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Reachable Set Bounding for Linear Discrete-Time Systems with Delays and Bounded Disturbances

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Abstract This paper addresses the problem of reachable set bounding for linear discrete-time systems that are subject to state delay and bounded disturbances. Based on the Lyapunov method, a sufficient condition for the existence of ellipsoid-based bounds of reachable sets of a linear uncertain discrete system is derived in terms of matrix inequalities. Here, a new idea is to minimize the projection distances of the ellipsoids on each axis with different exponential convergence rates, instead of minimization of their radius with a single exponential rate. A smaller bound can thus be obtained from the intersection of these ellipsoids. A numerical example is given to illustrate the effectiveness of the proposed approach.

Keywords Reachable set bounding · Interval time-varying delay · Lyapunov–Krasovskii functional · Projection distance

1 Introduction

It is well-known that time-varying delays and external disturbances are usually unavoidable in practical control systems due to data transformation, modeling inaccuracies, linearization approximations, unknown disturbances and measurement errors. Therefore, the design of any control schemes for dynamical systems subject to time

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delays and disturbances must take into account these influences on the closed-loop performance [1–4]. To obtain the available information about the maximum allowable time delay required to guarantee stability, extensive research has focused on determining delay-dependent stability or stabilization criteria for these systems. In this context, the problem of reachable set bounding for time-delay systems has received considerable attention in recent years, see, e.g., [5–11] and references therein. A reachable set is defined as a set of all the states that can be reached from the origin, in finite time, by input with bounded peak value and subject to uncertainties. Bounding reachable sets is of practical importance in the design of a suitable controller for these systems. Indeed, minimization of the reachable set bound can generally result in a controller with a larger gain to achieve better performance for the uncertain dynamical system under control [12]. When designing robust controllers for systems with time delays, which are known to cause system instability or performance degradation [13], information of the reachable set bound is of particular interest [7].

In [6], a delay-dependent condition for existence of an ellipsoidal bound of reachable sets of a linear system with time-varying delay and bounded peak input is obtained by applying the Lyapunov–Razumikhin approach. Based on the modified Lyapunov–Krasovskii functional, an improved condition for the reachable set bounding problem is proposed in [7] for time-delayed linear systems with bounded peak disturbances. A maximal Lyapunov–Krasovskii functional approach is used to derive conditions for reachable set bounding of linear systems with polytopic uncertainties and non-differential time-varying delays [8]. A further result for uncertain polytopic systems with interval time-varying delay is reported in [10]. For linear neutral systems, a recent result has been reported in [9] to determine the ellipsoidal bound. By using convex-hull properties and the Lyapunov method, sufficient conditions on reachable set bounding for uncertain dynamic systems with time-varying delays and bounded peak disturbances have also been established in terms of linear matrix inequalities (LMIs) [11].

The problem of finding the smallest possible ellipsoidal bound of reachable sets in the aforementioned papers is formulated into that of minimizing an ellipsoidal radius with all results being obtained for continuous time system. In the control design of practical systems, it is interesting to estimate the projection distances of the ellipsoid on all axes, since they reflect the physical magnitude of each coordinate of the system state trajectories. Moreover, using directly a discrete-time model also facilitates the control design, as nowadays the digital implementation is widely used for most control engineering applications. In this paper, dealing with discrete-time systems, we propose to minimize the projection distances of the ellipsoids on each axis of the coordinate systems with various convergence rates, instead of minimizing its radius with a single exponential rate, a smaller bound of reachable sets can thus be obtained from the intersection of these ellipsoids.

The paper is organized as follows. After the Introduction, the problem statement and preliminaries are introduced in Sect. 2. The main result is given in Sect. 3, followed by a numerical example, Sect. 4. Finally, a conclusion is drawn in Sect. 5.

General notation: Throughout this paper, $\mathbb{N}$ denotes the set of positive integer numbers; $\mathbb{R}^n$ denotes the n-dimensional space and the vector norm $\|.\|$; $\mathbb{R}^{n\times m}$ means the space of all matrices $(n \times m)$-dimension; A^T denotes the transpose of matrix A;

A is symmetric if $A = A^T$; I denotes the identity matrix and $(*)$ in a matrix means the symmetric term.

2 Problem Statement and Preliminaries

Consider the following linear discrete-time system:

$$\begin{aligned} x(k+1) &= A_0 x(k) + A_1 x\big(k-\tau(k)\big) + B\omega(k), \quad k > 0 \\ x(k) &\equiv 0, \quad k \in [-\tau_M, 0], \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state, A_0, A_1 and B are constant matrices with appropriate dimensions. The delay is time-varying, integer, belonging to a given interval $\tau(k) \in [\tau_m, \tau_M]$, and the disturbance $\omega(k) \in \mathbb{R}^p$ is assumed to be bounded:

$$\omega^T(k)\omega(k) \le \omega_m^2, \quad \forall k \ge 0, \tag{2}$$

where τ_m, τ_M are positive integers, and ω_m is a positive scalar.

