Problem Set 6

Math 565: Combinatorics and Graph Theory

Prof. Thomas Fun Yau Lam

Due Time: Nov 25 at 2:30p.m.

Author: **Yanzhi Li**

Collaborators: Lingwei Zhang

1 Problem 1

Problem 1.1.
(a) Let L be a lattice. Show that for any $x, y, z \in L$, we have

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

Solution:

Proof. By definition, we have:

$$\begin{aligned} x &\leq x \vee y \\ x &\leq x \vee z \\ y \wedge z &\leq y \leq x \vee y \\ y \wedge z &\leq z \leq x \vee z \end{aligned}$$

which implies that:

$$\begin{aligned} x &\leq (x \vee y) \wedge (x \vee z) \\ y \wedge z &\leq (x \vee y) \wedge (x \vee z) \\ \implies x \vee (y \wedge z) &\leq (x \vee y) \wedge (x \vee z) \end{aligned}$$

■

Problem 1.2.
(b) Let L be a finite lattice. We say that L is modular if for any $x, y, z \in L$ we have

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z))$$

Show that L is modular if and only if whenever $x \leq z$ we have

$$x \vee (y \wedge z) = (x \vee y) \wedge z$$

Solution:

Proof. ($\implies$): We have $\forall\, x, y, z \in L$:

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z))$$

substitute x by z and substitute z by x (switching the position of x and z), we have:

$$(z \wedge y) \vee (x \wedge z) = z \wedge (y \vee (x \wedge z))$$

Notice that if $x \leq z \implies x \wedge z = x,\ x \vee z = z$, then substitute such two equation in, we have:

$$x \vee (y \wedge z) = (x \vee y) \wedge z$$

($\impliedby$): Obviously, we have:

$$x \wedge z \leq x$$

so substitute into our giving condition we have:

$$\begin{aligned} (x \wedge z) \vee (x \wedge y) &= (y \vee (x \wedge z)) \wedge x \\ \iff (x \wedge y) \vee (x \wedge z) &= x \wedge (y \vee (x \wedge z)) \end{aligned}$$

which is intended. ■

2 Problem 2

Problem 2.1. Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^k$, and assume that these vectors span $\mathbb{R}^k$. Show that the set

$$\mathcal{M} := \{\{i_1, \ldots, i_k\} \mid v_{i_1}, \ldots, v_{i_k} \text{ is a basis for } \mathbb{R}^k\}$$

is a matroid of rank k on $[n]$.

Solution:

Proof. Let's fix $I, J \in \mathcal{M}$, and denote:

$$I := \{v_{e_1}, \ldots, v_{e_k}\}$$
$$J := \{v_{f_1}, \ldots, v_{f_k}\}$$

For $v_{e_j} \in I$ for some $j \in [k]$, see that:

$$\dim(span(\{v_{e_i} \mid i \neq j\})) = k - 1$$

Since $\{v_{e_1}, \ldots, v_{e_k}\}$ forms a basis for $\mathbb{R}^k$. But then by definition:

$$\dim(span(\{v_{f_1}, \ldots, v_{f_k}\})) = \dim(\mathbb{R}^k) = k$$

So there exists $v_{f_l} \in \mathbb{R}^k$ for some $l \in [k]$, such that $v_{f_l} \notin span(v_{e_i} \mid i \neq j)$. So $\{v_{e_i} \mid i \neq j\} \cup \{v_{f_l}\}$ is linearly independent, with cardinality to be k, thus be a basis. So we see:

$$(I - \{v_{e_j}\}) \cup \{v_{f_l}\} = \{v_{e_i} \mid i \neq j\} \cup \{v_{f_l}\} \in \mathcal{M}$$

which satisfy the exchange axiom, so $\mathcal{M}$ is a matroid. ■

3 Problem 3

Problem 3.1. Let $\mathcal{M}$ be a matroid of rank k on the set X. Recall that a set $A \subset X$ is **independent** if it is a subset of some basis of $\mathcal{M}$.

(a) Let $S \subset X$ be a nonempty subset. Show that the maximal (under inclusion) independent subsets of S all have the same cardinality.

Lemma 3.2. Let $\mathcal{I}$ to be the set of all the independent sets for matroid $\mathcal{M}$. If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 \backslash I_1$, such that $I_1 \cup e \in \mathcal{I}$.

Solution:

Proof of Lemma 3.2:

