Problem Set 5

Math 565: Combinatorics and Graph Theory

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1 Problem 1

Problem 1.1. Let G be a simple connected graph G on n vertices. Define a geometric lattice $L(G)$ whose points correspond to the edge set $E(G)$. The elements of L are all partitions Π of $V(G)$ such that the subgraph of G induced by each block of Π is connected.
Prove that the bases of $L(G)$ are exactly the edge sets of spanning trees in G.

Solution: We first state the proof for the following lemma:

Lemma 1.2. Given a finite simple connected graph $G = (V, E)$, given an arbitrary subgraph G' of G, it is a spanning tree if and only if G' is a minimal connective graph. i.e. $\forall\ e \in G'$, $G' \backslash \{e\}$ is not a connected graph.

Proof of the Lemma

Proof. We shall proceed the proof in two direction:

- ($\implies$): Given G' as an arbitrary spanning tree of G, thus we see that $V(G') = V(G)$ and there is no cycles within G'. Suppose that G' is not a minimal connective graph, then we see that there exists $e \in E'(V)$, such that $G' \backslash \{e\}$ is still a connective graph. Without losing generality, we denote $e = (u, v)$ for some $u, v \in V(G')$. Sine $G' \backslash \{e\}$ is still a connected graph, thus there exists a path l from u to v in the graph $G' \backslash \{e\}$. Let this path be the vertex sequence as $l = (u, v_1, v_2, \ldots, v_k, v)$, where all edges $(u, v_1), (v_1, v_2), \ldots, (v_k, v)$ are in $E(G' \backslash \{e\})$. But then l combines with e gives us a cycle in G', contradicting the fact that G' is a spanning tree $\lightning$. So G' is a minimal connective graph.
- ($\impliedby$): Given G' as an arbitrary minimal connective subgraph of G, suppose that G' is not a spanning tree, then there exists a cycle in G'. Arbitrarily pick an edge in that cycle, denoted as $e = (u, v)$, and remove it from G', since e is in a cycle, removing it will not affect the connectivity of the graph, so $G' \backslash \{e\}$ is still connective graph, but this contradict to the fact that G' is the minimal one that is connective $\lightning$. So we see G' is a spanning tree of G.

■

Then we give the proof to the Original Statement

Proof. First consider the minimum of such geometric lattics:

$$\hat{0} = \{\{v\} | v \in V(G)\}$$

which are essentially the partition over all the vertex. Each block only contains one vertex, and each vertex is vacuously connected. Then see that the atoms of such geometric lattice is defined as:

$$\Pi_{(v_i, v_j)} := \{\{v_i, v_j\} \cup \{v | v \in V(G), v \neq v_i, v_j\} | (v_i, v_j) \in E(G)\}$$

Since G is a simple and connected graph:

$$\hat{1} = \{V(G)\}$$

Given a basis of $L(G)$, denoted as $\mathcal{B}$. It is given by the minimum set of atoms (in this case, edges of G), such that their joins are $\hat{1}$:

$$\bigvee_{e \in \mathcal{B}} \Pi_e = \hat{1} \tag{1}$$

Let $\Pi_{\mathcal{B}} = \bigvee_{e \in \mathcal{B}} \Pi_e$. Let $e' = (x, y) \in E(G)$ and $\Pi' \in L(G)$, $\Pi' \vee \Pi_{(x,y)}$ equals to Π' if x, y is already in the same block of Π' and otherwise we merge the blocks of Π' where x, y in to get the new partition. Let $G_{\mathcal{B}} = (V, \mathcal{B})$ be the subgraph of G induced by the edge set $\mathcal{B}$. The join operation $\Pi_{\mathcal{B}}$ repeatedly merges vertex blocks. Therefore, two vertices u, v are in the same block of $\Pi_{\mathcal{B}}$ if and only if there exists a path between u

and v using only edges from $\mathcal{B}$. In other words, the blocks of $\Pi_{\mathcal{B}}$ are exactly the vertex sets of the connected components of the subgraph $G_{\mathcal{B}}$.