A reachable set for delay system (1) subject to bounded disturbance (2) is defined as

$$\mathcal{R}_x := \big\{x(k) \in \mathbb{R}^n \big| x(k), \omega(k) \text{ satisfy (1) and (2)}, \ k \ge 0\big\}. \tag{3}$$

For a positive-definite symmetric matrix $P > 0$, we define an ellipsoid $\varepsilon(P, 1)$ bounding the reachable set (3) as follows:

$$\varepsilon(P, 1) := \big\{x \in \mathbb{R}^n \big| x^T P x \le 1\big\}, \tag{4}$$

whose projection distance on the hth axis ($h = 1, 2, \dots, n$) is determined as

$$d_h(P) := \sup\big\{2|x_h| \big| x = [x_1, x_2 \dots, x_n]^T \in \varepsilon(P, 1)\big\}. \tag{5}$$

Our aim is to find a set, as small as possible, that bounds the reachable sets of system (1) subject to time-varying delays and bounded disturbances (2).

The following lemma is useful for our main results.

Lemma 2.1 *Let V be a positive-definite function and $V(0) = 0$. If there exists a scalar $r > 1$ such that*

$$\Delta V(k) + \big(1 - r^{-1}\big)V(k) - \frac{1 - r^{-1}}{\omega_m^2}\omega^T(k)\omega(k) \le 0, \tag{6}$$

then $V(k) < 1$, $\forall k \ge 0$.

Proof From (2) and (6), we have

$$\begin{aligned} V(k+1) &\le r^{-1}V(k) + \frac{1 - r^{-1}}{\omega_m^2}\omega^T(k)\omega(k) \\ &\le r^{-1}V(k) + 1 - r^{-1} \\ &\le r^{-2}V(k-1) + \big(1 - r^{-1}\big)\big(r^{-1} + 1\big) \\ &\quad \dots \\ &\le r^{-(k+1)}V(0) + \big(1 - r^{-(k+1)}\big). \end{aligned}$$

Since $r > 1$ and $V(0) = 0$, $V(k) < 1$, $\forall k \ge 0$. This completes the proof. □

From the above lemma, it can be seen that $\|V(k)-1\| \le r^{-k}\|(V(0)-1)\|, \forall k \ge 0$, thus $r > 1$ is a convergence rate. We are now ready to state our results for obtaining a sufficient condition for the existence of a smaller bound for the reachable sets of linear discrete-time systems with interval time-varying delays and bounded disturbances.

3 Main Results

The following notations are specifically used in our development. For an axis index $h = 1, 2, \dots, n$, we denote an integer number $\lambda_h \in (0, \tau_M - \tau_m]$; a positive scalar $r_h > 1$; symmetric positive-definite matrices $P_h, Q_{1h}, Q_{2h}, Q_{3h}, R_{1h}, R_{2h}, R_{3h} \in \mathbb{R}^{n\times n}$; $(2n \times 2n)$-matrices

$$X_h = \begin{bmatrix} X_{11h} & X_{12h} \\ \star & X_{22h} \end{bmatrix}, \qquad Y_h = \begin{bmatrix} Y_{11h} & Y_{12h} \\ \star & Y_{22h} \end{bmatrix}, \qquad Z_h = \begin{bmatrix} Z_{11h} & Z_{12h} \\ \star & Z_{22h} \end{bmatrix},$$

$$U_h = \begin{bmatrix} U_{11h} & U_{12h} \\ \star & U_{22h} \end{bmatrix}, \qquad V_h = \begin{bmatrix} V_{11h} & V_{12h} \\ \star & V_{22h} \end{bmatrix}, \qquad S_h = \begin{bmatrix} S_{11h} & S_{12h} \\ \star & S_{22h} \end{bmatrix};$$

and also $(n \times 2n)$-matrices

$$\begin{aligned} W_h^T &= \begin{bmatrix} W_{1h}^T & W_{2h}^T \end{bmatrix}, & K_h^T &= \begin{bmatrix} K_{1h}^T & K_{2h}^T \end{bmatrix}, \\ L_h^T &= \begin{bmatrix} L_{1h}^T & L_{2h}^T \end{bmatrix}, & M_h^T &= \begin{bmatrix} M_{1h}^T & M_{2h}^T \end{bmatrix}, \\ N_h^T &= \begin{bmatrix} N_{1h}^T & N_{2h}^T \end{bmatrix}, & T_h^T &= \begin{bmatrix} T_{1h}^T & T_{2h}^T \end{bmatrix}, \\ H_h^T &= \begin{bmatrix} H_{1h}^T & H_{2h}^T \end{bmatrix}, & E_h^T &= \begin{bmatrix} E_{1h}^T & E_{2h}^T \end{bmatrix}. \end{aligned}$$