Proof. Given I_1 and $I_2 \in \mathcal{I}$, with $|I_1| < |I_2|$. By definition of independent set, there exists $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{M}$, such that $I_1 \subseteq \mathcal{B}_1$ and $I_2 \subseteq \mathcal{B}_2$. We first try to exchange iteratively the elements in $\mathcal{B}_2 \backslash I_2$ into elements of I_1: By exchange axiom, for $b \in \mathcal{B}_2 \backslash I_2$, there exists $a \in \mathcal{B}_1$, such that $(\mathcal{B}_2 - \{a\}) \cup \{b\} \in \mathcal{M}$, evidently, $b \notin I_1 \cap I_2$. We iteratively doing this for all elements in $\mathcal{B}_2 \backslash I_2$, and get $\mathcal{B}_2' \in \mathcal{M}$, with the result that $\mathcal{B}_2' \backslash I_2 \subseteq \mathcal{B}_1$, and in particular, $\mathcal{B}_2' = (\mathcal{B}_2' \backslash I_2) \sqcup I_2$, and evidently, $(\mathcal{B}_2' \backslash I_2) \cap (I_1 \cap I_2) = \emptyset$. Now we apply exchange axiom for $\mathcal{B}_1$ and $\mathcal{B}_2'$. Since $|I_1| < |I_2|$ and $|\mathcal{B}_1| = |\mathcal{B}_2| \implies |\mathcal{B}_1 \backslash I_1| > |\mathcal{B}_2' \backslash I_2|$ and $\mathcal{B}_2' \backslash I_2 \subseteq \mathcal{B}_1$, by **Pigeonhole Principle**, there exists $i \in \mathcal{B}_1 \backslash I_1$, such that $i \notin \mathcal{B}_2' \backslash I_2$. We apply exchange axiom to such element $i \in \mathcal{B}_1$: i

can not exchange element $j \in \mathcal{B}_2' \backslash I_2$ since they are already in $\mathcal{B}_1$, i can not exchange element $j \in I_1 \cap I_2$ since $I_1 \cap I_2 \subseteq I_1 \subseteq \mathcal{B}_1$. So i can only exchange element $e \in I_2 - I_1$, and get $\mathcal{B}_1' \in \mathcal{M}$, evidently $e \notin I_1$. And we see $I_1 \cup \{e\} \subseteq \mathcal{B}_1' \in \mathcal{M} \implies I_1 \cup \{e\} \in \mathcal{I}$. ■

Proof of Problem 3.1:

Proof. Given $I_1, I_2 \in \mathcal{I}$ to be two distinct maximal independent subsets of S. Suppose that I_1, I_2 has different cardinality. Without losing generality, suppose that $|I_1| > |I_2|$, by **Lemma** 3.2, there exists $e \in I_1 \backslash I_2$, such that $I_2 \cup \{e\} \in \mathcal{I}$. But see that $I_2 \subseteq S, e \in I_1 \subseteq S \implies I_2 \cup \{e\} \subseteq S$, but $|I_2 \cup \{e\}| > |I_2|$, contradict to the maximality of the cardinality of I_2 $\lightning$. So we see all maximal independent subsets of S all have the same cardinality. ■

Problem 3.3. Same setting as **Problem** 3.1:

(b) Show that these maximal independent subsets form a matroid $\mathcal{M}|_S$ on S, that is, they satisfy the exchange axiom.

Solution:

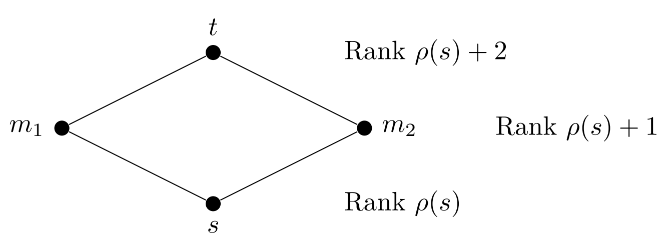
Proof. Lets fix $I, J \in \mathcal{I}$ to be two different maximal independent subset of S. $\forall\, i \in I$, define $I_i = I \backslash \{i\}$, with $|I_i| = k - 1$. See that $|J| = k > |I_i|$, then by **Lemma** 3.2, there exists $j \in J \backslash I_i$, such that $I_i \cup j \in \mathcal{I}$, with $|I_i \cup j| = k$ and $j \in J \subseteq S \implies I_i \cup j \in \mathcal{M}|_S$. But then $I_i \cup j = (I - \{i\}) \cup \{j\} \in \mathcal{M}|_S$, which satisfy the exchange axiom and thus $\mathcal{M}|_S$ is a matroid. ■

4 Problem 4

Problem 4.1. A poset P is **graded** (in the sense of the previous pset) if and only if we can assign an integer $\rho(x)$, called the rank, to each $x \in P$ so that if $x \lessdot y$ then $\rho(y) = \rho(x) + 1$. (You may assume this.) Let P be a finite graded poset with a $\hat{0}$ and $\hat{1}$. We say that P is **Eulerian** if each interval $[s, t]$ where $s < t$ has the same number of elements with odd rank as elements with even rank.

(a) What do intervals of length 2 (that is, $[s, t]$ where $\rho(t) = \rho(s) + 2$) in Eulerian posets look like?

Solution: Consider the following picture:



An interval of length 2 means the ranks of its bottom and top elements differ by two, i.e., $\rho(t) = \rho(s) + 2$. The elements properly between s and t must all lie at the intermediate rank, $\rho(s) + 1$. Let k be the number of these intermediate elements. The definition of an Eulerian poset states that the interval $[s, t]$ must have an equal number of elements with even rank and odd rank. Denote the elements in the interval $\{s, t\} \cup \{m_1, \ldots, m_k\}$. If $\rho(s)$ is even, then $\rho(t)$ is also even, and all k intermediate elements m_i have odd rank. This gives 2 elements (s, t) with even rank and k elements with odd rank. For the poset to be Eulerian, we must have $k = 2$. If $\rho(s)$ is odd, then $\rho(t)$ is also odd, and all k intermediate elements have even rank. This gives 2 elements (s, t)

with odd rank and k elements with even rank, which again implies $k = 2$. Thus, any interval of length 2 in an Eulerian poset must contain exactly two elements at the middle rank, forming the diamond-shaped lattice shown in the figure.