By **Equation** 1, $\Pi_{\mathcal{B}} = \hat{1}$, which means $\Pi_{\mathcal{B}}$ has only one block. Based on our clarification, this is equivalent to stating that the subgraph $G_{\mathcal{B}}$ is **connected**.

Furthermore, $\mathcal{B}$ is a **minimal** set satisfying this property. This means for any $e_0 \in \mathcal{B}$, the join of the remaining atoms is not $\hat{1}$. Let $\Pi_{\mathcal{B}\setminus\{e_0\}} := \bigvee_{e' \in \mathcal{B}\setminus\{e_0\}} \Pi_{e'}$. The minimality condition ensure $\Pi_{\mathcal{B}\setminus\{e_0\}} \neq \hat{1}$. This is equivalent to saying $|\Pi_{\mathcal{B}\setminus\{e_0\}}| > 1$, which, by our definition, means the subgraph $G_{\mathcal{B}\setminus\{e_0\}}$ (induced by the edge set $\mathcal{B} \setminus \{e_0\}$) is **not connected**.

So we see the subgraph $G_{\mathcal{B}}$ is actually a minimal connected graph of G (it is connected, but removing any edge $e_0 \in \mathcal{B}$ makes it disconnected). By **Lemma** 1.2, $G_{\mathcal{B}}$ is a spanning tree of G (Lemma apply since we can define geometric lattice on G, directly implies that G is finite graph, then so is any of its subgraph). Thus the basis $\mathcal{B}$ induced a spanning tree in G. So the bases of $L(G)$ are exactly the edge sets of spanning trees in G. ■

2 Problem 2

Problem 2.1. Let L be a geometric lattice, and $x \leq y$ be two elements. Show that the interval

$$[x, y] := \{z \in L \mid x \leq z \leq y\}$$

is again a geometric lattice.

Solution: We first state the proof for the following lemma:

Lemma 2.2. Given a geometric lattice L, the property of semimodularity and no infinite chain implies that $\forall\, x, y \in L$, if $x \wedge y \lessdot x \implies y \lessdot x \vee y$.

Proof of the Lemma

Proof. Given arbitrary $x, y \in L$, see that $x \wedge y \leq y$, then consider:

- if $x \wedge y = y$. It follows that $y \leq x$, $x \wedge y \lessdot x \implies y \lessdot x \implies y \lessdot x \vee y$.
- if $x \wedge y < y$. Since L contains no infinite chain, we can then construct a maximal chain from $x \wedge y$ to y, denoted as:

  $$x \wedge y = d_0 \lessdot \cdots \lessdot d_k = y$$

  and we want to see that $y \lessdot x \vee y$, i.e. $d_k \lessdot x \vee d_k$. We shall proceed the proof by induction on $i < k$ on such maximal chain to get stronger result that $d_i \lessdot x \vee d_i$, $\forall\, i \in \{1, \ldots, k\}$.

  - **Base case**: When $k = 0$, see that $d_0 \lessdot x \vee d_0$ is actually $x \wedge y \lessdot x \vee (x \wedge y) \iff x \wedge y \lessdot x$, which is exactly our assumption, so the base case holds.
  - **Inductive case**: Suppose that $d_i \lessdot x \vee d_i$, we want to see that $d_{i+1} \lessdot x \vee d_{i+1}$. See that since $d_i \lessdot d_{i+1}$ and $d_i \lessdot x \vee d_i \implies d_{i+1} \lessdot (x \vee d_i) \vee d_{i+1} \iff d_{i+1} \lessdot x \vee (d_i \vee d_{i+1})$. But since $d_i \lessdot d_{i+1} \implies (d_i \vee d_{i+1}) = d_{i+1}$. Then $d_{i+1} \lessdot x \vee d_{i+1}$. So the inductive case also follows.

So see that $d_k \lessdot x \vee d_k \iff y \lessdot x \vee y$. ■

Proof of the Original Statement

Proof. To see that $[x, y]$ is a geometric lattice, we need to verify that its join, meets exists for every elements, there is no infinite chain, and it is atomic and semimodular.

