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On the reachable set bounding of uncertain dynamic systems with time-varying delays and disturbances

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ABSTRACT

In this paper, improved conditions are proposed for finding an ellipsoidal bound of reachable sets which bound the state trajectories of uncertain dynamic systems with time-varying delays and bounded peak disturbances. A more generalized time-derivative condition is considered in comparison with previous work. By the use of convex-hull properties and application of the Lyapunov method, sufficient conditions for finding an ellipsoidal bound of reachable sets of the concerned systems are established in terms of linear matrix inequalities (LMIs). Two numerical examples are included to show the superiority and effectiveness of our results.

1. Introduction

Reachable set bounding was first considered in the late 1960s in the field of state estimation and it has subsequently received a lot of attention with regard to parameter estimation [3]. A reachable set is defined as a set which bounds the state trajectories for systems with disturbances. The ellipsoidal bounding of reachable sets is of practical importance in the design of a controller for dynamic systems with disturbances [6]. For example, as pointed in [6], minimization of an ellipsoidal bound of reachable sets for linear systems with saturating actuators can lead to the design of a controller with a larger gain and this can result in a better performance of the system [7].

On the other hand, it is well-known that occurrence of time-delay may cause the instability or poor performance of dynamic systems and the delay-dependent stability criteria are less conservative than the delay-independent ones. Thus, extensive researches on the delay-dependent stability or stabilization criteria for dynamic systems with time-delay, which give the available information about the maximum allowable time-delay required to guarantee stability, have been conducted by many researchers during the last decade. An overview and recent survey in this field can be obtained from [1,8–11,13,14] and the references therein. Recently, new Lyapunov–Krasovskii's functionals of the form of triple integrals were proposed in [1], and their stability conditions showed an improvement in the maximum delay bound which is an important index for checking of the conservatism of the delay-dependent stability criteria.

Since an LMI condition for an ellipsoid that bounds the reachable set of linear systems without time-delay was given by Boyd et al. [2], Fridman and Shaked [4] firstly proposed LMI criteria of an ellipsoid that bounds the reachable set of uncertain systems with time-varying delays and bounded peak input based on the Razumikhin theorem. More recently, by the use of

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the Lyapunov–Krasovskii functional method which is commonly used to derive delay-dependent stability criteria, improved conditions were investigated in [6]. However, when the upper bound of the time-derivative delay becomes larger, the results given in [6] are more conservative than the ones presented in [4]. Furthermore, when the information on the time derivative of the time-varying delays is unknown, the methods in [4] cannot be applied. Therefore, there is room for further improvement in the application to dealing with this problem. To the best of author's knowledge, the issue of finding an ellipsoidal bound of reachable sets for uncertain systems with the general time-derivative constraints of a time-varying delay has not been considered.

Motivated by this situation, improved LMI criteria will be derived in this paper to find an ellipsoidal bound of reachable sets for uncertain systems with time-varying delays and bounded disturbances. By the use of the Lyapunov–Krasovskii's functional and convex-hull properties, sufficient conditions for finding an ellipsoidal bound of reachable sets for the concerned systems with the more general constraints of a time-derivative delay are derived in terms of LMIs. The originality of the proposed method is in the utilization of the fully past information of the state $x(t)$ and in its applicability to systems with different values of the lower and upper bounds on the time-derivative constraint. Also, it will be shown that further larger feasible regions of application of the proposed criteria can be obtained by modification of the recently proposed the Lyapunov–Krasovskii's functionals in [1]. Through numerical examples, it will be shown that the method can provide improved results by comparison of the obtained results with the results given in [6,4].

Notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices X and Y, the notation $X > Y$ (respectively, $X \geqslant Y$) means that the matrix $X - Y$ is positive definite, (respectively, nonnegative). $diag\{\cdots\}$ denotes the block diagonal matrix. ★ represents the elements below the main diagonal of a symmetric matrix and the superscript 'T' represents the transpose.

2. Problem statements

Consider the uncertain dynamic systems with time-varying delays and disturbances:

$$\begin{aligned}&\dot{x}(t) = (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - h(t)) + (B + \Delta B(t))w(t),\\ &x(s) = 0, \quad s \in [-h_U, 0],\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^w$ is the disturbance input, A, A_d, and B are known constant matrices with appropriate dimensions, and $\Delta A(t)$, $\Delta A_d(t)$, and $\Delta B(t)$ are uncertainties expressed as a linear convex-hull of matrices A_i, A_{di}, B_i. Hence

$$\Delta A(t) = \sum_{i=1}^{N}\rho_i(t)A_i, \quad \sum_{i=1}^{N}\rho_i(t)A_{di}, \quad \Delta B(t) = \sum_{i=1}^{N}\rho_i(t)B_i, \tag{2}$$

where $\rho_i(t) \in [0,1]$, $(i = 1,\ldots,N)$, $\sum_{i=1}^{N}\rho_i(t) = 1,\ \forall t \geqslant 0$.

For the disturbances, it is assumed that

$$w^T(t)w(t) \leqslant w_m^2, \quad \forall t \geqslant 0, \tag{3}$$

where w_m is constant.

The delay, $h(t)$, is a time-varying continuous function which satisfies $0 \leqslant h(t) \leqslant h_U$. Defining $\dot{h}(t) = \frac{d}{dt}\{h(t)\}$. In this paper, three cases of time-delay derivative conditions are considered as follows:

$$\begin{aligned}\text{Case I} &: -\infty < \dot{h}(t) \leqslant \infty,\\ \text{Case II} &: -\infty < \dot{h}(t) \leqslant h_D,\\ \text{Case III} &: h_{Dl} \leqslant \dot{h}(t) \leqslant h_{Du},\end{aligned}$$

where $h_D > 0$, h_{Dl} and h_{Du} are constants.

Defining an ellipsoid ε which bounds the reachable sets of system (1) with the constraints of disturbance and time-varying delays, then

$$\boldsymbol{\varepsilon} = \left\{x \in \mathbb{R}^n : x^T P x \leqslant 1\right\}, \tag{4}$$

where $P > 0$.

The objective of this paper is to investigate delay-dependent LMI conditions for finding an ellipsoid ε (4) that bounds the reachable sets of the system (1).

Remark 1. In [4,6], the considered constraints of $\dot{h}(t)$ were $-\infty < \dot{h}(t) < \infty$ and $|\dot{h}(t)| \leqslant h_D < 1$, respectively. Therefore, Cases I–III of $\dot{h}(t)$ are more general than the ones in [4,6].

In order to derive LMI conditions for finding an ellipsoidal bound ε, the convex-hull representation and properties will be utilized in the main results. For the Case I and II, only $h(t)$ can be represented as convex hull, and for the Case III, whilst $\dot{h}(t)$ also can be considered as a convex hull.

For the condition $0 \leqslant h(t) \leqslant h_U$ and $h_{Dl} \leqslant \dot{h}(t) \leqslant h_{Du}$, ∇_d, and ∇_h are next defined in the set

$$\begin{aligned}\Phi_d &:= \left\{\nabla_d | \nabla_d \in Co\left\{\nabla_d^1 \nabla_d^2\right\}\right\},\\ \Phi_h &:= \left\{\nabla_{h1} | \nabla_{h1} \in Co\left\{\nabla_{h1}^1, \nabla_{h1}^2\right\}\right\},\end{aligned} \tag{5}$$

where Co denotes the convex hull, $\nabla_d^1 = h_{Dl}$, $\nabla_d^2 = h_{Du}$, $\nabla_h^1 = 0$, and $\nabla_h^2 = h_U$.

Then, there exist parameters β_l, and γ_l, where $\beta_l \geqslant 0$, $\gamma_l \geqslant 0$ for $l = 1,2$, $\sum_{l=1}^2 \beta_l = 1$, and $\sum_{l=1}^2 \gamma_l = 1$ such that $\dot{h}(t)$ and $h(t)$ can be expressed as a convex combination of the vertex values as follows:

$$\begin{aligned}\dot{h}(t) &= \sum_{l=1}^{2} \beta_l \nabla_d^l,\\ h(t) &= \sum_{l=1}^{2} \gamma_l \nabla_h^l.\end{aligned} \tag{6}$$

If matrices $M(\dot{h}(t))$ and $G(h(t))$ are affinely dependent on $\dot{h}(t)$ and $h(t)$ respectively, then $M(\dot{h}(t))$ and $G(h(t))$ can be expressed as a convex combination of the vertex values, respectively, i.e.,

$$\begin{aligned}M(\dot{h}(t)) &= \sum_{l=1}^{2} \beta_l M(\nabla_d^l),\\ G(h(t)) &= \sum_{l=1}^{2} \gamma_l G(\nabla_h^l).\end{aligned} \tag{7}$$

Before deriving the main results, the following lemmas will be stated.

