Problem Set 3

Math 565: Combinatorics and Graph Theory

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Due Time: Oct 7 at 2:30p.m.

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1 Problem 1

Problem 1.1. Let $(A_1, A_2, \ldots, A_n)$ be a family of sets with a system of distinct representatives. Let $a \in A_1$. Prove that there is an SDR containing a, but show by example that it may not be possible to find an SDR $(a_1, \ldots, a_n)$ in which $a_1 = a$.

Solution:

Proof. Since we know that $(A_1, A_2, \ldots, A_n)$ be a family of sets with a system of distinct representatives, which leads to the fact that **Hall's Condition** holds for $(A_1, A_2, \ldots, A_n)$. In particular, $\forall k \in [1, n]$, with $\forall i_1 < \ldots < i_k$, see that:

$$|A_{i_1} \cup \ldots \cup A_{i_k}| \geq k$$

Consider the following case of where is a:

- If a is in any kind of critical block, which is $\exists k_0$, s.t.

  $$|A_{j_1} \cup \ldots \cup A_{j_{k_0}}| = k_0$$

  with $a \in A_{j_1} \cup \ldots \cup A_{j_{k_0}}$, see that in this situation a must be in any SDR of $(A_1, \ldots, A_n)$, which is by the fact that any SDR for the entire family must assign k_0 distinct representatives to the subfamily $A_{j_1}, \ldots, A_{j_{k_0}}$, by definition, these representatives must be chosen from the union $U = A_{j_1} \cup \ldots \cup A_{j_{k_0}}$. Since the size of this union is exactly k_0, the set of the k_0 required representatives must be identical to the set U itself. Provided that $a \in U$, it follows that a must be in the SDR of $(A_1, \ldots, A_n)$.

- If a is not in any kind of critical block. We then consider $a \in A_1$, and we shall denote $A_i(a) := A_i - \{a\}$. Given $k \in [1, n-1]$, with $\forall 1 < i_1 < \ldots < i_k$, with the Hall's Condition followed by:

  $$|A_{i_1} \cup \ldots \cup A_{i_k}| \geq k$$

  If it is a critical block, we see:

  $$|A_{i_1} \cup \ldots \cup A_{i_k}| = k$$

  And after excluding a, it will still satisfy Hall's Condition, since a is not in any kinds of critical block:

  $$|A_{i_1}(a) \cup \ldots \cup A_{i_k}(a)| = k$$

  If it is not a critical block, see that:

  $$|A_{i_1} \cup \ldots \cup A_{i_k}| > k$$

  and thus:

  $$|A_{i_1}(a) \cup \ldots \cup A_{i_k}(a)| \geq k$$

  So Hall's Condition will holds for $(A_2, \ldots, A_n)$, and in particular it will attains a SDR as $(a_2, \ldots, a_n)$, and we now add back a in as the representative of A_1 and we shall thus find a SDR of $(A_1, \ldots, A_n)$ as $(a, a_2, \ldots, a_n)$.

We now give an example to show tha it may not be possible to find an SDR $(a_1, \ldots, a_n)$:

Consider A_1, A_2, A_3, A_4 defined as follows:

$$\begin{aligned} A_1 &= \{a, b\} \\ A_2 &= \{a, c\} \\ A_3 &= \{c, d\} \\ A_4 &= \{c, d\} \end{aligned}$$

For such cases all possible SDR will be $(a_1, a_2, a_3, a_4) = (b, a, c, d)$ or (b, a, d, c), and then one shall see that $a \in A_1$ but $a \neq a_1$ ∎

2 Problem 2

Problem 2.1. A *perfect* matching in a (possibly not bipartite) graph is a collection of edges so that every vertex is incident to one edge of the matching.

(a) Show that a finite regular bipartite graph of degree $d > 0$ has a perfect matching.

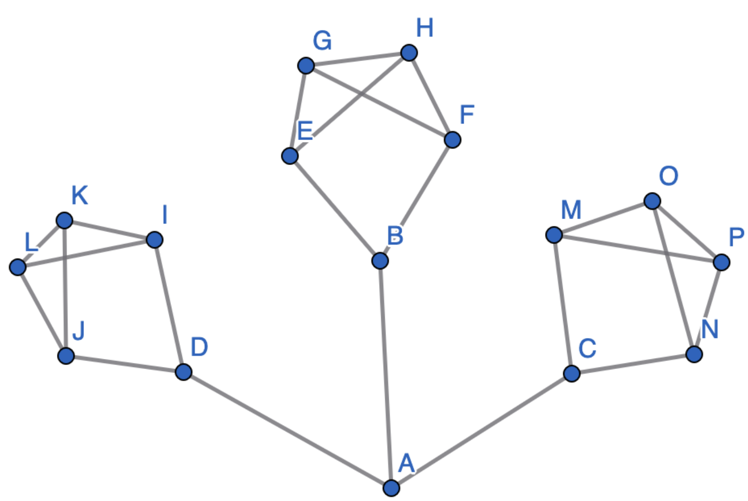
Solution:

Proof. We assume that $G = X \sqcup Y$ to be such finite regular bipartite graph of degree $d > 0$. We first want to see that $|X| = |Y|$. And we shall reasoning by contradiction, suppose that $|X| \neq |Y|$, the in degree of the component X or say edges connected to the component X will be $|Y| \cdot d$ by the definition of regular bipartite graph. Since the graph is regular, each vertex in $|X|$ will connect to $\frac{|Y| \cdot d}{|X|} \neq d$ ↯, which leads to contradiction. So we see that $|X| = |Y|$.

To see if there is a matching between X and Y, we shall see if $\forall A \subset X, |\Gamma(A)| \geq |A|$. We shall proof this by contradiction. Suppose there $\exists A \subset X$, s.t. $|\Gamma(A)| < |A|$, we denote $|A| = k$, $|\Gamma(A)| = g$. The total number of edges connected to $\Gamma(A)$ will be at least $k \cdot d$. But since by definition the total number of edges connected to $\Gamma(A)$ must be $g \cdot d$ by regularity, and $g \cdot d < k \cdot d$ ↯, this leads to contradiction. So we see that indeed $\forall A \subset X, |\Gamma(A)| \geq |A|$. By Hall's marriage theorem, we see there is a complete matching from X to Y, and since $|X| = |Y|$, such matching will be perfect. ■

Problem 2.2. (b) Find a simple graph, regular of degree 3, that does not have a perfect matching.

Solution:



Above is a simple graph, regular of degree 3, and we claim there is no way to find a perfect matching. Consider the vertex B, there will possibly 3 way of matching for B:

- Consider matching for (B, E) or (B, F), then there will be no matching with the polygon $BEGHF$ since exclude out two vertices will only have three vertices remain, which is odd number, and cannot have a matching.
- Consider matching for (B, A). Then the rest of the two polygon $DJLKI$ and $CNPOM$ will be isolated into two disjoint component, each of them consists of 5 vertices, which is odd number, and thus cannot have a matching.

So we see that the graph we given indeed exists no perfect matching.

Problem 2.3. (c) Suppose G is bipartite with parts X and Y. Further assume that every vertex in X has the same degree $s > 0$ and every vertex in Y has the same degree t. Prove that if $s \geq t$, then there is a complete matching M of X into Y.

Solution:

Proof. We want to see if $\forall A \subset X, |\Gamma(A)| \geq |A|$. We will give the reasoning by contradiction. Suppose that there exists $A \subset X$, s.t. $|\Gamma(A)| < A$ and suppose that $|A| = k, |\Gamma(A)| = g$, the total edges connected to $|\Gamma(A)|$ will be at least $s \cdot k$. Then we see:

$$t \cdot g < t \cdot k \leq s \cdot k$$

By Pigeonhole Principle, there must exists one vertex in $\Gamma(A)$ such that this vertex attains degree bigger than t, which leads to contradiction. So no such A exists, leading to $\forall A \subset X, |\Gamma(A)| \geq |A|$. By Hall's marriage theorem, we see there is a complete matching from X to Y. ■

3 Problem 3

Problem 3.1. Let S be the set $\{1, 2, \ldots, mn\}$. We partition S into m sets $A_1, \ldots, A_m$ of size n. We also partition S into m sets of $B_1, \ldots, B_m$ of size n. Show that the sets A_i can be renumbered so that $A_i \cap B_i \neq \emptyset$.

Solution:

Proof. We can construct a graph, whose vertices correspond to $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$, and we connect two vertices if their corresponding sets' intersection is not empty, and if their corresponding sets' intersection is empty, we don't connect. It immediately follows that $\forall i, j \in [1, m], i \neq j$:

$$A_i \cap A_j = \emptyset$$
$$B_i \cap B_j = \emptyset$$

since we construct such sets by partitioning S. So we see G is a bipartite graph, with $A = \{A_1, \ldots, A_m\}, B = \{B_1, \ldots, B_m\}$, and $G = A \sqcup B$. To see that the sets A_i can be renumbered so that $A_i \cap B_i \neq \emptyset$, it is equivalent to show that there exists a complete matching from A to B in this situation.

