Problem Set 1

Math 565: Combinatorics and Graph Theory

Prof. Thomas Fun Yau Lam

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Author: **Yanzhi Li**

Collaborators: Lingwei Zhang

1 Problem 1

Problem 1.1. Show that a connected graph on n vertices is a tree if and only if it has $n-1$ edges.

Solution:

Proof. Let $G = (V, E)$ be a connected graph with $|V| = n$ and $|E| = m$. We aim to show that G is a tree if and only if $m = n - 1$.

($\implies$): If G is a tree, then $m = n - 1$.

We proceed by induction on the number of vertices n.

- **Base Case:** For $n = 1$, the graph consists of a single isolated vertex. It has 0 edges. Since $n - 1 = 1 - 1 = 0$, the statement holds.
- **Inductive Hypothesis:** Assume that any tree with k vertices has $k - 1$ edges.
- **Inductive Step:** Let G be a tree with $n = k + 1$ vertices (where $k \geq 1$). Since G is a tree with at least 2 vertices, it must contain at least one leaf node (a vertex v with degree 1). Let v be a leaf and let e be the edge connecting v to the rest of the graph.

  Consider the graph $G' = G - \{v\}$ obtained by removing vertex v and the incident edge e.

  1. Since v was a leaf, removing it does not disconnect the rest of the graph. Thus, G' is connected.
  2. Since G had no cycles, G' (which is a subgraph of G) also has no cycles.

  Therefore, G' is a tree with k vertices. By the inductive hypothesis, G' has $k - 1$ edges. The number of edges in the original graph G is the number of edges in G' plus the one edge e we removed.

  $$|E(G)| = |E(G')| + 1 = (k - 1) + 1 = k$$

  Since $n = k + 1$, we have $k = n - 1$. Thus, G has $n - 1$ edges.

($\impliedby$): If G is connected and has $n - 1$ edges, then G is a tree.

We are given that G is connected and $m = n - 1$. To prove G is a tree, we only need to show that G contains no cycles (is acyclic).

We use proof by contradiction. Assume that G is connected, has $n - 1$ edges, but contains a cycle.

1. Since G contains a cycle, removing an edge from that cycle does not disconnect the graph.
2. We can repeatedly remove edges from cycles until no cycles remain. Let the resulting graph be G'.
3. Since we only removed edges that were part of cycles, G' remains connected.
4. Since G' has no cycles and is connected, G' is a tree (specifically, a spanning tree of G).

The graph G' still has n vertices. By the result proved in Direction 1, a tree with n vertices must have exactly $n - 1$ edges. However, to obtain G', we removed at least one edge from the original graph G (because we assumed G had a cycle). Therefore:

$$|E(G)| > |E(G')| = n - 1$$

This implies $|E(G)| > n - 1$, which contradicts the given condition that $|E(G)| = n - 1$.

Thus, the assumption that G contains a cycle must be false. G is connected and acyclic, so G is a tree. $\blacksquare$

2 Problem 2

Problem 2.1. The edges of K_n are colored red and blue in such a way that a red edge is in at most one red triangle. Show that there is a subgraph K_k with $k \geq \lfloor\sqrt{2n}\rfloor$ that contains no red triangle.

Solution:

Proof. We first extract out the K_k from K_n, with k be the largest integer, such that K_k contains no red triangle.

In particular, in this situation, adding any other vertex will generate a new red triangle in the graph.

We now consider the number of the rest of the vertices. Note that adding any the rest of the vertex back will form a red triangle. Since a red edge is in at most one red triangle, then with different vertices not in K_k added back in K_k will form red triangle with different edges in K_k, as if not the case, the property will be violated.

We see the number of the rest of the vertices p then satisfy that:

$$\begin{aligned} n &= k + p \\ p &\leq \#E(K_k) \end{aligned}$$

where $\#E(K_k)$ denote the number of edges in K_k. And since K_k is complete, we see that:

$$\#E(K_k) = \binom{k}{2} = \frac{k(k-1)}{2}$$

Then, we see:

$$\begin{aligned} n = k + p \leq \binom{k}{2} + k &= \frac{k(k+1)}{2} \\ \iff 2n &\leq k(k+1) < (k+1)^2 \\ \implies k &> \sqrt{2n} - 1 \\ \implies k &\geq \lfloor\sqrt{2n}\rfloor \end{aligned}$$

which proof the theorem. ∎

Alternative Method:

Moreover, consider remove all the blue edges, and consider the remaining edges (the blue edges is essentially useless in the context).

By **Thuran's theorem**, we see $\#E(K) \leq \frac{n^2}{4}$, then we consider the inequality as follows to solve, which gives a tighter bound:

$$p = n - k \leq \frac{k^2}{4} \iff \frac{k^2}{4} + k \geq n$$

Alternative Heuristic:

Consider the situation of piling up all the red triangles, and trim off one vertex in each triangle. We want to see that this way of trimming vertex removes the most number of vertices to eliminate the red triangle. And remain vertices number is $\frac{2}{3}n$. We see that for:

$$\frac{2}{3}n \geq \lfloor\sqrt{2}n\rfloor \iff 2n \geq 9$$

so for $n = 3, n = 4$ we shall draw the example to prove, for larger one, this holds by our way of eliminate the vertices.

3 Problem 3

Problem 3.1. A *tournament* on n vertices is an orientation of the edges of K_n. A *transitive tournament* is a tournament such that the vertices can be numbered in such a way that (i, j) is an edge if and only if $i < j$. Show that if $k \leq \log_2 n$, every tournament on n vertices has a transitive subtournament on k vertices.

