Math566: Combinatorial Theory

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Abstract

This is the note containning my personal thoughts as well as lecture notes, course content include some basic algebraic combinatorics. My course instructor is Prof. Shiyue Li.

Contents

Chapter 1

Linear Algebra

This chapter basically collect some linear algebra lemma that maybe helpful across the course.

Lemma 1.0.1. Given A to be a matrix, who has the eigenvalues to be $\lambda_1, \dots, \lambda_p$, then the eigenvalues of the matrix $A + c\,\mathrm{Id}$ where $c \in \mathbb{C}$ is $\lambda_1 + c, \dots, \lambda_p + c$.

Lemma 1.0.2. Let A be as above, the eigenvalues of A^l are $\lambda_1^l, \dots, \lambda_p^l$.

Lemma 1.0.3. Let $f \in \mathbb{C}[t]$, then $f(A)$ has eigenvalues being $\{f(\lambda_i)\}_{i=1}^p$.

Lemma 1.0.4. If A is a real and symmetric matrix, by spectral theorem for Hermitian product, we have $\lambda_i \in \mathbb{R}$.

Lemma 1.0.5. Let $\{\alpha_i\}_{i=1}^r$ and $\{\beta_i\}_{i=1}^s$ be non-zero complex numbers, if for every $l \geq 1$, we have:

$$\sum_{i=1}^{r} \alpha_i^l = \sum_{i=1}^{s} \beta_i^l$$

then $r = s$ and $\{\alpha_i\}$ is permutation of $\{\beta_i\}$.

Chapter 2

Walks in Graph

2.1 Graph Eigenvalues

To better phrasing a graph, we first need to define multiset so that we can express not only simple graph but also general graph in a mathematically rigourous way.

Definition 2.1.1 (Multiset). A multiset M on a set S is an unordered collection of elements in S, s.t.

1. $\forall\, x \in M, x \in S$.
2. The $\#$ of times for $x \in S$ to appear in M, denoted as $\mu_M(x)$, is ≥ 0.

Example 2.1.1. If $\mu_M(x) = 0, 1\ \forall x$, then M is a set.

Note that two multiset M, M' are said to be equal if $\forall x \in S, \mu_M(x) = \mu_{M'}(x)$.

Notation. Let S be a finite set of size p, then define:

$$\binom{S}{k} = \{k - \text{subsets of } S\}$$

note that

$$\left|\binom{S}{k}\right| = \binom{|S|}{k}$$

also define:

$$\left(\!\binom{S}{k}\!\right) = \{k - \text{subsets of } S\}$$

note that:

$$\left|\left(\!\binom{S}{k}\!\right)\right| = \binom{p+k-1}{k}$$

this is the case: consider rephrasing the combinatorial problem as possible assigning of numbers for p numbers $a_1, \ldots, a_p$ with $a_1 + \ldots + a_p = k$ and $a_i \in [\![0, k]\!]$.

Thus we can define the graph properly.

Definition 2.1.2 (Graph). A finite graph is a triple $G = (V, E, \varphi)$ with:

- $V = \{v_1, \ldots, v_p\}$.
- $E = \{e_1, \ldots, e_q\}$.
- φ is a function $E \to \left(\!\binom{V}{2}\!\right)$.

A finite simple graph is the same data with $\varphi : E \to \binom{V}{2}$

Definition 2.1.3 (Adjacency Matrix of Graph). The adjacency matrix of a graph G, denoted as $A(G)$, whose entries is defined by:

$$a_{ij} = \left|\varphi^{-1}(\{v_i, v_j\})\right|$$

In particular it counts the number of edges between two vertices v_i and v_j. Note that it is well-defined since if there is no edges between v_i and v_j, then the preimage of φ will be $\emptyset$, thus $a_{ij} = 0$.

Definition 2.1.4 (Walk). A walk of length k in a graph G is a non-empty finite sequence of vertices and edges

$$W = v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$$

such that for all $1 \leq i \leq k$, the edge e_i has **endpoints** v_{i-1} and v_i.