We also define $\tau_h := \tau_m + \lambda_h$; $\tau_{Mh} := \tau_M - \tau_h$; $\tau_{hm} := \tau_h - \tau_m$; $G_h := [G_{ij}]$, where

$$G_{ij} := \begin{cases} 1, & \text{if } i = j = h, \\ 0, & \text{otherwise,} \end{cases}$$

and matrices

$$\Omega_h = \begin{bmatrix} \Omega_{11h} & \Omega_{12h} & L_{1h} - M_{1h} & K_{1h} - L_{1h} & -W_{1h} & \Omega_{16h} \\ \star & \Omega_{22h} & L_{2h} - M_{2h} & K_{2h} - L_{2h} & -W_{2h} & \Omega_{26h} \\ \star & \star & -r_h^{-\tau_m} Q_{3h} & 0 & 0 & 0 \\ \star & \star & \star & -r_h^{-\tau_h} Q_{2h} & 0 & 0 \\ \star & \star & \star & \star & -r_h^{-\tau_M} Q_{1h} & 0 \\ \star & \star & \star & \star & \star & \Omega_{66h} \end{bmatrix}, \tag{7}$$

$$\Sigma_h = \begin{bmatrix} \Sigma_{11h} & \Sigma_{12h} & N_{1h} - T_{1h} & H_{1h} - E_{1h} & -H_{1h} & \Sigma_{16h} \\ \star & \Sigma_{22h} & N_{2h} - T_{2h} & H_{2h} - E_{2h} & -H_{2h} & \Sigma_{26h} \\ \star & \star & -r_h^{-\tau_m} Q_{3h} & 0 & 0 & 0 \\ \star & \star & \star & -r_h^{-\tau_h} Q_{2h} & 0 & 0 \\ \star & \star & \star & \star & -r_h^{-\tau_M} Q_{1h} & 0 \\ \star & \star & \star & \star & \star & \Sigma_{66h} \end{bmatrix}, \tag{8}$$

where

$$
\begin{aligned}
A &:= A_0 - I_n,\, O_{1h} := \tau_{Mh} R_{1h} + \tau_{hm} R_{2h} + \tau_m R_{3h}, \\
\Omega_{11h} &:= \left(1 - r_h^{-1}\right) P_h + A^T O_{1h} A + Q_{1h} + Q_{2h} + Q_{3h} + M_{1h} + M_{1h}^T + \tau_{Mh} X_{11h} \\
&\quad + \tau_{hm} Y_{11h} + \tau_m Z_{11h} + P_h A + A^T P_h^T + A^T P_h A, \\
\Omega_{12h} &:= P_h A_1 + A^T O_{1h} A_1 + A^T P_h A_1 + W_{1h} - K_{1h} + M_{2h}^T + \tau_{Mh} X_{12h} \\
&\quad + \tau_{hm} Y_{12h} + \tau_m Z_{12h}, \\
\Omega_{16h} &:= P_h B + A^T O_{1h} B + A^T P_h B, \\
\Omega_{22h} &:= A_1^T P_h A_1 + A_1^T O_{1h} A_1 + W_{2h} + W_{2h}^T - K_{2h} - K_{2h}^T + \tau_{Mh} X_{22h} \\
&\quad + \tau_{hm} Y_{22h} + \tau_m Z_{22h}, \\
\Omega_{26h} &:= A_1^T O_{1h} B + A_1^T P_h B, \\
\Omega_{66h} &:= B^T P_h B + B^T O_{1h} B - \frac{1 - r_h^{-1}}{\omega_m^2} I_p, \\
\Sigma_{11h} &:= \left(1 - r_h^{-1}\right) P_h + A^T O_{1h} A + Q_{1h} + Q_{2h} + Q_{3h} + T_{1h} + T_{1h}^T + \tau_{Mh} U_{11h} \\
&\quad + \tau_{hm} V_{11h} + \tau_m S_{11h} + P_h A + A^T P_h^T + A^T P_h A, \\
\Sigma_{12h} &:= P_h A_1 + A^T O_{1h} A_1 + A^T P_h A_1 + E_{1h} - N_{1h} + T_{2h}^T + \tau_{Mh} U_{12h} \\
&\quad + \tau_{hm} V_{12h} + \tau_m S_{12h}, \\
\Sigma_{16h} &:= P_h B + A^T O_{1h} B + A^T P_h B, \\
\Sigma_{22h} &:= A_1^T P_h A_1 + A_1^T O_{1h} A_1 + E_{2h} + E_{2h}^T - N_{2h} - N_{2h}^T + \tau_{Mh} U_{22h} \\
&\quad + \tau_{hm} V_{22h} + \tau_m S_{22h}, \\
\Sigma_{26h} &:= A_1^T O_{1h} B + A_1^T P_h B, \\
\Sigma_{66h} &:= B^T P_h B + B^T O_{1h} B - \frac{1 - r_h^{-1}}{\omega_m^2} I_p;
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{1h} &:= \begin{bmatrix} X_h & W_h \\ \star & r_h^{-\tau_M} R_{1h} \end{bmatrix}, \qquad \Psi_{2h} := \begin{bmatrix} X_h & K_h \\ \star & r_h^{-\tau_M} R_{1h} \end{bmatrix}, \\
\Psi_{3h} &:= \begin{bmatrix} Y_h & L_h \\ \star & r_h^{-\tau_h} R_{2h} \end{bmatrix}, \qquad \Psi_{4h} := \begin{bmatrix} Z_h & M_h \\ \star & r_h^{-\tau_m} R_{3h} \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{5h} &:= \begin{bmatrix} U_h & H_h \\ \star & r_h^{-\tau_M} R_{1h} \end{bmatrix}, \qquad \Psi_{6h} := \begin{bmatrix} V_h & E_h \\ \star & r_h^{-\tau_h} R_{2h} \end{bmatrix}, \\
\Psi_{7h} &:= \begin{bmatrix} V_h & N_h \\ \star & r_h^{-\tau_h} R_{2h} \end{bmatrix}, \qquad \Psi_{8h} := \begin{bmatrix} S_h & T_h \\ \star & r_h^{-\tau_m} R_{3h} \end{bmatrix}.
\end{aligned}
$$

