# Math566: Combinatorial Theory

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January 27, 2026

**Abstract**

This is the note containning my personal thoughts as well as lecture notes, course content include some basic algebraic combinatorics. My course instructor is Prof. Shiyue Li.

# Contents

# Chapter 1

# Linear Algebra

This chapter basically collect some linear algebra lemma that maybe helpful across the course.

**Lemma 1.0.1.** Given $A$ to be a matrix, who has the eigenvalues to be $\lambda_1, \ldots, \lambda_p$, then the eigenvalues of the matrix $A + c\,\mathrm{Id}$ where $c \in \mathbb{C}$ is $\lambda_1 + c, \ldots, \lambda_p + c$.

**Lemma 1.0.2.** Let $A$ be as above, the eigenvalues of $A^l$ are $\lambda_1^l, \ldots, \lambda_p^l$.

**Lemma 1.0.3.** Let $f \in \mathbb{C}[t]$, then $f(A)$ has eigenvalues being $\{f(\lambda_i)\}_{i=1}^p$.

**Lemma 1.0.4.** If $A$ is a real and symmetric matrix, by spectral theorem for Hermitian product, we have $\lambda_i \in \mathbb{R}$.

**Lemma 1.0.5.** Let $\{\alpha_i\}_{i=1}^r$ and $\{\beta_i\}_{i=1}^s$ be non-zero complex numbers, if for every $l \geq 1$, we have:

$$\sum_{i=1}^r \alpha_i^l = \sum_{i=1}^s \beta_i^l$$

then $r = s$ and $\{\alpha_i\}$ is permutation of $\{\beta_i\}$.

# Chapter 2

# Walks in Graph

## 2.1 Graph Eigenvalues

To better phrasing a graph, we first need to define multiset so that we can express not only simple graph but also general graph in a mathematically rigourous way.

**Definition 2.1.1 (Multiset).** A multiset $M$ on a set $S$ is an unordered collection of elements in $S$, s.t.

1. $\forall\, x \in M, x \in S$.
2. The $\#$ of times for $x \in S$ to appear in $M$, denoted as $\mu_M(x)$, is $\geq 0$.

**Example 2.1.1.** If $\mu_M(x) = 0, 1\ \forall x$, then $M$ is a set.

Note that two multiset $M, M'$ are said to be equal if $\forall x \in S, \mu_M(x) = \mu_{M'}(x)$.

**Notation.** Let $S$ be a finite set of size $p$, then define:

$$\binom{S}{k} = \{k-\text{subsets of } S\}$$

note that

$$\left|\binom{S}{k}\right| = \binom{|S|}{k}$$

also define:

$$\left(\!\!\binom{S}{k}\!\!\right) = \{k-\text{subsets of } S\}$$

note that:

$$\left|\left(\!\!\binom{S}{k}\!\!\right)\right| = \binom{p+k-1}{k}$$

this is the case: consider rephrasing the combinatorial problem as possible assigning of numbers for $p$ numbers $a_1, \ldots, a_p$ with $a_1 + \ldots + a_p = k$ and $a_i \in [\![0, k]\!]$.

Thus we can define the graph properly.

**Definition 2.1.2 (Graph).** A finite graph is a triple $G = (V, E, \varphi)$ with:

- $V = \{v_1, \ldots, v_p\}$.
- $E = \{e_1, \ldots, e_q\}$.
- $\varphi$ is a function $E \to \left(\!\binom{V}{2}\!\right)$.

A finite simple graph is the same data with $\varphi : E \to \binom{V}{2}$

**Definition 2.1.3 (Adjacency Matrix of Graph).** The adjacency matrix of a graph $G$, denoted as $A(G)$, whose entries is defined by:

$$a_{ij} = \left|\varphi^{-1}(\{v_i, v_j\})\right|$$

In particular it counts the number of edges between two vertices $v_i$ and $v_j$. Note that it is well-defined since if there is no edges between $v_i$ and $v_j$, then the preimage of $\varphi$ will be $\emptyset$, thus $a_{ij} = 0$.

**Definition 2.1.4 (Walk).** A walk of length $k$ in a graph $G$ is a non-empty finite sequence of vertices and edges

$$W = v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k$$

such that for all $1 \leq i \leq k$, the edge $e_i$ has **endpoints** $v_{i-1}$ and $v_i$.

