AAE 666 Notes

Martin Corless

School of Aeronautics & Astronautics
Purdue University
West Lafayette, Indiana
corless@purdue.edu

February 15, 2013

Contents

Chapter 1

Introduction

HI!

(How is it going?)

This is a course on the analysis and control of nonlinear systems.

Why study nonlinear systems?

• Most physical systems are nonlinear.
• Nonlinear systems exhibit many types of behavior which do not occur in linear systems: several equilibrium conditions; isolated periodic solutions; chaos.
• Some systems are not linearizable.
• Linearization yields local behavior only.

Text: Khalil, K. H., *Nonlinear Systems*, 2nd Edition, Prentice Hall, 1996.

References

Slotine J.-J. E., and Li, W., *Applied Nonlinear Control*, Prentice Hall, 1991.
Vidyasagar, M., *Nonlinear Systems Analysis*, 2nd Edition, Prentice Hall, 1993.
Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory*, SIAM, 1994.

Chapter 2

State space representation of dynamical systems

2.1 Examples

2.1.1 Continuous-time

A first example

Figure 2.1: The first nonlinear system

Consider the motion of a small block of mass m moving along a straight horizontal line. Suppose there is Coulomb (or dry) friction between the block and surface with coefficent of friction $\mu > 0$. Letting $t \in \mathbb{R}$ represent *time* and $v(t) \in \mathbb{R}$ represent the velocity of the block at time t, the motion of the block can be described by the following first order ordinary differential equation (ODE):

$$m\dot{v} = -\mu m g \operatorname{sgm}(v)$$

where the signum function, sgm, is defined by

$$\operatorname{sgm}(v) := \begin{cases} -1 & \text{if} \quad v < 0 \\ 0 & \text{if} \quad v = 0 \\ 1 & \text{if} \quad v > 0 \end{cases}$$

Figure 2.2: signum function

and g is the gravitational acceleration constant of the planet on which the block resides.

Introducing $x := v$ results in

$$\dot{x} = -\alpha \operatorname{sgm}(x)$$

and $\alpha := \mu g$.

Planar pendulum

Figure 2.3: Planar pendulum

$$J\ddot{\theta} + Wl \sin\theta = 0$$

Introducing the state variables,

$$x_1 := \theta \qquad \text{and} \qquad x_2 := \dot{\theta}\,,$$

results in the state space description:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 \end{aligned}$$

where $a := Wl/J > 0$.

Attitude dynamics of a rigid body

The equations describing the rotational dynamics of a rigid body are given by Euler's equations of motion. If we choose the axes of a body-fixed reference frame along the principal

Figure 2.4: Attitude dynamics

axes of inertia of the rigid body with origin at the center of mass, Euler's equations of motion take the simplified form

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \end{aligned}$$

where $\omega_1, \omega_2, \omega_3$ denote the components of the body angular velocity vector with respect to the body principal axes, and the positive scalars I_1, I_2, I_3 are the principal moments of inertia of the body with respect to its mass center.

Body in central force motion

Figure 2.5: Central force motion

$$\begin{array}{ccccccc} \ddot{r} & - & r\omega^2 & + & g(r) & = 0 \\ r\dot{\omega} & + & 2\dot{r}\omega & & & = 0 \end{array}$$

For the simplest situation in orbit mechanics (a "satellite" orbiting YFHB)

$$g(r) = \mu/r^2 \qquad \mu = GM$$

where G is the universal constant of gravitation and M is the mass of YFHB.

Ballistics in drag

Figure 2.6: Ballistics in drag

$$
\begin{aligned}
\dot{x} &= v \cos\gamma \\
\dot{y} &= v \sin\gamma \\
m\dot{v} &= -mg\sin\gamma - D(v) \\
mv\dot{\phi} &= -mg\cos\gamma
\end{aligned}
$$

Double pendulum on cart

Figure 2.7: Double pendulum on cart

The motion of the double pendulum on a cart can be described by

$$\begin{array}{rcccl}(m_0+m_1+m_2)\ddot{y}-m_1l_1\cos\theta_1\,\ddot{\theta}_1-m_2l_2\cos\theta_2\,\ddot{\theta}_2 & + & m_1l_1\sin\theta_1\dot{\theta}_1^2+m_2l_2\sin\theta_2\dot{\theta}_2^2 & = & u\\ -m_1l_1\cos\theta_1\,\ddot{y}+m_1l_1^2\ddot{\theta}_1 & + & m_1l_1g\sin\theta_1 & = & 0\\ -m_2l_2\cos\theta_2\,\ddot{y}+m_2l_2^2\ddot{\theta}_2 & + & m_2l_2g\sin\theta_2 & = & 0\end{array}$$

Figure 2.8: Double pendulum on cart

Two-link robotic manipulator

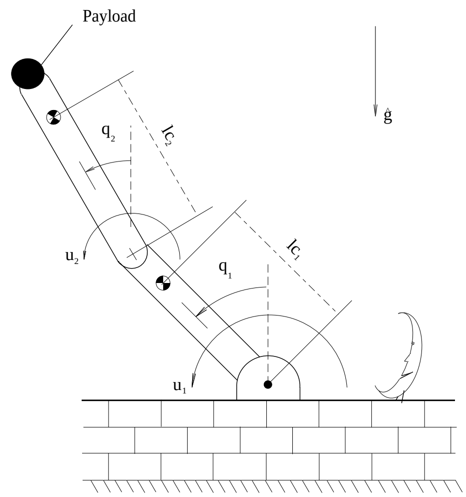

Figure 2.9: A simplified model of a two link manipulator

The coordinates q_1 and q_2 denote the angular location of the first and second links relative to the local vertical, respectively. The second link includes a payload located at its end. The masses of the first and the second links are m_1 and m_2, respectively. The moments of inertia of the first and the second links about their centers of mass are I_1 and I_2, respectively. The locations of the center of mass of links one and two are determined by lc_1 and lc_2, respectively; l_1 is the length of link 1. The equations of motion for the two arms are described by:

$$\begin{array}{llll} [m_1 lc_1^2 + m_2 l_1^2 + I_1]\ddot{q}_1 & +[m_2 l_1 lc_2 \cos(q_1 - q_2)]\ddot{q}_2 & +m_2 l_1 lc_2 \sin(q_1 - q_2)\dot{q}_2^2 - [m_1 lc_1 + m_2 l_1] g \sin(q_1) & = u_1 \\ [m_2 l_1 lc_2 \cos(q_1 - q_2)]\ddot{q}_1 & +[m_2 lc_2^2 + I_2]\ddot{q}_2 & -m_2 l_1 lc_2 \sin(q_1 - q_2)\dot{q}_1^2 - m_2 g lc_2 \sin(q_2) & = u_2 \end{array}$$

Traffic flow

Two roads connecting two points. Let x_1 and x_2 be the traffic flow rate (number of cars per hour) along the two routes.

$$\begin{aligned}\dot{x}_1 &= -\sigma(c_1(x) - c_2(x))x_1 + \sigma(c_2(x) - c_1(x))x_2 \\ \dot{x}_2 &= \sigma(c_1(x) - c_2(x))x_1 - \sigma(c_2(x) - c_1(x))x_2\end{aligned}$$

where

$$\sigma(y) = \begin{cases} y & \text{if} \quad y \geq 0 \\ 0 & \text{if} \quad y < 0 \end{cases}$$

$c_i(x)$ is the travel time along route i.

$$c_1(x) = x_1^2 + 1 \qquad c_2(x) = 2x_2$$

2.1.2 Discrete-time

Example 1 (Population dynamics)

$$x(k+1) = ax(k)$$

Example 2 (Computer control)

Example 3 (Iterative numerical algorithm) Recall Newton's algorithm for solving the equation $g(x) = 0$:

$$x(k+1) = x(k) - \frac{g(x(k))}{g'(x(k))}$$

Example 4 (Consensus algorithm)

2.2 General representation

2.2.1 Continuous-time

All of the preceding systems can be described by a bunch of first order ordinary differential equations of the form

$$\boxed{\begin{array}{rcl}\dot{x}_1 &=& f_1(x_1, x_2, \ldots, x_n)\\ \dot{x}_2 &=& f_2(x_1, x_2, \ldots, x_n)\\ &\vdots& \\ \dot{x}_n &=& f_n(x_1, x_2, \ldots, x_n)\end{array}}$$

where the scalars $x_i(t)$, $i = 1, 2, \ldots, n$, are called the state variables and the real scalar t is called the time variable.

Higher order ODE descriptions

Single equation. (Recall the planar pendulum.) Consider a system described by a single n^{th}- order differential equation of the form

$$F(q, \dot{q}, \ldots, q^{(n)}) = 0$$

where $q(t)$ is a scalar and $q^{(n)} := \frac{d^n q}{dt^n}$. To obtain a state space description of this system, we need to assume that we can solve for $q^{(n)}$ as a function of $q, \dot{q}, \ldots, q^{(n-1)}$. So suppose that the above equation is equivalent to

$$q^{(n)} = a(q, \dot{q}, \ldots, q^{(n-1)})\,.$$

Now let

$$\begin{array}{rcl}x_1 &:=& q\\ x_2 &:=& \dot{q}\\ &\vdots& \\ x_n &:=& q^{(n-1)}\end{array}$$

to obtain the following state space description:

$$\begin{array}{rcl}\dot{x}_1 &=& x_2\\ \dot{x}_2 &=& x_3\\ &\vdots& \\ \dot{x}_{n-1} &=& x_n\\ \dot{x}_n &=& a(x_1, x_2, \ldots, x_n)\end{array}$$

Multiple equations. (Recall the two link manipulator.) Consider a system described by m scalar differential equations in N variables:

$$
\begin{aligned}
F_1(q_1, \dot{q}_1, \ldots, q_1^{(n_1)},\ q_2, \dot{q}_2, \ldots, q_2^{(N_2)},\ \ldots,\ q_N, \dot{q}_N, \ldots, q_N^{(n_N)}\ ) &= 0 \\
F_2(q_1, \dot{q}_1, \ldots, q_1^{(n_1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2)},\ \ldots,\ q_N, \dot{q}_N, \ldots, q_N^{(n_N)}\ ) &= 0 \\
&\vdots \\
F_N(q_1, \dot{q}_1, \ldots, q_1^{(n_1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2)},\ \ldots,\ q_N, \dot{q}_N, \ldots, q_N^{(n_N)}) &= 0
\end{aligned}
$$

where $t, q_1(t), q_2(t), \ldots, q_N(t)$ are real scalars. Note that $q_i^{(n_i)}$ is the highest order derivative of q_i which appears in the above equations.

First solve for the highest order derivatives, $q_1^{(n_1)},\ q_2^{(n_2)},\ \ldots,\ q_N^{(n_N)}$, to obtain:

$$
\begin{aligned}
q_1^{(n_1)} &= a_1(\ q_1, \dot{q}_1, \ldots q_1^{(n_1-1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2-1)},\ \ldots,\ q_N, \dot{q}_N, \ldots, q_N^{(n_N-1)}\ ) \\
q_2^{(n_2)} &= a_2(\ q_1, \dot{q}_1, \ldots q_1^{(n_1-1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2-1)},\ \ldots,\ q_N, \dot{q}_N, \ldots, q_N^{(n_N-1)}\ ) \\
&\vdots \\
q_N^{(n_N)} &= a_N(\ q_1, \dot{q}_1, \ldots q_1^{(n_1-1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2-1)},\ \ldots,\ q_N, \dot{q}_N, \ldots, q_N^{(n_N-1)}\ )
\end{aligned}
$$

Now let

$$
\begin{array}{llll}
x_1 := q_1 & x_2 := \dot{q}_1 & \ldots & x_{n_1} := q_1^{(n_1-1)} \\
x_{n_1+1} := q_2, & x_{n_1+2} := \dot{q}_2 & \ldots & x_{n_1+n_2} := q_2^{(n_2-1)} \\
 & & \vdots & \\
x_{n_1+\ldots+n_{m-1}+1} := q_N & x_{n_1+\ldots+n_{m-1}+2} := \dot{q}_N & \ldots & x_n := q_N^{(n_N-1)}
\end{array}
$$

where

$$n := n_1 + n_2 + \ldots + n_N$$

to obtain

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n_1-1} &= x_{n_1} \\
\dot{x}_{n_1} &= a_1(x_1, x_2, \ldots, x_n) \\
\dot{x}_{n_1+1} &= x_{n_1+2} \\
&\vdots \\
\dot{x}_{n_1+n_2} &= a_2(x_1, x_2, \ldots, x_n) \\
&\vdots \\
\dot{x}_n &= a_n(x_1, x_2, \ldots, x_n)
\end{aligned}
$$

Example 5

$$
\begin{aligned}
\ddot{q}_1 + \dot{q}_2 + 2q_1 &= 0 \\
-\ddot{q}_1 + \dot{q}_1 + \dot{q}_2 + 4q_2 &= 0
\end{aligned}
$$

2.2.2 Discrete-time

A general state space description of a discrete-time system consists of a bunch of first order difference equations of the form

$$\boxed{\begin{array}{rcl} x_1(k+1) & = & f_1(x_1(k), x_2(k), \ldots, x_n(k)) \\ x_2(k+1) & = & f_2(x_1(k), x_2(k), \ldots, x_n(k)) \\ & \vdots & \\ x_n(k+1) & = & f_n(x_1(k), x_2(k), \ldots, x_n(k)) \end{array}}$$

where the scalars $x_i(k)$, $i = 1, 2, \ldots, n$, are called the state variables and the integer k is called the time variable.

Higher order DE descriptions

Story is similar to continuous-time.

2.3 Vectors

2.3.1 Vector spaces and $\mathbb{R}^n$

A scalar is a real or a complex number. The symbols $\mathbb{R}$ and $\mathbb{C}$ represent the set of real and complex numbers, respectively.

In this section all the definitions and results are given for real scalars. However, they also hold for complex scalars; to get the results for complex scalars, simply replace 'real' with 'complex' and $\mathbb{R}$ with $\mathbb{C}$.

Consider any positive integer n. A real n-vector x is an ordered n-tuple of real numbers, $x_1, x_2, \ldots, x_n$. This is is usually written as follows:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{or} \qquad x = \begin{pmatrix} x_1, & x_2, & \cdots, & x_n \end{pmatrix}$$

The real numbers $x_1, x_2, \ldots, x_n$ are called the scalar components of x; x_i is called the i-th component. The symbol $\mathbb{R}^n$ represents the set of ordered n-tuples of real numbers.

Addition. The addition of any two elements x, y of $\mathbb{R}^n$ yields an element of $\mathbb{R}^n$ and is defined by:

$$x + y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

Zero element of $\mathbb{R}^n$:

$$0 := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Note that we are using the same symbol, 0, for a zero scalar and a zero vector.

The negative of x:

$$-x := \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}$$

Properties of addition

(a) (Commutative). For each pair x, y in $\mathbb{R}^n$,

$$x + y = y + x$$

(b) (Associative). For each x, y, z in $\mathbb{R}^n$,

$$(x + y) + z = x + (y + z)$$

(c) There is an element 0 in $\mathbb{R}^n$ such that for every x in $\mathbb{R}^n$,

$$x + 0 = x$$

(d) For each x in $\mathbb{R}^n$, there is an element $-x$ in $\mathbb{R}^n$ such that

$$x + (-x) = 0$$

Scalar multiplication. The multiplication of an element x of $\mathbb{R}^n$ by a real scalar α yields an element of $\mathbb{R}^n$ and is defined by:

$$\alpha x \;=\; \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;:=\; \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

Properties of scalar multiplication

(a) For each scalar α and pair x, y in $\mathbb{R}^n$

$$\alpha(x + y) = \alpha x + \alpha y$$

(b) For each pair of scalars α, β and x in $\mathbb{R}^n$,

$$(\alpha + \beta)x = \alpha x + \beta x$$

(c) For each pair of scalars α, β, and x in $\mathbb{R}^n$,

$$\alpha(\beta x) = (\alpha\beta)x$$

(d) For each x in $\mathbb{R}^n$,

$$1x = x$$

Vector space. Consider *any* set $\mathcal{V}$ equipped with an addition operation and a scalar multiplication operation. Suppose the addition operation assigns to each pair of elements x, y in $\mathcal{V}$ a unique element $x + y$ in $\mathcal{V}$ and it satisfies the above four properties of addition (with $\mathbb{R}^n$ replaced by $\mathcal{V}$). Suppose the scalar multiplication operation assigns to each scalar α and element x in $\mathcal{V}$ a unique element αx in $\mathcal{V}$ and it satisfies the above four properties of scalar multiplication (with $\mathbb{R}^n$ replaced by $\mathcal{V}$). Then this set (along with its addition and scalar multiplication) is called a *vector space*. Thus $\mathbb{R}^n$ equipped with its definitions of addition and scalar multiplication is a specific example of a vector space. We shall meet other examples of vectors spaces later.

An element x of a vector space is called a *vector*.

A vector space with real (complex) scalars is called a *real (complex) vector space*.

Subtraction in a vector space is defined by:

$$x - y := x + (-y)$$

Hence in $\mathbb{R}^n$,

$$x - y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_n - y_n \end{pmatrix}$$

2.3.2 $\mathbb{R}^2$ and pictures

An element of $\mathbb{R}^2$ can be represented in a plane by a point or a directed line segment.

2.3.3 Derivatives

Suppose $x(\cdot)$ is a function of a real variable $t \in \mathbb{R}$ where $x(t)$ is in $\mathbb{R}^n$. Then

$$\dot{x} := \frac{dx}{dt} := \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}$$

2.4 Vector representation of dynamical systems

Recall the general description of a dynamical system given in Section 2.2.1. For a system with n state variables, $x_1, \cdots, x_n$, we define the state (vector) x to be the n-vector given by

$$x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} .$$

We also introduce the n-vector valued function f defined by

$$f(x) := \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{pmatrix} .$$

Continuous time The general representation of a continuous-time dynamical system can now be compactly described by the following *first order vector differential equation*:

$$\boxed{\dot{x} = f(x)} \tag{2.1}$$

where $x(t)$ is an n-vector and the scalar t is the time variable. A system described by the above equation is called autonomous or time-invariant because the right-hand side of the equation does not depend explicitly on time t. For the first part of the course, we will only concern ourselves with these systems.

However, one can have a system containing time-varying parameters or "disturbance inputs" which are time-varying. In this case the system might be described by

$$\dot{x} = f(t, x)$$

that is, the right-hand side of the differential depends explicitly on time. Such a system is called non-autonomous or time-varying. We will look at them later.

Discrete time The general representation of a discrete-time dynamical system can now be compactly described by the following *first order vector differential equation*:

$$\boxed{x(k+1) = f(x(k))} \tag{2.2}$$

where $x(k)$ is an n-vector and the scalar k is the time variable. A system described by the above equation is called autonomous or time-invariant because the right-hand side of the equation does not depend explicitly on time k. For the first part of the course, we will only concern ourselves with these systems.

However, one can have a system containing time-varying parameters or "disturbance inputs" which are time-varying. In this case the system might be described by

$$x(k+1) = f(k, x(k))$$

that is, the right-hand side of the differential depends explicitly on time. Such a system is called non-autonomous or time-varying. We will look at them later.

2.5 Solutions and equilibrium states

2.5.1 Continuous-time

A solution of (2.1) is any continuous function $x(\cdot)$ which is defined on some time interval and which satisfies

$$\dot{x}(t) = f(x(t))$$

for all t in the time interval.

An equilibrium solution is the simplest type of solution; it is constant for all time, that is, it satisfies

$$x(t) \equiv x^e$$

for some fixed state vector x^e. The state x^e is called an equilibrium state. Since an equilibrium solution must satisfy the above differential equation, all equilibrium states are given by:

$$\boxed{f(x^e) = 0}$$

or, in scalar terms,

$$\begin{array}{rcl} f_1(x_1^e, x_2^e, \ldots, x_n^e) & = & 0 \\ f_2(x_1^e, x_2^e, \ldots, x_n^e) & = & 0 \\ & \vdots & \\ f_n(x_1^e, x_2^e, \ldots, x_n^e) & = & 0 \end{array}$$

Sometimes an equilibrium state is referred to as an *equilibrium point*, a *stationary point*, or, a *singular point*.

Example 6 (Planar pendulum) Here, all equilibrium states are given by

$$x^e = \begin{pmatrix} m\pi \\ 0 \end{pmatrix}$$

where m is an arbitrary integer. Physically, there are only two distinct equilibrium states

$$x^e = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad x^e = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

An isolated equilibrium state is an equilibrium state with the property that there is a neighborhood of that state which contains no other equilibrium states.

Figure 2.10: Isolated equilibrium state

Example 7 The system

$$\dot{x} = x - x^3$$

has three equilibrium states: $0, -1, 1$. Clearly, these are isolated equilibrium states.

Example 8 Every state of the system

$$\dot{x} = 0$$

is an equilibrium state. Hence none of these equilibrium states are isolated.

Example 9 Consider the system described by

$$\dot{x} = \begin{cases} \sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

The origin is an equilibrium state, but, it is not isolated. All other equilibrium states are isolated.

Higher order ODEs

Consider a system described by an ordinary differential equation of the form

$$F(y, \dot{y}, \ldots, y^{(n)}) = 0$$

where $y(t)$ is a scalar. An equilibrium solution is a solution $y(\cdot)$ which is constant for all time, that is,

$$y(t) \equiv y^e$$

for some constant y^e. Hence,

$$F(y^e, 0, \ldots, 0) = 0 \tag{2.3}$$

For the state space description of this system introduced earlier, all equilibrium states are given by

$$x^e = \begin{pmatrix} y^e \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where y^e solves (2.3).

Multiple higher order ODEs Equilibrium solutions

$$y_i(t) \equiv y_i^e\,, \qquad i = 1, 2, \ldots, m$$

Hence

$$\begin{aligned}
F_1(y_1^e, 0, \ldots,\; y_2^e, 0, \ldots, \ldots,\; y_N^e, \ldots, 0\,) &= 0 \\
F_2(y_1^e, 0, \ldots,\; y_2^e, 0, \ldots, \ldots,\; y_N^e, \ldots, 0\,) &= 0 \\
&\;\vdots \\
F_N(y_1^e, 0, \ldots,\; y_2^e, 0, \ldots, \ldots,\; y_N^e, \ldots, 0\,) &= 0
\end{aligned} \tag{2.4}$$

For the state space description of this system introduced earlier, all equilibrium states are given by

$$x^e = \begin{pmatrix} y_1^e \\ 0 \\ \vdots \\ y_2^e \\ 0 \\ \vdots \\ \vdots \\ y_N^e \\ \vdots \\ 0 \end{pmatrix}$$

where $y_1^e, y_2^e, \ldots, y_N^e$ solve (2.4).