In a simple graph, where the edges are determined by their endpoints, a walk can be simplified to a sequence of vertices:

$$W = (v_0, v_1, \ldots, v_k) \quad \text{where } \{v_{i-1}, v_i\} \in E(G)$$

Note.

- In a walk, the both the edges and vertices can appear **repeatedly**.
- If $v_0 = v_k$, then such walk is called a **closed walk**.

Proposition 2.1.1. For any integer $l \geq 1$, the (i, j) entry of $(A(G))^l$, denoted as a_{ij}, is equal to the $\#$ of walks of length l in G starting from v_i to v_j.

Theorem 2.1.1. Let G be graph with $A(G)$ possessing eigenvalues $\lambda_1, \ldots, \lambda_p$, the $\#$ **closed walks** of length l is:

$$f_G(l) = \sum_{i=1}^{p} \lambda_i^l$$

The proof is straightforward combining the **Proposition** 2.1.1 and **Lemma** 1.0.2.

Definition 2.1.5 (Unitary Matrix). A complex square matrix U is unitary if $U^* = U^{-1}$.

Definition 2.1.6 (Orthogonal Matrices). An orthogonal matrix is a real square matrix O whose rows and cols are **orthonormal**, in particular equivalent to:

$$O^\intercal = O^{-1}$$

Theorem 2.1.2 (Spectral Theorem for Hermitian Matrices). A complex (real) square matrix A is Hermitian (real symmetric). Then there exists a unitary (orthogonal) U and a real diagonal matrix Λ, s.t.

$$A = U\Lambda U^{-1}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

In particular, U can be chosen to have the orthonormal eigenvectors of A as its column vectors.

Corollary 2.1.1. There are $(p-1)^l + (-1)^l(p-1)$ closed walks in K_p.

Notation.

- $\mathbb{J}$ is the all 1 matrix.
- $\mathbb{I}$ is the identity matrix.

Proof. For K_p, the adjacency matrix is given by $A(K_p) = \mathbb{J} - \mathbb{I}$, with the eigenvalue of $\mathbb{J}$ being $\{p, 0, \dots, 0\}$ and the eigenvalues of $\mathbb{I}$ being $\{1, \dots, 1\}$. By **Lemma** 1.0.3, here we have $f(x) = x - 1$, with $\mathbb{J}$ pluggined, thus yields the result. ■

Corollary 2.1.2. There are $\frac{1}{p}((p-1)^l - (-1)^l)$ non-closed walks of length l in K_p.

Proof. Consider the matrix $A(K_p)^l = (\mathbb{J} - \mathbb{I})^l$, since $\mathbb{J}$ and $\mathbb{I}$ commutes, one can expand it using the binomial theorem:

$$(\mathbb{J} - \mathbb{I})^l = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} \mathbb{J}^i$$

And note that:

$$\mathbb{J}^i = \begin{cases} \mathbb{J}^0 = \mathbb{I} & i = 0 \\ p^{i-1}\mathbb{J} & i > 0 \end{cases}$$

Then see that:

$$\begin{aligned} \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} \mathbb{J}^i &= \left(\sum_{i=0}^{p} (-1)^{l-i} \binom{l}{i} p^{i-1} \right) \mathbb{J} + (-1)^l \mathbb{I} \\ &= \left(\sum_{i=0}^{p} (-1)^{l-i} \binom{l}{i} p^{i-1} - (-1)^l \frac{1}{p} \right) \mathbb{J} + (-1)^l \mathbb{I} \\ &= \left(\frac{1}{p}(p-1)^l - (-1)^l \frac{1}{p} \right) \mathbb{J} + \underbrace{(-1)^l \mathbb{I}}_{\text{contribute nothing}} \end{aligned}$$

■

One can note that λ_i of $A(G)$ is completely determined by traces of $A(G)^l \quad \forall l \geq 1$ by **Lemma** 1.0.5. In particular if we know enough number of traces of $A(G)^l$ for multiple l, then we can calculate out the eigenvalues.

2.2 Radon Transform

We may define inner product space and orthogonal stuff first.

