Control of Serial Manipulators with elastic joints: using backstepping.

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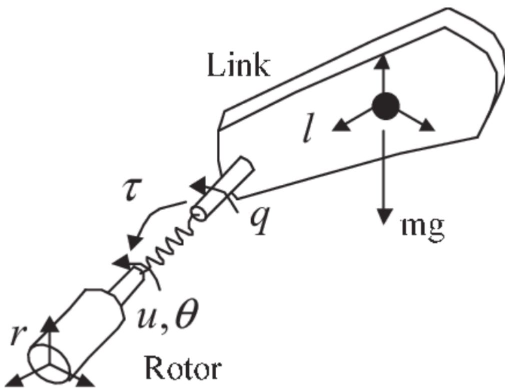
Underactuated Robotics project
Academic year 2018/2019

July 5, 2019



Abstract

Manipulator dynamics of two link planar serial robotic arm with flexible joints are derived to suitable form for the application of backstepping, applying backstepping on thus derived equations

Contents

1 Introduction

The use of harmonic drives in robots has introduced joint flexibility in robotic arms which wasn't of much pronounced effects when huge gear ratio transmissions were used wherein the non-linearities got divided by big numbers and could be effectively treated like linear. Thus problem arises when tasks requiring extreme precision and the joint elasticity can not be further neglected for adequate performance. Various techniques have been applied to treat the elastic and dissipative joints including[1],[2].We will be discussing the use of backstepping to control the robotic arm with elastic joints and with dissipation. The model of the elastic joint used is that of[2]. Backstepping is a technique that can be well applied to systems described in the chain structure[3]. To apply backstepping to serial manipulators we must first partial feedback linearize the equations without neglecting the elastic terms. In doing so we obtain equations of an underactuated system since the system can't assume instantaneous acceleration in any arbitrary direction[4]. And as we will see later the equations thus resulting will be in a form that needs some manipulations to go into suitable form before backstepping can be applied so we start with an approach that will be needed for such.

2 Non-regular static state feedback triangulation

Static state feedback linearization involves finding a diffeomorphism and control law thereby resulting in a feedback linearized system with state dimensions being equal to the relative degree of the system, however, the relationship between the transformed inputs and the original outputs still remains a nonlinear map[5].

A coordinate transformed system into a linear controllable system is static state feedback linearizable if $\exists$ a control law of the form $u = \alpha(x) + \beta(x)v$ with the matrix β being invertible. If , however, the matrix β is non-invertible i.e. it is square but singular or non-square then if $\exists u$ then the transformed system is non-regular static state feedback linearizable by a single control u [6]

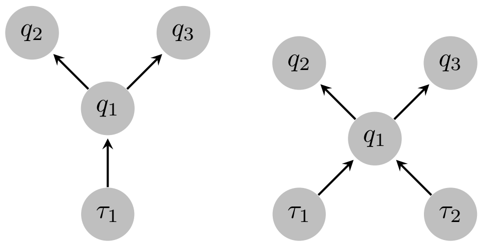


Figure 1: tree structure examples

On the other hand the non-regular static state feedback triangluation doesn't involve cancellation of useful nonlinear terms. Moreover, unlike the static state feedback linearization, it doesn't require coordinate transformation. NSSFT is essentially a method to convert a system of equations from tree structure, shown in the figure 2, to a chain structure to allow application of backstepping control.

Given an underactuated system written in partial feedback form and takes a from as below

$$\begin{aligned} \ddot{x} &= f(x, \dot{x}) + H(x, \dot{x})y + G(x, \dot{x})u \\ \ddot{y} &= u \end{aligned} \tag{1}$$

Can be converted to the chain structure, which has the form

$$\begin{aligned} \ddot{q}_i &= \Phi_i(q_i, .., q_{i+1}, \dot{q}_i, .., \dot{q}_{i+1}) \ , \ i = 1, .., n-1 \\ \ddot{q}_n &= \Phi_n(q, \dot{q}) + \gamma(q, \dot{q})v \end{aligned} \tag{2}$$

Assumption 1: $\exists P(x, \dot{x}) \in \mathbb{R}^{n \times n} > 0 \ni$

$$\begin{aligned} H(x, \dot{x}) &= P(x, \dot{x})H_0(x, \dot{x}) \\ G(x, \dot{x}) &= P(x, \dot{x})G_0(x, \dot{x}) \end{aligned} \tag{3}$$

with the $H_0(x, \dot{x})$ and $G_0(x, \dot{x})$ having and upper triangular and strictly upper triangular structure.

note : upper triangular matrix is the one which has elements defined only for $(i, j) \ni i \leqslant j$, *while the strictly upper triangular matrix has elements defined for* $(i, j) \ni i \leqslant j$.[7]

In the equation (1) we see the dynamics of the linear part being fed to the nonlinear part of the equation apart from the input u. An underactuated system for which the number of actuated DOFs are not less the unactuated DOFs takes on such a form of equation

2.1 Proof

If $m > n$, then let

$$\begin{aligned} u_{n+1} &= \alpha_{n+1} = y_{n+1} + \Psi_{n+1}(y_{n+1}) \\ &\vdots \\ u_{m-1} &= \alpha_{m-1} = y_m + \Psi_{m-1}(y_{n+1}, ..., ym - 1) \\ u_m &= \alpha_m = x_1 + \Psi_m(y_{n+1}, ..., ym) \end{aligned} \tag{4}$$

Rewrite equations in (1) as follows:

$$\begin{aligned} \ddot{x} &= f(x, \dot{x}) + P(x, \dot{x})H_o(x, \dot{x})y + P(x, \dot{x})G_o(x, \dot{x})u \\ P^{-1}(\bar{x})(\ddot{x} - f(\bar{x})) &= H_o(\bar{x})y + G_o(\bar{x})u \qquad \ni \quad \bar{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \end{aligned} \tag{5}$$

let $d_i(\bar{x}) = [d_{i,1}, .., d_{i,n}(\bar{x})]$ be the ith row of matrix $P_{-1}(\bar{x})$ using Assumption 1 and the equation (4) then the last equation in (5) will be

$$d_n(\bar{x})(\ddot{x} - f(\bar{x})) = \sum_{j=n}^{m} h_{n,j}(\bar{x})y_j + \sum_{j=n+1}^{m} g_{n,j}(\bar{x})\alpha_j \tag{6}$$

then

$$\ddot{x}_n = f_n + d_{n,n}^{-1}\left(\sum_{j=n}^{m} h_{n,j}(\bar{x})y_j + \sum_{j=n+1}^{m} g_{n,j}(\bar{x})\alpha_j - \sum_{j=1}^{n-1} d_{n,j}(\ddot{x}_j - f_j)\right) \tag{7}$$

next let us design u_i, $i = 2, n \ni$ x-dynamics have a chain structure i.e.

$$\begin{aligned} \ddot{x}_1 &= x_2 + \phi(\bar{y}, x_1) \\ &\vdots \\ \ddot{x}_{n-1} &= x_n + \phi_{n-1}(\bar{y}, x_1, ..., x_{n-1}) \\ \ni \quad \bar{y} &= [y_{n+1}, \dot{y}_{n+1}, ..., y_m, \dot{y}_m]^T \end{aligned} \tag{8}$$

substituting (8) into (7)

$$
\begin{aligned}
\ddot{x}_i &= x_{i+1} + \phi_i(\bar{y}, x_1, .., x_i) \quad \ni i = 1, ..., n-1 \\
\ddot{x}_n &= f_n + d_{n,n}^{-1}\left(\sum_{j=n}^{m} h_{n,j}y_j + \sum_{j=n+1}^{m} g_{n,j}\alpha_j - \sum_{j=1}^{n-1} d_{n,j}(x_{j+1} + \phi_j - f_j)\right) \\
&= d_{n,n}^{-1}(\bar{x})h_{n,n}(\bar{x})y_n + f_n + d_{n,n}^{-}1\left(\sum_{j=n+1}^{m} h_{n,j}y_j + g_{n,j}\alpha_j - \sum_{j=n}^{n-1} d_{n,j}(x_{j+1} + \phi_j - f_j)\right)
\end{aligned}
\tag{9}
$$

$$
\therefore \quad \ddot{x}_n = l(\bar{x})y_n + \kappa(\bar{x}, \bar{y})
$$

$$
\begin{aligned}
\ni \quad l(\bar{x}) &= d_{n,n}^{-1}(\bar{x})h_{n,n}(\bar{x})f \\
\kappa(\bar{x}, \bar{y}) &= f_n + d_{n,n}^{-1}\left(\sum_{j=n+1}^{m} h_{n,j}y_j + g_{n,j}\alpha_j - \sum_{j=n}^{n-1} d_{n,j}(x_{j+1} + \phi_j - f_j)\right)
\end{aligned}
$$

the fiirst $n-1$ equations of second equation in (5) are given by

$$
\begin{aligned}
d_1(\bar{x})(\ddot{x} - f(\bar{x}) &= \sum_{j=1}^{m} h_{1,j}(\bar{x})y_j + \sum_{j=n+1}^{m} g_{i,j}(\bar{x})\alpha_j + \boxed{\sum_{j=2}^{n} g_{i,j}(\bar{x})u_j} \rightarrow \text{check } G_o(x, \dot{x}) \\
&\vdots \\
d_{n-1}(\bar{x})(\ddot{x} - f(\bar{x}) &= \sum_{j=n-1}^{m} h_{n-1,j}(\bar{x})y_j + \sum_{j=n+1}^{m} g_{n-1,j}(\bar{x})\alpha_j + g_{n-1,n}(\bar{x})u_n \\
\ni \qquad g_{n-1,n}(\bar{x})u_n &= \sum_{j=n}^{n} g_{n-1,j}(\bar{x})u_j
\end{aligned}
\tag{10}
$$

From which we can solve for $u_i = 2, \ldots, n$ as

$$
\begin{aligned}
u_n = \alpha &= c_n(\bar{x}, \bar{y}, y_n) + e_n(\bar{x})y_{n-1} \\
u_{n-1} = \alpha_{n-1} &= c_{n-1}(\bar{x}, \bar{y}, y_{n-1}, y_n) + e_{n-}(\bar{x})y_{n-2} \\
&\vdots \\
u_2 = \alpha_2 &= c_2(\bar{x}, \bar{y}, y_2, .., y_n) + e_2(\bar{x})y_1
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
c_n &= g_{n-1,n}(\bar{x}) \left(d_{n-1}(\bar{x})(\sigma - f(\bar{x})) - \sum_{j=n}^{m} h_{n-1,j}(\bar{x}) y_j - \sum_{j=n+1}^{m} g_{n-1,j}(\bar{x}) \alpha_j \right) \\
e_n &= -g_{n-1,n}^{-1}(\bar{x}) h_{n-1,n-1}(\bar{x}) \\
c_2 &= g_{1,2}^{-1}(\bar{x}) \left(d_{-1}(\bar{x})(\sigma - f(\bar{x})) - \sum_{j=2}^{m} h_{i,j}(\bar{x}) y_j - \sum_{j=3}^{m} g_{i,j}(\bar{x}) \alpha_j \right) \\
e_2 &= -g_{1,2}^{-1}(\bar{x}) h_{1,1}(\bar{x}) \\
\sigma &= [x_2 + \phi_1(\bar{y}, x_1), ..., x_n + \phi(\bar{y}, x_1, ..., x_{n-1}), l(\bar{x} y_n + \kappa(\bar{x}, \bar{y})]^T
\end{aligned}
\tag{12}
$$

Finally let

$$
u_1 = v \tag{13}
$$

Gathering equations (4), (11), (13)

$$
u = \begin{bmatrix} 0 \\ c_2(\bar{x}, \bar{y}, y_2, ..., y_n) + e_2(\bar{x}) y_1 \\ \\ \vdots \\ c_n(\bar{x}, \bar{y}, y_n) + e_n(\bar{x}) y_{n-1} \\ y_{n+2} + \Psi_{n+1}(y_{n+1}) \\ \\ \vdots \\ x_1 + \Psi_m(y_{n+1}, \ldots, y_{m+1}, y_m) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} v \tag{14}
$$

where $\Psi_i, i = n+1, \ldots, m$ are any smooth functions of appropriate dimensions. Accordingly the overall system of (1) and (14) is given by

$$
\begin{cases}
\ddot{y}_{n+1} = y_{n+2} + \Psi_{n+1}(y_{n+1}) \\
\qquad \vdots \\
\ddot{y}_{m-1} = y_m + \Psi_{n+1}(y_{n+1}, \ldots, y_{m-1}) \\
\\
\ddot{x}_1 = x_2 + \phi_1(\kappa, x_1) \\
\qquad \vdots \\
\ddot{x}_{n-1} = x_n + \phi_{n-1}(\kappa, x_1, \ldots, x_{n-1}) \\
\ddot{x}_n = l(\bar{x}) y_n + \kappa(\bar{x}, \bar{y}) \\
\ddot{y}_n = e_n(\bar{x}) y_{n-1} + c_n(\bar{x}, \bar{y}, y_n) \\
\qquad \vdots \\
\ddot{y}_2 = e_2(\bar{x}) y_1 + c_2(\bar{x}, \bar{y}, y_2, \ldots, y_n) \\
\ddot{y}_1 = v
\end{cases}
\tag{15}
$$

note there are some flexibilities in the triangulating control law (14) and resulting system (15). In particular $\Psi_i i = n+1, \ldots, m$, $\Psi_i i = 1, \ldots, n-1$ are arbitrary smooth functions which can be choosen by the designer

3 Manipulator Equations

From the langrangian dynamcis the direct dynamics of the serial manipulator can be written in the familiar manipulator equations. Using the generalized coordinates ($q \epsilon \mathbb{R}^n$).