Example 10 Central force motion in inverse square gravitational field

$$\begin{aligned}
\ddot{r} \;-\; r\omega^2 \;+\; \mu/r^2 &= 0 \\
r\dot{\omega} \;+\; 2\dot{r}\omega &= 0
\end{aligned}$$

Equilibrium solutions

$$r(t) \equiv r^e\,, \qquad \omega(t) \equiv \omega^e$$

Hence,

$$\dot{r}, \ddot{r}, \dot{\omega} = 0$$

This yields

$$\begin{aligned}
-r^e(\omega^e)^2 \;+\; \mu/(r^e)^2 &= 0 \\
0 &= 0
\end{aligned}$$

Thus there are infinite number of equilibrium solutions given by:

$$\omega^e = \pm\sqrt{\mu/(r^e)^3}$$

where r^e is arbitrary. Note that, for this state space description, an equilibrium state corresponds to a circular orbit.

2.5.2 Discrete-time

2.6 Numerical simulation

2.6.1 MATLAB

```
>> help ode45

 ODE45  Solve non-stiff differential equations, medium order method.
    [T,Y] = ODE45('F',TSPAN,Y0) with TSPAN = [T0 TFINAL] integrates the
    system of differential equations y' = F(t,y) from time T0 to TFINAL with
    initial conditions Y0.  'F' is a string containing the name of an ODE
    file.  Function F(T,Y) must return a column vector.  Each row in
    solution array Y corresponds to a time returned in column vector T.  To
    obtain solutions at specific times T0, T1, ..., TFINAL (all increasing
    or all decreasing), use TSPAN = [T0 T1 ... TFINAL].

    [T,Y] = ODE45('F',TSPAN,Y0,OPTIONS) solves as above with default
    integration parameters replaced by values in OPTIONS, an argument
    created with the ODESET function.  See ODESET for details.  Commonly
    used options are scalar relative error tolerance 'RelTol' (1e-3 by
    default) and vector of absolute error tolerances 'AbsTol' (all
    components 1e-6 by default).

    [T,Y] = ODE45('F',TSPAN,Y0,OPTIONS,P1,P2,...) passes the additional
    parameters P1,P2,... to the ODE file as F(T,Y,FLAG,P1,P2,...) (see
    ODEFILE).  Use OPTIONS = [] as a place holder if no options are set.

    It is possible to specify TSPAN, Y0 and OPTIONS in the ODE file (see
    ODEFILE).  If TSPAN or Y0 is empty, then ODE45 calls the ODE file
    [TSPAN,Y0,OPTIONS] = F([],[],'init') to obtain any values not supplied
    in the ODE45 argument list.  Empty arguments at the end of the call list
    may be omitted, e.g. ODE45('F').

    As an example, the commands

        options = odeset('RelTol',1e-4,'AbsTol',[1e-4 1e-4 1e-5]);
        ode45('rigidode',[0 12],[0 1 1],options);

    solve the system y' = rigidode(t,y) with relative error tolerance 1e-4
    and absolute tolerances of 1e-4 for the first two components and 1e-5
    for the third.  When called with no output arguments, as in this
    example, ODE45 calls the default output function ODEPLOT to plot the
    solution as it is computed.

    [T,Y,TE,YE,IE] = ODE45('F',TSPAN,Y0,OPTIONS) with the Events property in
    OPTIONS set to 'on', solves as above while also locating zero crossings
```

```
    of an event function defined in the ODE file.  The ODE file must be
    coded so that F(T,Y,'events') returns appropriate information.  See
    ODEFILE for details.  Output TE is a column vector of times at which
    events occur, rows of YE are the corresponding solutions, and indices in
    vector IE specify which event occurred.

    See also ODEFILE and
        other ODE solvers:  ODE23, ODE113, ODE15S, ODE23S, ODE23T, ODE23TB
        options handling:   ODESET, ODEGET
        output functions:   ODEPLOT, ODEPHAS2, ODEPHAS3, ODEPRINT
        odefile examples:   ORBITODE, ORBT2ODE, RIGIDODE, VDPODE

>>
```

2.6.2 SIMULINK

Exercises

Exercise 1 By appropriate definition of state variables, obtain a first order state space description of the following systems where y_1 and y_2 are real scalars.

(i)

$$\begin{aligned} 2\ddot{y}_1 + \ddot{y}_2 + \sin y_1 &= 0 \\ \ddot{y}_1 + 2\ddot{y}_2 + \sin y_2 &= 0 \end{aligned}$$

(ii)

$$\begin{gathered} \ddot{y}_1 + \dot{y}_2 + y_1^3 = 0 \\ \dot{y}_1 + \dot{y}_2 + y_2^3 = 0 \end{gathered}$$

Exercise 2 By appropriate definition of state variables, obtain a first order state space description of the following system where q_1 and q_2 are real scalars.

$$\begin{gathered} q_1(k+2) + q_1(k) + 2q_2(k+1) = 0 \\ q_1(k+2) + q_1(k+1) + q_2(k) = 0 \end{gathered}$$

Exercise 3 Consider the Lorenz system described by

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= -bx_3 + x_1x_2 \end{aligned}$$

with $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$. Simulate this system with initial states

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 + eps \\ 0 \end{pmatrix}$$

where *eps* is the floating point relative accuracy in MATLAB. Comment on your results for the integration interval $[0 \quad 60]$.

Chapter 3

First order systems

The simplest type of nonlinear systems is one whose state can be described by a single scalar. We refer to such a system as a first order system.

3.1 Continuous time

A first order continuous-time system is described by

$$\dot{x} = f(x) \tag{3.1}$$

where $x(t)$ is a scalar. As an example, recall the first nonlinear system.

State portrait. As illustrated in Figure 3.1, one may readily determine the qualitative behavior of a first order system from the graph of f.

Figure 3.1: Qualitative behavior of a first order system

Example 11 Consider the first order system described by

$$\dot{x} = x - x^3$$

This system has three equilibrium states: -1, 0 and 1.

Figure 3.2: State portrait for $\dot{x} = x - x^3$

Linear systems. A linear time-invariant first order system is described by

$$\dot{x} = ax \tag{3.2}$$

where a is a constant real scalar. All solutions of this system are of the form

$$x(t) = ce^{at}$$

where c is an a constant real scalar. Thus the qualitative behavior of (3.2) is completely determined by the sign of a.

Linearization. Suppose x^e is an equilibrium state for system (3.1). Then

$$f(x^e) = 0\,.$$

Suppose that the function f is differentable at x^e with derivative $f'(x^e)$. When x is "close" to x^e, it follows from the definition of the derivative that

$$f(x) \approx f(x^e) + f'(x^e)(x - x^e) = f'(x^e)(x - x^e)\,.$$

If we introduce the "perturbed state",

$$\delta x = x - x^e,$$

then

$$\delta\dot{x} = f(x) \approx f'(x^e)\delta x\,.$$

So, the linearization of system (3.1) about an equilibrium state x^e is defined to be the following system:

$$\delta\dot{x} = a\delta x \qquad \text{where} \qquad a = f'(x^e)\,. \tag{3.3}$$

One can demonstrate the following result.

If $f'(x^e) \neq 0$, then the local behavior of the nonlinear system (3.1) about x^e is qualitatively the same as that of the linearized system about the origin.

Example 12

$$\dot{x} = x - x^3$$

Example 13

$$\dot{x} = ax^2$$

3.2 Solutions to scalar first order ODES

Consider

$$\dot{x} = f(x)$$

with initial condition $x(0) = x_0$. Suppose that $f(x_0) \neq 0$; then $f(x(t)) \neq 0$ for some interval $[0, t_1)$ and over this interval we have

$$\frac{1}{f(x(t))}\frac{dx}{dt} = 1\,.$$

Let

$$g(x) := \int_{x_0}^{x} \frac{1}{f(\eta)}\,d\eta$$

Then

$$\frac{d}{dt}\left(g(x(t))\right) = \frac{d}{dx}\left(g(x)\right)\frac{dx}{dt} = \frac{1}{f(x)}\frac{dx}{dt} = 1\,,$$

that is,

$$\frac{d}{dt}\left(g(x(t))\right) = 1\,.$$

Integrating over the interval $[0, t]$ and using the initial condition $x(0) = x_0$ yields $g(x(t)) = t$, that is,

$$\int_{x_0}^{x(t)} \frac{1}{f(\eta)}\,d\eta = t\,. \tag{3.4}$$

One now solves the above equation for $x(t)$.

Example 14 (Finite escape time) This simple example illustrates the concept of a finite escape time.

$$\dot{x} = x^2$$

Here,

$$\int_{x_0}^{x} \frac{1}{\eta^2}\,d\eta = -\frac{1}{x} + \frac{1}{x_0}\,.$$

Hence, all nonzero solutions satisfy

$$-\frac{1}{x(t)} + \frac{1}{x_0} = t\,.$$

Then, provided $x_0 t \leq 1$, the above equation can be uniquely solved for $x(t)$ to obtain

$$x(t) = \frac{x_0}{1 - x_0 t}\,. \tag{3.5}$$

Suppose $x_0 > 0$. Then $x(t)$ goes to infinity as t approaches $1/x_0$; the solution blows up in a finite time. This cannot happen in a linear system.

3.3 Discrete time

A first order continuous-time system is described by

$$x(k+1) = f(x(k))$$

Example 15

$$x(k+1) = \lambda x(k)(1 - x(k))$$

. Consider $\lambda > 3.56994$

Exercises

Exercise 4 Draw the state portrait of the first nonlinear system.

Exercise 5 Draw the state portrait for

$$\dot{x} = x^4 - x^2 \,.$$

Exercise 6 Obtain an explicit expression for all solutions of

$$\dot{x} = -x^3 \,.$$

Chapter 4

Second order systems

In this section, we consider systems whose state can be described by two real scalar variables x_1, x_2. We will refer to such systems as second order systems. They are described by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \tag{4.1}$$

- State plane, state portrait.
- Vector field f

Quite often we consider second order systems described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

In this case, the state plane and the state portrait are sometimes referred to as the phase plane and phase portrait, respectively.

4.1 Linear systems

A linear time-invariant second order system is described by

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned}$$

where each a_{ij} is a constant real scalar. The qualitative behavior of this system is determined by the eigenvalues λ_1, λ_2 of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

stable node	real	$\lambda_1, \lambda_2 < 0$
unstable node	real	$0 < \lambda_1, \lambda_2$
saddle	real	$\lambda_1 < 0 < \lambda_2$
stable focus	complex	$\Re(\lambda_1) < 0$
unstable focus	complex	$\Re(\lambda_1) > 0$
center	complex	$\Re(\lambda_1) = 0$

See Khalil.

4.2 Linearization

Suppose $x^e = (x_1^e, x_2^e)$ is an equilibrium state of system (4.1). Then the linearization of (4.1) about x^e is given by:

$$\delta\dot{x}_1 = a_{11}\delta x_1 + a_{12}\delta x_2 \tag{4.2a}$$

$$\delta\dot{x}_2 = a_{21}\delta x_1 + a_{22}\delta x_2 \tag{4.2b}$$

where

$$a_{ij} = \frac{\partial f_i}{\partial x_j}(x_1^e, x_2^e)$$

for $i, j = 1, 2$.

The behavior of a second order nonlinear system near an equilibrium state is qualitatively the same as the behavior of the corresponding linearized system about 0*, provided the eigenvalues of the linearized system have non-zero real part.*

Example 16 Van der Pol oscillator. Consider

$$\ddot{w} + \phi(\dot{w}) + w = 0$$

where ϕ is a nonlinear function. If one lets $y = \dot{w}$ and differentiates the above equation, one obtains

$$\ddot{y} + \phi'(y)\dot{y} + y = 0\,.$$

Considering $\phi(y) = \mu(y^3/3 - y)$ yields

$$\boxed{\ddot{y} - \mu(1 - y^2)\dot{y} + y = 0}$$

Introducing state variables $x_1 = y$ and $x_2 = \dot{y}$, this system can be described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu(1 - x_1^2)x_2 \end{aligned}$$

This system has a single equilibrium state at the origin. Linearization about the origin results in

$$\begin{aligned} \delta\dot{x}_1 &= \delta x_2 \\ \delta\dot{x}_2 &= -\delta x_1 + \mu\delta x_2\,. \end{aligned}$$

The eigenvalues λ_1, λ_2 of this linearized system are given by

$$\lambda_1 = \frac{\mu}{2} - \jmath\sqrt{1 - \frac{\mu^2}{4}} \qquad \text{and} \qquad \lambda_2 = \frac{\mu}{2} + \jmath\sqrt{1 - \frac{\mu^2}{4}}\,.$$

If we consider $0 < \mu < 2$, then λ_1 and λ_2 are complex numbers with positive real parts. Hence, the origin is an unstable focus of the linearized system. Thus, the origin is an unstable focus of the the original nonlinear system. Figure 4.1 contains some state trajectories of this system for $\mu = 0.3$.

Note that, although the system has a single equilibrium state and this is unstable, all solutions of the system are bounded. This behavior does not occur in linear systems. If the origin of a linear system is unstable, then the system will have some unbounded solutions.

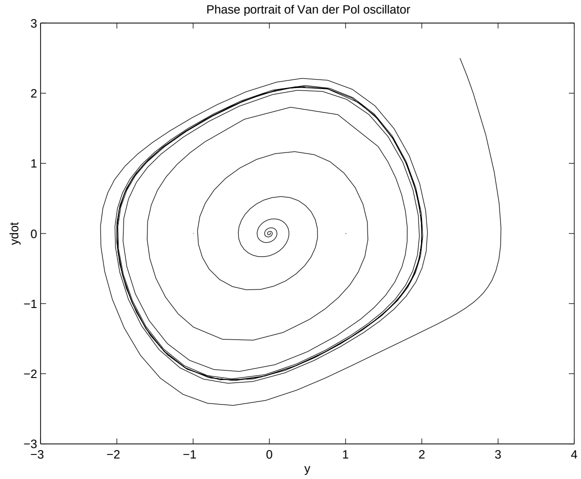

Figure 4.1: Van der Pol oscillator

Example 17 Duffing's equation Consider

$$\ddot{y} + \phi(y) = 0$$

where ϕ is a nonlinear function. With $\phi(y) = -y + y^3$, we have

$$\boxed{\ddot{y} - y + y^3 = 0}$$

The above differential equation has three equilibrium solutions:

$$y^e = 0, -1, 1\,.$$

Linearization about the zero solution yields

$$\delta\ddot{y} - \delta y = 0\,.$$

A state space description of this linearized system is given by

$$\begin{aligned} \delta\dot{x}_1 &= \delta x_2 \\ \delta\dot{x}_2 &= \delta x_1 \end{aligned}$$

The eigenvalues λ_1, λ_2 and corresponding eigenvectors v_1, v_2 associated with this linear system are given by

$$\lambda_1 = 1\,, \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -1\,, \qquad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So the origin of the linearized system is a saddle; see Figure 4.2. Hence the origin is a saddle for the original nonlinear system.

Linearization about either of the two non-zero equilibrium solutions results in

$$\delta\ddot{y} + 2\delta y = 0\,.$$

The eigenvalues associated with any state space realization of this system are

$$\lambda_1 = \jmath\sqrt{2} \qquad \lambda_1 = -\jmath\sqrt{2}\,.$$

Figure 4.2: Duffing saddle

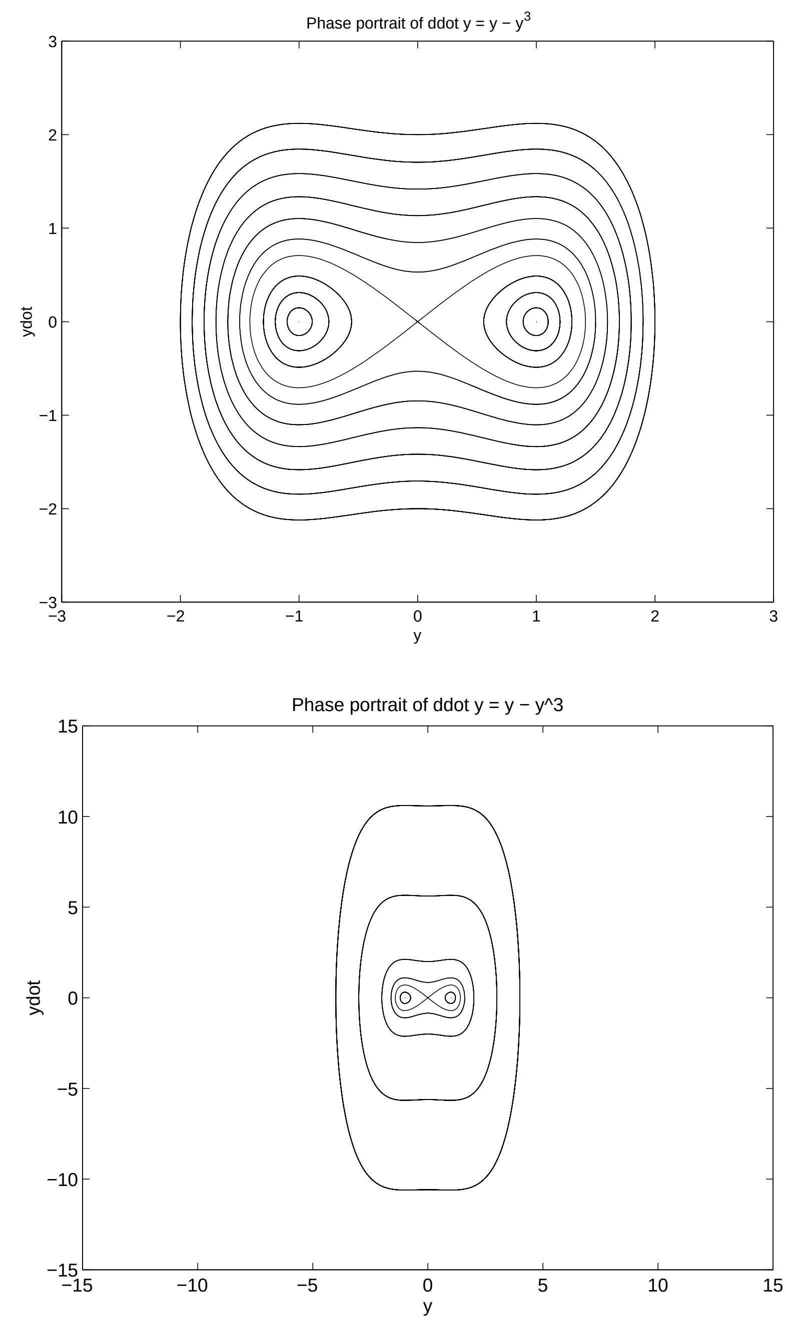

Figure 4.3: Phase portraits of Duffing system

4.3 Isocline method

See Khalil.

4.4 Periodic solutions and limit cycles

A solution $x(\cdot)$ of $\dot{x} = f(x)$ is a periodic solution if it is non-constant and there is a $T > 0$ such that

$$x(t+T) = x(t) \qquad \text{for all } t$$

The smallest T for which the above identity holds is called the period of the solution. Periodic solutions are sometimes called limit cycles.

Note that a limit cycle corresponds to a closed trajectory in the state plane. If a linear system has a periodic solution with period T, it will have an infinite number of periodic solutions with that period. However, an nonlinear system can have just a single periodic solution; recall the Van der Pol oscillator. Also, a nonlinear system may have an infinite number of periodic solutions, but, the period may be different for different amplitudes; this occurs for Duffing's equation.

Figure 4.4: limit cycle

In the analysis of second order systems, it is sometimes useful to introduce polar coordinates r, ϕ which are implicity defined by

$$x_1 = r\cos\phi \qquad \text{and} \qquad x_2 = r\sin\phi\,.$$

Note that

$$r = (x_1^2 + x_2^2)^{\frac{1}{2}}$$

and, when $(x_1, x_2) \neq (0, 0)$,

$$\phi = \tan^{-1}(x_2/x_1)$$

The scalar ϕ is not defined when $(x_1, x_2) = (0, 0)$.

Figure 4.5: Polar coordinates

One may show that

$$\dot{r} = (x_1\dot{x}_1 + x_2\dot{x}_2)/r \tag{4.3a}$$
$$\dot{\phi} = (x_1\dot{x}_2 - \dot{x}_1 x_2)/r^2 \tag{4.3b}$$

Example 18 The simple harmonic oscillator.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

Here

$$\begin{aligned} \dot{r} &= 0 \\ \dot{\phi} &= -1 \end{aligned}$$

Figure 4.6: Simple harmonic oscillator

Example 19 An isolated periodic solution

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2 - x_2^2) \end{aligned}$$

Here

$$\begin{aligned} \dot{r} &= r(1 - r)^2 \\ \dot{\phi} &= -1 \end{aligned}$$

Exercises

Exercise 7 Determine the nature of each equilibrium state of the damped duffing system

$$\ddot{y} + 0.1\dot{y} - y + y^3 = 0$$

Numerically obtain the phase portrait.

Exercise 8 Determine the nature (if possible) of each equilibrium state of the simple pendulum.

$$\ddot{y} + \sin y = 0$$

Numerically obtain the phase portrait.

Exercise 9 Numerically obtain a state portrait of the following system:

$$\begin{aligned}\dot{x}_1 &= x_2^2 \\ \dot{x}_2 &= -x_1 x_2\end{aligned}$$

Based on the state portrait, predict the stability properties of each equilibrium state.

Chapter 5

Some general considerations

Consider the differential equation

$$\dot{x} = f(x) \tag{5.1}$$

where x is an n-vector. By a solution of (5.1) we mean any continuous function $x(\cdot) : [0, t_1) \to \mathbb{R}^n$, with $t_1 > 0$, which satisfies (5.1). Consider an initial condition

$$x(0) = x_0\,, \tag{5.2}$$

where x_0 is some specified initial state. The corresponding initial value problem associated with (5.1) is that of finding a solution to (5.1) which satisfies the initial condition (5.2). If the system is linear then, for every initial state x_0, there is a solution to the corresponding initial value problem, this solution is unique and is defined for all $t \geq 0$, that is, $t_1 = \infty$. One cannot make the same statement for nonlinear systems in general.

5.1 Existence of solutions

Example 20 Example of nonexistence

$$\dot{x} = -s(x)\,, \qquad x(0) = 0$$

where

$$s(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Fact. *Continuity of f guarantees existence.*

The above function is not continuous at 0.