- **Joins, meets exist**: $\forall\, a, b \in [x, y]$, we see that $x \leq a \leq y$, $x \leq b \leq y$.

$$\begin{aligned} x \leq a, x \leq b &\implies x \leq a \wedge b \\ a \leq y, a \wedge b \leq a \leq y &\implies a \wedge b \leq y \\ &\implies a \wedge b \in [x, y] \end{aligned}$$

  thus $a \wedge b$ exists in $[x, y]$.

$$\begin{aligned} a \leq y, b \leq y &\implies a \vee b \leq y \\ x \leq a \implies x \leq a \leq a \vee b &\implies x \leq a \vee b \\ &\implies a \vee b \in [x, y] \end{aligned}$$

  thus $a \vee b$ exists in $[x, y]$.

- **Without infinite chain**: Since $[x, y] \subseteq L$, L has no infinite chain $\implies$ $[x, y]$ has no infinite chain, such statement is trivial to see by contraposition.

- **Semimodularity**: Since we've already seen that $a \vee b$ exists for all $a, b \in [x, y]$, semimodularity of $[x, y]$ is directly inherit from the semimodularity of L.

- **Atomic**: Without losing generality, denote the set of atoms of L as $\{l_1, \ldots, l_n\}$. We claim that $\{x \vee l_i \mid x \vee l_i \neq x,\ i \in \{1, \ldots, n\}\} =: A'$, is a subset of the atom set of $[x, y]$. i.e. $\forall a \in A'$, a is an atom of $[x, y]$. First note that $\forall z \in [x, y],\ x \leq z$, which means that x will be the minimum of $[x, y]$. It is sufficient to show that $\forall a \in A',\ x \lessdot a$. First note that $x \vee l_i \neq x \implies l_i \nleq x$. For those l_i such that $l_i \nleq x$, consider $l_i \wedge x$. Since $l_i \wedge x \leq l_i$, $l_i \wedge x \leq x$, but $l_i \nleq x$. So $l_i \wedge x \neq l_i$. But since l_i is the atom, then $\hat{\mathbf{0}} \lessdot l_i$ be the only element in the lattice that less than $l_i \implies l_i \wedge x = \hat{\mathbf{0}} \lessdot l_i$. Since we already verified that the semimodularity of $[x, y]$ holds, then by **Lemma** 2.2, see that $x \lessdot x \vee l_i$, in particular $x \vee l_i$ covers x, so A' is a subset of the atom set of $[x, y]$. But then $\forall\, l \in [x, y]$, since l_i is the atoms of L, then:

$$\begin{aligned} l &= \bigvee_{i \in I} l_i \\ \iff l \vee x &= \bigvee_{i \in I} l_i \vee x \\ \iff l \vee x &= \bigvee_{i \in I} (l_i \vee x) \end{aligned}$$

  and $l \vee x = l$ by the fact that $x \leq l$. So:

$$l = \bigvee_{i \in I} (l_i \vee x)$$

  so every element in $[x, y]$ can be expressed as the joins of a subset of the atom set of $[x, y]$, which means that $[x, y]$ is indeed atomic.

Thus we see $[x, y]$ is a gemoetric lattice. ■

3 Problem 3

Problem 3.1. Let $(X, \mathcal{F})$ be a combinatorial geometry. A **basis** B of a flat F is an independent set $B \subseteq F$ such that $\overline{B} = F$.
Now let F be a flat of a combinatorial geometry and suppose $B \subseteq F$ satisfies $\overline{B} = F$ and B is minimal satisfying this property. Show that B is a basis of F.

Solution: First note the definition of *independent set* and *closure* from the **Textbook**:

Definition 3.2. For a subset S of the points of a combinatorial geometry, the *closure* $\overline{S}$ is the intersection of all flats containing S, in particular:

$$\overline{S} := \bigcap_{F\in\mathcal{F}, S\subseteq F} F$$

Definition 3.3. A subset $S \subseteq X$ is a *independent set* when for each $x \in S, x \notin \overline{S\backslash\{x\}}$.

