.r.t.  $1_{M_n(R)} = \begin{pmatrix} 1_R & & 0_R \\ & \ddots & \\ 0_R & & 1_R \end{pmatrix}$ .

**Note.** If  $n \geq 2$ , even if  $R$  is commutative,  $M_n(R)$  is not commutative in general.

4. Given a family  $(R_i)_{i \in I}$  of rings, where  $I$  may not be finite, define the following by **Cartesian Prod.:**

$$\prod_{i \in I} R_i := \{(a_i)_{i \in I} \mid a_i \in R_i \forall i\}$$

define the operations **componentwise:**

$$\begin{aligned}(a_i)_{i \in I} + (b_i)_{i \in I} &:= (a_i + b_i)_{i \in I} \\ (a_i)_{i \in I} \cdot (b_i)_{i \in I} &:= (a_i \cdot b_i)_{i \in I}\end{aligned}$$

with  $0 = (0_{R_i})_{i \in I}$  and  $1 = (1_{R_i})_{i \in I}$ . If  $I = [n]$ , simply write:  $R_1 \times \cdots \times R_n$ .

**Proposition 1.1.1.** If  $R$  is a ring and  $a, b \in R$ , then:

1.  $a \cdot 0_R = 0_R = 0_R \cdot a$ .
2.  $-(ab) = (-a) \cdot b = a \cdot (-b)$ .

The proof follows quickly from distributivity and the fact that  $(R, +)$  is an abelian group.

**Note.** If  $R$  is a set with 1 element  $\star$ , then we can make it into a ring in a unique way, namely:

$$0_R = 1_R = \star$$

If  $R$  is a ring, then the following are equiv.:

1.  $\#R = 1$ .
2.  $R = \{0_R\}$ .
3.  $1_R = 0_R$ .

proof is also trivial.

**Definition 1.1.2 (Ring Homomorphism).** Let  $R, S$  be two rings, the ring homomorphism is a map  $f : R \rightarrow S$ , such that:

1.  $f(a + b) = f(a) + f(b) \forall a, b \in R$ .
2.  $f(a \cdot b) = f(a) \cdot f(b) \forall a, b \in R$ .
3.  $f(1_R) = 1_S$ .

**Remark 1.1.2.**

1. If  $f : R \rightarrow S$  is a ring homo., then  $f : (R, +) \rightarrow (S, +)$  is a group homomorphism, with  $f(0_R) = 0_S$ ,  $f(a - b) = f(a) - f(b) \forall a, b \in R$ .
2. However, in def of ring hom. condition 3 **does not** implied by 1 and 2.

**Example 1.1.2.** If  $R = \{0_R\}$ , then the only map  $f : R \rightarrow S$  that satisfies 1 and 2 in definition of ring homo. will satisfy:

$$f(0_R) = 0_S$$

however, this does not satisfy condition 3 if  $S \neq \{0_S\}$ .

**Remark 1.1.3.** In homework, we shall see if  $f : R \rightarrow S$ ,  $g : S \rightarrow T$  are ring homomorphisms, then  $g \circ f : R \rightarrow T$  is again a ring homomorphisms. In particular we have a **category** Rings:

- Objects: rings.
- Morphisms: ring homomorphisms.
- composition: usual function composition.

**Definition 1.1.3 (Ring Isomorphism).** If  $R, S$  are rings, a ring isomorphism  $R \rightarrow S$  is a ring homomorphism  $f : R \rightarrow S$ , s.t.  $\exists g : S \rightarrow R$  to be ring homomorphism, s.t.  $g \circ f = \text{Id}_R$ ,  $f \circ g = \text{Id}_S$ .

Such is equivalent that  $f : R \rightarrow S$  is an isomorphism in the category of Rings.

We say  $R$  and  $S$  are isomorphic, write  $R \cong S$  if  $\exists$  ring isomorphism  $R \rightarrow S$ .

**Proposition 1.1.2.** A ring isomorphism  $f : R \rightarrow S$  is an isomorphism if and only if  $f$  is bijective.

**Proof.** The only if part is trivial, consider the if part. We know  $f$  is homomorphism and bijection, we need to see  $f^{-1}$  is still a ring homomorphism. We already have corresponding results for group isomorphism for  $(R, +)$ :

$$f^{-1}(a + b) = f^{-1}(a) + f^{-1}(b)$$

Since  $f(1_R) = 1_S \Rightarrow f^{-1}(1_S) = 1_R$ . Remains to show:  $f^{-1}(ab) = f^{-1}(a)f^{-1}(b) \forall a, b \in S$ . Since  $f$  is injective, it is enough to show:

$$\underbrace{f(f^{-1}(ab))}_{=ab} = \underbrace{f(f^{-1}(a) \cdot f^{-1}(b))}_{=f(f^{-1}(a)) \cdot f(f^{-1}(b))=ab}$$

■

**Example 1.1.3 (Chinese Remainder Theorem).** Suppose  $m, n \in \mathbb{Z}_{>0}$  be relative primes:

$$f : \mathbb{Z} / mn\mathbb{Z} \rightarrow \mathbb{Z} / m\mathbb{Z} \times \mathbb{Z} / n\mathbb{Z}$$

$$[a]_{mn} \rightarrow ([a]_m, [a]_n)$$

easily seen before that such is well-defined and being a ring homomorphism in homework. In particular,  $\gcd(m, n) = 1 \Rightarrow \ker(f) = \{0\} \Rightarrow f$  is injective, thus  $\#LHS = mn = \#RHS$  thus it is surjective. We thus obtain a ring **isomorphism**.

## 1.2 Subrings and Ideals

We consider the subobjects of ring in this section. In particular, note that sometimes people define rings by unitary ring, in such case Ideals are **unitary ring**.

**Definition 1.2.1 (Subring).** Let  $R$  be a ring. A Subring of  $R$  is a subset  $S$ , s.t.  $(+), (\cdot)$  in  $R$  induce operations on  $S$  that make  $S$  a ring with unit  $1_R$ .

**Remark 1.2.1.** Definition of subrings implies:

1.  $\forall a, b \in S$ , we have  $a + b \in S$ .
2.  $\forall a, b \in S$ , we have  $a \cdot b \in S$ .
3. With respect to these operations,  $S$  is a ring with unit  $1_R$ .

**Proposition 1.2.1.** If  $R$  is a ring, a subset  $S \subseteq R$  is a subring if and only if:

1.  $a - b \in S, \forall a, b \in S$ .
2.  $ab \in S, \forall a, b \in S$ .
3.  $1_R \in S$ .

**Proof.** Only left to proof if 1,2,3 holds, then  $S$  is a ring with unit  $1_R$  w.r.t. the induced operations.

- $S$  is a subgroup w.r.t.  $(+)$ : By 3,  $S \neq \emptyset$ , hence by 1,  $S$  is a subgroup.  $R$  is abelian thus  $S$  is also abelian.
- $1_R \in S$ , this is the identity w.r.t. also in  $S$ .
- Associativity of  $(\cdot)$  and distributivity also holds in  $S$  because they hold in  $R$ .

■

**Example 1.2.1.**

1.  $\mathbb{Z} \subseteq \mathbb{Q}, \mathbb{Q} \subseteq \mathbb{R}, \mathbb{R} \subseteq \mathbb{C}$  are all subrings.

2.  $\{\text{even numbers}\} \subseteq \mathbb{Z}$  is not a subring since it doesn't contain 1.

**Proposition 1.2.2.** If  $f : R \rightarrow S$  is ring homomorphism, then  $\text{Im}(f) \subseteq S$  is a subring.

The proof is straightforward. With side note that  $f(1_R) = 1_S \in \text{Im}(f)$ .

**Definition 1.2.2 (Ideal).** Suppose  $R$  be a ring and  $I \subseteq R$  and  $I \neq \emptyset$ . Then

1.  $I$  is a left ideal (preserve multiplication on the **left**) if:

- $a + b \in I \forall a, b \in I$ .
- $\forall a \in R, b \in I \Rightarrow ab \in I$ .

2.  $I$  is a right ideal (preserve multiplication on the **right**) if:

- $a + b \in I \forall a, b \in I$ .
- $\forall a \in I, b \in R \Rightarrow ab \in I$ .

3.  $I$  is a two-sided ideal if it is both left and right ideal.

If  $R$  is **commutative**, then all the above definition coincide, so we simply say ideal in this case.

**Remark 1.2.2.**

1. Every (left/right) ideal is a subgroup.

- $I \neq \emptyset \Rightarrow \exists a \in I \Rightarrow 0a = 0 \in I$ .
- $\forall a \in I \Rightarrow -a \in I, -a = (-1)a = a \cdot (-1)$ .

2. If  $I$  is a left (or right) ideal and  $1 \in I$ , then  $I = R$ , since  $\forall a \in R, a = a \cdot 1 \in I$ .

Hence the only subring that is a left or right ideal is  $R$ .

**Example 1.2.2.**

1.  $R$  and  $\{0\}$  are always two-sided ideals in  $R$ .

2. Say  $R$  is **commutative** and  $a \in R$ , let  $(a)$  to be the subset of  $R$  which contain all multiples of  $a$ :

$$(a) := \{ab \mid b \in R\}$$

is an ideal in  $R$ , such ideal are called **Principal Ideals**.

- $(a) \neq \emptyset$  since  $a = a \cdot 1 \in (a)$ .
- $ab_1 + ab_2 = a(b_1 + b_2) \in (a)$ .
- $c \in R, (ab)c = a(bc) \in (a)$

**Proposition 1.2.3.** If  $f : R \rightarrow S$  is a ring homomorphism, then  $\ker(f) := \{a \in R \mid f(a) = 0\}$  is a two-sided ideal of  $R$ .

**Proof.** We know it is a subgroup of  $(R, +)$ , see that:

$$a \in \ker(f) \Rightarrow f(ba) = f(b) \cdot f(a) = f(b) \cdot 0 = 0 \Rightarrow ba \in \ker(f)$$

similarly,  $ab \in \ker(f) \forall b \in R$ . ■

Also note,  $f : R \rightarrow S$  be ring homomorphism, it is injective iff  $\ker(f) = \{0\}$ .



**Proposition 1.2.4.** Let  $R_1, R_2$  be a commutative ring, and let  $R = R_1 \times R_2$ . Then the ideal of  $R$  will take form of:

$$I_1 \times I_2$$

where  $I_1$  and  $I_2$  be some ideal of  $R_1, R_2$  respectively. Moreover, if  $P$  be prime ideal of  $R$ , then it will take form in:

- $R_1 \times P_2$  for some  $P_2$  be prime ideal in  $R_2$ .
- $P_1 \times R_2$  for some  $P_1$  be prime ideal in  $R_1$ .

**Sketch.** Consider the projection map  $\pi$  as the ring homomorphism, and see that the **images** are the ideal. As for the prime ideal case, write out the definition and one should observe that there are only two case for a prime ideal to take form:

- $P_1 \times P_2$ , actually not a prime ideal, one can see by verify the definition.
- $R_1 \times R_2$ , not the prime ideal since it is the whole ring.
- $R_1 \times P_2$ , ok.
- $P_1 \times R_2$ , ok.

■

## 1.3 Quotient Rings

In this section we construct quotient rings. Main heuristics is to follow the construction of quotient groups while maintaining the compatibility with multiplication, in particular, with **ring homomorphism**.

Let  $(R, +, \cdot)$  be a ring, if  $I \subseteq R$  be subgroup, then  $I$  is automatically normal since  $(R, +)$  is abelian. Thus we can construct  $R/I$  as a group:

$$R/I := R/\equiv \text{ mod } I \quad a \equiv b \text{ mod } I \text{ if } a - b \in I$$

Write  $a + I$  or simply  $\bar{a}$  or  $[a]$  for the image of  $a \in R$  in  $R/I$ . The group structure is **defined** s.t.:

$$\begin{aligned} \pi : R &\rightarrow R/I \\ a &\mapsto a + I \end{aligned}$$

is **group homomorphism**. which is:

$$\bar{a} + \bar{b} = \overline{a + b}$$

We then want to see that  $R/I$  to be not just a group, but make it a **ring**, which is:  $\pi$  to be a **ring homomorphism**.

Since  $\ker(\pi) = I$ , for the above to work, we need  $I \subseteq R$  is a **2-sided ideal**. So let's just assume  $I$  is a 2-sided ideal.

Since we want  $\pi$  to be a ring homomorphism, we have to define multiplication on  $R/I$ , which is by the most obvious way:

$$\bar{a} \cdot \bar{b} = \overline{ab}$$

The **key point** here is then to show that it is **well-defined**. And we need: if  $\bar{a} = \bar{a'}, \bar{b} = \bar{b'} \Rightarrow \bar{ab} = \bar{a'b'}$ . We know that  $a - a' \in I, b - b' \in I$  and we want  $ab - a'b' \in I$ , which is:

$$\begin{aligned} ab - a'b' &= (ab - ab') + (ab' - a'b); \\ &= \underbrace{a(b - b')}_{\in I \text{ since left ideal}} + \underbrace{(a - a')b'}_{\in I \text{ since right ideal}} \in I \end{aligned}$$

Once we know that multiplication is well-defined, need the following:

- multiplication is associative.

$$\underbrace{\underbrace{(\overline{ab})\overline{c}}_{=\overline{abc}=(\overline{ab})\overline{c}}}_{\text{by associativity in } R} = \underbrace{\overline{a}(\overline{bc})}_{=\overline{abc}=\overline{a}(\overline{bc})} \quad \forall \overline{a}, \overline{b}, \overline{c} \in R/I$$

- distributivity holds by similar argument as above.
- identity element for multiplication.

$$\begin{aligned} \overline{1}\overline{a} &= \overline{1a} = \overline{a} \\ \overline{a}\overline{1} &= \overline{a1} = \overline{a} \end{aligned}$$

- if  $R$  is commutative, then **so is**  $R/I$ .