5.2 Uniqueness of solutions

Example 21 Example of nonuniqueness: I'm falling

$$\dot{x} = \sqrt{2g|x|}\,, \qquad x(0) = 0$$

The above initial value problem has an infinite number of solutions. They are given by

$$x(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq T \\ \frac{g}{2}(t-T)^2 & \text{if } T \leq t \end{cases}$$

where $T \geq 0$ is arbitrary.

Fact. *Differentiability of f guarantees uniqueness.*

The f above is not differentiable at 0.

5.3 Indefinite extension of solutions

Example 22 Finite escape time

$$\dot{x} = 1 + x^2, \qquad x(0) = 0$$

Fact. *If a solution cannot be extended indefinitely, that is over $[0, \infty)$, then it must have a* finite escape time *T_e, that is, T_e is finite and*

$$\lim_{t \to T_e} x(t) = \infty.$$

Hence, if a solution is bounded on bounded time intervals then, it cannot have a finite escape time and, hence can be extended over $[0, \infty)$.

The following condition guarantees that solutions can be extended indefinitely. There are constants α and β such that

$$\|f(x)\| \leq \alpha\|x\| + \beta$$

for all x. Here one can show that when $\alpha \neq 0$

$$\|x(t)\| \leq e^{\alpha(t-t_0)}\|x(t_0)\| + \frac{\beta}{\alpha}(e^{\alpha t} - 1)$$

When $\alpha = 0$,

$$\|x(t)\| \leq \|x(t_0)\| + \beta(t - t_0)$$

Chapter 6

Stability and boundedness

Consider a general nonlinear system described by

$$\dot{x} = f(x) \tag{6.1}$$

where $x(t)$ is a real n-vector and t is a real scalar. By a solution of (6.1) we mean any continuous function $x(\cdot) : [0, t_1) \to \mathbb{R}^n$ with $t_1 > 0$, which satisfies $\dot{x}(t) = f(x(t))$ for $0 \leq t < t_1$.

6.1 Boundedness of solutions

Defn. (Boundedness) *A solution $x(\cdot)$ is* bounded *if there exists $\beta \geq 0$ such that*

$$||x(t)|| \leq \beta \qquad \textit{for all} \qquad t \geq 0$$

A solution is unbounded *if it is not bounded.*

It should be clear from the above definitions that a solution $x(\cdot)$ is bounded if and only if each component $x_i(\cdot)$ of $x(\cdot)$ is bounded. Also, a solution is unbounded if and only if at least one of its components is unbounded. So, $x(t) = [\ e^{-t} \quad e^{-2t}\ ]^T$ is bounded while $x(t) = [\ e^{-t} \quad e^{t}\ ]^T$ is unbounded.

Example 23 All solutions of

$$\dot{x} = 0$$

are bounded.

Example 24 All solutions of

$$\dot{x} = x - x^3$$

are bounded.

Example 25 All solutions (except the zero solution) of

$$\dot{x} = x$$

are unbounded.

Example 26 Consider

$$\dot{x} = x^2\,, \qquad x(0) = x_0$$

If $x_0 > 0$, the corresponding solution has a finite escape time and is unbounded. If $x_0 < 0$, the corresponding solution is bounded.

Example 27 Undamped oscillator.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1\end{aligned}$$

All solutions are bounded.

Boundedness and linear time-invariant systems. Consider a general LTI (linear time-invariant) system

$$\dot{x} = Ax \tag{6.2}$$

Recall that every solution of this system has the form

$$x(t) = \sum_{i=1}^{l} \sum_{j=0}^{n_i-1} t^j e^{\lambda_i t} v^{ij}$$

where $\lambda_1, \ldots, \lambda_l$ are the eigenvalues of A, n_i is the index of λ_i, and the constant vectors v^{ij} depend on initial state.

We say that an eigenvalue λ of A is non-defective if its index is one; this means that the algebraic multiplicity and the geometric multiplicity of λ are the same. Otherwise we say λ is defective.

Hence we conclude that all solutions of (6.2) are bounded if and only if for each eigenvalue λ_i of A:

(b1) $\Re(\lambda_i) \leq 0$ and

(b2) if $\Re(\lambda_i) = 0$ then λ_i is non-defective.

If there is an eigenvalue λ_i of A such that either

(u1) $\Re(\lambda_i) > 0$ or

(u2) $\Re(\lambda_i) = 0$ and λ_i is defective

then, the system has some unbounded solutions.

Example 28 Unattached mass

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 0\end{aligned}$$

Here

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has a single eigenvalue 0. This eigenvalue has algebraic multiplicity 2 but geometric multiplicity 1; hence some of the solutions of the system are unbounded. One example is

$$x(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

Example 29

$$\begin{aligned}\dot{x}_1 &= 0\\ \dot{x}_2 &= 0\end{aligned}$$

Here

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has a single eigenvalue 0. This eigenvalue has algebraic multiplicity 2 and geometric multiplicity 2. Hence all the solutions of the system are bounded. Actually every state is an equilibrium state and every solution is constant.

Example 30 (Resonance) Consider a simple linear oscillator subject to a sinusoidal input of amplitude W:

$$\ddot{q} + q = W\sin(\omega t+\phi)$$

Resonance occurs when $\omega = 1$. To see this, let

$$x_1 := q\,, \quad x_2 := \dot{q}\,, \quad x_3 := W\sin(\omega t+\phi)\,, \quad x_4 := \omega W\cos(\omega t+\phi)$$

to yield

$$\dot{x} = Ax$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & 0 \end{pmatrix}$$

If $\omega = 1$ then, A has eigenvalues $\jmath$ and $-\jmath$. These eigenvalues have algebraic multiplicity two but geometric multiplicity one; hence the system has unbounded solutions.

6.2 Stability of equilibrium states

Suppose x^e is an equilibrium state of the system $\dot{x} = f(x)$. Then, whenever $x(0) = x^e$, we have (assuming uniqueness of solutions) $x(t) = x^e$ for all $t \geq 0$. Roughly speaking, we say that x^e is a stable equilibrium state for the system if the following holds. If the initial state of the system is close to x^e, then the resulting solution is close to x^e. The formal definition is as follows.

Defn. (Stability) *An equilibrium state x^e is* stable *if for each $\epsilon > 0$ there exists $\delta > 0$ such that whenever $||x(0) - x^e|| < \delta$ one has $||x(t) - x^e|| < \epsilon$ for all $t \geq 0$.*

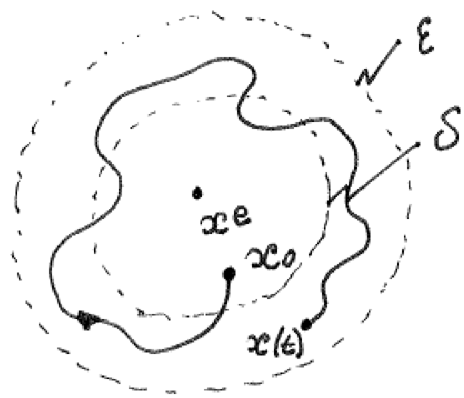

Figure 6.1: Stability

An equilibrium state x^e is said to be unstable *if it is not stable.*

Example 31

$$\dot{x} = 0$$

Every equilibrium state is stable. (Choose $\epsilon = \delta$.)

Example 32

$$\dot{x} = x$$

The origin is unstable.

Example 33

$$\dot{x} = x - x^3$$

The origin is unstable; the remaining equilibrium states 1 and -1 are stable.

Example 34 Undamped oscillator
The origin is stable.

Example 35 Simple pendulum

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1\end{aligned}$$

$(0,0)$ is stable; $(\pi,0)$ is unstable.

Example 36 Van der Pol oscillator

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1-x_1^2)x_2 - x_1\end{aligned}$$

Figure 6.2: Van der Pol oscillator

The origin is unstable. However, all solutions are bounded.

Stability of linear time-invariant systems. It can be shown that every equilibrium state of a LTI system (6.2) is stable if and only if all eigenvalues λ_i of A satisfy conditions (b1) and (b2) above. Hence every equilibrium state of a LTI system is stable if and only if all solutions are bounded.
It can also be shown that every equilibrium state is unstable if and only if there is an eigenvalue λ_i of A which satisfies condition (u1) or (u2) above. Hence every equilibrium state of a LTI system is unstable if and only if there are unbounded solutions.

6.3 Asymptotic stability

6.3.1 Global asymptotic stability

Defn. (Global asymptotic stability) *An equilibrium state* x^e *is* globally asymptotically stable (GAS) *if*

(a) It is stable

(b) Every solution $x(\cdot)$ *converges to* x^e *with increasing time, that is,*

$$\lim_{t\to\infty} x(t) = x^e$$

If x^e is a globally asymptotically stable equilibrium state, then there are no other equilibrium states and all solutions are bounded. In this case we say that the system $\dot{x} = f(x)$ is globally asymptotically stable.

Example 37 The system

$$\dot{x} = -x$$

is GAS.

Example 38 The system

$$\dot{x} = -x^3$$

is GAS.

6.3.2 Asymptotic stability

In asymptotic stability, we do not require that all solutions converge to the equilibrium state; we only require that all solutions which originate in some neighborhood of the equilibrium state converge to the equilibrium state.

Defn. (Asymptotic stability) *An equilibrium state* x^e *is* asymptotically stable (AS) *if*

(a) It is stable.

(b) There exists $R > 0$ *such that whenever* $||x(0)-x^e|| < R$ *one has*

$$\lim_{t\to\infty} x(t) = x^e \,. \tag{6.3}$$

The region of attraction of an equilibrium state x^e which is AS is the set of initial states which result in (13.3), that is it is the set of initial states which are attracted to x^e. Thus, the region of attraction of a globally asymptotically equilibrium state is the whole state space.

Figure 6.3: Asymptotic stability

Example 39

$$\dot{x} = -x$$

Example 40

$$\dot{x} = x - x^3$$

The equilibrium states -1 and 1 are AS with regions of attraction $(-\infty, 0)$ and $(0, \infty)$, respectively.

Example 41 Damped simple pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2 \end{aligned}$$

The zero state is AS but not GAS.

Example 42 Reverse Van der Pol oscillator

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= -(1 - x_1^2)x_2 + x_1 \end{aligned}$$

The zero state is AS but not GAS. Also, the system has unbounded solutions.

LTI systems. For LTI systems, it should be clear from the general form of the solution that the zero state is AS if and only if all the eigenvalues λ_i of A have negative real parts, that is,

$$\Re(\lambda_i) < 0$$

Also AS is equivalent to GAS.

Example 43 The system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

is GAS.

Example 44 The system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1 - x_2 \end{aligned}$$

is GAS.

Figure 6.4: Reverse Van der Pol oscillator

6.4 Exponential stability

We now present the "strongest" form of stability considered in this section.

DEFN. (Global exponential stability) *An equilibrium state* x^e *is* globally exponentially stable (GES) *with* rate of convergence $\alpha > 0$ *if there exists* $\beta > 0$ *such that every solution satisfies*

$$||x(t)-x^e|| \quad \leq \quad \beta||x(0)-x^e||\exp(-\alpha t) \qquad \textit{for all} \quad t \geq 0$$

Example 45

$$\dot{x} = -x$$

GES with rate $\alpha = 1$.

Note that global exponential stability implies global asymptotic stability, but, in general, the converse is not true. This is illustrated in the next example. For linear time-invariant systems, GAS and GES are equivalent.

Example 46

$$\dot{x} = -x^3$$

Solutions satisfy

$$x(t) = \frac{x_0}{\sqrt{1 + 2x_0^2 t}} \qquad \text{where} \qquad x_0 = x(0)\,.$$

GAS but not GES

Example 47 (General Example) Consider a general system

$$\dot{x} = f(x)$$

and suppose that there exists $\alpha > 0$ such that

$$x' f(x) \leq -\alpha \|x\|^2$$

for all x. Then this system is GES about zero with rate of convergence α.

As a specific example, consider

$$\dot{x} = -2x - 7x^3$$

As another specific example, consider

$$\begin{aligned}\dot{x}_1 &= -x_1 + \cos(x_1)x_2 \\ \dot{x}_2 &= -2x_2 - \cos(x_1)x_1\end{aligned}$$

DEFN. (Exponential stability) *An equilibrium state x^e is* exponentially stable (ES) *with* rate of convergence $\alpha > 0$ *if there exists $R > 0$ and $\beta > 0$ such that whenever $||x(0) - x^e|| < R$ one has*

$$||x(t) - x^e|| \quad \leq \quad \beta ||x(0) - x^e|| \exp(-\alpha t) \qquad \textit{for all} \quad t \geq 0$$

Note that exponential stability implies asymptotic stability, but, in general, the converse is not true.

Example 48

$$\dot{x} = -x^3$$

GAS but not even ES

Example 49

$$\dot{x} = -\frac{x}{1 + x^2}$$

GAS, ES, but not GES

Figure 6.5: Exponential stability

6.5 LTI systems

Consider a LTI system

$$\dot{x} = Ax \tag{6.4}$$

Recall that every solution of this system has the form

$$x(t) = \sum_{i=1}^{l} \sum_{j=0}^{n_i-1} t^j e^{\lambda_i t} v^{ij}$$

where $\lambda_1, \ldots, \lambda_l$ are the eigenvalues of A, the integer n_i is the index of λ_i, and the constant vectors v^{ij} depend on initial state. From this it follows that the stability properties of this system are completely determined by the location of its eigenvalues; this is summarized in the table below.

The following table summarizes the relationhip between the stability properties of a LTI system and the eigenproperties of its A-matrix. In the table, unless otherwise stated, a property involving λ must hold for all eigenvalues λ of A.

Stability properties	Eigenproperties
Global exponential stability and boundedness	$\Re(\lambda) < 0$
Stability and boundedness	$\Re(\lambda) \leq 0$ If $\Re(\lambda) = 0$ then λ is non-defective.
Instability and some unbounded solutions	There is an eigenvalue λ with $\Re(\lambda) > 0$ or $\Re(\lambda) = 0$ and λ is defective.

Example 50 The system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

is GES.

Example 51 The system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1 - x_2 \end{aligned}$$

is GES.

Example 52 The system

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= 0 \end{aligned}$$

is stable about every equilibrium point.

Example 53 The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{aligned}$$

is unstable about every equilibrium point.

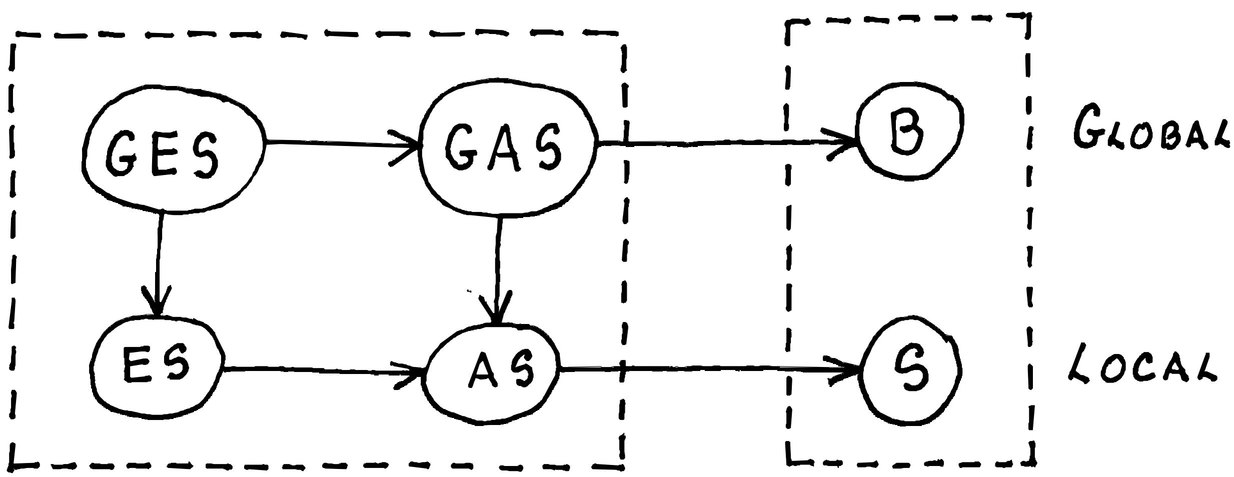

Figure 6.6: The big picture. The concepts in each dashed box are equivalent for linear systems.

6.6 Linearization and stability

Consider a nonlinear time-invariant system described by

$$\dot{x} = f(x)$$

where $x(t)$ is an n-vector at each time t. Suppose x^e is an equilibrium state for this system, that is, $f(x^e) = 0$, and consider the linearization of this system about x^e:

$$\delta\dot{x} = A\delta x \qquad \text{where} \quad A = \frac{\partial f}{\partial x}(x^e)\,.$$

The following results can be demonstrated using nonlinear Lyapunov stability theory.

Stability. *If all the eigenvalues of the A matrix of the linearized system have negative real parts, then the nonlinear system is* exponentially stable *about x^e.*

Instability. *If at least one eigenvalue of the A matrix of the linearized system has a positive real part, then the nonlinear system is* unstable *about x^e.*

Undetermined. *Suppose all the eigenvalues of the A matrix of the linearized system have non-positive real parts and at least one eigenvalue of A has zero real part. Then, based on the linearized system alone, one* cannot predict *the stability properties of the nonlinear system about x^e.*

Note that the first statement above is equivalent to the following statement. If the linearized system is exponentially stable, then the nonlinear system is exponentially stable about x^e.

Example 54 (Damped simple pendulum.) Physically, the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\sin x_1 - x_2\end{aligned}$$

has two distinct equilibrium states: $(0, 0)$ and $(\pi, 0)$. The A matrix for the linearization of this system about $(0, 0)$ is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

Since all the eigenvalues of this matrix have negative real parts, the nonlinear system is exponentially stable about $(0, 0)$. The A matrix corresponding to linearization about $(\pi, 0)$ is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Since this matrix has an eigenvalue with a positive real part, the nonlinear system is unstable about $(\pi, 0)$

The following example illustrates the fact that if the eigenvalues of the A matrix have non-positive real parts and there is at least one eigenvalue with zero real part, then, one cannot make any conclusions on the stability of the nonlinear system based on the linearization.

Example 55 Consider the scalar nonlinear system:

$$\dot{x} = ax^3$$

This origin is GAS if $a < 0$, unstable if $a > 0$ and stable if $a = 0$. However, the linearization of this system about zero, given by

$$\delta\dot{x} = 0$$

is independent of a and is stable.

The following example illustrates that instability of the linearized system does not imply instability of the nonlinear system.

Example 56 Using nonlinear techniques, one can show that the following system is GAS about the origin.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 - x_2^3 \end{aligned}$$

However, the linearization of this system, given by

$$\begin{aligned} \delta\dot{x}_1 &= \delta x_2 \\ \delta\dot{x}_2 &= 0\,, \end{aligned}$$

is unstable about the origin.

Exercises

Exercise 10 For each of the following systems, determine (from the state portrait) the stability properties of each equilibrium state. For AS equilibrium states, give the region of attraction.

(a)

$$\dot{x} = -x - x^3$$

(b)

$$\dot{x} = -x + x^3$$

(c)

$$\dot{x} = x - 2x^2 + x^3$$

Exercise 11 If possible, use linearization to determine the stability properties of each of the following systems about the zero equilibrium state.

(i)

$$\begin{aligned} \dot{x}_1 &= (1 + x_1^2)x_2 \\ \dot{x}_2 &= -x_1^3 \end{aligned}$$

(ii)

$$\begin{aligned} \dot{x}_1 &= \sin x_2 \\ \dot{x}_2 &= (\cos x_1)x_3 \\ \dot{x}_3 &= e^{x_1}x_2 \end{aligned}$$

Exercise 12 If possible, use linearization to determine the stability properties of each equilibrium state of the Lorenz system.

Chapter 7

Stability and boundedness: discrete time

7.1 Boundedness of solutions

Consider a general discrete-time nonlinear system described by

$$x(k+1) = f(x(k)) \tag{7.1}$$

where $x(k)$ is an n-vector and k is an integer. By a solution of (7.1) we mean a sequence $x(\cdot) = (\, x(0), x(1), x(2), \ldots\,)$ which satisfies (7.1) for all $k \geq 0$.

Defn. (Boundedness of solutions) A solution $x(\cdot)$ is bounded if there exists $\beta \geq 0$ such that

$$||x(k)|| \leq \beta \qquad \text{for all} \qquad k \geq 0\,.$$

A solution is unbounded if it is not bounded.

It should be clear from the above definitions that a solution $x(\cdot)$ is bounded if and only if each component $x_i(\cdot)$ of $x(\cdot)$ is bounded. Also, a solution is unbounded if and only if at least one of its components is unbounded. So, $x(k) = [\ (0.5)^k \quad (-0.5)^k\ ]^T$ is bounded while $x(k) = [\ (0.5)^k \quad 2^k\ ]^T$ is unbounded.

Example 57

$$x(k+1) = 0$$

All solutions are bounded.

Example 58

$$x(k+1) = x(k)$$

All solutions are bounded.

Example 59

$$x(k+1) = -2x(k)$$

All solutions (except the zero solution) are unbounded.

Linear time invariant (LTI) systems. All solutions of the LTI system

$$x(k+1) = Ax(k) \tag{7.2}$$

are bounded if and only if for each eigenvalue λ of A:

(b1) $|\lambda| \leq 1$ and

(b2) if $|\lambda| = 1$ then λ is non-defective.

If there is an eigenvalue λ of A such that either

(u1) $|\lambda| > 1$ or

(u2) $|\lambda| = 1$ and λ is defective.

then the system has some unbounded solutions.

Example 60 Discrete unattached mass. Here

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has a single eigenvalue 1. This eigenvalue has algebraic multiplicity 2 but geometric multiplicity 1; hence this eigenvalue is defective. So, some of the solutions of the system $x(k+1) = Ax(k)$ are unbounded. One example is

$$x(k) = \begin{pmatrix} k \\ 1 \end{pmatrix}$$

7.2 Stability of equilibrium states

Suppose x^e is an equilibrium state of the system $x(k+1) = f(x(k))$. Then, whenever $x(0) = x^e$, we have $x(k) = x^e$ for all $k \geq 0$. Roughly speaking, we say that x^e is a stable equilibrium state for the system if the following holds. If the initial state of the system is close to x^e, then the resulting solution is close to x^e. The formal definition is as follows.

DEFN. (Stability) An equilibrium state x^e is stable if for each $\epsilon > 0$ there exists $\delta > 0$ such that whenever $||x(0)-x^e|| < \delta$, one has $||x(k)-x^e|| < \epsilon$ for all $k \geq 0$.
x^e is unstable if it is not stable.

