Problem Set 4

Math 565: Combinatorics and Graph Theory

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1 Problem 1

Problem 1.1. Describe a bijective correspondence between symmetric Latin squares of order n in which all symbols appear on the main diagonal and symmetric Latin squares of order $n+1$ with all $(n+1)$'s on the diagonal.

Solution:

Proof. We denote $L_{(n+1)\times(n+1)} := \{$the symmetric Latin squares of order $n+1$ with all $(n+1)$'s on the diagonal$\}$, and $L_{n\times n} := \{$the symmetric Latin squares of order n in which all symbols appear on the main diagonal$\}$. For a specific Latin square l, we denote the element on its i row and j column as $l(i,j)$.

We now define a map φ as following:

$$\begin{aligned} \varphi : L_{(n+1)\times(n+1)} &\longrightarrow L_{n\times n} \\ l &\mapsto \varphi(l) \end{aligned}$$

such that

$$\varphi(l)(i,j) := \begin{cases} l(i,n+1), \text{ if } i=j \\ l(i,j), \text{ otherwise} \end{cases} \qquad \forall\, i,j \in [1,n]\cap\mathbb{Z}$$

We shall then check this function is actually well-defined and be a bijection, thus finish the proof.

1. **φ is well-defined**: we shall check that $\varphi(l)$ is indeed a $n\times n$ symmetric Latin square where all the diagonal elements are different. One shall see that $\varphi(l)$ is a $n\times n$ square, and since we inherit every elements from l except from the diagonal elements and we don't include the elements from $n+1$ row and $n+1$ columns, we see since l is a symmetric Latin square, $\varphi(l)$ will still be a symmetric square. Now consider the i-th row of $\varphi(l)$ its element set will be $\{l(i,j) \mid j\in[1,n]\cap\mathbb{Z},\ j\neq i\}\cup\{l(i,n+1)\}$. In the original Latin square l, its i-th row $\{l(i,j) \mid j\in[1,n+1]\cap\mathbb{Z}\}$ will be a permutation of $\{1,\dots,n+1\}$, since $l(i,i)=n+1$, then we see $\{l(i,j) \mid j\in[1,n+1]\cap\mathbb{Z},\ j\neq i\}=\{1,\dots,n\}$, which is exactly the element set of the i-th row of $\varphi(l)$. And similarly we can have the same results for any j-th column of $\varphi(l)$. So for $\varphi(l)(i,j)=x$, i,x fixed unique j and j,x fixed unique i. This leads to $\varphi(l)$ will be a $n\times n$ symmetric Latin square which all symbols appear on the main diagonal. $\longrightarrow$ **OK!**

2. **φ is injective**: Given $\varphi(l_1)=\varphi(l_2)$, we want to see that $l_1=l_2\in L_{(n+1)\times(n+1)}$, and we can reason it be contraposition, which is: if $l_1\neq l_2$, we see $\varphi(l_1)\neq\varphi(l_2)$. If $l_1\neq l_2$, this means that $l_1(i,j)\neq l_2(i,j)$ for some $i,j\in[1,n+1]\cap\mathbb{Z}$. Because of symmetry, we only consider the situation where $i,j\in[1,n+1]\cap\mathbb{Z}, i<j$. When $j=(n+1)$, we see $l_1(i,n+1)\neq l_2(i,n+1) \implies \varphi(l_1)(i,i)\neq\varphi(l_2)(i,i) \implies \varphi(l_1)\neq\varphi(l_2)$. If $j\neq(n+1)$, we see $l_1(i,j)\neq l_2(i,j) \implies \varphi(l_1)(i,j)\neq\varphi(l_2)(i,j) \implies \varphi(l_1)\neq\varphi(l_2)$. $\longrightarrow$ **OK!**

3. **φ is surjective**: Given $L\in L_{n\times n}$, we can construct a new Latin square l based on the following rules:

   - the new Latin square will be of size $(n+1)\times(n+1)$.
   - $l(i,i)=n+1,\ \forall i\in[1,n+1]\cap\mathbb{Z}$.
   - $l(i,j)=L(i,j),\ \forall i,j\in[1,n]\cap\mathbb{Z}, i\neq j$.
   - $l(i,n+1)=L(i,i),\ \forall i\in[1,n]\cap\mathbb{Z}$, $l(n+1,j)=L(j,j),\ \forall j\in[1,n]\cap\mathbb{Z}$

   Following similar pattern as we have reasoning in the way that φ is well-defined, we see that $l\in L_{(n+1)\times(n+1)}$, moreover, we see that $\varphi(l)=L$. So we see any $L\in L_{n\times n}$, there exists $l\in L_{(n+1)\times(n+1)}$, s.t. $\varphi(l)=L$, leading to surjectivity of φ. $\longrightarrow$ **OK!**

So we find such φ that is bijective, **Q.E.D.** ∎

2 Problem 2

Problem 2.1. Suppose that A is a symmetric Latin square of order n such that every symbol appears on the main diagonal. Prove that n is odd.