The **upshot** is:  $R/I$  is a ring and  $\pi : R \rightarrow R/I$  is a ring homomorphism, note that  $\pi(1_R) = 1_{R/I}$ .

**Proposition 1.3.1 (Universal Property of Quotient Rings).** Suppose  $R, I$  are as before, let  $f : R \rightarrow S$  be a ring homomorphism, s.t.  $I \subseteq \ker(f)$ . There is a **unique** ring homomorphism  $\overline{f} : R/I \rightarrow S$ , s.t. the following diagram is **commutative**:

$$\begin{array}{ccc} R & \xrightarrow{\pi} & R/I \\ f \downarrow & \swarrow \overline{f} & \\ S & & \end{array}$$

which is  $f = \overline{f} \circ \pi$ .

The main idea of the proof is to inherit from our idea for universal property of quotient groups and see that is compatible with ring multiplication.

**Proof.** The condition  $f = \overline{f} \circ \pi \Leftrightarrow \overline{f}(\overline{a}) = f(a) \forall a \in R$ , this implies uniqueness, since  $\pi$  is surjective, as it is explicitly defined for the whole domain  $R/I$ .

By the corresponding results for groups, there exists  $\overline{f} : R/I \rightarrow S$  to be group homomorphism, s.t.  $f = \overline{f} \circ \pi$ . Hence, it is enough to show:

- $\overline{f}(u \cdot v) = \overline{f}(u)\overline{f}(v) \forall u, v \in R/I$ .
- $\overline{f}(1_{R/I}) = 1_S$ .

the second assertion follows directly:

$$\overline{f}(1_{R/I}) = \overline{f}(\pi(1_R)) = f(1_R) = 1_S$$

for the first assertion, write  $u = \overline{a}, v = \overline{b}$  for some  $a, b \in R$ , then:

$$\overline{f}(uv) = \overline{f}(\overline{ab}) = f(ab) = f(a) \cdot f(b) = \overline{f}(\overline{a})\overline{f}(\overline{b}) = \overline{f}(u)\overline{f}(v)$$

■

## 1.4 Isomorphism Theorem

Follow similarly with group, there is also corresponding isomorphism for rings. One should notice that the quotient ring is quite restricted since it requires  $I$  to be a **two-sided ideals**, not either left or right ideal.

**Theorem 1.4.1 (Fundamental Isomorphism Theorem).** If  $f : R \rightarrow S$  is a **surjective** ring homomorphism, and  $I = \ker(f) \Rightarrow S \cong R/I$ .

**Note.** Note that the theorem implies that  $\text{Im}(f) \cong R/I$ .

**Remark 1.4.1.** If  $f$  be arbitrary ring homomorphism, then  $\text{Im}(f) \subseteq S$  is a subring.

**Proof.** Since  $I = \ker(f) \Rightarrow I$  is a two-sided ideal. Apply the universal property of  $R/I$ , the following diagram is commutative:

$$\begin{array}{ccc} R & \xrightarrow{\pi} & R/I \\ f \downarrow & \swarrow \bar{f} & \\ S & & \end{array}$$

there exists a unique ring homomorphism  $\bar{f} : R/I \rightarrow S$ , s.t.  $\bar{f} \circ \pi = f$ . In the context of group, we've shown that  $\bar{f}$  is a group isomorphism, so  $\bar{f}$  is bijective, thus it is a ring isomorphism. ■

We then consider the analog of the third isomorphism for groups. We want to describe the left/right/two-sided ideals of  $R/I$  in terms of the ones for  $R$ , and in fact we have the following proposition.

**Proposition 1.4.1.** We have an order preserving bijection:

$$\left\{ \begin{array}{c} \text{left/right/2-sided} \\ \text{ideals in } R/I \end{array} \right\} \xrightleftharpoons[\pi(I') \leftarrow I' \supseteq I]{J \mapsto \pi^{-1}(J)} \left\{ \begin{array}{c} \text{left/right/2-sided} \\ \text{ideals of } R \text{ containing } I \end{array} \right\}$$

where

$$\pi : R \rightarrow R/I$$

**Proof.** We have already seen these two maps given **mutual inverses** for corresponding in groups, to conclude, we only need to show:

- $J \subseteq R/I$  is a left/right/two-sided ideal, then so is  $\pi^{-1}(J) \subseteq R$ .
- $I' \subseteq R$  is a left/right/two-sided ideal, then so is  $\pi(I')$ .

It will be proved in homework. ■

**Notation.** if  $I' \subseteq I$  be ideal, we denote  $\pi(I')$  by  $I'/I$ .

**Theorem 1.4.2 (Third Isomorphism Theorem).** If  $R$  is a ring and  $I \subseteq I'$  are two-sided ideals, then:

$$R/I/I'/I \cong R/I'$$

Note that quotient rings doesn't make sense when  $I$  is left ideal or right ideal.

**Proof.** By the universal property of  $R/I$  for  $R \xrightarrow{p} R/I'$ , there exists a unique  $\bar{p} : R/I \rightarrow R/I'$ , s.t.  $\bar{p}(a+I) = a+I' \forall a \in R$ .

Easy to see that  $\bar{p}$  is surjective and  $\ker(\bar{p}) = I'/I$  as we've concluded in context of groups, then by the fundamental isomorphism theorem, yields the result. ■

**Example 1.4.1.** Let  $n \in \mathbb{Z}_{>0}$ , in  $\mathbb{Z}$ , we have ideal:

$$(n) := \{nk \mid k \in \mathbb{Z}\}$$

then  $\mathbb{Z}/(n)$  is exactly  $\mathbb{Z}/n\mathbb{Z}$ .  $d \in \mathbb{Z}_{>0}$ ,  $(d) \supseteq (n) \Leftrightarrow d|n$ . We have an ideal:

$$(\bar{d}) := \{\bar{d}a \mid a \in \mathbb{Z}/n\mathbb{Z}\} = (d)/(n)$$

and the theorem implies:

$$\mathbb{Z}/n\mathbb{Z}/(\bar{d}) \cong \mathbb{Z}/d\mathbb{Z}$$

## 1.5 Polynomial Ring and Formal Power Series Ring

In this section we define two important examples of commutative ring derived from a given commutative ring, namely the polynomial rings and formal power series ring. They are recursively define so one shall first define them for one variable.

**Definition 1.5.1.** Fix  $R$  to be a **commutative ring**, define:

$$R[X] := \{a_0 + a_1x + \cdots + a_nx^n \mid n \in \mathbb{Z}_{\geq 0}, a_0, \dots, a_n \in R\}$$

note that  $x$  which is the variable here is to help track how ring multiplication is defined.

Define the operations as:

1.  $\sum_{i=0}^n a_ix^i + \sum_{i=0}^n b_ix^i := \sum_{i=0}^n (a_i + b_i)x^i$ .
2.  $(\sum_{i=0}^n a_ix^i) \cdot (\sum_{j=0}^m b_jx^j) := \sum_{k=0}^{m+n} (\sum_{i+j=k} a_ib_j)x^k$ .

See that  $(R[X], +, \cdot)$  is a **commutative ring**, with

- zero element: 0, all coefficients being 0.
- unit element: 1, all coefficients of  $x^i$ ,  $i \geq 1$  are 0.

One shall see that we have a **injective** ring homomorphism:

$$\begin{aligned} R &\xrightarrow{i} R[X] \\ a &\mapsto a \end{aligned}$$

which yields the universal property of  $R[X]$ .

**Theorem 1.5.1 (Universal Property of  $R[X]$ ).** For every ring homomorphism  $\varphi : R \rightarrow S$  with  $R, S$  commutative and for every  $a \in S$ , there is a **unique** ring homomorphism  $\psi : R[X] \rightarrow S$ , s.t.

1. The following diagram is commutative:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow i & \searrow \psi & \\ R[X] & & \end{array}$$

i.e.  $\psi(b) = \varphi(b) \forall b \in R$ .

2.  $\psi(x) = a$ .

**Proof.** Suppose we have such  $\psi : \psi(x^i) = a^i \forall i > 0$ , then  $\psi \circ i = \varphi \Rightarrow$  if  $P = \alpha_0 + \alpha_1x + \cdots + \alpha_nx^n \Rightarrow$

$$\psi(P) = \underbrace{\varphi(\alpha_0) + \varphi(\alpha_1)a + \cdots + \varphi(\alpha_n)a^n}_{\text{denoted by } P(a)}$$

this is explicitly defined, yields uniqueness.

For existence, we use this formula to define  $\psi : R[X] \rightarrow S$  explicitly, thus property 1 and 2 is clear, only left to check that  $\psi$  is actually a ring homomorphism:

- $\psi(P + Q) = \psi(P) + \psi(Q)$  is straightforward.
- $\psi(1) = 1$  is also straightforward.
- $\psi(PQ) = \psi(P)\psi(Q) \forall P, Q$ . Suppose that  $P = \sum_{i=0}^n \alpha_i x^i$ ,  $Q = \sum_{j=0}^m \beta_j x^j$ , then:

$$\begin{aligned}
 PQ &= \sum_{k=0}^{m+n} \left( \sum_{i+j=k} \alpha_i \beta_j \right) x^k \\
 \Rightarrow \psi(PQ) &= \sum_{k=0}^{m+n} \varphi \left( \sum_{i+j=k} \alpha_i \beta_j \right) a^k \\
 &= \sum_{k=0}^{m+n} \left( \sum_{i+j=k} \varphi(\alpha_i) \cdot \varphi(\beta_j) \right) a^k \quad (\varphi \text{ is a ring homomorphism}) \\
 \text{And } \psi(P)\psi(Q) &= \left( \sum_{i=0}^n \varphi(\alpha_i) a^i \right) \cdot \left( \sum_{j=0}^m \varphi(\beta_j) a^j \right) \\
 &= \sum_{i=0}^n \sum_{j=0}^m \varphi(\alpha_i) \varphi(\beta_j) a^i a^j \\
 &= \sum_{k \geq 0} \left( \sum_{i+j=k} \varphi(\alpha_i) \varphi(\beta_j) \right) a^{i+j=k} \quad (R \text{ is commutative}) \\
 &= \psi(PQ)
 \end{aligned}$$

■

One can iterate this since  $R[X]$  is still a commutative ring, and thus get multi-variable polynomial ring over  $R$ , which is defined recursively by:

$$R[X_1, \dots, X_n] := (R[X_1, \dots, X_{n-1}])[X_n]$$

This is again a commutative ring.

**Theorem 1.5.2 (Universal Property of  $R[X_1, \dots, X_n]$ ).**  $\forall$  ring homomorphism  $\varphi : R \rightarrow S$ ,  $R, S$  commutative, and  $\forall a_1, \dots, a_n \in S$ , there exists a **unique** ring homomorphism  $\psi : R[X_1, \dots, X_n] \rightarrow S$ , s.t.

1. the following diagram is commutative:

$$\begin{array}{ccccccc}
 R & \longrightarrow & R[x_1] & \longrightarrow & R[x_1, x_2] & \longrightarrow & \dots \longrightarrow R[x_1, \dots, x_n] \\
 & \searrow \varphi & & & & & \nearrow \psi \\
 & & S & & & & 
 \end{array}$$

2.  $\psi(x_i) = a_i \forall i \in \llbracket 1, n \rrbracket$ .

**Example 1.5.1.**  $X_1^2 + X_1 X_3 + X_2^4 \in R[X_1, X_2, X_3]$

The proof is straightforward by using induction on  $n$  with the previous universal property of  $R[X]$ .

**Example 1.5.2.** If  $\sigma \in S_n \Rightarrow \exists!$  ring homomorphism, s.t. the following diagram is commutative:

$$\begin{array}{ccc} R & \xrightarrow{\quad} & R[X_1, \dots, X_n] \\ \downarrow & \nearrow f_\sigma & \\ R[X_1, \dots, X_n] & & \end{array}$$

and  $f_\sigma(x_i) = X_{\sigma(i)} \quad \forall i$ . In fact this is a ring isomorphism, thus be a automorphism, with inverse being  $f_{\sigma^{-1}}$ . In particular it shows that the process of constructing  $R[X_1, \dots, X_n]$  is just labelling and **doesn't matter with order** of  $X_1, \dots, X_n$ .