Lemma 1 [5]. *For any positive-definite matrix $M \in \mathbb{R}^{n\times n}$, a positive scalar γ, and a vector function $z : [0,\gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then*

$$\left(\int_0^{\gamma} z(s)ds\right)^T M\left(\int_0^{\gamma} z(s)ds\right) \leqslant \gamma \int_0^{\gamma} z^T(s)Mz(s)ds.$$

Lemma 2 [2]. *Let $V(x(0)) = 0$ and $w^T(t)w(t) \leqslant w_m^2$. If*

$$\dot{V}(x(t)) + \alpha V(x(t)) - \beta w^T(t)w(t) \leqslant 0, \quad \alpha > 0, \quad \beta > 0,$$

then,

$$V(x(t)) \leqslant \frac{\beta}{\alpha} w_m^2, \quad \forall t \geqslant 0.$$

Lemma 3 [12]. *Let $\zeta \in \mathcal{R}^n$, $\Phi = \Phi^T \in \mathcal{R}^{n\times n}$, and $B \in \mathcal{R}^{m\times n}$ such that rank(B) < n. The following statements are equivalent:*

(i) $\zeta^T \Phi \zeta < 0$, $\forall B\zeta = 0$, $\zeta \neq 0$,
(ii) $(B^{\perp})^T \Phi (B^{\perp}) < 0$ where $B^{\perp}$ is a right orthogonal complement of B.

Lemma 4. *For a positive matrix M, the following inequality holds:*

$$-\frac{(\alpha-\beta)^2}{2} \int_{\beta}^{\alpha} \int_{s}^{\alpha} x^T(u)Mx(u)du\,ds \leqslant -\left(\int_{\beta}^{\alpha} \int_{s}^{\alpha} x(u)du\,ds\right)^T M\left(\int_{\beta}^{\alpha} \int_{s}^{\alpha} x(u)du\,ds\right). \tag{8}$$

Proof. From Lemma 1, the following inequality holds:

$$-(\alpha - s)\int_s^{\alpha} x^T(u)Mx(u)du \leqslant -\left(\int_s^{\alpha} x(u)du\right)^T M\left(\int_s^{\alpha} x(u)du\right). \tag{9}$$

By the use of Schur's Complement [2], inequality (9) is equivalent to

$$\begin{bmatrix} -\int_s^{\alpha} x^T(u)Mx(u)du & \left(\int_s^{\alpha} x(u)du\right)^T \\ \star & -(\alpha - s)M^{-1} \end{bmatrix} \leqslant 0. \tag{10}$$

Integration of the inequality (10) from β to α yields

$$\begin{bmatrix} -\int_{\beta}^{\alpha}\int_{s}^{\alpha} x^T(u)Mx(u)duds & \left(\int_{\beta}^{\alpha}\int_{s}^{\alpha} x(u)duds\right)^T \\ \star & -\int_{\beta}^{\alpha}(\alpha-s)M^{-1}ds \end{bmatrix} \leqslant 0. \tag{11}$$

From Schur's Complement, inequality (11) is equivalent to inequality (8). This completes the Proof of Lemma 4. □

3. Main results

In this section, delay-dependent conditions for finding an ellipsoidal reachable sets that bounds the states of the system (1) will be derived by the Lyapunov stability method.

First, for the Case I, the following theorem is considered.

Theorem 1. *For given $\alpha > 0$ and $h_U > 0$, the reachable sets of the system (1) with $0 \leqslant h(t) \leqslant h_U$ and $-\infty < \dot{h}(t) < \infty$ are bounded by an ellipsoid ε defined in (4) if there exist positive matrices P, N_1, $\mathcal{G} = \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix}$, and any matrices L_1, L_2, L_3 satisfying the LMIs:*

$$\Sigma_1^i + \Omega^k < 0 \quad (i = 1,2,\ k = 1,\dots,N), \tag{12}$$

where

$$\begin{aligned}
&\Sigma_1^i = [\Sigma_{1(m,n)}^i], \quad m = 1,\dots,7, \quad n = 1,\dots,7,\\
&\Sigma_{1(1,1)}^i = \alpha P + N_1 + h_U^2 G_{11} + e^{-\alpha h_U}(-2 + h_U^{-1}\nabla_h^i)G_{22}, \quad \Sigma_{1(1,2)}^i = -e^{-\alpha h_U}(-2 + h_U^{-1}\nabla_h^i)G_{22},\\
&\Sigma_{1(1,3)}^i = 0, \quad \Sigma_{1(1,4)}^i = P + h_U^2 G_{12}, \quad \Sigma_{1(1,5)}^i = e^{-\alpha h_U}(-2 + h_U^{-1}\nabla_h^i)G_{12}^T, \quad \Sigma_{1(1,6)}^i = 0,\\
&\Sigma_{1(1,7)}^i = 0, \quad \Sigma_{1(2,2)}^i = e^{-\alpha h_U}(-2 + h_U^{-1}\nabla_h^i)G_{22} + e^{-\alpha h_U}(-1 - h_U^{-1}\nabla_h^i)G_{22},\\
&\Sigma_{1(2,3)}^i = -e^{-\alpha h_U}(-1 - h_U^{-1}\nabla_h^i)G_{22}, \quad \Sigma_{1(2,4)}^i = 0, \quad \Sigma_{1(2,5)}^i = -e^{-\alpha h_U}(-2 + h_U^{-1}\nabla_h^i)G_{12}^T,\\
&\Sigma_{1(2,6)}^i = e^{-\alpha h_U}(-1 - h_U^{-1}\nabla_h^i)G_{12}^T, \quad \Sigma_{1(2,7)}^i = 0, \Sigma_{1(3,3)}^i = -e^{-\alpha h_U}N_1 + e^{-\alpha h_U}(-1 - h_U^{-1}\nabla_h^i)G_{22},\\
&\Sigma_{1(3,4)}^i = 0, \quad \Sigma_{1(3,5)}^i = 0, \quad \Sigma_{1(3,6)}^i = -e^{-\alpha h_U}(-1 - h_U^{-1}\nabla_h^i)G_{12}^T, \quad \Sigma_{1(3,7)}^i = 0, \quad \Sigma_{1(4,4)}^i = h_U^2 G_{22},\\
&\Sigma_{1(4,5)}^i = 0, \quad \Sigma_{1(4,6)}^i = 0, \quad \Sigma_{1(4,7)}^i = 0, \quad \Sigma_{1(5,5)}^i = e^{-\alpha h_U}(-2 + h_U^{-1}\nabla_h^i)((-\alpha h_U^{-1}e^{\alpha h_U})N_2 + G_{11}),\\
&\Sigma_{1(5,6)}^i = 0, \quad \Sigma_{1(5,7)}^i = 0, \quad \Sigma_{1(6,6)}^i = e^{-\alpha h_U}(-1 - h_U^{-1}\nabla_h^i)G_{11}, \quad \Sigma_{1(6,7)}^i = 0, \quad \Sigma_{1(7,7)}^i = -(\alpha/w_m^2)I,
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\Omega^k = [\Omega_{(m,n)}^k], \quad m = 1,\dots,7, \quad n = 1,\dots,7,\\
&\Omega_{(1,1)}^k = L_1(A + A_k) + (A + A_k)^T L_1, \quad \Omega_{(1,2)}^k = L_1(A_d + A_{dk}) + (A + A_k)^T L_1^T,\\
&\Omega_{(1,4)}^k = -L_1 + (A + A_k)^T L_3^T, \quad \Omega_{(1,7)}^k = L_1(B + B_k), \quad \Omega_{(2,2)}^k = L_2(A_d + A_{dk}) + (A_d + A_{dk})^T L_2,\\
&\Omega_{(2,4)}^k = -L_2 + (A_d + A_{dk})^T L_3^T, \quad \Omega_{(2,7)}^k = L_2(B + B_k), \quad \Omega_{(4,4)}^k = -L_3 - L_3^T, \quad \Omega_{(4,7)}^k = L_3(B + B_k),\\
&\Omega_{(m,n)}^k = 0, \quad \text{(elsewhere)}.
\end{aligned} \tag{14}$$

Proof. For positive matrices P, N_1, $\mathcal{G} = \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix}$, consider the Lyapunov–Krasovskii functional candidate:

$$V(x(t)) = \sum_{i=1}^{3} V_i(x(t)) \tag{15}$$

where

$$\begin{aligned}
&V_1(x(t)) = x^T(t)Px(t),\\
&V_2(x(t)) = \int_{t-h_U}^{t} e^{\alpha(s-t)}x^T(s)N_1x(s)ds,\\
&V_3(x(t)) = h_U\int_{t-h_U}^{t}\int_{s}^{t} e^{\alpha(u-t)}\begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix}^T \mathcal{G}\begin{bmatrix} x(u) \\ \dot{x}(u) \end{bmatrix} du\,ds.
\end{aligned} \tag{16}$$