Solution:

Proof. **Heuristic:** Arbitrarily choose a vertex $v \in K_n$, and we divide the rest of the vertices as the following two sets:

$$S_1 = \{u \in V | (u, v) \in E\}$$
$$S_2 = \{u \in V | (v, u) \in E\}$$

In other words, S_1 denotes the set of vertices that directly connected to v, and S_2 denotes the set of vertices that directly connected by v. Observe that:

$$|S_1| + |S_2| = n - 1$$

By **Pigeonhole Principle**, we have that either:

$$|S_1| \geq \frac{n-1}{2}$$

or:

$$|S_2| \geq \frac{n-1}{2}$$

And we choose the set of points that attains larger cardinality, and proceed the algorithm. Each time of the iteration, the vertex we choose along with all the previous points chosen in each iteration is able to form a transitive subtournament since by definition of S_1 and S_2, all vertices in S_1 is able to be labeled with smaller number than v and all vertices in S_2 is able to be labeled with larger number than v. Then, all we need to ensure is that if $k \leq \log_2 n$, the graph with n vertices is guaranteed to iterate the following inequality for k times.

$$S_{k-1} \geq \frac{S_k - 1}{2}$$

where S_i is denoted as the biggest graph number that will ensure having a subtournament with i vertices. With $S_k = n$, we want to see that $S_1 \geq 1$.

$$\begin{aligned}
S_1 &\geq \frac{1}{2}S_2 - \frac{1}{2} \\
&\geq \frac{1}{4}S_3 - \frac{1}{4} - \frac{1}{2} \\
&\geq \frac{1}{8}S_4 - \frac{1}{8} - \frac{1}{4} - \frac{1}{2} \\
&\vdots \\
&\geq \frac{1}{2^{k-1}}S_k - 1 \\
&= \frac{n}{2^{k-1}} - 1 \\
&= \frac{2n}{2^{\log_2 n}} - 1 \\
&= 2 - 1 = 1
\end{aligned}$$

which is indeed the case that $S_1 \geq 1$, so the statement is proved. ■

4 Problem 4

Problem 4.1. The *degree* $\deg(v)$ of a vertex v is the number of edges incident to v, with the convention that a loop at v contributes two to the degree of v.

A graph G is called k-regular if all its vertices have degree exactly equal to k. Determine all pairs (k, n) such that there exists a k-regular simple graph on n vertices.

Solution: We consider the situation separately by divide situation and analysis on n. We first define the graph $G = (V, E)$.

- When $n \in \{2m + 1 | m \in \mathbb{N}\}$, in particular n is odd number.
  1. k **is odd number**. We see that all the odd k will not work, by that:

     $$\sum_{x \in V} \deg(x) = 2|E|$$

     while the *LHS* equals to $n \cdot k$ which is odd number, *RHS* is apparently even number. So such situation will not happen.
  2. k **is even number**. In particular, $k \in \{2m | m \in \mathbb{N}, 0 \leq m \leq \frac{n-1}{2}\}$. We consider label the n vertices by $0, 1, 2, \cdots, n-1$, and place them uniformly in order onto a circle. We then connect arbitrary point with label i by $i \pm 1, i \pm 2, \cdots, i \pm \frac{k}{2}$. Such strategy will result in a k-regular simple graph, which is intended.
- When $n \in \{2m | m \in \mathbb{N}\}$, in particular n is even number.
  1. k **is even number**. In particular $k \in \{2m | m \in \mathbb{N}, 0 \leq m \leq \frac{n}{2} - 1\}$. Again we consider label the n vertices by $0, 1, 2, \cdots, n-1$ and proceed the same process as previous done to get a k-regular simple graph.
  2. k **is odd number**. In particular $k \in \{2m + 1 | m \in \mathbb{N}, 0 \leq m \leq \frac{n}{2} - 1\}$. We consider first construct a $k-1$-regular subgraph by previous process, then for each of the point labeled with i, we can then connect the edge $(i, i + \frac{n}{2})$ to get the k-regular graph, in particular connect it with the corresponding diagonal vertex in the regular n-gon, which is intended.

To summarize, when n is odd, k can be taken as all even integers satisfying $0 \leq k \leq n$. when n is even, k can be taken as all integers satisfying $0 \leq k \leq n - 1$.

5 Problem 5

Problem 5.1. Let $G = (V, E)$ be a simple graph with no triangles. Prove that there is a partition $V = X \sqcup Y$ of the vertices into two disjoint parts, so that for any $x \in X$, we have $d(x) \leq |Y|$ and for any $y \in Y$, we have $d(y) \leq |X|$. Here $d(x)$ denotes the degree of a vertex x. Hint: Make a careful choice of a vertex x, and let Y be its set of neighbors.

Solution:

Proof. Let x_0 be the point with the highest degree, and let Y be the set of its neighbors. Since there is no triangles and it is a simple graph, any two vertex in the set Y are not connected to each other, and we then gather the rest of the vertices along with x_0 to be the set X.

Since $d(x_0) = |Y|$, then $\forall x \in X, d(x) \leq d(x_0) = |Y|$ by the definition of x_0. Then we want to see if $\forall y \in Y, d(y) \leq |X|$.

And we see indeed the case! Since vertices in Y are disjoint, they can only incident with vertices in X. And the most number vertices it can incident to is the size of the set X, which is $|X|$. In particular we have $d(y) \leq |X|$.

So we successfully find the set X and Y that satisfy the requirement. ■