International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

Limits of propriety for linear-quadratic regulator problems

C. D. JOHNSON [a]

[a] Electrical and Computer Engineering Department , University of Alabama in Huntsville , Huntsville, Alabama, 35899, U.S.A

Published online: 30 Jun 2010.

INT. J. CONTROL, 1987, VOL. 45, NO. 5, 1835–1846

Limits of propriety for linear–quadratic regulator problems

C. D. JOHNSON†

The linear–quadratic-regulator (LQR) theory, as presented in existing textbooks, restricts the Q-matrix to be positive-definite (or positive-semidefinite). In the present paper it is shown that this restriction on Q is unwarranted because there exist practical applications of LQR theory involving sign-indefinite (and even *negative-definite*!) Q-matrices that lead to well-defined well-behaved optimal solutions. A general class of scalar-control LQR problems is examined in detail and the fundamental theorem governing 'propriety' of such problems is derived. A procedure is then described for finding the set $\mathbf{Q}$ of *all* Q-matrices (including indefinite and negative-definite Q) that satisfy the propriety condition, i.e. lead to well-behaved optimal solutions. The procedure is illustrated by several worked examples, including explicit general solutions for $\mathbf{Q}$ for the cases $n = 1$ and $n = 2$.

1. Introduction

This paper is concerned with the so-called infinite-time linear–quadratic-regulator (LQR) problem for the real-valued time-invariant plant

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ x(0) &= x_0 \end{aligned}\right\} \tag{1}$$

where x is a real n-vector, u a real r-vector, A, B are real, the control u is unconstrained and (1) is assumed to be completely controllable. In particular, the control u in (1) is to be designed to regulate $x(t) \to 0$ as $t \to \infty$ and simultaneously *minimize* the quadratic performance index

$$J = \int_0^\infty [x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)]\, dt \tag{2}$$

where (Q, R) are respectively real $n \times n$ and $r \times r$ symmetric 'weighting' matrices with $R > 0$. In the typical textbook presentation of LQR theory (Athans and Falb 1966, Bryson and Ho 1969, Kwakernaak and Sivan 1972) the matrix Q in (2) is restricted to be positive-definite (or positive-semidefinite subject to an additional side-condition; see § 2). These conventional restrictions on Q appear to reflect a widely-held belief that the LQR minimization problem (1), (2) 'breaks down', or 'degenerates' in some manner (i.e. J has no greatest lower bound), when Q becomes sign-indefinite or negative-definite; see for example the remarks in Palm (1983, pp. 658, 659).

The purpose of this paper is to show that the LQR problem (1), (2) can, in fact, have well-behaved optimal solutions for a certain range of sign-indefinite (and even negative-definite) weighting matrices Q. Moreover, such indefinite and negative-definite Q-matrices can arise quite naturally in the course of applying modern control ideas and LQR theory to practical applications. In this paper we intro-

Received 23 July 1986.

† Electrical and Computer Engineering Department, University of Alabama in Huntsville, Huntsville, Alabama 35899, U.S.A.

duce the idea of a 'proper' LQR problem (1), (2) as a means for indicating when the solution of (1), (2) is well-behaved. Then we derive the fundamental theorem governing propriety of such LQR problems, for the scalar-control case ($r = 1$), and show how one can identify the set $\mathbf{Q}$ of *all* matrices Q (including those that are sign-indefinite and negative-definite) that lead to a proper LQR problem for that case. Such Q are called 'proper'. Several concrete examples are worked, and the geometry of the associated Q-proper sets $\mathbf{Q}$ is illustrated. These latter results clearly show that conventional LQR theory, with the usual restrictions $Q > 0$ (or $Q \geqslant 0$), fails to accommodate a broad range of 'proper' Q-matrices that lead to well-behaved optimal solutions of (1), (2).