Calculation of the time-derivative of $V_1(x(t))$ yields

$$\dot{V}_1(x(t)) = 2x^T(t)P\dot{x}(t). \tag{17}$$

The time-derivative of $V_2(x(t))$ can now be obtained as

$$\begin{aligned}\dot{V}_2(x(t)) &= \frac{d}{dt}\left\{e^{-\alpha t}\int_{t-h_U}^{t} e^{\alpha s}x^T(s)N_1x(s)ds\right\}\\ &= -\alpha e^{-\alpha t}\int_{t-h_U}^{t} e^{\alpha s}x^T(s)N_1x(s)ds + e^{-\alpha t}\left[x^T(t)(e^{\alpha t}N_1)x(t) - x^T(t-h_U)(e^{\alpha(t-h_U)}N_1)x(t-h_U)\right]\\ &= -\alpha V_2(x(t)) + x^T(t)N_1x(t) - x^T(t-h_U)e^{-\alpha h_U}N_1x(t-h_U).\end{aligned} \tag{18}$$

Calculation of $\dot{V}_3(x(t))$ leads to

$$\begin{aligned}\dot{V}_3(x(t)) &= \frac{d}{dt}\left\{h_U e^{-\alpha t}\int_{t-h_U}^{t}\int_{s}^{t} e^{\alpha u}\begin{bmatrix}x(u)\\ \dot{x}(u)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(u)\\ \dot{x}(u)\end{bmatrix}du\,ds\right\} = -\alpha V_3(x(t)) + e^{-\alpha t}\frac{d}{dt}\left\{h_U\int_{t-h_U}^{t}\int_{s}^{t} e^{\alpha u}\begin{bmatrix}x(u)\\ \dot{x}(u)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(u)\\ \dot{x}(u)\end{bmatrix}du\,ds\right\}\\ &= -\alpha V_3(x(t)) + e^{-\alpha t}\left\{(h_U^2 e^{\alpha t})\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} - h_U\int_{t-h_U}^{t} e^{\alpha s}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds\right\}\\ &\leqslant -\alpha V_3(x(t)) + (h_U^2)\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} - e^{-\alpha h_U}\left(h_U\int_{t-h_U}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds\right),\end{aligned} \tag{19}$$

where $-e^{\alpha(s-t)} \leqslant -e^{-\alpha h_U}$ was used in (19).

It should be noted that

$$-h_U\int_{t-h_U}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds = -h_U\int_{t-h(t)}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds - h_U\int_{t-h_U}^{t-h(t)}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds. \tag{20}$$

With $-h_U = -(h_U - h(t)) - h(t)$, $0 \leqslant h(t) \leqslant h_U$ and Lemma 1, an upper bound of the first integral term in the right side of Eq. (20) can be obtained as

$$\begin{aligned}-h_U\int_{t-h(t)}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds &= -(h_U - h(t))\int_{t-h(t)}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds - h(t)\int_{t-h(t)}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds\\ &\leqslant -h_U^{-1}(h_U - h(t))h(t)\int_{t-h(t)}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds - h(t)\int_{t-h(t)}^{t}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds\\ &\leqslant (-2 + h_U^{-1}h(t))\begin{bmatrix}\int_{t-h(t)}^{t}x(s)ds\\ x(t) - x(t-h(t))\end{bmatrix}^T \mathcal{G}\begin{bmatrix}\int_{t-h(t)}^{t}x(s)ds\\ x(t) - x(t-h(t))\end{bmatrix}.\end{aligned} \tag{21}$$

With the similar procedure to that used to obtain (21) and the use of Lemma 1, an upper bound of the second integral term in the right side of Eq. (20) can be estimated to be

$$\begin{aligned}-h_U\int_{t-h_U}^{t-h(t)}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds &= -(h_U - h(t))\int_{t-h_U}^{t-h(t)}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds - h_U^{-1}h(t)h_U\int_{t-h_U}^{t-h(t)}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds\\ &\leqslant -(h_U - h(t))\int_{t-h_U}^{t-h(t)}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds - h_U^{-1}h(t)(h_U - h(t))\int_{t-h_U}^{t-h(t)}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(s)\\ \dot{x}(s)\end{bmatrix}ds\\ &\leqslant (-1 - h_U^{-1}h(t))\begin{bmatrix}\int_{t-h_U}^{t-h(t)}x(s)ds\\ x(t) - x(t-h(t))\end{bmatrix}^T \mathcal{G}\begin{bmatrix}\int_{t-h_U}^{t-h(t)}x(s)ds\\ x(t) - x(t-h(t))\end{bmatrix},\end{aligned} \tag{22}$$

where $-h_U \leqslant -(h_U - h(t))$.

Therefore, from (21) and (22), an upper bound of $\dot{V}_3(x(t))$ can be further estimated as

$$\begin{aligned}\dot{V}_3(x(t)) \leqslant{}& -\alpha V_3(x(t)) + (h_U^2)\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} + e^{-\alpha h_U}\left(\left(-2 + h_U^{-1}h(t)\right)\begin{bmatrix}\int_{t-h(t)}^{t}x(s)ds\\ x(t) - x(t-h(t))\end{bmatrix}^T \mathcal{G}\begin{bmatrix}\int_{t-h(t)}^{t}x(s)ds\\ x(t) - x(t-h(t))\end{bmatrix}\right.\\ &\left. + (-1 - h_U^{-1}h(t))\begin{bmatrix}\int_{t-h_U}^{t-h(t)}x(s)ds\\ x(t-h(t)) - x(t-h_U)\end{bmatrix}^T \mathcal{G}\begin{bmatrix}\int_{t-h_U}^{t-h(t)}x(s)ds\\ x(t-h(t)) - x(t-h_U)\end{bmatrix}\right).\end{aligned}$$

In the upper bound of time-derivative of $V(x(t))$, the following equality is added with free variables L_i ($i = 1,2,3$) to be chosen as:

$$0 = 2\left[x^T(t)L_1 + x^T(t-h(t))L_2 + \dot{x}^T(t)L_3\right] \times [-\dot{x}(t) + (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t-h(t)) + (B + \Delta B(t))w(t)]. \tag{23}$$

Defining an $\zeta(t)$ as

$$\zeta^T(t) = \begin{bmatrix} x^T(t) & x^T(t-h(t)) & x^T(t-h_U) & \dot{x}^T(t) & \int_{t-h(t)}^{t} x^T(s)ds & \int_{t-h_U}^{t-h(t)} x^T(s)ds & w^T(t) \end{bmatrix}, \tag{24}$$

then, from (15)–(24),

$$\dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{w_m^2} w^T(t)w(t) \leqslant \zeta^T(t)(\Sigma_1 + \Omega)\zeta(t), \tag{25}$$