Solution:

Proof. Since A is a Latin square with order n, every symbol of this Latin square should appear n times. Given arbitrary a symbol of A, denote as a. Since every symbol should appears on the main diagonal, so every symbol will appear exactly once at the diagonal since we have in total n number of symbols. Then consider the situation of a appearing at the non-diagonal position: Since it is a symmetric Latin square, we see that such a should always in pair, in particular, if it appears at $A(i,j)$ for some $i,j \in [1,n], i \neq j$, it will definitely appear at $A(j,i)$. Since it appears in pair at the non-diagonal position, it will appear even times at non-diagonal position. So it will appear in odd times in the whole Latin square, given the fact that total number of appearing times equals to times of appearing in diagonal position plus times of appearing at non-diagonal position. Then we see that n is equal to the total number of appearing times and thus n is odd.
Q.E.D. ■

3 Problem 3

Problem 3.1. Let $(X = \{x_1, x_2, \dots, x_{n^2+n+1}\}, \mathcal{L} = \{L_1, L_2, \dots, L_{n^2+n+1}\})$ be a projective plane of order n. Define a $(n^2+n+1) \times (n^2+n+1)$ *incidence matrix* M by

$$m_{ij} = \begin{cases} 1 & \text{if } x_i \text{ lies on } L_j \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\det(M) = \pm(n+1)n^{(n^2+n)/2}$.

Solution:

Proof. We will first calculate MM^T. First see that by properties of transposed matrix:

$$(MM^T)_{ij} = \sum_{k=1}^{n^2+n+1} m_{ik} \cdot m_{kj} = \sum_{k=1}^{n^2+n+1} m_{ik} \cdot m_{jk}$$

Then consider the value that each entries of MM^T will take:

- When $i \neq j$: observe that $m_{ik} = 1$ iff x_i lies on L_k, $m_{jk} = 1$ iff x_j lies on L_k, so $m_{ik} \cdot m_{jk} = 1$ iff x_i, x_j both lies on L_k, by the property of finite projective plane, L_k will be the unique line that passing through both x_i and x_j, in particular only one such k out of the total n^2+n+1 will satisfy $m_{ik} \cdot m_{jk} = 1$, so:

$$(MM^T)_{ij} = 1, \quad \forall i,j \in [1,n] \cap \mathbb{Z},\ i \neq j$$

- When $i = j$: we have:

$$MM^T = \sum_{k=1}^{n^2+n+1} m_{ik} \cdot m_{ik} = \sum_{k=1}^{n^2+n+1} (m_{ik})^2 = n+1$$

  Given the fact that in a finite project plane with order n, each point will be contained in exactly $n+1$ lines.

So we see:

$$MM^T = \underbrace{\begin{pmatrix} n+1 & 1 & \cdots & 1 \\ 1 & n+1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & n+1 \end{pmatrix}}_{n^2+n+1 \text{ columns}} \left.\vphantom{\begin{pmatrix} 1\\1\\ \vdots \\ 1\end{pmatrix}}\right\} n^2+n+1 \text{ rows}$$

And now to compute $\det(MM^T)$, we first do elementary column operation by summing every columns to the first column, with the determinant value remain unchange, denote $w = n^2+n+1$, we get:

$$A = \underbrace{\begin{pmatrix} n+w & 1 & \cdots & 1 \\ n+w & n+1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ n+w & 1 & \cdots & n+1 \end{pmatrix}}_{n^2+n+1 \text{ columns}} \left.\vphantom{\begin{pmatrix} 1\\1\\ \vdots \\ 1\end{pmatrix}}\right\} n^2+n+1 \text{ rows}$$

and $\det(MM^T) = \det A$. We further see that:

$$A = (n+w) \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & n+1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & n+1 \end{pmatrix} =: (n+w)B$$

and then we operate on B by doing elementary row operation, subtracting every row by the first row, we then get:

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix}$$

and we see:

$$\begin{aligned} &\det B = \det C = n^{w-1} \\ \implies &\det A = (n+w)\det B = (n+w)n^{w-1} \\ \implies &\det(MM^T) = \det A \end{aligned}$$

Then we see:

$$\begin{aligned} &\det(MM^T) = \det M \cdot \det M^T = (\det M)^2 = (n+w)n^{w-1} \\ \implies &\det M = \sqrt{(n^2+2n+1)n^{n^2+n}} = \pm(n+1)n^{(n^2+n)/2} \end{aligned}$$

Q.E.D. ■

4 Problem 4

Problem 4.1. Let $(X, \mathcal{L})$ be a projective plane of order n. Suppose we are given a set of points $A \subseteq X$ such that no 3 points in A lie on a common line. Prove that $|A| \leq n+2$.