The manipulator equations:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q,\dot{q}) + G(q) = B(q)u \tag{16}$$

Where the $M(q)$ is positive definite inertial matrix,$C(q,\dot{q})$ includes all the Coriolis (Coriolis forces are due to the interaction of different links) and centrifugal terms while $G(q)$ include potential and Gravitational potential terms collected together. Note that the matrix B(q) is an invertible matrix.

let us also write a case of underactuation that results when actuators are missing at some joints and we have the eqution of the form:

$$\begin{aligned} M_1(q)\ddot{q_1} + M_2(q)\ddot{q_2} + D_1(q,\dot{q}) + C_1(q,\dot{q})\dot{q} + G_1(q) = 0 \\ M_2^T(q)\ddot{q_1} + M_3(q)\ddot{q_2} + D_2(q,\dot{q}) + C_2(q,\dot{q})\dot{q} + G_2(q) = B(q)\tau \end{aligned} \tag{17}$$

the $q_1 \epsilon \mathbb{R}^n$ of unactuated while $q_2 \epsilon \mathbb{R}^m$ of actuated DOFs.

Equation (17) is written out to be used as a comparison below. Since by consideration of elasticity at joints we essentially get underactuated system but is slightly different in form.

3.1 Joint elasticity consideration

When the joint elasticity is taken into consideration the generalized coordinated go from being $\mathbb{R}^n$ to $\mathbb{R}^{2n}$ due to the inclusion of elasticity in the dynamics as they further must include separate position for actuators and for the links.

Assuming elasticity between the actuators and the rigid links, such as due to presence of harmonic drives our new set of generalized coordinates are $q = (q_1, q_2)$ link positions and actuator positions respectively and the results in an underactuated system. For n links there are n actuators connected thorough n torsional springs and thus equation (16) can be expanded as follows[2][8][9] :

$$\begin{aligned} M_1(q_1)\ddot{q_1} + M_2\ddot{q_2} + D_1(q_1,\dot{q_1}) + C_1(q_1,\dot{q_1})\dot{q_1} + K_e(q_1 - q_2) + G_{g1}(q_1) = 0 \\ M_2^T\ddot{q_1} + M_3\ddot{q_2} + D_2(q_2,\dot{q_2}) + C_2(q_2,\dot{q_2})\dot{q_2} + K_e(q_2 - q_1) + G_{g2}(q_2) = \tau \end{aligned} \tag{18}$$

where:

$$K_e = \begin{bmatrix} k_1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & k_{n-1} & 0 \\ 0 & \ldots & 0 & k_n \end{bmatrix}$$

while $q_1 = [q_{l1}, q_{l2}, .., q_{ln}]$, $q_2 = [q_{m1}, q_{m2}, .., q_{mn}]$,the individual inertial terms belong to the matrix $M \ni M = M^T > 0$ and hence invertible with the off center terms being non-zero implying inertial coupling between the links, the $K_e = diag[k_1, k_2, .., k_n]$ is a diagonal vector of the torsional spring constants between the actuator and the link and lastly the G_{gi} are gravitational forces.

It is immediately seen that the equations of (18) are a special case of (17) since the number of unactuated DOFs is not less than the number of actuated joints[9] and there is exists interlink inertial coupling. This further means that this class of system falls in the category of systems discussed in section 2 that can be applied with non regular static state feedback linearization.

3.2 Application of Non-regular static state feedback triangulation (NSSFT) on manipulator equations with flexible joints

The manipulator equations are first written in partial feedback linearization form which is first done for the equation (17) and at a later step for equation (18) to apply regular or nonregular static state partial feedback triangluation. However, in what follows next it will be clear that for our case i.e. the elastic joint manipulators equation gives a special form of equations onto which NSSFT is applied for conversion to chain structure rather than regular static state feedback triangulation. Infact no form of regular static state feedback can be applied for the second case as per [9].

3.2.1 Underactuated Manipulator - Collocated partial feedback linearization

we proceed with collocated partial feedback linearization for underactuated manipulator. Remember this is the case of missing actuators at arbitrary joints. Also remember that collocated partial feedback linearization allows the dynamics of the actuated DOF to be expressed directly in terms of control while the unactuated will be controlled via the inertial coupling between the actuated and underactuated DOFs.

For the equation (17) picking $\ddot{q_1}$ as the subject of the first equation we have:

$$\ddot{q_1} = -M_1^{-1}(q)M_2(q)\ddot{q_2} - M_1^{-1}(q)D_1(q,\dot{q}) - M_1^{-1}(q)C_1(q,\dot{q})\dot{q} - M_1^{-1}(q)G_1(q) \tag{19}$$

substituting $\ddot{q_1}$ into the second equation, ignoring the dependencies for space saving

$$\begin{aligned}
M_2^T(-M_1^{-1}M_2\ddot{q_2} - M_1^{-1}D_1 - M_1^{-1}C_1\dot{q} - M_1^{-1}G_1)& \\
+M_3\ddot{q_2} + C_2\dot{q} + G_2 &= B\tau \\
-M_2^T M_1^{-1}M_2\ddot{q_2} - M_2^T M_1^{-1}D_1& \\
-M_2^T M_1^{-1}C_1\dot{q} - M_2^T M_1^{-1}G_1 + M_3\ddot{q_2} + C_2\dot{q} + G_2 &= B\tau \\
\ddot{q_2}(M_3 - M_2^T M_1^{-1}M_2) - M_2^T M_1^{-1}D_1& \\
-M_2^T M_1^{-1}C_1\dot{q} - M_2^T M_1^{-1}G_1 + C_2\dot{q} + G_2 &= B\tau
\end{aligned}$$

therefore by selecting τ as follows allows expressing control input u to our actuated joints in $\ddot{q_2}$

$$\begin{aligned}
\tau = \quad & B^{-1}[(M_3 - M_2^T M_1^{-1}M_2)u \\
& - M_2^T(M_1^{-1}D_1 + M_1^{-1}C_1\dot{q} \\
& + M_1^{-1}G_1) + C_2\dot{q} + G_2]
\end{aligned} \tag{20}$$

note: that $(M_3 - M_2^T M_1^{-1}M_2)$ is the schur complement of inertial matrix M which symmetric positive definite and so always invertible and can be used in the way it is used to cancel dynamics in τ

Thus we get the the system collocated partial feedback linearized as follows:

$$\begin{aligned}
\ddot{q_1} &= J(q)\ddot{q_2} + R(q,\dot{q}) \\
\ddot{q_2} &= u
\end{aligned} \tag{21}$$

where $J(q) = -M_1^{-1}(q)M_2(q)$ and $R(q,\dot{q}) = M_1^{-1}(q)D_1(q,\dot{q}) - M_1^{-1}(q)C_1(q,\dot{q})\dot{q} - M_1^{-1}(q)G_1(q)$.

The structure of the uncoupled dynamics after partial feedback linearization is highly coupled in a tree structure

3.2.2 Manipulator with elastic joints - Collocated partial feedback linearization & NSSFT

Our case of study is a case of underactuation different from the above case as we have no missing actuator at any link rather. each link is coupled to the actuator via a torsional spring which results in underactuation since an instantaneous acceleration in an arbitrary direction can't be assumed for the elastic joints. Having written the partial feed back linearized form for equation (17) in (21). We can do it for equation (18). Thus writing in collocated partial feedback linearization guided by the same procedure by choosing τ as:

$$\begin{aligned}\tau =& - M_2^T M_1^{-1}(q_1)(C_1(q,\dot{q}_1)\dot{q}_1 + K_e(q_1 - q_2)\\ &+ G_{g1}(q_1) + D_1(q_1,\dot{q}_1)) + K_e(q_2 - q_1)\\ &+ D_2(q_2,\dot{q}_2) + (M_3 - M_2 T M_1^{-1}(q_1)M_2)u\end{aligned} \tag{22}$$

we have:

$$\begin{aligned}\ddot{q}_1 =& - M_1^{-1}(q_1)M_2 u\\ &- M_1^{-1}(q_1)D_1(q_1,\dot{q}_1)\\ &- M_1^{-1}(q_1)C_1(q,\dot{q})\dot{q} - M_1^{-1}(q)G_1(q)\\ \ddot{q}_2 =& \ u\end{aligned} \tag{23}$$

which can be rearranged as follows

$$\begin{aligned}\ddot{q}_1 =& - M_1^{-1}(q_1)[C_1\dot{q}_1 + K_e q_1 + G_g + D_1]\\ &+ M_1^{-1}K_e q_2\\ &- M_1^{-1}M_2 u\\ \ddot{q}_2 =& u\end{aligned} \tag{24}$$

Again the structure of the uncoupled dynamics after partial feedback linearization is highly coupled in a tree structure but the difference in (21) from (24) is that we can explicitly see that regular static state feedback methods are inapplicable for the later rather it takes on the form as (1). Since $J(q,\dot{q}) = f(x,\dot{x}) = -M_1^{-1}(q_1)[C_1\dot{q}_1 + K_e q_1 + G_g + D_1]$, $H(q,\dot{q}) = M_1^{-1}K_e$ and $G(q,\dot{q}) = -M_1^{-1}M_2$ and hence as per the discussion above can be applied with non-regular static state feedback triangulation to convert it to the chain form

Reminding us that we are considering the above case to be for 2 link planar robotic arm and that

$$M_1^{-1}(q_1) = \begin{bmatrix} d_1(q_1) & d_2(q_1) \\ d_2(q_1) & d_3(q_1) \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & m_{12} \\ 0 & 0 \end{bmatrix}, \quad K_e = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

using the equation (24) and the data above we design u so that we have a single chain of integrators $\ddot{q}_{l1} = q_{l2}$

$$\begin{aligned}\ddot{q}_1 &= \begin{bmatrix} \ddot{q}_{l1} \\ \ddot{q}_{l2} \end{bmatrix} = \begin{bmatrix} J_1(q_1) \\ J_2(q_1) \end{bmatrix} + \begin{bmatrix} d_1(q_1) & d_2(q_1) \\ d_2(q_1) & d_3(q_1) \end{bmatrix} (K_e q_2 - M_2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix})\\ \ddot{q}_1 &= \begin{bmatrix} \ddot{q}_{l1} \\ \ddot{q}_{l2} \end{bmatrix} = \begin{bmatrix} J_1(q_1) \\ J_2(q_1) \end{bmatrix} + \begin{bmatrix} d_1(q_1) & d_2(q_1) \\ d_2(q_1) & d_3(q_1) \end{bmatrix} (K_e q_2 - \begin{bmatrix} 0 & m_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix})\\ \ddot{q}_1 &= \begin{bmatrix} \ddot{q}_{l1} \\ \ddot{q}_{l2} \end{bmatrix} = \begin{bmatrix} J_1(q_1) \\ J_2(q_1) \end{bmatrix} + \begin{bmatrix} d_1(q_1) & d_2(q_1) \\ d_2(q_1) & d_3(q_1) \end{bmatrix} K_e q_2 - \begin{bmatrix} d_1 m_{12} u_2 \\ d_2 m_{12} u_2 \end{bmatrix}\end{aligned}$$

meanwhile setting u_1 to be v for design latter and as stated earlier designing $u_2 \ni \ddot{q}_{l1} = q_{l2}$ then

$$u_2 = \alpha(q,\dot{q}) = \frac{1}{d_1(q_1)m_{12}}[J_1(q_1,\dot{q}_1) + \left[\begin{array}{cc} d_1(q_1) & d_2(q_1) \end{array}\right] K_e q_2 - q_{l2}] \tag{25}$$

substituting which we have in $\ddot{q}_1 = \left[\begin{array}{cc} \ddot{q}_{l1} & \ddot{q}_{l2} \end{array}\right]^T$

$$\ddot{q}_{l1} = q_{l2} \tag{26}$$

and processing $\ddot{q}_{l2}$ further with the substitution of u_2 chosen in (25)

$$\begin{aligned}
\ddot{q}_{l2} &= J_2(q_1) + \left[\begin{array}{cc} d_2(q_1) & d_3(q_1) \end{array}\right] K_e q_2 - d_2 m_{12} u_2 \\
&= J_2(q_1) + \frac{d_2(q_1)}{d_1(q_1)}\left(q_{l2} - J_1(q_1,\dot{q}_1)\right) + \left[\begin{array}{cc} d_2(q_1) & d_3(q_1) \end{array}\right] K_e q_2 - \frac{d_2(q_1)}{d_1(q_1)}\left[\begin{array}{cc} d_1(q_1) & d_2(q_1) \end{array}\right] K_e q_2 \\
&= J_2(q_1) + \frac{d_2(q_1)}{d_1(q_1)}(q_{l2} - J_1(q_1,\dot{q}_1)) + \left[\begin{array}{cc} d_2(q_1) - \frac{d_2(q_1)}{d_1(q_1)} d_1(q_1) & d_3(q_1) - \frac{d_2(q_1)}{d_1(q_1)} d_2(q_1) \end{array}\right] K_e q_2 \\
&= J_2(q_1) + \frac{d_2(q_1)}{d_1(q_1)}(q_{l2} - J_1(q_1,\dot{q}_1)) + \left[\begin{array}{cc} 0 & d_3(q_1) - \frac{d_2^2(q_1)}{d_1(q_1)} \end{array}\right] \left[\begin{array}{cc} K_1 & 0 \\ 0 & K_2 \end{array}\right] \left[\begin{array}{c} q_{m1} \\ q_{m2} \end{array}\right] \\
&= J_2(q_1) + \frac{d_2(q_1)}{d_1(q_1)}(q_{l2} - J_1(q_1,\dot{q}_1)) + \left[\begin{array}{cc} 0 & d_3(q_1) - \frac{d_2^2(q_1)}{d_1(q_1)} \end{array}\right] \left[\begin{array}{c} K_1 q_{m1} \\ K_2 q_{m2} \end{array}\right] \\
&= J_2(q_1) + \frac{d_2(q_1)}{d_1(q_1)}(q_{l2} - J_1(q_1,\dot{q}_1)) + \left(d_3(q_1) - \frac{d_2^2(q_1)}{d_1(q_1)}\right) K_2\, q_{m2}
\end{aligned} \tag{27}$$

we proceed with the $\ddot{q}_2 = \left[\begin{array}{cc} \ddot{q}_{m1} & \ddot{q}_{m2} \end{array}\right]^T$ i.e. the dynamics for the actuators.

