# Math566: Combinatorial Theory

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**Abstract**

This is the note containning my personal thoughts as well as lecture notes, course content include some basic algebraic combinatorics. My course instructor is Prof. Shiyue Li.

# Contents

# Chapter 1

# Linear Algebra

This chapter basically collect some linear algebra lemma that maybe helpful across the course.

**Lemma 1.0.1.** Given $A$ to be a matrix, who has the eigenvalues to be $\lambda_1, \dots, \lambda_p$, then the eigenvalues of the matrix $A + c\,\mathrm{Id}$ where $c \in \mathbb{C}$ is $\lambda_1 + c, \dots, \lambda_p + c$.

**Lemma 1.0.2.** Let $A$ be as above, the eigenvalues of $A^l$ are $\lambda_1^l, \dots, \lambda_p^l$.

**Lemma 1.0.3.** Let $f \in \mathbb{C}[t]$, then $f(A)$ has eigenvalues being $\{f(\lambda_i)\}_{i=1}^p$.

**Lemma 1.0.4.** If $A$ is a real and symmetric matrix, by spectral theorem for Hermitian product, we have $\lambda_i \in \mathbb{R}$.

**Lemma 1.0.5.** Let $\{\alpha_i\}_{i=1}^r$ and $\{\beta_i\}_{i=1}^s$ be non-zero complex numbers, if for every $l \geq 1$, we have:

$$\sum_{i=1}^r \alpha_i^l = \sum_{i=1}^s \beta_i^l$$

then $r = s$ and $\{\alpha_i\}$ is permutation of $\{\beta_i\}$.

# Chapter 2

# Walks in Graph

## 2.1 Graph Eigenvalues

To better phrasing a graph, we first need to define multiset so that we can express not only simple graph but also general graph in a mathematically rigourous way.

**Definition 2.1.1 (Multiset).** A multiset $M$ on a set $S$ is an unordered collection of elements in $S$, s.t.

1. $\forall\, x \in M, x \in S$.
2. The $\#$ of times for $x \in S$ to appear in $M$, denoted as $\mu_M(x)$, is $\geq 0$.

**Example 2.1.1.** If $\mu_M(x) = 0, 1\ \forall x$, then $M$ is a set.

Note that two multiset $M, M'$ are said to be equal if $\forall x \in S, \mu_M(x) = \mu_{M'}(x)$.

**Notation.** Let $S$ be a finite set of size $p$, then define:

$$\binom{S}{k} = \{k - \text{subsets of } S\}$$

note that

$$\left|\binom{S}{k}\right| = \binom{|S|}{k}$$

also define:

$$\left(\!\binom{S}{k}\!\right) = \{k - \text{subsets of } S\}$$

note that:

$$\left|\left(\!\binom{S}{k}\!\right)\right| = \binom{p+k-1}{k}$$

this is the case: consider rephrasing the combinatorial problem as possible assigning of numbers for $p$ numbers $a_1, \ldots, a_p$ with $a_1 + \ldots + a_p = k$ and $a_i \in [\![0, k]\!]$.

Thus we can define the graph properly.

**Definition 2.1.2 (Graph).** A finite graph is a triple $G = (V, E, \varphi)$ with:

- $V = \{v_1, \ldots, v_p\}$.
- $E = \{e_1, \ldots, e_q\}$.
- $\varphi$ is a function $E \to \left(\!\binom{V}{2}\!\right)$.

A finite simple graph is the same data with $\varphi : E \to \binom{V}{2}$

**Definition 2.1.3 (Adjacency Matrix of Graph).** The adjacency matrix of a graph $G$, denoted as $A(G)$, whose entries is defined by:

$$a_{ij} = \left|\varphi^{-1}(\{v_i, v_j\})\right|$$

In particular it counts the number of edges between two vertices $v_i$ and $v_j$. Note that it is well-defined since if there is no edges between $v_i$ and $v_j$, then the preimage of $\varphi$ will be $\emptyset$, thus $a_{ij} = 0$.