We then want to check if $\forall C \subset A, |\Gamma(C)| \geq |C|$. We then define that $D := \bigcup_{A_i \in C} A_i$, and we see that $|D| = |C| \cdot n$, by the fact that A_i are disjoint, with each of its cardinality to be n. Since $B_j, \forall j \in [1, m]$ are all disjoint, we will need at least $|C|$ number of block B_j to cover all these $|C| \cdot n$ elements, by the fact that B_j are also disjoint, with each of its cardinality to be n. In particular, we see that if we use at least $|C|$ number of B_j to cover these elements, we have for each B_j, $B_j \cap A_k \neq \emptyset$ for some $A_k \in C$, which means they are connected by an edge by our construction of the bipartite graph G. In particular we see that this leads to:

$$|\Gamma(C)| \geq |C|$$

So by Hall's marriage theorem, there exists a complete matching from A to B, so the proof is done. ■

4 Problem 4

Problem 4.1. Let A_i, $1 \leq i \leq k$ be distinct subsets of $\{1, 2, \ldots, n\}$. Suppose that $A_i \cap A_j \neq \emptyset$ for all i and j. Show that $k \leq 2^{n-1}$ and give an example where equality occurs.

Solution:

Proof. First see that there are in total 2^n number of subsets of the set $B := \{1, 2, \ldots, n\}$. Consider arbitrary a set $A \in B$, we see that $A \cap \overline{A} = \emptyset$. Lets say we have a collection of distinct subsets $A_i \in B$, such that $A_i \cap A_j \neq \emptyset$ for all i and j. If the size of the collection is larger than 2^{n-1}, by pigeonhole principle, we see that there must exists a set A_l in such collection, with $\overline{A_l}$ also in this collection. But we see this leads to contradiction, as $A_l \cap \overline{A_l} = \emptyset$. So we see that $k \leq 2^{n-1}$.

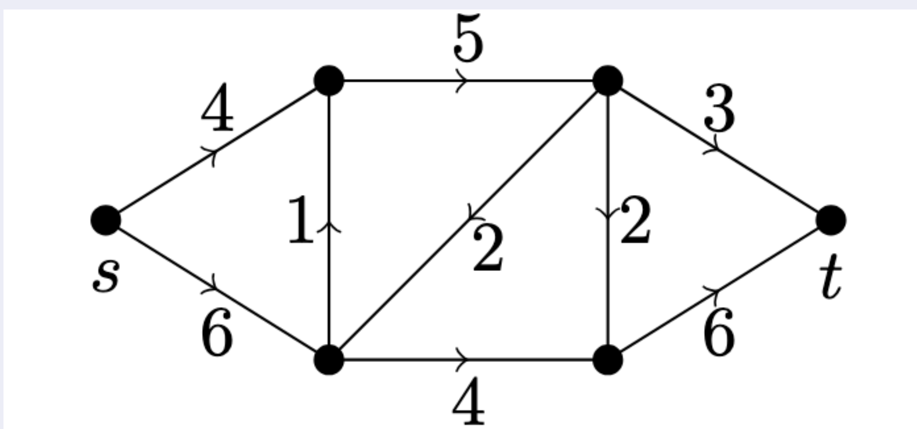
We now give an example on how such equality will occur. In the situation when n is odd number, we shall write it as $n = 2p+1$ for some $p \in \mathbb{N}$. We see that all subsets of B whose cardinality is bigger or equal to $p+1$ satisfy the requirement. We see that by pigeonhole's principle, any two of such sets will attain its intersection non-empty, and the total number of such sets are:

$$\binom{n}{p+1} + \binom{n}{p+2} + \ldots + \binom{n}{n} = 2^{n-1}$$

and we see this is an example where equality occurs. ∎

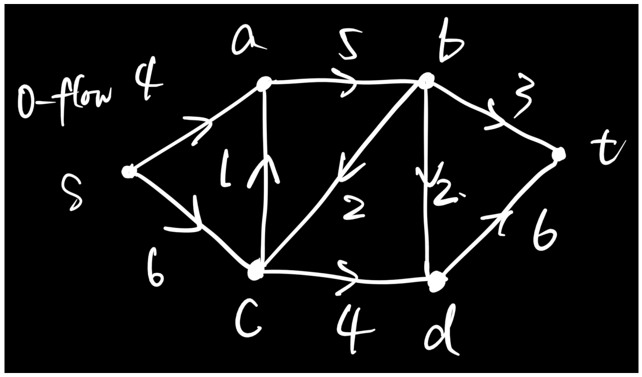
5 Problem 5

Problem 5.1. Find the max-flow and min-cut in the following network.