2. Brief review of conventional LQR theory for (1), (2)

The typical approach to the LQR problem (1), (2), as presented in existing textbooks, proceeds as follows. First one assumes that $Q > 0$, or that $Q \geqslant 0$ with the additional constraint that the pair (A, H) is completely observable where $Q = H^{\mathrm{T}} H$. Then, using the Hamilton–Jacobi–Bellman equation and Kalman's idea of assuming a positive-definite quadratic solution $V = x^{\mathrm{T}} P x$, $P = P^{\mathrm{T}} > 0$, one can show that the optimal (minimizing) control u^0 exists, is unique, and is given by

$$u^0 = -R^{-1} B^{\mathrm{T}} P x \tag{3}$$

where P is found as the unique symmetric positive-definite solution of the steady-state Riccati equation

$$PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + Q = 0 \tag{4}$$

Moreover, the corresponding closed-loop optimally controlled system (1), (3) is given by

$$\dot{x} = (A - BR^{-1} B^{\mathrm{T}} P)x \tag{5}$$

and it can be shown that all eigenvalues of the optimal closed-loop matrix

$$A_{\mathrm{c}} = (A - BR^{-1} B^{\mathrm{T}} P) \tag{6}$$

have *strictly negative* real parts, i.e. the state $x = 0$ is always a globally asymptotically stable equilibrium state for the optimal solution of (1), (2).

3. A critique of conventional LQR theory for (1), (2)

The conventional LQR theory with restrictions $Q > 0$ (or $Q \geqslant 0$), as just outlined, fails to accommodate a broad range of practical LQR problems (1), (2) that arise naturally in applications and that have well-defined well-behaved optimal solutions. These LQR problems, which are *un*accommodated by the conventional theory, arise from essentially two distinct sources.

Source 1: *Performance index decomposition in LQR problems*

In applications of modern control and optimization theories it is common to invoke a non-singular linear state-space transformation $x = Tz$, with T an $n \times n$ real matrix, as a means for simplifying the mathematical structure of the problem. When such a transformation is applied to the LQR problem (1), (2) the term $x^{\mathrm{T}} Q x$ in (2) is transformed to

$$x^{\mathrm{T}} Q x = z^{\mathrm{T}} \bar{Q} z \tag{7 a}$$

where

$$\bar{Q} = T^{\mathrm{T}} Q T \tag{7 b}$$

Now, if the transformation T is chosen appropriately, the transformed performance index (2)

$$J = \int_0^{\infty} [z^{\mathrm{T}}(t)\bar{Q}z(t) + u^{\mathrm{T}}(t)Ru(t)]\, dt \tag{8}$$

may allow one to directly integrate some of the component terms in $z^{\mathrm{T}}\bar{Q}z$, thereby decomposing (8) to the simplified but exactly equivalent form

$$J = V_1(z(0)) + \int_0^{\infty} [z^{\mathrm{T}}(t)\hat{Q}z(t) + u^{\mathrm{T}}(t)Ru(t)]\, dt \tag{9}$$

where V_1 is a well-defined quadratic function of z and the 'simplified' matrix $\hat{Q}$ may turn out, for instance, to be a diagonal matrix (Johnson 1986). A subtle fact about the simplification procedure leading to (9) is that *the positive-definite (semidefinite) property of Q in (2) does not necessarily carry over to the 'simplified' matrix $\hat{Q}$ in (9).* In particular, $\hat{Q}$ in (9) may turn out to be sign-indefinite, even though Q in (2) is positive-definite. This subtlety was in fact overlooked in Wonham and Johnson (1963) but was hand-corrected on all reprints mailed-out, or otherwise circulated, by the present author since late 1964; see also the more recent studies in Sirisena (1968), Bershchanskii (1976) and Johnson (1986). As a consequence of the foregoing observations, it is clear that the transformed and decomposed version (9) of the original LQR problem (1), (2) may lead to $Q \ngeq 0$ and thereby fall outside the range of problems for which conventional LQR theory applies—even though the original problem (1), (2) (and its simplification (9)) has a well-defined and well-behaved optimal solution.

Source 2: LQR problems of the net-flow type

In many potential practical applications of the LQR problem (1), (2) the structure and parameter values for the matrices (Q, R) are *not* something the control designer can choose arbitrarily, but rather are rigidly dictated by the problem 'physics'. For instance, the optimization of certain kinds of net commodity flows (net cash flow, net energy flow, net inventory flow, etc.) for linear dynamical systems (1) *naturally* leads to very specific forms and parameter values for (Q, R) in (2), which serve to express the 'physics' of the commodity flow process, i.e. the *integrand* of J in (2) expresses the instantaneous commodity flow *rate*, and the value of J expresses the overall net commodity flow. Since the flow rates and the optimum value of J in such problems can be positive, negative or zero, it is clear that the associated Q matrices dictated by the flow physics may not be positive-definite (or semidefinite)—even though the commodity-flow optimization problem (1), (2) has a well-defined well-behaved optimal solution. Thus the conventional textbook description of the performance index (2) as being a positive (or non-negative) quadratic weighting of 'error' $x(t)$ and 'cost of control' $u(t)$ is a narrow and misleading point-of-view, which diverts attention away from other well-posed, practical LQR problems (1), (2) involving 'non-error' integrands and for which $Q \ngeq 0$. One family of such problems is described in Jensen (1984).