**Definition 2.1.4 (Walk).** A walk of length $k$ in a graph $G$ is a non-empty finite sequence of vertices and edges

$$W = v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k$$

such that for all $1 \leq i \leq k$, the edge $e_i$ has **endpoints** $v_{i-1}$ and $v_i$.

In a simple graph, where the edges are determined by their endpoints, a walk can be simplified to a sequence of vertices:

$$W = (v_0, v_1, \dots, v_k) \quad \text{where } \{v_{i-1}, v_i\} \in E(G)$$

**Note.**

- In a walk, the both the edges and vertices can appear **repeatedly**.
- If $v_0 = v_k$, then such walk is called a **closed walk**.

**Proposition 2.1.1.** For any integer $l \geq 1$, the $(i, j)$ entry of $(A(G))^l$, denoted as $a_{ij}$, is equal to the $\#$ of walks of length $l$ in $G$ starting from $v_i$ to $v_j$.

**Theorem 2.1.1.** Let $G$ be graph with $A(G)$ possessing eigenvalues $\lambda_1, \dots, \lambda_p$, the $\#$ **closed walks** of length $l$ is:

$$f_G(l) = \sum_{i=1}^{p} \lambda_i^l$$

The proof is straightforward combining the **Proposition** 2.1.1 and **Lemma** 1.0.2.

**Definition 2.1.5 (Unitary Matrix).** A complex square matrix $U$ is unitary if $U^* = U^{-1}$.

**Definition 2.1.6 (Orthogonal Matrices).** An orthogonal matrix is a real square matrix $O$ whose rows and cols are **orthonormal**, in particular equivalent to:

$$O^\intercal = O^{-1}$$

**Theorem 2.1.2 (Spectral Theorem for Hermitian Matrices).** A complex (real) square matrix $A$ is Hermitian (real symmetric). Then there exists a unitary (orthogonal) $U$ and a real diagonal matrix $\Lambda$, s.t.

$$A = U\Lambda U^{-1}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

In particular, $U$ can be chosen to have the orthonormal eigenvectors of $A$ as its column vectors.

**Corollary 2.1.1.** There are $(p-1)^l + (-1)^l(p-1)$ closed walks in $K_p$.

**Notation.**

- $\mathbb{J}$ is the all 1 matrix.
- $\mathbb{I}$ is the identity matrix.

**Proof.** For $K_p$, the adjacency matrix is given by $A(K_p) = \mathbb{J} - \mathbb{I}$, with the eigenvalue of $\mathbb{J}$ being $\{p, 0, \dots, 0\}$ and the eigenvalues of $\mathbb{I}$ being $\{1, \dots, 1\}$. By **Lemma** 1.0.3, here we have $f(x) = x - 1$, with $\mathbb{J}$ plugined, thus yields the result. ■

**Corollary 2.1.2.** There are $\frac{1}{p}((p-1)^l - (-1)^l)$ non-closed walks of length $l$ in $K_p$.

**Proof.** Consider the matrix $A(K_p)^l = (\mathbb{J} - \mathbb{I})^l$, since $\mathbb{J}$ and $\mathbb{I}$ commutes, one can expand it using the binomial theorem:

$$(\mathbb{J} - \mathbb{I})^l = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} \mathbb{J}^i$$

And note that:

$$\mathbb{J}^i = \begin{cases} \mathbb{J}^0 = \mathbb{I} & i = 0 \\ p^{i-1}\mathbb{J} & i > 0 \end{cases}$$