Problem 4.2. Same setting as **Problem** 4.1:

(b) Verify that the Boolean algebra B_n is Eulerian.

Solution:

Proof. The posets of the Boolean Algebra B_n, denoted as $P(B_n)$, with the entries be all the subset of $[n]$, and $\hat{0} = \emptyset$, $\hat{1} = [n]$, and the partial order given by $\subseteq$. We want to see such poset is graded and Eulerian.

- $\mathbf{P(B_n)}$ **is graded:** Given $A, B \in P(B_n)$, with $A \subseteq B$. Given a maximal chain from B to A, if the length of it is less than $|B \backslash A| + 1$, it means that there exists some $e \in B \backslash A$, such that it is not in any elements on such maximal chain by Pigeonhole Principle, thus we can extend the length of the chain by adding an element who is the union of e and the element that is connected to B on the chain and get a new chain, which contradict to maximality of the maximal chain $\lightning$. Now suppose that the length of the maximal chain is bigger than $|B \backslash A| + 1$, by Pigeonhole Principle, it exceeds the number of total elements of $|B \backslash A|$, thus cannot be a chain from A to B $\lightning$. So any maximal chain from A to B has same length, this shows that $P(B_n)$ is graded.

- $\mathbf{P(B_n)}$ **is Eulerian:** Since $P(B_n)$ is graded, so we can assign rank function to the entries in $P(B_n)$, which define as the cardinality of the entry:

$$\rho(X) = |X|$$

  Now given $[S, T]$ to be arbitrary interval in $P(B_n)$, see that $S \subseteq T$. There are four kinds of situation:

  - $\mathbf{S, T}$ **are both even numbers:** see that the cardinality difference is even in this case, suppose $|T| - |S| = 2k$ for some $k \in \mathbb{Z}$. Then:

$$\begin{aligned} \#\text{ elem. of even rank: } & \binom{2k}{0} + \binom{2k}{2} + \ldots + \binom{2k}{2k} \\ \#\text{ elem. of odd rank: } & \binom{2k}{1} + \binom{2k}{3} + \ldots + \binom{2k}{2k-1} \end{aligned}$$

  - $\mathbf{S, T}$ **are both odd numbers:** see that the cardinality difference is even in this case, suppose $|T| - |S| = 2k$ for some $k \in \mathbb{Z}$. Then:

$$\begin{aligned} \#\text{ elem. of odd rank: } & \binom{2k}{0} + \binom{2k}{2} + \ldots + \binom{2k}{2k} \\ \#\text{ elem. of even rank: } & \binom{2k}{1} + \binom{2k}{3} + \ldots + \binom{2k}{2k-1} \end{aligned}$$

  - **S is odd number, T is even number:** see that the cardinality difference is odd number this case, suppose $|T| - |S| = 2k + 1$ for some $k \in \mathbb{Z}$. Then:

$$\begin{aligned} \#\text{ elem. of even rank: } & \binom{2k+1}{1} + \binom{2k+1}{3} + \ldots + \binom{2k+1}{2k+1} \\ \#\text{ elem. of odd rank: } & \binom{2k+1}{0} + \binom{2k+1}{2} + \ldots + \binom{2k+1}{2k} \end{aligned}$$

- **S is even number, T is odd number:** see that the cardinality difference is odd in this case, suppose $|T| - |S| = 2k+1$ for some $k \in \mathbb{Z}$. Then:

$$\# \text{ elem. of odd rank: } \binom{2k+1}{1} + \binom{2k+1}{3} + \ldots + \binom{2k+1}{2k+1}$$
$$\# \text{ elem. of even rank: } \binom{2k+1}{0} + \binom{2k+1}{2} + \ldots + \binom{2k+1}{2k}$$

It left to proof that for all $k \in \mathbb{Z}$, the following holds:

$$\binom{2k+1}{1} + \binom{2k+1}{3} + \ldots + \binom{2k+1}{2k+1} = \binom{2k+1}{0} + \binom{2k+1}{2} + \ldots + \binom{2k+1}{2k} \tag{1}$$

$$\binom{2k}{0} + \binom{2k}{2} + \ldots + \binom{2k}{2k} = \binom{2k}{1} + \binom{2k}{3} + \ldots + \binom{2k}{2k-1} \tag{2}$$

Consider the following function:

$$\begin{aligned} f(x) &= (1+x)^{2k} \\ &= \binom{2k}{0} + \binom{2k}{1}x + \ldots + \binom{2k}{2k}x^{2k} \\ &= \sum_{i=0}^{2k} \binom{2k}{i} x^i \end{aligned}$$

Then consider the value of $f(-1)$:

$$\begin{aligned} f(-1) = (1-1)^{2k} &= 0 \\ &= \sum_{i=0}^{2k} \binom{2k}{i} (-1)^i \end{aligned}$$

$$\implies \sum_{i=0}^{2k} \binom{2k}{i} (-1)^i = 0$$

$$\iff \binom{2k}{0} + \binom{2k}{2} + \ldots + \binom{2k}{2k} = \binom{2k}{1} + \binom{2k}{3} + \ldots + \binom{2k}{2k-1}$$

which proves **Equation** 2. Similarly we consider:

$$\begin{aligned} g(x) &= (1+x)^{2k+1} \\ &= \binom{2k+1}{0} + \binom{2k+1}{1}x + \ldots + \binom{2k+1}{2k+1}x^{2k+1} \\ &= \sum_{i=0}^{2k+1} \binom{2k+1}{i} x^i \end{aligned}$$