4. The notion of propriety for the LQR problem (1), (2)

In order to overcome the tradition of benign neglect of indefinite (and negative-definite) Q-matrices in the LQR textbook literature, it is useful to first introduce the notion of a 'proper' LQR problem (1), (2). For this purpose, we propose the following.

Definition 1

An LQR problem (1), (2) is *proper* if and only if the optimal minimizing control $u^0(t)$ results in $x(t)\to 0$ as $t\to\infty$.

Since the regulation condition $x(t)\to 0$ as $t\to\infty$ forms part of the given LQR problem *specifications* (i.e. boundary condition) in (1), (2), this definition appears to be innocuous and rather useless. However, it prompts the following interesting concepts.

Definition 2

A symmetric matrix Q is said to be *proper*, with respect to (1) and a given symmetric matrix R, if and only if the pair (Q, R) forms a proper LQR problem (1), (2).

Definition 3

The collection of all proper matrices Q, with respect to (1) and a given R, is defined as the *Q-proper set* $\mathbf{Q}(R)$ for (1), (2).

In the light of these definitions, it is natural to ask what special features are required of Q-matrices in order that they be proper? Of course, the conventional LQR theory (3)–(6) provides a rigorous proof that the set of *all* matrices Q satisfying $Q>0$ (or $Q\geqslant 0$ with (A, H) completely observable) are proper provided that $R>0$ (Kwakernaak and Sivan 1972). But this set is only a *subset* of $\mathbf{Q}$. The fact is the Q-proper set $\mathbf{Q}(R)$ also typically contains a dense family (continuum) of sign-indefinite Q and even negative-definite Q! This fact is demonstrated in the examples that follow.

5. Propriety analysis of the general LQR problem (1), (2) with $r=1$

The general solution of the LQR problem (1), (2) under the restrictions $Q>0$ (or $Q\geqslant 0$ with (A, H) completely observable) and $R>0$ is of course well-known and well-documented. However, the general analysis of (1), (2) from the *propriety* point-of-view, considering the widest range of Q-matrices for which (1), (2) is proper, has apparently received little, if any, attention. In this section we examine the general scalar-control ($r=1$) case of (1), (2) and develop the necessary and sufficient condition for propriety of that case. By this means, a procedure is given whereby one can identify the set $\mathbf{Q}$ of all Q-matrices for which (1), (2) is proper. Several concrete examples are worked to illustrate the procedure.

The general scalar-control case of (1), (2) can be written

$$\dot{x}=Ax+bu, \quad u \text{ scalar} \tag{10 a}$$

$$J=\int_0^{\infty}[x^{\mathrm{T}}(t)Qx(t)+\rho u^2]\,dt, \quad \rho \text{ scalar} \tag{10 b}$$

The propriety analysis of (10) is simplified if one first transforms (10) to a certain canonical form. For this purpose, recall (Wonham and Johnson 1963, Appendix 1) that since the pair (A, b) is assumed completely controllable, there exists a real non-singular linear-transformation matrix T such that

$$x = Tz \tag{11}$$

takes (10) to the phase-variable (control) canonical form

$$\dot{z} = A_0 z + b_0 u \tag{12 a}$$

$$J = \int_0^\infty [z^{\mathrm{T}}(t)\bar{Q}z(t) + \rho u^2]\, dt \tag{12 b}$$

where

$$A_0 = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}, \quad b_0 = T^{-1}b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \bar{Q} = T^{\mathrm{T}}QT \tag{12 c}$$

Effective recipes for generating T(and T^{-1}) are given in Wonham and Johnson (1963), Johnson and Wonham (1964 a, 1966) and Johnson (1971, §§ 6–8). Now if one applies Pontryagin's maximum principle to the LQR problem (12), as was done for example in Johnson and Wonham 1964 b, it is readily found that the $2n$th degree characteristic equation $\mathscr{P}(\lambda)$ of the associated Pontryagin canonical equations is given by

$$\mathscr{P}(\lambda) = \det\left\{\lambda I - \left[\begin{array}{c|c} A_0 & (2\rho)^{-1}b_0 b_0^{\mathrm{T}} \\ \hline 2\bar{Q} & -A_0^{\mathrm{T}} \end{array}\right]\right\} = 0 \tag{13}$$