Then see that:

$$\begin{aligned} \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} \mathbb{J}^i &= \left( \sum_{i=0}^{p} (-1)^{l-i} \binom{l}{i} p^{i-1} \right) \mathbb{J} + (-1)^l \mathbb{I} \\ &= \left( \sum_{i=0}^{p} (-1)^{l-i} \binom{l}{i} p^{i-1} - (-1)^l \frac{1}{p} \right) \mathbb{J} + (-1)^l \mathbb{I} \\ &= \left( \frac{1}{p}(p-1)^l - (-1)^l \frac{1}{p} \right) \mathbb{J} + \underbrace{(-1)^l \mathbb{I}}_{\text{contribute nothing}} \end{aligned}$$

■

One can note that $\lambda_i$ of $A(G)$ is completely determined by traces of $A(G)^l \quad \forall l \geq 1$ by **Lemma** 1.0.5. In particular if we know enough number of traces of $A(G)^l$ for multiple $l$, then we can calculate out the eigenvalues.

## 2.2 Radon Transform

We may define inner product space and orthogonal stuff first.

**Definition 2.2.1 (Inner product space).** An inner product space is a vector space over $\mathbb{C}$ together with an inner product $\langle -, - \rangle : V \times V \to \mathbb{C}$, s.t. $x, y, z \in V, \quad a, b \in \mathbb{C}$:

1. Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
2. Linearity with the first entry: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
3. Positivity: if $x \neq 0$, then $\langle x, x \rangle > 0$.

In this section we shall assume all $V$ to be inner product space.

**Definition 2.2.2.** $x, y \in V$ are orthogonal if $\langle x, y \rangle = 0$.

**Lemma 2.2.1.** If $\langle x, y \rangle = 0$, then $x, y$ are linearly independent.

**Definition 2.2.3 (Kronecker Delta).** Let $S$ be a set, the kronecker delta function on $S^2$ is given by:

$$\delta_{uv} = \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}$$

Let $\mathbb{Z}_2$ be the cyclic group of order 2, i.e. $(\{0,1\},+)$.
Let $\mathbb{Z}_2^n$ be the $n$-folde product of $\mathbb{Z}_2$, called an $n$-cube:

$$\mathbb{Z}_2^n = \{(a_1,\ldots,a_n) \mid a_i \in \mathbb{Z}_2\}$$

such set has some properties if viewed as a vector space, for example, $ZZ_2^n$ has a dot product defined by:

$$\begin{aligned} \langle -,- \rangle : \mathbb{Z}_2^n \times \mathbb{Z}_2^n &\to \mathbb{Z}_2 \\ \langle y,z \rangle &\mapsto \underbrace{\sum y_i z_i}_{\text{group adding}} \in \mathbb{Z}_2 \end{aligned}$$

**Lemma 2.2.2.** $\forall u,v,w \in ZZ_2^n$, see that $u+v=w \Leftrightarrow u+w=v \Leftrightarrow v+w=u$.

The proof directly follows once realize that in this group $(-)$ is the same as $(+)$.
For $u \in \mathbb{Z}_2^n$, the weight $|u| = \sum u_i$ which count the number of entries that is non-zero.

### 2.2.1 Counting in Cube

In this section we shall use Radon Transform to help counting the closed walks in a cube. For other special graphs, one can similarly embed the structure of it into a specific group and construct the corresponding Radon Transform to count. It may rely on specific kinds of symmetry but really provide and powerful and convenient tool for counting.

**Proposition 2.2.1.** Let $V$ be all functions $f : \mathbb{Z}_2^n \to \mathbb{C}$. This is a vector space over $\mathbb{C}$, we have the following fact:

1. $\dim_{\mathbb{C}} V = 2^n$, with the basis given by:

$$i_u(v) = \begin{cases} 1, & \text{if } u = v \\ 0, & \text{if } u \neq v \end{cases}$$

2. $V$ has a inner product space structure over $\mathbb{C}$:

$$\langle f,g \rangle = \sum_{u\in\mathbb{Z}_2^n} f(y)\overline{g(u)}$$

3. $V$ has basis:

$$\begin{aligned} \mathcal{B}_1 &= \{f_u :\ u \in \mathbb{Z}_2^n,\ f_u(v) = \delta_{uv}\} \\ \mathcal{B}_2 &= \{\chi_u :\ \chi_u(v) = (-1)^{uv}\} \end{aligned}$$