Then consider the value of $g(-1)$:

$$\begin{aligned} g(-1) = (1-1)^{2k+1} &= 0 \\ &= \sum_{i=0}^{2k+1} \binom{2k+1}{i} (-1)^i \end{aligned}$$

$$\implies \sum_{i=0}^{2k+1} \binom{2k+1}{i} (-1)^i = 0$$

$$\iff \binom{2k+1}{1} + \binom{2k+1}{3} + \ldots + \binom{2k+1}{2k+1} = \binom{2k+1}{0} + \binom{2k+1}{2} + \ldots + \binom{2k+1}{2k}$$

which proves **Equation** 1. Thus we see in all the cases, there are same number of elements of odd rank as the elements with even rank in the interval, which shows that $P(B_n)$ is Eulerian.

So we see that the Boolean Algebra B_n is Eulerian. ■

Problem 4.3. Same setting as **Problem** 4.1:

(c) Prove that a poset is Eulerian if and only if the Mobius function is given by $\mu(s,t) = (-1)^{\rho(t)-\rho(s)}$.

Solution:

Proof. ($\Longrightarrow$): Suppose that P is a Eulerian poset. Notice that by definition of Eulerian, any interval of P will also be Eulerian. Given a interval $[s,t]$ of P, since P is Eulerian, we can assign rank function to it. We shall proceed the prove by induction on $\rho(t)-\rho(s)$ for arbitrary interval:

- **Base case**: when $\rho(t)-\rho(s) = 1$, there is only 2 elements in the interval, easy to verify that $\mu(s,t) = (-1)^{\rho(t)-\rho(s)}$ in the case.
- **Inductive case**: suppose that for any interval $[s',t']$ such that $1 \leq \rho(t')-\rho(s') < n$, we have $\mu(s',t') = (-1)^{\rho(t')-\rho(s')}$. Now consider the interval $[s,t]$, such that $\rho(s)-\rho(t) = n$. See that:

$$\mu(s,t) = -\sum_{s\leq z<t} \mu(s,z) \qquad s<t$$

Now apply the induction hypothesis, we have:

$$\begin{aligned}\mu(s,t) &= -\sum_{s\leq z<t} (-1)^{\rho(z)-\rho(s)} \\ &= -(-1)^{-\rho(s)} \sum_{s\leq z<t} (-1)^{\rho(z)}\end{aligned}$$

Since $[s,t]$ is Eulerian, there are same number of elements with odd rank as elements with even rank, see that:

$$\sum_{s\leq z\leq t} (-1)^{\rho(z)} = 0 \tag{3}$$

which implies that:

$$\begin{aligned}\sum_{s\leq z<t} (-1)^{\rho(z)} + (-1)^{\rho(t)} = 0 \\ \Longrightarrow \sum_{s\leq z<t} (-1)^{\rho(z)} = -(-1)^{\rho(t)}\end{aligned}$$

So we see:

$$\begin{aligned}\mu(s,t) &= -(-1)^{-\rho(s)} \sum_{s\leq z<t} (-1)^{\rho(z)} \\ &= -(-1)^{-\rho(s)} \cdot -(-1)^{\rho(t)} \\ &= (-1)^{\rho(s)-\rho(t)}\end{aligned}$$

which yields the inductive case.

($\Longleftarrow$): Given that for any interval $[s,t]$:

$$\begin{aligned}\mu(s,t) &= (-1)^{\rho(t)-\rho(s)} \\ &= -\sum_{s\leq z<t} \mu(s,z) \\ \Longrightarrow \sum_{s\leq z\leq t} \mu(s,z) &= 0\end{aligned}$$

Since the Mobius function holds for any interval, we then see:

$$\begin{aligned}
&\implies \sum_{s \le z \le t} (-1)^{\rho(z)-\rho(t)} = 0 \\
&\iff (-1)^{-\rho(t)} \cdot \sum_{s \le z \le t} (-1)^{\rho(z)} = 0 \\
&\iff \sum_{s \le z \le t} (-1)^{\rho(z)} = 0
\end{aligned}$$

which means that there are same number of elements with odd rank as elements with even rank with the interval $[s, t]$. Such holds for any interval in the poset, so such poset is Eulerian by definition. ■

5 Problem 5

Problem 5.1. Let $\mathcal{A}$ be the hyperplane arrangement consisting of the n hyperplanes $x_i = 0$ in $\mathbb{R}^n$, for $i = 1, 2, \ldots, n$.

(a) Show that the intersection poset $L(\mathcal{A})$ is isomorphic to the Boolean algebra.