Example 61

$$x(k+1) = -x(k)\,.$$

The origin is stable. (Consider $\delta = \epsilon$.)

Example 62

$$x(k+1) = -2x(k)\,.$$

The origin is unstable.

Example 63

$$x(k+1) = -x(k)^3\,.$$

The single equilibrium at the origin is stable, but, the system has unbounded solutions.

LTI systems. Every equilibrium state of a LTI system (7.2) is stable if and only if all eigenvalues λ of A satisfy conditions (b1) and (b2) above. Hence every equilibrium state of a LTI system is stable if and only if all solutions are bounded.

Every equilibrium state is unstable if and only if there is an eigenvalue λ of A which satisfies condition (u1) or (u2) above. Hence every equilibrium state of a LTI system is unstable if and only if there are unbounded solutions.

7.3 Asymptotic stability

7.3.1 Global asymptotic stability

DEFN. (Global asymptotic stability (GAS)) An equilibrium state x^e is globally asymptotically stable (GAS) if

(a) It is stable

(b) Every solution $x(\cdot)$ converges to x^e, that is,

$$\lim_{k\to\infty} x(k) = x^e\,.$$

If x^e is a globally asymptotically stable equilibrium state, then there are no other equilibrium states. In this case we say *the system (7.1) is globally asymptotically stable.*

Example 64

$$x(k+1) = \frac{1}{2}x(k)$$

GAS

Example 65

$$x(k+1) = \frac{x(k)}{2 + x(k)^2}$$

7.3.2 Asymptotic stability

DEFN. (Asymptotic stability (AS)) An equilibrium state x^e is asymptotically stable (AS) if

(a) It is stable

(b) There exists $R > 0$ such that whenever $||x(0) - x^e|| < R$ one has

$$\lim_{k\to\infty} x(k) = x^e \tag{7.3}$$

The region of attraction of an equilibrium state x^e which is AS is the set of initial states which result in (7.3), that is it is the set of initial states which are attracted to x^e. Thus, the region of attraction of a globally asymptotically equilibrium state is the whole state space.

Example 66

$$x(k+1) = x(k)^3$$

Origin is AS with region of attraction $[-1 \quad 1]$.

LTI systems. For LTI systems, it should be clear from the general form of the solution that the zero state is AS if and only if all the eigenvalues λ_i of A have magnitude less than one, that is,

$$|\lambda_i| < 1\,.$$

Also AS is equivalent to GAS.

7.4 Exponential stability

We now present the "strongest" form of stability considered in this section.

DEFN. (Global exponential stability) *An equilibrium state* x^e *is* globally exponentially stable (GES) *with there exists* $0 \le \lambda < 1$ *and* $\beta > 0$ *such that every solution satisfies*

$$||x(k)-x^e|| \quad \le \quad \beta\lambda^k||x(0)-x^e|| \qquad \textit{for all} \quad k \ge 0$$

Example 67

$$x(k+1) = \frac{1}{2}x(k)$$

GES with $\gamma = \frac{1}{2}$.

Note that global exponential stability implies global asymptotic stability, but, in general, the converse is not true. This is illustrated in the next example. For linear time-invariant systems, GAS and GES are equivalent.

Lemma 1 *Consider a system described by* $x(k+1) = f(x(k))$ *and suppose that for some scalar* $\lambda \geq 0$,

$$\|f(x)\| \leq \lambda \|x\|$$

for all x. *Then, every solution* $x(\cdot)$ *of the system satisfies*

$$\|x(k)\| \leq \lambda^k \|x(0)\|$$

for all $k \geq 0$. *In particular, if* $\lambda < 1$ *then, the system is globally exponentially stable.*

Example 68

$$x(k+1) = \frac{x(k)}{\sqrt{1 + 2x(k)^2}}$$

Solutions satisfy

$$x(k) = \frac{x_0}{\sqrt{1 + 2kx_0^2}} \qquad \text{where} \qquad x_0 = x(0)\,.$$

GAS but not GES

Defn. (Exponential stability) *An equilibrium state* x^e *is* exponentially stable (ES) *if there exists* $R > 0$, $0 \leq \lambda < 1$ *and* $\beta > 0$ *such that whenever* $||x(0) - x^e|| < R$ *one has*

$$||x(k) - x^e|| \quad \leq \quad \beta\lambda^k ||x(0) - x^e|| \qquad \textit{for all} \quad k \geq 1$$

Note that exponential stability implies asymptotic stability, but, in general, the converse is not true.

Example 69

$$x(k+1) = \frac{x(k)}{\sqrt{1 + 2x(k)^2}}$$

Solutions satisfy

$$x(k) = \frac{x_0}{\sqrt{1 + 2kx_0^2}} \qquad \text{where} \qquad x_0 = x(0)\,.$$

GAS but not even ES

7.5 LTI systems

The following table summarizes the relationhip between the stability properties of a LTI system and the eigenproperties of its A-matrix. In the table, unless otherwise stated, a property involving λ must hold for all eigenvalues λ of A.

Stability properties	eigenproperties
Asymptotic stability and boundedness	$\lvert\lambda\rvert < 1$
Stability and boundedness	$\lvert\lambda\rvert \leq 1$ If $\lvert\lambda\rvert = 1$ then λ is non-defective
Instability and some unbounded solutions	There is an eigenvalue of A with $\lvert\lambda\rvert > 1$ or $\lvert\lambda\rvert = 1$ and λ is defective

7.6 Linearization and stability

Consider a nonlinear time-invariant system described by

$$x(k+1) = f(x(k))$$

where $x(k)$ is an n-vector at each time t. Suppose x^e is an equilibrium state for this system, that is, $f(x^e) = x^e$, and consider the linearization of this system about x^e:

$$\delta x(k+1) = A\delta x(k) \qquad \text{where} \quad A = \frac{\partial f}{\partial x}(x^e)\,.$$

The following results can be demonstrated using nonlinear Lyapunov stability theory.

Stability. *If all the eigenvalues of the A matrix of the linearized system have magnitude less than one, then the nonlinear system is* exponentially stable *about x^e.*

Instability. *If at least one eigenvalue of the A matrix of the linearized system has magnitude greater than one, then the nonlinear system is* unstable *about x^e.*

Undetermined. *Suppose all the eigenvalues of the A matrix of the linearized system have magnitude less than or equal to one and at least one eigenvalue of A has magnitude one. Then, based on the linearized system alone, one* cannot predict *the stability properties of the nonlinear system about x^e.*

Note that the first statement above is equivalent to the following statement. If the linearized system is exponentially stable, then the nonlinear system is exponentially stable about x^e.

Example 70 (Newton's method) Recall that Newton's method for a scalar function can be described by

$$x(k+1) = x(k) - \frac{g(x(k))}{g'(x(k))}$$

Here

$$f(x) = x - \frac{g(x)}{g'(x)}$$

So,

$$f'(x) = 1 - 1 + \frac{g(x)g''(x)}{g'(x)^2} = \frac{g(x)g''(x)}{g'(x)^2}\,.$$

At an equilibrium state x^e, we have $g(x^e) = 0$; hence $f'(x^e) = 0$ and the linearization about any equilibrium state is given by

$$\delta x(k+1) = 0\,.$$

Thus, every equilibrium state is exponentially stable.

Chapter 8

Basic Lyapunov theory

Suppose we are interested in the stability properties of the system

$$\dot{x} = f(x) \tag{8.1}$$

where $x(t)$ is a real n-vector at time t. If the system is linear, we can determine its stability properties from the properties of the eigenvalues of the system matrix. What do we do for a nonlinear system? We could linearize about each equilibrium state and determine the stability properties of the resulting linearizations. Under certain conditions this will tell us something about the local stability properties of the nonlinear system about its equilibrium states. However there are situations where linearization cannot be used to deduce even the local stability properties of the nonlinear system. Also, linearization tells us nothing about the global stability properties of the nonlinear system.

In general, we cannot explicitly obtain solutions for nonlinear systems. However, Lyapunov theory allows to say something about the stability properties of a system without knowing the form or structure of its solutions. Lyapunov theory is based on Lyapunov functions which are scalar-valued functions of the system state.

Suppose V is a scalar-valued function of the state, that is $V : \mathbb{R}^n \to \mathbb{R}$. If V is continuously differentiable then, at any time t, the derivative of V along a solution $x(\cdot)$ of system (8.1) is given by

$$\begin{aligned} \frac{dV}{dt}(x(t)) &= DV(x(t))\dot{x}(t) \\ &= DV(x(t))f(x(t)) \end{aligned}$$

where $DV(x)$ is the derivative of V at x and is given by

$$DV(x) = \begin{pmatrix} \frac{\partial V}{\partial x_1}(x) & \frac{\partial V}{\partial x_2}(x) & \dots & \frac{\partial V}{\partial x_n}(x) \end{pmatrix}$$

Note that

$$DV(x)f(x) = \frac{\partial V}{\partial x_1}(x)f_1(x) + \frac{\partial V}{\partial x_2}(x)f_2(x) + \ldots + \frac{\partial V}{\partial x_n}(x)f_n(x)$$

In what follows, if a condition involves DV, then it is implicitly assumed that V is continuously differentiable. Sometimes DV is denoted by

$$\frac{\partial V}{\partial x} \qquad \text{or} \qquad \nabla V(x)^T .$$

Also, when the system under consideration is fixed, DVf is sometimes denoted by

$$\dot{V}$$

This a slight but convenient abuse of notation.

In the following sections, we use Lyapunov functions to present sufficient conditions for the stability and boundedness concepts previously introduced. To guarantee a specific type of stability using a Lyapunov function, one has to obtain a Lyapunov function V which has certain properties and whose "time-derivative" $\dot{V}$ also has certain properties.

8.1 Lyapunov conditions for stability

8.1.1 Locally positive definite functions

Defn. (Locally positive definite function) *A function V is* locally positive definite (lpd) about a point x^e *if*

$$V(x^e) = 0$$

and there is a scalar $R > 0$ such that

$$V(x) > 0 \qquad \textit{whenever} \quad x \neq x^e \quad \textit{and} \quad ||x - x^e|| < R$$

Basically, a function is lpd about a point x^e if it is zero at x^e has a strict local minimum at x^e.

Figure 8.1: A locally positive definite function

Example 71 (Scalar x) The following functions are lpd about zero.

$$\begin{aligned} V(x) &= x^2 \\ V(x) &= 1 - e^{-x^2} \\ V(x) &= 1 - \cos x \\ V(x) &= x^2 - x^4 \end{aligned}$$

Example 72

$$\begin{aligned} V(x) &= ||x||^2 \\ &= x_1^2 + x_2^2 + \ldots + x_n^2 \end{aligned}$$

Lpd about the origin.

Example 73

$$\begin{aligned} V(x) &= ||x||_1 \\ &= |x_1| + |x_2| + \ldots + |x_n| \end{aligned}$$

Lpd about the origin.

Quadratic forms. Suppose P is a real $n \times n$ symmetric matrix and is positive definite. Consider the quadratic form defined by

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij} x_i x_j \,.$$

Clearly $V(0) = 0$. Recalling the definition of a positive definite matrix, it follows that $V(x) = x^T P x > 0$ for all nonzero x. Hence V is locally positive definite about the origin.

• The second derivative of V at x is the square symmetric matrix given by:

$$D^2V(x) := \begin{pmatrix} \frac{\partial^2 V}{\partial^2 x_1}(x) & \frac{\partial^2 V}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 V}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 V}{\partial^2 x_2}(x) & \cdots & \frac{\partial^2 V}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1}(x) & \frac{\partial^2 V}{\partial x_n \partial x_2}(x) & \cdots & \frac{\partial^2 V}{\partial^2 x_n}(x) \end{pmatrix}$$

that is,

$$D^2V(x)_{ij} = \frac{\partial^2 V}{\partial x_i \partial x_j}(x)$$

Sometimes $D^2V(x)$ is written as $\frac{\partial^2 V}{\partial x^2}(x)$ and is referred to as the Hessian of V.

The following lemma is sometimes useful in demonstrating that a function is lpd.

Lemma 2 *Suppose V is twice continuously differentiable and*

$$\begin{aligned} V(x^e) &= 0 \\ DV(x^e) &= 0 \\ D^2V(x^e) &> 0 \end{aligned}$$

Then V is a locally positive definite about x^e.

Example 74 Consider $V(x) = x^2 - x^4$ where x is a scalar. Since $V(0) = DV(0) = 0$ and $D^2V(0) = 2 > 0$, V is lpd about zero.

Example 75 Consider

$$V(x) = 1 - \cos x_1 + \frac{1}{2} x_2^2$$

Clearly

$$V(0) = 0$$

Since

$$DV(x) = \begin{pmatrix} \sin x_1 & x_2 \end{pmatrix}$$

we have

$$DV(0) = 0$$

Also,

$$D^2V(x) = \begin{pmatrix} \cos x_1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence,

$$D^2V(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > 0$$

Since V satisfies the hypotheses of the previous lemma with $x^e = 0$, it is lpd about zero.

8.1.2 A stability result

If the equilibrium state of a nonlinear system is stable but not asymptotically stable, then one cannot deduce the stability properties of the equilibrium state of the nonlinear system from the linearization of the nonlinear system about that equilibrium state. The following Lyapunov result is useful in demonstrating stability of an equilibrium state for a nonlinear system.

Theorem 1 (Stability) *Suppose there exists a function V and a scalar $R > 0$ such that V is locally positive definite about x^e and*

$$DV(x)f(x) \leq 0 \qquad \textit{for} \quad ||x - x^e|| < R$$

Then x^e is a stable equilibrium state.

If V satisfies the hypotheses of the above theorem, then V is said to be a Lyapunov function which guarantees the stability of x^e.

Example 76

$$\dot{x} = 0$$

Consider

$$V(x) = x^2$$

as a candidate Lyapunov function. Then V is lpd about 0 and

$$DV(x)f(x) = 0$$

Hence (it follows from Theorem 1 that) the origin is stable.

Example 77 (Undamped linear oscillator.)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_1 \qquad\qquad k > 0 \end{aligned}$$

Consider the total *energy*,

$$V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2$$

as a candidate Lyapunov function. Then V is lpd about the origin and

$$DV(x)f(x) = 0$$

Hence the origin is stable.

Example 78 (Simple pendulum.)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1\end{aligned}$$

Consider the total energy,

$$V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2$$

as a candidate Lyapunov function. Then, as we have allready shown, V is lpd about the origin; also

$$DV(x)f(x) = 0$$

Hence the origin is stable.

Example 79 (Stability of origin for attitude dynamics system.) Recall

$$\begin{aligned}\dot{x}_1 &= \frac{(I_2 - I_3)}{I_1}x_2x_3 \\ \dot{x}_2 &= \frac{(I_3 - I_1)}{I_2}x_3x_1 \\ \dot{x}_3 &= \frac{(I_1 - I_2)}{I_3}x_1x_2\end{aligned}$$

where

$$I_1, I_2, I_3 > 0$$

Consider the kinetic energy

$$V(x) = \frac{1}{2}\left(I_1x_1^2 + I_2x_2^2 + I_3x_3^2\right)$$

as a candidate Lyapunov function. Then V is lpd about the origin and

$$DV(x)f(x) = 0$$

Hence the origin is stable.

Example 80 (Undamped Duffing system)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3\end{aligned}$$

As candidate Lyapunov function for the equilibrium state $x^e = [\ 1 \quad 0\ ]^T$ consider the total energy

$$V(x) = \frac{1}{4}x_1^4 - \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}\,.$$

Since

$$DV(x) = \left(\ x_1^3 - x_1 \quad x_2\ \right) \quad \text{and} \qquad D^2V(x) = \begin{pmatrix} 3x_1^2 - 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we have $V(x^e) = 0$, $DV(x^e) = 0$ and $D^2V(x^e) > 0$, and it follows that V is lpd about x^e. One can readily verify that $DV(x)f(x) = 0$. Hence, x^e is stable.

Exercises

Exercise 13 Determine whether or not the following functions are lpd. (a)

$$V(x) = x_1^2 - x_1^4 + x_2^2$$

(b)

$$V(x) = x_1 + x_2^2$$

(c)

$$V(x) = 2x_1^2 - x_1^3 + x_1x_2 + x_2^2$$

Exercise 14 (Simple pendulum with Coulomb damping.) By appropriate choice of Lyapunov function, show that the origin is a stable equilibrium state for

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - c\ \mathrm{sgm}\,(x_2) \end{aligned}$$

where $c > 0$.

Exercise 15 By appropriate choice of Lyapunov function, show that the origin is a stable equilibrium state for

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 \end{aligned}$$

Note that the linearization of this system about the origin is unstable.

Exercise 16 By appropriate choice of Lyapunov function, show that the origin is a stable equilibrium state for

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_1^3 \end{aligned}$$

8.2 Asymptotic stability

The following result presents conditions which guarantee that an equilibrium state is asymptotically stable.

Theorem 2 (Asymptotic stability) *Suppose there exists a function V and a scalar $R > 0$ such that V is locally positive definite about x^e and*

$$DV(x)f(x) < 0 \qquad \textit{for} \quad x \neq x^e \quad \textit{and} \quad ||x - x^e|| < R$$

Then x^e is an asymptotically stable equilibrium state for $\dot{x} = f(x)$.

Example 81

$$\dot{x} = -x^3$$

Consider

$$V(x) = x^2$$

Then V is lpd about zero and

$$DV(x)f(x) = -2x^4 < 0 \qquad \text{for} \quad x \neq 0$$

Hence the origin is AS.

Example 82

$$\dot{x} = -x + x^3$$

Consider

$$V(x) = x^2$$

Then V is lpd about zero and

$$DV(x)f(x) = -2x^2(1 - x^2) < 0 \qquad \text{for} \quad |x| < 1, \ x \neq 0$$

Hence the origin is AS. Although the origin is AS, there are solutions which go unbounded in a finite time.

Example 83

$$\dot{x} = -\sin x$$

Consider

$$V(x) = x^2$$

Then V is lpd about zero and

$$DV(x)f(x) = -2x\sin(x) < 0 \qquad \text{for} \quad |x| < \pi, \ x \neq 0$$

Hence the origin is AS.

Example 84 *(Simple pendulum with viscous damping.)* Intuitively, we expect the origin to be an asymptotically stable equilibrium state for the damped simple pendulum:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\sin x_1 - cx_2\end{aligned}$$

where $c > 0$ is the damping coefficient. If we consider the total mechanical energy

$$V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2$$

as a candidate Lyapunov function, we obtain

$$DV(x)f(x) = -cx_2^2\,.$$

Since $DV(x)f(x) \le 0$ for all x, we have stability of the origin. Since $DV(x)f(x) = 0$ whenever $x_2 = 0$, it follows that $DV(x)f(x) = 0$ for points arbitrarily close to the origin; hence V does not satisfy the requirements of the above theorem for asymptotic stability.

Suppose we modify V to

$$V(x) = \frac{1}{2}\lambda c^2 x_1^2 + \lambda c x_1 x_2 + \frac{1}{2}x_2^2 + 1 - \cos x_1$$

where λ is any scalar with $0 < \lambda < 1$. Letting

$$P = \frac{1}{2}\begin{pmatrix} \lambda c^2 & \lambda c\\ \lambda c & 1\end{pmatrix}$$

note that $P > 0$ and

$$\begin{aligned}V(x) &= x^T P x + 1 - \cos x_1\\ &\ge x^T P x\end{aligned}$$

Hence V is lpd about zero and we obtain

$$DV(x)f(x) = -\lambda c x_1 \sin x_1 - (1-\lambda)cx_2^2 < 0 \qquad \text{for} \quad ||x|| < \pi,\ x \neq 0$$

to satisfy the requirements of above theorem; hence the origin is AS.

Exercises

Exercise 17 By appropriate choice of Lyapunov function, show that the origin is an asymptotically stable equilibrium state for

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1^5 - x_2\end{aligned}$$

Exercise 18 By appropriate choice of Lyapunov function, show that the origin is a asymptotically stable equilibrium state for

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + x_1^3 - x_2\end{aligned}$$

8.3 Boundedness

8.3.1 Radially unbounded functions

DEFN. *A scalar valued function V of the state x is said to be* radially unbounded *if*

$$\boxed{\lim_{x\to\infty} V(x) = \infty}$$

Example 85

$$\begin{array}{rcll} V(x) &=& x^2 & \text{yes} \\ V(x) &=& 1-e^{-x^2} & \text{no} \\ V(x) &=& x^2-x & \text{yes} \\ V(x) &=& x^4-x^2 & \text{yes} \\ V(x) &=& x\sin x & \text{no} \\ V(x) &=& x^2(1-\cos x) & \text{no} \end{array}$$

Example 86 Suppose P is a real $n\times n$ matrix and is positive definite symmetric and consider the quadratic form defined by

$$V(x) = x^T P x$$

Since P is real symmetric,

$$x^T P x \geq \lambda_{\min}(P)||x||^2$$

for all x, where $\lambda_{\min}(P)$ is the minimum eigenvalue of P. Since P is positive definite, $\lambda_{\min}(P) > 0$. From this it should be clear that V is radially unbounded.

The following lemma can be useful for guaranteeing radial unboundedness.

Lemma 3 *Suppose V is twice continuously differentiable, and there is a positive definite symmetric matrix P and a scalar $R \geq 0$ such that*

$$D^2V(x) \geq P \qquad \text{for} \quad ||x|| \geq R$$

Then, V is radially unbounded.

8.3.2 A boundedness result

Theorem 3 *Suppose there exists a radially unbounded function V and a scalar $R \geq 0$ such that*

$$DV(x)f(x) \leq 0 \qquad \text{for} \quad ||x|| \geq R$$

Then all solutions of $\dot{x} = f(x)$ are bounded.

Note that, in the above theorem, V does not have to be positive away from the origin; it only has to be radially unbounded.

Example 87 Recall

$$\dot{x} = x - x^3 .$$

Consider

$$V(x) = x^2$$

Since V is radially unbounded and

$$DV(x)f(x) = -2x^2(x^2 - 1)$$

the hypotheses of the above theorem are satisfied with $R = 1$; hence all solutions are bounded. Note that the origin is unstable.

Example 88 Duffing's equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 \end{aligned}$$

Consider

$$V(x) = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$$

It should be clear that V is radially unbounded; also

$$DV(x)f(x) = 0 \leq 0 \qquad \text{for all} \quad x$$

So, the hypotheses of the above theorem are satisfied with any R; hence all solutions are bounded.

Example 89

$$\dot{x} = -x - x^3 + w$$

where $|w| \leq \beta$.