   See that:

$$g(v) = \sum_{u\in\mathbb{Z}_2^n} g(v)\delta_{uv} = \sum_{u\in\mathbb{Z}_2^n} g(v) f_u(v)$$

and

$$\begin{aligned}\langle \chi_u, \chi_v \rangle &= \sum_{w \in \mathbb{Z}_2^n} \chi_u(w)\overline{\chi_v(w)} \\ &= \sum_{w \in \mathbb{Z}_2^n} (-1)^{(u+v)\cdot w} \\ &= \begin{cases} 2^n, & \text{if } u = v \Leftrightarrow 2u = 0 \\ 0, & \text{if } u \neq v \end{cases}\end{aligned} \tag{2.1}$$

which means that $\chi_u$ and $\chi_v$ are orthogonal and thus linearly independent. The part that leads to 0 can be deduce by pairing vector $w$ on the entry that $(u+v)$ is 1.

**Definition 2.2.4 (Radon Transform).** For a subset $\Gamma \in \mathbb{Z}_2^n$ and a function $f \in V : \mathbb{Z}_2^n \to \mathbb{C}$, the discrete radon transform on $\Gamma$ of $\mathbb{Z}_2^n$ is:

$$\Phi_\Gamma(f)(v) := \sum_{w \in \Gamma} f(v+w)$$

this is a linear transformation $V \to V$.

**Theorem 2.2.1.** The eigenvectors of $\Phi_\Gamma$ are $\{\chi_u\}_{u \in \mathbb{Z}_2^n}$, with eigenvalues being $\lambda_u = \sum_{w \in \Gamma} (-1)^{u \cdot v}$.

**Proof.** See that:

$$\begin{aligned}\Phi_\Gamma(\chi_u)(v) &= \sum_{w \in \Gamma} \chi_u(v+w) \\ &= \sum_{w \in \Gamma} (-1)^{u(v+w)} \\ &= (-1)^{uv} \sum_{w \in \Gamma} (-1)^{uw} \\ &= \chi_u(v) \cdot \lambda_u\end{aligned}$$

■

**Definition 2.2.5.** The $n$-cube graph $C_n$ is the graph with:

- $V(G) = \mathbb{Z}_2^n$.
- $E(G) = \{(u, v) \mid u, v \text{ differ in only 1 coordinate}\}$, for example the 3-dimensional cube.

The adjacency matrix of this kinds of graph is rather complicated and hard to comprehend and compute its eigenvalue by simply looking at the matrix.

We now propose a simpler way to obtain the eigenvalue and eigenvector of adjacency matrix of $n$-cube graph, basically choosing special subset to do radon transform and see that the matrix is exactly the same.

Let $\Delta$ be the set:

$$\Delta := \{\delta_i \mid \delta_i \text{ is the i-th unit vector in } \mathbb{Z}_2^n\} \subseteq \mathbb{Z}_2^n$$

Let $\Phi_\Delta : V \to V$, and $[\Phi_\Delta]$ be the matrix of $\Phi_\Delta$ w.r.t. $\mathcal{B}_1 = \{f_u : \ u \in \mathbb{Z}_2^n\}$.