Solution:

Proof. First denote that:

$$H_i := \{(x_1, \ldots, x_n) \mid x_i = 0,\ i \in [n]\}$$

to be the n hyperplane in $\mathcal{A}$. We then define a function as follow, and we intend to see it is an isomorphism:

$$\begin{aligned}
\varphi : L(\mathcal{A}) &\longrightarrow 2^{[n]} \\
H_I := \bigcap_{i \in I, I \subseteq [n]} H_i &\longmapsto I \subseteq [n]
\end{aligned}$$

- **φ is well-defined:** it is clear that φ is well-defined, since if $I \neq J$, then $H_I \neq H_J$ by our definition.
- **φ is injective:** given that:

  $$\begin{aligned}
  &\varphi(H_I) = \varphi(H_J) \\
  &\iff I = J \\
  &\implies H_I = H_J
  \end{aligned}$$

  so φ is injective.
- **φ is surjective:** by definition, it is clear that given $I \subseteq [n]$, there exists H_I, with $\varphi(H_I) = I$.
- **φ is homomorphism w.r.t. the poset:** we want to see that for $A, B \in L(\mathcal{A})$:

  $$A \le B \iff \varphi(A) \le \varphi(B)$$

  we write:

  $$\begin{aligned}
  A &:= \bigcap_{i \in I, I \subseteq [n]} H_i = H_I \\
  B &:= \bigcap_{i \in J, J \subseteq [n]} H_i = H_J
  \end{aligned}$$

then:

$$\begin{aligned} &A \leq B \\ \iff & B \subseteq A \\ \iff & \bigcap_{i \in J, J \subseteq [n]} H_i \subseteq \bigcap_{i \in I, I \subseteq [n]} H_i \\ \iff & I \subseteq J \\ \iff & \varphi(A) \subseteq \varphi(B) \\ \iff & \varphi(A) \leq \varphi(B) \end{aligned}$$

So we see φ preserve the poset structure between $L(\mathcal{A})$ and $2^{[n]}$.

So we see there exists an isomorphism φ between $L(\mathcal{A})$ and the Boolean algebra B_n. ■

Problem 5.2. Same setting as **Problem** 5.1:

(b) Compute the Mobius function of $L(\mathcal{A})$.

Solution: We've proven in **Problem** 5.1 that $L(\mathcal{A})$ is isomorphic with Boolean algebra B_n, so they will share the same Mobius function. We proven in **Problem** 4.2 that Boolean algebra B_n is actually an Eulerian poset, then so is $L(\mathcal{A})$. And we've proven in **Problem** 4.3, that a poset is Eulerian if and only if its Mobius function is given by $\mu(s,t) = (-1)^{\rho(t)-\rho(s)}$. So we see the Mobius function of $L(\mathcal{A})$ is also given by $\mu(s,t) = (-1)^{\rho(t)-\rho(s)}$.

Problem 5.3. Same setting as **Problem** 5.1:

(c) Compute the characteristic polynomial of $\mathcal{A}$.

Solution: By definition, we see:

$$\chi_{\mathcal{A}}(t) = \sum_{X \in L(\mathcal{A})} \mu(X) \cdot t^{\dim X}$$

Since $L(\mathcal{A})$ is Eulerian poset, the Mobius function of an element X of $L(\mathcal{A})$ is given by:

$$\mu(X) = (-1)^{\rho(X)-\rho(\hat{0})} = (-1)^{\rho(X)}$$

Since we see in Boolean algebra is isomorphic to $L(\mathcal{A})$, see that $\rho(X) = |\varphi(X)|$ and $\dim X = n - |\varphi(X)|$, so:

$$\begin{aligned} \chi_{\mathcal{A}}(t) &= \sum_{X \in L(\mathcal{A})} (-1)^{|\varphi(X)|} \cdot t^{n-|\varphi(X)|} \\ &= \sum_{\varphi(X) \in P(B_n)} (-1)^{|\varphi(X)|} \cdot t^{n-|\varphi(X)|} \\ &= \sum_{k \in P(B_n)} (-1)^{|k|} \cdot t^{n-|k|} \\ &= \sum_{i=0}^{n} \binom{n}{i} (-1)^i \cdot t^{n-i} \\ &= (t-1)^n \end{aligned}$$

6 Problem 6

Problem 6.1. Let G be a simple graph on $[n]$ and let $\mathcal{A}_G$ denote the corresponding graphical arrangement in $\mathbb{R}^n$. Prove that when G has no cycles, the poset $L(\mathcal{A}_G)$ is isomorphic to a Boolean algebra. Deduce a formula for the number of regions and bounded regions in $\mathcal{A}_G$ in this case.

Solution:

Proof. We denote the hyperplanes in the graphical arrangment in $\mathbb{R}^n$ as follow:

$$H_{(i,j)} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j,\ (i,j) \in E(G)\}$$

We denote $|E(G)| = m$. We then define a function as follow, and we intend to see it is an isomorphism:

$$\begin{aligned} \varphi : B_m &\longrightarrow L(\mathcal{A}_G) \\ I \subseteq E(G) &\longmapsto \bigcap_{e \in I} H_e =: H_{G_I} \end{aligned}$$

where G_I is the subgraph of G induced by edges in the edge set I, and the ground set of B_m is $E(G)$, which is the edge set of the simple graph G.