Theorem 3.1 *For each $h \in \{1, 2, \ldots, n\}$, with given positive integers τ_m, τ_M and a positive scalar ω_m, if there exist an integer number $\lambda_h \in [0, \tau_M - \tau_m]$, scalars $r_h > 1$, $\delta_h > 0$, symmetric positive-definite matrices P_h, Q_{1h}, Q_{2h}, Q_{3h}, R_{1h}, R_{2h}, R_{3h}, X_h, Y_h, Z_h, U_h, V_h, S_h, and matrices W_h, K_h, L_h, M_h, H_h, E_h, N_h, T_h, such that the following conditions hold*:

$$
P_h > \delta_h G_h, \tag{9}
$$

$$\Omega_h < 0, \quad \Sigma_h < 0, \tag{10}$$

$$\Psi_{jh} \geq 0 \quad \forall j \in \{1, \dots, 8\}; \tag{11}$$

then the reachable sets of the system (1) *are bounded by*

$$\bigcap_{h=1}^{n} \varepsilon(P_h, 1). \tag{12}$$

Moreover, for each $h = 1, 2, \dots, n$, *the projection distance of the ellipsoid* $\varepsilon(P_h, 1)$ *on the hth axis is*

$$d_h(P_h) = \frac{2}{\sqrt{\delta_h}}. \tag{13}$$

Proof From (1), by defining $y(k) := x(k+1) - x(k)$, we have

$$y(k) := (A_0 - I_n)x(k) + A_1 x\bigl(k - \tau(k)\bigr) + B\omega(k).$$

For an axis index $h \in \{1, 2, \dots, n\}$, note that, for any $k \in \mathbb{N}$, we have either $\tau(k) \in [\tau_m, \tau_h]$ or $\tau(k) \in (\tau_h, \tau_M]$. Let us define two sets

$$\Pi_{1h} := \bigl\{k \in \mathbb{N} \big| \tau(k) \in (\tau_h, \tau_M]\bigr\},$$

and

$$\Pi_{2h} := \bigl\{k \in \mathbb{N} \big| \tau(k) \in [\tau_m, \tau_h]\bigr\}.$$

Consider the following Lyapunov–Krasovskii functional:

$$V_h = V_{1h} + V_{2h} + V_{3h}, \tag{14}$$

where

$$\begin{aligned} V_{1h} &= x^T(k) P_h x(k), \\ V_{2h} &= \sum_{s=k-\tau_M}^{k-1} x^T(s) Q_{1h} x(s) + \sum_{s=k-\tau_h}^{k-1} x^T(s) Q_{2h} x(s) + \sum_{s=k-\tau_m}^{k-1} x^T(s) Q_{3h} x(s), \\ V_{3h} &= \sum_{s=-\tau_M}^{-\tau_h-1} \sum_{v=k+s}^{k-1} y^T(v) R_{1h} y(v) + \sum_{s=-\tau_h}^{-\tau_m-1} \sum_{v=k+s}^{k-1} y^T(v) R_{2h} y(v) \\ &\quad + \sum_{s=-\tau_m}^{-1} \sum_{v=k+s}^{k-1} y^T(v) R_{3h} y(v). \end{aligned}$$

