Math566: Combinatorial Theory

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Abstract

This is the note containning my personal thoughts as well as lecture notes, course content include some basic algebraic combinatorics. My course instructor is Prof. Shiyue Li.

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Chapter 1

Lemma

This chapter basically collect some linear algebra lemma that maybe helpful across the course.

Lemma 1.0.1. Given A to be a matrix, who has the eigenvalues to be $\lambda_1, \ldots, \lambda_p$, then the eigenvalues of the matrix $A + c\,\mathrm{Id}$ where $c \in \mathbb{C}$ is $\lambda_1 + c, \ldots, \lambda_p + c$.

Lemma 1.0.2. Let A be as above, the eigenvalues of A^l are $\lambda_1^l, \ldots, \lambda_p^l$.

Lemma 1.0.3. Let $f \in \mathbb{C}[t]$, then $f(A)$ has eigenvalues being $\{f(\lambda_i)\}_{i=1}^p$.

Lemma 1.0.4. If A is a real and symmetric matrix, by spectral theorem for Hermitian product, we have $\lambda_i \in \mathbb{R}$.

Lemma 1.0.5. Let $\{\alpha_i\}_{i=1}^r$ and $\{\beta_i\}_{i=1}^s$ be non-zero complex numbers, if for every $l \geq 1$, we have:

$$\sum_{i=1}^{r} \alpha_i^l = \sum_{i=1}^{s} \beta_i^l$$

then $r = s$ and $\{\alpha_i\}$ is permutation of $\{\beta_i\}$.

Chapter 2

Walks in Graph

2.1 Graph Eigenvalues

To better phrasing a graph, we first need to define multiset so that we can express not only simple graph but also general graph in a mathematically rigourous way.

Definition 2.1.1 (Multiset). A multiset M on a set S is an unordered collection of elements in S, s.t.

1. $\forall\, x \in M, x \in S$.
2. The $\#$ of times for $x \in S$ to appear in M, denoted as $\mu_M(x)$, is ≥ 0.

Example 2.1.1. If $\mu_M(x) = 0, 1\ \forall x$, then M is a set.

Note that two multiset M, M' are said to be equal if $\forall x \in S, \mu_M(x) = \mu_{M'}(x)$.

Notation. Let S be a finite set of size p, then define:

$$\binom{S}{k} = \{k - \text{subsets of } S\}$$

note that

$$\left|\binom{S}{k}\right| = \binom{|S|}{k}$$

also define:

$$\left(\!\binom{S}{k}\!\right) = \{k - \text{subsets of } S\}$$

note that:

$$\left|\left(\!\binom{S}{k}\!\right)\right| = \binom{p+k-1}{k}$$

this is the case: consider rephrasing the combinatorial problem as possible assigning of numbers for p numbers $a_1, \ldots, a_p$ with $a_1 + \ldots + a_p = k$ and $a_i \in [\![0, k]\!]$.

Thus we can define the graph properly.

Definition 2.1.2 (Graph). A finite graph is a triple $G = (V, E, \varphi)$ with:

- $V = \{v_1, \ldots, v_p\}$.
- $E = \{e_1, \ldots, e_q\}$.
- φ is a function $E \to \left(\!\binom{V}{2}\!\right)$.

A finite simple graph is the same data with $\varphi : E \to \binom{V}{2}$

Definition 2.1.3 (Adjacency Matrix of Graph). The adjacency matrix of a graph G, denoted as $A(G)$, whose entries is defined by:

$$a_{ij} = \left|\varphi^{-1}(\{v_i, v_j\})\right|$$

In particular it counts the number of edges between two vertices v_i and v_j. Note that it is well-defined since if there is no edges between v_i and v_j, then the preimage of φ will be $\emptyset$, thus $a_{ij} = 0$.

Definition 2.1.4 (Walk). A walk of length k in a graph G is a non-empty finite sequence of vertices and edges

$$W = v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$$

such that for all $1 \leq i \leq k$, the edge e_i has **endpoints** v_{i-1} and v_i.

In a simple graph, where the edges are determined by their endpoints, a walk can be simplified to a sequence of vertices:

$$W = (v_0, v_1, \ldots, v_k) \quad \text{where } \{v_{i-1}, v_i\} \in E(G)$$

Note.

- In a walk, the both the edges and vertices can appear **repeatedly**.
- If $v_0 = v_k$, then such walk is called a **closed walk**.

Proposition 2.1.1. For any integer $l \geq 1$, the (i, j) entry of $(A(G))^l$, denoted as a_{ij}, is equal to the # of walks of length l in G starting from v_i to v_j.