Example 90

$$\dot{x} = Ax + Bw$$

where A is Hurwitz and $|w| \leq \beta$.

Exercises

Exercise 19 Determine whether or not the the following function is radially unbounded.

$$V(x) = x_1 - x_1^3 + x_1^4 - x_2^2 + x_2^4$$

Exercise 20 (Forced Duffing's equation with damping.) Show that all solutions of the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 - cx_2 + 1 \qquad c > 0 \end{aligned}$$

are bounded.
Hint: Consider

$$V(x) = \frac{1}{2}\lambda c^2 x_1^2 + \lambda c x_1 x_2 + \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$$

where $0 < \lambda < 1$. Letting

$$P = \frac{1}{2}\begin{pmatrix} \lambda c^2 & \lambda c \\ \lambda c & 1 \end{pmatrix}$$

note that $P > 0$ and

$$\begin{aligned} V(x) &= x^T P x - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \\ &\geq x^T P x - \frac{1}{4} \end{aligned}$$

Exercise 21 Recall the Lorenz system

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= rx_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -bx_3 + x_1 x_2 \end{aligned}$$

with $b > 0$. Prove that all solutions of this system are bounded. (Hint: Consider $V(x) = rx_1^2 + \sigma x_2^2 + \sigma(x_3 - 2r)^2$.)

8.4 Global asymptotic stability

8.4.1 Positive definite functions

Defn. (Positive definite function.) *A function V is* positive definite (pd) *if*

$$\begin{array}{rcll} V(0) & = & 0 & \\ V(x) & > & 0 & \textit{for all} \quad x \neq 0 \\ \lim_{x\to\infty} V(x) & = & \infty & \end{array}$$

In the above definition, note the requirement that V be *radially unbounded.*

Example 91 Scalar x

$$\begin{array}{rcll} V(x) & = & x^2 & \text{pd} \\ V(x) & = & 1 - e^{-x^2} & \text{lpd but not pd} \\ V(x) & = & 1 - \cos x & \text{lpd but not pd} \end{array}$$

Example 92

$$V(x) = x_1^4 + x_2^2$$

Example 93

$$\begin{array}{rcl} V(x) & = & ||x||^2 \\ & = & x_1^2 + x_2^2 + \ldots + x_n^2 \end{array}$$

Example 94 Suppose P is a real $n \times n$ matrix and is positive definite symmetric and consider the quadratic form defined by

$$V(x) = x^T P x$$

V is a positive definite function.

The following lemma can be useful in demonstrating that a function has the first two properties needed for a function to be pd.

Lemma 4 *Suppose V is twice continuously differentiable and*

$$\begin{array}{rcll} V(0) & = & 0 & \\ DV(0) & = & 0 & \\ D^2V(x) & > & 0 & \textit{for all} \quad x \end{array}$$

Then $V(x) > 0$ for all x.

If V satisfies the hypotheses of the above lemma, it is not guaranteed to be radially unbounded, hence it is not guaranteed to be positive definite. Lemma 3 can be useful for guaranteeing radial unboundedness. We also have the following lemma.

Lemma 5 *Suppose V is twice continuously differentiable,*

$$\begin{aligned} V(0) &= 0 \\ DV(0) &= 0 \end{aligned}$$

and there is a positive definite symmetric matrix P such that

$$D^2V(x) \geq P \qquad \text{for all} \quad x$$

Then

$$V(x) \geq \frac{1}{2}x^T P x$$

for all x.

8.4.2 A result on global asymptotic stability

Theorem 4 (Global asymptotic stability) *Suppose there exists a positive definite function V such that*

$$DV(x)f(x) < 0 \qquad \text{for all} \quad x \neq 0$$

Then the origin is a globally asymptotically stable equilibrium state for $\dot{x} = f(x)$.

Example 95

$$\dot{x} = -x^3$$

$$V(x) = x^2$$

$$\begin{aligned} DV(x)f(x) &= -2x^4 \\ &< 0 \qquad \text{for all} \quad x \neq 0 \end{aligned}$$

We have GAS. Note that linearization of this system about the origin cannot be used to deduce the asymptotic stability of this system.

Example 96 The first nonlinear system

$$\dot{x} = -\text{sgm}\,(x)\,.$$

This system is not linearizable about its unique equilibrium state at the origin. Considering

$$V(x) = x^2$$

we obtain

$$\begin{aligned} DV(x)f(x) &= -2|x| \\ &< 0 \qquad \text{for all} \quad x \neq 0\,. \end{aligned}$$

Hence, we have GAS.

Example 97 Consider

$$\begin{aligned}\dot{x}_1 &= -x_1 + \beta\sin(x_1 - x_2)\\ \dot{x}_1 &= -2x_2 - \beta\sin(x_1 - x_2)\end{aligned}$$

We will show that this system is GAS provided β is small enough. Considering the positive definite function

$$V(x) = x_1^2 + x_2^2\,.$$

as a candidate Lyapunov function, we have

$$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$

Example 98 *(Linear globally stabilizing controller for inverted pendulum.)*

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \sin x_1 + u\end{aligned}$$

Consider

$$u = -k_1x_1 - k_2x_2 \qquad \text{with} \quad k_1 > 1,\ k_2 > 0$$

Closed loop system:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -k_1x_1 + \sin x_1 - k_2x_2\end{aligned}$$

Consider

$$V(x) = \frac{1}{2}\lambda k_2^2x_1^2 + \lambda k_2x_1x_2 + \frac{1}{2}x_2^2 + \frac{1}{2}k_1x_1^2 + \cos x_1 - 1$$

where $0 < \lambda < 1$. Then V is pd (apply lemma 5) and

$$DV(x)f(x) = \lambda k_2(-k_1x_1^2 + x_1\sin x_1) - (1-\lambda)k_2x_2^2$$

Since

$$|\sin x_1| \leq |x_1| \qquad \text{for all} \quad x_1$$

it follows that

$$x_1 \sin x_1 \leq x_1^2 \qquad \text{for all} \quad x_1$$

hence

$$DV(x)f(x) \leq -\lambda k_2(k_1 - 1)x_1^2 - (1-\lambda)k_2 x_2^2 \quad < 0 \qquad \text{for all} \quad x \neq 0$$

The closed loop system is GAS.

Example 99 (Stabilization of origin for attitude dynamics system.)

$$\begin{aligned}
\dot{x}_1 &= \frac{(I_2 - I_3)}{I_1} x_2 x_3 &+& \frac{u_1}{I_1} \\
\dot{x}_2 &= \frac{(I_3 - I_1)}{I_2} x_3 x_1 &+& \frac{u_2}{I_2} \\
\dot{x}_3 &= \frac{(I_1 - I_2)}{I_3} x_1 x_2 &+& \frac{u_3}{I_3}
\end{aligned}$$

where

$$I_1, I_2, I_3 > 0$$

Consider any linear controller of the form

$$u_i = -kx_i \quad i = 1, 2, 3, \qquad k > 0$$

Closed loop system:

$$\begin{aligned}
\dot{x}_1 &= \left(\frac{I_2 - I_3}{I_1}\right) x_2 x_3 &-& \frac{kx_1}{I_1} \\
\dot{x}_2 &= \left(\frac{I_3 - I_1}{I_2}\right) x_3 x_1 &-& \frac{kx_2}{I_2} \\
\dot{x}_3 &= \left(\frac{I_1 - I_2}{I_3}\right) x_1 x_2 &-& \frac{kx_3}{I_3}
\end{aligned}$$

Consider the kinetic energy

$$V(x) = \frac{1}{2}\left(I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2\right)$$

as a candidate Lyapunov function. Then V is pd and

$$\begin{aligned}
DV(x)f(x) &= -k(x_1^2 + x_2^2 + x_3^2) \\
&< 0 \qquad \text{for all} \quad x \neq 0
\end{aligned}$$

Hence the origin is GAS for the closed loop system.

Exercises

Exercise 22 Determine whether or not the following function is positive definite.

$$V(x) = x_1^4 - x_1^2 x_2 + x_2^2$$

Exercise 23 Consider any scalar system described by

$$\dot{x} = -g(x)$$

where g has the following properties:

$$\begin{aligned} g(x) > 0 \quad &\text{for} \quad x > 0 \\ g(x) < 0 \quad &\text{for} \quad x < 0 \end{aligned}$$

Show that this system is GAS.

Exercise 24 Recall the Lorenz system

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -b x_3 + x_1 x_2 \end{aligned}$$

Prove that if

$$b > 0 \qquad \text{and} \qquad 0 \le r < 1\,,$$

then this system is GAS about the origin. (Hint: Consider $V(x) = x_1^2 + \sigma x_2^2 + \sigma x_3^2$.)

Exercise 25 *(Stabilization of the Duffing system.)* Consider the Duffing system with a scalar control input $u(t)$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 + u \end{aligned}$$

Obtain a linear controller of the form

$$u = -k_1 x_1 - k_2 x_2$$

which results in a closed loop system which is GAS about the origin. Numerically simulate the open loop system ($u = 0$) and the closed loop system for several initial conditions.

8.5 Exponential stability

8.5.1 Global exponential stability

Theorem 5 (Global exponential stability) *Suppose there exists a function V and positive scalars α, β_1, β_2 such that for all x,*

$$\beta_1 ||x - x^e||^2 \le V(x) \le \beta_2 ||x - x^e||^2$$

and

$$DV(x)f(x) \leq -2\alpha V(x)\,.$$

Then, for the system $\dot{x} = f(x)$*, the state* x^e *is a globally exponentially stable equilibrium state with rate of convergence* α*. In particular, all solutions* $x(\cdot)$ *of the system satisfy*

$$\|x(t)-x^e\| \leq \sqrt{\beta_2/\beta_1}\,\|x(0)-x^e\|\,e^{-\alpha t} \qquad \textit{for all} \qquad t \geq 0\,.$$

PROOF. See below.

Example 100

$$\dot{x} = -x$$

Considering

$$V(x) = x^2$$

we have

$$\begin{aligned} DV(x)f(x) &= -2x^2 \\ &= -2V(x) \end{aligned}$$

Hence, we have GES with rate of convergence 1.

Example 101

$$\dot{x} = -x - x^3$$

Considering

$$V(x) = x^2$$

we have

$$\begin{aligned} DV(x)f(x) &= -2x^2 - 2x^4 \\ &\leq -2V(x) \end{aligned}$$

Hence, we have GES with rate of convergence 1.

8.5.2 Proof of theorem 6

Consider any solution $x(\cdot)$ and let $v(t) = V(x(t))$. Then

$$\dot{v} \leq -2\alpha v\,.$$

8.5.3 Exponential stability

Theorem 6 (Exponential stability) *Suppose there exists a function V and positive scalars R, α, β_1, β_2 such that, whenever $\|x - x^e\| \leq R$, one has*

$$\beta_1 ||x - x^e||^2 \leq V(x) \leq \beta_2 ||x - x^e||^2$$

and

$$DV(x)f(x) \leq -2\alpha V(x)$$

Then, for system $\dot{x} = f(x)$, the state x^e is an exponentially stable equilibrium state with rate of convergence α.

Exercise 26 Consider the scalar system

$$\dot{x} = -x + x^3$$

As a candidate Lyapunov function for exponential stability, consider $V(x) = x^2$. Clearly, the condition on V is satisfied with $\beta_1 = \beta_2 = 1$. Noting that

$$\dot{V} = -2x^2 + 2x^4 = -2(1 - x^2)x^2\,,$$

and considering $R = 1/2$, we obtain that whenever $|x| \leq R$, we have $\dot{V} \leq -2\alpha V$ where $\alpha = 3/4$. Hence, we have ES.

8.5.4 A special class of GES systems

Consider a system described by

$$\dot{x} = f(x) \tag{8.2}$$

and suppose that there exist two positive definite symmetric matrices P and Q such that

$$\boxed{x^T P f(x) \leq -x^T Q x\,.}$$

We will show that the origin is GES with rate

$$\boxed{\alpha := \lambda_{\min}(P^{-1}Q)}$$

where $\lambda_{\min}(P^{-1}Q) > 0$ is the smallest eigenvalue of $P^{-1}Q$.

As a candidate Lyapunov function, consider

$$V(x) = x^T P x\,.$$

Then

$$\lambda_{\min}(P)||x||^2 \leq V(x) \leq \lambda_{\max}(P)||x||^2$$

that is

$$\beta_1||x||^2 \leq V(x) \leq \beta_2||x||^2$$

with

$$\beta_1 = \lambda_{\min}(P) > 0 \qquad \text{and} \qquad \beta_2 = \lambda_{\max}(P) > 0\,.$$

Now note that

$$\begin{aligned} DV(x)f(x) &= 2x^T P f(x) \\ &\leq -2x^T Q x \end{aligned}$$

For any two positive-definite matrices P and Q, one can show that all the eigenvalues of $P^{-1}Q$ are real positive and

$$x^T Q x \geq \lambda_{\min}(P^{-1}Q)\, x^T P x$$

where $\lambda_{\min}(P^{-1}Q) > 0$ is the smallest eigenvalue of $P^{-1}Q$. Thus,

$$\begin{aligned} DV(x)f(x) &\leq -2\lambda_{\min}(P^{-1}Q) x^T P x \\ &= -2\alpha V(x) \end{aligned}$$

Hence GES with rate α.

Example 102 Recall example 99.
Closed loop system:

$$\begin{aligned} \dot{x}_1 &= \left(\frac{I_2 - I_3}{I_1}\right) x_2 x_3 && - \frac{k x_1}{I_1} \\ \dot{x}_2 &= \left(\frac{I_3 - I_1}{I_2}\right) x_3 x_1 && - \frac{k x_2}{I_2} \\ \dot{x}_3 &= \left(\frac{I_1 - I_2}{I_3}\right) x_1 x_2 && - \frac{k x_3}{I_3} \end{aligned}$$

Considering

$$P = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

we have

$$x^T P f(x) = -k x^T x$$

that is, $Q = kI$. Hence, GES with rate

$$\begin{aligned} \alpha &= \lambda_{\min}(P^{-1}Q) \\ &= \lambda_{\min}(kP^{-1}) \\ &= k\lambda_{\min}(P^{-1}) \\ &= \frac{k}{\lambda_{\max}(P)} \\ &= \frac{k}{\max\{I_1, I_2, I_3\}} \end{aligned}$$

8.5.5 Summary

The following table summarizes the results of this chapter for stability about the origin.

V	$\dot{V}$	$\Rightarrow$	stability properties
lpd	≤ 0 for $\|\|x\|\| \leq R$	$\Rightarrow$	S
lpd	< 0 for $\|\|x\|\| \leq R$, $x \neq 0$	$\Rightarrow$	AS
ru	≤ 0 for $\|\|x\|\| \geq R$	$\Rightarrow$	B
pd	< 0 for $x \neq 0$	$\Rightarrow$	GAS
$\beta_1\|\|x\|\|^2 \leq V(x) \leq \beta_2\|\|x\|\|^2$	$\leq -2\alpha V(x)$	$\Rightarrow$	GES
$\beta_1\|\|x\|\|^2 \leq V(x) \leq \beta_2\|\|x\|\|^2$ for $\|\|x\|\| \leq R$	$\leq -2\alpha V(x)$ for $\|\|x\|\| \leq R$	$\Rightarrow$	ES

Figure 8.2: Lyapunov table

Chapter 9

Basic Lyapunov theory: discrete time

Suppose we are interested in the stability properties of the system,

$$x(k+1) = f(x(k)) \tag{9.1}$$

where $x(k) \in \mathbb{R}^n$ and $k \in \mathbb{R}$. If the system is linear, we can determine its stability properties from the properties of the eigenvalues of the system matrix. What do we for a nonlinear system? We could linearize about each equilibrium state and determine the stability properties of the resulting linearizations. Under certain conditions (see later) this will tell us something about the local stability properties of the nonlinear system about its equilibrium states. However there are situations where linearization cannot be used to deduce even the local stability properties of the nonlinear system. Also, linearization tells us nothing about the global stability properties of the nonlinear system.

In general, we cannot explicitly obtain solutions for nonlinear systems. Lyapunov theory allows to say something about the stability properties of a system without knowing the form or structure of the solutions.

• In this chapter, V is a scalar-valued function of the state, i.e. $V : \mathbb{R}^n \to \mathbb{R}$. At any time k, the one step change in V along a solution $x(\cdot)$ of system (9.1) is given by

$$V(x(k+1)) - V(x(k)) = \Delta V(x(k))$$

where

$$\boxed{\Delta V(x) := V(f(x)) - V(x)}$$

9.1 Stability

Theorem 7 (Stability) *Suppose there exists a locally positive definite function V and a scalar $R > 0$ such that*

$$\Delta V(x) \leq 0 \qquad \textit{for} \quad ||x|| < R$$

Then the origin is a stable equilibrium state.

• If V satisfies the hypotheses of the above theorem, then V is said to be a *Lyapunov function* which guarantees the stability of origin.

Example 103

$$x(k+1) = x(k)$$

Consider

$$V(x) = x^2$$

as a *candidate Lyapunov function.* Then V is a lpdf and

$$\Delta V(x) = 0$$

Hence (it follows from theorem 7 that) the origin is stable.

9.2 Asymptotic stability

Theorem 8 (Asymptotic stability) *Suppose there exists a locally positive definite function V and a scalar $R > 0$ such that*

$$\Delta V(x) < 0 \qquad \textit{for} \quad x \neq 0 \quad \textit{and} \quad ||x|| < R$$

Then the origin is an asymptotically stable equilibrium state.

Example 104

$$x(k+1) = \frac{1}{2}x(k)$$

Consider

$$V(x) = x^2$$

Then V is a lpdf and

$$\begin{aligned} \Delta V(x) &= (\frac{1}{2}x)^2 - x^2 \\ &= -\frac{3}{4}x^2 \\ &< 0 \qquad \text{for} \quad x \neq 0 \end{aligned}$$

Hence the origin is AS.

Example 105

$$x(k+1) = \frac{1}{2}x(k) + x(k)^2$$

Consider

$$V(x) = x^2$$

Then V is a lpdf and

$$\begin{aligned} \Delta V(x) &= (\frac{1}{2}x + x^2)^2 - x^2 \\ &= -x^2(\frac{3}{2} + x)(\frac{1}{2} - x) \\ &< 0 \qquad \text{for} \quad |x| < \frac{1}{2},\ x \neq 0 \end{aligned}$$

Hence the origin is AS.

9.3 Boundedness

Theorem 9 *Suppose there exists a radially unbounded function V and a scalar $R \geq 0$ such that*

$$\Delta V(x) \leq 0 \qquad \text{for} \quad ||x|| \geq R$$

Then all solutions of (9.1) are bounded.

Note that, in the above theorem, V does not have to positive away from the origin; it only has to be radially unbounded.

Example 106

$$x(k+1) = \frac{2x(k)}{1 + x(k)^2}$$

Consider

$$V(x) = x^2$$

Since V is radially unbounded and

$$\begin{aligned} \Delta V(x) &= \left(\frac{2x}{1+x^2}\right)^2 - x^2 \\ &= -\frac{x^2(x^2+3)(x^2-1)}{(x^2+1)^2} \\ &\leq 0 \qquad \text{for} \quad |x| \geq 1 \end{aligned}$$

the hypotheses of the above theorem are satisfied with $R = 1$; hence all solutions are bounded. Note that the origin is unstable.

9.4 Global asymptotic stability

Theorem 10 (Global asymptotic stability) *Suppose there exists a positive definite function V such that*

$$\Delta V(x) < 0 \qquad \textit{for all} \quad x \neq 0$$

Then the origin is a globally asymptotically stable equilibrium state.

Example 107

$$x(k+1) = \frac{1}{2}x(k)$$

Example 108

$$x(k+1) = \frac{x(k)}{1+x(k)^2}$$

Consider

$$V(x) = x^2$$

Then

$$\begin{aligned}\Delta V(x) &= \left(\frac{x}{1+x^2}\right)^2 - x^2 \\ &= -\frac{2x^4+x^6}{(1+x^2)^2} \\ &< 0 \qquad \text{for all} \quad x \neq 0\end{aligned}$$

Hence this system is GAS about zero.

Exercise 27 Consider any scalar system described by

$$x(k+1) = g(x(k))$$

where g has the following properties:

$$|g(x)| < |x| \quad \text{for} \quad x \neq 0$$

Show that this system is GAS.

9.5 Exponential stability

Theorem 11 (Global exponential stability.) *Suppose there exists a function V and scalars α, β_1, β_2 such that for all x,*

$$\beta_1 ||x||^2 \leq V(x) \leq \beta_2 ||x||^2 \qquad \beta_1, \beta_2 > 0$$

and

$$V(f(x)) \;\leq\; \alpha^2 V(x) \qquad 0 \leq \alpha < 1$$

Then, every solution satisfies

$$||x(k)|| \leq \sqrt{\frac{\beta_2}{\beta_1}} \alpha^k ||x(0)|| \qquad \textit{for} \quad k \geq 0$$

Hence, the origin is a globally exponentially stable equilibrium state with rate of convergence α.

PROOF.

Example 109

$$x(k+1) = -\frac{1}{2} x(k)$$

Considering

$$V(x) = x^2$$

we have

$$\begin{aligned} V(f(x)) &= \frac{1}{4}x^2 \\ &= (\frac{1}{2})^2 V(x) \end{aligned}$$

Hence, we have GES with rate of convergence $\alpha = \frac{1}{2}$.

Example 110

$$x(k+1) = \frac{1}{2}\sin(x(k))$$

Considering

$$V(x) = x^2$$

we have

$$\begin{aligned} V(f(x)) &= (\frac{1}{2}\sin x)^2 \\ &= (\frac{1}{2})^2 |\sin x|^2 \\ &\leq (\frac{1}{2})^2 |x|^2 \\ &= (\frac{1}{2})^2 V(x) \end{aligned}$$

Hence, we have GES with rate of convergence $\alpha = \frac{1}{2}$

Chapter 10

Lyapunov theory for linear time-invariant systems

The main result of this section is contained in Theorem 12.

10.1 Positive and negative (semi)definite matrices

10.1.1 Definite matrices

For any square matrix P we can define an associated quadratic form:

$$x'Px = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}x_i'x_j$$

If P is hermitian ($P' = P$) then the scalar $x'Px$ is real for all $x \in \mathbb{C}^n$.

Defn. A hermitian matrix P is positive definite (pd) if

$$x'Px > 0$$

for all nonzero x. We denote this by $P > 0$. The matrix P is negative definite (nd) if $-P$ is positive definite; we denote this by $P < 0$.