By taking the difference of functional (11), we obtain

$$\begin{aligned} \Delta V_{1h}(k) &= x^T(k)\bigl(1 - r_h^{-1}\bigr) P_h x(k) + 2x^T(k) P_h y(k) + y^T(k) P_h y(k) \\ &\quad + \bigl(r_h^{-1} - 1\bigr) V_{1h}(k), \\ \Delta V_{2h}(k) &= x^T(k)(Q_{1h} + Q_{2h} + Q_{3h})x(k) - r_h^{-\tau_M} x^T(k - \tau_M) Q_{1h} x(k - \tau_M) \\ &\quad - r_h^{-\tau_h} x^T(k - \tau_h) Q_{2h} x(k - \tau_h) - r_h^{-\tau_m} x^T(k - \tau_m) Q_{3h} x(k - \tau_m) \\ &\quad + \bigl(r_h^{-1} - 1\bigr) V_{2h}(k), \end{aligned}$$

$$\begin{aligned}\Delta V_{3h}(k) &= y^T(k)\big[(\tau_M-\tau_h)R_{1h}+(\tau_h-\tau_m)R_{2h}+\tau_m R_{3h}\big]y(k)\\&\quad-\sum_{s=k-\tau_M}^{k-\tau_h-1} r_h^{s-k}y^T(s)R_{1h}y(s)-\sum_{s=k-\tau_h}^{k-\tau_m-1} r_h^{s-k}y^T(s)R_{2h}y(s)\\&\quad-\sum_{s=k-\tau_m}^{k-1} r_h^{s-k}y^T(s)R_{3h}y(s)+\big(r_h^{-1}-1\big)V_{3h}(k).\end{aligned}$$

From (11) and by noting that $r_h^s > 1$, $\forall s > 0$, we have

$$\begin{aligned}\Delta V_h(k) &\le x^T(k)\big(1-r_h^{-1}\big)P_h x(k)+2x^T(k)P_h y(k)+y^T(k)P_h y(k)\\&\quad+x^T(k)(Q_{1h}+Q_{2h}+Q_{3h})x(k)\\&\quad-r_h^{-\tau_M}x^T(k-\tau_M)Q_{1h}x(k-\tau_M)-r_h^{-\tau_h}x^T(k-\tau_h)Q_{2h}x(k-\tau_h)\\&\quad-r_h^{-\tau_m}x^T(k-\tau_m)Q_{3h}x(k-\tau_m)\\&\quad+y^T(k)\big[(\tau_M-\tau_h)R_{1h}+(\tau_h-\tau_m)R_{2h}+\tau_m R_{3h}\big]y(k)\\&\quad-\sum_{s=k-\tau_M}^{k-\tau_h-1} r_h^{-\tau_M}y^T(s)R_{1h}y(s)-\sum_{s=k-\tau_h}^{k-\tau_m-1} r_h^{-\tau}y^T(s)R_{2h}y(s)\\&\quad-\sum_{s=k-\tau_m}^{k-1} r_h^{-\tau_m}y^T(s)R_{3h}y(s)+\big(r_h^{-1}-1\big)V_h(k).\end{aligned}\tag{15}$$

We distinguish two sub-intervals of the time delay in the following cases.

Case I For $k \in \Pi_{1h}$, i.e. the time delay $\tau(k) \in (\tau_h, \tau_M]$, we have

$$-\sum_{s=k-\tau_M}^{k-\tau_h-1} y^T(s)R_{1h}y(s) = -\sum_{s=k-\tau_M}^{k-\tau(k)-1} y^T(s)R_{1h}y(s)-\sum_{s=k-\tau(k)}^{k-\tau_h-1} y^T(s)R_{1h}y(s).\tag{16}$$

By denoting $\xi^T(k) = [x^T(k)\ x^T(k-\tau(k))]$, $\eta^T(k) = [x^T(k)\ x^T(k-\tau(k))\ x^T(k-\tau_m)\ x^T(k-\tau_h)\ x^T(k-\tau_M)\ \omega^T(k)]$, and $\zeta^T(k,s) = [\xi^T(k)\ y^T(s)]$ for any matrices W_h, K_h, L_h and M_h, the following equations always hold:

$$2\xi^T(k)W_h\Bigg[x\big(k-\tau(k)\big)-x(k-\tau_M)-\sum_{s=k-\tau_M}^{k-\tau(k)-1} y(s)\Bigg]=0,\tag{17}$$

$$2\xi^T(k)K_h\Bigg[x(k-\tau_h)-x\big(k-\tau(k)\big)-\sum_{s=k-\tau(k)}^{k-\tau_h-1} y(s)\Bigg]=0,\tag{18}$$

$$2\xi^T(k)L_h\Bigg[x(k-\tau_m)-x(k-\tau_h)-\sum_{s=k-\tau_h}^{k-\tau_m-1} y(s)\Bigg]=0,\tag{19}$$

$$2\xi^T(k)M_h\Bigg[x(k)-x(k-\tau_m)-\sum_{s=k-\tau_m}^{k-1} y(s)\Bigg]=0.\tag{20}$$