Solution:

Proof. Given a point set $A \subseteq X$ such that no 3 points in A lie on a common line. Suppose that $|A| > n+2$,

we want to see that there will always have 3 points lie on a common line and thus leads to contradiction. In particular we only consider the case where $|A| = n+3$, since if every $|A| = n+3$ will always contain 3 points lie on a common line, then any point set with bigger cardinality will also have. Let's arbitrarily choose $a \in A$, and then arbitrarily pick $n+1$ points that is different from a from A, and connect them with a, and we thus obtain $(n+1)$ lines, since any two distinct point will fix a unique line in a finite projective plane. In particular we see all of these $(n+1)$ lines will contain a. Since $(X, \mathcal{L})$ is a finite projective plane of order n, we see any point will contained in exactly $n+1$ number of lines. And we see the $(n+1)$ lines we just constructed will be exactly all the lines that contain a. Now consider the remaining one point $b \in A$, we connect it with a, and thus obtain a line. By Pigeonhole Principle, it has to be one of the $(n+1)$ lines we constructed just now. And this leads to that a, b, and the point we used to constructed the line that agree with $\overline{ab}$ we be on the common line, which leads to contradiction that A will have no 3 points lie on a common line $\lightning$. So we see $|A| \leq n+2$.
Q.E.D. ■

5 Problem 5

Problem 5.1. An affine plane is a pair $(X, \mathcal{L})$ of points and lines satisfying:

- **(A0)**: there exist 3 points not all on one line.
- **(A1)**: any two points belong to a unique line.
- **(A2)**: given a point p not on a line L, there exists a unique L' passing through p such that $L \cap L' = \emptyset$.

Convince yourself that $\mathbb{R}^2$ is an affine plane.

Problem 5.2. (i) : For any projective plane, show that one can construct an affine plane by removing one of the lines and all the points on it.

Solution:

Proof. Given a projective plane $(X, \mathcal{L})$, we arbitrarily remove one of the lines and all the points on it, we shall then verify that this gives us an affine plane, i.e. it will satisfy **(A0), (A1), (A2)** as above.

- Verify **(A0)**: Based on **(P0)** property of projective plane, we will have a 4-element subset $F := \{p_1, p_2, p_3, p_4\} \subseteq X$, such that $|L \cap F| \leq 2$, $\forall L \in \mathcal{L}$. Since we remove one line, denote the removed line as L_∞, we see that at most 2 elements of F will be removed. Without losing generality, we suppose that p_3 and p_4 will be removed. We now want to find another point p_5, such that p_1, p_2, p_5 are not all on one line. And we define $p_5 \in \overline{p_1p_3} \cap \overline{p_2p_4}$ with $|\overline{p_1p_3} \cap \overline{p_2p_4}| = 1$. First note that by the property of projective plane, we have $L_\infty = \overline{p_3p_4}$, then we see that $p_5 \notin L_\infty = \overline{p_3p_4}$, otherwise we will have p_1, p_2, p_3, p_4 both on L_∞, contradict to the definition of F $\lightning$. So p_5 will not be removed after the removal of L_∞. Then we see that p_1, p_2, p_5 will not all on one line, otherwise we will have p_1, p_2, p_5, p_3, p_4 all on the same line, again contradict to the definition of F $\lightning$. So we find p_1, p_2, p_5 in the affine plane, such that they are not all on one line. $\longrightarrow$ **OK!**
- Verify **(A1)**: This property directly inherit from the fact that in a projective plane, for distinct point $x_1, x_2 \in X$, there exists a unique $L \in \mathcal{L}$, s.t. $x_1, x_2 \in L$. $\longrightarrow$ **OK!**
- Verify **(A2)**: Since in a projective plane, for distinct $L_1, L_2 \in \mathcal{L}$, $|L_1 \cap L_2| = 1$, we see that if we remove a whole line along with every point on that line, denote the removed line as L_∞, every line will lose exactly one point. Now given a point p not on a line L, with $L \neq L_\infty$, then suppose that $a \in L \cap L_\infty$, with $|L \cap L_\infty| = 1$, then after the removal of L_∞, $a \notin L$, so we have $\overline{pa} \cap L = \emptyset$, where before the removal of L_∞, $\overline{pa} \cap L = \{a\}$. We then see if $\overline{pa}$ is the unique line that satisfy such property: Suppose that there exists another line $\overline{pb}$, such that $\overline{pb} \cap L = \emptyset$. Before the removal of L_∞, we will have, say $\overline{pb} \cap L := \{c\}$,

by the property of projective plane. By our assumption, we see $\overline{pb} \neq \overline{pa} \implies a \neq c$. Now after the removal, we have $\overline{pb} \cap L = \emptyset$, this means that $c \in L_\infty$, so we see that $a \in L_\infty, c \in L_\infty \implies \overline{ac} = L_\infty$. But we also have that $a \in L, c \in L \implies \overline{ac} = L \implies L = L_\infty$, contradict to the fact that $L \neq L_\infty$ $\lightning$. So this means that $a = c \implies \overline{pa} = \overline{pb}$, in particular we see $\overline{pa}$ is unique. $\longrightarrow$ **OK!**

By verification on (A0), (A1), (A2), we see we construct an affine plane by removing arbitrary one lines and all points on the line of any projective plane. **Q.E.D.** ■

Problem 5.3. (*ii*): Prove that a finite affine plane has $n^2 + n$ lines and n^2 points, for some integer n.