where

$$\begin{aligned}
&\Sigma_1 = [\Sigma_{1(m,n)}], \quad m = 1,\dots,7, \quad n = 1,\dots,7,\\
&\Sigma_{1(1,1)} = \alpha P + N_1 + h_U^2 G_{11} + e^{-\alpha h_U}(-2 + h_U^{-1}h(t))G_{22}, \quad \Sigma_{1(1,2)} = -e^{-\alpha h_U}(-2 + h_U^{-1}h(t))G_{22},\\
&\Sigma_{1(1,3)} = 0, \quad \Sigma_{1(1,4)} = P + h_U^2 G_{12}, \quad \Sigma_{1(1,5)} = e^{-\alpha h_U}(-2 + h_U^{-1}h(t))G_{12}^T, \quad \Sigma_{1(1,6)} = 0,\\
&\Sigma_{1(1,7)} = 0, \quad \Sigma_{1(2,2)} = e^{-\alpha h_U}(-2 + h_U^{-1}h(t))G_{22} + e^{-\alpha h_U}(-1 - h_U^{-1}h(t))G_{22},\\
&\Sigma_{1(2,3)} = -e^{-\alpha h_U}(-1 - h_U^{-1}h(t))G_{22}, \quad \Sigma_{1(2,4)} = 0, \quad \Sigma_{1(2,5)} = -e^{-\alpha h_U}(-2 + h_U^{-1}h(t))G_{12}^T,\\
&\Sigma_{1(2,6)} = e^{-\alpha h_U}(-1 - h_U^{-1}h(t))G_{12}^T, \quad \Sigma_{1(2,7)} = 0, \quad \Sigma_{1(3,3)} = -e^{-\alpha h_U}N_1 + e^{-\alpha h_U}(-1 - h_U^{-1}h(t))G_{22},\\
&\Sigma_{1(3,4)} = 0, \quad \Sigma_{1(3,5)} = 0, \quad \Sigma_{1(3,6)} = -e^{-\alpha h_U}(-1 - h_U^{-1}h(t))G_{12}^T, \quad \Sigma_{1(3,7)} = 0, \quad \Sigma_{1(4,4)} = h_U^2 G_{22},\\
&\Sigma_{1(4,5)} = 0, \quad \Sigma_{1(4,6)} = 0, \quad \Sigma_{1(4,7)} = 0, \quad \Sigma_{1(5,5)} = e^{-\alpha h_U}(-2 + h_U^{-1}h(t))((-\alpha h_U^{-1}e^{\alpha h_U})N_2 + G_{11}),\\
&\Sigma_{1(5,6)} = 0, \quad \Sigma_{1(5,7)} = 0, \quad \Sigma_{1(6,6)} = e^{-\alpha h_U}(-1 - h_U^{-1}h(t))G_{11}, \quad \Sigma_{1(6,7)} = 0, \quad \Sigma_{1(7,7)} = -(\alpha/w_m^2)I
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&\Omega = [\Omega_{(m,n)}], \quad m = 1,\dots,7, \quad n = 1,\dots,7,\\
&\Omega_{(1,1)} = L_1(A + \Delta A(t)) + (A + \Delta A(t))^T L_1, \quad \Omega_{(1,2)} = L_1(A_d + \Delta A_d(t)) + (A + \Delta A(t))^T L_1^T,\\
&\Omega_{(1,4)} = -L_1 + (A + \Delta A(t))^T L_3^T, \quad \Omega_{(1,7)} = L_1(B + \Delta B(t)),\\
&\Omega_{(2,2)} = L_2(A_d + \Delta A_d(t)) + (A_d + \Delta A_d(t))^T L_2, \quad \Omega_{(2,4)} = -L_2 + (A_d + \Delta A_d(t))^T L_3^T,\\
&\Omega_{(2,7)} = L_2(B + \Delta B(t)), \quad \Omega_{(4,4)} = -L_3 - L_3^T, \quad \Omega_{(4,7)} = L_3(B + \Delta B(t)),\\
&\Omega_{(m,n)} = 0 \quad (\text{elsewhere}).
\end{aligned} \tag{27}$$

It should be noted that $\Sigma_1 + \Omega < 0$, which means that the inequality $\dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{w_m^2} w^T(t)w(t) < 0$, is affinely dependent on $h(t)$, $\Delta A(t)$, $\Delta A_d(t)$, and $\Delta B(t)$. From the properties of a convex-hull, if the LMI (12) holds, then $\Sigma_1 + \Omega < 0$ is satisfied. Thus, $V(x(t)) \leqslant \frac{\alpha/w_m^2}{\alpha} w_m^2 = 1$ by Lemma 2. Also, $V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) \geqslant V_1(x(t)) = x^T(t)Px(t)$ since $V_2(x(t)) + V_3(x(t)) \geqslant 0$. Therefore, if the LMI (12) holds, then $x^T(t)Px(t) \leqslant V(x(t)) \leqslant 1$. This completes the proof. □

Remark 2. As mentioned in [4,6], a simple approximation to minimize the volume ε of Eq. (4) in Theorem 1 can be used as follows:

$$\begin{aligned}
&\text{minimize} \quad \bar{\delta},\\
&\text{subject to} \quad \begin{bmatrix} \bar{\delta}I & I \\ I & P \end{bmatrix} \geqslant 0 \quad \text{and} \quad \text{LMI (12)}.
\end{aligned}$$

Here it should be noted that Theorem 1 does not impose any conditions on the time-derivative of $h(t)$ in system (1). If $\dot{h}(t)$ in system (1) satisfies $-\infty < \dot{h}(t) \leqslant h_D$ where $h_D > 0$, then we have the following theorem.

Theorem 2. *For given $\alpha > 0$, $h_U > 0$, $h_D > 0$, the reachable sets of the system (1) with $0 \leqslant h(t) \leqslant h_U$ and $-\infty < \dot{h}(t) \leqslant h_D$ are bounded by an ellipsoid ε as defined in (4) providing that there exist positive matrices P, N_1, N_2, $\mathcal{G} = \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix}$, and any matrices L_1, L_2, L_3 satisfying the following LMIs:*

$$\Sigma_1^i + \Sigma_2^i + \Omega^k < 0 \quad (i = 1,2,\ k = 1,\dots,N), \tag{28}$$

$$\begin{bmatrix} -\alpha h_U^{-1} e^{\alpha h_U} N_2 + G_{11} & G_{22} \\ \star & G_{22} \end{bmatrix} > 0, \tag{29}$$

where

$$\begin{aligned}&\Sigma_2^i=[\Sigma_{2(m,n)}],\quad m=1,\ldots,7,\quad n=1,\ldots,7,\\&\Sigma^i_{2(1,1)}=N_2,\quad \Sigma^i_{2(2,2)}=-(1-h_D)N_2,\quad \Sigma^i_{2(5,5)}=(2-h_U^{-1}\nabla_h^i)\alpha h_U^{-1}N_2,\\&\Sigma^i_{2(m,n)}=0\quad(\text{elsewhere}).\end{aligned}\tag{30}$$

and Σ_1^i *and* Ω^k *have already been defined in (13) and (14), respectively.*

Proof. For positive matrices P, N_1, N_2, and $\mathcal{G}$, consider the Lyapunov–Krasovskii functional candidate:

$$V(x(t))=\sum_{i=1}^{4}V_i(x(t)),\tag{31}$$

where

$$V_4(x(t))=\int_{t-h(t)}^{t}x^T(s)N_2x(s)ds\tag{32}$$

and $V_i(x(t))$ ($i=1,2,3$) are defined in (31).

Calculation of $\dot V_4(x(t))$ establishes the relationship

$$\dot V_4(x(t))=x^T(t)N_2x(t)-(1-\dot h(t))x^T(t-h(t))N_2x(t-h(t))\leqslant x^T(t)N_2x(t)-(1-h_D)x^T(t-h(t))N_2x(t-h(t)).\tag{33}$$

Since the upper bound of $\dot V_4(x(t))$ does not contain $-\alpha\, V_4(x(t))$ as shown in (33), $\alpha V_4(x(t))$ is added to the left side of the condition to produce

$$\dot V(x(t))+\alpha V(x(t))-\frac{\alpha}{w_m^2}w^T(t)w(t)<0.\tag{34}$$

Since the form of $\int_{t-h(t)}^{t}x^T(s)Nx(s)ds(N>0)$ exists in the upper bound of $\dot V_3(t)$, the term $\alpha V_4(x(t))=\alpha\int_{t-h(t)}^{t}x^T(s)N_2x(s)ds$ can be incorporated in the upper bound of $\dot V_3(x(t))$ so that

$$\begin{aligned}\dot V_3(x(t))+\alpha V_4(x(t))&\leqslant-\alpha V_3(x(t))+(h_U^2)\begin{bmatrix}x(t)\\\dot x(t)\end{bmatrix}^T\mathcal{G}\begin{bmatrix}x(t)\\\dot x(t)\end{bmatrix}-e^{-\alpha h_U}\left(h_U\int_{t-h_U}^{t}\begin{bmatrix}x(s)\\\dot x(s)\end{bmatrix}^T\mathcal{G}\begin{bmatrix}x(s)\\\dot x(s)\end{bmatrix}ds\right)+\alpha\int_{t-h(t)}^{t}x^T(s)N_2x(s)ds\\&=-\alpha V_3(x(t))+(h_U^2)\begin{bmatrix}x(t)\\\dot x(t)\end{bmatrix}^T\mathcal{G}\begin{bmatrix}x(t)\\\dot x(t)\end{bmatrix}-e^{-\alpha h_U}\left(h_U\int_{t-h(t)}^{t}\begin{bmatrix}x(s)\\\dot x(s)\end{bmatrix}^T\begin{bmatrix}-\alpha h_U^{-1}e^{\alpha h_U}N_2+G_{11}&G_{12}\\\star&G_{22}\end{bmatrix}\begin{bmatrix}x(s)\\\dot x(s)\end{bmatrix}ds\right)\\&\quad-e^{-\alpha h_U}\left(h_U\int_{t-h_U}^{t-h(t)}\begin{bmatrix}x(s)\\\dot x(s)\end{bmatrix}^T\mathcal{G}\begin{bmatrix}x(s)\\\dot x(s)\end{bmatrix}ds\right).\end{aligned}\tag{35}$$