$$\begin{aligned}
\ddot{q}_2 &= \left[\begin{array}{c} \ddot{q}_{m1} \\ \ddot{q}_{m2} \end{array}\right] = \left[\begin{array}{c} u_1 \\ u_2 \end{array}\right] = \left[\begin{array}{c} v \\ u_2 \end{array}\right] \\
&= \left[\begin{array}{c} v \\ \frac{1}{d_1(q_1)m_{12}}[J_1(q_1,\dot{q}_1) - q_{l2} + \left[\begin{array}{cc} d_1(q_1) & d_2(q_1) \end{array}\right] K_e q_2] \end{array}\right] \\
&= \left[\begin{array}{c} v \\ \frac{1}{d_1(q_1)m_{12}}[J_1(q_1,\dot{q}_1) - q_{l2} + \left[\begin{array}{cc} d_1(q_1) & d_2(q_1) \end{array}\right] \left[\begin{array}{cc} K_1 & 0 \\ 0 & K_2 \end{array}\right] \left[\begin{array}{c} q_1 \\ q_2 \end{array}\right] \end{array}\right] \\
&= \left[\begin{array}{c} v \\ \frac{1}{d_1(q_1)m_{12}}(J_1(q_1,\dot{q}_1) - q_{l2} + d_2(q_1)K_2 q_{m2}) + \frac{K_1}{m_{12}} q_{m1} \end{array}\right] \\
&= \left[\begin{array}{c} v \\ \frac{1}{d_1(q_1)m_{12}}(J_1(q_1,\dot{q}_1) - q_{l2} + d_2(q_1)K_2 q_{m2}) + \frac{K_1}{m_{12}} q_{m1} \end{array}\right] \\
&= \left[\begin{array}{c} v \\ \frac{1}{d_1(q_1)m_{12}}(J_1(q_1,\dot{q}_1) - q_{l2} + d_2(q_1)K_2 q_{m2}) + \frac{K_1}{m_{12}} q_{m1} \end{array}\right]
\end{aligned} \tag{28}$$

Equations (26), (27) and (28) are combined into the equation (29) which is just a compact representation into one matrix as follows (equation (29) will be simplified after substituting data to get (33) :

$$
\ddot{q} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} \ddot{q}_{l1} \\ \ddot{q}_{l2} \\ \ddot{q}_{m1} \\ \ddot{q}_{m2} \end{bmatrix} = \begin{bmatrix} q_{l2} \\ J_2(q_1) + \frac{d_2(q_1)}{d_1(q_1)}(q_{l2} - J_1(q_1,\dot{q}_1)) + \left(d_3(q_1) - \frac{d_2^2(q_1)}{d_1(q_1)}\right) + K_2\, q_{m2} \\ v \\ \frac{1}{d_1(q_1)m_{12}}(J_1(q_1,\dot{q}_1) - q_{l2} + d_2(q_1)K_2 q_{m2}) + \frac{K_1}{m_{12}} q_{m1} \end{bmatrix}
$$
$$
= \begin{bmatrix} q_{l2} \\ \rho_1(q_1,\dot{q}_1) + \eta_1(q_1)q_{m2} \\ v \\ \rho_2(q_1,\dot{q}_1,q_{m2}) + \eta_2 q_{m1} \end{bmatrix} \tag{29}
$$

where

$$
\begin{aligned}
\rho_1(q_1,\dot{q}_1) &= J_2(q_1) + \frac{d_2(q_1)}{d_1(q_1)}(q_{l2} - J_1(q_1,\dot{q}_1)) \\
\eta_1(q_1) &= \left(d_3(q_1) - \frac{d_2^2(q_1)}{d_1(q_1)}\right) K_2 \\
\rho_2(q_1,\dot{q}_1,q_{m2}) &= \frac{1}{d_1(q_1)m_{12}}(J_1(q_1,\dot{q}_1) - q_{l2} + d_2(q_1)K_2 q_{m2}) \\
\eta_{2m1} &= \frac{K_1}{m_{12}}
\end{aligned}
$$

The τ that solves the nonregular static state feedback triangulation problem is given as follows:

$$
\begin{aligned}
\tau = &- M_2^T M_1^{-1}(q_1)[C_1(q_1,\dot{q}_1)\dot{q}_1 + K_e(q_1 - q_2) + G_g(q_1) + D_1(q_1,\dot{q}_1)] \\
&+ K_e(q_2 - q_1) + D_2(q_2,\dot{q}_2) \\
&+ (M_3 - M_2^T M_1^{-1}(q)M_2)\begin{bmatrix} 0 \\ u_2 \end{bmatrix} + M_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} v
\end{aligned} \tag{30}
$$

3.3 Case study

To the symbolic equations we add specificity by assigining values for now in term of $p_1, p_2, \ldots, p_7$ coefficients and K_1, K_2 for spring constants. Then in what follows next, pre-computing blocks to write (29) (30) for conducting simulations.

$$
\begin{aligned}
M_1(q_1) &= \begin{bmatrix} p_1 + p_2 cos^2(q_{l2}) & 0 \\ 0 & 0 \end{bmatrix} \\
M_2 &= \begin{bmatrix} 0 & P_a \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} P_5 & 0 \\ 0 & p_6 \end{bmatrix} \\
C_1(q_1, \dot{q}_1) &= \begin{bmatrix} -0.5 p_2 \dot{q}_{l2} sin 2 q_{l2} & -0.5 P_2 q_{l1} sin 2 q_{l2} \\ 0.5 P_2 \dot{q}_{l1} sin 2 q_{l2} & 0 \end{bmatrix} \\
G_g &= \begin{bmatrix} 0 \\ p_7 cos q_{l2} \end{bmatrix} \\
M_2^T M_1^{-1} &= \begin{bmatrix} 0 & 0 \\ p_4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{p_1 + p_2 cos^2 q_{l1}} & 0 \\ 0 & \frac{1}{p_3} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ \frac{p_4}{p_1 + p_2 cos^2 q_{l1}} & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M_2^T M_1^{-1} M_2 &= \begin{bmatrix} 0 & 0 \\ \frac{p_4}{p_1 + p_2 cos^2 q_{l1}} & 0 \end{bmatrix} \begin{bmatrix} 0 & p_4 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \frac{p_4^2}{p_1 + p_2 cos^2 q_{l1}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
C_1 \dot{q}_1 + K_e q_1 + G_g + D &= \begin{bmatrix} -0.5 p_2 \dot{q}_{l2} \dot{q}_{l1} sin 2(q_{12}) - 0.5 p_2 \dot{q}_{l1} \dot{q}_{l2} sin 2(q_{12}) \\ 0.5 p_2 \dot{q}_{l1}^2 sin 2(q_{12}) \end{bmatrix} + \begin{bmatrix} K_1 q_{l1} \\ K_2 q_{l2} \end{bmatrix} + \begin{bmatrix} 0 \\ p_7 cos q_{l2} \end{bmatrix} \\
&= \begin{bmatrix} -p_2 \dot{q}_{l2} \dot{q}_{l1} sin 2(q_{12}) + K_1 q_{l1} \\ 0.5 p_2 \dot{q}_{l1}^2 sin 2(q_{12}) + K_2 q_{l2} + p_7 cos q_{l2} \end{bmatrix}
\end{aligned}
$$

$$
M_1^{-1}(q_1) z = \begin{bmatrix} d_1(q_1) & d_2(q_1) \\ d_2(q_1) & d_3(q_1) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{p_1 + p_2 cos^2 q_{l2}} & 0 \\ 0 & \frac{1}{p_3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = - \begin{bmatrix} \frac{z_2}{p_1 + p_2 cos^2 q_{l2}} \\ \frac{z_2}{p_3} \end{bmatrix}
$$

$$
\begin{aligned}
J = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} &= -M_1^{-1} [C_1 \dot{q}_1 + K_e q_1 G_{g1} + D_1] \\
&= \begin{bmatrix} \frac{1}{p_1 + p_2 cos^2 q_{l2}} & 0 \\ 0 & \frac{1}{p_3} \end{bmatrix} \begin{bmatrix} -p_2 \dot{q}_{l2} \dot{q}_{l1} sin 2(q_{12}) + K_1 q_{l1} \\ 0.5 p_2 \dot{q}_{l1}^2 sin 2(q_{12}) + K_2 q_{l2} + p_7 cos q_{l2} \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{1}{p_1 + p_2 cos^2 q_{l2}} \right) - p_2 \dot{q}_{l2} \dot{q}_{l1} sin 2(q_{12}) + K_1 q_{l1} \\ \left(\frac{1}{p_3} \right) 0.5 p_2 \dot{q}_{l1}^2 sin 2(q_{12}) + K_2 q_{l2} + p_7 cos q_{l2} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
u_2 = \alpha_2 &= \frac{1}{d_1 m_{12}}\left(J_1 - q_{l2} + \left[\begin{array}{cc} d_1 & d_2 \end{array}\right]\left[\begin{array}{cc} K_1 & 0 \\ 0 & K_2 \end{array}\right]\left[\begin{array}{c} q_{m1} \\ q_{m2} \end{array}\right]\right) \\
&= \frac{1}{d_1 m_{12}}(J_1 - q_{l2} + d_1 K_1 q_{m1} + \cancelto{0}{d_2} K_2 q_{m2}) \\
&= \frac{p_1 + p_2 cos^2 q_{l2}}{p_4}\left(\frac{p_2 \dot{q}_{l1}\dot{q}_{l2} sin2q_{l2} - K_1 q_{l1}}{p_1 + p_2 cos^2 ql2} - q_{l2} + \frac{K_1 q_{m2}}{P_1 + p_2 cos^2 q_{l2}}\right) \\
&= \frac{1}{p_4}(p_2 \dot{q}_{l1}\dot{q}_{l2} sin2q_{l2} - K_1 q_{21} - (p_z + p_2 cos^2(q_{l2}) q_{l2}))
\end{aligned}
$$

$$
\begin{aligned}
(M_3 - M_2^T M_1^{-1} M_2)\left[\begin{array}{c} 0 \\ \alpha_2 \end{array}\right] &= \left[\begin{array}{cc} p_5 & 0 \\ 0 & p_6 - \frac{p_4^2}{P_1 + P_2 cos^2 q_{l2}} \end{array}\right]\left[\begin{array}{c} 0 \\ \alpha_2 \end{array}\right] \\
&= \left[\begin{array}{c} 0 \\ \left(p_6 - \frac{p_4^2}{P_1 + P_2 cos^2 q_{l2}}\right)\alpha_2 \end{array}\right] \\
&= \left[\begin{array}{c} 0 \\ \left(\frac{p_6(P_1 + P_2 cos^2 q_{l2}) - p_4^2}{P_1 + P_2 cos^2 q_{l2}}\right)\alpha_2 \end{array}\right]
\end{aligned}
$$

$$
M_3\left[\begin{array}{c} 1 \\ 0 \end{array}\right] v = \left[\begin{array}{cc} p_5 & 0 \\ 0 & p_6 \end{array}\right]\left[\begin{array}{c} 1 \\ 0 \end{array}\right] v = \left[\begin{array}{c} p_5 \\ 0 \end{array}\right] v = \left[\begin{array}{c} p_5 v \\ 0 \end{array}\right]
$$

$$
K_e(q_2 - q_1) = \left[\begin{array}{cc} K_1 & 0 \\ 0 & K_2 \end{array}\right]\left[\begin{array}{c} q_{m1} - q_{l1} \\ q_{m2} - q_{l2} \end{array}\right] = \left[\begin{array}{c} K_1(q_{m1} - q_{l1}) \\ K_2(q_{m2} - q_{l2}) \end{array}\right]
$$

$$
K_e(q_1 - q_2) = \left[\begin{array}{cc} K_1 & 0 \\ 0 & K_2 \end{array}\right]\left[\begin{array}{c} q_{l1} - q_{m1} \\ q_{l2} - q_{m2} \end{array}\right] = \left[\begin{array}{c} K_1(q_{l1} - q_{m1}) \\ K_2(q_{l2} - q_{m2}) \end{array}\right]
$$

$$
C_1\dot{q}_1 + K_e(q_1 - q_2) + G_{g1} + D_1 = \left[\begin{array}{c} -p_2 \dot{q}_{l2}\dot{q}_{l1} sin2(q_{12}) + K_1(q_{l1} - q_{m1}) \\ 0.5 p_2 \dot{q}_{l1}^2 sin2(q_{12}) + K_2(q_{l2} - q_{m2}) + p_7 cos q_{l2} \end{array}\right]
$$

$$
-M_2^T M_1^{-1} z = \left[\begin{array}{c} 0 \\ \frac{p_4}{p_1 + p_2 cos^2 q_{l1}} z_1 \end{array}\right]
$$

$$
-M_2^T M_1^{-1}(C_1\dot{q}_1 + K_e(q_1 - q_2) + G_{g1} + D_1) = \left[\begin{array}{c} 0 \\ \frac{p_4}{p_1 + p_2 cos^2 q_{l1}}(p_2 \dot{q}_{l1}\dot{q}_{l2} sin2q_{l2} + K_1(q_{m1} - q_{l1})) \end{array}\right]
$$