In arriving at (13), one must assume $\rho \neq 0$, for otherwise optimal motions $x(t)$ exist, in the usual sense, only on certain 'singular' hyperplanes in state-space; see Wonham and Johnson (1963), Johnson and Wonham (1964 b) and Johnson (1986). Following the procedure used in Wonham and Johnson (1963, equation (4.3)) and in Johnson and Wonham (1964 b, equation (A.1)), one can easily expand the determinant (13) to obtain $\mathscr{P}(\lambda)$ explicitly as the real even-powered polynomial

$$\mathscr{P}(\lambda) = \sum_1^n \hat{q}_{kk}\lambda^{k-1}(-\lambda)^{k-1} + \rho\left[\lambda^n - \sum_1^n a_k\lambda^{k-1}\right]\left[(-\lambda)^n - \sum_1^n a_k(-\lambda)^{k-1}\right] \tag{14 a}$$

where the $\hat{q}_{kk}$ are defined by the expression (recall that $\bar{q}_{jk} = \bar{q}_{kj}$)

$$\sum_1^n \hat{q}_{kk}\lambda^{k-1}(-\lambda)^{k-1} = \sum_1^n \sum_1^n \bar{q}_{jk}\lambda^{j-1}(-\lambda)^{k-1} \tag{14 b}$$

The solution of (14 b) is (Johnson 1986):

$$\hat{q}_{kk} = \bar{q}_{kk} + 2\sum_1^{k-1} \bar{q}_{k-s,k+s}(-1)^s \tag{14 c}$$

The n quantities $(\hat{q}_{11}, \dots, \hat{q}_{nn})$ in (14) are fundamentally important parameters in the LQR theory for (10) because, as shown in Johnson (1986), they represent the n *essential degrees of freedom* that actually exist among the $\frac{1}{2}n(n+1)$ 'arbitrary' elements q_{ij} of Q. In other words, (14 a) shows that $\bar{Q}$ in (12) is always 'LQR equivalent' to the *diagonal* matrix $\bar{Q} = \text{diag}(\hat{q}_{11}, \ldots, \hat{q}_{nn})$. For this reason, the $\hat{q}_{kk}$ are hereinafter referred to as the 'essential cost parameters' for the LQR problem (10).

The $2n$ eigenvalues of (14) obviously occur in pairs $(\lambda_i, -\lambda_i)$, $i = 1, 2, \ldots, n$, and, moreover, complex λ_i occur in conjugate pairs $(\lambda_i, \bar{\lambda}_i)$, as is well known. Therefore, if $\text{Re}(\lambda_i) \neq 0$ for all i (see Theorem 1 below) one can factor (14 *a*) uniquely into two nth-degree factors of the form

$$\bar{\mathscr{P}}(\lambda) = \pi(\lambda)\pi(-\lambda) \tag{15 a}$$

where $\text{Re}(\lambda_i) < 0$ for each root λ_i of the factor

$$\pi(\lambda) = c_1 + c_2\lambda + \ldots + c_n\lambda^{n-1} + c_{n+1}\lambda^n \tag{15 b}$$

and where (15 *a*) determines the c_i in (15 *b*) *up to* a common factor of -1. That sign ambiguity can be removed by normalizing one of the c_i signs, as demonstrated in Examples 2 and 3 below.

The usual textbook presentation of LQR theory for (10) describes the optimal motions $x(t)$ as those corresponding to the n (left-half-plane) roots λ_i of $\pi(\lambda)$ in (15). However, that description is incomplete. The complete description of propriety for (10) must also include the subtle, but *critical*, requirement that $\pi(\lambda)$ in (15) have all *real-valued* coefficients c_i, $i = 1, 2, \ldots, n+1$. The latter requirement is rarely mentioned in texts because it just happens to be automatically satisfied in the two popular textbook cases where: (i) the $\hat{q}_{kk}$ in (14) are *all positive*, or (ii) one makes the traditional assumption $Q > 0$ or $Q \geqslant 0$ as explained above (3). However, *neither* of those cases is necessary for the realness and asymptotic stability of $\pi(\lambda)$! Moreover, if one chooses $Q > 0$ or $Q \geqslant 0$ in (10) it is possible for some of the essential cost parameters $\hat{q}_{kk}$ in (14) to turn-out *negative*. This latter possibility is the point overlooked in Wonham and Johnson (1963, equation (7)); see Johnson (1986) for further elaboration.