If the inequality (29) in Theorem 2 holds, then, from (21) and (22), Eq. (35) can be further estimated as

$$\begin{aligned}\dot V_3(x(t))\leqslant&-\alpha V_3(x(t))+(h_U^2)\begin{bmatrix}x(t)\\\dot x(t)\end{bmatrix}^T\mathcal{G}\begin{bmatrix}x(t)\\\dot x(t)\end{bmatrix}+e^{-\alpha h_U}\left((-2+h_U^{-1}h(t))\begin{bmatrix}\int_{t-h(t)}^{t}x(s)ds\\x(t)-x(t-h(t))\end{bmatrix}^T\begin{bmatrix}-\alpha h_U^{-1}e^{\alpha h_U}N_2+G_{11}&G_{12}\\\star&G_{22}\end{bmatrix}\right.\\&\left.\times\begin{bmatrix}\int_{t-h(t)}^{t}x(s)ds\\x(t)-x(t-h(t))\end{bmatrix}+(-1-h_U^{-1}h(t))\begin{bmatrix}\int_{t-h_U}^{t-h(t)}x(s)ds\\x(t-h(t))-x(t-h_U)\end{bmatrix}^T\mathcal{G}\begin{bmatrix}\int_{t-h_U}^{t-h(t)}x(s)ds\\x(t-h(t))-x(t-h_U)\end{bmatrix}\right).\end{aligned}\tag{36}$$

From (15)–(24), (31)–(33), and (35), (36), it is found that

$$\dot V(x(t))+\alpha V(x(t))-\frac{\alpha}{w_m^2}w^T(t)w(t)<\zeta^T(t)(\Sigma_1+\Sigma_2+\Omega)\zeta(t).\tag{37}$$

Here, Σ_1 and Ω are as defined in (26) and (27), respectively. Also Σ_2 is given by

$$\begin{aligned}&\Sigma_2=[\Sigma_{2(m,n)}],\quad m=1,\ldots,7,\quad n=1,\ldots,7,\\&\Sigma_{2(1,1)}=N_2,\quad \Sigma_{2(2,2)}=-(1-h_D)N_2,\quad \Sigma_{2(5,5)}=(2-h_U^{-1}h(t))\alpha h_U^{-1}N_2,\\&\Sigma_{2(m,n)}=0\quad(\text{elsewhere}).\end{aligned}\tag{38}$$

From the properties of the convex-hull, if the LMIs (28) and (29) hold, then $\Sigma_1+\Sigma_2+\Omega<0$, which implies that $\dot V(x(t))+\alpha V(x(t))-\frac{\alpha}{w_m^2}w^T(t)w(t)<0$. Therefore, from Lemma 2, if the LMIs (28) and (29) hold, then $x^T(t)Px(t)\leqslant V(x(t))\leqslant 1$ since $V(x(t))\geqslant x^T(t)Px(t)$. This completes the proof. □

Remark 3. In practical systems, it is more realistic to consider the condition of $\dot{h}(t)$ which has a lower bound and upper bound such as $h_{Dl} \leqslant \dot{h}(t) \leqslant h_{Du}$. For this case, if an LMI condition is affinely dependent on $\dot{h}(t)$, then by utilizing convex-hull properties, an improved condition for finding an ellipsoidal reachable set for system (1) can be obtained by considering the equation

$$V_5(x(t)) = \begin{bmatrix} x(t) \\ \int_{t-h(t)}^{t} x(s)ds \\ \int_{t-h_U}^{t-h(t)} x(s)ds \end{bmatrix}^T \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ \star & R_{22} & R_{23} \\ \star & \star & R_{33} \end{bmatrix} \begin{bmatrix} x(t) \\ \int_{t-h(t)}^{t} x(s)ds \\ \int_{t-h_U}^{t-h(t)} x(s)ds \end{bmatrix}, \tag{39}$$

which will be introduced in Theorem 3.

Theorem 3. *For a given $\alpha > 0$, $h_U > 0$, h_{Dl}, and h_{Du}, the reachable sets of the system (1) with $0 \leqslant h(t) \leqslant h_U$ and $h_{Dl} \leqslant \dot{h}(t) \leqslant h_{Du}$ are bounded by an ellipsoid ε as defined in (4) if there exist positive matrices P, N_1, N_2, $\mathcal{G} = \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix}$, $\mathcal{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ \star & R_{22} & R_{23} \\ \star & \star & R_{33} \end{bmatrix}$, and any matrices L_1, L_2, L_3 satisfying the following LMIs:*

$$\Sigma_1^i + \Sigma_2^{i,j} + \Sigma_3^j + \Omega^k < 0 \quad (i = 1,2,\ j = 1,2,\ k = 1,\ldots,N), \tag{40}$$

$$\begin{bmatrix} -\alpha h_U^{-1} e^{\alpha h_U} N_2 + G_{11} & G_{22} \\ \star & G_{22} \end{bmatrix} > 0, \tag{41}$$

where

$$\begin{aligned}
&\Sigma_2^{i,j} = \left[\Sigma_{2(m,n)}^{i,j}\right], \quad m = 1,\ldots,7, \quad n = 1,\ldots,7,\\
&\Sigma_{2(1,1)}^{i,j} = N_2, \quad \Sigma_{2(2,2)}^{i,j} = -(1-\nabla_d^j)N_2, \quad \Sigma_{2(5,5)}^{i,j} = (2 - h_U^{-1}\nabla_h^i)\alpha h_U^{-1} N_2,\\
&\Sigma_{2(m,n)}^{i,j} = 0 \quad (otherwise),\\
&\Sigma_3^j = [\Sigma_{3(m,n)}], \quad m = 1,\ldots,7, \quad n = 1,\ldots,7,\\
&\Sigma_{3(1,1)}^j = \alpha R_{11} + R_{12} + R_{12}^T, \quad \Sigma_{3(1,2)}^j = -(1-\nabla_d^j)R_{12} + (1-\nabla_d^j)R_{13}, \quad \Sigma_{3(1,3)}^j = -R_{13},\\
&\Sigma_{3(1,4)}^j = R_{11}, \quad \Sigma_{3(1,5)}^j = \alpha R_{12} + R_{22}, \quad \Sigma_{3(1,6)}^j = \alpha R_{13} + R_{23}, \quad \Sigma_{3(1,7)}^j = 0, \quad \Sigma_{3(2,2)}^j = 0, \quad \Sigma_{3(2,3)}^j = 0,\\
&\Sigma_{3(2,4)}^j = 0, \quad \Sigma_{3(2,5)}^j = -(1-\nabla_d^j)R_{22} + (1-\nabla_d^j)R_{23}^T, \quad \Sigma_{3(2,6)}^j = -(1-\nabla_d^j)R_{23} + (1-\nabla_d^j)R_{33},\\
&\Sigma_{3(2,7)}^j = 0, \quad \Sigma_{3(3,3)}^j = 0, \quad \Sigma_{3(3,4)}^j = 0, \quad \Sigma_{3(3,5)}^j = -R_{23}^T, \quad \Sigma_{3(3,6)}^j = -R_{33}, \quad \Sigma_{3(3,7)}^j = 0, \quad \Sigma_{3(4,4)}^j = 0,\\
&\Sigma_{3(4,5)}^j = R_{12}, \quad \Sigma_{3(4,6)}^j = R_{13}, \quad \Sigma_{3(4,7)}^j = 0, \quad \Sigma_{3(5,5)}^j = \alpha R_{22}, \quad \Sigma_{3(5,6)}^j = \alpha R_{23}, \quad \Sigma_{3(5,7)}^j = 0,\\
&\Sigma_{3(6,6)}^j = \alpha R_{33}, \quad \Sigma_{3(6,7)}^j = 0, \quad \Sigma_{3(7,7)}^j = 0
\end{aligned} \tag{42}$$

and Σ_1^i and Ω^k are as already defined in (13) and (14), respectively.

Proof. For positive matrices P, N_1, N_2, $\mathcal{G}$, $\mathcal{R}$, let us take Lyapunov–Krasovskii functional candidate:

$$V(x(t)) = \sum_{i=1}^{5} V_i(x(t)), \tag{43}$$

where

$$V_5(x(t)) = \begin{bmatrix} x(t) \\ \int_{t-h(t)}^{t} x(s)ds \\ \int_{t-h_U}^{t-h(t)} x(s)ds \end{bmatrix}^T \mathcal{R} \begin{bmatrix} x(t) \\ \int_{t-h(t)}^{t} x(s)ds \\ \int_{t-h_U}^{t-h(t)} x(s)ds \end{bmatrix} \tag{44}$$

and $V_i(x(t))$ $(i = 1,\ldots,4)$ are as defined in (31).