Explicitly writing τ in (30) as follows:

$$
\tau = \left[\begin{array}{c} \tau_1 \\ \tau_2 \end{array}\right] \tag{31}
$$

with the following being the expressions for τ_1 and τ_2

$$\tau_1 = K_1(q_{m1} - q_{l1}) + p_5 v$$

$$\begin{aligned}
\tau_2 = \; & \frac{1}{p_1 + p_2 cos^2 q_{l2}} (p_2 p_4 \dot{q}_{l1} \dot{q}_{l2} sin 2q_{l2} - K_1 p_4 q_{l1} + K_1 p_4 q_{m1}) \\
& + K_2 q_{m2} - K_2 q_{l2} \\
& + \frac{p_6(p_1 + p_2 cos^2 q_{l2}) - p_4^2}{p_1 + p_2 cos^2 q_{l2}} \frac{1}{p_4} ((p_2 \dot{q}_{l1} \dot{q}_{l2} sin 2q_{l2} - K_1 q_{l1} + K_1 q_{m1}) \\
& - (p_1 + p_2 cos^2 q_{l2}) q_{l2})
\end{aligned}$$

$$\begin{aligned}
= \; & K_2 q_{m2} - K_2 q_{l2} \\
& + \frac{1}{p_1 + p_2 cos^2 q_{l2}} (p_2 p_4 \dot{q}_{l1} \dot{q}_{l2} sin 2q_{l2} - K_1 p_4 q_{l1} + K_1 p_4 q_{m1}) \\
& + \left(\frac{p_6 \cancel{(p_1 + p_2 cos^2 q_{l2})}}{p_4 \cancel{(p_1 + p_2 cos^2 q_{l2})}} - \frac{p_4^{\cancel{2}}}{\cancel{p_4} (p_1 + p_2 cos^2 q_{l2})} \right) ((p_2 \dot{q}_{l1} \dot{q}_{l2} sin 2q_{l2} - K_1 q_{l1} + K_1 q_{m1}) \\
& - (p_1 + p_2 cos^2 q_{l2}) q_{l2})
\end{aligned} \tag{32}$$

$$\begin{aligned}
= \; & K_2 q_{m2} - K_2 q_{l2} + p_4 q_{l2} \\
& + \cancel{\frac{1}{p_1 + p_2 cos^2 q_{l2}} (p_2 p_4 \dot{q}_{l1} \dot{q}_{l2} sin 2q_{l2} - K_1 p_4 q_{l1} + K_1 p_4 q_{m1})} \\
& - \cancel{\frac{1}{p_1 + p_2 cos^2 q_{l2}} (p_2 p_4 \dot{q}_{l1} \dot{q}_{l2} sin 2q_{l2} - K_1 p_4 q_{l1} + K_1 p_4 q_{m1})} \\
& + \left(\frac{p_6}{p_4} \right) \big((p_2 \dot{q}_{l1} \dot{q}_{l2} sin 2q_{l2} - K_1 q_{l1} + K_1 q_{m1}) - (p_1 + p_2 cos^2 q_{l2}) q_{l2} \big)
\end{aligned}$$

$$\begin{aligned}
= \; & K_2 q_{m2} - K_2 q_{l2} + p_4 q_{l2} \\
& + \frac{1}{p_4} \big(p_2 p_6 \dot{q}_{l1} \dot{q}_{l2} sin 2q_{l2} - K_1 p_6 q_{l1} + K_1 p_6 q_{m1} - p_1 p_6 q_{l2} - p_2 p_6 q_{l2} cos^2 q_{12} \big)
\end{aligned}$$

$$\begin{aligned}
= \; & \frac{1}{p_4} \big(p_2 p_6 q_{l2} cos^2 q_{l2} - p_2 p_6 \dot{q}_{l1} \dot{q}_{l2} sin 2q_{l2} + K_2 p_4 q_{l2} - p_1 p_6 q_{l2} - K_2 p_4 q_{m2} \\
& + K_1 p_6 q_{l1} - K_1 p_6 q_{m1} - p_4^2 q_{l2} \big)
\end{aligned}$$

Starting with the equation (29) and the computed blocks above we start its simplification as follow:

$$
\begin{aligned}
\ddot{q}_{l2} &= J_2(q,\dot{q}_1) + \frac{d_2(q_1)}{d_1(q_1)}(q_{l2} - J_1(q,\dot{q}_1) + \left(d_3(q_1) - \frac{d_2^2(q_1)}{d_1(q_1)}\right) K_2 q_{m2} \\
&= J_2(q,\dot{q}_1) + \frac{\cancelto{0}{d_2(q_1)}}{d_1(q_1)}(q_{l2} - J_1(q,\dot{q}_1) + \left(\cancelto{\frac{1}{p_3}}{d_3}(q_1) - \cancelto{0}{\frac{d_2^2(q_1)}{d_1(q_1)}}\right) K_2 q_{m2} \\
&= J_2(q,\dot{q}_1) + \frac{1}{p_3} K_2 q_{m2} \\
&= \left(\frac{1}{p_3}(-0.5 p_2 \dot{q}_{l1}^2 sin2q_{l2} - K_2 q_{l2} - p_7 cosq_{l2})\right) + \frac{1}{p_3} K_2 q_{m2} \\
&= \frac{K_2}{p_3} q_{m2} - \frac{1}{2p_3}\left(p_2 \dot{q}_{l1}^2 sin2q_{l2} + 2K_2 q_{l2} + 2p_7 cosq_{l2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{q}_{m2} &= \frac{1}{d_1(q_1)\cancelto{p_4}{m_{12}}}(J_1(q_1,\dot{q}_1) - q_{l2} + \cancelto{0}{d_2}(q_1) K_2 q_{m2}) + \frac{K_1}{\cancelto{p_4}{m_{12}} q_{m1}} \\
&= \frac{1}{d_1(q_1) p_4}(J_1(q_1,\dot{q}_1) - q_{l2}) + \frac{K_1}{p_4 q_{m1}} \\
&\text{inserting values of } d_1 = \frac{1}{p_1 + p_2 cos^2 q_{l2}} \text{ and } J_1 = \frac{1}{p_1 + p_2 cos^2 q_{l2}}(p_2 \dot{q}_{l1}\dot{q}_{l2} sin2q_{l2} - K_1 q_{l1}) \\
&= \frac{1}{p_4}(p_2 \dot{q}_{l1}\dot{q}_{l2} sin2q_{l2} - K_1 q_{l1}) - \frac{1}{p_4}(p_1 + p_2 cos^2 q_{l2}) q_{l2}) + \frac{K_1}{p_4} q_{m1} \\
&= \frac{K_1}{p_4} q_{m1} - \frac{1}{p_4}\left(-p_2 \dot{q}_{l1}\dot{q}_{l2} sin2q_{l2} + K_1 q_{l1} + p_1 q_{l2} + p_2 cos^2 q_{l2} q_{l2}\right)
\end{aligned}
$$

Thus finally obtaining (33)

$$
\begin{aligned}
\ddot{q} &= \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} \\
&= \begin{bmatrix} \ddot{q}_{l1} \\ \ddot{q}_{l2} \\ \ddot{q}_{m1} \\ \ddot{q}_{m2} \end{bmatrix} = \begin{bmatrix} q_{l2} \\ \frac{K_2}{p_3} q_{m2} - \frac{1}{2p_3}\left(p_2 \dot{q}_{l1}^2 sin2q_{l2} + 2K_2 q_{l2} + 2p_7 cosq_{l2}\right) \\ v \\ \frac{K_1}{p_4} q_{m1} - \frac{1}{p_4}\left(-p_2 \dot{q}_{l1}\dot{q}_{l2} sin2q_{l2} + K_1 q_{l1} + p_1 q_{l2} + p_2 cos^2 q_{l2} q_{l2}\right) \end{bmatrix}
\end{aligned} \tag{33}
$$

The equilibrium point can be found by setting $\ddot{q} = [0,0,0,0]^T$ thus we have

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} q_{l2} \\ \frac{K_2}{p_3} q_{m2} - \frac{1}{2p_3}\left(p_2 \dot{q}_{l1}^2 sin2q_{l2} + 2K_2 q_{l2} + 2p_7 cosq_{l2}\right) \\ v \\ \frac{K_1}{p_4} q_{m1} - \frac{1}{p_4}\left(-p_2 \dot{q}_{l1}\dot{q}_{l2} sin2q_{l2} + K_1 q_{l1} + p_1 q_{l2} + p_2 cos^2 q_{l2} q_{l2}\right) \end{bmatrix}
$$

we immediately see $q_{l2} = 0$ using which

$$\frac{K_2}{p_3}q_{m2} - \frac{1}{2p_3}\left(p_2\dot{q}_{l1}^2 sin2\cancelto{0}{q_{l2}} + 2K_2\cancelto{0}{q_{l2}} + 2p_7cos\cancelto{0}{q_{l2}}\right) = 0$$
$$\frac{K_2}{p_3}q_{m2} - \frac{p_7}{p_3} = 0$$
$$\therefore \quad q_{m2} = \frac{p_7}{k_2}$$

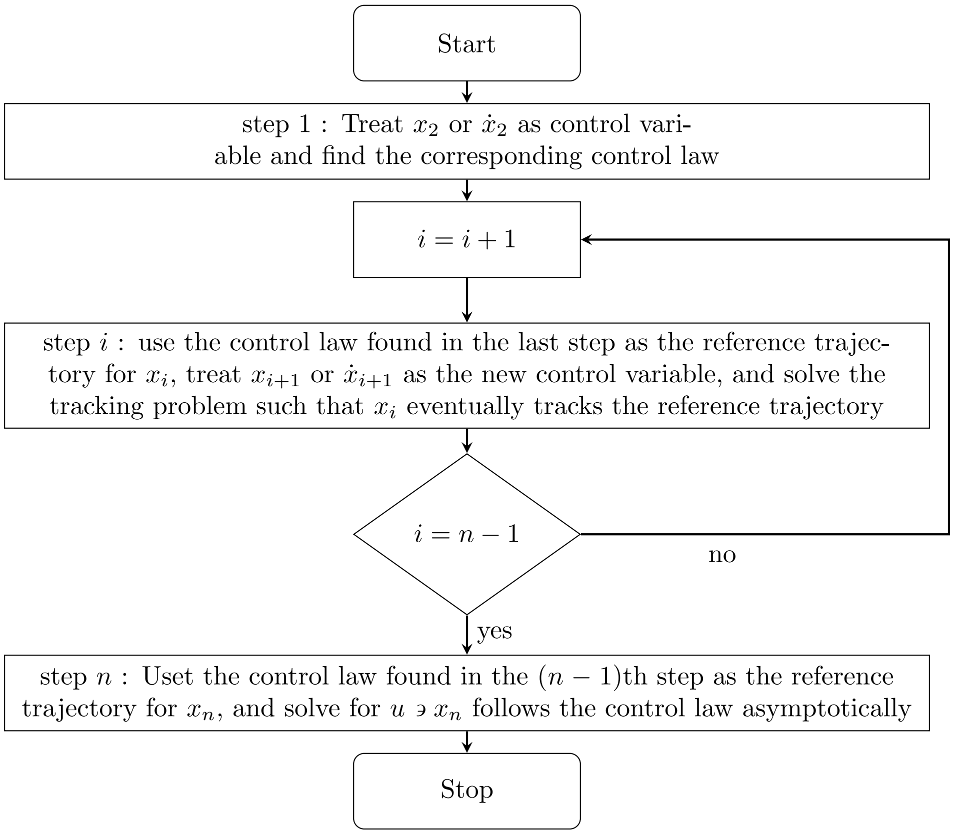
$$\frac{K_1}{p_4}q_{m1} - \frac{1}{p_4}\left(-p_2\dot{q}_{l1}\dot{q}_{l2}sin2\cancelto{0}{q_{l2}} + K_1q_{l1} + p_1\cancelto{0}{q_{l2}} + p_2cos^2q_{l2}\cancelto{0}{q_{l2}}\right) = 0$$
$$K_1q_{m1} - K_1q_{l1} = 0$$
$$\therefore q_{m1} = q_{l1}$$

note that $q_{m1} = q_{l1}$ is only at zero state because base frame is angle free

4 Backstepping Control - simulations and results

4.1 Overview of backstepping

The general way to go about applying backstepping to a nonlinear system control can be to input-output feedback linearize systems and instead of designing the control objective directly, we tend to first stabilize the residual dynamics by picking the transformed input control through lyapunov that then while stabilizing the system stabilizes the linearized dynamics to zero. Backstepping is shown below in terms of an easy to follow flow chart:



4.2 Applying backstepping

As shown in the flow chart above, we will follow the backstepping technique on (33) for the system to converge to track the desired input. The equations first are transformed from second order to frist order chain of integrators

$$
\begin{cases}
x_1 = q_{l1} \\
x_2 = \dot{q_{l1}} \\
x_3 = q_{l2} \\
x_4 = \dot{q_{l2}} \\
x_5 = q_{m2} \\
x_6 = \dot{q_{m2}} \\
x_7 = q_{m1} \\
x_8 = \dot{q_{m1}}
\end{cases}
\implies
\begin{cases}
\dot{x_1} = x_2 \\
\dot{x_2} = x_3 \\
\dot{x_3} = x_4 \\
\dot{x_4} = \dfrac{K_2}{p_3} q_{m2} - \dfrac{1}{2p_3}\left(p_2 \dot{q}_{l1}^2 sin2q_{l2} + 2K_2 q_{l2} + 2p_7 cos q_{l2}\right) \\
\dot{x_5} = x_6 \\
\dot{x_6} = \dfrac{K_1}{p_4} q_{m1} - \dfrac{1}{p_4}\left(-p_2 \dot{q}_{l1}\dot{q}_{l2} sin2q_{l2} + K_1 q_{l1} + p_1 q_{l2} + p_2 cos^2 q_{l2} q_{l2}\right) \\
\dot{x_7} = x_2 \\
\dot{x_8} = v
\end{cases}
\tag{34}
$$

we choose simple quadratic lyapunov functions for the purpose of choosing successive inputs. For better performance different lyapunov functions can be chosen. We keep lyapunov functions the system equations side by side, at places for better viewing. Auxiliary variables are given as e_i where $i = 2, \ldots, 8$

$$\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = x_4$$

$$
\begin{array}{rl|l}
x_2 = & -x_1 + e_2 \rightarrow e_2 = x_1 + x_2 & V_1 = \frac{1}{2}x_1^2 \\
\therefore \dot{x}_1 = & -x_1 + e_2 & \dot{V}_1 = x_1\dot{x}_1 = -x_1^2 + x_1 e_2 \\
\dot{e}_2 = & \dot{x}_1 + \dot{x}_2 & V_2 = \frac{1}{2}(x_1^2 + e_2^2) \\
\therefore \dot{e_2} = & -x_1 - e_2 + e_3 & \dot{V}_2 = x_1\dot{x}_1 + e_2\dot{e}_2 \\
x_3 = & -2e_2 + e_3 & \dot{V}_2 = -x_1^2 + \cancel{x_1 e_2} - \cancel{x_1 e_2} + e_2^2 + x_3 e_2 \\
\therefore \dot{e}_2 = & -x_1 - e_2 + e_3 & \dot{V}_2 = -x_1^2 - e_2^2 + e_2 e_3 \\
e_3 = & x_3 + 2e_2 & V_3 = \frac{1}{2}(x_1^2 + e_2^2 + e_3^2) \\
\dot{e}_3 = & \dot{x}_3 + 2\dot{e}_2 & \dot{V}_3 = x_1 x_1 + e_2\dot{e}_2 + e_3\dot{e}_3 \\
\therefore \dot{e}_3 = & x_4 - 2x_1 - 2e_2 + 2e_3 & \dot{V}_3 = x_1^2 - e_2^2 + e_2 e_3 + x_4 e_3 - 2x_1 e_3 - 2e_2 e_3 + 2e_3^2 \\
x_4 = & e_2 + 2x_1 - 3e_3 + e_4 & \dot{V}_3 = -x_1^2 - e_2^2 + \cancel{e_2 e_3} - \cancel{e_2 e_3} - e_3^2 + e_3 e_4 \\
\therefore \dot{e}_3 = & -e_2 - e_3 + e_4 & V_4 = \frac{1}{2}(x_1^2 + e_2^2 + e_3^2 + e_4^2) \\
e_4 = & x_4 + 3e_3 - e_2 - 2x_1 & \dot{V}_4 = x_1\dot{x}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 \\
\dot{e_4} = & \dot{x}_4 + 3\dot{e}_3 - \dot{e}_2 - 2\dot{x}_1 &
\end{array}
$$

$$
\begin{aligned}
\dot{x}_4 &= \frac{K_2}{p_3}x_5 - \frac{p_2}{2p_3}x_2^2 sin2x_3 - \frac{2K_2}{2p_3}x_3 - \frac{2K_2}{2p_3}cosx_3\\
\dot{x}_5 &= x_6\\
\dot{x}_6 &= \frac{K_1}{p_4}x_7 + \frac{p_2}{p_4}x_2x_4 sin2x_3 - \frac{K_1}{p_4}x_1 - \frac{p_1}{p_4}x_3 - \frac{p_2}{p_4}x_3cos^2x_3\\
\dot{x}_7 &= x_8\\
\dot{x}_8 &= v\\
\dot{e}_4 &= (-3e_2 - 3e_3 + 3e_4) + (x_1 + e_2 - e_3) + (2x_1 - 2e_2) +\\
&\quad \left(\frac{K_2}{p_3}x_5 - \frac{p_2}{2p_3}(-x_1+e_2)^2 sin(-4e_2+2e_3) - \frac{K_2}{p_3}(-2e_2+e_3) - \frac{p_7}{p_3}cos(-2e_2+e_3)\right)\\
\dot{e}_4 &= \frac{K_2}{p_3}x_5 + 3e_4 - 4e_3 - \frac{K_2}{p_3}e_3 - 4e_2 + \frac{2K_2}{p_3} + 3x_1 - \frac{p_7}{p_3}cos(-2e_2+e_3) -\\
&\quad \frac{p_2}{2p_3}(-x_1+e_2)^2 sin(-4e_2+2e_3)\\
\dot{V}_4 &= (-x_1^2 - e_2^2 - e_3^2 + \cancel{e_3e_4}) - \overset{3}{\cancel{4}}e_3e_4 + \left(\frac{K_2}{p_3}x_5e_4 + 3e_4^2 - \frac{K_2}{p_7}e_3e_4 - 4e_2e_4 + \frac{2K_2}{p_3}e_2e_4\right.\\
&\quad \left. + 3x_1e_4 - \frac{p_7}{p_3}e_4cos(-2e_2+e_3) - \frac{p_2}{2p_3}e_4(-x_1+e_2)^2 sin(-4e_2+2e_3)\right)\\
x_5 &= \frac{p_3}{K_2}(e_5 - 4e_4 + (3 + \frac{K_2}{p_3})e_3 + (4 - \frac{2K_2}{p_3})e_2 - 3x_1 + \frac{p_7}{p_3}cos(-2e_2+e_3)\\
&+ \frac{p_2}{2p_3}(-x_1+e_2)^2 sin(-4e_2+2e_3))\\
\dot{V}_4 &= -x_1^2 - e_2^2 - e_3^2 - e_4^2 + e_4e_5\\
\dot{e}_4 &= \quad e_5 - 4e_4 + (3 + \frac{K_2}{p_3})e_3 + (4 - \frac{2K_2}{p_3})e_2 - 3x_1 + \frac{p_7}{p_3}cos(-2e_2+e_3)\\
&\quad + \frac{p_2}{2p_3}(-x_1+e_2)^2 sin(-4e_2+2e_3) +\\
&\quad\ 3e_4 - (4 + \frac{K_2}{p_3})e_3 - (4 - \frac{2K_2}{p_3})e_2 + 3x_1\\
&\quad - \frac{p_7}{p_3}cos(-2e_2+e_3) - \frac{p_2}{2p_3}(-x_1+e_2)^2 sin(-4e_2+2e_3)\\
\therefore \dot{e}_4 &= e_5 - e_4 - e_3\\
e_5 &= \frac{K_2}{p_3}x_5 + 4e_4 - (3 + \frac{K_2}{p_3})e_3 - (4 - \frac{2K_2}{p_3})e_2 + 3x_1 - \frac{p_7}{p_3}cos(-2e_2+e_3)\\
&\quad - \frac{p_2}{2p_3}(-x_1+e_2)^2 sin(-4e_2+2e_3)\\
\dot{e}_5 &= \frac{K_2}{p_3}\dot{x}_5 + 4\dot{e}_4 - (3 + \frac{K_2}{p_3})\dot{e}_3 - (4 - \frac{2K_2}{p_3})\dot{e}_2 + 3\dot{x}_1 + \frac{p_7}{p_3}sin(-2e_2+e_3)(-2\dot{e}_2+\dot{e}_3)\\
&- \frac{p_2}{2p_3}(-x_1+e_2)^2 cos(-4e_2+2e_3)(-4\dot{e}_2+2\dot{e}_3)\\
&- \frac{p_2}{p_3}(-x_1+e_2) sin(-4e_2+2e_3)(-\dot{x}_1+\dot{e}_2)
\end{aligned}
$$

$$
\begin{aligned}
\dot{e}_5 &= \frac{K_2}{p_3}x_6 + 4(e_5 - e_4 - e_3) - (3 + \frac{K_2}{p_3})(e_4 - e_3 - e_2) - (4 - \frac{2K_2}{p_3})(e_3 - e_2 - x_1) \\
&\quad + 3(e_2 - x_1) + \frac{p_7}{p_3}sin(-2e_{2e3})(-2e_3 + 2e_2 + 2x_1 + e_4 - e_3 - e_2) \\
&\quad - \frac{p_2}{2p_3}(-x_1 + e_2)^2 cos(-4e_2 + 2e_3)(-4e_3 + 4e_2 + 4x_1 + 2e_4 - 2e_3 - 2e_2) \\
&\quad - \frac{p_2}{p_2}(-x_1 + e_2)sin(-4e_2 + 2e_3)(-e_2 + x_1 + e_3 - e_2 - x_1) \\
\dot{e}_5 &= \frac{K_2}{p_3}x_6 + 4e_5 - (7 + \frac{K_2}{p_3})e_4 = (5 - \frac{3K_2}{p_3}) + (10 - \frac{K_2}{p_3})e_2 + (1 - \frac{2K_2}{p_3})x_1 \\
&\quad + \frac{p_7}{p_3}sin(-2e_2 + e_3)(e_4 - 3e_3 + e_2 + 2x_1) \\
&\quad - \frac{p_2}{2p_3}(-x_1 + e_2)^2(2e_4 - 6e_3 + 2e_2 + 4x_1)cos(-4e_2 + 2e_3) \\
&\quad - \frac{p_2}{p_3}(-x_1 + e_2)(e_3 - 2e_2)sin(-4e_2 + 2e_3) \\
V_5 &= \frac{1}{2}(x_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2) \\
\dot{V}_5 &= x_1\dot{x}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 \\
\dot{V}_5 &= -x_1^2 - e_2^2 - e_3^2 - e_4^2 + \cancel{e_4e_5} + e_5\left(\frac{K_2}{p_3}x_6 + 4e_5 - \left(\overset{6}{\cancel{7}} + \frac{K_2}{p_3}\right)e_4 + ...\right) \\
x_6 &= \frac{p_3}{K_2}(e_6 - 5e_5 + (6 + \frac{K_2}{p_3}e_4 + (5 - \frac{3K_2}{p_3})e_3 - (10 - \frac{K_2}{p_3})e_2 - (1 - \frac{2K_2}{p_3})x_1 \\
&\quad - \frac{p_7}{p_3}sin(-2e_2 + e_3)(e_4 - 3e_3 + e_2 + 2x_1) \\
&\quad + \frac{p_2}{2p_3}(-x_1 + e_2)^2(2e_4 - 6e_3 + 2e_2 + 4x_1)cos(-4e_2 + 2e_3) \\
&\quad + \frac{p_2}{p_3}(-x_1 + e_2)(e_3 - 2e_2)sin(-4e_2 + 2e_3)) \\
\therefore \dot{e}_5 &= e_6 - e_5 - e_4 \\
\dot{V}_5 &= x_1^2 - e_2^2 - e_3^2 - e_4^2 - e_5^2 - e_5e_6 \\
e_6 &= \frac{K_2}{p_3}x_6 + 5e_5 - (6 + \frac{K-2}{p_3})e_4 - (5 - \frac{3K_2}{p_3})e_3 + (10 - \frac{K_2}{p_3})e_2 + (1 - \frac{2K_2}{p_3})x_1 \\
&\quad + \frac{p_7}{p_3}sin(-2e_2 + e_3)(e_4 - 3e_3 + e_2 + 2x_1) - \frac{p_2}{p_3}(-x_1 + e_2)(e_3 - 2e_2)sin(-4e_2 + 2e_2) \\
&\quad - \frac{p_2}{\cancel{2}p_3}(-x_1 + e_2)^2(\cancel{e_4} - \overset{3}{\cancel{6}}e_3 + 2\cancel{2}e_2 + \overset{2}{\cancel{4}}x_1)cos(-4e_2 + 2e_3)
\end{aligned}
$$

$$
\begin{aligned}
\dot{e}_6 = &\frac{K_1K_2}{P_3P_4}x_7 + 5e_6 - (11 + \frac{K_2}{p_3})e_5 - (4 - \frac{4K_2}{p_3})e_4 + (21 - \frac{3K_2}{p_3} - \frac{p_1K_2}{p_3p_4})e_3 \\
&- (4 + \frac{4K_2}{p_3} + \frac{2p_1K_2}{p_3p_4})e_2 - (11 - \frac{3K_2}{p_3} + \frac{K_1K_2}{P_3p_4})x_1 + \Phi
\end{aligned}
$$