Solution:

Proof. Given a finite affine plane $(X, \mathcal{L})$, we first claim that each point is contained in same number of lines. Given $x_1, x_2 \in X, x_1 \neq x_2$, then by (A2), there will be a unique line contain both x_1, x_2, denote as $\overline{x_1x_2}$. Consider a line L different from $\overline{x_1x_2}$ that contain x_1, we see that $x_2 \notin L$, otherwise $L = \overline{x_1x_2}$. Then by (A2), there exists a unique line L' that pass through x_2, such that $L \cap L' = \emptyset$, thus form a parity relation. We define $\mathcal{L}_1 := \{L \in \mathcal{L} \mid x_1 \in L\}$, $\mathcal{L}_2 := \{L \in \mathcal{L} \mid x_2 \in L\}$. By above reasoning, we will have a injective map from $\mathcal{L}_1$ to $\mathcal{L}_2$. And similarly, interchange the position of x_1 and x_2 in the above reasoning, we will also obtain a injective map from $\mathcal{L}_2$ to $\mathcal{L}_1$. And thus there exists a bijection between $\mathcal{L}_1$ and $\mathcal{L}_2$. In particular we see $|\mathcal{L}_1| = |\mathcal{L}_2|$, which proved our first claim.

We then suppose that each point will be contained in $(n+1)$ number of lines, where n be some integer. We then claim that any line will contain exactly n points in this case. Given an arbitrary line $L \in \mathcal{L}$, we see that we can always find a point $p \notin L$. By (A0), there exists $F := \{p_1, p_2, p_3\} \subseteq X$, such that not all of them on one line, so at least one of the elements in F will not be in L and thus such case always hold. By (A2), there will be a unique L' passing through p, such that $L \cap L' = \emptyset$. Previously, we have proven that there will be exactly $(n+1)$ lines passing through p, denote such lines set as $\mathcal{L}_3$. We then see that $L' \in \mathcal{L}_3$, $|\mathcal{L}_3| = n+1$. And since L' is the only unique line that satisfy $L' \cap L = \emptyset$, this just means that:

$$\forall L'' \in \mathcal{L}_3, L'' \neq L' \implies L'' \cap L \neq \emptyset \implies |L'' \cap L| \geq 1$$

We want to see that actually $|L'' \cap L| = 1$ in such case. Suppose that there exists $\overline{pab} \in \mathcal{L}_3$, $\overline{pab} \neq L'$, s.t. $\{a, b\} \in \overline{pab} \cap L$ ($|\overline{pab} \cap L| > 1$), in particular, we have $a \in L, b \in L$ and $a \in \overline{pab}, b \in \overline{pab}$, by (A1), we see that $\overline{pab} = L \implies p \in L$, contradict to the setting that $p \notin L$ $\lightning$. So we see $|L'' \cap L| = 1$, and by that $|\mathcal{L}_3 - \{L'\}| = n$, we see that the lines in $\mathcal{L}_3$ will induce n distinct point on L. And there will be no other point, since if there is a point, say a, not induced by lines in $\mathcal{L}_3$, we can obtain a unique line $\overline{pa}$ by (A1), and $\overline{pa}$ will be in one of $\mathcal{L}_3$, so a is still induced by lines in $\mathcal{L}_3$, leads to contradiction. So we see there will be exactly n distinct point contained in L, which proved the claim.

We then want to see that $|X| = n^2$. Given a line $L \in \mathcal{L}$, we can find a point p that is not on L. We denote the n points on L by $a_1, \ldots, a_n$. Clearly, $p \notin L = \{a_1, \ldots, a_n\}$, by (A1), we defined $L_i := \overline{px_i}$, and define L_{n+1}, such that L_{n+1} to be the unique line, such that $L_{n+1} \cap L = \emptyset$. We see that:

$$\bigcup_{i=1}^{n} L_i \cup L_{n+1} = X \tag{1}$$

given that $\forall x \in X,\ p \neq x$, then we see $\overline{px}$ will be one of $L_1, \ldots, L_n, L_{n+1}$. We then want to count the number of $|\bigcup_{i=1}^{n} L_i \cup L_{n+1}|$, with $|\bigcup_{i=1}^{n} L_i| = n^2 - n + 1$ and $|L_{n+1}| = n$, then:

$$\begin{aligned}
\implies |\bigcup_{i=1}^{n} L_i \cup L_{n+1}| &= |\bigcup_{i=1}^{n} L_i| + |L_{n+1}| - |\bigcup_{i=1}^{n} L_i \cap L_{n+1}| \\
&= |\bigcup_{i=1}^{n} L_i| + |L_{n+1}| - |\{p\}| \\
&= n^2 - n + 1 + n - 1 = n^2 \\
\implies |X| = n^2 &
\end{aligned}$$

We then want to see that $|\mathcal{L}| = n^2 + n$. Given $L_1, L_2 \in \mathcal{L}$, we say that $L_1 \parallel L_2$ iff either $L_1 = L_2$ or $L_1 \cap L_2 = \emptyset$. We then want to see that $\parallel$ is a equivalence relation:

- **$\parallel$ is reflexive**: Clearly we see that $L_1 = L_1 \implies L_1 \parallel L_1$.
- **$\parallel$ is symmetric**: Given $L_1 \parallel L_2$, either $L_1 = L_2 \iff L_2 = L_1$, or $L_1 \cap L_2 = \emptyset \iff L_2 \cap L_1 = \emptyset$, which implies that $L_2 \parallel L_1$.
- **$\parallel$ is transitive**: Suppose that $L_1 \parallel L_2$ and $L_2 \parallel L_3$, then there are four kinds of possibilities:
  1. $L_1 = L_2$ and $L_2 = L_3 \implies L_1 = L_3 \implies L_1 \parallel L_3$.
  2. $L_1 = L_2$ and $L_2 \cap L_3 = \emptyset \implies L_1 \cap L_3 = \emptyset \implies L_1 \parallel L_3$.
  3. $L_1 \cap L_2 = \emptyset$ and $L_2 = L_3 \implies L_1 \cap L_2 = \emptyset \implies L_1 \parallel L_3$.
  4. $L_1 \cap L_2 = \emptyset$ and $L_2 \cap L_3 = \emptyset$. Suppose that $L_1 \nparallel L_3$, then it means that $L_1 \cap L_3 \neq \emptyset$. Let $p \in L_1 \cap L_3$, lets consider the relationship between L_2 and p: if $p \in L_2$, we see that since $p \in L_1 \implies p \in L_1 \cap L_2 \implies L_1 \cap L_2 \neq \emptyset$, contradict to the fact that $L_1 \parallel L_2$ $\lightning$. So we see that $p \notin L_2$. Then by (A2), we see that actually $L_1 = L_3 \implies L_1 \parallel L_3$, leads to contradiction $\lightning$. So we see that $L_1 \parallel L_3$.

Thus we see that $\parallel$ is a equivalence relation, so it induce a partition on $\mathcal{L}$. We now want to investigate how many classes are there such partition, and how many lines are contained in each classes. We first arbitrarily pick a point $p \in X$, we've seen before, there will be exactly $(n+1)$ lines passing through p, denote them as $l_1, \dots, l_{n+1}$. Now since $p \in l_i \cap l_j \implies l_i \cap l_j \neq \emptyset,\ \forall\ i, j \in [1, n+1] \cap \mathbb{Z},\ i \neq j$, we see each l_i belongs to different equivalence classes, and thus there will be at least $n+1$ equivalence classes. Now given an arbitrary line $\bar{l} \in \mathcal{L}$, if $p \in \bar{l}$, then $\bar{l}$ will be one of l_i, if $p \notin \bar{l}$, then by (A2), there exists a unique line l_p passing through p, such that $l_p \cap \bar{l} = \emptyset \implies l_p$ and $\bar{l}$ will be in the same equivalence classes, and l_p in this case will be one of l_i and thus will be in one of the $(n+1)$ equivalence classes, so we see here are exactly $(n+1)$ equivalence classes in $\mathcal{L}$. We then arbitrarily pick a equivalence class, denote it as $\mathcal{C}$. We see that by (A0), not every lines are parallel to each other, so we can find $l_{\mathcal{C}} \notin \mathcal{C}$. By definition of $\parallel$, we see $\forall\ l \in \mathcal{C},\ l \cap l_{\mathcal{C}} \neq \emptyset$. Since we know that there are exactly n points on $l_{\mathcal{C}}$, we denote such points as $q_1, \dots, q_n$. For each q_i, by (A2), there will be a unique line passing through q_i, that is parallel to some line in $\mathcal{C}$, and thus such line will be also in $\mathcal{C}$. So we can construct n lines in this way, all of them in $\mathcal{C}$, denote as $L'_1, \dots, L'_n$. Note that all of them are distinct: if $L'_i = L'_j$ $(i \neq j)$, we see that $q_i, q_j \in L'_i = L'_j \implies L'_i = L'_j = l_{\mathcal{C}}$, but $l_{\mathcal{C}} \notin \mathcal{C}$. So we see $|\mathcal{C}| \geq n$. Suppose there are another line $l'_{\mathcal{C}} \in \mathcal{C}$, it will also intersect with $l_{\mathcal{C}}$ by definition. But we see that there are only n points on $l_{\mathcal{C}}$, and they already induced $L'_1, \dots, L'_n$, leading to $l'_{\mathcal{C}}$ will be one of $L'_1, \dots, L'_n$. So we see $|\mathcal{C}| = n$. Such arguments works for any equivalence classes, so we see:

$$\mathcal{L} = (n+1)|\mathcal{C}| = n^2 + n$$

Above reasoning holds for any finite affine plane, so we see a finite affine plane has $n^2 + n$ lines and n^2 points, for some integer n. **Q.E.D.**