**Lemma 2.2.3.** The matrix $[\Phi_\Delta] = A(C_n)$.

**Proof.** Basically we want to see that all entries on the left hand side is the same as the right hand side, so we consider the $(u, v)$-entry of $[\Phi_\Delta]$, let $z \in \mathbb{Z}_2^n$, and compute: $\Phi_\Delta f_u(v)$,

$$\begin{aligned}
\Phi_\Delta f_u(z) &= \sum_{w\in\Delta} f_u(z+w) \\
&= \sum_{w\in\Delta} f_{u+w}(z) \\
\Rightarrow \Phi_\Delta f_u &= \sum_{u\in\Delta} f_{u+w} \\
\Rightarrow [\Phi_\Delta]_{u,v} &= \begin{cases} 1, & \text{if } u+w=v \\ 0, & \text{if } u+w\neq v \end{cases} \\
\Leftrightarrow [\Phi_\Delta]_{u,v} &= \begin{cases} 1, & \text{if } u+v=w\in\Delta \\ 0, & \text{otherwise} \end{cases} \\
\Leftrightarrow [\Phi_\Delta]_{u,v} &= \begin{cases} 1, & \text{if } u,v\in E(C_n) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

■

**Corollary 2.2.1.** The eigenvectors of $A(C_n)$ are:

$$\mathcal{E}_u = \sum_{v\in\mathbb{Z}_2^n} (-1)^{u\cdot v} v$$

With eigenvalues being:

$$\lambda_u = n - 2|u|$$

In particular, $A(C_n)$ has $\binom{n}{k}$ eigenvalues equal to $n-2k$.

**Proof.** For $g \in V$, see that $g = \sum_{v\in\mathbb{Z}_2^n} g(v)\cdot f_v$, which implies that:

$$g(u) = \sum_{v\in\mathbb{Z}_2^n} g(v) f_v(u) = \sum_{v\in\mathbb{Z}_2^n} g(v)\delta_{uv}$$

Recall that $[\Phi_\Delta]_{\mathcal{B}_1}$ has eigenvectors $\{\chi_u(v) = (-1)^{u\cdot v}\}$, then:

$$\chi_u = \sum_{v\in\mathbb{Z}_2^n} \chi_u(v) f_v = \sum_{v\in\mathbb{Z}_2^n} (-1)^{u\cdot v} f_v$$

with eigenvalues being:

$$\lambda_u = \sum_{w\in\Delta} (-1)^{u\cdot w} = \sum_{i\in[n]} (-1)^{u\cdot\delta_i} = n - 2|u|$$

Notice that those $\mathcal{E}_u$ with the **Hamming Weight** of $u$ being the same share the same eigenvalue, total number of such $u$ will be $\binom{n}{k}$ if $|u| = k$. ■

**Proposition 2.2.2.** If $G$ is a graph with $p$ vertices and eigenvalues $\lambda_1, \ldots, \lambda_p$, then there exists $c_1, \ldots, c_p \in \mathbb{R}$, s.t.

$$A(G)^l_{ij} = \sum_{k=1}^{p} c_k \lambda_k^l$$

And in particular, if $U$ is an **orthogonal** matrix ($A(G) = U\Lambda U^{-1}$), then:

$$A(G)^l_{ij} = \sum_{k=1}^{n} U_{ik} U_{jk} \lambda_k^l$$

**Proof.** By **Spectral Theorem** 2.1.2, there exists an orthogonal matrix $U$, s.t.

$$A = U\Lambda U^{-1}$$

Let $l \geq 1$, then:

$$A^l = U\Lambda^l U^{-1}$$

take the $(i,j)$-entry of right hand side and see that it equal to $\sum_{k=1}^n U_{ik}U_{jk}\lambda_k^l$. ∎

**Corollary 2.2.2.** Let $u, v \in \mathbb{Z}_2^n$, supppose that $|u+v| = k$, i.e. $u$ and $v$ **differ** in $k$ places, then:

$$A(C_n)^l_{uv} = \frac{1}{2^n}\sum_{i=1}^n\sum_{j=1}^k(-1)^k\binom{k}{j}\binom{n-k}{i-j}(n-2i)^l \qquad (i \geq j)$$