And then we shall have the following small Lemma:

Lemma 3.4. Given two subsets $S \subseteq S' \subseteq X$, we have:

$$\overline{B} \subseteq \overline{B'}$$

and

$$\overline{B} = \overline{\overline{B}}$$

Proof of the Lemma:

Proof. $\forall\, b \in B$, since $B \subseteq B'$, see that:

$$\begin{aligned} b \in \bigcap_{F\in\mathcal{F}, B\subseteq F} F \implies b \in \bigcap_{F\in\mathcal{F}, B'\subseteq F} F = \overline{B'} \\ \implies b \in \overline{B'} \end{aligned}$$

So we see $\overline{B} \subseteq \overline{B'}$.

Clearly by definition, we have $\forall\ S \subseteq X$, $S \subseteq \overline{S}$. Denote $K := \overline{B}$, and we want to see $\overline{K} = K$. Clearly, we have $K \subseteq \overline{K}$, and we see:

$$\overline{K} = \bigcap_{F\in\mathcal{F}, K=\overline{B}\subseteq F} F$$

But we see by the axiom of combinatorial geometry, $K = \overline{B}$ it self is a flat, since it is the intersection of several flats. So:

$$\overline{K} = \bigcap_{F\in\mathcal{F}, K=\overline{B}\subseteq F} F = \bigcap_{F\in\mathcal{F}, K=\overline{B}\subsetneq F} F \cap \overline{B} \implies \overline{K} \subseteq \overline{B} = K$$

So $\overline{K} = K$, which is intended. ∎

Proof of the Original Statement:

Proof. Given F to be the flat of a combinatorial geometry, and suppose $B \subseteq F$ satisfies that $\overline{B} = F$ and B is the minimal satisfying this property. Now suppose that B is a dependent set, then there exists $b \in B$, such that $b \in \overline{B\backslash\{b\}}$. Denote $B' := B\backslash\{b\}$. Clearly we have $B' \subseteq \overline{B'}, b \in \overline{B'}$, which implies that $B = B' \cup \{b\} \subseteq \overline{B'}$. Thus by **Lemma** 3.4, $\overline{B} \subseteq \overline{\overline{B'}} = \overline{B'}$, and $B' \subseteq B \implies \overline{B'} \subseteq \overline{B}$. So we see $\overline{B'} = \overline{B} = F$. But since $B' = B\backslash\{b\}$, which contradicts to the minimality of B $\lightning$. So we see B is indeed an independent set, and thus a basis of flat F. ∎

4 Problem 4

Problem 4.1. Let L be a finite geometric lattice. Prove that L is **graded**: for any $x \leq y$ in L, any two maximal chains from x to y have the same length. You must prove this directly from the axioms of a geometric lattice; do not use the correspondence with combinatorial geometries or matroids proved in class.

Solution:

Proof. It is sufficient to proceed the induction on the following statement $P(n)$: for any interval $[x, y]$ in L, if there exists one maximal chain from x to y whose length is n, then all the maximal chain from x to y attain their length to be n.