Calculation of $\dot{V}_5(x(t))$ leads to

$$\dot{V}_5(x(t)) = 2\begin{bmatrix} x(t) \\ \int_{t-h(t)}^{t} x(s)ds \\ \int_{t-h_U}^{t-h(t)} x(s)ds \end{bmatrix}^T \mathcal{R} \begin{bmatrix} \dot{x}(t) \\ x(t) - (1-\dot{h}(t))x(t-h(t)) \\ (1-\dot{h}(t))x(t-h(t)) - x(t-h_U) \end{bmatrix}. \tag{45}$$

From (15)–(24), (31)–(33), (35), (36) and (43)–(45), the inequality

$$\dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{w_m^2} w^T(t)w(t) < \zeta^T(t)(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Omega)\zeta(t) \tag{46}$$

is obtained where Σ_2 is the same one as in (38) except $\Sigma_{2(2,2)} = -(1-\dot{h}(t))N_2$,

$$\begin{aligned}
&\Sigma_3 = [\Sigma_{3(m,n)}], \quad m = 1,\dots,7, \quad n = 1,\dots,7,\\
&\Sigma_{3(1,1)} = \alpha R_{11} + R_{12} + R_{12}^T, \quad \Sigma_{3(1,2)} = -(1-\dot{h}(t))R_{12} + (1-\dot{h}(t))R_{13}, \quad \Sigma_{3(1,3)} = -R_{13},\\
&\Sigma_{3(1,4)} = R_{11}, \quad \Sigma_{3(1,5)} = \alpha R_{12} + R_{22}, \quad \Sigma_{3(1,6)} = \alpha R_{13} + R_{23}, \quad \Sigma_{3(1,7)} = 0, \quad \Sigma_{3(2,2)} = 0, \quad \Sigma_{3(2,3)} = 0,\\
&\Sigma_{3(2,4)} = 0, \quad \Sigma_{3(2,5)} = -(1-\dot{h}(t))R_{22} + (1-\dot{h}(t))R_{23}^T, \quad \Sigma_{3(2,6)} = -(1-\dot{h}(t))R_{23} + (1-\dot{h}(t))R_{33},\\
&\Sigma_{3(2,7)} = 0, \quad \Sigma_{3(3,3)} = 0, \quad \Sigma_{3(3,4)} = 0, \quad \Sigma_{3(3,5)} = -R_{23}^T, \quad \Sigma_{3(3,6)} = -R_{33}, \quad \Sigma_{3(3,7)} = 0, \quad \Sigma_{3(4,4)} = 0,\\
&\Sigma_{3(4,5)} = R_{12}, \quad \Sigma_{3(4,6)} = R_{13}, \quad \Sigma_{3(4,7)} = 0, \quad \Sigma_{3(5,5)} = \alpha R_{22}, \quad \Sigma_{3(5,6)} = \alpha R_{23}, \quad \Sigma_{3(5,7)} = 0,\\
&\Sigma_{3(6,6)} = \alpha R_{33}, \quad \Sigma_{3(6,7)} = 0, \quad \Sigma_{3(7,7)} = 0
\end{aligned} \tag{47}$$

and Σ_1, and Ω are as defined in (13) and (14), respectively. By applying a procedure similar to the proof of Theorem 2, if the LMIs (40) and (41) hold, then $x^T(t)Px(t) \leqslant V(x(t)) \leqslant 1$. This completes the proof. □

Remark 4. Very recently, an improved condition for finding an ellipsoidal volume (4) for system (1) with the condition $0 \leqslant h(t) \leqslant h_U$ and $\left|\dot{h}(t)\right| \leqslant h_D$ was proposed in [6]. Here it is noted that the considered condition for time-derivative delay in Theorem 3 is $h_{Dl} \leqslant \dot{h}(t) \leqslant h_{Du}$, which is a more general case than $|\dot{h}(t)| \leqslant h_D$. In [6], the proposed Lyapunov–Krasovskii's functional only has the form of $\int_{t-h(t)}^{t}\int_s^t x^T(u)G_{11}x(u)du\,ds$ and the term $\int_{t-h_U}^{t-h(t)}\int_s^t x^T(u)G_{11}x(u)du\,ds$ was not considered. It is also noted that the $V_3(x(t))$ term in (15) contains the form of $\int_{t-h_U}^{t}\int_s^t x^T(u)G_{11}x(u)du\,ds$. Therefore, the proposed $V_3(x(t))$ utilizes fully past information on $x(t)$ for $0 \leqslant h(t) \leqslant h_U$. Also, in obtained time-derivative of $V(x(t))$ of [6], the term including $\dot{h}(t)$ was estimated using the well-known fact $2a^Tb \leqslant a^TW^{-1} + b^TW\,b(W > 0)$. However, Theorem 3 does not use this fact to estimate the terms which include $\dot{h}(t)$. These two differences cause the proposed Theorem 3 to improve the feasible region of an ellipsoidal reachable set for the same system. In Section 4, the results obtained will be compared with the results of [6].

Remark 5. In order to improve the feasible region of the stability condition of the time-delay systems, the form of the triple integral Lyapunov–Krasovskii's functional was proposed in [1]. An important question is that the form of triple integral Lyapunov–Krasovskii's functional can be applied to find an ellipsoidal volume (4) for system (1) with the condition $0 \leqslant h(t) \leqslant h_U$ and $|\dot{h}(t)| \leqslant h_D$. Motivated by this aim, in Theorem 4, by consideration of the following Lyapunov–Krasovskii's functional

$$V_6(x(t)) = (h_U^2/2)\int_{t-h_U}^{t}\int_s^t\int_u^t e^{\alpha(v-t)}\dot{x}^T(v)G_3\dot{x}(v)dv\,du\,ds, \tag{48}$$

a further improved condition will be introduced.

Theorem 4. *For given $\alpha > 0$, $h_U > 0$, h_{Dl}, and h_{Du}, the reachable sets of the system (1) with $0 \leqslant h(t) \leqslant h_U$ and $h_{Dl} \leqslant \dot{h}(t) \leqslant h_{Du}$ are bounded by an ellipsoid ε defined in (4) provided that there exist positive matrices P, N_1, N_2, $\mathcal{G} = \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22}\end{bmatrix}$, $\mathcal{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ \star & R_{22} & R_{23} \\ \star & \star & R_{33}\end{bmatrix}$, G_3, and any matrices L_1, L_2, L_3 satisfying the following LMIs:*

$$\Sigma_1^i + \Sigma_2^{i,j} + \Sigma_3^j + \Sigma_4 + \Omega^k < 0 \quad (i = 1,2,\ j = 1,2,\ k = 1,\dots,N), \tag{49}$$

$$\begin{bmatrix} -\alpha h_U^{-1}e^{\alpha h_U}N_2 + G_{11} & G_{22} \\ \star & G_{22}\end{bmatrix} > 0, \tag{50}$$

where

$$\begin{aligned}
&\Sigma_4 = [\Sigma_{4(m,n)}], \quad m = 1,\dots,7, \quad n = 1,\dots,7,\\
&\Sigma_{4(1,1)} = -h_U^2 e^{-\alpha h_U}G_3, \quad \Sigma_{4(1,5)} = h_U e^{-\alpha h_U}G_3, \quad \Sigma_{4(1,6)} = h_U e^{-\alpha h_U}G_3, \quad \Sigma_{4(4,4)} = (h_U/4^4)G_3,\\
&\Sigma_{4(5,5)} = -e^{-\alpha h_U}G_3, \quad \Sigma_{4(5,6)} = e^{-\alpha h_U}G_3, \quad \Sigma_{4(6,6)} = -e^{-\alpha h_U}G_3,\\
&\Sigma_{4(m,n)} = 0 \quad \text{(elsewhere)}
\end{aligned} \tag{51}$$

and other notations are as defined in Theorem 3.

Proof. For positive matrices P, N_1, N_2, $\mathcal{G}$, $\mathcal{R}$, and G_3, taking the Lyapunov–Krasovskii functional candidate:

$$V(x(t)) = \sum_{i=1}^{6} V_i(x(t)), \tag{52}$$

where

$$V_6(x(t)) = (h_U^2/2)\int_{t-h_U}^{t}\int_{s}^{t}\int_{u}^{t} e^{\alpha(v-t)}\dot{x}^T(v)G_3\dot{x}(v)dv\,du\,ds \tag{53}$$

and $V_i(x(t))$ $(i = 1,\ldots,5)$ are the same ones as in (43).