**Notation.** Every element of  $R[X_1, \dots, X_n]$  can be written as

$$f = \sum_{u=(u_1, \dots, u_n) \in \mathbb{Z}_{\geq 0}^n} a_u X^u$$

where  $X^u = X_1^{u_1} \cdots X_n^{u_n}$  with  $a_u \in R$  which is a monomial.

**Example 1.5.3.**

$$f(x, y) = 3x^2y + 5xy^2 + 7$$

We then define the ring for formal power series, basically it allows infinite sum in this case.

**Definition 1.5.2 (Ring of Formal Power Series).** Suppose  $R$  a commutative ring, define the ring of formal power series as:

$$R[[X]] := \left\{ \sum_{i \geq 0} a_i x^i \mid a_i \in R, \forall i \geq 0 \right\}$$

with the operations defined:

- addition:  $\sum_{i \geq 0} a_i x^i + \sum_{i \geq 0} b_i x^i := \sum_{i \geq 0} (a_i + b_i) x^i$ .
- multiplication:

$$\left( \sum_{i \geq 0} a_i x^i \right) \cdot \left( \sum_{j \geq 0} b_j x^j \right) := \sum_{k \geq 0} c_k x^k$$

where  $c_k = \sum_{i+j=k} a_i b_j \in R$

See that  $(R[[X]], +, \cdot)$  is again a **commutative ring**, s.t. we have  $R[X] \subseteq R[[X]]$  being a subring.

## 1.5.1 R-Algebra

**Definition 1.5.3 (R-Algebra).** Suppose that  $R$  is a commutative ring, an R-Algebra is a ring  $S$  together with a ring homomorphism  $R \xrightarrow{\varphi} S$ , s.t.  $\varphi(a)b = b\varphi(a) \quad \forall a \in R, b \in S$ .

**Example 1.5.4.**

1.  $R[X], R[[X]]$  have natural structures of  $R$ -Algebras  $R[X_1, \dots, X_n]$ .

2. here is a non-commutative ring example:  $M_n(R)$  with the ring homomorphism defined as:

$$R \rightarrow M_n(R)$$

$$a \mapsto \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}$$

See that we can derive a category of  $R$ -Algebras, with objects being the  $R$ -algebras and the morphisms are given by the ring homomorphism that makes the following diagram commutative:

$$\begin{array}{ccc} R & \longrightarrow & S_1 \\ & \searrow & \downarrow u_1 \\ & & S_2 \end{array}$$

such category is w.r.t. the usual function composition.

## 1.6 Fields and Integral Domain

It will be better to think fields and integral domain as very special ring, as they are already endowed with relatively complex structure, thus they are more closer to our intuition sometimes, and easier to construct examples from  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ .

**Definition 1.6.1 (Invertible).** Fix a ring  $R$ ,  $a \in R$  is invertible if there exists  $b \in R$ , s.t.  $ab = 1_R = ba$ .  $b$  is the inverse of  $a$  and denoted as  $a^{-1}$ .

**Definition 1.6.2 (Field).** A ring  $R$  is a field if;

1.  $R$  is commutative.
2.  $1_R \neq 0_R$ , namely it is not a 0 ring.
3. Every  $a \in R \setminus \{0\}$  is invertible.

**Remark 1.6.1.**

- Note that for  $I$  being an ideal of  $R$ ,  $I = R$  if and only if  $I$  contains an unit of  $R$ .
- This directly shows that  $\mathbb{K}$  being a field if and only if it attains no non-trivial ideal. In particular, see that the ideal structure of a field is **trivial**.
- One result from this is that given any non-zero ring homomorphism  $\varphi : \mathbb{K} \rightarrow R$ , such ring homomorphism is always **injective** as  $\ker(f)$  being a ideal of  $\mathbb{K}$  and thus can only be  $\{0\}$  for a non-zero map.

**Example 1.6.1.**  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$  are fields,  $\mathbb{Z}$  is not a field.

**Definition 1.6.3 (Zero-Divisor).** If  $R$  is a commutative ring,  $a \in R$  is a zero-divisor if  $\exists b \neq 0$  in  $R$ , s.t.  $ab = 0$ . Otherwise, we say  $a$  is a non-zero-divisor.

**Definition 1.6.4 (Integral Domain).** A ring  $R$  is an integral domain or simply a domain if:

1.  $R$  is commutative.
2.  $1_R \neq 0_R$ .

3. Every  $a \neq 0$  is a non-zero-divisor. Or it is equivalent to say:

$$\forall a, b \in R, ab = 0 \Rightarrow a = 0 \text{ or } b = 0$$

**Remark 1.6.2.** If  $R$  is a domain, then we have **cancellation rule w.r.t. multiplication**. Namely if  $ab = bc$ ,  $a, b, c \in R$ ,  $a \neq 0 \Rightarrow b = c$ .

**Proof.**  $a(b - c) = 0 \Rightarrow b - c = 0$ . ■

**Example 1.6.2.** If  $n > 0$ , then  $\mathbb{Z} / n\mathbb{Z}$  is a domain if and only if  $n$  is **prime number**.

**Proof.** Suppose  $\bar{a}, \bar{b} \in \mathbb{Z} / n\mathbb{Z}$ , with  $\bar{a}, \bar{b} \neq 0 \Leftrightarrow a \nmid a, n \nmid b$ , and  $\bar{a}\bar{b} = 0 \Leftrightarrow n \mid ab$ . Now if  $n$  is prime number, then  $n \nmid a, n \nmid b \Rightarrow n \nmid ab$ , hence  $\mathbb{Z} / n\mathbb{Z}$  is a domain.

Now if  $n$  is not a prime, then  $n = n_1 \cdot n_2$  for some  $n_1, n_2 > 1$ , which means  $\bar{n}_1, \bar{n}_2 \neq 0$ , but  $\bar{n}_1 \cdot \bar{n}_2 = 0$  in  $\mathbb{Z} / n\mathbb{Z}$ . ■

**Proposition 1.6.1.** If  $\mathbb{K}$  is a field, then  $\mathbb{K}$  is an integral domain.

**Proof.**  $\mathbb{K}$  is commutative with  $1_{\mathbb{K}} \neq 0_{\mathbb{K}}$ . Suppose that  $a, b \in \mathbb{K}$ ,  $ab = 0$ ,  $a \neq 0$  means that it will attain an inverse by field property, denote it as  $a^{-1}$ . Thus we have:

$$\begin{aligned} b &= (a^{-1}a)b = a^{-1}(ab) = a^{-1}0 = 0 \\ \Rightarrow b &= 0 \end{aligned}$$

■

**Proposition 1.6.2.** If  $R$  is a finite domain, then  $R$  is a field.

**Proof.**  $R$  being a domain means that  $R$  is commutative and  $1_R \neq 0_R$ .

Now fix  $a \in R$ ,  $a \neq 0$ , and consider the function given by:

$$\begin{aligned} f : R &\rightarrow R \\ f(b) &= ab \end{aligned}$$

By cancellation w.r.t. multiplication, since  $a \neq 0$ , this function is thus injective. But  $R$  is finite, meaning  $f$  is also surjective, and thus bijective. So there exists  $b \in R$ , s.t.  $ab = 1 \Rightarrow a$  is invertible, thus being a field. ■

**Example 1.6.3.** If  $n \in \mathbb{Z}_{>0}$ , then  $\mathbb{Z} / n\mathbb{Z}$  is field if and only if  $n$  is prime.

**Remark 1.6.3.** If  $R$  is a domain, then every subring of  $R$  is a domain. In particular, every subring of a field is a domain.

Our goal then now switch to focus on  $R$  being a domain implies that  $R[X]$  is also a domain, for formal power series, the proof is almost the same.

**Definition 1.6.5 (Degree of  $R[X]$ ).** Fix  $R$  to be a commutative ring. If  $f \in R[X]$ ,  $f \neq 0$ , write:

$$f = a_0 + a_1x + \cdots + a_nx^n$$

s.t.  $a_n \neq 0$ , then the degree of  $f$  is  $\deg(f) = n$ . And we follow the convention that  $\deg(0) = -\infty$ .

**Remark 1.6.4.**  $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$



**Proposition 1.6.3.** If  $R$  is a domain, and  $f, g \in R[X]$  are non-zero, we have:

$$\deg(f \cdot g) = \deg(f) + \deg(g)$$

In particular,  $f \cdot g \neq 0$  thus being a domain by contraposition. Note that if it is not a domain, it is not generally true as one can cancel out the highest degree coefficient by product.

**Proof.** Suppose that:

$$\begin{aligned} f &= a_0 + a_1x + \cdots + a_mx^m & a_m \neq 0 & \deg(f) = m \\ g &= b_0 + b_1x + \cdots + b_nx^n & b_n \neq 0 & \deg(g) = n \end{aligned}$$

then:

$$\begin{aligned} fg &= \sum_{k \geq 0} \left( \sum_{i+j=k} a_i b_j \right) x^k \\ &= \underbrace{a_m b_n}_{\neq 0} x^{m+n} + \text{lower degree monomials} \end{aligned}$$

Since  $R$  is a domain, then  $a_m b_n \neq 0 \Rightarrow \deg(f \cdot g) = m + n$ . ■

**Corollary 1.6.1.** If  $n \geq 1$ , then  $R$  is a domain if and only if  $R[X_1, \dots, X_n]$  is a domain.

**Proof.** Arguing by induction on  $n$ , and it is enough to treat  $n = 1$ .  $R$  being a domain implies that  $R[X]$  also be a domain. And if  $R[X]$  being a domain, we have a injective ring homomorphism  $R \hookrightarrow R[X]$ , thus it is a subring of a domain, thus be a domain. ■

## 1.7 Ring Fraction

In this section, we want to construt the ring fraction. Our goal is to show that starting with a domain, we want to have fraction field. More generally, we don't require  $R$  to be a domain, and start with arbitrary "set of denominators", just like from  $\mathbb{Z}$  to get  $\mathbb{Q}$ .

**Definition 1.7.1 (Multiplicative System).** Fix  $R$  be commutative ring,  $S \subseteq R$  be a multiplicative system if:

1.  $1 \in S$ .
2. If  $s_1, s_2 \in S \Rightarrow s_1 \cdot s_2 \in S$ .

We can make an attempt to construct ring fraction:

Consider pairs  $(a, s)$  where  $a \in R, s \in S$ , up to equivalence relation, we want to see:

$$\begin{aligned} (a_1, s_1) \sim (a_2, s_2) &\Leftrightarrow s_2 a_1 = s_1 a_2 \\ \text{denoted as} \quad &\frac{a_1}{s_1} = \frac{a_2}{s_2} \end{aligned}$$

The issue is that in general this is not an equivalence relation, as it will fail **transitivity**: Let

$$\begin{aligned} (a, s) \sim (a', s') \quad (a', s') \sim (a'', s'') \\ \Leftrightarrow s' a = s a' \quad s'' a' = s' a'' \end{aligned}$$

We want to see that  $(a, s) \sim (a'', s'') \Leftrightarrow s'' a = s a''$ . And see that:

$$\textcolor{blue}{s' s''} a = s'' s a' = s s' a'' = \textcolor{red}{s' s} a''$$

And it is not clear that the blue one is equal to the red one by definition. Thus we make some modification to the definition.

**Definition 1.7.2.** Let  $R$  be a commutative ring and  $S \subseteq R$  be a multiplication system. Consider pairs  $a, s$  where  $a \in R, s \in S$ , write  $(a, s) \sim (a', s')$  if there exists  $t \in S$ , s.t.  $t(s'a - sa') = 0$ .  
Note that 0 is not necessarily in  $S$ .

**Claim.** Above definition is a **equivalence relation**.

**Notation.** Write  $\frac{a}{s}$  denote the equivalence class of  $(a, s)$ .

**Proof of Claim.**

- Reflexive and symmetric is straightforward.
- Consider Transitivity:

$$(a, s) \sim (a', s') \quad (a', s') \sim (a'', s'') \\ \Rightarrow t_1(s'a - sa') = 0 \quad t_2(s''a' - s'a'') = 0 \quad \text{for some } t_1, t_2 \in S$$

We now interest in:

$$\begin{aligned} t_1 t_2 s' (s''a - sa'') &= t_2 s'' \underbrace{t_1 (s'a - sa')}_{=0} + \underbrace{t_2 t_1 s'' sa' - t_1 t_2 s' sa''}_{=t_1 s \underbrace{t_2 (s''a' - s'a'')}_{=0}} \\ &= 0 \\ &\Rightarrow (a, s) \sim (a'', s'') \end{aligned}$$

note that  $t_1 t_2 s' \in S$  since  $s', t_1, t_2 \in S$  and  $S$  being a multiplication system.

■

And thus we denote:

$$S^{-1}R := \{(a, s) \mid a \in R, s \in S\}$$

We want to then define the  $(+)$  and  $(\cdot)$  operations on it to make it a ring.