Definition 2.2.1 (Inner product space). An inner product space is a vector space over $\mathbb{C}$ together with an inner product $\langle -, - \rangle : V \times V \to \mathbb{C}$, s.t. $x, y, z \in V, \quad a, b \in \mathbb{C}$:

1. Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
2. Linearity with the first entry: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
3. Positivity: if $x \neq 0$, then $\langle x, x \rangle > 0$.

In this section we shall assume all V to be inner product space.

Definition 2.2.2. $x, y \in V$ are orthogonal if $\langle x, y \rangle = 0$.

Lemma 2.2.1. If $\langle x, y \rangle = 0$, then x, y are linearly independent.

Definition 2.2.3 (Kronecker Delta). Let S be a set, the kronecker delta function on S^2 is given by:

$$\delta_{uv} = \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}$$

Let $\mathbb{Z}_2$ be the cyclic group of order 2, i.e. $(\{0,1\},+)$.
Let $\mathbb{Z}_2^n$ be the n-folde product of $\mathbb{Z}_2$, called an n-cube:

$$\mathbb{Z}_2^n = \{(a_1,\ldots,a_n) \mid a_i \in \mathbb{Z}_2\}$$

such set has some properties if viewed as a vector space, for example, ZZ_2^n has a dot product defined by:

$$\begin{aligned} \langle -,- \rangle : \mathbb{Z}_2^n \times \mathbb{Z}_2^n &\to \mathbb{Z}_2 \\ \langle y,z \rangle &\mapsto \underbrace{\sum y_i z_i}_{\text{group adding}} \in \mathbb{Z}_2 \end{aligned}$$

Lemma 2.2.2. $\forall u,v,w \in ZZ_2^n$, see that $u+v=w \Leftrightarrow u+w=v \Leftrightarrow v+w=u$.

The proof directly follows once realize that in this group $(-)$ is the same as $(+)$.
For $u \in \mathbb{Z}_2^n$, the weight $|u| = \sum u_i$ which count the number of entries that is non-zero.

2.2.1 Counting in Cube

In this section we shall use Radon Transform to help counting the closed walks in a cube. For other special graphs, one can similarly embed the structure of it into a specific group and construct the corresponding Radon Transform to count. It may rely on specific kinds of symmetry but really provide and powerful and convenient tool for counting.

Proposition 2.2.1. Let V be all functions $f : \mathbb{Z}_2^n \to \mathbb{C}$. This is a vector space over $\mathbb{C}$, we have the following fact:

1. $\dim_{\mathbb{C}} V = 2^n$, with the basis given by:

$$i_u(v) = \begin{cases} 1, & \text{if } u = v \\ 0, & \text{if } u \neq v \end{cases}$$

2. V has a inner product space structure over $\mathbb{C}$:

$$\langle f,g \rangle = \sum_{u \in \mathbb{Z}_2^n} f(y)\overline{g(u)}$$

3. V has basis:

$$\begin{aligned} \mathcal{B}_1 &= \{f_u : \ u \in \mathbb{Z}_2^n, \ f_u(v) = \delta_{uv}\} \\ \mathcal{B}_2 &= \{\chi_u : \ \chi_u(v) = (-1)^{uv}\} \end{aligned}$$

See that:

$$g(v) = \sum_{u \in \mathbb{Z}_2^n} g(v)\delta_{uv} = \sum_{u \in \mathbb{Z}_2^n} g(v) f_u(v)$$

and

$$
\begin{aligned}
\langle \chi_u, \chi_v \rangle &= \sum_{w \in \mathbb{Z}_2^n} \chi_u(w)\overline{\chi_v(w)} \\
&= \sum_{w \in \mathbb{Z}_2^n} (-1)^{(u+v)\cdot w} \\
&= \begin{cases} 2^n, & \text{if } u = v \Leftrightarrow 2u = 0 \\ 0, & \text{if } u \neq v \end{cases}
\end{aligned} \tag{2.1}
$$

which means that χ_u and χ_v are orthogonal and thus linearly independent. The part that leads to 0 can be deduce by pairing vector w on the entry that $(u + v)$ is 1.