- **φ is well-defined:** it is clear that φ is well-defined since $\mathcal{A}_G$ is the graphical arrangement induced by the graph G.
- **φ is injective:** since G is simple and has no cycle, meaning that the graph is forest and thus independent, so different subspaces constructed by the intersection of different hyperplanes induced by different edge sets are different, thus for any $S_1, S_2 \in E(G)$:

  $$S_1 \neq S_2 \implies \bigcap_{e \in S_1} H_e \neq \bigcap_{e \in S_2} H_e$$

  which yields the injectivity of φ by contraposition.
- **φ is surjective:** given $H \in \mathcal{A}_G$, it can be written as $\bigcap_{e \in K} H_e$ for some $K \subseteq E(G)$ since it is a graphical arrangement. Then we see $H = \varphi(K)$, which yields the surjectivity of φ.
- **φ is homomorphism w.r.t. to poset:** we want to see that for $I, J \subseteq E$:

  $$I \leq J \iff \varphi(J) \geq \varphi(I)$$

  see that:

  $$\begin{aligned} &I \leq J \\ \iff &I \subseteq J \\ \implies &\bigcap_{e \in J} H_e \subseteq \bigcap_{e \in I} H_e \end{aligned}$$

  But since G is a simple graph with no cycle:

  $$\bigcap_{e \in J} H_e \subseteq \bigcap_{e \in I} H_e \implies I \subseteq J$$

  then:

  $$\begin{aligned} &\bigcap_{e \in J} H_e \subseteq \bigcap_{e \in I} H_e \iff I \subseteq J \\ \implies &\varphi(J) \subseteq \varphi(I) \iff I \subseteq J \\ \implies &\varphi(J) \geq \varphi(I) \iff I \leq J \end{aligned}$$

  So we see φ preserve the poset structure between $L(\mathcal{A}_G)$ and B_m.

So we see that when G be simple graph that has no cycles, the poset $L(\mathcal{A}_G)$ is isomorphic to a Boolean algebra, in this case is B_m where m is the size of the edge set of G. ■

Deduction of $\mathbf{r}(\mathcal{A}_\mathbf{G})$ and $\mathbf{b}(\mathcal{A}_\mathbf{G})$:

Since $L(\mathcal{A}_G)$ is isomorphic to B_m which is a Boolean algebra, by **Problem** 5.3, we've seen that hyperplane arrangement which are isomorphic to B_n has the characteristic polynomial to be:

$$\chi_{\mathcal{A}}(t) = (t-1)^n$$

For $I \subseteq E$, the corresponding subspace attains dimension to be $n - |I|$. So we see for graphical arrangment $\mathcal{A}_G$, the characteristic polynomial is given by:

$$\begin{aligned}
\chi_{\mathcal{A}_G} &= \sum_{k=0}^{m} \binom{m}{k} (-1)^k t^{n-k} \\
&= t^{n-m} \sum_{k=0}^{m} (-1)^k t^{m-k} \\
&= t^{n-m} (t-1)^m
\end{aligned}$$

And we see for any hyperplane arrangement $\mathcal{A}$:

$$\begin{aligned}
r(\mathcal{A}) &= (-1)^n \chi_{\mathcal{A}}(-1) \\
b(\mathcal{A}) &= (-1)^{rank(\mathcal{A})} \chi_{\mathcal{A}}(1)
\end{aligned}$$

So for $\mathcal{A}_G$, we have:

$$\begin{aligned}
r(\mathcal{A}_G) &= (-1)^n \chi_{\mathcal{A}_G}(-1) = (-1)^n \cdot (-1)^{n-m} \cdot (1-(-1))^m = 2^m \\
b(\mathcal{A}_G) &= (-1)^{rank(\mathcal{A}_G)} \chi_{\mathcal{A}_G}(1) = (1-1)^m = 0
\end{aligned}$$

7 Problem 7

Problem 7.1. Let $\mathcal{A}$ be the hyperplane arrangement in $\mathbb{R}^n$ consisting of all hyperplanes $x_i = x_j$ for $i \neq j$ and the hyperplanes $x_i = 0$ for $i = 1, 2, \ldots, n$. Prove that

$$\chi_{\mathcal{A}}(t) = (t-1)(t-2)(t-3)\cdots(t-n)$$

Solution:

Proof. We denote the hyperplane arrangement in $\mathbb{R}^n$ consisting of all hyperplanes $x_i = x_j$ for $i \neq j$ and the hyperplanes $x_i = 0$ for $i \in [n]$ as $\mathcal{A}^n$. And we define the hyperplanes of $\mathcal{A}^n$ as:

$$\begin{aligned}
H^n_{i,j} &:= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\} \\
H^n_i &:= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0\}
\end{aligned}$$

We shall proceed the prove for:

$$\chi_{\mathcal{A}^n}(t) = (t-1)(t-2)(t-3)\cdots(t-n)$$

by induction on the dimension of the endian space $\mathbb{R}^n$.