Thus the propriety of (10) reduces to the requirement of realness *and* asymptotic stability of the factor $\pi(\lambda)$ in (15). A fundamentally important result concerning this requirement is given by the following theorem, which appears to be a new result in LQR theory.

Theorem 1

The roots $(\lambda_1, \ldots, \lambda_n; -\lambda_1, \ldots, -\lambda_n)$ of the real polynomial $\bar{\mathscr{P}}(\lambda)$ in (14) satisfy $\text{Re}(\lambda_i) \neq 0$, $i = 1, \ldots, n$, *and* the coefficients c_t, $t = 1, 2, \ldots, n+1$ in (15 *b*) are all real and non-zero if and only if

$$\bar{\mathscr{P}}(j(v)^{1/2}) > 0 \quad \text{for all real } v \geqslant 0;\ j = (-1)^{1/2} \tag{16}$$

Proof

It is clear that $\text{Re}(\lambda_i) \neq 0$, $i = 1, \ldots, n$, if and only if

$$\bar{\mathscr{P}}(j(v)^{1/2}) \neq 0 \quad \text{for all real } v \geqslant 0 \tag{17}$$

From (14), (15) the value of $\bar{\mathscr{P}}(j0^{1/2})$ is computed to be

$$\bar{\mathscr{P}}(j0^{1/2}) = \hat{q}_{11} + \rho a_1^2 = c_1^2 \tag{18}$$

It follows from (18) that c_1 is real and non-zero, and consequently $(c_2, \ldots, c_{n+1})$ are real also, if and only if

$$\bar{\mathscr{P}}(j0^{1/2}) > 0 \tag{19}$$

As a consequence of (19) and the continuity of $\bar{\mathscr{P}}(j(v)^{1/2})$, condition (17) is realized, and consequently $(c_2, \ldots, c_{n+1})$ are all non-zero, if and only if (16) is satisfied. This completes the proof. □

Remarks

(i) $\bar{\mathscr{P}}(j(v)^{1/2})$ is a real nth-degree polynomial in v which can be tested for positivity with respect to real roots $v \geqslant 0$ (i.e. tested for condition (16)) by using the modified Sturm-sequence methods of Fuller (1955), Siljak (1970), Jury (1974) or Meerov (1965). In this paper we shall use a special Routh–Hurwitz method introduced in Johnson (1986).

(ii) The two popular textbook cases described below (15) are sufficient, but *not* necessary conditions for satisfaction of (16).

(iii) Condition (16) is *different from*, but obviously related to, Kalman's well-known frequency-domain inequality (Kalman 1964) characterizing optimal linear feedback control laws.

(iv) As $\rho \to 0$ Theorem 1 reduces to a result in 'singular control' theory previously established in Sirisena (1968) and Bershchanskii (1976); see also Johnson (1986).

Thus condition (16) is seen to be *the* necessary and sufficient condition for propriety of the LQR problem (10). The Q-proper set $\mathbf{Q}$ for (10) can therefore be characterized as follows.

Theorem 2

For given (A, b, ρ), the Q-proper set $\mathbf{Q}$ for (10) consists of the set of all matrices Q that by means of $\tilde{Q} = T^{\mathrm{T}} Q T$ and (14 *b*, *c*), produce elements $\hat{q}_{11}, \hat{q}_{22}, \ldots, \hat{q}_{nn}$ leading to satisfaction of (16), i.e. leading to *real* coefficients $c_1, c_2, \ldots, c_{n+1}$ in (15 *b*) satisfying the Routh–Hurwitz stability conditions for $\mathrm{Re}\,(\lambda_i) < 0$.

This result shows that the crux of the Q-propriety problem for (10) is the identification of the set $\hat{\mathbf{Q}}$ of all element sets $\hat{q}_{11}, \ldots, \hat{q}_{nn}$ in (14) that satisfy (16). Then the corresponding set $\mathbf{Q}$ of all Q-proper matrices in (10) can be calculated using (14 *b*, *c*) and the inverse relation $Q = (T^{\mathrm{T}})^{-1} \tilde{Q} T^{-1}$. It will now be demonstrated how $\hat{\mathbf{Q}}$ can be calculated from (14 *a*) in particular cases, using a Routh–Hurwitz equivalent of (16).