Definition 2.2.4 (Radon Transform). For a subset $\Gamma \in \mathbb{Z}_2^n$ and a function $f \in V : \mathbb{Z}_2^n \to \mathbb{C}$, the discrete radon transform on Γ of $\mathbb{Z}_2^n$ is:

$$
\Phi_\Gamma(f)(v) := \sum_{w \in \Gamma} f(v + w)
$$

this is a linear transformation $V \to V$.

Theorem 2.2.1. The eigenvectors of Φ_Γ are $\{\chi_u\}_{u \in \mathbb{Z}_2^n}$, with eigenvalues being $\lambda_u = \sum_{w \in \Gamma}(-1)^{u \cdot v}$.

Proof. See that:

$$
\begin{aligned}
\Phi_\Gamma(\chi_u)(v) &= \sum_{w \in \Gamma} \chi_u(v + w) \\
&= \sum_{w \in \Gamma} (-1)^{u(v+w)} \\
&= (-1)^{uv} \sum_{w \in \Gamma} (-1)^{uw} \\
&= \chi_u(v) \cdot \lambda_u
\end{aligned}
$$

■

Definition 2.2.5. The n-cube graph C_n is the graph with:

- $V(G) = \mathbb{Z}_2^n$.
- $E(G) = \{(u, v) \mid u, v \text{ differ in only 1 coordinate}\}$, for example the 3-dimensional cube.

The adjacency matrix of this kinds of graph is rather complicated and hard to comprehend and compute its eigenvalue by simply looking at the matrix.

We now propose a simpler way to obtain the eigenvalue and eigenvector of adjacency matrix of n-cube graph, basically choosing special subset to do radon transform and see that the matrix is exactly the same.

Let Δ be the set:

$$
\Delta := \{\delta_i \mid \delta_i \text{ is the i-th unit vector in } \mathbb{Z}_2^n\} \subseteq \mathbb{Z}_2^n
$$

Let $\Phi_\Delta : V \to V$, and $[\Phi_\Delta]$ be the matrix of Φ_Δ w.r.t. $\mathcal{B}_1 = \{f_u : \ u \in \mathbb{Z}_2^n\}$.

Lemma 2.2.3. The matrix $[\Phi_\Delta] = A(C_n)$.

Proof. Basically we want to see that all entries on the left hand side is the same as the right hand side, so we consider the (u, v)-entry of $[\Phi_\Delta]$, let $z \in \mathbb{Z}_2^n$, and compute: $\Phi_\Delta f_u(v)$,

$$\begin{aligned}
\Phi_\Delta f_u(z) &= \sum_{w\in\Delta} f_u(z+w) \\
&= \sum_{w\in\Delta} f_{u+w}(z) \\
\Rightarrow \Phi_\Delta f_u &= \sum_{u\in\Delta} f_{u+w} \\
\Rightarrow [\Phi_\Delta]_{u,v} &= \begin{cases} 1, & \text{if } u+w=v \\ 0, & \text{if } u+w\neq v \end{cases} \\
\Leftrightarrow [\Phi_\Delta]_{u,v} &= \begin{cases} 1, & \text{if } u+v=w\in\Delta \\ 0, & \text{otherwise} \end{cases} \\
\Leftrightarrow [\Phi_\Delta]_{u,v} &= \begin{cases} 1, & \text{if } u,v\in E(C_n) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

■

Corollary 2.2.1. The eigenvectors of $A(C_n)$ are:

$$\mathcal{E}_u = \sum_{v\in\mathbb{Z}_2^n} (-1)^{u\cdot v} v$$

With eigenvalues being:

$$\lambda_u = n - 2|u|$$

In particular, $A(C_n)$ has $\binom{n}{k}$ eigenvalues equal to $n-2k$.