We also have

$$(\tau_M-\tau_h)\xi^T(k)X_h\xi(k)-\sum_{s=k-\tau_M}^{k-\tau(k)-1}\xi^T(k)X_h\xi(k)-\sum_{s=k-\tau(k)}^{k-\tau_h-1}\xi^T(k)X_h\xi(k)=0, \tag{21}$$

$$(\tau_h-\tau_m)\xi^T(k)Y_h\xi(k)-\sum_{s=k-\tau_h}^{k-\tau_m-1}\xi^T(k)Y_h\xi(k)=0, \tag{22}$$

$$\tau_m\xi_h^T(k)Z_h\xi(k)-\sum_{s=k-\tau_m}^{k-1}\xi_h^T(k)Z_h\xi(k)=0. \tag{23}$$

Thus, from (15)–(23), we obtain

$$\begin{aligned}
&\Delta V_h(k)+\left(1-r_h^{-1}\right)V_h(k)-\frac{1-r_h^{-1}}{\omega_m^2}\omega^T(k)\omega(k)\\
&\le\eta^T(k)\Omega_h\eta(k)-\sum_{s=k-\tau_M}^{k-\tau(k)-1}\zeta^T(k,s)\Psi_{1h}\zeta(k,s)\,ds\\
&\quad-\sum_{s=k-\tau(k)}^{k-\tau_h-1}\zeta^T(k,s)\Psi_{2h}\zeta(k,s)\,ds\\
&\quad-\sum_{s=k-\tau_h}^{k-\tau_m-1}\zeta^T(k,s)\Psi_{3h}\zeta(k,s)\,ds\\
&\quad-\sum_{s=k-\tau_m}^{k-1}\zeta^T(k,s)\Psi_{4h}\zeta(k,s)\,ds\\
&<0.
\end{aligned}$$

Case II For $k\in\Pi_{2h}$, i.e. the time delay $\tau(k)\in[\tau_m,\tau_h]$, we have

$$-\sum_{s=k-\tau_h}^{k-\tau_m-1}y^T(s)R_{2h}y(s)=-\sum_{s=k-\tau_h}^{k-\tau(k)-1}y^T(s)R_{2h}y(s)-\sum_{s=k-\tau(k)}^{k-\tau_m-1}y^T(s)R_{2h}y(s).$$

Similarly, we also obtain

$$\begin{aligned}
&\Delta V_h(k)+\left(1-r_h^{-1}\right)V_h(k)-\frac{1-r_h^{-1}}{\omega_m^2}\omega^T(k)\omega(k)\\
&\le\eta^T(k)\Sigma_h\eta(k)-\sum_{s=k-\tau_M}^{k-\tau(k)-1}\zeta^T(k,s)\Psi_{5h}\zeta(k,s)\,ds\\
&\quad-\sum_{s=k-\tau(k)}^{k-\tau_h-1}\zeta^T(k,s)\Psi_{6h}\zeta(k,s)\,ds
\end{aligned}$$

$$\begin{aligned} &- \sum_{s=k-\tau_h}^{k-\tau_m-1} \zeta^T(k,s)\Psi_{7h}\zeta(k,s)\,ds \\ &- \sum_{s=k-\tau_m}^{k-1} \zeta^T(k,s)\Psi_{8h}\zeta(k,s)\,ds \\ < 0. \end{aligned}$$

By using Lemma 2.1, we have $V_h(k) < 1$, $\forall k \geq 0$, which yields $x^T(k)P_h x(k) < 1$, $\forall k \geq 0$. Thus, the reachable sets of the system (1) for $k \geq 0$ is bounded by ellipsoid $\varepsilon(P_h, 1)$, according to definition (4). Moreover, a smaller bound of the reachable sets of the system (1) can be obtained from the intersection of the ellipsoids given in (12). From condition (9), the projection distances can then be obtained as (13). The proof is completed. □

Remark 3.1 It should be noted that the matrix inequalities (9) and (10) are not LMIs to be computationally solved with the use of Matlab's LMI toolbox. However, when λ_h, r_h, and δ_h are fixed, matrix inequalities (9) and (10) can reduce to LMIs. Therefore, we can combine a three-dimensional search method with Matlab's LMI Toolbox to solve them.

Remark 3.2 In the case of a single convergence rate $r_h = r$, $\delta_h = \delta$, $G_h = I_n$, $P_h = P$, and $Q_{jh} = Qj$, $R_{jh} = R_j$ for $j = 1, 2, 3$; $h = 1, 2, \ldots, n$, our approach is similar to the ones proposed in [6–11]. The reachable set (3) of system (1) is then bounded by ellipsoid $\varepsilon(P, 1)$, whose volume, as suggested therein, can be minimized by solving the following optimization problem for a scalar $\delta > 0$:

$$\min\left(\frac{1}{\delta}\right) \text{ s.t. } \begin{cases} (a) & P \geq \delta I \\ (b) & (9) \text{ and } (10). \end{cases}$$

Remark 3.3 It can be seen from Theorem 3.1 that the constraints $P_h \geq \delta_h G_h$ are less conservative than $P \geq \delta I$. Thus, the projection distances of the ellipsoid $\varepsilon(P_h, 1)$, obtained with respect to the hth axis, is always smaller than the diameter of ellipsoid $\varepsilon(P, 1)$. Furthermore, by taking the intersection of the ellipsoids, we can even further improve the bound for the reachable set as the intersection of the ellipsoids $\varepsilon = (\bigcap_{h=1}^n \varepsilon(P_h, 1)) \bigcap \varepsilon(P, 1)$.