6. Propriety analysis for particular cases of (10)

The general propriety analysis of (10) involves determining the set of all sets $\hat{q}_{11}, \ldots, \hat{q}_{nn}$ that satisfy (16). For this purpose, effective procedures for testing (16) are available (Siljak 1970, Jury 1974, Meerov 1965), but they are elaborate and somewhat abstract. For simple problems it is more instructive to replace (16) by an equivalent Routh–Hurwitz condition, which we shall now demonstrate.

Example 1

Propriety analysis of the general first-order case of (10)

Suppose $n = 1$ in (10). Then (14 *a*) has the particular form

$$\bar{\mathscr{P}}(\lambda) = \hat{q}_{11} + \rho[\lambda - a_1][-\lambda - a_1] \tag{20}$$

$$= -\rho\lambda^2 + \hat{q}_{11} + \rho a_1^2 \tag{21}$$

where $\hat{q}_{11} = \bar{q}_{11} = qb^2$. Setting (21) equal to the $n = 1$ version of (15) yields

$$-\rho\lambda^2 + \hat{q}_{11} + \rho a_1^2 = (c_2\lambda + c_1)(-c_2\lambda + c_1) \tag{22}$$

and solving (22) for the 'unknowns' (c_1, c_2) yields

$$c_1^2 = \hat{q}_{11} + \rho a_1^2 \tag{23 a}$$

$$c_2^2 = \rho \tag{23 b}$$

It follows from (23) that

$$(\hat{q}_{11} + \rho a_1^2) \geqslant 0 \tag{24 a}$$

$$\rho \geqslant 0 \tag{24 b}$$

are *the* necessary and sufficient conditions for the realness of c_1 and c_2. Under those conditions

$$c_1 = \pm(\hat{q}_{11} + \rho a_1^2)^{1/2} \tag{25 a}$$

$$c_2 = \pm\rho^{1/2} \tag{25 b}$$

Next, the Routh–Hurwitz conditions must be invoked for the factor $\pi(\lambda)$ in (15), where in this example

$$\pi(\lambda) = c_2\lambda + c_1 \tag{26}$$

It follows that the necessary and sufficient conditions for Re $(\lambda_i) < 0$ in (26) are

$$c_1 c_2 > 0 \tag{27}$$

In view of (25), condition (27) implies

$$[\rho(\hat{q}_{11} + \rho a_1^2]^{1/2} > 0 \tag{28}$$

and therefore the simultaneous satisfaction of (24), (28) requires

$$\hat{q}_{11} + \rho a_1^2 > 0 \tag{29 a}$$

$$\rho > 0 \tag{29 b}$$

Conditions (29) are *the* necessary and sufficient conditions for (12) to be a proper LQR problem when $n = 1$. Therefore those conditions are also necessary and sufficient for satisfaction of the positivity condition (16) for (21), as may be readily verified. It is interesting to note from (29) that proper $\hat{q}_{11}$ must satisfy

$$\hat{q}_{11} > -\rho a_1^2, \quad \text{where } \rho > 0 \tag{30}$$

and therefore *a certain range of negative values of* $\hat{q}_{11}$ (as well as $\hat{q}_{11} = 0$) *are proper*—provided only that $a_1 \neq 0$. Since $\hat{q}_{11} = \bar{q}_{11} = q_{11}b^2$ in this example, it follows that (30) with $\hat{q}_{11}$ replaced by $q_{11}b^2$ defines the Q-proper set for the general $n = 1$ case of (10). The $(\hat{q}_{11}, \rho)$ propriety domain defined by (30) is illustrated in Fig. 1, which should be compared with the (grossly incorrect) remarks in Palm 1983, pp. 658, 659).

Example 2

Propriety analysis of the general 2nd-order case of (10)

Suppose $n = 2$ in (10). Then (14 *a*) becomes

$$\mathscr{P}(\lambda) = \rho\lambda^4 + \lambda^2(-\hat{q}_{22} - 2\rho a_1 - \rho a_2^2) + (\hat{q}_{11} + \rho a_1^2) \tag{31}$$

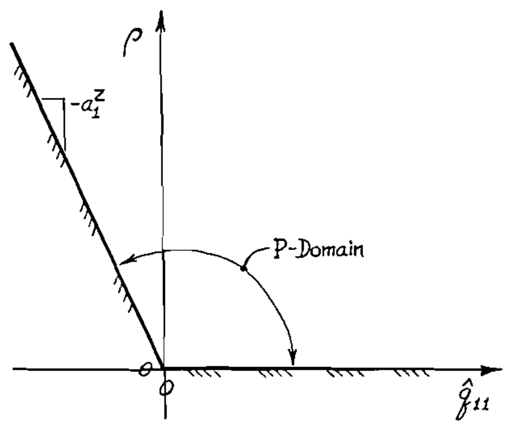

Figure 1. Propriety domain for Example 1.