Proof. For $g \in V$, see that $g = \sum_{v\in\mathbb{Z}_2^n} g(v)\cdot f_v$, which implies that:

$$g(u) = \sum_{v\in\mathbb{Z}_2^n} g(v) f_v(u) = \sum_{v\in\mathbb{Z}_2^n} g(v)\delta_{uv}$$

Recall that $[\Phi_\Delta]_{\mathcal{B}_1}$ has eigenvectors $\{\chi_u(v) = (-1)^{u\cdot v}\}$, then:

$$\chi_u = \sum_{v\in\mathbb{Z}_2^n} \chi_u(v) f_v = \sum_{v\in\mathbb{Z}_2^n} (-1)^{u\cdot v} f_v$$

with eigenvalues being:

$$\lambda_u = \sum_{w\in\Delta} (-1)^{u\cdot w} = \sum_{i\in[n]} (-1)^{u\cdot\delta_i} = n - 2|u|$$

Notice that those $\mathcal{E}_u$ with the **Hamming Weight** of u being the same share the same eigenvalue, total number of such u will be $\binom{n}{k}$ if $|u| = k$. ■

Proposition 2.2.2. If G is a graph with p vertices and eigenvalues $\lambda_1, \ldots, \lambda_p$, then there exists $c_1, \ldots, c_p \in \mathbb{R}$, s.t.

$$A(G)^l_{ij} = \sum_{k=1}^{p} c_k \lambda_k^l$$

And in particular, if U is an **orthogonal** matrix ($A(G) = U\Lambda U^{-1}$), then:

$$A(G)^l_{ij} = \sum_{k=1}^{n} U_{ik} U_{jk} \lambda_k^l$$

Proof. By **Spectral Theorem** 2.1.2, there exists an orthogonal matrix U, s.t.

$$A = U\Lambda U^{-1}$$

Let $l \geq 1$, then:

$$A^l = U\Lambda^l U^{-1}$$

take the (i, j)-entry of right hand side and see that it equal to $\sum_{k=1}^{n} U_{ik}U_{jk}\lambda_k^l$. ■

Corollary 2.2.2. Let $u, v \in \mathbb{Z}_2^n$, suppose that $|u + v| = k$, i.e. u and v **differ** in k places, then:

$$A(C_n)_{uv}^l = \frac{1}{2^n}\sum_{i=1}^{n}\sum_{j=1}^{k}(-1)^k\binom{k}{j}\binom{n-k}{i-j}(n-2i)^l \qquad (i \geq j)$$

Proof. Consider the eigenvector of $A(C_n)$, given by:

$$\mathcal{E}_u = \sum_{v\in\mathbb{Z}_2^n}(-1)^{u\cdot v}v$$

By **Equation** 2.1, $|\mathcal{E}_u| = 2^{\frac{n}{2}}$, so one needs to normalize it to get the orthonormal basis:

$$\mathcal{E}_u' = \frac{1}{2^{\frac{n}{2}}}\mathcal{E}_u$$

Then by **Proposition** 2.2.2, we have:

$$A(C_n)_{uv}^l = \frac{1}{2^n}\sum_{w\in\mathbb{Z}_2^n}\mathcal{E}_{uw}'\mathcal{E}_{vw}'\lambda_w^l$$

One can then consider what is $\mathcal{E}_{uw}'$, written in canonical basis $v \in \mathbb{Z}_2^n$, it is exactly:

$$\mathcal{E}_{uw}' = (-1)^{u\cdot w}$$

Thus we have:

$$A(C_n)_{uv}^l = \frac{1}{2^n}\sum_{w\in\mathbb{Z}_2^n}(-1)^{(u+v)\cdot w}(n-2|w|)^l$$

One may consider the red part, with the number of vectors w of **Hamming Weight** i which have j 1's in common with $u + v$ is:

$$\binom{k}{j}\binom{n-k}{i-j}$$

it is simply the way of how we choose w, so that the dot product result $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{j}$. As w runs over all elements in $\mathbb{Z}_2^n$, where k is just $|u + v|$ to be the **upper bound** of j, see that:

$$A(C_n)_{uv}^l = \frac{1}{2^n}\sum_{i=1}^{n}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\binom{n-k}{i-j}(n-2i)^l \quad (i = |w|)$$

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Appendix