where Φ represents terms yet to be differentiated for ease of computation

$$
\begin{aligned}
\Phi = &\frac{K_3p_2 - p_4p_2}{p_3p_4}(e_2 - x_1)(e_4 - 3e_3 + e_2 + 2x_1)sin(2e_3 - 4e_2) \\
&- \frac{K_2p_3}{p_3p_4}(e_3 - 2e_2)cos^2(e_3 - 2e_2) + \frac{p_7}{p_3}(e_4 - 3e_3 + e_2 + 2x_1)^2 cos(e_3 - 2e_2) \\
&+ \frac{p_7}{p_3}(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)sin(e_3 - 2e_2) \\
&+ \frac{4p_2}{p_3}(e_2 - x_1)(e_3 - 2e_2)(e_4 - 3e_3 + e_2 + 2x_1)cos(2e_2 - 4e_2) \\
&- \frac{p_2}{p_3}(e_3 - 2e_2)^2 sin(2e_3 - 4e_3) \\
&+ \frac{2p_2}{p_3}(e_2 - x_1)^2(e_4 - 3e_3 + e_2 + 2x_1)^2 sin(2e_3 - 4e_2) \\
&- \frac{p_2}{p_3}(e_2 - x_1)^2(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)cos(2e_3 - 4e_2)
\end{aligned}
$$

$$
V_6 = \frac{1}{2}(x_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2)
$$

$$
\dot{V}_6 = x_1\dot{x}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 + e_6\dot{e}_6
$$

$$
\begin{aligned}
\dot{V}_6 = &-x_1^2 - e_2^2 - e_3^2 - e_4^2 - e_5^2 \cancel{+ e_5e_6} + e_6\left(\Phi + \frac{K_1K_2}{p_3p_4}x_7 + 5e_6 - \left(\overset{10}{\cancel{11}} + \frac{K_2}{p_3}\right)e_5\right. \\
&- \left(4 - \frac{4K_2}{p_3}\right)e_4 + \left(21 - \frac{3K_2}{p_3} - \frac{p_1K_2}{p_3p_4}\right)e_3 - \left(4 + \frac{4K_2}{p_3} - \frac{2p_1K_2}{p_3p_4}\right)e_2 \\
&\left. - \left(11 - \frac{3K_2}{p_3} + \frac{K_1K_2}{p_3p_4}\right)x_1\right)
\end{aligned}
$$

$$
\begin{aligned}
x_7 = &\frac{p_3p_4}{K_1K_2}\left(e_7 - 6e_6 + \left(10 + \frac{K_2}{p_3})e_5\right) + \left(4 - \frac{4K_2}{p_3}\right)e_4 - \left(21 - \frac{3K_2}{p_3} - \frac{p_1K_2}{P_3p_4}\right)e_3\right. \\
&\left. + \left(4 + \frac{4K_2}{p_3} - \frac{2P_1K_2}{P_3p_4}\right)e_2 + \left(11 - \frac{3K_2}{p_3} + \frac{K_1K_2}{p_3p_4}\right)x_1 - \Phi\right)
\end{aligned}
$$

$$
\dot{V}_6 = -x_1^2 - e_2^2 - e_3^2 - e_4^2 - e_5^2 - e_6^2 - e_6e_7
$$

$$
\therefore \dot{e}_6 = e_7 - e_6 - e_5
$$

$$
\begin{aligned}
e_7 = &\frac{K_2K_2}{p_3p_4}x_7 + 6e6 - \left(10 + \frac{K_2}{p_3}\right)e_5 - \left(4 - \frac{4K_2}{p_3}\right)e_4 + \left(21 - \frac{3K_2}{p_3} - \frac{p_1K_2}{p_3p_4}\right)e_3 \\
= &-\left(4 + \frac{4K_2}{p_3} - \frac{2p_1K_2}{p_3p_4}\right)e_2 - \left(11 - \frac{3K_2}{p_3} + \frac{K_1K_2}{p_3p_4}\right)x_1 + \Phi
\end{aligned}
$$

$$
\begin{aligned}
\dot{e}_7 = {}& \frac{K_1K_2}{p_3p_4}x_8 + 6(e_7 - e_6 - e_5) - \left(10 + \frac{k_2}{p_3}\right)(e_6 - e_5 - e_4) - \left(4 - \frac{4K_2}{p_3}\right)(e_5 - e_4 - e_3) \\
& + \left(21 - \frac{3K_2}{p_3} - \frac{p_2K - 2}{p_3p_4}\right)(e_4 - e_3 - e_2) - \left(4 + \frac{4K_2}{p_3} - \frac{2P_1K_2}{p_3p_4}\right)(e_3e_2 - x_1) \\
& - \left(11 - \frac{3K_2}{p_3} + \frac{K_1K_2}{p_3p_4}\right)(e_2 - x_1) + \frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
V_7 = {}& \frac{1}{2}(x_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2) \\
\dot{V}_7 = {}& x_1\dot{x}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 + e_6\dot{e}_6 + e_7\dot{e}_7 \\
\dot{V}_7 = {}& -x_1^2 - e_2^2 - e_3^2 - e_4^2 - e_5^2 - e_6^2 - \cancel{e_6^2e_7^2} + e_7\left(\frac{K_1K_2}{p_3p_4}x_8 + 6e_7 - \left(\overset{15}{\cancel{16}} + \frac{K_2}{p_3}\right)e_6 + ...\right) \\
x_8 = {}& \frac{p_3p_4}{K_1K_2}\left(e_8 - 7e_7 + \left(15 + \frac{K_2}{p_3}\right)e_6 - ...\right) \\
\dot{V}_7 = {}& -x_1^2 - e_2^2 - e_3^2 - e_4^2 - e_5^2 - e_6^2 - e_7^2 + e_7^2e_8^2 \\
\therefore \dot{x}_7 = {}& e_8 - e_7 - e_6 \\
e_8 = {}& \frac{K_1K_2}{p_3p_4}x_8 + 7e_7 - \left(15 + \frac{K_2}{p_3}\right)e_6 + \frac{5K_2}{p_3}e_5 + \left(35 - \frac{6K_2}{p_3} - \frac{p_1K_2}{p_3p_4}\right)e_4 \\
& - \left(21 + \frac{5K_2}{p_3} - \frac{3p_1K_2}{p_3p_4}\right)e_3 - (28 - \frac{10K_2}{p_3} + \frac{p_1K_2}{p_3p_4} + \frac{K_1K_2}{p_3p_4})e_2 \\
& + \left(15 + \frac{K_2}{p_3} - \frac{2p_1K_2}{p_3p_4} + \frac{K_1K_2}{p_3p_4}\right)x_1 + \frac{\mathrm{d}\Phi}{\mathrm{d}t} \\
\dot{e}_8 = {}& \frac{K_1K_2}{p_3p_4}v + 7(e_8 - e_7 - e_6) - \left(15 + \frac{K_2}{p_3}\right)(e_7 - e_6 - e_5) + \frac{5K_2}{p_3}(e_6 - e_5 - e_4) \\
& + \left(35 - \frac{6K_2}{p_3} - \frac{p_1K_2}{p_3p_4}(e_5 - e_4 - e_3)\right) - \left(21 + \frac{5K_2}{p_3} - \frac{3p_1K_2}{p_3p_4}\right)(e_4 - e_3 - e_2) \\
& - \left(28 - \frac{10K_2}{p_3} + \frac{p_1K_2}{p_3p_4} + \frac{K_1K_2}{p_3p - 4}\right)(e_3 - e_2 - x_1) + \\
& + \left(15 + \frac{K_2}{p_3} - \frac{2p_1K_2}{p_3p_4} + \frac{K_1K_2}{p_3p_4}\right)(e_2 - x_1) + \frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} \\
\dot{e}_8 = {}& \frac{K_1K_2}{p_3p_4}v + 7e_8 - \left(22 + \frac{K_2}{p_3}\right)e_7 + \left(8 + \frac{6K_2}{p_3}\right)e_6 \\
& + \left(50 - \frac{10K_2}{p_3} - \frac{p_1K_2}{p_3p_4}\right)e_5 - \left((56 + \frac{4K_2}{p_3} - \frac{4p_1K_2}{p_3p_4}\right)e_4 \\
& - \left(42 - \frac{21K_2}{p_3} + \frac{3p_1K_2}{p_3p_4} + \frac{K_1K_2}{p_3p_4}\right)e_3 + (64-) \\
& + \left(13 - \frac{11K_2}{p_3} + \frac{3p_1K_2}{p_3p_4}\right)x_1 + \frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} \\
V_8 = {}& \frac{1}{2}(x_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2) \\
\dot{V}_8 = {}& x_1\dot{x}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 + e_5\dot{e}_5 + e_6\dot{e}_6 + e_7\dot{e}_7 + e_8\dot{e}_8 \\
\dot{V}_8 = {}& -x_1^2 - e_2^2 - e_3^2 - e_4^2 - e_5^2 - e_6^2 - e_7^2 + \cancel{e_7e_8}\left(\frac{K_1K_2}{p_3p_4}v + 7e_8 - \left(\overset{21}{\cancel{22}} + \frac{K_2}{p_3}\right)e_7 + ...\right)
\end{aligned}
$$

$$
\begin{aligned}
v = \frac{p_3p_4}{K_1K_2}\Bigg(&-8e_8 + \left(21+\frac{K_2}{p_3}\right)e_7 - \left(8+\frac{6K_2}{p_3}\right)e_6 - \left(50-\frac{10K_2}{p_3}-\frac{P_1K_2}{p_3p_4}\right)e_5 \\
&+\left(56+\frac{4K_2}{p_3}-\frac{4p_1K_2}{p_3p_4}\right)e_4 + \left(42-\frac{21K_2}{p_3}+\frac{3P_1K_2}{p_3p_4}+\frac{K_1K_2}{p_3p_4}\right)e_3 \\
&-\left(64-\frac{4K_2}{p_3}-\frac{4p_1K_2}{p_3p_4}+\frac{2K_1K_2}{p_3p_4}\right)e_2 - \left(13-\frac{11K_2}{p_3}+\frac{3p_1K_2}{p_3p_4}\right)x_1 - \frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}\Bigg) \\
\therefore \dot{e}_8 = -e_8 - e_7
\end{aligned}
$$

Next computing $\frac{\mathrm{d}\Phi}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}$ to be put into the controller v expression.

$$
\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}t} = &\, 2\frac{K_2p_2-P_4P_2}{p_3p_4}(e_2-x_1)(e_4-3e_3+e_2+2x_1)^2 cos(2e_3-4e_2) \\
&+\frac{K_2p_2-P_4P_2}{p_3p_4}(e_2-x_1)(e_5-4e_4+3e_3+4e_2-3x_1)sin(2e_3-4e_2) \\
&+\frac{K_2p_2-P_4P_2}{p_3p_4}(e_3-2e_2)(e_4-3e_3+2x_1)sin(2e_{34}e_2) \\
&+\frac{K_2P_2}{p_3p_4}(e_3-2e_2)(e_4-3e_3+2x_1)sin(2e_{34}e_2) \\
&-\frac{K_2P_2}{p_3p_4}(e_3-2e_2)(e_4-3e_3+e_2+2x_1)sin(2e_3-4e_2) \\
&-\frac{p_7}{p_3}(e_4-3e_3+e_2+e_2+2x_1)^3 sin(e_3-2e_2) \\
&+2\frac{p_7}{p_3}(e_4-3e_3+e_{22}x_1)(e_5-4e_4+3e_3+4e_2-3x_1)cos(e_3-2e_2) \\
&+\frac{p_7}{p_3}(e_5-4e_4+3e_3+4e_3-3x_1)(e_4-3e_3+e_2+2x_1)cos(e_3-2e_2) \\
&+\frac{p_7}{p_3}(e_6-5e_5+6e_4+5e_3-10e_2-x_1)sin(e_3-2e_2) \\
&+\frac{8p_2}{p_3}(e_2-x_1)(e_3-2e_2)(e_5-4e_4+3e_3+4e_2-3x_1)cos(2e_3-4e_2) \\
&-\frac{4p_2}{p_3}(e_2-x_1)(e_3-2e_2)(e_5-4e_4+3e_3+4e_2-3x_1)cos(2e_3-4e_2) \\
&-\frac{4p_2p_4}{p_3p_4}(e_2-x_1)(e_4-3e_3+e_2+2x_1)^2 cos(2e_3-4e_2) \\
&-\frac{4p_2}{p_3}(e_3-2e_2)^2(e_4-3e_3+e_2+2x_1)cos(2e_3-4e_2)
\end{aligned}
$$

$$= -2\frac{p_2}{p_3}(e_3 - 2e_2)^2(e_4 - 3e_3 + e_2 + 2x_1)cos(2e_3 - 4e_2)$$
$$= -2\frac{p_2 p_4}{p_3 p_4}(e_3 - 2e_2)(e_4 - 3e_3 + e_2 + 2x_1)sin(2e_3 - 4e_2)$$
$$= +4\frac{p_2}{p_3}(e_2 - x_1)^2(e_4 - 3e_3 + e_2 + 2x_1)^3 cos(2e_3 - 4e_2)$$
$$= +4\frac{p_2}{p_3}(e_2 - x_1)^2(e_4 - 3e_2 + e_2 + 2x_1)(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)sin(2e_3 - 4e_2)$$
$$= +4\frac{p_2}{p_3}(e_2 - x_1)(e_3 - 2e_2)(e_4 - 3e_3 + e_2 - 2x_1)^2 sin(2e_3 - 4e_2)$$
$$= +2\frac{p_2}{p_3}(e_2 - x_1)^2(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)(e_4 - 3e_3 + e_2 + 2x_1)sin(2e_3 - 4e_2)$$
$$= -\frac{p_2}{p_3}(e_2 - x_2)^2(e_6 - 5e_5 + 6e_4 + 5e_3 - 10e_2 - x_1)cos(2e_3 - 4e_2)$$
$$= 2\frac{p_2}{p_3}(e_2 - x_1)(e_3 - 2e_2)(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)cos(2e_3 - 4e_2)$$