- **Base case:** when $n = 2$, we have the $L(\mathcal{A}^2)$ as follows:

$$\begin{tikzcd}
& \{\mathbf{0}\} & \\
H_1^2 \arrow[ur, dash] & H_2^2 \arrow[u, dash] & H_{1,2}^2 \arrow[ul, dash] \\
& \mathbb{R}^2 \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

  See that:

$$\begin{aligned}
\mu(\mathbb{R}^2) &= 1 \\
\mu(H_1^2) &= -\mu(\mathbb{R}^2) = -1 \\
\mu(H_2^2) &= -1 \\
\mu(H_{1,2}^2) &= -1 \\
\implies \mu(\{0\}) &= -(\mu(\mathbb{R}^2) + \mu(H_1^2) + \mu(H_2^2) + \mu(H_{1,2}^2)) \\
&= -(1 + (-1) + (-1) + (-1)) = 2
\end{aligned}$$

  Then by:

$$\begin{aligned}
\chi_{\mathcal{A}}(t) &= \sum_{X \in L(\mathcal{A})} \mu(X) \cdot t^{\dim X} \\
\implies \chi_{\mathcal{A}^2}(t) &= t^2 - 3t + 2 = (t-1)(t-2)
\end{aligned}$$

  which yields the base case.

- **Inductive case:** Suppose that:

$$\chi_{\mathcal{A}^k}(t) = (t-1)(t-2)\cdots(t-k)$$

  for $k \geq 2$, now we consider $\mathcal{A}^{k+1}$. We shall remove the hyperplanes $H_0 := H_{k+1}^{k+1}$, $H_1 := H_{1,k+1}^{k+1}, H_2 := H_{2,k+1}^{k+1} \ldots, H_k := H_{k,k+1}^{k+1}$ one by one in order, and denote the result hyperplane arrangement as $\mathcal{A}_k$. We will remove $k+1$ hyperplanes one by one in order in this case, and we will denote the hyperplane arrangement after the i-th removal by $\mathcal{A}_{i-1}$, for example, $\mathcal{A}^{k+1} \backslash H_{k+1}^{k+1} =: \mathcal{A}_0$ and $\mathcal{A}^{k+1} \backslash (H_{k+1}^{k+1} \cup \{H_{j,k+1}^{k+1} \mid j \in [i],\ i \leq k\}) =: \mathcal{A}_i$. Notice that we have:

$$\begin{aligned}
\chi_{\mathcal{A}}(t) &= \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t) \\
\implies \chi_{\mathcal{A}'}(t) &= \chi_{\mathcal{A}}(t) + \chi_{\mathcal{A}''}(t)
\end{aligned}$$

  By our removal idea, see that:

$$\begin{aligned}
\chi_{\mathcal{A}_k}(t) &= \chi_{\mathcal{A}^{k+1}}(t) + \sum_{i=0}^{k-1} \chi_{\mathcal{A}_i''}(t) + \chi_{(\mathcal{A}^{k+1})''}(t) \\
\implies \chi_{\mathcal{A}^{k+1}}(t) &= \chi_{\mathcal{A}_k}(t) - (\sum_{i=0}^{k-1} \chi_{\mathcal{A}_i''}(t) + \chi_{(\mathcal{A}^{k+1})''}(t))
\end{aligned} \tag{4}$$

  where $(\mathcal{A}^{k+1})''$ denotes the hyperplane arrangement that is formed by contracting H_0 from $\mathcal{A}^{k+1}$, and $\mathcal{A}_i''$ denotes the hyperplane arrangement that is formed by contracting H_{i+1} from $\mathcal{A}_i$.

  We first **claim** that:

$$\begin{aligned}
(A^{k+1})'' &\simeq \mathcal{A}^k \\
\mathcal{A}_i'' &\simeq \mathcal{A}^k \qquad 0 \leq i \leq k-1
\end{aligned} \tag{5}$$

In particular, our claim means that contracting any of the $k+1$ hyperplanes involving x_{k+1} results in an arrangement isomorphic to $\mathcal{A}^k$.

First consider $(\mathcal{A}^{k+1})''$ which is formed by contracting H_0 from $\mathcal{A}^{k+1}$. See that:

$$H_0 = H_{k+1}^{k+1} = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_{k+1} = 0\}$$

after the contraction, the hyperplane of $(\mathcal{A}^{k+1})''$ can be grouped into the following three classes:

$$\begin{aligned}
A &:= \{H_i^{k+1} \cap H_{k+1}^{k+1}\}_{i\in[k]} = \{\{x \mid x_i = x_{k+1} = 0\}\}_{i\in[k]} \\
B &:= \{H_{i,j}^{k+1} \cap H_{k+1}^{k+1}\}_{i,j\in[k], i\neq j} = \{\{x \mid x_{k+1} = 0,\ x_i = x_j\}\}_{i,j\in[k], i\neq j} \\
C &:= \{H_{i,k+1}^{k+1} \cap H_{k+1}^{k+1}\}_{i\in[k]} = \{\{x \mid x_{k+1} = x_i = 0\}\}_{i\in[k]} \\
\implies A &= C \\
\implies (\mathcal{A}^{k+1})'' &= A \sqcup B
\end{aligned}$$

See that in this case $(\mathcal{A}^{k+1})''$ shares the same lattice structure as well as the dimension structure as $\mathcal{A}^k$ since we always have $x_{k+1} = 0$. In particular, we see $(\mathcal{A}^{k+1})'' \simeq \mathcal{A}^k$.