Another proof idea about showing $|\mathcal{L}| = \mathbf{n^2 + n}$

We want to see that $|\mathcal{L}| = n^2 + n$. Given $L \in \mathcal{L}$, we can find a point $p \notin L$, and thus obtain a unique line L' passing through p, such that $L' \cap L = \emptyset$. We can then denote points on L as $\alpha_1, \dots, \alpha_n$, and points on L' as $\beta_1, \dots, \beta_n$. Thus we can then obtain n^2 distinct lines, defined by $\overline{\alpha_i\beta_j},\ \forall\ i, j \in [1, n] \cap \mathbb{Z}$, which are also different from L, L' by (A1). Then we pick one of those n^2 lines, say $\overline{\alpha_1\beta_1}$, there will be another $n-2$ points $\zeta_1, \dots, \zeta_{n-2}$ on it, which are different from α_1, β_1. Each of them will induced a unique line passing through it, such that its intersection with L will be $\emptyset$. We denote such $(n-2)$ lines as $l_1, \dots, l_{n-2}$. See that by (A1) and (A2), all of these lines will be different from each other and different from L and L', and we define $\mathcal{L}_4 := \{l_1, \dots, l_{n-1}\}$. We want to see that:

$$\mathcal{L} = \mathcal{L}_4 \sqcup \{L, L'\} \sqcup \{\overline{\alpha_i\beta_j} \mid \forall i, j \in [1, n] \cap \mathbb{Z}\} =: \mathcal{L}'$$

Given arbitrary $l \in \mathcal{L}$. If $l \cap L \neq \emptyset$, without losing generality, we suppose $l \cap L = \{\alpha_1\}$ in this case. Since there are already $(n+1)$ lines passing through α_1, which are $L, \overline{\alpha_1\beta_1}, \dots, \overline{\alpha_1\beta_n}$, so l must be one of them. In particular $l \in \{L, L'\} \sqcup \{\overline{\alpha_i\beta_j} \mid \forall i, j \in [1, n] \cap \mathbb{Z}\} \implies l \in \mathcal{L}'$. If $l \cap L = \emptyset$, one can immediately observe that $\alpha_1, \dots, \alpha_n \notin l$. If $b_j \in l$ for some $j \in [1, n] \cap \mathbb{Z}$, then we immediately see that $L' = l$, since we already see that

L' will be the only line passing through any b_j whose intersection with L is $\emptyset$ by our previous construction. So when $l \neq L'$, we also see $\beta_1, \ldots, \beta_n \notin l$. We then note that $|\overline{\alpha_i\beta_j} \cap l| \leq 1,\ \forall i, j \in [1, n] \cap \mathbb{Z}$, otherwise, by (A1), we will have $l = \overline{\alpha_i\beta_j}$ for some $i, j \in [1, n] \cap \mathbb{Z}$, which implies that $l \cap L \neq \emptyset$, leading to contradiction $\lightning$. So we have $|\overline{\alpha_i\beta_j} \cap l| \leq 1,\ \forall i, j \in [1, n] \cap \mathbb{Z}$. As we have proven before in **Equation** 1, we see that $X = \bigcup_{i=1}^{n} \overline{\alpha_i\beta_1} \cup L' \implies l \subseteq \bigcup_{i=1}^{n} \overline{\alpha_i\beta_1}$. By (A1), $\bigcap_{i=1}^{n} \overline{\alpha_i\beta_1} = \{\beta_1\}$, thus we see:

$$\begin{aligned}
\bigcup_{i=i}^{n} \overline{\alpha_i\beta_1} &= \{\alpha_1, \ldots, \alpha_n\} \sqcup \{\beta_1\} \sqcup \bigsqcup_{i=1}^{n} (\overline{\alpha_i\beta_1} - \{\alpha_i, \beta_1\})) \\
\implies \bigcup_{i=i}^{n} \overline{\alpha_i\beta_1} - (\{\alpha_1, \ldots, \alpha_n\} \sqcup \{\beta_1\}) &= \bigsqcup_{i=1}^{n} (\overline{\alpha_i\beta_1} - \{\alpha_i, \beta_1\})) \\
\implies |\bigsqcup_{i=1}^{n} (\overline{\alpha_i\beta_1} - \{\alpha_i, \beta_1\}))| &= n(n-2)
\end{aligned}$$

Since $\beta_1, \alpha_i \notin l,\ \forall\ i$, we see $|(\overline{\alpha_i\beta_1} - \{\alpha_i, \beta_1\}) \cap l| \leq 1$ and $l \subseteq \bigsqcup_{i=1}^{n} (\overline{\alpha_i\beta_1} - \{\alpha_i, \beta_1\}))$. Since $|l| = n$, $|\bigsqcup_{i=1}^{n} (\overline{\alpha_i\beta_1} - \{\alpha_i, \beta_1\}))| = n(n-2)$, by **Pigeonhole Principle**, we see $|(\overline{\alpha_i\beta_1} - \{\alpha_i, \beta_1\}) \cap l| = 1$, which implies that $|(\overline{\alpha_1\beta_1} - \{\alpha_i, \beta_1\}) \cap l| = 1 \iff \zeta_i \in l$ for some $i \in [1, n-2] \cap \mathbb{Z}$, which means that $l \in \mathcal{L}_4 \implies l \in \mathcal{L}'$. So we see that $\mathcal{L} \subseteq \mathcal{L}'$, but obviously we shall have $\mathcal{L}' \subseteq \mathcal{L}$, so we see that $\mathcal{L} = \mathcal{L}'$. Then:

$$\begin{aligned}
|\mathcal{L}'| &= |\mathcal{L}_4 \sqcup \{L, L'\} \sqcup \{\overline{\alpha_i\beta_j} \mid \forall i, j \in [1, n] \cap \mathbb{Z}\}| \\
&= (n-2) + 2 + n^2 \\
&= n^2 + n \\
\implies |\mathcal{L}| &= n^2 + n
\end{aligned}$$

■

6 Problem 6

Problem 6.1. Prove that the Fano plane cannot be embedded in the Euclidean plane, that is, one cannot find 7 points and 7 lines in $\mathbb{R}^2$ with three points in each line, three lines through each point, and such that each pair of lines intersect at one of the points, and each pair of points lies on one of the lines.