In a simple graph, where the edges are determined by their endpoints, a walk can be simplified to a sequence of vertices:

$$W = (v_0, v_1, \dots, v_k) \quad \text{where } \{v_{i-1}, v_i\} \in E(G)$$

**Note.**

- In a walk, the both the edges and vertices can appear **repeatedly**.
- If $v_0 = v_k$, then such walk is called a **closed walk**.

**Proposition 2.1.1.** For any integer $l \geq 1$, the $(i, j)$ entry of $(A(G))^l$, denoted as $a_{ij}$, is equal to the # of walks of length $l$ in $G$ starting from $v_i$ to $v_j$.

**Theorem 2.1.1.** Let $G$ be graph with $A(G)$ possessing eigenvalues $\lambda_1, \dots, \lambda_p$, the # **closed walks** of length $l$ is:

$$f_G(l) = \sum_{i=1}^{p} \lambda_i^l$$

The proof is straightforward combining the **Proposition** 2.1.1 and **Lemma** 1.0.2.

**Definition 2.1.5 (Unitary Matrix).** A complex square matrix $U$ is unitary if $U^* = U^{-1}$.

**Definition 2.1.6 (Orthogonal Matrices).** An orthogonal matrix is a real square matrix $O$ whose rows and cols are **orthonormal**, in particular equivalent to:

$$O^\intercal = O^{-1}$$

**Theorem 2.1.2 (Spectral Theorem for Hermitian Matrices).** A complex (real) square matrix $A$ is Hermitian (real symmetric). Then there exists a unitary (orthogonal) $U$ and a real diagonal matrix $\Lambda$, s.t.

$$A = U\Lambda U^{-1}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

In particular, $U$ can be chosen to have the orthonormal eigenvectors of $A$ as its column vectors.

**Corollary 2.1.1.** There are $(p-1)^l + (-1)^l(p-1)$ closed walks in $K_p$.

**Notation.**

- $\mathbb{J}$ is the all 1 matrix.
- $\mathbb{I}$ is the identity matrix.

**Proof.** For $K_p$, the adjacency matrix is given by $A(K_p) = \mathbb{J} - \mathbb{I}$, with the eigenvalue of $\mathbb{J}$ being $\{p, 0, \ldots, 0\}$ and the eigenvalues of $\mathbb{I}$ being $\{1, \ldots, 1\}$. By **Lemma** 1.0.3, here we have $f(x) = x - 1$, with $\mathbb{J}$ plugined, thus yields the result. ∎

**Corollary 2.1.2.** There are $\frac{1}{p}((p-1)^l - (-1)^l)$ non-closed walks of length $l$ in $K_p$.

**Proof.** Consider the matrix $A(K_p)^l = (\mathbb{J} - \mathbb{I})^l$, since $\mathbb{J}$ and $\mathbb{I}$ commutes, one can expand it using the binomial theorem:

$$(\mathbb{J} - \mathbb{I})^l = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} \mathbb{J}^i$$

And note that:

$$\mathbb{J}^i = \begin{cases} \mathbb{J}^0 = \mathbb{I} & i = 0 \\ p^{i-1}\mathbb{J} & i > 0 \end{cases}$$

Then see that:

$$\begin{aligned} \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} \mathbb{J}^i &= \left( \sum_{i=0}^{p} (-1)^{l-i} \binom{l}{i} p^{i-1} \right) \mathbb{J} + (-1)^l \mathbb{I} \\ &= \left( \sum_{i=0}^{p} (-1)^{l-i} \binom{l}{i} p^{i-1} - (-1)^l \frac{1}{p} \right) \mathbb{J} + (-1)^l \mathbb{I} \\ &= \left( \frac{1}{p}(p-1)^l - (-1)^l \frac{1}{p} \right) \mathbb{J} + \underbrace{(-1)^l \mathbb{I}}_{\text{contribute nothing}} \end{aligned}$$

∎

One can note that $\lambda_i$ of $A(G)$ is completely determined by traces of $A(G)^l \quad \forall l \geq 1$ by **Lemma** 1.0.5. In particular if we know enough number of traces of $A(G)^l$ for multiple $l$, then we can calculate out the eigenvalues.

## 2.2 Radon Transform

We may define inner product space and orthogonal stuff first.