- **Base case**:
  - $\mathbf{P(0)}$: If there exists a maximal chain with length 0 from x to y, it just means that $x = y$, statement trivially holds.
  - $\mathbf{P(1)}$: If there exists a maximal chain with length 1, which means that $x \lessdot y$, see that any maximal chain from x to y will be at least $x \lessdot y$. But if its length is bigger than 1, say $x < z < y$, then contradicting to the fact that y is a cover of x $\lightning$. So $x \lessdot y$ is the only maximal chain, with length 1, so $P(1)$ holds.
- **Inductive case**: Suppose that $P(k)$ holds for all $k < n$, with $n \geq 2$, we want to see that $P(n)$ also holds. Given two arbitrary maximal chain from x to y, denoted as C_1 and C_2, with length to be n and m respectively:
$$C_1 : x = c_0 \lessdot \ldots \lessdot c_n = y$$
$$C_2 : x = d_0 \lessdot \ldots \lessdot d_m = y$$
  And we see either $c_1 = d_1$ or $c_1 \neq d_1$:
  - If $\mathbf{c_1 = d_1}$: Then we see $c_1 \lessdot \ldots \lessdot c_n = y$ is a maximal chain in $[c_1, y]$ with length $n - 1$. By inductive hypothesis, every maximal chain in $[c_1, y]$ will have the length to be $n-1$, and notice that $d_1 \lessdot \ldots \lessdot d_m = y$ is also a maximal chain in $[c_1, y]$, with length $m-1 \implies m-1 = n-1 \implies m = n$.
  - If $\mathbf{c_1 \neq d_1}$: Notice that $x \lessdot c_1$ and $x \lessdot d_1$, by semimodularity, we see $c_1 \lessdot c_1 \vee d_1$ and $d_1 \lessdot c_1 \vee d_1$. Consider the interval $[c_1 \vee d_1, y]$, choose arbitarry a maximal chain from $c_1 \vee d_1$ to y in $[c_1 \vee d_1, y]$, with length to be l, and denote it as C_3:
  $$C_3 : c_1 \vee d_1 = e_0 \lessdot e_1 \lessdot \ldots \lessdot e_l = y$$
    Now we construct to maximal chain from x to y, by concatenating x, c_1 and C_3 and x, d_1 and C_3:
  $$C_a : x \lessdot c_1 \lessdot e_0 \lessdot \ldots \lessdot e_l = y$$
  $$C_b : x \lessdot d_1 \lessdot e_0 \lessdot \ldots \lessdot e_l = y$$
    We then consider the the interval $[c_1, y]$, since we already have the maximal chain $c_1 \lessdot \ldots \lessdot c_n = y$, with length to be $n-1$, but we also have the maximal chain $c_1 \lessdot e_0 \lessdot \ldots \lessdot e_l = y$, with length $l+1$. By induction hypothesis, these two maximal chain has the same length, see that $n - 1 = 1 + l$.
    Then consider the interval $[d_1, y]$, similarly we shall see that $m - 1 = l + 1$.
    Thus we see:
  $$m = n$$

  In either case, we see $m = n$, which shows that the inductive case $P(n)$ also holds.

And $P(n)$ holds for all $n \in \mathbb{N}$, since we've seen before in **Problem** 2.1, $[x, y]$ is a geometric lattice, thus there is no infinite chain in $[x, y]$, thus no infinite maximal chain, so our statement directly give the proof for the original statement. ∎

5 Problem 5

Problem 5.1. A collection $\mathcal{M}$ of k-element subsets of $[n]$, satisfies the **exchange axiom** if: given $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that $(I - \{i\} \cup \{j\}) \in \mathcal{M}$. (Thus $\mathcal{M}$ is the bases of a matroid.) Suppose $\mathcal{M}$ satisfies the exchange axiom. Show that $\mathcal{M}$ satisfies the **dual exchange axiom**: if $I, J \in \mathcal{M}$ and $j \in J$ there exists $i \in I$ such that $(I - \{i\} \cup \{j\}) \in \mathcal{M}$.

Solution: We state the following definition and shall prove the following lemma one by one:

Definition 5.2. A circuit is a subset $S \subseteq [n]$ that is dependent (= not independent), and minimal under inclusion. And we define:

$$\mathcal{C} := \{C \subseteq [n] \mid C \text{ is a circuit for } \mathcal{M}\}$$
$$\mathcal{I} := \{I \subseteq [n] \mid I \text{ is an independent set for } \mathcal{M}\}$$

Lemma 5.3. If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 \backslash I_1$, such that $I_1 \cup e \in \mathcal{I}$.

Lemma 5.4. Given $\mathcal{C}$ to be the set containing all the circuits for $\mathcal{M}$.

- If C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.
- Furthermore, if C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$, such that $C_3 \subseteq (C_1 \cup C_2) \backslash \{e\}$.

