Math494: Honors Algebra II

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Abstract

This is the note containning my personal thoughts as well as lecture notes. My course instructor is Prof. Mircea Immanuel Mustaţă.

Contents

Chapter 1

Ring Theory

We've learnt about group theories which represents the symmetry for objects, which is kind of abstract. Rings are groups with extra structures, it is naturally more complicated, however it is closer to our intuition due to the same reason.

1.1 Basic Definition

Definition 1.1.1 (Ring). A Ring is a tuple $(R, +, \cdot)$ being a set R endowed with 2 binary operations $(+)$ and $(\cdot)$, s.t.:

1. $(R, +)$ is an **abelian** group, with identity element 0_R or 0.
2. $(\cdot)$ is associative, and has an identity element 1_R or 1.
3. It satisfy distributivity:
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c), \ \forall\, a, b, c \in R$.
   - $(b + c) \cdot a = (b \cdot a) + (c \cdot a), \ \forall\, a, b, c \in R$.

Notation.

1. Usually write ab for $a \cdot b$.
2. If we don't use parentheses, the order of operations is First $(\cdot)$ then $(+)$.
3. If $(+), (\cdot)$ are understood, simply denote the ring by R.

Remark 1.1.1.

1. As always, with identity elements $0_R, 1_R$ are unique.
2. For every $a \in R$, we have a unique inverse w.r.t $(+)$, denoted by $-a$.
3. In general, don't require $xy = yx \ \forall\, x, y \in R$, if this is the case, then R is a commutative ring.
4. Sometimes the definition of a ring does not require existence of 1_R, such is defined as unitary ring.

Example 1.1.1.

1. $\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}$ are rings w.r.t. $(+), (\cdot)$.

2. If $n \in \mathbb{Z}_{>0}$, then $\mathbb{Z}/n\mathbb{Z}$ carries two operations:

$$[a] + [b] := [a+b]$$
$$[a] \cdot [b] := [ab]$$

where $[a] := a + n\mathbb{Z}$, this is well-defined since operations holds regardless of the choice of representatives This is a ring with $0_{\mathbb{Z}/n\mathbb{Z}} = [0]$ and $1_{\mathbb{Z}/n\mathbb{Z}} = [1]$.

3. Let R be any ring, then:

$$M_n(R) := \{A = (a_{ij})_{1 \leq i,j \leq n} \mid a_{ij} \in R \ \forall\, i, j\}$$

with "usual" addition and mult. for matrices:

$$(a_{ij}) + (b_{ij}) := (a_{ij} + b_{ij})$$
$$(a_{ij}) \cdot (b_{ij}) := (c_{ij}) \rightsquigarrow c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

then $(M_n(R), +, \cdot)$ is a ring with w.r.t. $1_{M_n(R)} = \begin{pmatrix} 1_R & & 0_R \\ & \ddots & \\ 0_R & & 1_R \end{pmatrix}$.

> **Note.** If $n \geq 2$, even if R is commutative, $M_n(R)$ is not commutative in general.

4. Given a family $(R_i)_{i \in I}$ of rings, where I may not be finite, define the following by **Cartesian Prod.**:

$$\prod_{i \in I} R_i := \{(a_i)_{i \in I} \mid a_i \in R_i \ \forall\, i\}$$

define the operations **componentwise**:

$$(a_i)_{i \in I} + (b_i)_{i \in I} := (a_i + b_i)_{i \in I}$$
$$(a_i)_{i \in I} \cdot (b_i)_{i \in I} := (a_i \cdot b_i)_{i \in I}$$

with $0 = (0_{R_i})_{i \in I}$ and $1 = (1_{R_i})_{i \in I}$. If $I = [n]$, simly write: $R_1 \times \cdots \times R_n$.

Proposition 1.1.1. If R is a ring and $a, b \in R$, then:

1. $a \cdot 0_R = 0_R = 0_R \cdot a$.
2. $-(ab) = (-a) \cdot b = a \cdot (-b)$.

The proof follows quickly from distributivity and the fact that $(R, +)$ is an abelian group.

Note. If R is a set with 1 element $\star$, then we can make it into a ring in a unique way, namely:

$$0_R = 1_R = \star$$

If R is a ring, then the following are equiv.:

1. $\#R = 1$.
2. $R = \{0_R\}$.
3. $1_R = 0_R$.

proof is also trivial.

Definition 1.1.2 (Ring Homomorphism).

Appendix