Then we consider $\mathcal{A}_i''$ which is formed by conrtacting H_{i+1} from $\mathcal{A}_i$, where $\mathcal{A}_i = \mathcal{A}^{k+1} \backslash \{H_0, \ldots, H_i\}$. See that:

$$H_i = H_{i,k+1}^{k+1} = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_i = x_{k+1}\} \qquad i \in [k]$$

after the contraction, the hyperplanes of $\mathcal{A}_i''$ can be grouped into the following five classes:

$$\begin{aligned}
I &:= \{H_j \cap H_{i+1}\}_{j>i+1, i,j\in[k]} = \{\{x \mid x_j = x_{k+1} = x_{i+1}\}\}_{j>i+1,i,j\in[k]} \\
J &:= \{H_{i+1,j}^{k+1} \cap H_{i+1}\}_{j\neq i+1, i,j\in[k]} = \{\{x \mid x_j = x_{i+1} = x_{k+1}\}\}_{j\neq i+1,i,j\in[k]} \\
K &:= \{H_{j,l}^{k+1} \cap H_{i+1}\}_{j\neq l\neq i+1, i,j,l\in[k]} = \{\{x \mid x_j = x_l, x_{i+1} = x_{k+1}\}\}_{j\neq l\neq i+1,i,j,l\in[k]} \\
L &:= \{H_j^{k+1} \cap H_{i+1}\}_{j\neq i+1, j\in[k]} = \{\{x \mid x_j = 0,\ x_{i+1} = x_{n+1}\}\}_{j\neq i+1,j\in[k]} \\
O &:= \{H_{i+1}^{k+1} \cap H_{i+1}\} = \{\{x \mid x_{i+1} = x_{n+1} = 0\}\} \\
\implies I &\subseteq J \\
\implies \mathcal{A}_i'' &= J \sqcup K \sqcup L \sqcup O
\end{aligned}$$

For $\mathcal{A}^k$, denote:

$$\begin{aligned}
X &:= \{\{x \in \mathbb{R}^k \mid x_i = x_j,\ i \neq j,\ i,j \in [k]\}\} \\
Y &:= \{\{x \in \mathbb{R}^k \mid x_i = 0,\ i \in [k]\}\}
\end{aligned}$$

by definition, see that:

$$\mathcal{A}^k = X \sqcup Y$$

Then notice that for $\mathcal{A}_i'' = J \sqcup K \sqcup L \sqcup O$, the condition of $x_{i+1} = x_{k+1}$ drops the dimension by 1, and further compare the indices and dimension, see that:

$$\begin{aligned}
X &\simeq L \sqcup O \\
Y &\simeq J \sqcup K
\end{aligned}$$

In particular, we then see that:

$$\mathcal{A}^k \simeq \mathcal{A}_i''$$

which yields our **Claim** 5. Then by **Claim** 5 and the **Induction Hypothesis**, see that:

$$\begin{aligned}
\chi_{\mathcal{A}^k}(t) &= (t-1)(t-2)\cdots(t-k) \\
&= \chi_{\mathcal{A}_i''}(t) \qquad \forall\, 0 \leq i \leq k-1 \\
&= \chi_{(\mathcal{A}^{k+1})''}(t)
\end{aligned} \tag{6}$$

We then **claim** that:

$$\chi_{\mathcal{A}_k}(t) = t(t-1)(t-2)\cdots(t-k) \tag{7}$$

we write:

$$X' := \{\{x \in \mathbb{R}^{k+1} \mid x_i = x_j,\ i \neq j,\ i,j \in [k]\}\}$$
$$Y' := \{\{x \in \mathbb{R}^{k+1} \mid x_i = 0,\ i \in [k]\}\}$$

and see that:

$$\mathcal{A}_k = X' \sqcup Y'$$

Notice that $L(\mathcal{A}^k)$ and $L(\mathcal{A}_k)$ shares the same lattice structure, in particular $L(\mathcal{A}^k) \simeq L(\mathcal{A}_k)$, so they have the same Mobius function. However, deleting all hyperplanes involving x_{k+1} leaves an arrangement equivalent to $\mathcal{A}^k \times \mathbb{R}$, in particular, every elements in $L(\mathcal{A}_k)$ is exactly 1 dimension bigger than the corresponding elements in $L(\mathcal{A}^k)$. By the fact that:

$$\chi_{\mathcal{A}}(t) = \sum_{X \in L(\mathcal{A})} \mu(X) \cdot t^{\dim X}$$

see that also by our **Inductive Hypothesis**:

$$\chi_{\mathcal{A}_k}(t) = t \cdot \chi_{\mathcal{A}_k}(t) = t(t-1)(t-2)\cdots(t-k) \tag{8}$$

which yields our **Claim** 7.

Now by **Equation** 4, **Equation** 6 and **Equation** 8:

$$\begin{aligned}
\chi_{\mathcal{A}^{k+1}}(t) &= \chi_{\mathcal{A}_k}(t) - (\sum_{i=0}^{k-1} \chi_{\mathcal{A}_i''}(t) + \chi_{(\mathcal{A}^{k+1})''}(t)) \\
&= t(t-1)(t-2)\cdots(t-k) - [k \cdot (t-1)(t-2)\cdots(t-k) + (t-1)(t-2)\cdots(t-k)] \\
&= t(t-1)(t-2)\cdots(t-k) - (k+1)\cdot(t-1)(t-2)\cdots(t-k) \\
&= (t-1)(t-2)\cdots(t-k)(t-(k+1))
\end{aligned}$$

which yields the inductive case.

■

8 Appendix