Define:

$$\begin{aligned} \frac{a_1}{s_1} + \frac{a_2}{s_2} &:= \frac{s_2 a_1 + s_1 a_2}{s_1 s_2} \\ \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} &:= \frac{a_1 a_2}{s_1 s_2} \end{aligned}$$

See that it is well-defined: suppose  $\frac{a_1}{s_1} = \frac{b_1}{t_1}$  and  $\frac{a_2}{s_2} = \frac{b_2}{t_2}$ , we want:

$$\frac{s_2 a_1 + s_1 a_2}{s_1 s_2} = \frac{t_2 b_1 + t_1 b_2}{t_1 t_2} \quad (1.1)$$

$$\frac{a_1 a_2}{s_1 s_2} = \frac{b_1 b_2}{t_1 t_2} \quad (1.2)$$

**Proof of Equation 1.1.** By our hypothesis, there exists  $u, v \in S$ , s.t.:

$$\begin{aligned} u(t_1 a_1 - s_1 b_1) &= 0 \\ v(t_2 a_2 - s_2 b_2) &= 0 \end{aligned}$$

Consider:

$$t_1 t_2 (s_2 a_1 + s_1 a_2) - s_1 s_2 (t_2 b_1 + t_1 b_2) = t_2 s_2 (t_1 a_1 - s_1 b_1) + t_1 s_1 (t_2 a_2 - s_2 b_2)$$

If we multiply with  $uv \in S$ , we get 0, which shows that they are in the same equivalence class thus equal. ■

**Proof of Equation 1.2.** Similarly:

$$t_1 t_2 a_1 a_2 - s_1 s_2 b_1 b_2 = t_2 a_2 (t_1 a_1 - s_1 b_1) + s_1 b_1 (t_2 a_2 - s_2 b_2)$$

Multiply  $uv \in S$ , we get 0. ■

**Proposition 1.7.1.** With  $(+)$  and  $(\cdot)$ ,  $S^{-1}R$  is a **commutative ring**. This is the ring of fraction of “ $R$  with denominator in  $S$ ” or the “localization of  $R$  w.r.t.  $S$ ”.

**Sketch of Proof.** It's easy to see that both  $(+)$  and  $(\cdot)$  are commutative.

The 0 element is given by  $\frac{0}{1}$ , see that:

$$\frac{0}{1} + \frac{a}{s} = \frac{0 \cdot s + 1 \cdot a}{1 \cdot s} = \frac{a}{s}$$

and the inverse of  $\frac{a}{s}$  is  $-\frac{a}{s}$ .

The 1 element is given by  $\frac{1}{1}$ .

Associativity of  $(+)$ :

$$\begin{aligned} \left( \frac{a_1}{s_1} + \frac{a_2}{s_2} \right) + \frac{a_3}{s_3} &= \frac{s_2 a_1 + a_2 s_1}{s_1 s_2} + \frac{a_3}{s_3} \\ &= \frac{s_3 s_2 a_1 + s_3 a_2 s_1 + a_3 s_1 s_2}{s_1 s_2 s_3} \\ &= \frac{a_1}{s_1} + \left( \frac{a_2}{s_2} + \frac{a_3}{s_3} \right) \quad \text{by symmetry} \end{aligned}$$

Associativity of  $(\cdot)$  is clear, and distributivity is similar manner. ■

**Remark 1.7.1.**  $S^{-1}R$  has a canonical structure of  $R$ -Algebra with the following canonical ring homomorphism:

$$\begin{aligned} \varphi : R &\rightarrow S^{-1}R \\ \varphi(r) &= \frac{r}{1} \end{aligned}$$

Note that it is not injective in general, we care whether it is injective because we want not to lose information.

**Remark 1.7.2.**  $a \in \ker(\varphi) \Leftrightarrow \frac{a}{1} = \frac{0}{1} \Leftrightarrow \exists s \in S, \text{ s.t. } sa = 0$ .

Hence:  $\varphi$  is not injective if and only if  $\exists s \in S$ , which is a **zero divisor**.

**Remark 1.7.3.**  $S^{-1}R = \{0\}$  iff  $0 \in S$ , it tells us in general we don't care about the case where  $0 \in S$ .

**Example 1.7.1.** Let  $R$  be a integral domain, and  $S = R \setminus \{0\}$ , then  $\varphi : R \rightarrow S^{-1}R$  is injective, see that  $S^{-1}R$  in this case is a field: it is not 0, it is commutative, and if  $\frac{a}{s} \neq 0 (\Leftrightarrow a \neq 0) \Rightarrow$  this has the multiplicative inverse  $\frac{s}{a}$  since:

$$\frac{a}{s} \cdot \frac{s}{a} = \frac{as}{as} = \frac{1}{1}$$

We then give some example on how things are constructed:

1. If  $R = \mathbb{Z} \rightsquigarrow \mathbb{Q}$ .
2. If  $F$  be a field, and  $R = F[X_1, \dots, X_n] \rightsquigarrow$  field of rational function  $F(X_1, \dots, X_n)$  which is quotients of polynomials.

**Note.** In this case, by property of integral domain and property of  $S$  that  $0 \notin S$ ,  $\frac{a_1}{s_1} = \frac{a_2}{s_2}$  if and only if  $s_2 a_1 - s_1 a_2 = 0$ .

**Example 1.7.2.** Let  $f \in R$  and  $S = \{1, f, f^2, \dots\} = \{f^n \mid n \in \mathbb{Z}_{>0}\}$  be a multiplicative system, then  $S^{-1}R$  is denoted by  $R_f$ . There is a **universal property** of  $S^{-1}R$ :

Suppose  $S \subseteq R$  is a multiplicative system and  $\varphi : R \rightarrow S^{-1}R$  is the canonical ring homomorphism, then:

1.  $\forall s \in S$ ,  $\varphi(s)$  is invertible.
2.  $S^{-1}R$  is universal with the following property: if  $R \xrightarrow{\psi} T$  is commute  $R$ -Algebra, s.t.  $\psi(s)$  is invertible  $\forall s \in S$ , then there exists a **unique**  $R$ -Algebra homomorphism  $S^{-1}R \xrightarrow{\tilde{f}} T$ , s.t. the following diagram is commutative;

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S^{-1}R \\ & \searrow \psi & \downarrow \tilde{f} \\ & & T \end{array}$$

**Note.** Proof manner is very similar to what we do for those universal property: We first suppose that it exists, try to prove uniqueness, in such process we may be able to write out the explicit formula of such morphism, so we can then proof the well-definedness and so on to see the existence.

**Proof.**

- $\varphi(s) = \frac{s}{1}$  with inverse  $\frac{1}{s}$ .
- First uniqueness then existence:
  - **Uniqueness:** Suppose  $f : S^{-1}R \rightarrow T$  is a morphism of  $R$ -Algebra, s.t.  $f(\frac{a}{1}) = \psi(a) \forall a \in R$ . Given any  $\frac{a}{s} \in S^{-1}R$ , we have  $\frac{a}{s} \cdot \frac{s}{1} = \frac{a}{1}$ , see that since  $f$  is a ring homomorphism:

$$\begin{aligned} f\left(\frac{a}{s}\right) \cdot \underbrace{f\left(\frac{s}{1}\right)}_{\psi(s)} &= \underbrace{f\left(\frac{a}{1}\right)}_{\psi(a)} \\ \Rightarrow f\left(\frac{a}{s}\right) &= \psi(a)\psi(s)^{-1} \quad (\psi(s) \text{ is invertible by hypothesis.}) \end{aligned}$$

Hence  $f$  is unique, as we have it a formula, and clearly it is unique.

- **Existence:** Define  $f : S^{-1}R \rightarrow T$  by  $f(\frac{a}{s}) = \psi(a)\psi(s)^{-1}$ , need to check the following:
  1.  $f$  is well-defined: Suppose  $\frac{a}{s} = \frac{b}{t}$  then there exists  $u \in S$ , s.t.  $u(ta - sb) = 0$ . Apply  $\psi$  to both sides we get:

$$\psi(u)(\psi(t)\psi(a) - \psi(s)\psi(b)) = 0$$

Multiply by  $\psi(u)^{-1}\psi(s)^{-1}\psi(t)^{-1}$  on both sides:

$$\psi(a)\psi(s)^{-1} - \psi(b)\psi(t)^{-1} = 0$$

Thus definition is unique.

2.  $f \circ \varphi = \psi$ :  $f(\frac{a}{1}) = \psi(a)\psi(1)^{-1} = \psi(a)$ .
3.  $f$  is a ring homomorphism:

$$\begin{aligned} f\left(\frac{a}{s} + \frac{b}{t}\right) &= f\left(\frac{ta + sb}{st}\right) \\ &= \psi(ta + sb)\psi(st)^{-1} \\ &= (\psi(t)\psi(a) + \psi(s)\psi(b))\psi(s)^{-1}\psi(t)^{-1} \\ &= \psi(a)\psi(s)^{-1} + \psi(b)\psi(t)^{-1} \\ &= f\left(\frac{a}{s}\right) + f\left(\frac{b}{t}\right) \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a}{s} \cdot \frac{b}{t}\right) &= f\left(\frac{ab}{st}\right) \\ &= \psi(ab)\psi(st)^{-1} \\ &= \psi(a)\psi(s)^{-1}\psi(b)\psi(t)^{-1} \\ &= f\left(\frac{a}{s}\right) \cdot f\left(\frac{b}{t}\right) \end{aligned}$$

and

$$f(1) = 1$$

■

## 1.8 Prime Ideals and Maximal Ideals

In this section we shall discuss prime ideals and maximal ideals

leave some overview!

## 1.8.1 Prime Ideals

**Definition 1.8.1 (Prime Ideal).** Let  $R$  be a commutative ring, an ideal  $P \subseteq R$  is a prime ideal if:

1.  $P \neq R$ .
2. If  $x, y \in R$  are s.t.  $xy \in P \Rightarrow x \in P$  or  $y \in P$ .

**Parenthesis.** If  $P$  is a prime ideal, then  $S = R - P$  is a **multiplicative system**, in this case  $S^{-1}R$  is denoted as  $R_P$ , which is called local ring.

**Proposition 1.8.1.** An ideal  $P \subseteq R$  is prime ideal if and only if  $R/P$  is an integral domain.

**Proof.** See that  $R/P$  is always commutative.  $R/P \neq \{0\} \Leftrightarrow P \neq R$ .

(Let  $\bar{x}, \bar{y} \neq 0 \in R/P \Rightarrow \bar{x} \cdot \bar{y} \neq 0$ )  $\Leftrightarrow (\forall x, y \in R, x, y \notin P \Rightarrow xy \notin P)$ , which, LHS is definition of integral domain, and RHS is definition of prime ideal. ■

**Example 1.8.1.** If  $R = \mathbb{Z}$ , then

1.  $\{0\}$  is a prime ideal ( $\mathbb{Z}$  is an integral domain).
2. If  $n \in \mathbb{Z}_{>0}$ , then  $(n)$  is a prime ideal if and only if  $\mathbb{Z}/n\mathbb{Z}$  is an integral domain if and only if  $n$  is prime number. Namely  $n\mathbb{Z}$  is prime ideal if and only if  $n$  is prime number.

**Note.** If  $I$  is an ideal in  $R$ , then all ideals in  $R/I$  are of the form  $P/I$  where  $I \subseteq P$  is an ideal. See that by Isomorphism theorem:

$$R/I / P/I \cong R/P$$

Hence  $P/I$  is prime ideal if and only if  $P$  is prime ideal.

**Example 1.8.2.** The following are equivalent:

- $R$  is a domain.
- $(x) := \{xf \mid f \in R[X]\}$  inside  $R[X]$  is a prime ideal.

**Proof.** This follows if we show the following, which gives the result by **Proposition 1.8.1**:

$$R[X] / (x) \cong R$$

Consider the  $R$ -algebra homomorphism:

$$\begin{aligned} R[X] &\xrightarrow{\varphi} R \\ \varphi(x) &= 0 \\ a_0 + a_1x + \cdots + a_nx^n &\rightarrow a_0 \end{aligned}$$

This is a surjective homomorphism, with kernel being:

$$\ker(\varphi) = (x)$$

Then by isomorphism theorem 1.4.1, yields the result. ■

**Question 1.8.1.** What about now consider  $(x - a) \subseteq R[X]$ ?

**Note.** There exists a  $R$ -Algebra isomorphism:

$$\begin{aligned} f : R[x] &\rightarrow R[x] \\ f(x) &= (x - a) \end{aligned}$$

So see that  $(x)$  is prime ideal if and only if  $(x - a)$  is prime ideal, if and only if  $R$  is a domain, thus if and only if  $(x - a)$  is also a prime ideal.

By the universal property of  $R[x]$ : there exists a unique such morphism  $f$  of  $R$ -Algebra, and exists a unique morphism of  $R$ -Algebra  $R[X] \xrightarrow{g} R[X]$ ,  $x \mapsto x + a$ , thus  $g = f^{-1}$ , and we can use the universal property to show that the composition  $f \circ g$  and  $g \circ f$  are identity, which yields isomorphism property.