4 Numerical Example

Consider system (1) with the following matrices:

$$A_0 = \begin{bmatrix} 0.8 & -0.01 \\ -0.5 & 0.09 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -0.02 & 0 \\ -0.1 & -0.01 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.01 \\ 0.15 \end{bmatrix}, \qquad \tau(k) \in [0, 15] \text{ and } \omega(k) = \sin(7k).$$

By solving LMIs (9)–(11) of Theorem 3.1, we find

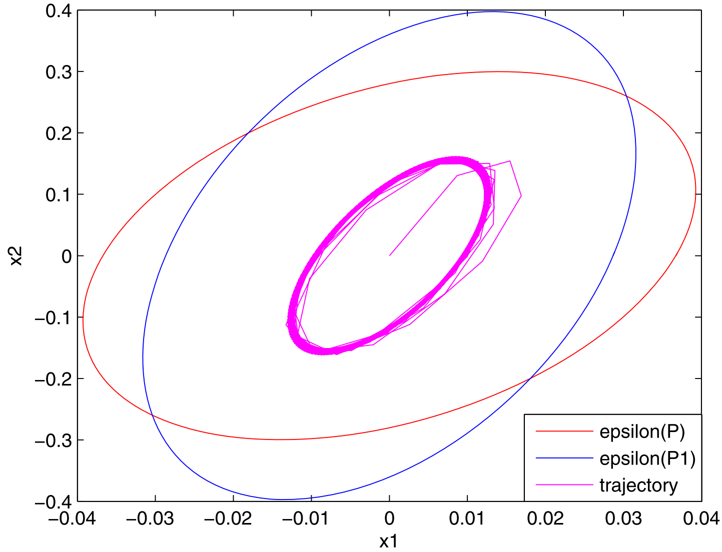

Fig. 1 Ellipsoidal bounds of the reachable set by $\varepsilon(P,1)$ and $\varepsilon(P_1,1)$

$$P = \begin{bmatrix} 741.3221 & -34.4791 \\ -34.4791 & 12.7422 \end{bmatrix}, \qquad P_1 = 10^3 \times \begin{bmatrix} 1.2159 & -0.0407 \\ -0.0407 & 0.0077 \end{bmatrix},$$
$$P_2 = 10^3 \times \begin{bmatrix} 1.0440 & -0.0608 \\ -0.0608 & 0.0202 \end{bmatrix}$$

and convergence rates $r = 1.118$, $r_1 = 1.150$, and $r_2 = 1.155$.

The ellipsoidal bound by $\varepsilon(P,1)$ as obtained from [6–11], $\varepsilon(P_1,1)$ and $\varepsilon(P_2,1)$ for system (1) proposed in this paper are, respectively, depicted in Fig. 1 and Fig. 2. By using Theorem 3.1 and Remark 3.3, a smaller bound is obtained from intersection of these ellipsoids as $\varepsilon = (\bigcap_{h=1}^{n} \varepsilon(P_h,1)) \bigcap \varepsilon(P,1)$ for system (1) as depicted in Fig. 3, whereby it can be seen that the system trajectory is prescribed in a smaller bound. Moreover, in order to design a robust controller, the information that the coordinates x_1 and x_2 of the trajectory are respectively bounded by $d_1 = 0.06$ and $d_2 = 0.48$, obtained from this paper, may be more favorable to the control design than by $d = 0.6$, obtained from $\varepsilon(P,1)$.

5 Conclusion

This paper has proposed a sufficient condition for the existence of a smaller bound of the reachable sets for linear discrete-time systems subject to disturbances and time-varying delays. By minimizing the projection distances of the ellipsoids on each coordinate axis with different convergence rates, an improved bound can be obtained as compared to the bound determined by minimizing the ellipsoidal radius with a single convergence rate. A smaller bound can be further obtained from the intersection of these ellipsoids. A numerical example is given to illustrate the effectiveness of the proposed approach.