Proceeding as in (22), (23), it is found that

$$c_1^2 = \hat{q}_{11} + \rho a_1^2 \tag{32 a}$$

$$c_2^2 = 2c_1c_3 + \hat{q}_{22} + 2\rho a_1 + \rho a_2^2 \tag{32 b}$$

$$c_3^2 = \rho \tag{32 c}$$

It follows from (32) that (c_1, c_2, c_3) are real if and only if

$$\hat{q}_{11} + \rho a_1^2 \geqslant 0 \tag{33 a}$$

$$2c_1c_3 + \hat{q}_{22} + 2\rho a_1 + \rho a_2^2 \geqslant 0 \tag{33 b}$$

$$\rho \geqslant 0 \tag{33 c}$$

It is convenient at this point to invoke a universal sign normalization of the $(c_1, \dots, c_{n+1})$ in (15 *b*) (see remarks below (15 *b*)) by agreeing to always set

$$c_{n+1} = +(\rho)^{1/2} \tag{34}$$

The choice (34) involves no loss of generality, and will force $(c_1, c_2, \dots, c_n)$ to be *positive* also. Thus, for the example at hand

$$c_3 = +(\rho)^{1/2} \tag{35 a}$$

and therefore the sign normalized (c_1, c_2) are given by

$$c_1 = +(\hat{q}_{11} + \rho a_1^2)^{1/2} \tag{35 b}$$

$$c_2 = +\{2[\rho(\hat{q}_{11} + \rho a_1^2)]^{1/2} + \hat{q}_{22} + 2\rho a_1 + \rho a_2^2\}^{1/2} \tag{35 c}$$

Now, by invoking the corresponding Routh–Hurwitz conditions $c_1 > 0, c_2 > 0, c_3 > 0$ for

$$\pi(\lambda) = c_1 + c_2\lambda + c_3\lambda^2 \tag{36}$$

one obtains the general necessary and sufficient conditions for $n = 2$ propriety as

$$\rho > 0 \tag{37 a}$$

$$\hat{q}_{11} + \rho a_1^2 > 0 \tag{37 b}$$

$$2[\rho(\hat{q}_{11} + \rho a_1^2)]^{1/2} + \hat{q}_{22} > -\rho(2a_1 + a_2^2) \tag{37 c}$$

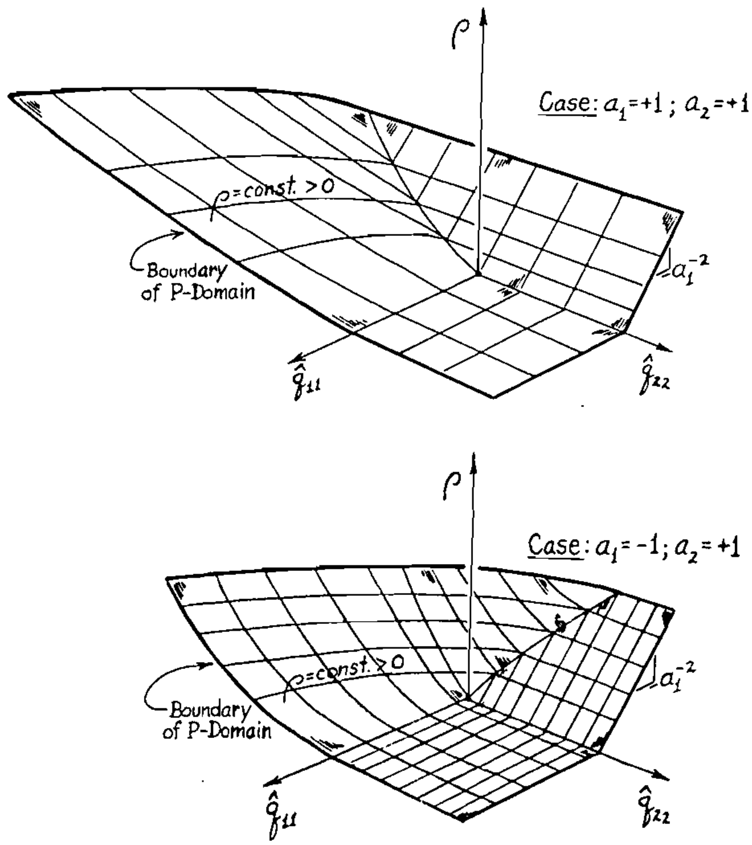

Figure 2. Propriety domains for two cases of Example 2.