**Proof.** Consider the eigenvector of $A(C_n)$, given by:

$$\mathcal{E}_u = \sum_{v\in\mathbb{Z}_2^n}(-1)^{u\cdot v}v$$

By **Equation** 2.1, $|\mathcal{E}_u| = 2^{\frac{n}{2}}$, so one needs to normalize it to get the orthonormal basis:

$$\mathcal{E}'_u = \frac{1}{2^{\frac{n}{2}}}\mathcal{E}_u$$

Then by **Proposition** 2.2.2, we have:

$$A(C_n)^l_{uv} = \frac{1}{2^n}\sum_{w\in\mathbb{Z}_2^n}\mathcal{E}'_{uw}\mathcal{E}'_{vw}\lambda_w^l$$

One can then consider what is $\mathcal{E}'_{uw}$, written in canonical basis $v \in \mathbb{Z}_2^n$, it is exactly:

$$\mathcal{E}'_{uw} = (-1)^{u\cdot w}$$

Thus we have:

$$A(C_n)^l_{uv} = \frac{1}{2^n}\sum_{w\in\mathbb{Z}_2^n}(-1)^{(u+v)\cdot w}(n-2|w|)^l$$

One may consider the red part, with the number of vectors $w$ of **Hamming Weight** $i$ which have $j$ 1's in common with $u+v$ is:

$$\binom{k}{j}\binom{n-k}{i-j}$$

it is simply the way of how we choose $w$, so that the dot product result $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{j}$. As $w$ runs over all elements in $\mathbb{Z}_2^n$, where $k$ is just $|u+v|$ to be the **upper bound** of $j$, see that:

$$A(C_n)^l_{uv} = \frac{1}{2^n}\sum_{i=1}^n\sum_{j=1}^k(-1)^j\binom{k}{j}\binom{n-k}{i-j}(n-2i)^l \quad (i = |w|)$$

∎

## 2.3 Matrix-Tree Theorem

**Definition 2.3.1 (Spanning Subgraph).** A spanning subgraph of a graph $G$ is a graph $H$ with the same vertex set as $G$ and $E(H) \subseteq E(G)$.

(If $G$ has $q$ edges, there are $2^q$ spanning subgraph, in particular, choose whether each edges are included in.)

**Definition 2.3.2 (Spanning Tree).** A spanning tree is a spanning subgraph that is a **tree**. The # of spanning tress in a graph $G$ is called complexity of $G$, denoted as $\kappa(G)$.

**Definition 2.3.3 (Laplacian of a Graph).** The laplacian of a graph $G$ is given by:

$$\mathbb{L}(G) = \begin{cases} -\left|\varphi^{-1}(v_i, v_j)\right|, & \text{if } v_i \neq v_j \\ \deg(v_i), & \text{otherwise} \end{cases}$$

In short, the diagonal of such matrix will be the degree of the vertex, and the rest represent the number of edges connect between the vertices mult with $-1$.

**Note.** The laplacian actually can also be given by:

$$\mathbb{L}(G) = \operatorname{diag}\{\deg(v_1), \dots, \deg(v_n)\} - A(G)$$

**Definition 2.3.4 ((r,c)-Cofactor).** Given a matrix $A$, the (r,c)-cofactor of $A$ is the:

$$A_{r,c} := (-1)^{r+c} \det A_{\hat{r},\hat{c}}$$

where $\hat{r}$ represent deleting the $r$-th row, and $\hat{c}$ represent deleting the $c$-th column.

**Definition 2.3.5 (Orientation).** An orientation of a graph $G$ is an assignment of an order on every edge: which is, for every edge $e \in E(G)$, for example, it connects $u$ and $v$, then choose one of the ordered pair $(u, v)$ or $(v, u)$ as direction. Such orientation is usually denoted as $\mathfrak{o}$.