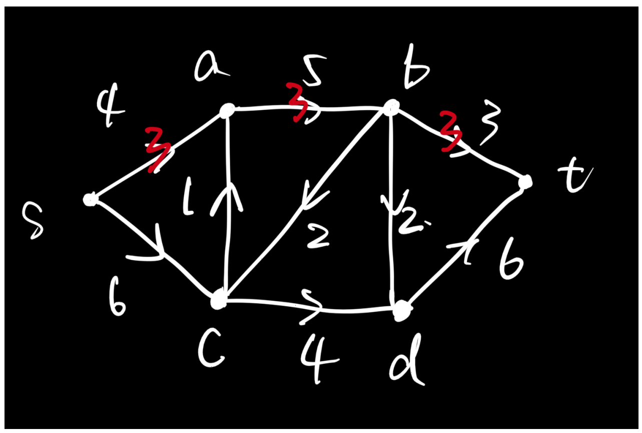


Solution: We will proceed the algorithm introduce in the lecture to find the max-flow and min-cut in the transportation network.

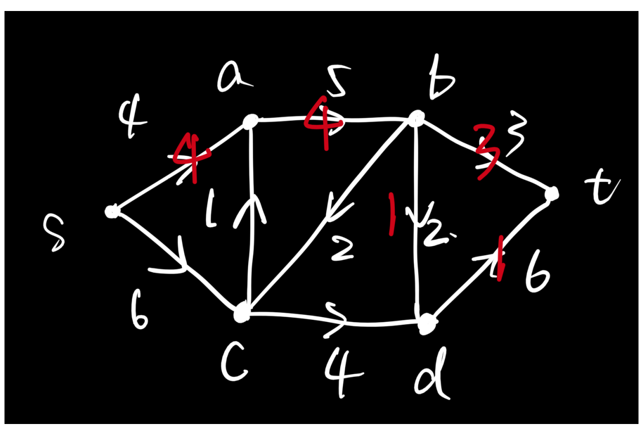
First we start with a 0-flow, and we label the vertices in the transportation network as the picture below:



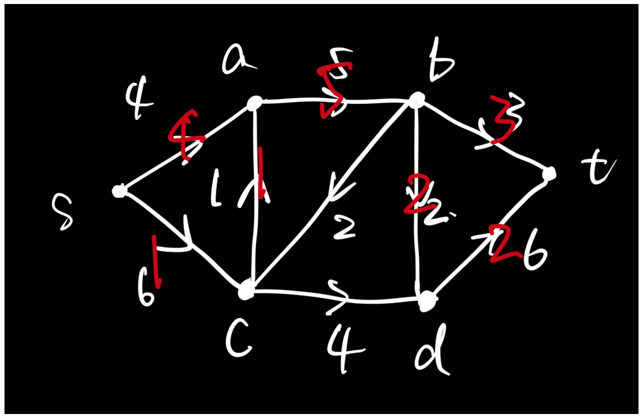
We start with the special path $s \to a \to b \to t$, and notice that in this case $\alpha_1 = 4, \alpha_2 = 5, \alpha_3 = 3 \implies \alpha = 3$, so we see $|f| = 0 + 3 = 3$, and then obtain the transportation network below, note that red number here denote the flow through corresponding edges.



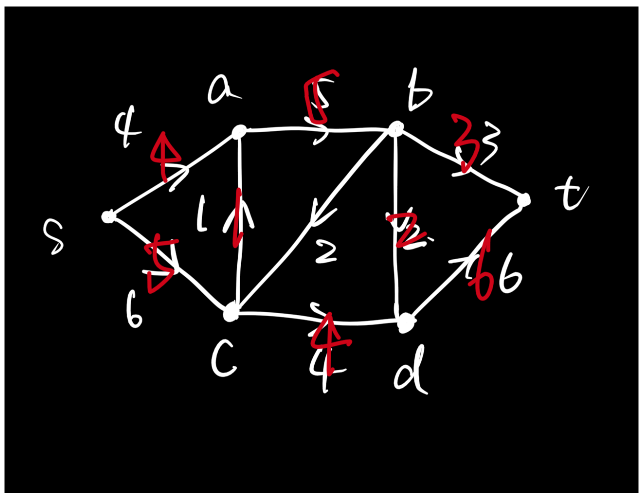
Choose the special path $s \to a \to b \to d \to t$, we have $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 2, \alpha_4 = 6 \implies \alpha = 1$, then we see $|f| = 3 + 1 = 4$, and obtain the transportation network below:



Choose the special path $s \to c \to a \to b \to d \to t$, we have $\alpha_1 = 6, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = 1, \alpha_5 = 5 \implies \alpha = 1$, then $|f| = 4 + 1 = 5$, and obtain the transportation network below:



Choose the special path $s \to c \to d \to t$, we have $\alpha_1 = 5, \alpha_2 = 4, \alpha_3 = 4 \implies \alpha = 4$, so $|f| = 5 + 4 = 9$, and obtain the transportation network below:



And we see that at this stage, there is no special path from s to t, and we obtain a max flow, with $|f| = 9$. We see that the vertices that are reachable by s through special paths are a, c, b, so we see we have a min cut to be $(\{s, a, b, c\}, \{t, d\})$, with $C(\{s, a, b, c\}, \{t, d\}) = 9$, and another min cut $(\{s, a, b, c, d\}, \{t\})$ with $C(\{s, a, b, c, d\}, \{t\}) = 9$.

6 Problem 6

Problem 6.1. Let (X_1, Y_1) and (X_2, Y_2) be two mincuts. Show that $(X_1 \cup X_2, Y_1 \cap Y_2)$ is also a mincut.

Solution:

Proof. Suppose that (X_1, Y_1) and (X_2, Y_2) are two cut separating s and t which are the source vertex and sink vertex respectively, we will have $s \in X_1$, $s \in X_2 \implies s \in X_1 \cap X_2$ and $t \in Y_1$, $t \in Y_2 \implies t \in Y_1 \cap Y_2$.

First we need to see that $(X_1 \cup X_2, Y_1 \cap Y_2)$ is a valid cut. Suppose that $D = X_1 \cup Y_1 = X_2 \cup Y_2$. And let $X_3 := X_1 \cup X_2$ and $Y_3 := Y_1 \cap Y_2$. Since $s \in X_1 \cap X_2 \implies s \in X_1 \cup X_2$ and $t \in Y_1 \cap Y_2$, and $D - Y_3 = D - (Y_1 \cap Y_2) = (D - Y_1) \cup (D - Y_2) = X_1 \cup X_2$, we see that it is indeed a valid cut separating s and t.

Given a maxflow f on such graph, we see that by maxflow-mincut theorem, $C(X_1, Y_1) = C(X_2, Y_2) = |f|$. And by the fact that $|f| \leq C(X, Y)$ for some cut (X, Y) and by:

$$|f| = f(X, Y) - f(Y, X)$$

we see that on the between the mincut (X_1, Y_1):

- for $e = (x, y)$ where $x \in X_1, y \in Y_1$, we have $f(e) = c(e)$.
- for $e = (y, x)$ where $x \in X_1, y \in Y_1$, we have $f(e) = 0$.
- In other words, those edges from X_1 into Y_1 are saturated by its flow, and those edges from Y_1 to X_1 attains 0-flow.

And we see such analogous statement also holds for mincut (X_2, Y_2).

We then consider the situation for the cut $(X_1 \cup X_2, Y_1 \cap Y_2)$:

- for $e = (x, y)$ where $x \in X_1 \cup X_2, y \in Y_1 \cap Y_2$, we see that either $x \in X_1$ or $x \in X_2$:
  - $x \in X_1$, we see that $y \in Y_1 \cap Y_2$, so $y \in Y_1$, which implies that $f(e) = c(e)$.
  - $x \in X_2$, we see that $y \in Y_1 \cap Y_2$, so $y \in Y_2$, which still implies that $f(e) = c(e)$.
  - In other words, those edges from $X_1 \cup X_2$ into $Y_1 \cap Y_2$ are saturated by its flow.
- for $e = (y, x)$ where $x \in X_1 \cup X_2, y \in Y_1 \cap Y_2$, we see that either $x \in X_1$ or $x \in X_2$:
  - $x \in X_1$, we see that $y \in Y_1 \cap Y_2$, so $y \in Y_1$, which implies that $f(e) = 0$.
  - $x \in X_2$, we see that $y \in Y_1 \cap Y_2$, so $y \in Y_2$, which still implies that $f(e) = 0$.
  - In other words, those edges from $Y_1 \cap Y_2$ into $X_1 \cup X_2$ attains 0-flow.