Example 111 For

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

we have (note the completion of squares trick)

$$\begin{aligned} x'Px &= x_1'x_1 - x_1'x_2 - x_2'x_1 + 2x_2'x_2 \\ &= (x_1 - x_2)'(x_1 - x_2) + x_2'x_2 \\ &= |x_1 - x_2|^2 + |x_2|^2 \end{aligned}$$

Clearly, $x'Px \geq 0$ for all x. If $x'Px = 0$, then $x_1 - x_2 = 0$ and $x_2 = 0$; hence $x = 0$. So, $P > 0$.

Fact 1 *The following statements are equivalent for any hermitian matrix P.*

(a) P is positive definite.

(b) All the eigenvalues of P are positive.

(c) All the leading principal minors of P are positive.

Example 112 Consider

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Since $p_{11} = 1 > 0$ and $\det(P) = 1 > 0$, we must have $P > 0$.

```
>> eig(P)
ans =
    0.3820
    2.6180
```

Note the positive eigenvalues.

10.1.2 Semi-definite matrices

Defn. A hermitian matrix P is positive semi-definite (psd) if

$$x'Px \geq 0$$

for all non-zero x. We denote this by $P \geq 0$

P is *negative semi-definite (nsd)* if $-P$ is positive semi-definite; we denote this by $P \leq 0$

Fact 2 *The following statements are equivalent for any hermitian matrix P.*

(a) P is positive semi-definite.

(b) All the eigenvalues of P are non-negative.

(c) All the leading minors of P are positive.

Example 113 This example illustrates that non-negativity of the leading principal minors of P is not sufficient for $P \geq 0$.

$$P = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

We have $p_{11} = 0$ and $\det(P) = 0$. However,

$$x'Px = -|x_2|^2$$

hence, P is not psd. Actually, P is nsd.

Fact 3 *Consider any $m \times n$ complex matrix M and let $P = M'M$. Then*

(a) P is hermitian and $P \geq 0$

(b) $P > 0$ iff rank $M = n$.

Example 114

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Since

$$P = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

and

$$\text{rank} \begin{pmatrix} 1 & 1 \end{pmatrix} = 1$$

$P \geq 0$ but P is not pd.

Exercise 28 (Optional)
Suppose P is hermitian and T is invertible. Show that $P > 0$ iff $T'PT > 0$.

Exercise 29 (Optional)
Suppose P and Q are two hermitian matrices with $P > 0$. Show that $P + \lambda Q > 0$ for all real λ sufficiently small; i.e., there exists $\bar{\lambda} > 0$ such that whenever $|\lambda| < \bar{\lambda}$, one has $P + \lambda Q > 0$.

10.2 Lyapunov theory

10.2.1 Asymptotic stability results

The scalar system

$$\dot{x} = ax$$

is asymptotically stable if and only if $a+a' < 0$. Consider now a general linear time-invariant system described by

$$\dot{x} = Ax \tag{10.1}$$

The generalization of $a + a' < 0$ to this system is $A + A' < 0$. We will see shortly that this condition is sufficient for asymptotic stability; however, as the following example illustrates, it is not necessary.

Example 115 The matrix

$$A = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix}$$

is asymptotically stable. However, the matrix

$$A + A' = \begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}$$

is not negative definite.

Since stability is invariant under a similarity transformation T, i.e., the stability of A and $T^{-1}AT$ are equivalent for any nonsingular T, we should consider the more general condition

$$T^{-1}AT + T'A'T^{-\prime} < 0$$

Introducing the hermitian matrix $P := T^{-\prime}T^{-1}$, and pre- and post-multiplying the above inequality by $T^{-\prime}$ and T^{-1}, respectively, yields

$$P > 0 \tag{10.2a}$$

$$PA + A'P < 0 \tag{10.2b}$$

We now show that the existence of a hermitian matrix P satisfying these conditions guarantees asymptotic stability.

Lemma 6 *Suppose there is a hermitian matrix P satisfying (10.2). Then system (10.1) is asymptotically stable*

Proof. Suppose there exists a hermititian matrix P which satisfies inequalities (10.2). Consider any eigenvalue λ of A. Let $v \neq 0$ be an eigenvector corresponding to λ, i.e.,

$$Av = \lambda v$$

Then

$$v'PAv = \lambda v'Pv \; ; \qquad v'A'Pv = \bar{\lambda}v'Pv$$

Pre- and post-multiplying inequality (10.2b) by v' and v, respectively, yields

$$\lambda v'Pv + \bar{\lambda} v'Pv < 0$$

i.e.,

$$2Re(\lambda)v'Pv < 0$$

Since $P > 0$, we must have $v'Pv > 0$; hence $Re(\lambda) < 0$. Since the above holds for every eigenvalue of A, system (10.1) is asymptotically stable. ■

Remark 1 A hermitian matrix P which satisfies inequalities (10.2) will be referred to as a *Lyapunov matrix* for (10.1) or A.

Remark 2 Conditions (10.2) are referred to as *linear matrix inequalities (LMI's)*. In recent years, efficient numerical algorithms have been developed to solve LMI's or determine that a solution does not exist.

Lyapunov functions. To obtain an alternative interpretation of inequalities (10.2), consider the quadratic function V of the state defined by

$$V(x) := x'Px$$

Since $P > 0$, this function has a strict global minimum at zero. Using inequality (10.2b), it follows that along any non-zero solution $x(\cdot)$ of (10.1),

$$\begin{aligned}
\frac{dV(x(t))}{dt} &= \dot{x}'Px + x'P\dot{x} \\
&= x'\left(A'P + PA\right)x \\
&< 0
\end{aligned}$$

i.e., $V(x(\cdot))$ is strictly decreasing along any non-zero solution. Intuitively, one then expects every solution to asymptotically approach the origin. The function V is called a *Lyapunov function* for the system. The concept of a Lyapunov function readily generalizes to nonlinear systems and has proved a very useful tool in the analysis of the stability properties of nonlinear systems.

Remark 3 Defining the hermitian matrix $S := P^{-1}$, inequalities (10.2) become

$$S > 0 \qquad (10.3a)$$

$$AS + SA' < 0 \qquad (10.3b)$$

Hence the above lemma can be stated with S replacing P and the preceding inequalities replacing (10.2).

So far we have shown that if a LTI system has a Lyapunov matrix, then it is AS. Is the converse true? That is, does every AS LTI system have a Lyapunov matrix? And if this is true how does one find a Lyapunov matrix? To answer this question note that satisfaction of inequality (10.2b) is equivalent to

$$\boxed{PA + A'P + Q = 0} \tag{10.4}$$

where Q is a hermitian positive definite matrix. This linear matrix equation is known as the Lyapunov equation. So one approach to looking for Lyapunov matrices could be to choose a pd hermitian Q and determine whether the Lyapunov equation has a pd hermitian solution for P.

We first show that if the system $\dot{x} = Ax$ is asymptotically stable and the Lyapunov equation (10.4) has a solution then, the solution is unique. Suppose P_1 and P_2 are two solutions to (10.4). Then,

$$(P_2 - P_1)A + A'(P_2 - P_1) = 0\,.$$

Hence,

$$e^{A't}(P_2 - P_1)Ae^{At} + e^{A't}A'(P_2 - P_1)e^{At} = 0$$

that is,

$$\frac{d\left(e^{A't}(P_2 - P_1)e^{At}\right)}{dt} = 0$$

This implies that, for all t,

$$e^{A't}(P_2 - P_1)e^{At} = e^{A'0}(P_2 - P_1)e^{A0} = P_2 - P_1\,.$$

Since $\dot{x} = Ax$ is asymptotically stable, $\lim_{t\longrightarrow\infty} e^{At} = 0$. Hence

$$P_2 - P_1 = \lim_{t\longrightarrow\infty} e^{A't}(P_2 - P_1)e^{At} = 0\,.$$

From this it follows that $P_2 = P_1$.

The following lemma tells us that every AS LTI system has a Lyapunov matrix and a Lyapunov matrix can be obtained by solving the Lyapunov equation with any pd hermitian Q.

Lemma 7 *Suppose system (10.1) is asymptotically stable. Then for every matrix Q, the Lyapunov equation (10.4) has a unique solution for P. If Q is positive definite hermitian then this solution is positive definite hermitian.*

PROOF. Suppose system (10.1) is asymptotically stable and consider any matrix Q. Let

$$\boxed{P := \int_0^\infty e^{A't}Qe^{At}dt}$$

This integral exists because each element of e^{At} is exponentially decreasing.

To show that P satisfies the Lyapunov equation (10.4), use the following properties of e^{At},

$$e^{At}A = \frac{de^{At}}{dt} \qquad A'e^{A't} = \frac{de^{A't}}{dt}$$

to obtain

$$
\begin{aligned}
PA + A'P &= \int_0^\infty \left(e^{A't} Q e^{At} A + A' e^{A't} Q e^{At} \right) \, dt \\
&= \int_0^\infty \left(e^{A't} Q \frac{de^{At}}{dt} + \frac{de^{A't}}{dt} Q e^{At} \right) \, dt \\
&= \int_0^\infty \frac{d\left(e^{A't} Q e^{At} \right)}{dt} \, dt \\
&= \lim_{t\to\infty} \int_0^t \frac{de^{A't} Q e^{At}}{dt} \, dt \\
&= \lim_{t\to\infty} e^{A't} Q e^{At} \;-\; Q \\
&= -Q
\end{aligned}
$$

We have already demonstrated uniqueness of solutions to (10.4),
Suppose Q is pd hermitian. Then it should be clear that P is pd hermitian. ■.

Using the above two lemmas, we can now state the main result of this section.

Theorem 12 *The following statements are equivalent.*

(a) The system $\dot{x} = Ax$ is asymptotically stable.

(b) There exist positive definite hermitian matrices P and Q satisfying the Lyapunov equation (10.4).

(c) For every *positive definite hermitian matrix Q, the Lyapunov equation (10.4) has a unique solution for P and this solution is hermitian positive definite.*

PROOF. The first lemma yields (b) $\Longrightarrow$ (a). The second lemma says that (a) $\Longrightarrow$ (c). Hence, (b) $\Longrightarrow$ (c).

To see that (c) $\Longrightarrow$ (b), pick any positive definite hermitian Q. So, (b) is equivalent to (c). Also, (c) $\Longrightarrow$ (a); hence (a) and (c) are equivalent.

■.

Example 116

$$
\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= -x_1 + cx_2
\end{aligned}
$$

Here

$$
A = \begin{pmatrix} 0 & 1 \\ -1 & c \end{pmatrix}
$$

Choosing

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and letting

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

(note we have taken $p_{21} = p_{12}$ because we are looking for symmetric solutions) the Lyapunov equation results in

$$\begin{aligned} -2p_{12} + 1 &= 0 \\ p_{11} + cp_{12} - p_{22} &= 0 \\ 2p_{12} + 2cp_{22} + 1 &= 0 \end{aligned}$$

We consider three cases:
Case i) $c = 0$. No solution; hence no GAS.
Case ii) $c = -1$. A single solution; this solution is pd; hence GAS.
Case iii) $c = 1$. A single solution; this solution is not pd; hence no GAS.

Remark 4 One can state the above theorem replacing P with S and replacing (10.4) with

$$AS + SA' + Q = 0 \tag{10.5}$$

10.2.2 MATLAB.

```
>> help lyap

 LYAP   Lyapunov equation.
        X = LYAP(A,C) solves the special form of the Lyapunov matrix
        equation:

                A*X + X*A' = -C

        X = LYAP(A,B,C) solves the general form of the Lyapunov matrix
        equation:

                A*X + X*B = -C

        See also DLYAP.
```

Note. In MATLAB, A' is the complex conjugate transpose of A (i.e. A'). Hence, to solve (10.4) one must use P= lyap(A', Q).

```
>> q=eye(2);

>> a=[0 1; -1 -1];

>> p=lyap(a', q)
p =
    1.5000    0.5000
    0.5000    1.0000

>> p*a + a'*p
ans =
   -1.0000   -0.0000
         0   -1.0000

>> det(p)
ans =
    1.2500

>> eig(p)
ans =
    1.8090
    0.6910
```

Exercise 30 Consider the Lyapunov equation

$$PA + A'P + 2\alpha P + Q = 0$$

where A is a square matrix, Q is some positive definite hermitian matrix and α is some positive scalar. *Show* that this equation has a unique solution for P and this solution is positive definite hermitian if and only if for every eigenvalue λ of A,

$$Re(\lambda) < -\alpha$$

10.2.3 Stability results

For stability, the following lemma is the analog of lemma 10.2.

Lemma 8 *Suppose there is a hermitian matrix P satisfying*

$$\begin{aligned} P &> 0 && (10.6a) \\ PA + A'P &\leq 0 && (10.6b) \end{aligned}$$

Then system (10.1) is stable.

We do not have the exact analog of lemma 7. However the following can be shown.

Lemma 9 *Suppose system (10.1) is stable. Then there exist a positive definite hermitian matrix P and a positive semi-definite hermitian matrix Q satisfying the Lyapunov equation (10.4).*

The above lemma does *not* state that for every positive semi-definite hermitian Q the Lyapunov equation has a solution for P. Also, when the Lyapunov equation has a solution, it is not unique. This is illustrated in the following example.

Example 117

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

With

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we obtain

$$PA + A'P = 0$$

In this example the Lyapunov equation has a pd solution with $Q = 0$; this solution is not unique; any matrix of the form

$$P = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$$

(where p is arbitrary) is also a solution.

If we consider the psd matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

the Lyapunov equation has no solution.

Combining the above two lemmas results in the following theorem.

Theorem 13 *The following statements are equivalent.*

(a) The system $\dot{x} = Ax$ is stable.

(b) There exist a positive definite hermitian matrix P and a positive semi-definite matrix Q satisfying the Lyapunov equation (10.4).

10.3 Mechanical systems

Example 118 A simple structure

Recall

$$\begin{array}{ccccccccccc} m_1\ddot{q}_1 & + & (c_1+c_2)\dot{q}_1 & - & c_2\dot{q}_2 & + & (k_1+k_2)q_1 & - & k_2q_2 & = & 0 \\ m_2\ddot{q}_2 & - & c_2\dot{q}_1 & + & c_2\dot{q}_2 & - & k_2q_1 & + & k_2q_2 & = & 0 \end{array}$$

Letting

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

this system can be described by the following second order vector differential equation:

$$M\ddot{q} + C\dot{q} + Kq = 0$$

where the symmetric matrices M, C, K are given by

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad C = \begin{pmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix} \qquad K = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$

Since $m_1, m_2 > 0$, we have $M > 0$.

If $k_1, k_2 > 0$ then $K > 0$ (why?) and if $c_1, c_2 > 0$ then $C > 0$.

Note also that

$$\begin{array}{lclcl} \text{kinetic energy} & = & \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 & = & \frac{1}{2}\dot{q}'M\dot{q} \\ \text{potential energy} & = & \frac{1}{2}k_1q_1^2 + \frac{1}{2}k_2(q_2-q_1)^2 & = & \frac{1}{2}q'Kq \end{array}$$

Consider now a general mechanical system described by

$$\boxed{M\ddot{q} + C\dot{q} + Kq = 0}$$

where $q(t)$ is a real N-vector of generalized coordinates which describes the configuraton of the system. The real matrix M is the *inertia matrix* and satisfies

$$M' = M > 0$$

We call K and C the *'stiffness' matrix* and *'damping' matrix* respectively. The *kinetic energy* of the system is given by

$$\frac{1}{2}\dot{q}'M\dot{q}$$

With $x = (q', \dot{q}')'$, this system has a state space description of the form $\dot{x} = Ax$ with

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}$$

A first Lyapunov matrix. Suppose

$$K' = K > 0 \qquad \text{and} \qquad C' = C \geq 0\,.$$

The *potential energy* of the system is given by

$$\frac{1}{2}q'Kq$$

Consider the following *candidate Lyapunov matrix*

$$P = \frac{1}{2}\begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix}$$

Then $P' = P > 0$ and

$$PA + A'P + Q = 0$$

where

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$$

It should be clear that

$$PA + A'P \leq 0$$

iff

$$C \geq 0$$

Hence,

$$\boxed{K' = K > 0 \text{ and } C' = C \geq 0 \text{ imply stability.}}$$

Note that

$$\begin{aligned} V(x) &= x'Px = \frac{1}{2}\dot{q}'M\dot{q} + \frac{1}{2}q'Kq = \text{total energy} \\ \dot{V}(x) &= -\dot{q}'C\dot{q} \end{aligned}$$

A second Lyapunov matrix.

$$P = \frac{1}{2}\begin{pmatrix} K+\lambda C & \lambda M \\ \lambda M & M \end{pmatrix} \quad = \quad \frac{1}{2}\begin{pmatrix} K & 0 \\ 0 & M \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} C & M \\ M & 0 \end{pmatrix}$$

For sufficiently small λ, the matrix P is positive definite.

Now,

$$PA + A'P + Q = 0$$

where

$$Q = \begin{pmatrix} \lambda K & 0 \\ 0 & C - \lambda M \end{pmatrix}$$

For sufficiently small $\lambda > 0$, the matrix $C - \lambda M$ is pd and, hence, Q is pd. So,

$$\boxed{K' = K > 0 \text{ and } C' = C > 0 \text{ imply asymptotic stability.}}$$

10.4 Rates of convergence for linear systems

Theorem 14 *Consider an asymptotically stable linear system described by $\dot{x} = Ax$ and let α be any real number satisfying*

$$0 < \alpha < \bar{\alpha}$$

where

$$\bar{\alpha} := -\max\{\Re(\lambda) : \lambda \text{ is an eigenvalue of } A \}\,.$$

Then $\dot{x} = Ax$ is GES with rate α.

PROOF: Consider any α satisfying $0 < \alpha < \bar{\alpha}$. As a consequence of the definition of $\bar{\alpha}$, all the eigenvalues of the matrix $A+\alpha I$ have negative parts. Hence, the Lyapunov equation

$$P(A+\alpha I) + (A+\alpha I)'P + I = 0\,.$$

has a unique solution for P and $P = P' > 0$. As a candidate Lyapunov for the system $\dot{x} = Ax$, consider $V(x) = x'Px$. Then,

$$\begin{aligned}\dot{V} &= x'P\dot{x} + \dot{x}'Px \\ &= x'PAx + (Ax)'Px \\ &= x'(PA + A'P)x\,.\end{aligned}$$

From the above Lyapunov matrix equation, we obtain that

$$PA + A'P = -2\alpha P - I\,;$$

hence,

$$\begin{aligned}\dot{V} &= -2\alpha x'Px - x'x \\ &\leq -2\alpha V(x)\,.\end{aligned}$$

Hence the system is globally exponentially stable with rate of convergence α.

10.5 Linearization and exponential stability

Consider a nonlinear system

$$\dot{x} = f(x) \tag{10.7}$$

and suppose that x^e is an equilibrium state for this system. We assume that f is differential at x^e and let

$$Df(x^e) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x^e) & \frac{\partial f_1}{\partial x_2}(x^e) & \cdots & \frac{\partial f_1}{\partial x_n}(x^e) \\ \frac{\partial f_2}{\partial x_1}(x^e) & \frac{\partial f_2}{\partial x_2}(x^e) & \cdots & \frac{\partial f_2}{\partial x_n}(x^e) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^e) & \frac{\partial f_n}{\partial x_2}(x^e) & \cdots & \frac{\partial f_n}{\partial x_n}(x^e) \end{pmatrix}$$

where $x = (x_1, x_2, \cdots, x_n)$. Then the linearization of $\dot{x} = f(x)$ about x^e is the linear system defined by

$$\delta\dot{x} = A\delta x \qquad \text{where} \quad A = Df(x^e)\,. \tag{10.8}$$

Theorem 15 *Suppose x^e is an equilibrium state of a nonlinear system of the form $\dot{x} = f(x)$ and the corresponding linearization is exponentially stable. Then the nonlinear system is exponentially stable about x^e.*

PROOF. Suppose the linearization is exponentially stable and let

$$\bar{\alpha} := -\max\{\Re(\lambda) \,:\, \lambda \text{ is an eigenvalue of A }\}\,.$$

Since all the eigenvalues have negative real parts, we have $\bar{\alpha} > 0$. Consider now any α satisfying $0 < \alpha < \bar{\alpha}$. As a consequence of the definition of $\bar{\alpha}$, all the eigenvalues of the matrix $A+\alpha I$ have negative parts. Hence, the Lyapunov equation

$$P(A+\alpha I) + (A+\alpha I)'P + I = 0\,.$$

has a unique solution for P and $P = P' > 0$. As a candidate Lyapunov function for the nonlinear system, consider

$$V(x) = (x-x^e)'P(x-x^e)\,.$$

Recall that

$$f(x) = f(x^e) + Df(x^e)(x-x^e) + o(x-x^e)$$

where the "remainder term" has the following property:

$$\lim_{x\to x^e,\, x\neq x^e} \frac{o(x-x^e)}{\|x-x^e\|} = 0\,.$$

Hence

$$f(x) = A(x-x^e) + o(x-x^e)$$

and

$$\begin{aligned}\dot{V} &= 2x'P\dot{x} \\ &= 2x'Pf(x) \\ &= 2x'PAx + 2x'Po(x) \\ &\le x'(PA + A'P)x + 2\|x\|\|P\|\|o(x)\|\,.\end{aligned}$$

From the above Lyapunov matrix equation, we obtain that

$$PA + A'P = -2\alpha P - I\,;$$

hence,

$$\begin{aligned}\dot{V} &= -2\alpha x'Px - x'x + 2\|x\|\|P\|\|o(x)\| \\ &= -2\alpha V(x) - \|x\|^2 + 2\|x\|\|P\|\|o(x)\|\,.\end{aligned}$$

As a consequence of the properties of o, there is a scalar $R > 0$ such that

$$2\|P\|\|o(x)\| \le \|x\| \text{ when } \|x\| \le R\,.$$

Hence, whenever $\|x\| \le R$ we obtain that

$$\dot{V} \le -2\alpha V(x)\,.$$

Hence the nonlinear system is exponentially stable about x^e with rate of convergence α.

Chapter 11

Quadratic stability

11.1 Introduction

[3] In this chapter, we introduce numerical techniques which are useful in finding Lyapunov functions for certain classes of globally exponentially stable nonlinear systems. We restrict consideration to quadratic Lyapunov functions. For specific classes of nonlinear systems, we reduce the search for a Lyapunov function to that of solving LMIs (Linear matrix inequalities).

The results in this chapter are also useful in proving stability of switching linear systems.