**Definition 2.2.1 (Inner product space).** An inner product space is a vector space over $\mathbb{C}$ together with an inner product $\langle -, - \rangle : V \times V \to \mathbb{C}$, s.t. $x, y, z \in V, \quad a, b \in \mathbb{C}$:

1. Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
2. Linearity with the first entry: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
3. Positivity: if $x \neq 0$, then $\langle x, x \rangle > 0$.

In this section we shall assume all $V$ to be inner product space.

**Definition 2.2.2.** $x, y \in V$ are orthogonal if $\langle x, y \rangle = 0$.

**Lemma 2.2.1.** If $\langle x, y \rangle = 0$, then $x, y$ are linearly independent.

**Definition 2.2.3 (Kronecker Delta).** Let $S$ be a set, the kronecker delta function on $S^2$ is given by:

$$\delta_{uv} = \begin{cases} 1, & u = v \\ 0, & u \neq v \end{cases}$$

Let $\mathbb{Z}_2$ be the cyclic group of order 2, i.e. $(\{0,1\},+)$.
Let $\mathbb{Z}_2^n$ be the $n$-folde product of $\mathbb{Z}_2$, called an $n$-cube:

$$\mathbb{Z}_2^n = \{(a_1,\dots,a_n) \mid a_i \in \mathbb{Z}_2\}$$

such set has some properties if viewed as a vector space, for example, $ZZ_2^n$ has a dot product defined by:

$$\begin{aligned} \langle -,- \rangle : \mathbb{Z}_2^n \times \mathbb{Z}_2^n &\to \mathbb{Z}_2 \\ \langle y, z\rangle &\mapsto \underbrace{\sum y_i z_i}_{\text{group adding}} \in \mathbb{Z}_2 \end{aligned}$$

**Lemma 2.2.2.** $\forall u, v, w \in ZZ_2^n$, see that $u + v = w \Leftrightarrow u + w = v \Leftrightarrow v + w = u$.

The proof directly follows once realize that in this group $(-)$ is the same as $(+)$.
For $u \in \mathbb{Z}_2^n$, the weight $|u| = \sum u_i$ which count the number of entries that is non-zero.

### 2.2.1 Counting in Cube

In this section we shall use Radon Transform to help counting the closed walks in a cube. For other special graphs, one can similarly embed the structure of it into a specific group and construct the corresponding Radon Transform to count. It may rely on specific kinds of symmetry but really provide and powerful and convenient tool for counting.

**Proposition 2.2.1.** Let $V$ be all functions $f : \mathbb{Z}_2^n \to \mathbb{C}$. This is a vector space over $\mathbb{C}$, we have the following fact:

1. $\dim_{\mathbb{C}} V = 2^n$, with the basis given by:

$$i_u(v) = \begin{cases} 1, & \text{if } u = v \\ 0, & \text{if } u \neq v \end{cases}$$

2. $V$ has a inner product space structure over $\mathbb{C}$:

$$\langle f, g\rangle = \sum_{u \in \mathbb{Z}_2^n} f(y)\overline{g(u)}$$

3. $V$ has basis:

$$\begin{aligned} \mathcal{B}_1 &= \{f_u : \ u \in \mathbb{Z}_2^n, \ f_u(v) = \delta_{uv}\} \\ \mathcal{B}_2 &= \{\chi_u : \ \chi_u(v) = (-1)^{uv}\} \end{aligned}$$

See that:

$$g(v) = \sum_{u \in \mathbb{Z}_2^n} g(v)\delta_{uv} = \sum_{u \in \mathbb{Z}_2^n} g(v) f_u(v)$$

and

$$\begin{aligned}\langle \chi_u, \chi_v \rangle &= \sum_{w\in\mathbb{Z}_2^n} \chi_u(w)\overline{\chi_v(w)} \\ &= \sum_{w\in\mathbb{Z}_2^n} (-1)^{(u+v)\cdot w} \\ &= \begin{cases} 2^n, & \text{if } u = v \Leftrightarrow 2u = 0 \\ 0, & \text{if } u \neq v \end{cases}\end{aligned} \tag{2.1}$$

which means that $\chi_u$ and $\chi_v$ are orthogonal and thus linearly independent. The part that leads to 0 can be deduce by pairing vector $w$ on the entry that $(u + v)$ is 1.

**Definition 2.2.4 (Radon Transform).** For a subset $\Gamma \in \mathbb{Z}_2^n$ and a function $f \in V : \mathbb{Z}_2^n \to \mathbb{C}$, the discrete radon transform on $\Gamma$ of $\mathbb{Z}_2^n$ is:

$$\Phi_\Gamma(f)(v) := \sum_{w\in\Gamma} f(v + w)$$

this is a linear transformation $V \to V$.