more compactly

$$\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}t} = &-\frac{K_2 p_3}{p_3 p_4}(e_4 - 3e_3 + e_2 + 2x_1)cos^(e_3 - 2e_2) \\
&- \frac{p_7}{p_3}(e_4 - 3e_3 + e_2 + 2x_1)^3 sin(e_3 - 2e_2) \\
&+ \frac{p_7}{p_3}(e_4 - 3e_3 + e_2 + 2x_1)^3 sin(e_3 - 2e_2) \\
&+ 3\frac{p_7}{p_3}(e_4 - 3e_3 + e_2 + 2x_1)(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)cos(e_3 - 2e_2) \\
&+ \frac{2K_2 p_2 - 3p_4 p_2}{p_3 p_4}(e_3 - 2e_2)(e_4 - 3e_3 + e_2 + 2x_1)sin(2e_3 - 4e_2) \\
&+ \frac{K_2 p_2 - p_4 p_2}{p_3 p_4}(e_2 - x_1)(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)sin(2e_3 - 4e_2) \\
&+ 12\frac{p_2}{p_3}(e_2 - x_1)(e_3 - 2e_2)(e_4 - 3e_3 + e_2 + 2x_1)^2 sin(2e_3 - 4e_2) \\
&+ 6\frac{p_2}{p_3}(e_2 - x_1)^2(e_4 - 3e_2 + e_2 + 2x_1)(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)sin(2e_3 - 4e_2) \\
&+ 4\frac{p_2}{p_3}(e_2 - x_1)^2(e_4 - 3e_3 + e_2 + 2x_1)^3 cos(2e3 - 4e_2) \\
&- \frac{p_2}{p_3}(e_2 - x_1)^2(e_6 - 5e_5 + 6e_4 + 5e_3 - 10e_2 - x_1)cos(2e_3 - 4e_2) \\
&+ \frac{2K_2 p_2 - 6p_4 p_2}{p_3 p_4}(e_2 - x_1)(e_4 - 3e_3 + e_{22}x_1)^2 cos(2e_3 - 4e_2) \\
&- 6\frac{p_2}{p_3}(e_2 - x_1)(e_3 - 2e_2)(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)cos(2e_3 - 4e_2) \\
&- 6\frac{p_2}{p_3}(e_2 - x_1)(e_3 - 2e_2)(e_5 - 4e_4 + 3e_3 + 4e_2 - 3x_1)cos(2e_3 - 4e_2) \\
&- 6\frac{p_2}{p_3}(e_3 - 2e_2)^2(e_4 - 3e_3 + e_2 + 2x_1)cos(2e_3 - 4e_2)
\end{aligned}$$

$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}$ is given as follow:

$$
\begin{aligned}
\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} = &-\frac{K_2p_2}{p_3p_4}(e_5-4e_4+3e_3+4e_2-3x_1)cos^2(e_3-2e_2)\\
&-\frac{p_7}{p_3}(e_4-3e_3+e_2x_1)^4cos(e_3-2e_2)\\
&+3\frac{p_7}{p_3}(e_5-4e_4+3e_3+4e_2-3x_1)^2cos(e_3-2e_2)\\
&+4\frac{p_7}{p_3}(e_4-3e_3+e_2+2x_1)(e_6-5e_5+6e_4+5e_3-10e_2-x_1)cos(e_3-2e_2)\\
&-6\frac{p_7}{p_3}(e_4-3e_3+e_2+2x_1)^2(e_5-4e_4+3e_3+4e_2-3x_1)sin(e_3-2e_2)\\
&+\frac{p_7}{p_3}(e_7-6e_6+10e_5+4e_4-21e_3+4e_2+11x_1)sin(e_3-2e_2)\\
&\frac{3K_2p_2-3p_4p_2}{p_3p_4}(e_4-3e_3+e_2+2x_1)^2sin(2e_3-4e_2)\\
&+\frac{3K_2p_2-4p_4p_2}{p_3p_4}(e_3-2e_2)(e_5-4e_4+3e_3+4e_2-3x_1)sin(2e_3-4e_2)\\
&+\frac{6K_2p_2-24p_4p_2}{p_3p_4}(e_3-2e_2)(e_4-3e_3+e_2+2x_1)^2cos(2e_3-4e_2)\\
&+\frac{K_2p_2-p_4p_2}{p_3p_4}(e_2-x_1)(e_6-5e_5+6e_4+5e_3-10e_2-x_1)sin(2e_3-4e_2)\\
&+\frac{48p_2}{p_3}(e_2-x_1)(e_3-2e_2)(e_4-3e_3+e+2+2x_1)(e_5-4e_4+3e_3+4e_2-3x_1)sin(2e_3+4e_2)\\
&-\frac{4K_2p_2-24p_2p_4}{p_3p_4}(e_2-x_1)(e_4-3e_3+e_2+2x_1)^3sin(2e_2-4e_2)\\
&+\frac{24p_2}{p_3}(e_3-2e_2)^2(e_4-3e_3+e_2+2x_1)^2sin(2e_3-4e_2)\\
&+\frac{8p_2}{p_3}(e_2-x_1)^2(e_4-3e_3+e_2+2x_1)(e_6-5e_5+6e_4+5e_3-10e_2-x_1)sin(2e_3-4e_2)\\
&+\frac{6p_2}{p_3}(e_2-x_1)^2(e_5-4e_4+3e_3+4e_2-3x_1)^2sin(2e_3-4e_2)\\
&-\frac{8p_2}{p_3}(e_2-x_1)^2(e_4-3e_3+e_2+2x_1)^4sin(2e_3-4e_2)\\
&+\frac{6K_2p_2-20p_4p_2}{p_3p_4}(e_2-x_1)(e_4-3e_3+e_2+2x_1)(e_5-4e_4+3e_3+4e_2-3x_1)(cos(2e_3-4e_2)\\
&+\frac{32p_2}{p_3}(e_2-x_1)(e_3-2e_2)(e_4-3e_3+e_2+2x_1)^3cos(2e_3-4e_2)\\
&+\frac{24p_2}{p_3}(e_2-x_1)^2(e_4-3e_3+e_2+2x_1)^2(e_5-4e_4+3e_3+4e_2-3x_1)cos(2e_3-4e_2)\\
&-\frac{p_2}{p_3}(e_2-x_1)^2(e_7-6e_6+10e_5+4e_4-21e_3+4e_2+11x_1)cos(2e_3-4e_2)\\
&-\frac{8p_2}{p_3}(e_2-x_1)(e_3-2e_2)(e_6-5e_5+6e_4+5e_3-10e_2-x_1)cos(2e_3-4e_2)\\
&-\frac{12p_2}{p_3}(e_3-2e_2)^2(e_5-4e_4+3e_3+4e_2-3x_1)cos(2e_3-4e_2)
\end{aligned}
$$

4.3 simulation

For simulation we substitute the data given below into the symbolic expressions

$$\begin{aligned} \left[\begin{array}{ccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{array}\right] &= \left[\begin{array}{ccccccc} 1.5 & 0.5 & 2 & 0.2 & 1 & 1 & 50 \end{array}\right] \\ \left[\begin{array}{cc} K_1 & K_2 \end{array}\right] &= \left[\begin{array}{cc} 100 & 200 \end{array}\right] \end{aligned} \tag{35}$$

Assuming the manipulator with the initial conditions $q(0) = [0, \pi/4, 0, \pi/8]^T$ and $\dot{q}(0) = [0, 0, 0, 0]^T$. It can be seen that the link positions converge to the equilibrium points at an exponential rate The simulink setup used is shown below

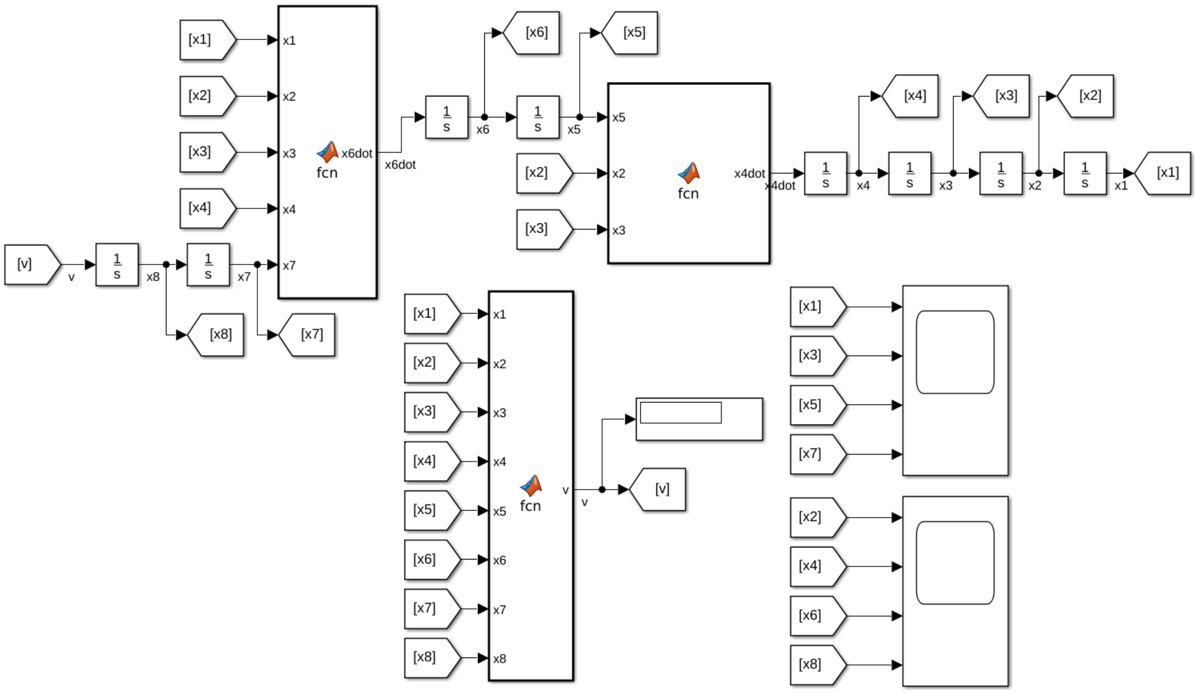


Figure 2: simulation architecture

referring to the figure 2, lets call the top left corner function block as block A, the top right as block B and the bottom as block C, then the respective code for the blocks can be found in the Appendix section. Running the simulation and checking the signals x1, x3, x5, x7 i.e. $[q_{l1}, q_{l2}, q_{m2}, q_{m1}]$ in upper part of figure 3 and x2, x4, x6, x8 i.e. $[\dot{q}_{l1}, \dot{q}_{l2}, \dot{q}_{m2}, \dot{q}_{m1}]$ in second part of figure 3 we see that all the signals converge asymptotically to zero.

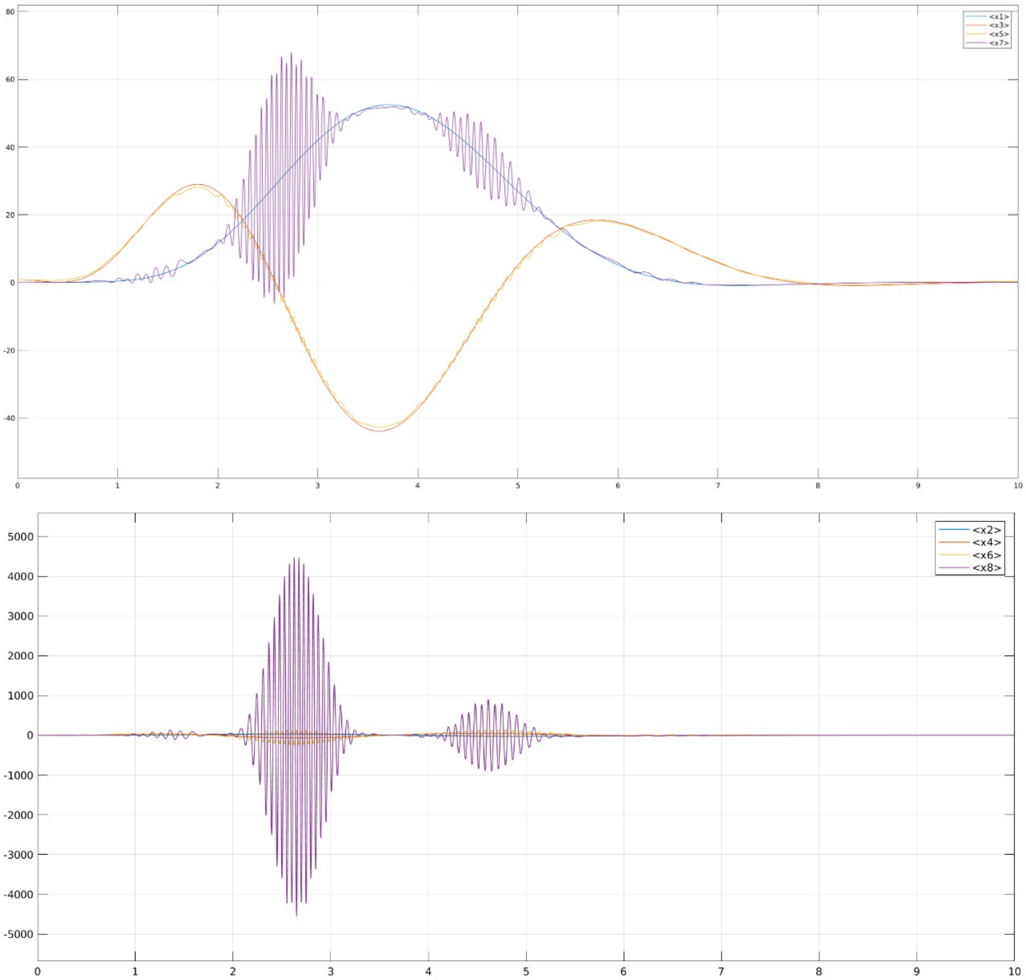


Figure 3: q_{l1}, q_{l2}, q_{m2}, q_{m1},$\dot{q}_{l1}$, $\dot{q}_{l2}$, $\dot{q}_{m2}$, $\dot{q}_{m1}$

5 Summary

The manipulator equations where adapted to include consideration for flexible joints for the case of a two link planar revolute joint manipulator. noticing the inertial coupling between the two links no regular static state feedback laws could be used to linearize the dynamics for the application of backstepping. Hence the technique of non-regular static state feedback triangulation was applied that allowed the application of backstapping haveing changed the highly coupled tree structured manipulator equations written in partial feedback form. Simulation was performed using matlab and the results are shown for the link positions to converge exponentially to the critical points and using backstepping for stabilizing reference trajectories.