To establish the results in this chapter, recall that a system

$$\dot{x} = f(x) \tag{11.1}$$

is globally exponentially stable about the origin if there exist positive definite symmetric matrices P and Q which satisfy

$$x^T P f(x) \leq -x^T Q x$$

for all x. When this is the case we call P a Lyapunov matrix for system (11.1).

11.2 Polytopic nonlinear systems

Here we consider nonlinear systems described by

$$\boxed{\dot{x} = A(x)x} \tag{11.2}$$

where the n-vector x is the state. The state dependent matrix $A(x)$ has the following structure

$$\boxed{A(x) = A_0 + \psi(x)\Delta A} \tag{11.3}$$

where A_0 and ΔA are constant $n \times n$ matrices and ψ is a scalar valued function of the state x which is bounded above and below, that is

$$\boxed{a \leq \psi(x) \leq b} \tag{11.4}$$

for some constants a and b. Examples of functions satisfying the above conditions are given by $\psi(x) = g(c(x))$ where $c(x)$ is a scalar and $g(y)$ is given by $\sin y, \cos y, e^{-y^2}$ or $\mathrm{sgm}\,(y)$. The signum function is useful for modeling switching systems.

Example 119 *Inverted pendulum under linear feedback.* Consider an inverted pendulum under linear control described by

$$\dot{x}_1 = x_2 \tag{11.5a}$$
$$\dot{x}_2 = -2x_1 - x_2 + \gamma \sin x_1 \qquad \gamma > 0 \tag{11.5b}$$

This system can be described by (11.2)-(11.4) with

$$A_0 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \qquad \Delta A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$\psi(x) = \begin{cases} \gamma \sin x_1 / x_1 & \text{if} \quad x_1 \neq 0 \\ \gamma & \text{if} \quad x_1 = 0 \end{cases}$$

Since $|\sin x_1| \leq |x_1|$, we have

$$-\gamma \leq \psi(x) \leq \gamma;$$

hence $a = -\gamma$ and $b = \gamma$.

The following theorem provides a sufficient condition for the global exponential stability of system (11.2)-(11.4). These conditions are stated in terms of linear matrix inequalities and the two matrices corresponding to the extreme values of $\psi(x)$, namely

$$A_1 := A_0 + a\Delta A \qquad \text{and} \qquad A_2 := A_0 + b\Delta A$$

Theorem 16 *Suppose there exists a positive-definite symmetric matrix P which satisfies the following linear matrix inequalities:*

$$\boxed{\begin{array}{rcl} PA_1 + A_1^T P & < & 0 \\ PA_2 + A_2^T P & < & 0 \end{array}} \tag{11.6}$$

Then system (11.2)-(11.4) is globally exponentially stable (GES) about the origin with Lyapunov matrix P.

Proof. As a candidate Lyapunov function for GES, consider $V(x) = x^T P x$. Then

$$\begin{aligned} \dot{V} &= 2x^T P \dot{x} \\ &= 2x^T P A_0 x + 2\psi(x) x^T P \Delta A x \,. \end{aligned}$$

For each fixed x, the above expression for $\dot{V}$ is a linear affine function of the scalar $\psi(x)$. Hence an upper bound for $\dot{V}$ occurs when $\psi(x) = a$ or $\psi(x) = b$ which results in

$$\dot{V} \leq 2x^T P A_1 x = x^T (PA_1 + A_1^T P) x$$

or

$$\dot{V} \leq 2x^T P A_2 x = x^T (PA_2 + A_2^T P) x$$

respectively. As a consequence of the matrix inequalities (11.6), there exist positive scalars α_1 and α_2 such that

$$\begin{aligned} PA_1 + A_1^T P &\le -2\alpha_1 P \\ PA_2 + A_2^T P &\le -2\alpha_2 P \end{aligned}$$

Letting $\alpha = \min\{\alpha_1, \alpha_2\}$, we now obtain that

$$\dot{V} \le -2\alpha x^T P x = -2\alpha V\,.$$

This guarantees the system is GES with rate α. ∎

Exercise 31 Prove the following result: Suppose there exists a positive-definite symmetric matrix P and a positive scalar α which satisfy

$$PA_1 + A_1^T P + 2\alpha P \le 0 \tag{11.7a}$$

$$PA_2 + A_2^T P + 2\alpha P \le 0 \tag{11.7b}$$

where $A_1 := A_0 + a\Delta A$ and $A_2 := A_0 + b\Delta A$. Then system (11.2)-(11.4) is globally exponentially stable about the origin with rate of convergence α.

11.2.1 LMI Control Toolbox

Recall the pendulum of Example 119. We will use the Matlab LMI Control Toolbox to see if we can show exponential stability of this system. First note that the existence of a positive definite symmetric P satisfying inequalities (11.6) is equivalent to the existence of another symmetric matrix P satisfying

$$\begin{aligned} PA_1 + A_1^T P &< 0 \\ PA_2 + A_2^T P &< 0 \\ P &> I \end{aligned}$$

For $\gamma = 1$, we determine the feasibility of these LMIs using the following Matlab program.

```
% Quadratic stability of inverted pendulum  under linear PD control.
%
gamma=1;
A0 = [0 1; -2 -1];
DelA = [0 0; 1 0];
A1 = A0 - gamma*DelA;
A2 = A0 + gamma*DelA;
%
%
setlmis([])
%
p=lmivar(1, [2,1]);
%
```

```
lmi1=newlmi;
lmiterm([lmi1,1,1,p],1,A1,'s')
%
lmi2=newlmi;
lmiterm([lmi2,1,1,p],1,A2,'s')
%
Plmi= newlmi;
lmiterm([-Plmi,1,1,p],1,1)
lmiterm([Plmi,1,1,0],1)
%
lmis = getlmis;
%
[tfeas, xfeas] = feasp(lmis)
%
P = dec2mat(lmis,xfeas,p)
```

Running this program yields the following output

```
 Solver for LMI feasibility problems L(x) < R(x)
    This solver minimizes  t  subject to  L(x) < R(x) + t*I
    The best value of t should be negative for feasibility

 Iteration   :    Best value of t so far

     1                        0.202923
     2                       -0.567788

 Result:  best value of t:    -0.567788
          f-radius saturation:  0.000% of R =  1.00e+09


tfeas =   -0.5678

xfeas = 5.3942
        1.0907
        2.8129

P = 5.3942    1.0907
    1.0907    2.8129
```

Hence the above LMIs are feasible and the pendulum is exponentially stable with Lyapunov matrix

$$P = \begin{pmatrix} 5.3942 & 1.0907 \\ 1.0907 & 2.8129 \end{pmatrix}$$

By iterating on γ, the largest value for which LMIs (11.6) were feasible was found to be $\gamma \approx 1.3229$. However, from previous considerations, we have shown that this system is exponentially stable for $\gamma < 2$. Why the difference?

Exercise 32 What is the supremal value of $\gamma > 0$ for which Theorem 16 guarantees that the following system is guaranteed to be stable about the origin?

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + \gamma e^{-x_1^2} x_2 \\ \dot{x}_2 &= -x_1 - 3x_2 - \gamma e^{-x_1^2} x_1\end{aligned}$$

Exercise 33 Consider the pendulum system of Example 119 with $\gamma = 1$. Obtain the largest rate of exponential convergence that can be obtained using the results of Exercise 31 and the LMI toolbox.

11.2.2 Generalization

One can readily generalize the results of this section to systems described by

$$\dot{x} = A(x)x \tag{11.8}$$

where the state dependent matrix $A(x)$ has the following structure

$$A(x) = A_0 + \psi_1(x)\Delta A_1 + \cdots + \psi_l(x)\Delta A_l\,, \tag{11.9}$$

where $A_0, \Delta A_1, \cdots, \Delta A_l$ are constant $n \times n$ matrices and each ψ_i is a scalar valued function of the state x which satisfies

$$a_i \leq \psi_i(x) \leq b_i \tag{11.10}$$

for some known constants a_i and b_i.

To obtain a sufficient condition for exponential stability, introduce the following set of 2^l "extreme matrices":

$$\mathcal{A} = \{A_0 + \psi_1\Delta A_1 + \cdots + \psi_l\Delta A_l \;:\; \psi_i = a_i \text{ or } b_i \text{ for } i = 1, \cdots, l\}$$

For example, with $l = 2$, $\mathcal{A}$ consists of the following four matrices:

$$\begin{array}{c} A_0 + a_1\Delta A_1 + a_2\Delta A_2 \\ A_0 + a_1\Delta A_1 + b_2\Delta A_2 \\ A_0 + b_1\Delta A_1 + a_2\Delta A_2 \\ A_0 + b_1\Delta A_1 + b_2\Delta A_2 \end{array} \tag{11.11}$$

We have now the following result.

Theorem 17 *Suppose there exists a positive-definite symmetric matrix P which satisfies the following linear matrix inequalities:*

$$PA + A^TP < 0 \qquad \textit{for all } A \textit{ in } \mathcal{A} \tag{11.12}$$

Then system (11.8)-(11.10) is globally exponentially stable about the origin with Lyapunov matrix P.

Exercise 34 Consider the double inverted pendulum described by

$$\begin{aligned} \ddot{\theta}_1 + 2\dot{\theta}_1 - \dot{\theta}_2 + 2k\theta_1 - k\theta_2 - \sin\theta_1 &= 0 \\ \ddot{\theta}_2 - \dot{\theta}_1 + \dot{\theta}_2 - k\theta_1 + k\theta_2 - \sin\theta_2 &= 0 \end{aligned}$$

Using the results of Theorem 17, obtain a value of the spring constant k which guarantees that this system is globally exponentially stable about the zero solution.

11.2.3 Robust stability

Consider an uncertain system described by

$$\dot{x} = f(x, \delta) \tag{11.13}$$

where δ is some uncertain parameter vector. suppose that the only information we have on δ is its set of possible values, that is,

$$\delta \in \Delta$$

where the set Δ is known. So we do not know what δ is, we only know a set of possible values of δ. We say that the above system is robustly stable if it is stable for all allowable values of δ, that is, for all $\delta \in \Delta$.

We say that the uncertain system is quadratically stable if there are positive definite symmetric matrices P and Q such

$$x^T P f(x, \delta) \leq -x^T Q x \tag{11.14}$$

for x and $\delta \in \Delta$. Clearly, quadratic stability guarantees robust exponentially stability about zero.

One of the useful features of the above concept, is that one can sometimes guarantee that (11.14) holds for all δ by just checking that is holds for a finite number of δ. For, example consider the linear uncertain system described by

$$\dot{x} = A(\delta)x\,.$$

Suppose that the uncertain matrix $A(\delta)$ has the following structure

$$A(\delta) = A_0 + \delta_1 \Delta A_1 + \cdots + \delta \Delta A_l\,, \tag{11.15}$$

where $A_0, \Delta A_1, \cdots, \Delta A_l$ are constant $n \times n$ matrices and each δ_i is a scalar which satisfies

$$a_i \leq \delta_i \leq b_i \tag{11.16}$$

Introduce the following set of 2^l "extreme matrices":

$$\mathcal{A} = \{A_0 + \psi_1 \Delta A_1 + \cdots + \psi_l \Delta A_l \;:\; \psi_i = a_i \text{ or } b_i \text{ for } i = 1, \cdots, l\}$$

Suppose there exists a positive-definite symmetric matrix P which satisfies the following linear matrix inequalities:

$$PA + A^T P < 0 \qquad \text{for all } A \text{ in } \mathcal{A} \tag{11.17}$$

Then, for all allowable δ, the uncertain system is globally exponentially stable about the origin with Lyapunov matrix P.

11.3 Another special class of nonlinear systems

Here we consider nonlinear systems which have the following structure:

$$\boxed{\dot{x} = Ax + B\phi(Cx)} \tag{11.18}$$

where the n-vector $x(t)$ is the state; B and C are constant matrices with dimensions $n \times m$ and $p \times n$, respectively; and ϕ is a function which satisfies

$$\boxed{||\phi(z)|| \leq \gamma ||z||} \tag{11.19}$$

for some $\gamma \geq 0$.

Note that if we introduce a fictitious output $z = Cx$ and a fictitious input $w = \phi(Cx)$, the nonlinear system (11.18) can be described as a feedback combination of a linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx \end{aligned}$$

and a memoryless nonlinearity

$$w = \phi(z)$$

This is illustrated in Figure 11.1.

Figure 11.1: A feedback system

Example 120 *Inverted pendulum under linear feedback.* Recall the inverted pendulum of the previous example described by (11.5). This system can be described by the above general description with

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \phi(z) = \gamma \sin z$$

Also,

$$||\phi(z)|| = |\gamma \sin z| \leq \gamma |z| = \gamma ||z||$$

The following theorem provides a sufficient condition for the global exponential stability of system (11.18).

Theorem 18 *Suppose there exists a positive-definite symmetric matrix P which satisfies the following* quadratic matrix inequality (QMI)*:*

$$PA + A^T P + \gamma^2 PBB^T P + C^T C < 0 \tag{11.20}$$

Then system (11.18)-(11.19) is globally exponentially stable about the origin with Lyapunov matrix P.

PROOF. Along any solution of system (11.18)-(11.19) we have

$$2x^TP\dot{x} = 2x^TPAx + 2x^TPB\phi = x^T(PA + A^TP)x + 2x^TPB\phi\,. \tag{11.21}$$

Note that for any two real numbers a and b,

$$2ab = a^2 + b^2 - (a-b)^2\,;$$

hence

$$2ab \le a^2 + b^2\,.$$

Considering the last term in (11.21), we now obtain

$$\begin{aligned} 2x^TPB\phi &\le 2||B^TPx||||\phi|| \\ &\le 2||B^TPx||\gamma||Cx|| \\ &\le \gamma^2||B^TPx||^2 + ||Cx||^2 \\ &= \gamma^2x^TPBB^TPx + x^TC^TCx\,. \end{aligned} \tag{11.22}$$

Combining (11.21) and (11.22), it now follows that

$$2x^TP\dot{x} \le -2x^TQx$$

where $-2Q = PA + A^TP + \gamma^2PBB^TP + C^TC$. By assumption, the matrix Q is positive definite. Hence system (11.18)-(11.19) is globally exponentially stable about the origin with Lyapunov matrix P. ■

A linear matrix inequality (LMI). Using a Schur complement result (see Appendix at the end of this chapter), one can show that satisfaction of quadratic matrix inequality (11.20) is equivalent to satisfaction of the following matrix inequality:

$$\boxed{\begin{pmatrix} PA + A^TP + C^TC & PB \\ B^TP & -\gamma^{-2}I \end{pmatrix} < 0} \tag{11.23}$$

Note that this inequality is linear in P and γ^{-2}. Suppose one wishes to compute the supremal value $\bar{\gamma}$ of γ which guarantees satisfaction of the hypotheses of Theorem 18 and hence stability of system (11.32). This can be achieved by solving the following optimization problem:

$$\text{minimize } \beta \text{ subject to}$$

$$\begin{aligned} \begin{pmatrix} PA + A^TP + C^TC & PB \\ B^TP & -\beta I \end{pmatrix} &< 0 \\ 0 &< P \\ 0 &< \beta \end{aligned}$$

and then letting $\bar{\gamma} = \bar{\beta}^{-\frac{1}{2}}$ where $\bar{\beta}$ is the infimal value of β. The following program uses the LMI toolbox to compute $\bar{\gamma}$ for the inverted pendulum example.

```
%Quadratic stability radius of inverted pendulum
%
A = [0 1; -2 -1];
B=[0;1];
C=[1 0];
%
setlmis([])
%
pvar=lmivar(1,[2,1]);                %P
betavar=lmivar(1,[1,1]);             %beta
%
lmi1=newlmi;
lmiterm([lmi1 1 1 pvar],1,A,'s');   %PA+A'P
lmiterm([lmi1 1 1 0],C'*C);         %C'C
lmiterm([lmi1 1 2 pvar],1,B);       %PB
lmiterm([lmi1 2 2 betavar],-1,1);   %-beta I
%
lmi2=newlmi;
lmiterm([-lmi2 1 1 pvar],1,1)       %P
%
lmi3=newlmi;
lmiterm([-lmi3 1 1 betavar],1,1);   %beta
 %
lmis=getlmis;
%
c=mat2dec(lmis,0,1);                %specify weighting
options=[1e-5 0 0 0 0];
[copt,xopt]=mincx(lmis,c,options)   %optimize
%
gamma=1/sqrt(copt)
```

Running this program yields the following output:

```
 Solver for linear objective minimization under LMI constraints

 Iterations   :    Best objective value so far

     1                   0.862772
     2                   0.807723
     3                   0.675663
***                 new lower bound:    -0.031634
     4                   0.645705
***                 new lower bound:     0.148578
     5                   0.626163
***                 new lower bound:     0.284167
```

```
     6                    0.598055
***                 new lower bound:     0.394840
     7                    0.580987
***                 new lower bound:     0.535135
     8                    0.572209
***                 new lower bound:     0.551666
     9                    0.571637
***                 new lower bound:     0.567553
    10                    0.571450
***                 new lower bound:     0.570618
    11                    0.571438
***                 new lower bound:     0.571248
    12                    0.571429
***                 new lower bound:     0.571391
    13                    0.571429
***                 new lower bound:     0.571420
    14                    0.571429
***                 new lower bound:     0.571426

 Result:  feasible solution of required accuracy
          best objective value:     0.571429
          guaranteed absolute accuracy:  2.86e-06
          f-radius saturation:  0.000% of R =  1.00e+09

copt = 0.5714

xopt = 1.1429
       0.2857
       0.5714
       0.5714

gamma = 1.3229
```

Thus $\bar{\gamma} = 1.3229$ which is the same as that achieved previously.

A Riccati equation. It should be clear that P satisfies the above QMI if and only if it satisfies the following Riccati equation for some positive definite symmetric matrix Q:

$$PA + A^T P + \gamma^2 PBB^T P + C^T C + Q = 0 \tag{11.24}$$

Using properties of QMI (11.20) (see Ran and Vreugdenhil, 1988), one can demonstrate the following.

Lemma 10 *There exists a positive-definite symmetric matrix P which satisfies the quadratic matrix inequality (11.20) if and only if for* any *positive-definite symmetric matrix Q there* *is an* $\bar{\epsilon} > 0$ *such that for all* $\epsilon \in (0, \bar{\epsilon}]$ *the following* Riccati equation *has a positive-definite symmetric solution for* P

$$PA + A^T P + \gamma^2 PBB^T P + C^T C + \epsilon Q = 0 \tag{11.25}$$

Using this corollary, the search for a Lyapunov matrix is reduced to a one parameter search.

Exercise 35 Prove the following result: Suppose there exist a positive-definite symmetric matrix P and a positive scalar α which satisfy

$$\begin{pmatrix} PA + A^T P + C^T C + 2\alpha P & PB \\ B^T P & -\gamma^{-2} I \end{pmatrix} \leq 0 \, . \tag{11.26}$$

Then system (11.18)-(11.19) is globally exponentially stable about the origin with rate of convergence α.

11.3.1 The circle criterion: a frequency domain condition

In many situations, one is only interested in whether a given nonlinear system is stable or not; one may not actually care what the Lyapunov matrix is. To this end, the following frequency domain characterization of quadratic stability is useful. Consider system (11.18) and define the transfer function $\hat{G}$ by

$$\hat{G}(s) = C(sI - A)^{-1} B \tag{11.27}$$

and let

$$\|\hat{G}\|_\infty := \sup_{\omega \in \mathbb{R}} \|\hat{G}(j\omega)\| \tag{11.28}$$

Then we have the following result from the control literature.

Lemma 11 (KYP1) *[6, 4] There exists a positive-definite symmetric matrix* P *which satisfies the linear matrix inequality (11.23) if and only if*

(i) A *is asymptotically stable and*

(ii)

$$\boxed{\gamma \|\hat{G}\|_\infty < 1} \tag{11.29}$$

The last lemma and Theorem 18 tell us that $\bar{\gamma}$, the supremal value of γ for stability of system (11.18)-(11.19) via the method of this section, is given by

$$\bar{\gamma} = 1/||\hat{G}||_\infty$$

Example 121 Consider Example 120 again. Here the matrix A is asymptotically stable and

$$\hat{G}(s) = 1/(s^2 + s + 2)$$

One may readily compute that

$$\|\hat{G}\|_\infty = 2/\sqrt{7} < 1$$

Hence, this nonlinear system is exponentially stable. Also, $\bar{\gamma} = 1/||\hat{G}||_\infty = \sqrt{7}/2 \approx 1.3229$. This is the same as before.

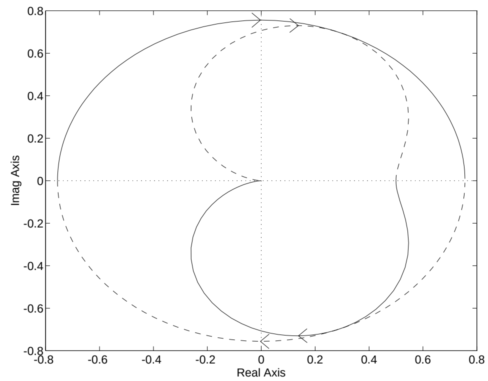

Figure 11.2: A nice figure

11.3.2 Sector bounded nonlinearities

Suppose ϕ is a scalar valued function of a scalar variable ($m = p = 1$). Then the bound (11.19) is equivalent to

$$-\gamma z^2 \leq z\phi(z) \leq \gamma z^2$$

That is, ϕ is a function whose graph lies in the sector bordered by the lines passing through the origin and having slopes γ and $-\gamma$; see Figure 11.3. Consider now a function ϕ whose

Figure 11.3: A norm bounded nonlinearity

graph lies in the sector bordered by the lines passing through the origin and having slopes a and b, with $a < b$; see Figure 11.4. Analytically, this can be expressed as

Figure 11.4: A general sector bounded nonlinearity

$$az^2 \leq z\phi(z) \leq bz^2 \tag{11.30}$$

Example 122 The saturation function which is defined below is illustrated in Figure 11.5.

$$\phi(z) = \begin{cases} z & \text{if} \quad |z| \leq 1 \\ -1 & \text{if} \quad z < -1 \\ 1 & \text{if} \quad z > 1 \end{cases}$$

Since $0 \leq z\phi(z) \leq z^2$, the saturation function is sector bounded with $a = 0$ and $b = 1$.

Figure 11.5: Saturation

The following result is sometimes useful in showing that a particular nonlinear function is sector bounded.

Fact 4 *Suppose ϕ is a differentiable scalar valued function function of a scalar variable, $\phi(0) = 0$ and for all z*

$$a \leq \phi'(z) \leq b$$

for some scalars a and b. Then ϕ satisfies (11.30) for all z.