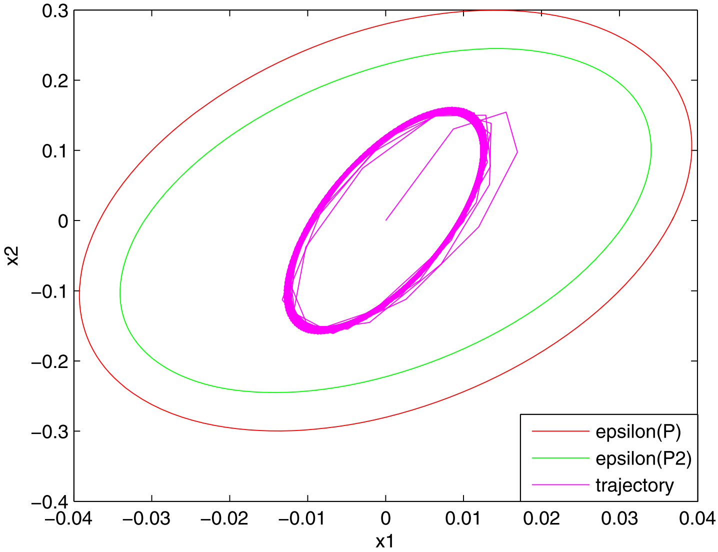

Fig. 2 Ellipsoidal bounds of the reachable set by $\varepsilon(P, 1)$ and $\varepsilon(P_2, 1)$

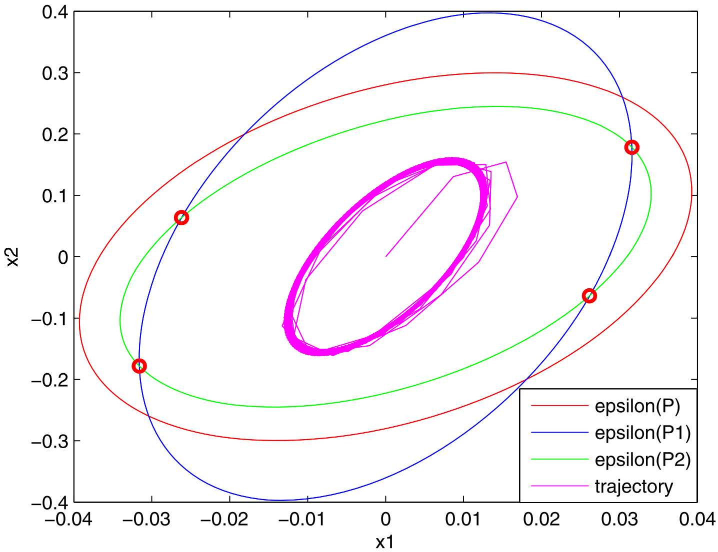

Fig. 3 An improved bound of the reachable set by $\varepsilon = (\bigcap_{h=1}^{n} \varepsilon(P_h, 1)) \bigcap \varepsilon(P, 1)$

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References

1. Udwadia, F.E., von Bremen, H., Phohomsiri, P.: Time-delayed control design for active control of structures: principles and applications. Struct. Control Health Monit. **14**, 27–61 (2007)
2. Phat, V.N., Niamsup, P.: Stabilization of linear nonautonomous systems with norm-bounded controls. J. Optim. Theory Appl. **131**, 135–149 (2006)
3. Gao, H., Chen, T.: New results on stability of discrete-time systems with time-varying state delay. IEEE Trans. Autom. Control **52**, 328–334 (2007)
4. Meng, X., Lam, J., Du, B., Gao, H.: A delay-partitioning approach to the stability analysis of discrete-time systems. Automatica **46**, 610–614 (2010)
5. Durieu, C., Walter, E., Polyak, B.: Multi-input multi-output ellipsoidal state bounding. J. Optim. Theory Appl. **111**, 273–303 (2001)
6. Fridman, E., Shaked, U.: On reachable sets for linear systems with delay and bounded peak inputs. Automatica **39**, 2005–2010 (2003)
7. Kim, J.H.: Improved ellipsoidal bound of reachable sets for time-delayed linear systems with disturbances. Automatica **44**, 2940–2943 (2008)
8. Zuo, Z., Ho, D.W.C., Wang, Y.: Reachable set bounding for delayed systems with polytopic uncertainties: the maximal Lyapunov–Krasovskii functional approach. Automatica **46**, 949–952 (2010)
9. Shen, C., Zhong, S.: The ellipsoidal bound of reachable sets for linear neutral systems with disturbances. J. Franklin Inst. **348**, 2570–2585 (2011)
10. Nam, P.T., Pathirana, P.N.: Further result on reachable set bounding for linear uncertain polytopic systems with interval time-varying delays. Automatica **47**, 1838–1841 (2011)
11. Kwon, O.M., Lee, S.M., Park, J.H.: On the reachable set bounding of uncertain dynamic systems with time-varying delays and disturbances. Inf. Sci. **181**, 3735–3748 (2011)
12. Corless, M., Leitmann, G.: Bounded controllers for robust exponential convergence. J. Optim. Theory Appl. **76**, 1–12 (1993)
13. Peng, C., Tian, Y.C.: Delay-dependent robust H_∞ control for uncertain systems with time-varying delay. Inf. Sci. **179**, 3187–3197 (2009)