By calculation of $\dot{V}_6(x(t))$, it is found that

$$\begin{aligned}
\dot{V}_6(x(t)) &= -\alpha V_6(x(t)) + e^{-\alpha t}\frac{d}{dt}\left\{(h_U^2/2)\int_{t-h_U}^{t}\int_{s}^{t}\int_{u}^{t} e^{\alpha v}\dot{x}^T(v)G_3\dot{x}(v)dv\,du\,ds\right\} \\
&= -\alpha V_6(x(t)) + e^{-\alpha t}\left\{(h_U^4/4)\dot{x}^T(t)(e^{\alpha t}G_3)\dot{x}(t) - (h_U/2^2)\int_{t-h_U}^{t}\int_{s}^{t} e^{\alpha u}\dot{x}^T(u)G_3\dot{x}(u)du\,ds\right\} \\
&\leqslant -\alpha V_6(x(t)) + (h_U^4/4)\dot{x}^T(t)G_3\dot{x}(t) - (h_U^2/2)e^{-\alpha h_U}\int_{t-h_U}^{t}\int_{s}^{t}\dot{x}^T(u)G_3\dot{x}(u)du\,ds \\
&\leqslant -\alpha V_6(x(t)) + (h_U^4/4)\dot{x}^T(t)G_3\dot{x}(t) - e^{-\alpha h_U}\left(\int_{t-h_U}^{t}\int_{s}^{t}\dot{x}(u)du\,ds\right)^T G_3\left(\int_{t-h_U}^{t}\int_{s}^{t}\dot{x}(u)du\,ds\right) \\
&= -\alpha V_6(x(t)) + (h_U^4/4)\dot{x}^T(t)G_3\dot{x}(t) - e^{-\alpha h_U}\left(h_U x(t) - \int_{t-h(t)}^{t} x(s)ds - \int_{t-h_U}^{t-h(t)} x(s)ds\right)^T G_3 \\
&\quad\times\left(h_U x(t) - \int_{t-h(t)}^{t} x(s)ds - \int_{t-h_U}^{t-h(t)} x(s)ds\right)
\end{aligned} \tag{54}$$

where Lemma 4 was used in (54).

By use of the procedure in the proof of Theorem 2, if the LMIs (49) and (50) hold, then $x^T(t)Px(t) \leqslant V(x(t)) \leqslant 1$. This completes the proof. □

Remark 6. In Theorems 1–4, free variables L_i $(i = 1,2,3)$ are used. However, by the use of Lemma 3, this can be eliminated as Corollary 1. For simplicity, only Theorem 1 without L_i $(i = 1,2,3)$ will be shown in Corollary 1.

Corollary 1. *For given $\alpha > 0$ and $h_U > 0$, the reachable sets of the system (1) with $0 \leqslant h(t) \leqslant h_U$ and $-\infty < \dot{h}(t) < \infty$ are bounded by an ellipsoid ε defined in (4) if there exist positive matrices P, N_1, and $\mathcal{G} = \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22}\end{bmatrix}$ satisfying the following LMIs:*

$$(N_\Gamma^k)^T \Sigma_1^i (N_\Gamma^k) < 0 \quad (i = 1,2,\ k = 1,\ldots,N), \tag{55}$$

where N_Γ^k is the orthogonal complement of Γ^k,

$$\Gamma^k = [A + A_k \quad A_d + A_{dk} \quad 0 \quad -I \quad 0 \quad 0 \quad B + B_k] \tag{56}$$

and Σ_1^i has been defined in Theorem 1.

Proof. Instead of (23), consider the zero equation

$$0 = \Gamma\zeta(t), \tag{57}$$

where

$$\Gamma = [A + \Delta A(t) \quad A_d + \Delta A_d(t) \quad 0 \quad -I \quad 0 \quad 0 \quad B + \Delta B(t)]. \tag{58}$$

By utilization of Lemma 3 and convex-hull properties, the proof of Corollary 1 follows straightforwardly from the proof of Theorem 1. So, the proof is not repeated here. □

Remark 7. In [4,6], the tuning parameters are five and one, respectively. The proposed Theorems 1–4 has only the one tuning parameter α. Therefore, as mentioned in [6], a feasibility check of Theorems 1–4 can be numerically tractable.

4. Numerical examples

Example 1. Consider the uncertain time-varying delayed system (1) utilized in [4,6] with the parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9+\rho \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1+0.5\rho \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \tag{59}$$

where $0 \leqslant h(t) \leqslant h_U$, $\left|\dot{h}(t)\right| \leqslant h_D$, $|\rho| \leqslant 0.2$ and $w^T(t)w(t) \leqslant w_m^2 = 1$.

Then, matrices A_i, A_{di} and B_i can be obtained in the form

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad B_i(i=1,2) = 0. \tag{60}$$

From Remark 1, by applying Theorems 1–4 to the system (59) when h_U = 0.7 and h_U = 0.75, the sizes $(\bar{\delta})$ of the ellipsoidal bound of the reachable sets obtained for different h_D are listed as Tables 1 and 2, respectively. From Tables 1 and 2, even if the most restrictive condition $-\infty < \dot{h}(t) < \infty$, which means that the time-derivative information of $\dot{h}(t)$ is unknown, is considered in Theorem 1, the results obtained are less conservative than those of [4,6]. Also, by applying Corollary 1 to the system (59), it can be seen that the sizes $\bar{\delta}$ are the same ones as in Theorem 1 in Tables 1 and 2. It should be noted that the number of decision variables of Corollary 1 was 17, whilst the number of variable in [6] was 23. This situation meant that Corollary 1 with fewer decision variables than those of [6] gives larger feasible regions than are found in the results presented in [6] in spite of the most restrictive constraints of $\dot{h}(t)$ being taken into consideration.

When $-\infty < \dot{h}(t) \leqslant h_D$ is considered in Theorem 2, improved results are obtained as shown in Tables 1 and 2. When the time-derivative condition $|\dot{h}(t)| \leqslant h_D$ is considered in Theorem 3, it can be seen that less conservative results than those of Theorems 1 and 2 are obtained in Theorem 3. Lastly, by considering a triple integral form of the Lyapunov–Krasovskii's functionals (53) when $|\dot{h}(t)| \leqslant h_D$, slightly improved results can be obtained.

Example 2. Consider the practical system which is the satellite system [15] shown in Fig. 1. The satellite system has two rigid bodies joined by a flexible link. The dynamic equations of this system are

Table 1
The sizes $(\bar{\delta})$ of ellipsoidal bound with h_U = 0.7 and different h_D (Example 1).

h_D	Kim [6] $(\lvert\dot{h}(t)\rvert \leqslant h_D)$	Fridman [4] (unknown)	Theorem 1 (unknown)	Theorem 2 $(-\infty < \dot{h}(t) \leqslant h_D)$	Theorem 3 $(\lvert\dot{h}(t)\rvert \leqslant h_D)$	Theorem 4 $(\lvert\dot{h}(t)\rvert \leqslant h_D)$
0	2.97	–	–	1.9151	1.7142	1.7103
0.1	3.30	–	–	1.9475	1.7795	1.7728
0.2	3.85	–	–	1.9818	1.8088	1.8032
0.3	4.85	–	–	2.0182	1.8251	1.8199
0.4	6.93	–	–	2.0558	1.8349	1.8344
0.5	12.84	–	–	2.0901	1.8535	1.8498
0.6	53.86	–	–	2.1123	1.8703	1.8674
0.7	–	–	–	2.1139	1.8901	1.8881
0.8	–	–	–	2.1139	1.9142	1.9131
0.9	–	–	–	2.1139	1.9438	1.9433
Unknown	–	19.71	2.1139	–	–	–

Table 2
The sizes $(\bar{\delta})$ of ellipsoidal bound with h_U = 0.75 and different h_D (Example 1).

h_D	Kim [6] $(\lvert\dot{h}(t)\rvert \leqslant h_D)$	Fridman [4] (unknown)	Theorem 1 (unknown)	Theorem 2 $(-\infty < \dot{h}(t) \leqslant h_D)$	Theorem 3 $(\lvert\dot{h}(t)\rvert \leqslant h_D)$	Theorem 4 $(\lvert\dot{h}(t)\rvert \leqslant h_D)$
0	3.34	–	–	2.3199	2.0137	2.0100
0.1	3.79	–	–	2.3762	2.0953	2.0866
0.2	4.53	–	–	2.4287	2.1306	2.1229
0.3	5.88	–	–	2.4845	2.1541	2.1464
0.4	8.85	–	–	2.5441	2.1770	2.1692
0.5	18.36	–	–	2.6035	2.2022	2.1947
0.6	127.70	–	–	2.6520	2.2318	2.2250
0.7	–	–	–	2.6654	2.2674	2.2625
0.8	–	–	–	2.6654	2.3130	2.3101
0.9	–	–	–	2.6654	2.3738	2.3725
Unknown	–	65.42	2.6654	–	–	–