Lemma 5.5. Suppose that I s an independent set in a matroid $\mathcal{M}$ and e is an element of the ground set of $\mathcal{M}$ such that $I \cup \{e\}$ is dependent. Then $\mathcal{M}$ has a unique circuit contained in $I \cup \{e\}$, and this circuit contains e.

Proof of the first Lemma 5.3:

Proof. Given I_1 and $I_2 \in \mathcal{I}$, with $|I_1| < |I_2|$. By definition of independent set, there exists $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{M}$, such that $I_1 \subseteq \mathcal{B}_1$ and $I_2 \subseteq \mathcal{B}_2$. We first try to exchange iteratively the elements in $\mathcal{B}_2 \backslash I_2$ into elements of I_1: By exchange axiom, for $b \in \mathcal{B}_2 \backslash I_2$, there exists $a \in \mathcal{B}_1$, such that $(\mathcal{B}_2 - \{a\}) \cup \{b\} \in \mathcal{M}$, evidently, $b \notin I_1 \cap I_2$. We iteratively doing this for all elements in $\mathcal{B}_2 \backslash I_2$, and get $\mathcal{B}_2' \in \mathcal{M}$, with the result that $\mathcal{B}_2' \backslash I_2 \subseteq \mathcal{B}_1$, and in particular, $\mathcal{B}_2' = (\mathcal{B}_2' \backslash I_2) \sqcup I_2$, and evidently, $(\mathcal{B}_2' \backslash I_2) \cap (I_1 \cap I_2) = \emptyset$. Now we apply exchange axiom for $\mathcal{B}_1$ and $\mathcal{B}_2'$. Since $|I_1| < |I_2|$ and $|\mathcal{B}_1| = |\mathcal{B}_2| \implies |\mathcal{B}_1 \backslash I_1| > |\mathcal{B}_2' \backslash I_2|$ and $\mathcal{B}_2' \backslash I_2 \subseteq \mathcal{B}_1$, by **Pigeonhole Principle**, there exists $i \in \mathcal{B}_1 \backslash I_1$, such that $i \notin \mathcal{B}_2' \backslash I_2$. We apply exchange axiom to such element $i \in \mathcal{B}_1$: i can not exchange element $j \in \mathcal{B}_2' \backslash I_2$ since they are already in $\mathcal{B}_1$, i can not exchange element $j \in I_1 \cap I_2$ since $I_1 \cap I_2 \subseteq I_1 \subseteq \mathcal{B}_1$. So i can only exchange element $e \in I_2 - I_1$, and get $\mathcal{B}_1' \in \mathcal{M}$, evidently $e \notin I_1$. And we see $I_1 \cup \{e\} \subseteq \mathcal{B}_1' \in \mathcal{M} \implies I_1 \cup \{e\} \in \mathcal{I}$. ■

Proof of the second Lemma 5.4:

Proof. Given $C_1, C_2 \in \mathcal{C}$, $C_1 \subseteq C_2$, then $\forall\, x \in C_2 - C_1$, by definition of circuits, see that $C_2 \backslash \{x\}$ is independent since C_2 is the maximal dependent set. But since $x \notin C_1 \implies C_1 \subseteq C_2 \backslash \{x\}$, right hand side is independent while the left hand side is dependent, leading to a contradiction $\lightning$. So $C_2 \backslash C_1 = \emptyset \implies C_1 = C_2$.

Let C_1, C_2, e be as the setting in **Lemma** 5.4. Suppose that $(C_1 \cup C_2) \backslash \{e\}$ does not contain any circuit. Then see that $(C_1 \cup C_2) \backslash \{e\}$ is an independent set. Since C_1, C_2 are distinct, by what we proven just now in the same Lemma, we see $C_2 - C_1$ is non-empty, so we can choose an element f from this set. As C_2 is a minimal dependent set, $C_2 \backslash \{f\} \in \mathcal{I}$. Now choose a subset I of $C_1 \cup C_2$ which is maximal with the properties that it contains $C_2\backslash\{f\}$ and it is independent. Evidently, see that $f \notin I$. Moreover, as C_1 is a circuit, some element $g \in C_1$ is not in I. As $f \in C_2 - C_1$, see that $f \neq g$. Hence we have:

$$|I| \leq |(C_1 \cup C_2) - \{f, h\}| = |(C_1 \cup C_2)| - 2 < |(C_1 \cup C_2)\backslash\{e\}|$$