**Theorem 2.2.1.** The eigenvectors of $\Phi_\Gamma$ are $\{\chi_u\}_{u\in\mathbb{Z}_2^n}$, with eigenvalues being $\lambda_u = \sum_{w\in\Gamma}(-1)^{u\cdot v}$.

**Proof.** See that:

$$\begin{aligned}\Phi_\Gamma(\chi_u)(v) &= \sum_{w\in\Gamma} \chi_u(v + w) \\ &= \sum_{w\in\Gamma}(-1)^{u(v+w)} \\ &= (-1)^{uv}\sum_{w\in\Gamma}(-1)^{uw} \\ &= \chi_u(v) \cdot \lambda_u\end{aligned}$$

■

**Definition 2.2.5.** The $n$-cube graph $C_n$ is the graph with:

- $V(G) = \mathbb{Z}_2^n$.
- $E(G) = \{(u, v) \mid u, v \text{ differ in only 1 coordinate}\}$, for example the 3-dimensional cube.

The adjacency matrix of this kinds of graph is rather complicated and hard to comprehend and compute its eigenvalue by simply looking at the matrix.

We now propose a simpler way to obtain the eigenvalue and eigenvector of adjacency matrix of $n$-cube graph, basically choosing special subset to do radon transform and see that the matrix is exactly the same.

Let $\Delta$ be the set:

$$\Delta := \{\delta_i \mid \delta_i \text{ is the i-th unit vector in } \mathbb{Z}_2^n\} \subseteq \mathbb{Z}_2^n$$

Let $\Phi_\Delta : V \to V$, and $[\Phi_\Delta]$ be the matrix of $\Phi_\Delta$ w.r.t. $\mathcal{B}_1 = \{f_u : u \in \mathbb{Z}_2^n\}$.

**Lemma 2.2.3.** The matrix $[\Phi_\Delta] = A(C_n)$.

**Proof.** Basically we want to see that all entries on the left hand side is the same as the right hand side, so we consider the $(u,v)$-entry of $[\Phi_\Delta]$, let $z \in \mathbb{Z}_2^n$, and compute: $\Phi_\Delta f_u(v)$,

$$\begin{aligned}
\Phi_\Delta f_u(z) &= \sum_{w\in\Delta} f_u(z+w) \\
&= \sum_{w\in\Delta} f_{u+w}(z) \\
\Rightarrow \Phi_\Delta f_u &= \sum_{u\in\Delta} f_{u+w} \\
\Rightarrow [\Phi_\Delta]_{u,v} &= \begin{cases} 1, & \text{if } u+w=v \\ 0, & \text{if } u+w \neq v \end{cases} \\
\Leftrightarrow [\Phi_\Delta]_{u,v} &= \begin{cases} 1, & \text{if } u+v=w\in\Delta \\ 0, & \text{otherwise} \end{cases} \\
\Leftrightarrow [\Phi_\Delta]_{u,v} &= \begin{cases} 1, & \text{if } u,v\in E(C_n) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

■

**Corollary 2.2.1.** The eigenvectors of $A(C_n)$ are:

$$\mathcal{E}_u = \sum_{v\in\mathbb{Z}_2^n} (-1)^{u\cdot v} v$$

With eigenvalues being:

$$\lambda_u = n - 2|u|$$

In particular, $A(C_n)$ has $\binom{n}{k}$ eigenvalues equal to $n-2k$.

**Proof.** For $g \in V$, see that $g = \sum_{v\in\mathbb{Z}_2^n} g(v)\cdot f_v$, which implies that:

$$g(u) = \sum_{v\in\mathbb{Z}_2^n} g(v) f_v(u) = \sum_{v\in\mathbb{Z}_2^n} g(v)\delta_{uv}$$

Recall that $[\Phi_\Delta]_{\mathcal{B}_1}$ has eigenvectors $\{\chi_u(v) = (-1)^{u\cdot v}\}$, then:

$$\chi_u = \sum_{v\in\mathbb{Z}_2^n} \chi_u(v) f_v = \sum_{v\in\mathbb{Z}_2^n} (-1)^{u\cdot v} f_v$$

with eigenvalues being:

$$\lambda_u = \sum_{w\in\Delta} (-1)^{u\cdot w} = \sum_{i\in[n]} (-1)^{u\cdot\delta_i} = n - 2|u|$$