**Definition 1.8.2 (Coprime Ideal).** Given  $R$  to be a commutative ring, let  $I_1, I_2$  be two ideal s.t.  $I_1 \neq I_2$ , these two ideals are coprime if:

$$I_1 + I_2 = R$$

**Theorem 1.8.1 (Generalized Chinese Remainder Theorem).** Let  $I_1, \dots, I_n$  be ideals in  $R$  such that  $I_i + I_j = R$  for all  $i \neq j$ . Then:

$$R / I_1 \cap \dots \cap I_n \cong \prod_{i=1}^n R / I_i$$

**Sketch.** One should first prove the case that when  $I_1 + I_2 = R$ , then:

$$R / I_1 \cap I_2 \cong R / I_1 \times R / I_2$$

By first isomorphism theorem, then show that:

$$I_1 + (I_2 \cap I_3) = R$$

by noticing that one can have such decomposition:

$$\begin{aligned} 1 &= (x + y) \cdot (u + v) \\ &= xu + yu + xv + yv \\ &\text{where } x, u \in I_1, y \in I_2, v \in I_3 \\ &\Rightarrow xu \in I_1, yu \in I_1 \cap I_2, xv \in I_1 \cap I_3, yv \in I_2 \cap I_3 \\ &\text{with } xu + yu + xv \in I_1, yv \in I_2 \cap I_3 \end{aligned}$$

and proceed on induction on  $n$ . ■

## 1.8.2 Maximal Ideals

**Definition 1.8.3 (Maximal Ideal).** An ideal  $M \subseteq R$  is a maximal ideal if:

1.  $M \neq R$ .
2. If  $M \subseteq I \subseteq R$  and  $I$  be an ideal, then  $I = M$  or  $I = R$ .

Notably, it is not necessarily for a ring to possess a maximal ideal unless  $1_R \neq 0_R$ . One can make such ring by taking abelian groups that possess no maximal subgroup, for example  $\mathbb{Q}$ , and just set the multiplication into the trivial case  $ab = 0 \forall a, b$ . **Theorem 1.8.2** states that when  $1_R \neq 0_R$  there's always maximal ideal. Hence any result related to maximal ideal forces the ring to be non-zero.

**Lemma 1.8.1.** If  $R$  is a commutative ring, then  $R$  is a field if and only if  $\{0\}$  is a maximal ideal.

**Proof.**

- Suppose that  $R$  is a field, then  $R \neq \{0\}$ . If  $I \subseteq R$  is an ideal, and  $I \neq \{0\}$ . Let  $a \in I \setminus \{0\}$ , since  $R$  is a field, see that  $a$  is **invertible**. Then:

$$\forall b \in R, \quad b = (ba^{-1})a \in I \Rightarrow I = R$$

- If  $\{0\}$  is a maximal ideal, then  $R \neq \{0\}$ .  $\forall a \in R$  with  $a \neq 0$ , then  $a \in (a) \neq \{0\} \Rightarrow (a) = R \Rightarrow \exists b \in R$ , s.t.  $ab = 1 \Rightarrow a$  is invertible.

■

**Corollary 1.8.1.** An ideal  $M \subseteq R$  is maximal if and only if  $R/M$  is a field.

**Proof.** By correspondance between ideals of  $R/M$  and ideals in  $R$  **containing**  $M$ , this follows from **Lemma 1.8.1**. ■

In particular this corollary basically tells us how to construct some fields, we later shall use it to construct all kinds of finite fields, by taking quotients on the ring  $\mathbb{Z}[x]$ .

**Example 1.8.3.**

- Let  $R$  be the ring of all functions from  $[0, 1] \rightarrow \mathbb{R}$ , see that for each  $a \in [0, 1]$  define:

$$M_a = \{f \in R \mid f(a) = 0\}$$

and since evaluation is a surjective ring homomorphism from  $R \rightarrow \mathbb{R}$ , thus:

$$R/M_a \cong \mathbb{R}$$

and thus  $M_a$  being a maximal ideal.

- $(x) \in \mathbb{Z}[x]$  is not a maximal ideal. One can see that from either  $(x) \subsetneq (2, x) \subseteq \mathbb{Z}[x]$  or  $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$  with  $\mathbb{Z}$  not being a field.

**Corollary 1.8.2.** Every maximal ideal is prime ideal.

**Proof.** This follows since **every field is a domain**. And by **Corollary 1.8.1** and **Proposition 1.8.1**. ■

**Example 1.8.4.** The following are ideals that are prime but not maximal:

1.  $\{0\} \subseteq \mathbb{Z}$  is a prime ideal, but not maximal ideal.
2.  $(x) \subseteq \mathbb{Z}[x]$  is a prime ideal, but not maximal ideal.

why?

**Theorem 1.8.2.** If  $I \subsetneq R$  is a proper ideal in a commutative ring  $R$ , then there exists  $M$  being a maximal ideal, s.t.  $I \subseteq M$ .

To prove it we'll need the famous **Zorn's Lemma**.

**Lemma 1.8.2 (Zorn's Lemma).** If  $(A, \leq)$  is a non-empty partially ordered set, s.t. every totally ordered subset  $B \subseteq A$  has an upper bound in  $A$  ( $\exists a \in A$ , s.t.  $b \leq a \forall b \in B$ ), then  $A$  has at least a maximal element. ( $\exists a \in A$ , s.t. if  $a \leq a' \in A \Rightarrow a = a'$ )

**Proof uses Zorn's Lemma 1.8.2.** Fix  $I$  be in the theorem, and let  $\mathcal{J} = \{J \subseteq R \text{ is ideal} \mid I \subseteq J\}$ . See that it is ordered by inclusion:  $J \leq J' \Leftrightarrow J \subseteq J'$ . Note that  $\mathcal{J} \neq \emptyset$  since  $I \in \mathcal{J}$ .



So our basic task is to check it satisfies the hypothesis in **Zorn's Lemma 1.8.2**.

Let  $\mathcal{J}' \subseteq \mathcal{J}$  be a totally ordered subset: namely if  $J_1, J_2 \in \mathcal{J}' \Rightarrow (J_1 \subseteq J_2) \vee (J_2 \subseteq J_1)$ . Now let:

$$J := \bigcup_{J' \in \mathcal{J}'} J'$$

The **key point** is that  $J$  is an ideal. Suppose  $a, b \in J$ , let  $J', J'' \in \mathcal{J}'$  are s.t.  $a \in J', b \in J''$ . If  $J' \subseteq J'' \Rightarrow a \in J'' \Rightarrow a + b \in J'' \Rightarrow a + b \in J$ . Similarly for  $J'' \subseteq J'$ .

If  $x \in J$  and  $\lambda \in R$ , then there exists  $J' \in \mathcal{J}'$ , s.t.  $x \in J' \Rightarrow \lambda x \in J' \subseteq J$ .

See that  $J \neq \emptyset$  since  $0 \in J$ . So the **conclusion** is  $J$  is an ideal. And it is clear that  $I \subseteq J$ .

See that  $J \neq R$ , otherwise  $1 \in J \Rightarrow 1 \in J'$  for some  $J' \in \mathcal{J}'$ , contradict to the fact that  $J' \neq R$ .

It is clear that  $J' \leq J \forall J' \in \mathcal{J}' \Rightarrow J$  is the upperbound for  $\mathcal{J}'$ . The Apply **Zorn's Lemma 1.8.2**: there exists  $M \in \mathcal{J}$  to be the maximal element, and this is a maximal ideal that containing  $I$ . ■

**Corollary 1.8.3.** If  $R \neq \{0\}$  is a commutative ring, then there exists a maximal ideals in  $R$ , in particular, there exists a prime ideal.

**Proof.** Apply the theorem with  $I = \{0\}$  shows the existence of maximal ideal. ■

**Proposition 1.8.2.** Let  $R$  be a commutative ring, and  $\mathfrak{p}_1, \mathfrak{p}_2$  be two distinct maximal ideal, then:

$$\mathfrak{p}_1 + \mathfrak{p}_2 = R$$

**Sketch.** It is straightforward once realized that  $\mathfrak{p}_1 + \mathfrak{p}_2$  is actually an ideal by definition. ■

Notably for  $\mathbb{Z}$  there is **no difference** between the notion of maximal ideal and prime ideal,  $(p)$  is always maximal. Consider  $(p_1, p_2)$  by Bezout's Theorem one shall notice that  $1 \in (p_1, a)$  and they must be coprime if distinct and thus directly be  $\mathbb{Z}$ .

## 1.9 Local Ring

In this section we shall discuss local rings.

leave some overview!

**Definition 1.9.1 (Local Ring).** A commutative ring  $R$  is a local ring if  $R$  has a **unique** maximal ideal.

**Proposition 1.9.1.** For a commutative ring  $R$ , the following are equivalent:

1.  $R$  is a local ring. (with maximal ideal  $M = \{a \in R \mid a \text{ is not invertible}\}$ )
2.  $R \neq \{0\}$  and for all  $a, b \in R$ , s.t.  $a + b = 1$ , either  $a$  or  $b$  is invertible.

**Proof.** Suppose that  $R$  is a local ring with maximal ideal  $M$ , then  $M \subseteq \{a \in R \mid a \text{ is not invertible}\}$  since  $M \neq R$ . If  $a \in R$  is not invertible, then  $(a) \neq R$ , by **Theorem 1.8.2**, it is contained in a maximal ideal  $\Rightarrow (a) \subseteq M \Rightarrow a \in M$ .

In this case,  $R \neq \{0\}$  since  $M \neq R$ . If  $a + b = 1$ , since  $1 \notin M$  and  $M$  being a subgroup, then either  $a \notin M$  or  $b \notin M \Rightarrow a$  is invertible or  $b$  is invertible.

Define  $M = \{a \in R \mid a \neq \text{invertible}\}$ . We claim that  $M$  is an ideal:

- $0 \in M$  since  $R \neq \{0\}$ .
- If  $a \in M, \lambda \in R \Rightarrow \lambda a \in M$ , otherwise  $\exists \mu \in R$ , s.t.  $(\mu\lambda)a = 1 \Rightarrow a$  is invertible, leading to contradiction  $\nexists$ .
- If  $a, b \in M \Rightarrow c := a + b \in M$ , otherwise, If  $c$  is invertible, then:

$$(a + b)c^{-1} = ac^{-1} + bc^{-1} = 1$$

then this implies that  $ac^{-1}$  or  $bc^{-1}$  is invertible, then  $a = (ac^{-1})c$  is also invertible, similar for  $b$  is invertible, leading to contradiction  $\nmid$ .

So we see that  $M$  is an ideal, remains to check that it is the only maximal ideal.

- $1 \notin M \Rightarrow M \neq R$ .
- If  $I \neq R$  is an ideal, then  $I \subseteq M$ :  $I \neq R \Rightarrow 1 \notin I$ , thus  $\forall a \in I$ , see that  $a$  is not invertible, since  $(a) \subseteq I \neq R \Rightarrow 1 \notin (a) \Rightarrow a \in M$ .

Since we know  $R$  has an maximal ideal by **Corollary 1.8.3**, then  $M$  is a maximal ideal, and in fact the unique one. See that any maximal ideal  $M' \subseteq M \Rightarrow M' = M$ . ■

### Example 1.9.1.

1.  $\mathbb{K}$  is a field  $\Rightarrow \mathbb{K}$  is a local ring.
2. Let  $R$  be a commutative ring,  $P \subseteq R$  be a prime ideal, define  $S = R - P$ , and thus be a multiplicative system. Define  $R_p := S^{-1}R$ . By HW #3:

$$\{\text{Prime ideal in } S^{-1}R\} \xleftrightarrow{\text{order preserving bij.}} \{\text{Prime ideal } q \text{ in } R \text{ with } S \cap q = \emptyset \Leftrightarrow q \subseteq P\}$$

where order preserving means the bijection is compatible with inclusion. This implies that  $S^{-1}R$  is a local ring with maximal ideal:

$$S^{-1}P = \left\{ \frac{a}{s} \in R_p \mid a \in P \right\}$$

**Example 1.9.2.** If  $p \in \mathbb{Z}_{>0}$  is a prime integer, then:

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, (p \nmid b) \Leftrightarrow (S = \mathbb{Z} - (p)) \right\}$$

with the maximal ideal being:

$$\left\{ \frac{pa}{b} \mid a, b \in \mathbb{Z}, p \nmid b \right\} \quad \text{by } pa \in (p)$$

The point is that if we want to study the property of the ring, we can sometimes go to the local ring and use their properties.

## 1.10 Radical Ideals

In this section we shall discuss radical rings.

leave some overview!

**Definition 1.10.1 (Radical Ideal).** Let  $R$  be commutative ring, and  $I \subseteq R$  be an ideal.  $I$  is a radical ideal if  $\forall a \in R$ , s.t.  $a^n \in I$  for some  $n \geq 1 \Rightarrow a \in I$ .

**Definition 1.10.2 (Reduced Ring).** Let  $R$  commutative ring,  $R$  is called a reduced ring if  $\{0\}$  is a radical ideal.

**Theorem 1.10.1.** Let  $R$  be commutative ring, and  $I \subseteq R$  be an ideal.  $I$  is a radical ideal if and only if  $R/I$  is a reduced ring.

The proof is straightforward by just formally write out the definition.