**Definition 2.3.6 (Incidence Matrix).** The incident matrix of a directed graph $G$ with respect to the orientation $\mathfrak{o}$ is the $|V| \times |E|$ matrix whose $(i, j)$-entry is given by:

$$\mathbb{M}(G) = \begin{cases} -1, & \text{if the edge } e_j \text{ has initial vertex } v_i \\ 1, & \text{if the edge } e_j \text{ has final vertex } v_i \\ 0, & \text{otherwise} \end{cases}$$

**Lemma 2.3.1.**

$$\mathbb{L}(G) = \mathbb{M}(G)\mathbb{M}(G)^\intercal$$

**Proof.** The proof is quite straightforward once consider the situation of matrix manipulation, first denote $\mathbb{M}(G) = (m_{ij})$:

$$(\mathbb{M}(G)\mathbb{M}(G)^\intercal)_{ij} = \sum_{e_k \in E(G)} m_{ik} m_{jk}$$

Consider the situation of $i = j$ and $i \neq j$:

- When $i \neq j$: One should only consider the case when $e_k$ connects $i$ and $j$, otherwise one of them will be 0 and thus the product of them will be 0. So consider the case where $i$ and $j$ are connected by $e_k$. In this case one vertex must be the source and one vertex must be the sink, thus one of them will be 1 and the other will be $-1$. Summing through all edges in the graph gives us the result to be $-\left|\varphi^{-1}(v_i, v_j)\right|$.

- When $i = j$: In this case one will still only consider the case where $e_k$ touches $v_i = v_j$, otherwise one of $m_{ik}$ or $m_{jk}$ will be 0. Whether it is the source or sink it doesn't matter, the result should still yields to be 1. Summing through all edges in the graph gives us $\deg(v_i = v_j)$.

It follows that this is exactly the **Definition** 2.3.3 of Laplacian. ■

**Lemma 2.3.2.** Let $|V| = p$. Let $S$ be a subset of $p-1$ edges. Let $v$ be any vertex of $G$. Then:

$$\det\left(\underbrace{\mathbb{M}_{\hat{v}}[S]}_{(p-1)\times(p-1)}\right) = \begin{cases} \pm 1, & \text{if } S \text{ forms a spanning tree of } G \\ 0, & \text{otherwise} \end{cases}$$

**Proof.** One can prove the theorem by deviding the case of whether $S$ forms a spanning tree of $G$:

- If $S$ doesn't forms a spanning tree. Then there must exsist a cycle $C$ in the graph, denoted the edge of the cycle as $f_1, \dots, f_s$. The point is we want to make it as a oriented cycle, so that the column vector representing $f_1, \dots, f_s$ are linearly dependent. For convenience one can interchange the column vector of $f_1, \dots, f_s$ into in order and adacent ones on the matrix, the determinant still remains the same. One can choose an directed orientation on such edges set, for example in the sense that $1 \to 2 \to \cdots \to s$, and for those edges with the same orientation with the chosen one, we multiply the column vectors with 1 and for those who are not, we multiply with $-1$ for the column vectors. One can see that by definition, summing over all such column vectors with the result leads to 0, thus $f_1, \dots, f_s$ are linearly dependent, thus the determinant will be 0.

- If $S$ did forms a spanning tree. By degree handshake theorem, one can see that the tree always have a leave vertex, namely a vertex that only connects with a unique edge. We now choose a leave node $v$ in the spanning tree and the corresponding edge $e$, by definition of incidence matrix, for the row represented by $v$ attains only $(v, e)$-entry which is non-zero, and it will be $\pm 1$. By property of determinant, abuse the notation of $S$ for the whole spanning tree, see that $\det(S) = \det(S \setminus \{v\})$. This is true as we also deleted the corresponding edges as we deleted the vertex. And we induct on the leavenode to reduct on the whole graph, yields the result to be $\pm 1$.

■

**Theorem 2.3.1 (Matrix-Tree Theorem).** Let $G$ be a finite connected graph without loops. The # of spanning trees of $G$ is:

$$\kappa(G) = L_{r,c}(G) \qquad \forall\, r, c$$

where $L_{r,c}$ is the $(r, c)$-cofactor of $\mathbb{L}(G)$.

# Appendix