Notice that those $\mathcal{E}_u$ with the **Hamming Weight** of $u$ being the same share the same eigenvalue, total number of such $u$ will be $\binom{n}{k}$ if $|u| = k$. ■

**Proposition 2.2.2.** If $G$ is a graph with $p$ vertices and eigenvalues $\lambda_1, \ldots, \lambda_p$, then there exists $c_1, \ldots, c_p \in \mathbb{R}$, s.t.

$$A(G)^l_{ij} = \sum_{k=1}^{p} c_k \lambda_k^l$$

And in particular, if $U$ is an **orthogonal** matrix ($A(G) = U\Lambda U^{-1}$), then:

$$A(G)^l_{ij} = \sum_{k=1}^{n} U_{ik} U_{jk} \lambda_k^l$$

**Proof.** By **Spectral Theorem** 2.1.2, there exists an orthogonal matrix $U$, s.t.

$$A = U\Lambda U^{-1}$$

Let $l \geq 1$, then:

$$A^l = U\Lambda^l U^{-1}$$

take the $(i,j)$-entry of right hand side and see that it equal to $\sum_{k=1}^{n} U_{ik}U_{jk}\lambda_k^l$. ∎

**Corollary 2.2.2.** Let $u, v \in \mathbb{Z}_2^n$, supppose that $|u+v| = k$, i.e. $u$ and $v$ **differ** in $k$ places, then:

$$A(C_n)^l_{uv} = \frac{1}{2^n}\sum_{i=1}^{n}\sum_{j=1}^{k}(-1)^k\binom{k}{j}\binom{n-k}{i-j}(n-2i)^l \qquad (i \geq j)$$

**Proof.** Consider the eigenvector of $A(C_n)$, given by:

$$\mathcal{E}_u = \sum_{v\in\mathbb{Z}_2^n}(-1)^{u\cdot v}v$$

By **Equation** 2.1, $|\mathcal{E}_u| = 2^{\frac{n}{2}}$, so one needs to normalize it to get the orthonormal basis:

$$\mathcal{E}'_u = \frac{1}{2^{\frac{n}{2}}}\mathcal{E}_u$$

Then by **Proposition** 2.2.2, we have:

$$A(C_n)^l_{uv} = \frac{1}{2^n}\sum_{w\in\mathbb{Z}_2^n}\mathcal{E}'_{uw}\mathcal{E}'_{vw}\lambda_w^l$$

One can then consider what is $\mathcal{E}'_{uw}$, written in canonical basis $v \in \mathbb{Z}_2^n$, it is exactly:

$$\mathcal{E}'_{uw} = (-1)^{u\cdot w}$$

Thus we have:

$$A(C_n)^l_{uv} = \frac{1}{2^n}\sum_{w\in\mathbb{Z}_2^n}(-1)^{(u+v)\cdot w}(n-2|w|)^l$$

One may consider the red part, with the number of vectors $w$ of **Hamming Weight** $i$ which have $j$ 1's in common with $u+v$ is:

$$\binom{k}{j}\binom{n-k}{i-j}$$

it is simply the way of how we choose $w$, so that the dot product result $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{j}$. As $w$ runs over all elements in $\mathbb{Z}_2^n$, where $k$ is just $|u+v|$ to be the **upper bound** of $j$, see that:

$$A(C_n)^l_{uv} = \frac{1}{2^n}\sum_{i=1}^{n}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\binom{n-k}{i-j}(n-2i)^l \quad (i = |w|)$$

∎

## 2.3 Matrix-Tree Theorem

**Definition 2.3.1 (Spanning Subgraph).** A spanning subgraph of a graph $G$ is a graph $H$ with the same vertex set as $G$ and $E(H) \subseteq E(G)$.

(If $G$ has $q$ edges, there are $2^q$ spanning subgraph, in particular, choose whether each edges are included in.)

**Definition 2.3.2 (Spanning Tree).** A spanning tree is a spanning subgraph that is a **tree**. The # of spanning tress in a graph $G$ is called complexity of $G$, denoted as $\kappa(G)$.

