Math566: Combinatorial Theory

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January 8, 2026

Abstract

This is the note containning my personal thoughts as well as lecture notes, course content include some basic algebraic combinatorics. My course instructor is Prof. Shiyue Li.

Contents

Chapter 1

Linear Algebra Lemma

This chapter basically collect some linear algebra lemma that maybe helpful across the course.

Lemma 1.0.1. Given A to be a matrix, who has the eigenvalues to be $\lambda+1, \ldots, \lambda_p$, then the eigenvalues of the matrix $A + c\,\mathrm{Id}$ where $c \in \mathbb{C}$ is $\lambda_1 + c, \ldots, \lambda_p + c$.

Lemma 1.0.2. Let A be as above, the eigenvalues of A^l are $\lambda_1^l, \ldots, \lambda_p^l$.

Chapter 2

Walks in Graph

2.1 Graph Eigenvalues

To better phrasing a graph, we first need to define multiset so that we can express not only simple graph but also general graph in a mathematically rigourous way.

Definition 2.1.1 (Multiset). A multiset M on a set S is an unordered collection of elements in S, s.t.

1. $\forall\, x \in M, x \in S$.
2. The $\#$ of times for $x \in S$ to appear in M, denoted as $\mu_M(x)$, is ≥ 0.

Example 2.1.1. If $\mu_M(x) = 0, 1\ \forall x$, then M is a set.

Note that two multiset M, M' are said to be equal if $\forall x \in S, \mu_M(x) = \mu_{M'}(x)$.

Notation. Let S be a finite set of size p, then define:

$$\binom{S}{k} = \{k - \text{subsets of } S\}$$

note that

$$\left|\binom{S}{k}\right| = \binom{|S|}{k}$$

also define:

$$\left(\!\binom{S}{k}\!\right) = \{k - \text{subsets of } S\}$$

note that:

$$\left|\left(\!\binom{S}{k}\!\right)\right| = \binom{p+k-1}{k}$$

this is the case: consider rephrasing the combinatorial problem as possible assigning of numbers for p numbers $a_1, \ldots, a_p$ with $a_1 + \ldots + a_p = k$ and $a_i \in [\![0, k]\!]$.

Thus we can define the graph properly.

Definition 2.1.2 (Graph). A finite graph is a triple $G = (V, E, \varphi)$ with:

- $V = \{v_1, \ldots, v_p\}$.
- $E = \{e_1, \ldots, e_q\}$.
- φ is a function $E \to \left(\!\binom{V}{2}\!\right)$.

A finite simple graph is the same data with $\varphi : E \to \binom{V}{2}$

Definition 2.1.3 (Adjacency Matrix of Graph). The adjacency matrix of a graph G, denoted as $A(G)$, whose entries is defined by:

$$a_{ij} = \left|\varphi^{-1}(\{v_i, v_j\})\right|$$

In particular it counts the number of edges between two vertices v_i and v_j. Note that it is well-defined since if there is no edges between v_i and v_j, then the preimage of φ will be $\emptyset$, thus $a_{ij} = 0$.

Definition 2.1.4 (Walk). A walk of length k in a graph G is a non-empty finite sequence of vertices and edges

$$W = v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k$$

such that for all $1 \leq i \leq k$, the edge e_i has **endpoints** v_{i-1} and v_i.

In a simple graph, where the edges are determined by their endpoints, a walk can be simplified to a sequence of vertices:

$$W = (v_0, v_1, \dots, v_k) \quad \text{where } \{v_{i-1}, v_i\} \in E(G)$$

Note.

- In a walk, the both the edges and vertices can appear **repeatedly**.
- If $v_0 = v_k$, then such walk is called a **closed walk**.

Proposition 2.1.1. For any integer $l \geq 1$, the (i, j) entry of $(A(G))^l$, denoted as a_{ij}, is equal to the $\#$ of walks of length l in G starting from v_i to v_j.

Theorem 2.1.1. Let G be graph with $A(G)$ possessing eigenvalues $\lambda_1, \dots, \lambda_p$, the $\#$ **closed walks** of length l is:

$$f_G(l) = \sum_{i=1}^{p} \lambda_i^l$$

The proof is straightforward combining the **Proposition** 2.1.1 and **Lemma** 1.0.2.

Appendix