### Example 1.10.1.

1.  $I$  is a prime ideal  $\Rightarrow I$  is a radical ideal. As one can consider  $a^n \in I \Rightarrow (a^{n-1})a \in I \Rightarrow$  either  $a^{n-1} \in I$  or  $a \in I$  and doing this repeatedly eventually leads to  $a \in I$ .
2. If  $n \in \mathbb{Z}_{>0}$ , then  $(n)$  is radical ideal if and only if  $n$  is square free, namely if  $n = p_1^{a_1} \cdots p_r^{a_r}$  to be the prime decomposition, then  $a_i = 1 \forall i$ .

**Sketch of Proof.** One can consider the prime factorization of  $n$  as:

$$n = p_{i_1}^{a_{i_1}} \cdots p_{i_r}^{a_{i_r}}$$

and consider arbitrary element  $a \in \mathbb{Z}$  such that  $a^k \in (n)$ , the prime factorization of  $a^k$  given by:

$$a^k = p_{j_1}^{b_{j_1}} \cdots p_{j_l}^{b_{j_l}} = cn \quad \text{for some } c \in \mathbb{Z}$$

and since  $c$  is integer it means that LHS should cancel out all the prime factors of  $n$ , in particular, this means that:

$$\{i_1, \dots, i_r\} \subseteq \{j_1, \dots, j_l\}$$

and cancel things out one can still write  $a = dn$  for some  $d \in \mathbb{Z}$ . ■

### Remark 1.10.1.

1. We showed in HW#2, that:

$$\text{rad}(I) = \{a \in R \mid a^n \in I \text{ for some } n \geq 1\}$$

is an ideal in  $R$ . See that  $I \subseteq \text{rad}(I)$ , with equality if and only if  $I$  is a **radical ideal**.

**Sketch of Proof.** Its straightforward to see that  $I \subseteq \text{rad}(I)$ . When  $I$  is a radical ideal, this means that  $a^n \in I \Rightarrow a \in I \Rightarrow \text{rad}(I) \subseteq I \Rightarrow I = \text{rad}(I)$ . When  $I = \text{rad}(I)$ , then  $\text{rad}(I) \subseteq I \Rightarrow a^n \in I \Rightarrow a \in I$ . ■

2.  $\text{rad}(I)$  is a radical ideal. Just check above proof.
3.  $\text{rad}(I)$  is the **smallest** radical ideal containing  $I$ .
- 4.

**Parenthesis.** If  $(I_\alpha)_\alpha$  is a family of left/right/two-sided ideals in **any ring**  $R$ , then:

$$\bigcap_{\alpha} I_{\alpha} \text{ has the same property.}$$

5. If each  $I_\alpha$  is a radical ideal, then  $\bigcap_{\alpha} I_\alpha$  is also a radical ideal. The proof is quite straightforward.

**Note.** This is **false** for prime ideals, see that in  $\mathbb{Z}$ ,  $(2) \cap (3) = (6)$ , where  $(6)$  is not a prime ideal.

However, if the family is a **totally ordered set characterized by set inclusion**, then this statement is true. See homework related to Zorn's Lemma, this also gives us the statement that **every prime ideals have a minimal prime ideal**.

**Proposition 1.10.1.** For every ideal  $I \subseteq R$ , see that:

$$\text{rad}(I) = \bigcap_{P \supseteq I, P \text{ prime ideal}} P$$

**Proof.** First note that:

$$I \subseteq \underbrace{\bigcap_{I \subseteq P, P \text{ prime ideal}} P}_{\text{radical, since prime is radical and intersection of radical is radical}} \Rightarrow \text{rad}(I) \subseteq \bigcap_{I \subseteq P, P \text{ prime ideal}} P$$

since  $\text{rad}(I)$  is the smallest radical ideal that contains  $I$ .

Suppose  $f \in \bigcap_{I \subseteq P, P \text{ prime ideal}} P$ , we want to see that  $f^n \in I$  for some  $n \geq 1$ .

The general idea here is to replace  $(R, I, f)$  by  $(R/I, \{0\}, \bar{f})$ , as one can see that:

$$\bar{f}^n = 0 \Leftrightarrow f^n \in I$$

big picture is that the property  $f^n \in I$  is **carried** by ring homomorphism, and one will make it easier to consider in quotient ring, and further to fraction it out using the multiplicative system  $S = \{1, f, f^2, \dots\}$ .

May assume  $I = \{0\}$ , thus  $f \in P$  for all prime ideal  $P$ . The **tricks** here is to consider  $R_f = S^{-1}R$  where  $S$  is defined as above. The prime ideals in  $R_f$  is the same as the prime ideals  $P$  in  $R$ , s.t.  $S \cap P = \emptyset \Leftrightarrow f \notin P$ . And see that there are no such prime ideals in  $R$ , and so there will be no such prime ideal in  $S^{-1}R$ , but we've seen in **Theorem 1.8.2** that every commutative ring who have a proper ideal should have a maximal ideal, and maximal ideal is prime ideal, and itself is an ideal, it follows that  $R_f = \{0\}$ . So:

$$R_f = \{0\} \Rightarrow \frac{0}{1} = \frac{1}{1} \Leftrightarrow \exists n, \text{ s.t. } f^n = 0$$

■

**Corollary 1.10.1.** An ideal is a radical ideal if and only if it is the intersection of all prime ideals who contains it.

**Proof.** As prime ideals are radical ideal, the forward direction trivially holds. The reverse direction directly yields combining the proposition and the fact that  $I$  is radical if and only if  $I = \text{rad}(I)$ . ■

**Corollary 1.10.2.** A ring  $R$  is reduced if and only if

$$\bigcap_{\mathfrak{p} \in \text{Spec } R} \mathfrak{p} = \{0\}$$

## 1.11 Operations with Ideals

In this section we will see several operators to help us construct more and more ideals from existing ideals.

### 1.11.1 Sum of Ideals

Let  $R$  be any ring, then we've seen that the intersection of ideals are ideals, we now define the sum of ideals for  $(I_\alpha)_{\alpha \in \Lambda}$ .

Let  $I_\alpha$  be left/right/2-sided ideal in  $R$ , define the sum of them as:

$$\sum_{\alpha \in \Lambda} I_\alpha := \bigcap_{I \text{ be such ideal } I_\alpha \subseteq I \forall \alpha} I$$

This is the unique **smallest** ideal containing all  $I_\alpha$ . Note that we consider **finite sum** here, if it is infinite sum, then we put finitely of them that are non-zero.

such here means corresponding left/right/2-sided

**Proposition 1.11.1 (Equivalence def. of Sum of Ideals).**

$$\sum_{\alpha \in \Lambda} I_{\alpha} = \left\{ \sum_{\alpha \in \Lambda} a_{\alpha} \mid a_{\alpha} \in I_{\alpha} \forall \alpha, \text{ only finitely many } a_{\alpha} \text{ are } \neq 0 \right\}$$

**Example 1.11.1.**

$$I_1 + I_2 = \{a + b \mid a \in I_1, b \in I_2\}$$

**Sketch of Proof.** It is straightforward to verify that the RHS is an ideal, and it contains all  $I_{\alpha}$ , thus “ $\subseteq$ ” part directly yields.

On the other hand, if  $I$  is an ideal, s.t.  $I_{\alpha} \subseteq I \forall \alpha$ , then  $\text{RHS} \subseteq I$ , which then yields “ $\supseteq$ ” part. ■

More generally, given any subset  $A \subseteq R$ , may consider the smallest left/right/2-sided ideal **generated** by  $A$ :

$$\bigcap_{I \text{ be such ideal } A \subseteq I} I$$

If  $R$  is commutative, write  $(A)$  for this ideal.

**Example 1.11.2.** If  $A = \{a\}$ , then the left ideal generated by  $A$  is:

$$Ra = \{\lambda a \mid \lambda \in R\}$$

One can also write the following for the corresponding ideal, the definition is easy to understand by the notation:

$$RA := \{r_1 a_1 + \cdots + r_n a_n \mid n \in \mathbb{Z}_{>0}, a_i \in A, r_i \in R \forall i \in [1, n]\}$$

On one hand, since  $R$  contains the identity elements, thus we have  $A \subseteq R$ , one the other hand, any left ideal containing  $A$  should contain all the finite sums of the element in the form  $ra$  with  $r \in R, a \in A$ , thus  $RA$  is **precisely** the left ideal generated by  $A$ . Similarly, things happen for  $AR$  and  $RAR$  which corresp. to the right/2-sided ideal generated by  $A$ . Note that when  $R$  is not commutative, the set  $\{ras \mid r, s \in R\}$  is not necessarily the two-sided ideal generated by  $a$  as it is not closed under addition.

**Remark 1.11.1.**

1. For any  $A$ , we have left/right ideal generated by  $A$  is:

$$\sum_{a \in A} Ra \quad (\text{resp.} \quad \sum_{a \in A} aR)$$

2. If  $A = \{a_1, \dots, a_n\}$  and  $R$  be a commutative ring, then the ideal generated by  $A$  is denoted as  $(a_1, \dots, a_n)$ , which is:

$$(a_1, \dots, a_n) := \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_1, \dots, \lambda_n \in R\}$$

3. We say  $A$  is a **system of generators** of  $I$ , where  $I$  being a left/right/2-sided ideal, if  $I$  is such ideal generated by  $A$ .

**Example 1.11.3.** If  $R$  is commutative, a principal ideal in  $R$  is an ideal generated by 1 element:  $(a), a \in R$ .

**Proposition 1.11.2.** Given  $a, b \in R$ , see that the following are equivalent:

- $b \in (a)$ .
- $(b) \subseteq (a)$ .

The proof is quite straightforward, one thing to note is that Thus containment relations between ideals, in particular between principal ideals, is seen to capture some of the **arithmetic** of general commutative rings. Notably, the construction of  $\langle a \rangle$  is quite like what we did in group theory to construct cyclic subgroup from an element, which shows the usage of single or multiple generators.

### 1.11.2 Product of Ideals

Let  $R$  be commutative ring,  $I_1, \dots, I_n \subseteq R$  be ideals, then define:

$$I_1 \cdots I_n = \text{ideal generated by } \{a_1 a_2 \cdots a_n \mid a_j \in I_j \forall j\}$$

This means that it can be written as:

$$I_1 \cdots I_n := \left\{ \sum_{k=1}^d a_{k1} a_{k2} \cdots a_{kn} \mid d \in \mathbb{Z}_{>0}, a_{kj} \in I_j \forall j \right\}$$

**Example 1.11.4.** For example:

$$I_1 \cdot I_2 = \left\{ \sum_{i=1}^d a_i b_i \mid d \in \mathbb{Z}_{>0}, a_i \in I_1, b_i \in I_2 \right\}$$

Note that the sum here runs over all finite sum.

**Note.** It should be stated clear that

$$I_1 \cdot I_2 \neq \{ab \mid a \in I_1, b \in I_2\}$$

such set is in general not closed under addition, thus **not** being an **ideal**.

Suppose that  $f : R \rightarrow S$  be a ring homomorphism of commutative rings. If  $I \subseteq R$  be an ideal, what can we say about  $f(I)$ ?

- $f(I) \subseteq S$  is a subgroup.
- $f(I)$  is not necessarily an ideal, as it is if and only if  $f$  is **surjective**.

Thus we can look into the ideal generated by  $f(I)$ , which is denoted by  $IS$  or  $I \cdot S$ :

$$IS := \left\{ \sum_{j=1}^n a_j f(b_j) \mid n \in \mathbb{Z}_{>0}, a_j \in S, b_j \in I \right\} \quad (1.3)$$

**Example 1.11.5.** Suppose that  $S = T^{-1}R$  where  $T \subseteq R$  be a multiplicative system. Let  $I \subseteq R$  be an ideal. See that:

$$IS = T^{-1}I = \left\{ \frac{a}{s} \mid a \in I, s \in T \right\}$$

## 1.12 Spectrum of a Commutative Ring

This connects closely on topology. Basically it allow us to glue several ring to get some geometric shape.

**Definition 1.12.1.** Given a commutative ring  $R$ , define:

$$\text{Spec } R := \{P \subset R \mid P \text{ is prime ideal.}\}$$

For every ideal (not necessarily prime)  $I \subset R$ , let

$$V(I) := \{P \in \text{Spec } R \mid I \subseteq P\}$$

Note that we consider it as some sort of topology with closed set being  $V(I)$  for some  $I \subset R$  being ideal.

**Proposition 1.12.1 (Zariski Topology).** We have a topology on  $\text{Spec } R$ , s.t. the **closed sets** are the  $V(I)$  for  $I \subseteq R$ . The topology it forms is called Zariski Topology.

**Proof.** To verify topology property we basically need to check:

1.  $\text{Spec } R = V(I)$  for some  $I$ .

This follows by taking  $I = \{0\}$ .

2.  $\emptyset = V(I)$  for some  $I$ .

This follows by taking  $I = R$ .

3.  $\forall (I_\alpha)$  be ideals in  $R$ , we have

$$\bigcap_{\alpha \in \Lambda} V(I_\alpha) = V(J) \quad \text{for some } J$$

Consider let  $P \in \bigcap_{\alpha} V(I_\alpha) \Leftrightarrow P \supseteq I_\alpha \forall \alpha \Leftrightarrow P \subseteq \sum_{\alpha \in \Lambda} I_\alpha$ . So this follows by taking  $J = \sum_{\alpha} I_\alpha$ .