6 Appendix

6.1 Matlab Code

6.1.1 Block A

```
function x6dot = fcn(x1,x2,x3,x4,x7)
```

Block A code

the interpretation of the code is given below

Variables

The function takes as input x1, x2, x3, x4, x7 which essentially are the q_{l1}, $\dot{q}_{l1}$, q_{l2}, $\dot{q}_{l2}$ and q_{m1}. These then are used to compute the expressions fo x6dot which is the expression $\ddot{q}_{m2}$

```
p1 = 1.5; p2 = 0.5; p3 = 2; p4 = 0.2; p5 = 1; p6 = 1; p7 = 50;
k1 = 100; k2 = 200;

x6dot = (k1/p4)*x7 - (1/p4)*(-p2*x2*x4*sin(2*x3) + ...
        k1*x1 + p1*x3 + p2*x3*cos(x3)^2);
```

6.1.2 Block B

see below

Block B code

the interpretation of the code is given below

Variables

The function takes as input x5, x2 and x3 which essentially are the q_{m2}, $\dot{q}_{l1}$, q_{l2}. While outputs the expressions for x4dot which is essentially $\ddot{q}_{l2}$

```
function x4dot = fcn(x5,x2,x3)

p1 = 1.5; p2 = 0.5; p3 = 2; p4 = 0.2; p5 = 1; p6 = 1; p7 = 50;
k1 = 100; k2 = 200;

x4dot = (k2/p3)*x5 - (1/(2*p3))*(p2*x2^2*sin(2*x3) +...
        2*k2*x3 + 2*p7*cos(x3));
```

6.1.3 Block C

see below

Block C code

the interpretation of the code is given below

Variables

The function takes as input x1, x2 and x3, x4, x5, x6, x7, x8 which is the complete set of inputs to the backstepping algorithm implemented below. We already know that the inputs to this matlab function, listed above are basically q_{l1}, $\dot{q}_{l1}$, q_{l2}, $\dot{q}_{l2}$, q_{m2}, $\dot{q}_{m2}$, q_{m1}, $\dot{q}_{m1}$ The algorithm outputs the expression for the controller v

```
function v = fcn(x1,x2,x3,x4,x5,x6,x7,x8)
p1 = 1.5; p2 = 0.5; p3 = 2; p4 = 0.2; p5 = 1; p6 = 1; p7 = 50;
k1 = 100; k2 = 200;

e2 = x2 + x1;
e3 = x3 + 2*e2;
e4 = x4 + 3*e3 - e2 - 2*x1;
e5 = (k2/p3)*x5 + 4*e4 - (3+(k2/p3))*e3 - (4-2*(k2/p3))*e2 + 3*x1 ...
     - (p7/p3)*cos(e3-2*e2) ...
     - (p2/(2*p3))*(e2-x1)^2*sin(2*(e3-2*e2));
e6 = (k2/p3)*x6 + 5*e5 - (6+(k2/p3))*e4 - (5-3*(k2/p3))*e3 ...
     + (10-(k2/p3))*e2 + (1-2*(k2/p3))*x1 ...
     + (p7/p3)*(e4-3*e3+e2+2*x1)*sin(e3-2*e2) ...
     - (p2/p3)*(e2-x1)*(e3-2*e2)*sin(2*(e3-2*e2)) ...
     - (p2/p3)*(e2-x1)^2*(e4-3*e3+e2+2*x1)*cos(2*(e3-2*e2));

phi = (k2*p2/(p3*p4))*(e2-x1)*(e4-3*e3+e2+2*x1)*sin(2*(e3-2*e2)) ...
     - (k2*p2/(p3*p4))*(e3-2*e2)*cos(e3-2*e2)^2 ...
     + (p7/p3)*(e4-3*e3+e2+2*x1)^2*cos(e3-2*e2) ...
     + (p7/p3)*(e5-4*e4+3*e3+4*e2-3*x1)*sin(e3-2*e2) ...
     - (2*p2/p3)*(e2-x1)*(e3-2*e2)*(e4-3*e3+e2+2*x1)*cos(2*(e3-2*e2)) ...
     - (p2/p3)*(e2-x1)*(e4-3*e3+e2+2*x1)*sin(2*(e3-2*e2)) ...
     - (p2/p3)*(e3-2*e2)^2*sin(2*(e3-2*e2)) ...
     + (2*p2/p3)*(e2-x1)^2*(e4-3*e3+e2+2*x1)^2*sin(2*(e3-2*e2)) ...
     - (p2/p3)*(e2-x1)^2*(e5-4*e4+3*e3+4*e2-3*x1)*cos(2*(e3-2*e2)) ...
     - (2*p2/p3)*(e2-x1)*(e3-2*e2)*(e4-3*e3+e2+2*x1)*cos(2*(e3-2*e2));

e7 = (k1*k2/(p3*p4))*x7 + 6*e6 - (10+(k2/p3))*e5 - (4-4*(k2/p3))*e4 ...
     + (21-3*(k2/p3)-(p1*k2/(p3*p4)))*e3 ...
     - (4+4*(k2/p3)-2*(p1*k2/(p3*p4)))*e2 ...
     - (11-3*(k2/p3)+(k1*k2/(p3*p4)))*x1 ...
     + phi;
```

```
phidot = -(k2*p2/(p3*p4))*(e4-3*e3+e2+2*x1)*cos(e3-2*e2)^2 ...
     - (p7/p3)*(e4-3*e3+e2+2*x1)^3*sin(e3-2*e2) ...
     + (p7/p3)*(e6-5*e5+6*e4+5*e3-10*e2-x1)*sin(e3-2*e2) ...
     + (3*p7/p3)*(e4-3*e3+e2+2*x1)*(e5-4*e4+3*e3+4*e2-3*x1)*...
     cos(e3-2*e2)...
     + ((2*k2*p2-3*p4*p2)/(p3*p4))*(e3-2*e2)*(e4-3*e3+e2+2*x1)*...
     sin(2*(e3-2*e2))...
     + ((k2*p2-p4*p2)/(p3*p4))*(e2-x1)*(e5-4*e4+3*e3+4*e2-3*x1)*...
     sin(2*(e3-2*e2)) ...
     + (12*p2/p3)*(e2-x1)*(e3-2*e2)*(e4-3*e3+e2+2*x1)^2*sin(2*(e3-2*e2))...
     + (6*p2/p3)*(e2-x1)^2*(e4-3*e3+e2+2*x1)*(e5-4*e4+3*e3+4*e2-3*x1)*...
     sin(2*(e3-2*e2)) ...
     + (4*p2/p3)*(e2-x1)^2*(e4-3*e3+e2+2*x1)^3*cos(2*(e3-2*e2)) ...
     - (p2/p3)*(e2-x1)^2*(e6-5*e5+6*e4+5*e3-10*e2-x1)*cos(2*(e3-2*e2)) ...
     + ((2*k2*p2-6*p4*p2)/(p3*p4))*(e2-x1)*(e4-3*e3+e2+2*x1)^2*...
     cos(2*(e3-2*e2)) ...
     - (6*p2/p3)*(e2-x1)*(e3-2*e2)*(e5-4*e4+3*e3+4*e2-3*x1)*...
     cos(2*(e3-2*e2))...
     - (6*p2/p3)*(e3-2*e2)^2*(e4-3*e3+e2+2*x1)*cos(2*(e3-2*e2));

e8 = (k1*k2/(p3*p4))*x8 + 7*e7 - (15+(k2/p3))*e6 + 5*(k2/p3)*e5 ...
     + (35-6*(k2/p3)-(p1*k2/(p3*p4)))*e4 ...
     - (21+5*(k2/p3)-3*(p1*k2/(p3*p4)))*e3 ...
     - (28-10*(k2/p3)+(p1*k2/(p3*p4))+(k1*k2/(p3*p4)))*e2 ...
     + (15+(k2/p3)-2*(p1*k2/(p3*p4))+(k1*k2/(p3*p4)))*x1 ...
     + phidot;

phiddot = -(k2*p2/(p3*p4))*(e5-4*e4+3*e3+4*e2-3*x1)*cos(e3-2*e2)^2 ...
     - (p7/p3)*(e4-3*e3+e2+2*x1)^4*cos(e3-2*e2) ...
     + (3*p7/p3)*(e5-4*e4+3*e3+4*e2-3*x1)^2*cos(e3-2*e2) ...
     + (4*p7/p3)*(e4-3*e3+e2+2*x1)*(e6-5*e5+6*e4+5*e3-10*e2-x1)*...
     cos(e3-2*e2) ...
     - (6*p7/p3)*(e4-3*e3+e2+2*x1)^2*(e5-4*e4+3*e3+4*e2-3*x1)*...
     sin(e3-2*e2) ...
     + (p7/p3)*(e7-6*e6+10*e5+4*e4-21*e3+4*e2+11*x1)*sin(e3-2*e2) ...
     + ((3*k2*p2-3*p4*p2)/(p3*p4))*(e4-3*e3+e2+2*x1)^2*sin(2*(e3-2*e2)) ...
     + ((3*k2*p2-4*p4*p2)/(p3*p4))*(e3-2*e2)*(e5-4*e4+3*e3+4*e2-3*x1)*...
     sin(2*(e3-2*e2)) ...
     + ((6*k2*p2-24*p4*p2)/(p3*p4))*(e3-2*e2)*(e4-3*e3+e2+2*x1)^2*...
     cos(2*(e3-2*e2)) ...
     + ((k2*p2-p4*p2)/(p3*p4))*(e2-x1)*(e6-5*e5+6*e4+5*e3-10*e2-x1)*...
     sin(2*(e3-2*e2)) ...
```

```
    + (48*p2/p3)*(e2-x1)*(e3-2*e2)*(e4-3*e3+e2+2*x1)*...
    (e5-4*e4+3*e3+4*e2-3*x1)*sin(2*(e3-2*e2)) ...
    - ((4*k2*p2-24*p4*p2)/(p3*p4))*(e2-x1)*...
    (e4-3*e3+e2+2*x1)^3*sin(2*(e3-2*e2)) ...
    + (24*p2/p3)*(e3-2*e2)^2*(e4-3*e3+e2+2*x1)^2*sin(2*(e3-2*e2)) ...
    + (8*p2/p3)*(e2-x1)^2*(e4-3*e3+e2+2*x1)*(e6-5*e5+6*e4+5*e3-10*e2-x1)*...
    sin(2*(e3-2*e2)) ...
    + (6*p2/p3)*(e2-x1)^2*(e5-4*e4+3*e3+4*e2-3*x1)^2*sin(2*(e3-2*e2)) ...
    - (8*p2/p3)*(e2-x1)^2*(e4-3*e3+e2+2*x1)^4*sin(2*(e3-2*e2)) ...
    + ((6*k2*p2-20*p4*p2)/(p3*p4))*(e2-x1)*(e4-3*e3+e2+2*x1)*...
    (e5-4*e4+3*e3+4*e2-3*x1)*cos(2*(e3-2*e2)) ...
    + (32*p2/p3)*(e2-x1)*(e3-2*e2)*(e4-3*e3+e2+2*x1)^3*cos(2*(e3-2*e2))...
    + (24*p2/p3)*(e2-x1)^2*(e4-3*e3+e2+2*x1)^2*(e5-4*e4+3*e3+4*e2-3*x1)*...
    cos(2*(e3-2*e2)) ...
    - (p2/p3)*(e2-x1)^2*(e7-6*e6+10*e5+4*e4-21*e3+4*e2+11*x1)*...
    cos(2*(e3-2*e2)) ...
    - (8*p2/p3)*(e2-x1)*(e3-2*e2)*(e6-5*e5+6*e4+5*e3-10*e2-x1)*...
    cos(2*(e3-2*e2)) ...
    - (12*p2/p3)*(e3-2*e2)^2*(e5-4*e4+3*e3+4*e2-3*x1)*cos(2*(e3-2*e2));

v = (p3*p4/(k1*k2))*(-8*e8 + (21+(k2/p3))*e7 - (8+6*(k2/p3))*e6 ...
    - (50-10*(k2/p3)-(p1*k2/(p3*p4)))*e5 ...
    + (56+4*(k2/p3)-4*(p1*k2/(p3*p4)))*e4 ...
    + (42-21*(k2/p3)+3*(p1*k2/(p3*p4))+(k1*k2/(p3*p4)))*e3 ...
    - (64-4*(k2/p3)-4*(p1*k2/(p3*p4))+2*(k1*k2/(p3*p4)))*e2 ...
    - (13-11*(k2/p3)+3*(p1*k2/(p3*p4)))*x1 ...
    - phiddot);
```

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