So we see that $|f| = f(X_1 \cup X_2, Y_1 \cap Y_2) - f(Y_1 \cap Y_2, X_1 \cup X_2) = f(X_1 \cup X_2, Y_1 \cap Y_2) = C(X_1 \cup X_2, Y_1 \cap Y_2)$, which then leads to:

$$C(X_1, Y_1) = C(X_2, Y_2) = C(X_1 \cup X_2, Y_1 \cap Y_2)$$

so we see that $(X_1 \cup X_2, Y_1 \cap Y_2)$ is a mincut. ■

7 Problem 7

Problem 7.1. Prove Hall's theorem from the maxflow-mincut theorem.

Solution:

Proof. Given a bipartite graph $G = X \sqcup Y$, it is obvious to see that if there is a complete matching from X to Y in G, we have $|\Gamma(A)| \geq |A|$ for every $A \subset X$.

So we intend to see when $|\Gamma(A)| \geq |A|$ for every $A \subset X$, we will have a complete matching from X to Y. We now build a transportation network by following rules:

- Adding a vertex s as the source vertex, and connect the direct edges by $(s, x), \forall x \in X$, we give the capacity of such edges to be 1.
- Adding a vertex t as the sink vertex, and connect the direct edges by $(y, t), \forall y \in Y$, we give the capacity of such edges to be 1.

- For edges between the two component X and Y of the original bipartite graph G, we directed the directions of the edges by (x, y) where $x \in X, y \in Y$, we assign the capacity of such edges to be $|X| + 1$.

Hence we successfully build up a transportation network, denoted as (D, s, t). We now want to see that having a matching from X to Y with size k in G if and only if there exists a flow with $|f| = k$ in transportation network (D, s, t).

Suppose we have a matching from X to Y with size k in G, we can directly set the corresponding edges of the matching in (D, s, t) with their flows to be 1. Given (x, y) in the k-matching, we set the flow of the (s, x) and (y, t) to be 1. All the rest of the edges' flows will be set to 0. Such construction satisfy the conservation of flows, and thus is a valid flow. One can easily check that such flow's strength is k.

Suppose there exists a flow with $|f| = k$ in transportation network (D, s, t). Observe that $D = \{s\} \sqcup \{t\} \sqcup X \sqcup Y$, by conservation of flow we see that:

$$|f| = k = f(\{s\}, \{t\} \cup X \cup Y) = f(\{s\} \cup X, Y \cup \{t\}) = f(\{s\} \cup X \cup Y, \{t\})$$

Since the capacity of edges between $\{s\}$ and X are all set to 1, it follows that there are in total k edges with flow between such two sets. And thus by conservation of flow, there are also k edges between X and Y, no two touches same vertices followed by the fact that there are still k edges with flow between Y and $\{t\}$. So we see such k edges with flow equal to 1 between X and Y are the intended k-matching.

We then want to proof that when $|\Gamma(A)| \geq |A|$, for every $A \subset X$, there exists a maximum flow $|f| = |X|$, which is, by maxflow-mincut theorem, exists a mincut $C(S, T) = |X|$ in the transportation network (D, s, t). We shall proceed the proof by contraposition, suppose that there exists a mincut (S, T), such that $C(S, T) < |X|$, we want to see that there exists $A \subset X$, such that $|\Gamma(A)| < |A|$. We define a set as follow:

$$P = (X \cap T) \sqcup (Y \cap S)$$

Since all the edges between X and Y attains capacity to be $|X| + 1$, and $C(S, T) = |X| < |X| + 1$, so for a mincut to hold, there exists no edges $e = (x, y)$, where $x \in X \cap S$ and $y \in Y \cap T$. So for all the edges $e = (x, y)$, if $x \in X \cap S$, then $y \in Y \cap S$. So it follows that:

$$|\Gamma(X \cap S)| \leq |Y \cap S| \tag{1}$$

and we see all the edges $e = (x, y)$ from S to T will be either $x = s, y \in X \cap T$ or $x \in Y \cap S, y = t$. By our construction of the transportation network, we see:

$$C(S, T) = |P| = |X \cap T| + |Y \cap S| < |X| \tag{2}$$

And with the fact that:

$$|X \cap S| + |X \cap T| = |X| \tag{3}$$

Combining **Equation** 1, **Equation** 2 and **Equation** 3, it follows that:

$$|\Gamma(X \cap S)| < |X \cap S|$$

so we find the intended subset of X by $A = X \cap S$. So by contraposition the original statement holds.

■

8 Appendix