4.  $\forall$  ideals  $I_1, I_2 \subseteq R$ , we have:

$$V(I_1) \cup V(I_2) = V(J) \quad \text{for some } J$$

We try to show that  $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$ , the " $\subseteq$ " part is clear simply by the fact that  $I_1 \cap I_2 \subseteq I_1$  and  $I_1 \cap I_2 \subseteq I_2$ . Now suppose that  $P \in V(I_1 \cap I_2)$ , then see that  $I_1 \cap I_2 \subseteq P$ . If  $I_1 \not\subseteq P, I_2 \not\subseteq P \Rightarrow \exists x_1 \in I_1 - P, x_2 \in I_2 - P$ , s.t.  $x_1 x_2 \in I_1 \cap I_2$  (follows by ideal property) but  $x_1 x_2 \notin P$  since  $P$  is prime ideal, and reason by contraposition. This leads to contradiction  $\nexists$ . Hence by contradiction, either  $P \supseteq I_1$  or  $P \supseteq I_2$ , thus  $P \in V(I_1) \cup V(I_2)$ .

**Note.** Note that the formula only holds for ideal  $I_1, I_2 \subseteq R$ :

$$V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$$

**Example 1.12.1.** Let  $R = \mathbb{Z} \Rightarrow$  the prime ideals are **precisely**  $\{\{0\}, p\mathbb{Z} \text{ for } p \text{ prime.}\}$ . Then:

- $\{p\mathbb{Z}\}$  are closed.
- $\overline{\{0\}} = \overline{\{0\}} = \text{Spec}(\mathbb{Z})$ .

In fact, we can define  $\text{Spec}$  as **functor** as follows:

$$\text{CommutativeRings} \longrightarrow \text{Top}^\circ$$

where LHS is the category of commutative rings, and the RHS is the dual of the category of topological spaces.

First recall the definition of a functor:

**Definition 1.12.2 (Functor).** If  $\mathcal{C}$  and  $\mathcal{D}$  are categories, then a functor:

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

see next time every ideal is principal in this case

is given by:

1. For every  $X \in \text{Ob}(\mathcal{C})$ , we have  $F(X) \in \text{Ob}(\mathcal{D})$ .
2. For all  $X, Y \in \text{Ob}(\mathcal{C})$ , we have a map

$$\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

s.t.

- (a)  $F(1_X) = 1_{F(X)} \quad \forall X \in \text{Ob}(\mathcal{C})$ .
- (b)  $\forall u \in \text{Hom}_{\mathcal{C}}(X, Y), v \in \text{Hom}_{\mathcal{C}}(Y, Z)$ , preserving the composition structures of the morphisms.

$$F(u \circ v) = F(v) \circ F(u)$$

If  $f : R \rightarrow S \supseteq P$  to be a ring homomorphism between commutative rings, one can define:

$$\begin{aligned} f^\# : \text{Spec}(S) &\rightarrow \text{Spec}(R) \\ f^\#(P) &= f^{-1}(P) \end{aligned}$$

We know that  $f^{-1}(P) \subseteq R$  is an ideal, it's "inverse" is "surjective", notably to see that it is actually a **prime ideal**:

$$\text{If } xy \in f^{-1}(P) \Rightarrow f(xy) = f(x)f(y) \in P \Rightarrow f(x) \in P \text{ or } f(y) \in P \Rightarrow x \in f^{-1}(P) \text{ or } y \in f^{-1}(P)$$

**Claim.**  $f^\#$  is **continuous**.

**Proof.** It is enough to show that  $(f^\#)^{-1}(\text{closed set})$  is also closed.

Fix  $I \subseteq R$  be an ideal, then:

$$(f^\#)^{-1}(V(I)) = \{P \subseteq S \mid P \text{ prime. } f^\#(P) \supseteq I \Leftrightarrow f^{-1}(P) \supseteq I \Leftrightarrow P \supseteq f(I) \Leftrightarrow P \supseteq IS \text{ 1.3}\}$$

Thus we can conclude:  $(f^\#)^{-1}(V(I)) = V(IS)$  which is closed. ■

To check that it is a functor, need:

1.  $(\text{Id}_R)^\# = \text{Id}_{\text{Spec } R}$ , which is clear.
2. The following is true since  $(g \circ f)^{-1}(P) = f^{-1}(g^{-1}(P))$ :

$$\begin{aligned} R &\xrightarrow{f} S \xrightarrow{g} T \\ (g \circ f)^\# &= f^\# \circ g^\# \end{aligned}$$

We can actually also do similar things for maximal ideal!

**Remark 1.12.1.** May consider:

$$\text{MaxSpec}(R) := \{P \subset R \mid P \text{ maximal ideal.}\} \subseteq \text{Spec}(R) \text{ w.r.t. subspace topology.}$$

- It **doesn't** give a functor in general.
- It indeed **well-behaved** and useful in geometry if  $R$  is a quotient of a polynomial ring, namely:  $\mathbb{K}[x_1, \dots, x_n] / I$  where  $\mathbb{K}$  be a field, and in this case, it indeed **gives a functor**.

## 1.13 Noetherian Rings

In this section we focus on the context of  $R$  being any ring instead of being commutative ring.



**Definition 1.13.1 (Noetherian Rings and Artinian Rings).** Let  $R$  be any ring. Then  $R$  is left/right Noetherian ring if there exists **no** strictly increasing sequence:

$$I_1 \subsetneq I_2 \subsetneq \cdots \text{ of left/right ideals.}$$

$R$  is left/right Artinian ring if there exists **no** strictly decreasing sequence

$$I_1 \supsetneq I_2 \supsetneq \cdots \text{ of left/right ideals.}$$

**Note.** Artinian ring is a stronger property, most of the rings doesn't satisfy the property of artinian ring.

**Example 1.13.1.**

1. A field is both noetherian and artinian.
2. Every finite ring is both noetherian and artinian.
3.  $\mathbb{Z}$  is not a artinian ring, because there exists the following strictly decreasing ideal sequence:

$$2\mathbb{Z} \supsetneq 2^2\mathbb{Z} \supsetneq 2^3\mathbb{Z} \supsetneq \cdots$$

4.  $\mathbb{Z}$  is a Noetherian ring. One can directly use 1 to finitely generate it. Notably, one may use the following as construction for counterexample:

$$(2) \subseteq (2, 3) \subseteq (2, 3, 5) \subseteq (2, 3, 5, 7) \subseteq \cdots$$

But actually by **Bezout's theorem**,  $2x + 3y = 1$  and thus see that  $1 \in (2, 3) \Rightarrow (2, 3) = \mathbb{Z}$ .

**Proposition 1.13.1.** For a ring  $R$ , **TFAE**:

1.  $R$  is left/right Noetherian ring.
2. Every non-empty family  $\mathcal{I}$  of left/right ideals of  $R$  contains a maximal element. The maximal element namely is:

$$\exists I \in \mathcal{I} \text{ s.t. if } I \subseteq J, J \in \mathcal{I} \Rightarrow I = J$$

3. Every left/right ideal of  $R$  is finitely generated. Namely:

$$\exists a_1, \dots, a_n \in R, \text{ s.t. } I = \sum_{i=1}^n Ra_i \quad \left( I = \sum_{i=1}^n a_i R \right)$$

**Proof.**

- $(2) \Rightarrow (1)$ : This is clear as one cannot find family of ideals that form a strictly increasing sequence family given the condition of existence of maximal element.
- $(1) \Rightarrow (2)$ : One can reason by contraposition, non-existing maximal element means we can always find ideal that is strictly bigger than the current one, and thus construct a infinite increasing sequence, fail the property of Noetherian ring.
- $(1) \Rightarrow (3)$ : Given a left ideal  $I$  that is not finitely generated. We can construct inductively  $a_1, a_2, \dots \in I$ , s.t.:

$$a_{n+1} \notin \sum_{i=1}^n Ra_i \quad \forall n$$

This implies that the sequence of ideals  $(\sum_{i=1}^n Ra_i)$  is strictly increasing and infinite, thus fail the property of Noetherian ring.

- (3)  $\Rightarrow$  (1): Suppose we have  $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$  is a sequence of left ideals. Let:

$$I := \bigcup_{n \geq 1} I_n$$

we've seen before in the proof of **Theorem 1.8.2**,  $I$  is also a left ideal given that  $\{I_\alpha\}$  is a total ordered set. Now **by (3)**,  $I$  is finitely generated, so we can write:

$$I = \sum_{i=1}^n Ra_i \text{ for some } a_1, \dots, a_n \in R$$

Suppose that  $a_i \in I_{r_i}$  for  $1 \leq i \leq n$ , let  $r = \max\{r_i\}$ , this implies that:

$$\sum_{i=1}^n Ra_i \subseteq I_r \subseteq I = \sum_{i=1}^n Ra_i \quad \forall s \geq r$$

thus  $I_s = I_r \quad \forall s \geq r$ . So we see that the sequence **stabilize to some ideal**, leading to the property of Noetherian ring. ■

The following theorem gives us a basic idea that in fact a lot of rings are actually Noetherian, because the property can inherit from some other ring. So we can build up Noetherian ring from the existing one.

**Theorem 1.13.1 (Hilbert's Basis Theorem).** If  $R$  is a **Noetherian commutative ring**, then  $R[X]$  is also a Noetherian ring.

In fact this proof is quite influential and this is actually the first result in Commutative Algebra. Commutative property here basically only help us construct the polynomial ring.

**Proof.** We will show basically every ideal  $I \subseteq R[X]$  is finitely generated, and yield that  $R[X]$  is Noetherian by **Proposition 1.13.1**.

Suppose  $I$  is not finitely generated. Then we can inductively construct a sequence of element as follow, each time of iteration, the condition on the degree of such element is a little bit stronger than before.

$$I \neq \{0\} \Rightarrow \text{chose } f_1 \in I \setminus \{0\} \text{ of minimal degree } d_1$$

$$I \neq (f_1) \Rightarrow \text{chose } f_2 \in I \setminus (f_1) \text{ of minimal degree } d_2$$

We repeat this to construct  $f_1, f_2, \dots$ , s.t.:

$$\forall n \geq 0, f_{n+1} \in I \setminus (f_1, f_2, \dots, f_n)$$

and  $\deg(f_{n+1}) = d_{n+1}$  is minimal among such  $f_{n+1}$ . The reason why we can construct  $f_i$  in this way is that  $I$  is not finitely generated, so there is always some element between  $(f_1, \dots, f_n) \subset I$ .

**Note.** By construction and **minimality** of  $d_n$ , we have:

$$d_1 \leq d_2 \leq \dots$$

We then want to make contradiction to this minimality. For every  $n \geq 1$ , let's write:

$$f_n = a_n x^{d_n} + \text{lower degree monomials}$$

Let  $J \subseteq R$  be the ideal generated by  $\{a_n \mid n \geq 1\}$ . Since  $R$  is a Noetherian ring,  $J$  is then **finitely generated**. So we can write:

$$J = (b_1, \dots, b_k)$$

We can write each  $b_i$  as an linear combination of  $a_j$ 's, by the fact that  $J$  be the ideal generated by the whole sequence  $\{a_n \mid n \geq 1\}$ . Namely:

$$b_i = \sum_{j=1}^{m_i} \lambda_{ij} a_j \quad \lambda_{ij} \in R$$

Now if we let  $m = \max\{m_i\}$ , then  $(a_1, \dots, a_m) \supseteq (b_1, \dots, b_k) = (a_1, a_2, \dots)$ , and it is clear that  $(a_1, \dots, a_m) \subseteq (b_1, \dots, b_k)$ . Thus we conclude.

**Conclusion 1.13.1.**  $J = (a_1, \dots, a_m)$ .

Thus see that  $a_{m+1} \in (a_1, \dots, a_m) \Rightarrow$  we can write  $a_{m+1} = \lambda_1 a_1 + \dots + \lambda_m a_m$ . One can then define:

$$g = \underbrace{f_{m+1}}_{=a_{m+1}x^{d_{m+1}} + \text{l.d.m.}} - \sum_{i=1}^m \underbrace{\lambda_i x^{d_{m+1}-d_i} f_i}_{\substack{\text{by non-dec.} \\ \in R[x] \\ \lambda_i a_i x^{d_{m+1}} + \text{l.d.m.}}}$$

$$g = f_{m+1} - \sum_{i=1}^m \lambda_i x^{d_{m+1}-d_i} f_i$$

l.d.m.  
stands for  
lower de-  
gree mono-  
mials.

**Note.**

1.  $\deg(g) < d_{m+1} (*)$ .
2.  $g \in I$  and  $g \notin (f_1, \dots, f_n)$ , otherwise, see that:

$$f_{m+1} = g + \sum_{i=1}^m \lambda_i x^{d_{m+1}-d_i} f_i \in (f_1, \dots, f_m)$$

which is **not** ok.

Thus see that  $(*)$  implies a contradiction with minimality of  $d_{m+1}$ . ■

**Corollary 1.13.1.** If  $\mathbb{K}$  is a field, then  $\mathbb{K}[x_1, \dots, x_n]$  is Noetherian ring for all  $n \geq 1$ .