Lemma 6.2. Every finite set of points in the Euclidean plane has a line that passes through exactly two of the points or a line that passes through all of them. A line that contains exactly two of a set of points is known as an *ordinary line.*

Solution: We shall first give proof to **Lemma** 6.2

Proof. Given a finite point set S, if all points in S lies on a common line, then we are done. Now suppose that not all points in S are collinear. Now define a connecting line to be a line that passing through at least 2 of the points in S. Since S is a finite set, then there must exists a point p and a connecting line l, with positive distance apart, such that no other point-line pair has smaller distance between them. We want to see that l is actually the intended ordinary line.

Suppose that l is not an ordinary line, then there must exists three point of S, such that they all lie on the common line l. Denote them as $\{a, b, c\}$. We now draw the perpendicular projection of p to l, denoted as p'. Then by pigeonhole principle, at least 2 of the points $\{a, b, c\}$ will lie on the same side of p', say, a, b. And consider b to be the point that goes closer to p', and possibly coincide with p'. And now give the connecting line l' that passing through p and a, namely, $\overline{pa} = l'$, we see that the distance between the point pair b and l' is smaller than the distance between the original smallest-distanced point pair p and l, since if we draw the perpendicular projection of b onto l', denoted as p'', we see that $\triangle bap''$ is fully contained in $\triangle app'$ with common vertex a, and thus $p''b < pp'$, leading to contradiction $\lightning$. So we find l as such ordinary line. **Q.E.D.** ∎

We then shall give the proof of the **Original Statement**.

Proof. Suppose that Fano plane $(X, \mathcal{L})$ can be embedded in the Euclidean plane, we see such embedding gives us a finite points set. But then clearly in such embedding we cannot find a line passing through exactly 2 points by definition of Fano plane (In a Fano plane, there are three points on every line), which contradict to our **Lemma** 6.2 $\lightning$. So we see Fano plane cannot be embedded in the Euclidean plane. **Q.E.D.** ∎

7 Problem 7

Problem 7.1. Let X be a set with n^2+n+1 elements, where $n \geq 2$. Let $\mathcal{L}$ a collection of n^2+n+1 subsets of X, such that each $L \in \mathcal{L}$ has size $n+1$. Suppose that for any two distinct $L, L' \in \mathcal{L}$, we have $|L \cap L'| \leq 1$. Prove that $(X, \mathcal{L})$ is a projective plane of order n.

Solution:

Proof. To see $(X, \mathcal{L})$ is a projective plane of order n, we shall verify the three properties of projective plane.

- Verify **(P2)**: We first want to prove that any pair of points will be in at most one line. Given $\{a, b\} \subseteq X$, suppose that there exists two distinct lines $L, L' \in \mathcal{L}$, such that $\{a, b\} \subseteq L$, $\{a, b\} \subseteq L' \implies \{a, b\} \subseteq L \cap L' \implies |L \cap L'| \geq 2$, which contradicts to $|L \cap L'| \leq 1$, $\forall L, L' \in \mathcal{L}$, $L \neq L'$ $\lightning$. So we see in this case $L = L'$, so ant pair of points will be in at most one line.

  We then consider whether every pair of point is in one line. We will proceed the proof via **double counting** on how many pair of points are there in X, and we denote such number as N. First we directly count via point set:
  $$N = \binom{|X|}{2} = \binom{n^2+n+1}{2} = \frac{(n^2+n+1)(n^2+n)}{2}$$
  Then we count via points on different lines, since there are $(n+1)$ different points on a line, we see each line will consists $\binom{n+1}{2} = \frac{n^2+n}{2}$ number of point pairs. Observe that we've seen that any point pair will occur on at most one line, so this means that the point pairs consists in different lines will be different from each other. So we then try to count the total number of point pairs contributed by all of the lines, given that $|\mathcal{L}| = n^2+n+1$, and denote such number by N':
  $$N' = |\mathcal{L}| \times \binom{n+1}{2} = (n^2+n+1)\frac{n^2+n}{2}$$
  $$\implies N = N'$$
  So we see that the total number of point pairs in X is exactly equal to the total number of point pairs that lie in a common line. So given any two points, such points will be lie in a line, and such line will be unique by our previous proof on uniqueness. So we see that (P2) holds in this case. $\longrightarrow$ **OK!**