Now apply **Lemma** 5.3, taking I_1 and I_2 to be I and $(C_1 \cup C_2)\backslash\{e\}$ respectively. The resulting independent set will be $I \cup \{e\} \in \mathcal{I}$, contradicting to the maximality of I $\lightning$. ■

Proof of the third Lemma 5.5:

Proof. Clearly $I \cup \{e\}$ contains a circuit, and all such circuits must contain e. Let C and C' be distinct such circuits. Then by **Lemma** 5.4, we see $(C \cup C')\backslash\{e\}$ contains a circuit. As $(C \cup C')\backslash\{e\} \subseteq I \implies (C \cup C')\backslash\{e\} \in \mathcal{I}$, thus cannot contain any dependent subset and in particular does not contain any circuit, which is a contradiction $\lightning$. So C is unique. ■

Proof of the Original Statement:

Proof. First note that exchange axiom $\implies$ **Lemma** 5.3 $\implies$ **Lemma** 5.4 $\implies$ **Lemma** 5.5.

Given $I, J \in \mathcal{M}$, given $j \in J$, since $I \in \mathcal{I}$ and $I \cup \{j\}$ is dependent, thus by **Lemma** 5.5, $I \cup \{j\}$ contains a unique circuit, denoted as $C(j, I)$. As $C(j, I)$ is dependent and J is independent, see that $C(j, I) - J$ is non-empty. Let $i \in (C(j, I) - J)$. Evidently we see that $i \in I \cup \{j\}, i \neq j \implies i \in (I - J)$. Moreover, $(I - \{i\}) \cup \{j\}$ is independent since it does not contain $C(j, I)$, which is the only possible circuit that can be contained in any subset of $I \cup \{j\} \implies C(j, I) \nsubseteq (I - \{i\}) \cup \{j\}$. And since $|(I - \{i\}) \cup \{j\}| = |I|$, so it follows that $(I - \{i\}) \cup \{j\} \in \mathcal{M}$. In particular, we have proven $\forall\ I, J \in \mathcal{M}$, $\forall\ j \in J - I$, there exists an element $i \in I - J$, such that $(I - \{i\}) \cup \{j\} \in \mathcal{M}$, which implies the **dual exchange axiom**. ■

6 Problem 6

Problem 6.1. Let $M = U_{k,n}$ be the uniform matroid of rank k on $[n] = \{1, 2, \ldots, n\}$. Thus the bases of M are all k-element subsets of $[n]$. Find the independent sets, circuits, rank function, closure operator, and flats of $U_{k,n}$.

Solution: See that $U_{k,n}$ is the uniform matroid, whose bases are all k-elements of $[n]$. Which is:

$$\mathcal{M} = U_{k,n} = \binom{[n]}{k}$$

- **Independent sets**: The independent sets given by:
$$\mathcal{I} = \{I \subseteq [n] \mid |I| \leq k\}$$
- **Circuits**: The circuits given by:
$$\mathcal{C} = \{C \subseteq [n] \mid |C| = k + 1\}$$
- **Rank functions**: The rank functions given by:
$$rk(S) = \begin{cases} |S|\ , & \text{if } |S| \leq k \\ k\ , & \text{if } |S| > k \end{cases}$$
$$= \min(|S|, k)$$
- **Closure operator**: The closure operator is given by:
$$\overline{S} = \begin{cases} [n]\ , & \text{if } |S| \geq k \\ S\ , & \text{if } |S| < k \end{cases}$$

- **Flats**: The flats are given by:

$$\mathcal{F} = \{F \subseteq [n] \mid |F| < k\} \cup \{[n]\}$$

7 Appendix