**Proof.** Basically use **Theorem 1.13.1** and induction on  $n$  and the fact that  $\mathbb{K}$  is a Noetherian ring. ■

## 1.14 PIDs and Euclidean Domains

**Definition 1.14.1 (PID).** Let  $R$  be a domain, then  $R$  is a Principal ideal domain (PID) if every ideal in  $R$  is principal: namely it is of the form  $(a)$  for some  $a \in R$ .

**Example 1.14.1 (Non-PID Example).**

1.  $(2, x) \subseteq \mathbb{Z}[x]$  is not a principal ideal, and thus  $\mathbb{Z}[x] \neq \text{PID}$ .

**Proof.** Suppose that  $(2, x) = (f)$  for some  $f \in \mathbb{Z}[x]$ , then  $2 \in (f) \Rightarrow 2 = fg$  for some  $g \in \mathbb{Z}[x] \Rightarrow \deg(f) = 0$ . Thus see that  $f = n \in \mathbb{Z} \setminus \{0\}$ , and see that  $x = ng \Rightarrow n = \pm 1$  by considering the fact that the coefficient of  $x$  is 1, and in the domain  $\mathbb{Z}$ , the only way to get 1 is by multiplying  $\pm 1$ .

Hence  $(2, x) = \mathbb{Z}[x]$  by the fact that  $\pm 1$  is in the ideal. Then it means that there exists  $P, Q \in \mathbb{Z}[x]$ , s.t.  $1 = 2P + xQ$ .

$$\begin{aligned} P &= a_0 + a_1x + \cdots \\ Q &= b_0 + b_1x + \cdots \end{aligned}$$

$\Rightarrow 1 = 2a_0$ , but it cannot happen since  $a_0 \in \mathbb{Z}$ . ■

2. If  $\mathbb{K}$  is a field, then  $(x, y) \subseteq \mathbb{K}[x, y]$  is not finitely generated.

exer!

**Definition 1.14.2 (Euclidean Domain).** A domain  $R$  is an Euclidean Domain if there exists  $N : R \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$  being **arbitrary** functions, such that,  $\forall a, b \in R$  with  $b \neq 0$ , there exists  $q, r \in R$  (**not necessarily unique**), such that:

- $a = bq + r$ .
- Either  $r = 0$  or  $(r \neq 0 \text{ and } N(r) < N(b))$ .

**Proposition 1.14.1.** Every Euclidean Domain  $R$  is a PID.

**Proof.** Let  $I \subseteq R$  be an ideal. If  $I = \{0\}$ , clearly that  $I$  is principl. If  $I \neq \{0\}$  and  $N$  is as in the definition of Euclidean domain. Let  $b \in I \setminus \{0\}$  be s.t.  $N(b)$  is minimal.

**Claim.** We claim that  $I = (b)$ .

“ $\supseteq$ ” part is clear as  $b \in I$ . For “ $\subseteq$ ” part, Suppose that  $a \in I$ , with  $b \neq 0$ , then  $\exists q, r \in R$ , s.t.  $a = bq + r$  and:

- $r = 0 \Rightarrow a \in (b)$  which is ok.
- $r \neq 0$  and  $N(r) < N(b)$ , thus  $r = a - bq \in I$  by the ideal property, which contradict to the minimality assumption of  $N(b)$ . So this case cannot happen.

**Example 1.14.2.**

1.  $\mathbb{Z}$  is an Euclidean domain, with:

$$\begin{aligned} N : \mathbb{Z} \setminus \{0\} &\rightarrow \mathbb{Z}_{\geq 0} \\ N(a) &= |a| \end{aligned}$$

**Proof.** Suppose  $b \in \mathbb{Z} \setminus \{0\}$ , then there are two cases:

- $b > 0$ : we know  $a = qb + r$ ,  $0 \leq r < b = N(b)$  for some  $q, r \in \mathbb{Z}$  with the usual division algorithm.
- $b < 0$ : do the same for  $-a, -b$ . Have  $-a = (-b)q + r \Leftrightarrow a = bq - r$  for  $0 \leq r < -b$ , with  $N(r) = |r| = r < -b = |b| = N(b)$ .

2.  $\mathbb{Z}[i]$  is a Euclidean domain, with:

$$N : \mathbb{Z}[i] \rightarrow \mathbb{Z}_{\geq 0}$$

$$N(a + bi) = a^2 + b^2$$

**Proof.** Let  $a + bi, c + di \neq 0 \in \mathbb{Z}[i]$ , in  $\mathbb{Q}[i]$ , have:

$$\frac{a + bi}{c + di} = p + qi \quad p, q \in \mathbb{Q}$$

$$= (\alpha + \beta i) + (\gamma + \delta i)$$

where  $\alpha, \beta \in \mathbb{Z}, \gamma, \delta \in \mathbb{Q}, |\gamma|, |\delta| \leq \frac{1}{2}$ .

This implies that:

$$\underbrace{a + bi}_{\in \mathbb{Z}[i]} = \underbrace{(c + di)}_{\in \mathbb{Z}[i]} \cdot \underbrace{(\alpha + \beta i)}_{\in \mathbb{Z}[i]} + r$$

where  $r = (c + di)(\gamma + \delta i)$   
 $\Rightarrow r \in \mathbb{Z}[i]$

thus:

$$N(r) = N(c + di) \cdot \underbrace{N(\gamma + \delta i)}_{=\gamma^2 + \delta^2 \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} \quad \text{by } (**)$$

thus either  $r = 0$  or  $N(r) < N(c + di)$  since  $N(c + di) \neq 0$ . ■

3. If  $\mathbb{K}$  is a field, then  $\mathbb{K}[x]$  is an Euclidean domain, with:

$$N : \mathbb{K}[x] \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$$

$$N(f) = \deg(f)$$

We verify the condition by showing the following proposition.

**Proposition 1.14.2.** Let  $R$  be any commutative ring,  $f, g \in R[x]$  and  $g \neq 0$ . Let

$$g = a_n x^n + \cdots + a_1 x + a_0$$

s.t.  $a_n$  is **invertible**, then there exists **unique**  $q, r \in R[x]$ , s.t.  $f = gq + r$  and either  $r = 0$  or  $r \neq 0$  and  $\deg(r) < n = \deg(g)$ .

**Proof of Existence is Enough.** Proceed the proof by contradiction. If for given  $g$ , there are  $f$ 's that don't satisfy this condition. Let's choose such  $f$  of minimal degree, clearly we have  $m := \deg(f) \geq n \Rightarrow f = b_m x^m + \text{lower order term}$ . Consider

$$f' = f - b_m a_n^{-1} x^{m-n} g$$

see that  $\deg(f') < \deg(f) \Rightarrow \exists q', r'$ , s.t.  $f' = q'g + r' \Rightarrow f = (b_m a_n^{-1} x^{m-n} + q')g + r'$ , which leads to contradiction  $\nexists$ , as we see that since  $f$  is minimal case that don't satisfy the condition, meaning we have either  $r' = 0$  or  $r' \neq 0$  and  $\deg(r') < n = \deg(g)$ . ■

**Note.** To show the uniqueness, show that  $\deg(gh) = \deg(g) + \deg(h) \forall h$ . We've proceed this for the case of domain, and it is crucial that  $a_n$  is invertible here.

**Remark 1.14.1.** It is very hard to say a domain is not an euclidean domain, because it's very hard to tell that  $N$  doesn't exists for sure.

## 1.15 The Rings $\mathbb{Z}[\sqrt{d}]$ and $\mathbb{Q}[\sqrt{d}]$

**Definition 1.15.1.** Fix  $d \in \mathbb{Z}$ , with  $|d|$  not a square: namely  $\nexists n \in \mathbb{Z}$ , s.t.  $n^2 = d$ . This implies that  $\sqrt{|d|} \notin \mathbb{Q}$ . We write  $\sqrt{|d|} = \sqrt{-d}$  if  $d < 0$ . Thus define:

$$\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$$

$$\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\} \subseteq \mathbb{C}$$

### Remark 1.15.1.

1. Since  $\sqrt{d} \notin \mathbb{Q}$ , thus if  $a + b\sqrt{d} = a' + b'\sqrt{d}$  for some  $a, a', b, b' \in \mathbb{Q} \Rightarrow a = a', b = b'$ .
2.  $\mathbb{Z}[\sqrt{d}]$  and  $\mathbb{Q}[\sqrt{d}]$  are subring of  $\mathbb{C}$ , thus they are **domains**.
3.  $\mathbb{Q}[\sqrt{d}]$  is a field, as one can multiply by its conjugate. If  $u = a + b\sqrt{d} \Rightarrow u \cdot (a - b\sqrt{d}) = a^2 - b^2d \in \mathbb{Q} \setminus \{0\}$ , thus:

$$u^{-1} = \frac{a}{a^2 - db^2} - \frac{b}{a^2 - db^2} \sqrt{d}$$

4. Since  $\mathbb{Z}[\sqrt{d}] \subseteq \mathbb{Q}[\sqrt{d}]$ , by the universal property of fraction field  $\mathbb{K}$  of  $\mathbb{Z}[\sqrt{d}]$  gives us:  $f : \mathbb{K} \rightarrow \mathbb{Q}[\sqrt{d}]$  as a ring homomorphism.
  - This is injective since  $\mathbb{K}$  is a field, since clearly the map is induced by  $\mathbb{Z}[\sqrt{d}] \subseteq \mathbb{Q}[\sqrt{d}]$  so it is not a zero map. The field has no non-trivial ideals and  $\ker(f)$ , thus the only case would be  $\ker(f) = \{0\}$ , yields injectivity.
  - This is surjective as for  $a + b\sqrt{d} \in \mathbb{Q}[\sqrt{d}]$ , one can always write:

$$a + b\sqrt{d} = \frac{p_1q_2 + p_2q_1\sqrt{d}}{q_1q_2} \text{ with } p_1q_2 + p_2q_1\sqrt{d} \text{ and } q_1q_2 \in \mathbb{Z}[\sqrt{d}]$$

Thus yields to be **isomorphism**. Note here the multiplicative system yields to be  $\mathbb{Z}[\sqrt{d}] \setminus \{0\}$  which is the only case can be constructed as field for  $\mathbb{K}$ .

In particular this tells us **up to isomorphism**, the fraction field of  $\mathbb{Z}[\sqrt{d}]$  is  $\mathbb{Q}[\sqrt{d}]$ .

5. Define:

$$\mathbb{Z}[\sqrt{d}] \xrightarrow{\varphi} \mathbb{Z}[\sqrt{d}] \varphi(a + b\sqrt{d}) = a - b\sqrt{d}$$

when  $d$  is negative, this is exactly the conjugation for complex number. See that  $\varphi$  is actually a **ring homomorphism** and in particular  $\varphi \circ \varphi = \text{Id} \Rightarrow \varphi$  is actually a ring isomorphism.

6. Define:

$$N : \mathbb{Z}[\sqrt{d}] \rightarrow \mathbb{Z}[\sqrt{d}]$$

$$N(u) = u \cdot \varphi(u)$$

$$N(a + b\sqrt{d}) = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - db^2 \in \mathbb{Z}$$

### Note.

$$N(uv) = N(u)N(v) \quad \forall u, v \quad (**)$$

since  $\varphi(uv) = \varphi(u)\varphi(v)$ . But similar result doesn't holds for addition. Thus it is not a ring homomorphism.

7. One can view  $\mathbb{Z}[\sqrt{d}]$  as being the quotient ring  $\mathbb{Z}[x] / (x^2 - d)$ . Since  $\mathbb{Z}$  is a Noetherian ring, by **Hilbert Basis Theorem 1.13.1**,  $\mathbb{Z}[x]$  is also a Noetherian ring, and thus  $\mathbb{Z}[\sqrt{d}]$  is a **Noetherian ring**. Alternatively, one can see that  $\mathbb{Z}[\sqrt{d}] = (1, \sqrt{d})$ , so it's finitely generated, and being a Noetherian ring.

**Example 1.15.1.**  $u = a + b\sqrt{d}$  is invertible in  $\mathbb{Z}[\sqrt{d}]$  if and only if  $N(u) = \pm 1$ .

**Proof.** If  $u$  is invertible, then  $uv = 1$  for some  $v \in \mathbb{Z}[\sqrt{d}]$ , thus:

$$N(u)N(v) = N(uv) = N(1) = 1$$

Since  $N(u), N(v)$  are integers, this implies that  $N(u) = \pm 1$ .

Conversely, if  $u \cdot \varphi(u) = N(u) = \pm 1 \Rightarrow u^{-1} = \pm \varphi(u)$ . ■

**Example 1.15.2 (Gauss Integers:  $d = -1$ ).** The Gauss Integers is given as:

$$\mathbb{Z}[i] := \{a + bi \mid a, b \in \mathbb{Z}\}$$

and  $N(a + bi) = a^2 + b^2$ . We see  $a + bi$  is a unit (invertible elements in a ring) if and only if  $a^2 + b^2 = 1 \Rightarrow$  the units are exactly  $\pm 1, \pm i$ .

# Appendix