Theorem 2.1.1. Let G be graph with $A(G)$ possessing eigenvalues $\lambda_1, \ldots, \lambda_p$, the # **closed walks** of length l is:

$$f_G(l) = \sum_{i=1}^{p} \lambda_i^l$$

The proof is straightforward combining the **Proposition** 2.1.1 and **Lemma** 1.0.2.

Corollary 2.1.1. There are $(p-1)^l + (-1)^l(p-1)$ closed walks in K_p.

Notation.

- $\mathbb{J}$ is the all 1 matrix.
- $\mathbb{I}$ is the identity matrix.

Proof. For K_p, the adjacency matrix is given by $A(K_p) = \mathbb{J} - \mathbb{I}$, with the eigenvalue of $\mathbb{J}$ being $\{p, 0, \ldots, 0\}$ and the eigenvalues of $\mathbb{I}$ being $\{1, \ldots, 1\}$. By **Lemma** 1.0.3, here we have $f(x) = x - 1$, with $\mathbb{J}$ pluggined, thus yields the result. ∎

Corollary 2.1.2. There are $\frac{1}{p}((p-1)^l - (-1)^l)$ non-closed walks of length l in K_p.

Proof. Consider the matrix $A(K_p)^l = (\mathbb{J} - \mathbb{I})^l$, since $\mathbb{J}$ and $\mathbb{I}$ commutes, one can expand it using the binomial theorem:

$$(\mathbb{J} - \mathbb{I})^l = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} \mathbb{J}^i$$

And note that:

$$\mathbb{J}^i = \begin{cases} \mathbb{J}^0 = \mathbb{I} & i = 0 \\ p^{i-1}\mathbb{J} & i > 0 \end{cases}$$

Then see that:

$$\begin{aligned}
\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}\mathbb{J}^i &= \left(\sum_{i=0}^{p}(-1)^{l-i}\binom{l}{i}p^{i-1}\right)\mathbb{J} + (-1)^l\mathbb{I} \\
&= \left(\sum_{i=0}^{p}(-1)^{l-i}\binom{l}{i}p^{i-1} - (-1)^l\frac{1}{p}\right)\mathbb{J} + (-1)^l\mathbb{I} \\
&= \left(\frac{1}{p}(p-1)^l - (-1)^l\frac{1}{p}\right)\mathbb{J} + \underbrace{(-1)^l\mathbb{I}}_{\text{contribute nothing}}
\end{aligned}$$

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One can note that λ_i of $A(G)$ is completely determined by traces of $A(G)^l$ $\forall l \geq 1$ by **Lemma** 1.0.5. In particular if we know enough number of traces of $A(G)^l$ for multiple l, then we can calculate out the eigenvalues.

2.2 Radon Transform

We may define inner product space and orthogonal stuff first.

Definition 2.2.1 (Inner product space). An inner product space is a vector space over $\mathbb{C}$ together with an inner product $\langle -, - \rangle : V \times V \to \mathbb{C}$, s.t. $x, y, z \in V$, $a, b \in \mathbb{C}$:

1. Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
2. Linearity with the first entry: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
3. Positivity: if $x \neq 0$, then $\langle x, x \rangle > 0$.

In this section we shall assume all V to be inner product space.

Definition 2.2.2. $x, y \in V$ are orthogonal if $\langle x, y \rangle = 0$.

Lemma 2.2.1. If $\langle x, y \rangle = 0$, then x, y are linearly independent.

Definition 2.2.3 (Kronecker Delta). Let S be a set, the kronecker delta function on S^2 is given by:

$$\delta_{uv} = \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}$$

Let $\mathbb{Z}_2$ be the cyclic group of order 2, i.e. $(\{0, 1\}, +)$.
Let $\mathbb{Z}_2^n$ be the n-folde product of $\mathbb{Z}_2$, called an n-cube:

$$\mathbb{Z}_2^n = \{(a_1, \dots, a_n) \mid a_i \in \mathbb{Z}_2\}$$

such set has some properties if viewed as a vector space, for example, ZZ_2^n has a dot product defined by:

$$\begin{aligned}
\langle -, - \rangle : \mathbb{Z}_2^n \times \mathbb{Z}_2^n &\to \mathbb{Z}_2 \\
\langle y, z \rangle &\mapsto \underbrace{\sum y_i z_i}_{\text{group adding}} \in \mathbb{Z}_2
\end{aligned}$$

Lemma 2.2.2. $\forall u, v, w \in ZZ_2^n$, see that $u + v = w \Leftrightarrow u + w = v \Leftrightarrow v + w = u$.

The proof directly follows once realize that in this group $(-)$ is the same as $(+)$.
For $u \in \mathbb{Z}_2^n$, the weight $|u| = \sum u_i$ which count the number of entries that is non-zero.

Appendix