**Definition 2.3.3 (Laplacian of a Graph).** The laplacian of a graph $G$ is given by:

$$\mathbb{L}(G) = \begin{cases} -\left|\varphi^{-1}(v_i, v_j)\right|, & \text{if } v_i \neq v_j \\ \deg(v_i), & \text{otherwise} \end{cases}$$

In short, the diagonal of such matrix will be the degree of the vertex, and the rest represent the number of edges connect between the vertices mult with $-1$.

**Note.** The laplacian actually can also be given by:

$$\mathbb{L}(G) = \operatorname{diag}\{\deg(v_1), \dots, \deg(v_n)\} - A(G)$$

**Definition 2.3.4 ((r,c)-Cofactor).** Given a matrix $A$, the (r,c)-cofactor of $A$ is the:

$$A_{r,c} := (-1)^{r+c} \det A_{\hat{r},\hat{c}}$$

where $\hat{r}$ represent deleting the $r$-th row, and $\hat{c}$ represent deleting the $c$-th column.

**Definition 2.3.5 (Orientation).** An orientation of a graph $G$ is an assignment of an order on every edge: which is, for every edge $e \in E(G)$, for example, it connects $u$ and $v$, then choose one of the ordered pair $(u, v)$ or $(v, u)$ as direction. Such orientation is usually denoted as $\mathfrak{o}$.

**Definition 2.3.6 (Incidence Matrix).** The incident matrix of a directed graph $G$ with respect to the orientation $\mathfrak{o}$ is the $|V| \times |E|$ matrix whose $(i, j)$-entry is given by:

$$\mathbb{M}(G) = \begin{cases} -1, & \text{if the edge } e_j \text{ has initial vertex } v_i \\ 1, & \text{if the edge } e_j \text{ has final vertex } v_i \\ 0, & \text{otherwise} \end{cases}$$

**Lemma 2.3.1.**

$$\mathbb{L}(G) = \mathbb{M}(G)\mathbb{M}(G)^\intercal$$

**Proof.** The proof is quite straightforward once consider the situation of matrix manipulation, first denote $\mathbb{M}(G) = (m_{ij})$:

$$(\mathbb{M}(G)\mathbb{M}(G)^\intercal)_{ij} = \sum_{e_k \in E(G)} m_{ik} m_{jk}$$

Consider the situation of $i = j$ and $i \neq j$:

- When $i \neq j$: One should only consider the case when $e_k$ connects $i$ and $j$, otherwise one of them will be 0 and thus the product of them will be 0. So consider the case where $i$ and $j$ are connected by $e_k$. In this case one vertex must be the source and one vertex must be the sink, thus one of them will be 1 and the other will be $-1$. Summing through all edges in the graph gives us the result to be $-\left|\varphi^{-1}(v_i, v_j)\right|$.

- When $i = j$: In this case one will still only consider the case where $e_k$ touches $v_i = v_j$, otherwise one of $m_{ik}$ or $m_{jk}$ will be 0. Whether it is the source or sink it doesn't matter, the result should still yields to be 1. Summing through all edges in the graph gives us $\deg(v_i = v_j)$.

It follows that this is exactly the **Definition** 2.3.3 of Laplacian. ■

**Lemma 2.3.2.** Let $|V| = p$. Let $S$ be a subset of $p-1$ edges. Let $v$ be any vertex of $G$. Then:

$$\det\left(\underbrace{\mathbb{M}_{\hat{v}}[S]}_{(p-1)\times(p-1)}\right) = \begin{cases} \pm 1, & \text{if } S \text{ forms a spanning tree of } G \\ 0, & \text{otherwise} \end{cases}$$

**Proof.** One can prove the theorem by deviding the case of whether $S$ forms a spanning tree of $G$:

- If $S$ doesn't forms a spanning tree. Then there must exsist a cycle $C$ in the graph, denoted the edge of the cycle as $f_1, \ldots, f_s$. The point is we want to make it as a oriented cycle, so that the column vector representing $f_1, \ldots, f_s$ are linearly independent. For convenience one can interchange the column vector of $f_1, \ldots, f_s$ into in order and adacent ones on the matrix, the determinant still remains the same.

■

**Theorem 2.3.1 (Matrix-Tree Theorem).** Let $G$ be a finite connected graph without loops. The # of spanning trees of $G$ is:

$$\kappa(G) = L_{r,c}(G) \qquad \forall r, c$$

where $L_{r,c}$ is the $(r, c)$-cofactor of $\mathbb{L}(G)$.

# Appendix