- Verify **(P1)**: Suppose that there exists $L_1, L_2 \in \mathcal{L}$, $L_1 \neq L_2$, such that $L_1 \cap L_2 = \emptyset$, then there will be $(n+1)$ distinct point on L_1, L_2 respectively, we denote them as $\{\alpha_0, \dots, \alpha_n\}$ and $\{\beta_0, \dots, \beta_n\}$ respectively. And now define $l_i := \overline{b_0 a_i}$, $\forall i \in [0, n] \cap \mathbb{Z}$. By (P2), we now count the total number of vertices in this case:
  $$\begin{aligned} |\bigcup_{i=0}^{n} \overline{b_0 a_i} \cup L_1 \cup L_2| &= |\bigcup_{i=0}^{n} \overline{b_0 a_i} \cup L_1| \\ &= |\bigcup_{i=0}^{n} \overline{b_0 a_i}| + |L_1| - |(\bigcup_{i=0}^{n} \overline{b_0 a_i}) \cap L_1| \end{aligned}$$

$$\implies |\bigcup_{i=0}^{n} \overline{b_0 a_i} \cup L_1 \cup L_2| = ((n+1)^2 - n) + (n+1) - 1$$
$$= n^2 + 2n + 1 > n^2 + n + 1 = |X| \lightning$$

So we see $\forall L_1, L_2 \in \mathcal{L},\ L_1 \neq L_2,\ |L_1 \cap L_2| = 1$. So (P1) holds. $\longrightarrow$ **OK!**

- Verify **(P0)**: We start with picking arbitrary two distinct lines $L_1, L_2 \in \mathcal{L}$, by (P1), we define $|L_1 \cap L_2| := \{\pi\}$, and we denote the rest points on $L_1,\ L_2$ as $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ respectively. And we now have n^2 number of lines, which is:

$$\mathcal{L}_0 := \{\overline{\alpha_i \beta_j} \mid i, j \in [1, n] \cap \mathbb{Z}\}$$

So far, we have $n^2 + 2$ number of lines, so there are $(n-1)$ number of lines remaining. We claim that all these $(n-1)$ lines should passing through π. Suppose that this is not the case, then we can find a line l, such that $l \notin \mathcal{L}_0 \cup \{L_1, L_2\}$ and $a \notin l$. We see that $|l \cap L_1| = 1,\ |l \cap L_2| = 1$ and $a \notin l \cap L_1,\ a \notin l \cap L_2$, which implies that $l \cap L_1 = \{\alpha_i\},\ l \cap L_2 = \{\beta_j\}$ for some integer i, j. But this leads to that $\overline{\alpha_i \beta_j} = l \implies l \in \mathcal{L}_0$, leads to contradiction $\lightning$. So our claim holds.

We then define:

$$\mathcal{L}_1 := \{l \in \mathcal{L} \mid a \in l,\ l \notin \mathcal{L}_0 \cup \{L_1, L_2\}\}$$

Arbitrarily pick a line $l_0 \in \mathcal{L}_1$, and pick another point $\zeta \in l_0$, such that $\zeta \neq \pi$ and $\zeta \notin \overline{\alpha_0 \beta_0} \cap l_0$, there are in total $(n+1)$ number of points on l_0, so we can always do such choosing. We claim that $F := \{\pi, \zeta, \alpha_0, \beta_0\}$ is the 4-element points set that satisfying (P0). Note that:

$$\mathcal{L} = \mathcal{L}_0 \sqcup \mathcal{L}_1 \sqcup \{L_1, L_2\} \tag{2}$$

  - For $l' \in \mathcal{L}_0$, we see that $\pi \notin l'$, otherwise $l' \in \{L_1, L_2\}$. And by our choosing we see $\zeta \notin \overline{\alpha_0 \beta_0}$. So we see that when $l' = \overline{\alpha_0 \beta_0}$, we have $\zeta \notin l',\ \pi \notin l' \implies |l' \cap F| \leq 2$. When $l' \neq \overline{\alpha_0 \beta_0}$, we have at most one element of $\{\alpha_0, \beta_0\}$ will be in l', combined with that $\pi \notin l'$, still yeilds that $|l' \cap F| \leq 2$.
  - For $l' \in \mathcal{L}_1$, when $l' = l_0$, we see that $\zeta \in l',\ \pi \in l'$, however we see $\alpha_0, \beta_0 \notin l'$, otherwise we will deduce that $\overline{\alpha_0 \beta_0} = l'$, leading to contradiction $\lightning$, which implies that in this case $|l' \cap F| = 2$. When $l' \neq l_0$, we see at most one element of $\{\alpha_0, \beta_0\}$ will be in l', combined with that $\zeta \notin l'$, still yields that $|l' \cap F| \leq 2$.
  - For $l' \in L_1$ or $l' \in L_2$, we only consider the case $l' \in L_1$, the other case is exactly the same. In this case, we have $\alpha_0 \in L_1, \pi \in L_1$, but we see $\zeta \notin L_1, \beta_0 \notin L_1 \implies |L_1 \cap F| = 2 \implies |l' \cap F| \leq 2$.

  Then by **Equation** 2, we see that $\forall\, l' \in \mathcal{L},\ |l' \cap F| \leq 2$, which yields (p0). $\longrightarrow$ **OK!**

Combining above reasoning, we see that it is indeed a finite projective plane of order n. **Q.E.D.** $\blacksquare$

8 Appendix