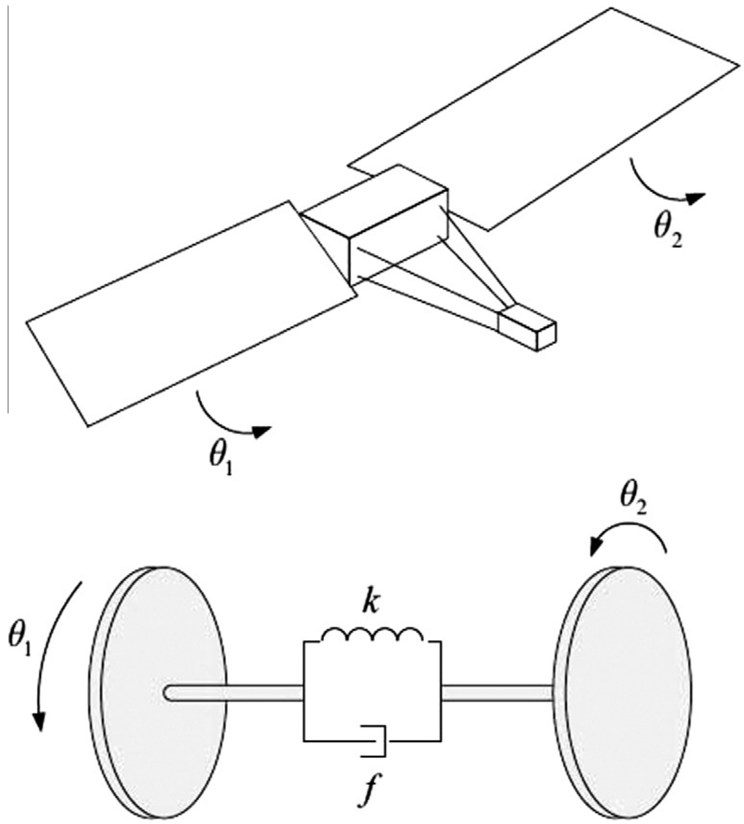

Fig. 1. Satellite system.

$$\begin{aligned} &J_1\ddot{\theta}_1(t) + f\left(\dot{\theta}_1(t) - \dot{\theta}_2(t)\right) + k(\theta_1(t) - \theta_2(t)) = u(t) + w(t), \\ &J_2\ddot{\theta}_2(t) + f\left(\dot{\theta}_1(t) - \dot{\theta}_2(t)\right) + k(\theta_1(t) - \theta_2(t)) = 0, \end{aligned} \tag{61}$$

where J_i (i = 1,2) are the moments of inertia of the two bodies (the main body and the instrumentation module), f is a viscous damping, k is a torque constant, $\theta_i(t)$ (i = 1,2) are the yaw angles for the two bodies, $u(t)$ is a control input and $w(t)$ is a disturbance. Assume J_i (i = 1,2) = 1, k = 0.09, f = 0.004, and state vector $x^T(t) = [x_1(t)x_2(t)x_3(t)x_4(t)]^T = [\theta_1(t)\theta_2(t)\dot{\theta}_1(t)\dot{\theta}_2(t)]^T$. Let us choose the control law as $u(t) = Kx(t - h(t))$ where $K = [-3.3092 \quad -0.7443 \quad -2.5909 \quad -8.0395]$ used in [15]. Then, system (61) can be represented by

$$\dot{x}(t) = Ax(t) + A_d x(t - h(t)) + Bw(t), \tag{62}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0900 & 0.0900 & -0.0040 & 0.0040 \\ 0.0900 & -0.0900 & 0.0040 & -0.0040 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3.3092 & -0.7443 & -2.5909 & -8.0395 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

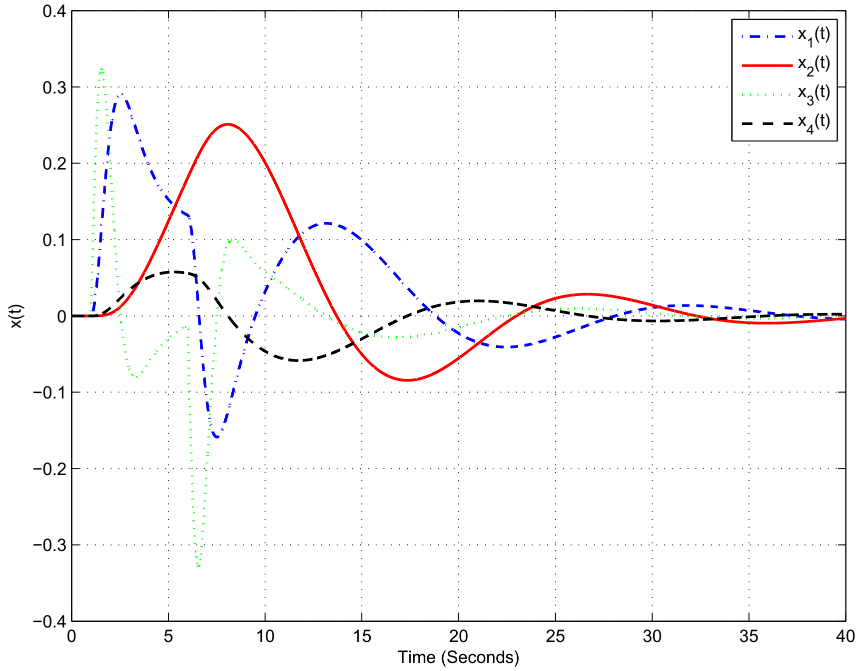

Fig. 2. State response of satellite system (Example 2).

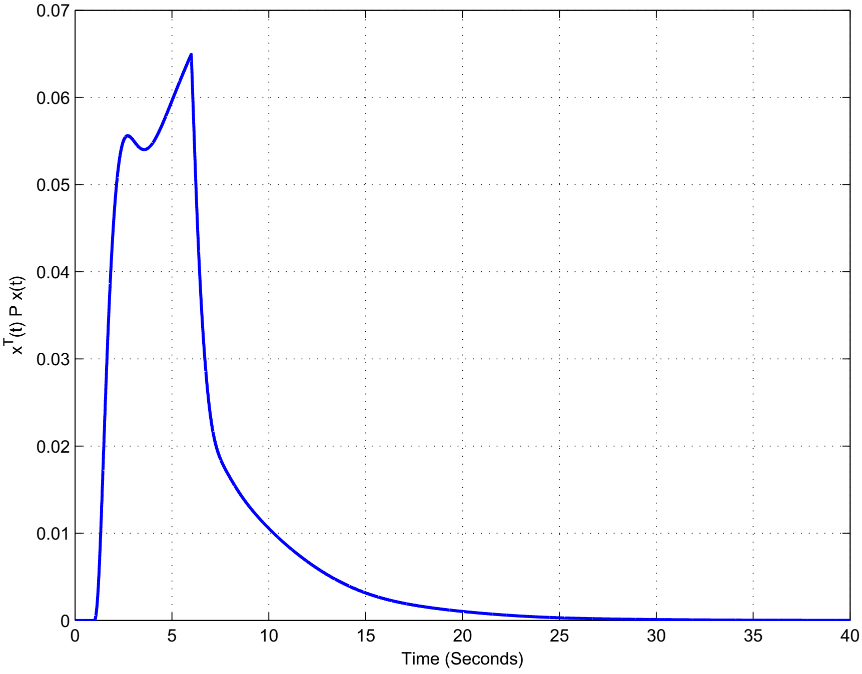

Fig. 3. Response of $x^T(t)Px(t)$ (Example 2).

Assuming that h_D is unknown. Application of Theorem 1 to system (61) when $h_U = 0.3$ yields $\bar{\delta} = 17.9171$ and the corresponding matrix P is

$$P = \begin{bmatrix} 0.4367 & 0.1507 & 0.1046 & 1.0434 \\ 0.1507 & 0.3075 & 0.0214 & 0.5640 \\ 0.1046 & 0.0214 & 0.0867 & 0.2812 \\ 1.0434 & 0.5640 & 0.2812 & 4.5204 \end{bmatrix}. \tag{63}$$

In order to check the validity of Theorem 1, it is assumed that the following $w(t)$

$$w(t) = \begin{cases} 1, & 1 \leqslant t \leqslant 6, \\ 0, & \text{otherwise} \end{cases} \tag{64}$$

is applied to the system (61). In the numerical simulations, it is assumed further that $h(t)$ is $0.3\,sin^2(100t)$ and that fourth-order Runge–Kutta method is applied to solve the systems with a time step size of 0.0001. Figs. 2 and 3 show the state response of the system that was obtained (61) and the plot of $x^T(t)Px(t)$, respectively. From Fig. 3, it can be seen that $x^T(t)Px(t)$ is always less than one, which verifies the validity of the proposed Theorem 1.

5. Conclusion

In this paper, improved LMI conditions for finding an ellipsoidal bound of the reachable set of uncertain systems with time-varying delays and disturbances have been proposed. The constraints of $\dot{h}(t)$ that were considered are more general ones than the conditions published in the existing literature up to now. In order to show the improved feasible regions arising with the proposed methods, the numerical example used in [4,6] was considered and showed an improvement of the presented LMI conditions even when information on $\dot{h}(t)$ is unknown. Also the satellite system was considered to show the effectiveness of the proposed methods.

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