Setting $\hat{q}_{11}=\bar{q}_{11}$, $\dot{\hat{q}}_{22}=\bar{q}_{22}-2\bar{q}_{13}$ in (37), one obtains the corresponding general $n=2$ propriety conditions for $\bar{Q}$ in (12). Finally, setting $Q=(T^{\mathrm{T}})^{-1}\bar{Q}T^{-1}$, one obtains the general $n=2$ propriety conditions for Q in (10). It is interesting to observe that for each fixed value of $\rho>0$ the propriety conditions (37) allow a *potentially large range* (continuum) *of sign-indefinite and negative-definite* (and zero) *'proper' choices for* $(\hat{q}_{11}, \hat{q}_{22})$! Some example plots of those 'sets of proper $\hat{q}_{11}$, $\hat{q}_{22}$' as functions of ρ are shown in Fig. 2 for several values of a_1, a_2.

Example 3

Propriety analysis of the general 3rd-order case of (10)

Suppose $n=3$ in (10). Then, following the same procedure used in Example 2, including (34), it is found that (c_1, c_2, c_3, c_4) are given by the semi-implicit expressions

$$c_1=+(\hat{q}_{11}+\rho a_1^2)^{1/2} \tag{38 a}$$

$$c_2=+[2c_3(\hat{q}_{11}+\rho a_1^2)^{1/2}-\rho(2a_1a_3-a_2^2)+\hat{q}_{22}]^{1/2} \tag{38 b}$$

$$c_3=+[2c_2(\rho)^{1/2}+\rho(a_3^2+2a_2)+\hat{q}_{33}]^{1/2} \tag{38 c}$$

$$c_4=+(\rho)^{1/2} \tag{38 d}$$

Now, the Routh–Hurwitz conditions

$$c_1 > 0, \quad c_2 > 0, \quad c_3 > 0, \quad c_4 > 0 \tag{39 a}$$

$$c_2 c_3 > +[\rho(\hat{q}_{11} + \rho a_1^2)]^{1/2} \tag{39 b}$$

for the polynomial

$$\pi(\lambda) = c_1 + c_2\lambda + c_3\lambda^2 + c_4\lambda^3 \tag{40}$$

can be invoked, using (38), and one can thereby obtain the necessary and sufficient conditions for propriety of $(\hat{q}_{11}, \hat{q}_{22}, \hat{q}_{33})$. For this purpose, it is desirable to solve (38 *b*, *c*) explicitly. The corresponding propriety conditions for Q in (10) can then be found as in Examples 1 and 2. The propriety conditions obtained from (38), (39) allow a potentially broad range of *indefinite* and *negative-definite* choices of $(\hat{q}_{11}, \hat{q}_{22}, \hat{q}_{33})$—depending on the values of a_1, a_2, a_3 and ρ. In fact, unlike the results in Figs. 1 and 2, this third-order case allows one of the $\hat{q}_{ii}$ $(= \hat{q}_{22})$ to be negative *even* as $\rho \to 0$; see Johnson (1986, Example 2).

7. Summary and conclusions

The infinite-time linear–quadratic-regulator (LQR) theory as traditionally presented in textbooks restricts the choice of Q to either $Q > 0$, or $Q \geqslant 0$ subject to an additional constraint. In this paper it has been shown that this traditional restriction on Q is unwarranted because it *leaves out* a broad range of sign-indefinite and negative-definite Q-matrices that may arise in applications and that produce well-behaved (i.e. 'proper') optimal motions. The necessary and sufficient condition for 'propriety' of the general scalar-control infinite-time LQR problem has been derived in this paper (Theorem 1), and it has been demonstrated by examples that the propriety condition admits a potentially broad range (continuum) of sign-indefinite and negative-definite Q-matrices.

In view of the expanding domain of socio-economic and other non-traditional control problems currently being addressed from the LQR point of view, it seems imperative that textbook and A/V presentations of general LQR theory should consider the *widest* range of Q-matrices for which well-behaved LQR solutions exist. The 'limits-of-propriety' analysis developed in this paper, and the companion analysis developed in Johnson (1986) for the case $\rho \to 0$, provide a basis for filling that need.

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