We can treat sector bounded nonlinearities using the results of the previous section by introducing some transformations. Specifically, if we introduce

$$\tilde{\phi}(z) := \phi(z) - kz \qquad \text{where} \qquad k := (a+b)/2\,,$$

then,

$$-\gamma z^2 \leq z\tilde{\phi}(z) \leq \gamma z^2 \qquad \text{with} \qquad \gamma := (b-a)/2$$

and system (11.32) is described by

$$\dot{x} = \tilde{A}x + B\tilde{\phi}(Cx) \tag{11.31}$$

with $\tilde{A} = A + kBC$. Since $|\tilde{\phi}(z)| \leq \gamma|z|$, we can apply the results of the previous section to system (11.31) to obtain stability results for the original nonlinear system, which contains the nonlinearity satisfying (11.30).

11.3.3 Generalization

Here we consider systems whose nonlinearity is characterized by several nonlinear functions, $\phi_i : \mathbb{R}^{p_i} \to \mathbb{R}^{m_i},\ i = 1, 2, \ldots, l$:

$$\dot{x} = Ax + B_1\phi(C_1x) + \ldots + B_l\phi_l(C_lx) \tag{11.32}$$

where A_0 and B_i, C_i, $i = 1, 2, \cdots, l$ are constant matrices of appropriate dimensions. Each ϕ_i satisfies

$$||\phi_i(z_i)|| \leq \gamma ||z_i|| \tag{11.33}$$

for some $\gamma \geq 0$.

If we introduce fictitious outputs $z_i = C_ix$ and a fictitious inputs $w_i = \phi_i(C_ix)$, the above nonlinear system can be described as a feedback combination of a linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + B_1w_1 + \ldots + B_lw_l \\ z_1 &= C_1x \\ &\vdots \\ z_l &= C_lx \end{aligned}$$

and multiple memoryless nonlinearities:

$$w_i = \phi_i(z_i) \qquad i = 1, 2, \cdots, l$$

This is illustrated in Figure 11.6.

Figure 11.6: A multi-loop feedback system

We will use the results of the previous section to obtain results for the systems of this section. Letting $m = m_1 + \cdots + m_l$ and $p = p_1 + \cdots + p_l$, suppose we introduce a function $\phi : R^p \to R^m$ defined by

$$\phi(z) = \begin{pmatrix} \phi_1(z_1) \\ \phi_2(z_2) \\ \vdots \\ \phi_l(z_l) \end{pmatrix} \qquad \text{where} \qquad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_l \end{pmatrix} \text{ with } z_i \in \mathbb{R}^{p_i} \tag{11.34}$$

Then system (11.32) can also be described by (11.18) with

$$B := \begin{pmatrix} B_1 & B_2 & \dots & B_l \end{pmatrix}, \qquad C := \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_l \end{pmatrix}$$

Also,

$$||\phi(z)||^2 = \sum_{i=1}^{l} ||\phi_i(z_i)||^2 \leq \sum_{i=1}^{l} \gamma^2 ||z_i||^2 = \gamma^2 ||z||^2$$

that is $||\phi(z)|| \leq \gamma||z||$. We can now apply the results of the previous section to obtain sufficient conditions for global exponential stability. However, these results will be overly conservative since they do not take into account the special structure of ϕ; the function ϕ is not just any function satisfying the bound $\|\phi(z)\| \leq \gamma \|z\|$; it also has the structure indicated in (11.34). To take this structure into account, consider any l positive scalars $\lambda_1, \lambda_2, \dots, \lambda_l$ and introduce the nonlinear function

$$\tilde{\phi}(z) = \begin{pmatrix} \lambda_1 \phi_1(\lambda_1^{-1} z_1) \\ \lambda_2 \phi_2(\lambda_2^{-1} z_2) \\ \vdots \\ \lambda_l \phi_l(\lambda_l^{-1} z_l) \end{pmatrix} \qquad \text{where} \qquad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_l \end{pmatrix} \text{ with } z_i \in \mathbb{R}^{p_i}$$

Then system (11.32) can also be described by (11.18), specifically, it can be described by

$$\dot{x} = Ax + \tilde{B}\tilde{\phi}(\tilde{C}x) \tag{11.35}$$

with

$$\tilde{B} := \begin{pmatrix} \lambda_1^{-1} B_1 & \lambda_2^{-1} B_2 & \dots & \lambda_l^{-1} B_l \end{pmatrix}, \qquad \tilde{C} := \begin{pmatrix} \lambda_1 C_1 \\ \lambda_2 C_2 \\ \vdots \\ \lambda_l C_l \end{pmatrix}$$

Noting that $||\lambda_i \phi_i(\lambda_i^{-1} z_i)|| = \lambda_i ||\phi(\lambda_i^{-1} z_i)|| \leq \lambda_i \gamma ||\lambda_i^{-1} z_i|| = \gamma ||z_i||$, we obtain

$$||\tilde{\phi}(z)||^2 = \sum_{i=1}^{l} ||\lambda_i \phi(\lambda_i^{-1} z_i)||^2 \leq \gamma^2 \sum_{i=1}^{l} ||z_i||^2 = \gamma^2 ||z||^2$$

that is, $||\tilde{\phi}(z)|| \leq \gamma ||z||$.

To obtain sufficient conditions for stability of (11.32), we simply apply the results of the previous section to (11.35). Using Theorem 18 one can readily obtain the following result:

Theorem 19 *Suppose there exists a positive-definite symmetric matrix P and l positive scalars $\mu_1, \mu_2, \dots, \mu_l$ which satisfy the following matrix inequality:*

$$PA + A^T P + \sum_{i=1}^{l} \left(\mu_i^{-1} \gamma^2 P B_i {B_i}^T P + \mu_i C_i^T C_i \right) < 0 \tag{11.36}$$

Then system (11.32) is globally exponentially stable about the origin with Lyapunov matrix P.

PROOF. If we let $\lambda_i = \sqrt{\mu_i}$ for $i = 1, 2, \cdots, l$, then (11.36) can be written as

$$PA + A^T P + \gamma^2 P\tilde{B}\tilde{B}^T P + \tilde{C}^T\tilde{C} < 0$$

Since $||\tilde{\phi}(z)|| \leq \gamma||z||$, it follows from representation (11.35) and Theorem 18 that system (11.32) is globally exponentially stable about the origin with Lyapunov matrix P.

An LMI. Note that, using a Schur complement result, inequality (11.36) is equivalent to the following inequality which is linear in P and the scaling parameters $\mu_1, \cdots, \mu_l$:

$$\begin{pmatrix} PA + A^T P + \sum_{i=1}^{l} \mu_i C_i' C_i & \gamma PB_1 & \gamma PB_2 & \cdots & \gamma PB_l \\ \gamma B_1^T P & -\mu_1 I & 0 & \cdots & 0 \\ \gamma B_2^T P & 0 & -\mu_2 I & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \gamma B_l^T P & 0 & 0 & \cdots & -\mu_l I \end{pmatrix} < 0$$

It should be clear that one may also obtain a sufficient condition involving a Riccati equation with scaling parameters μ_i using Corollary 10 and a H_∞ sufficient condition using Lemma 11.

Exercise 36 Recall the double inverted pendulum of Example 34. Using the results of this section, obtain a value of the spring constant k which guarantees that this system is globally exponentially stable about the zero solution.

11.4 Yet another special class

Here we consider systems described by

$$\dot{x} = Ax - B\phi(Cx) \tag{11.37}$$

where

$$z'\phi(z) \geq 0 \tag{11.38}$$

for all z. Examples of ϕ include $\phi(z) = z, z^3, z^5, \text{sat}(z), \text{sgm}\,(z)$.

Figure 11.7: A first and third quadrant nonlinearity

Note that such a system can also be regarded as a feedback combination of the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx \end{aligned}$$

and a memoryless nonlinear system

$$w = -\phi(z)\,.$$

Theorem 20 *Suppose there exists a positive-definite symmetric matrix P which satisfies*

$$\boxed{\begin{aligned} PA + A'P &< 0 \\ B'P &= C \end{aligned}} \tag{11.39}$$

Then system (11.37)-(11.38) is globally exponentially stable about the origin with Lyapunov matrix P.

PROOF.

$$\begin{aligned} 2x'P\dot{x} &= x'(PA + A'P)x - 2x'PB\phi(Cx) \\ &= x'(PA + A'P)x - 2(Cx)'\phi(Cx) \\ &\leq x'(PA + A'P)x\,. \end{aligned}$$

Hence

$$2x'P\dot{x} \leq -2x^TQx$$

where $2Q = -PA - A^T P$. Since Q is positive definite, we have global exponential stability about zero. ■

Now note that $P > 0$ satisfies (11.39) if and only if the following optimization problem has a minimum of zero:

Minimize β subject to

$$\begin{aligned} PA + A'P &< 0 \\ P &> 0 \end{aligned}$$

$$\begin{pmatrix} \beta I & B'P - C \\ PB - C' & \beta I \end{pmatrix} \geq 0$$

Example 123 Consider

$$\begin{array}{rcrrr} \dot{x}_1 & = & -3x_1 & +x_2 & \\ \dot{x}_2 & = & x_1 & -3x_2 & -x_2^3 \end{array}$$

Here

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

and $\phi(z) = z^3$. Clearly $z\phi(z) \geq 0$ for all z. Also, conditions (11.39) are assured with $P = I$. Hence, the system of this example is globally exponentially stable. Performing the above optimization with the LMI toolbox yields a minimum of β to equal to "zero" and

$$P = \begin{pmatrix} 22.8241 & 0 \\ 0 & 1 \end{pmatrix}$$

This P also satisfies conditions (11.39).

Exercise 37 Prove the following result: Suppose there exists a positive-definite symmetric matrix P and a positive scalar α which satisfy which satisfies

$$PA + A'P + 2\alpha P \leq 0 \tag{11.40a}$$

$$B'P = C \tag{11.40b}$$

Then system (11.37)-(11.38) is globally exponentially stable about the origin with rate α and with Lyapunov matrix P.

11.4.1 Generalization

Here we consider systems described by

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \\ w &= -\phi(z) \end{aligned} \tag{11.41}$$

where

$$z'\phi(z) \geq 0 \tag{11.42}$$

for all z. In order for the above system to be well-defined, one must have a unique solution to the equation

$$z + D\phi(z) = Cx$$

for each x.

Theorem 21 *Consider a system described by (11.41) and satisfying (11.42). Suppose there exist a positive-definite symmetric matrix P and a positive scalar α which satisfy*

$$\boxed{\begin{pmatrix} PA + A'P + 2\alpha P & PB - C' \\ B'P - C & -(D + D') \end{pmatrix} \leq 0} \tag{11.43}$$

Then system (11.41) is globally exponentially stable about the origin with rate α and Lyapunov matrix P.

Proof.

$$2x'P\dot{x} \quad = \quad x'(PA + A'P)x + 2x'PBw \quad = \quad \begin{pmatrix} x \\ w \end{pmatrix}' \begin{pmatrix} PA + A'P & PB \\ B'P & 0 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

We also have

$$2z'\phi(z) = 2(Cx + Dw)'w \quad = \quad \begin{pmatrix} x \\ w \end{pmatrix}' \begin{pmatrix} 0 & C' \\ C & D + D' \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

It now follows from (11.43) that

$$x'(PA + A'P + 2\alpha P)x + 2x'PBw - 2(Cx + Dw)'w \leq 0\,,$$

that is,

$$2x^T P\dot{x} + 2\alpha x^T Px + z'\phi(z) \leq 0\,.$$

Hence

$$\begin{aligned} 2x'P\dot{x} &\leq -2\alpha x'Px - z'\phi(z) \\ &\leq -2\alpha x'Px\,. \end{aligned}$$

■

• Note that (11.43) is linear in P.
  If $D + D' > 0$, then (11.43) is equivalent to

$$PA + A'P + 2\alpha P + (PB - C')(D + D')^{-1}(B'P - C) \leq 0$$

11.4.2 Strictly positive real transfer functions and a frequency domain condition

We consider here square transfer functions $\hat{G}$ given by

$$\hat{G}(s) = C(sI - A)^{-1}B + D \tag{11.44}$$

where A, B, C, D are real matrices of dimensions $n \times n, n \times m, m \times n$ and $m \times m$. We will say the $\hat{G}$ is stable if all its poles have negative real parts. Clearly, if A is Hurwitz (all its eigenvalues have negative real parts) then $\hat{G}$ is stable.

Definition 1 (SPR) *A square transfer function $\hat{G}$ is* strictly positive real (SPR) *if there exists a real scalar $\epsilon > 0$ such that $\hat{G}$ has no poles in a region containing the set $\{s \in \mathbb{C} : \Re(s) \geq -\epsilon\}$ and*

$$\hat{G}(\jmath\omega - \epsilon) + \hat{G}(\jmath\omega - \epsilon)' \geq 0 \qquad \textit{for all} \qquad \omega \in \mathbb{R}\,. \tag{11.45}$$

We say that $\hat{G}$ is regular if $\det[\hat{G}(\jmath\omega) + \hat{G}(\jmath\omega)']$ is not identically zero for all $\omega \in \mathbb{R}$.

Requirement (11.45) is not very satisfactory; it involves an apriori unknown parameter ϵ. How do we get rid of ϵ?

Lemma 12 *[5] A transfer function $\hat{G}$ is SPR and regular if and only if the following conditions hold.*

(a) [Stability] $\hat{G}$ is stable.

(b) [Dissipativity]

$$\hat{G}(\jmath\omega) + \hat{G}(\jmath\omega)' > 0 \qquad \textit{for all} \qquad \omega \in \mathbb{R} \tag{11.46}$$

(c) [Asymptotic side condition]

$$\lim_{|\omega|\to\infty} \omega^{2\rho} \det[\hat{G}(\jmath\omega) + \hat{G}(\jmath\omega)'] \neq 0 \tag{11.47}$$

where ρ is the nullity of $D + D'$.

In either case, the above limit is positive.

Note that $D = \hat{G}(\infty) := \lim_{|\omega|\to\infty} \hat{G}(\jmath\omega)$ and the nullity of $D + D'$ is the dimension of the null space of $D + D'$.

Remark 5 Suppose $\hat{G}(s) = C(sI - A)^{-1}B + D$ is a scalar valued transfer function. Then $\rho = 0$ if $D \neq 0$ and $\rho = 1$ is $D = 0$. Hence, the above side condition reduces to

$$D \neq 0 \qquad \text{or} \qquad \lim_{|\omega|\to\infty} \omega^2[\hat{G}(\jmath\omega) + \hat{G}(\jmath\omega)'] \neq 0$$

When $D = 0$,

$$\lim_{|\omega|\to\infty} \omega^2[\hat{G}(\jmath\omega) + \hat{G}(\jmath\omega)'] = -CAB$$

Example 124 Consider the scalar transfer function

$$\hat{g}(s) = \frac{s+1}{s^2+bs+1}\,.$$

We claim that this transfer function is SPR if and only if $b>1$. This is illustrated in Figures 11.8 and 11.9 for $b=2$ and $b=0.5$ respectively.

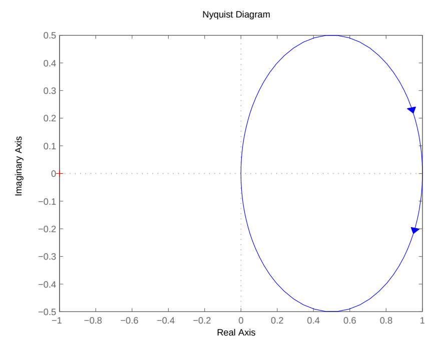

Figure 11.8: Nyquist plot of $\frac{s+1}{s^2+2s+1}$

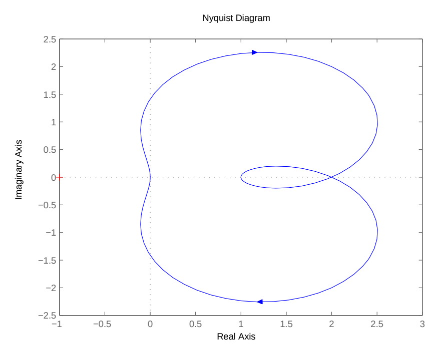

Figure 11.9: Nyquist plot of $\frac{s+1}{s^2+0.5s+1}$

To prove the above claim, we first note that $\hat{g}$ is stable if and only if $b>0$. Now note that

$$\hat{g}(\jmath\omega) = \frac{1+\jmath\omega}{1-\omega^2+\jmath b\omega}$$

Hence

$$\hat{g}(\jmath\omega)+\hat{g}(\jmath\omega)' = \frac{2[1+(b-1)\omega^2]}{(1-\omega^2)^2+(b\omega)^2}$$

It should be clear from the above expression that $\hat{g}(\jmath\omega)+\hat{g}(\jmath\omega)'>0$ for all finite ω if and only if $b\geq 1$. Also,

$$\lim_{\omega\to\infty}\omega^2[\hat{g}(\jmath\omega)+\hat{g}(\jmath\omega)'] = b-1\,.$$

Thus, in order for the above limit to be positive, it is necessary and sufficient that $b>1$.

The next lemma tells us why SPR transfer functions are important in stability analysis of the systems under consideration in the last section.

Lemma 13 (KYPSPR) *Suppose (A, B) is controllable and (C, A) is observable. Then there exists a matrix $P = P' > 0$ and scalar $\alpha > 0$ such that LMI (11.43) holds if and only if the transfer function $\hat{G}(s) = D + C(sI - A)^{-1}B$ is strictly positive real.*

Corollary 1 *Consider a system described by (11.41) and satisfying (11.42) and suppose that (A, B) is controllable, (C, A) is observable and the transfer function $C(sI - A)^{-1}B + D$ is SPR. Then this system is globally exponentially stable about the origin.*

Remark 6 Sometimes the KYPSPR Lemma is stated as follows: *Suppose (A, B) is controllable and (C, A) is observable. Then $\hat{G}(s) = D + C(sI - A)^{-1}B$ is strictly positive real if and only if there exists a matrix $P = P' > 0$, a positive scalar ϵ and matrices W and L such that*

$$\begin{aligned} PA + A'P &= -L'L - \epsilon P && (11.48)\\ PB &= C' - L'W && (11.49)\\ W'W &= D + D' && (11.50) \end{aligned}$$

To see that this result is equivalent to Lemma 13 above, let $\alpha = \epsilon/2$ and rewrite the above equations as

$$\begin{aligned} PA + A'P + 2\alpha P &= -L'L \\ PB - C' &= -L'W \\ -(D + D') &= -W'W\,, \end{aligned}$$

that is,

$$M := \begin{pmatrix} PA + A'P + 2\alpha P & PB - C' \\ B'P - C & -(D + D') \end{pmatrix} = -\begin{pmatrix} L'L & L'W \\ W'L & W'W \end{pmatrix} = -\begin{pmatrix} L \\ W \end{pmatrix}' \begin{pmatrix} L \\ W \end{pmatrix}$$

The existence of matrices L and W satisfying

$$M = -\begin{pmatrix} L \\ W \end{pmatrix}' \begin{pmatrix} L \\ W \end{pmatrix}$$

is equivalent to $M \leq 0$.

Exercise 38 Consider the transfer function

$$\hat{g}(s) = \frac{\beta s + 1}{s^2 + s + 2}$$

Using Lemma 12, determine the range of β for which this transfer function is SPR. Verify your results with the KYPSPR lemma.

Appendix: A Schur complement result

The following result is useful in manipulating matrix inequalities. Suppose

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

is a hermitian 2×2 block matrix. Then

$$Q > 0$$

if and only if

$$\boxed{Q_{22} > 0 \qquad \text{and} \qquad Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} > 0}$$

To see this, note that $Q_{22}' = Q_{22}$, $Q_{21}' = Q_{12}$ and

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ Q_{22}^{-1}Q_{21} & I \end{pmatrix}' \begin{pmatrix} Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} & 0 \\ 0 & Q_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ Q_{22}^{-1}Q_{21} & I \end{pmatrix} \tag{11.51}$$

Note that Q_{11} and Q_{22} do not have to have the same dimensions.

Clearly, we can readily obtain the following result:

$$Q < 0$$

if and only if

$$Q_{22} < 0 \qquad \text{and} \qquad Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} < 0$$

The following result also follows from (11.51). Suppose $Q_{22} > 0$. Then

$$Q \geq 0$$

if and only if

$$Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} \geq 0$$

Bibliography

[1] Boyd, S. and El Ghaoui, L. and Feron, E. and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

[2] Gahinet, P., Nemirovski, A., Laub, A.J., and Chilali, M., *LMI Contol Toolbox Users Guide*, The MathWorks Inc., Natick, Massachussets, 1995.

[3] Corless, M., "Robust Stability Analysis and Controller Design With Quadratic Lyapunov Functions," in *Variable Structure and Lyapunov Control,* A. Zinober, ed., Springer-Verlag, 1993.

[4] Açıkmeşe, A.B., and Corless, M., "Stability Analysis with Quadratic Lyapunov Functions: Some Necessary and Sufficient Multiplier Conditions,“ *Systems and Control Letters*, Vol. 57, No. 1, pp. 78-94, January 2008.

[5] Corless, M. and Shorten, R., "A correct characterization of strict positive realness", submitted for publication.

[6] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma," *Systems & Control Letters*, vol. 28, pp. 7-10, 1996.

[7] Shorten, R., Corless, M., Wulff, K., Klinge, S. and R. Middleton, R., "Quadratic Stability and Singular SISO Switching Systems," submitted for publication.

[8] Ran, A.C.M., R. Vreugdenhil, R. 1988, Existence and comparison theorems for algebraic Riccati equations for continuous- and discrete-time systems, *Linear Alg. Appl.* Vol. 99, pp. 63–83, 1988

Chapter 12

Invariance results

Recall the simple damped oscillator described by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(k/m)x_1 - (c/m)x_2\end{aligned}$$

with m, c, k positive. This system is globally asymptotically stable about the origin. If we consider the total mechanical energy, $V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$, of this system as a candidate Lyapunov function, then along any solution we obtain

$$\dot{V}(x) = -cx_2^2 \leq 0\,.$$

Using our Lyapunov results so far, this will only guarantee stability; it will not guarantee asymptotic stability, because, regardless of the value of x_1, we have $\dot{V}(x) = 0$ whenever $x_2 = 0$. However, one can readily show that there are no non-zero solutions for which $x_2(t)$ is identically zero. Is this sufficient to guarantee asymptotic stability? We shall shortly see that it is sufficient; hence we can use the energy of this system to prove asymptotic stability.

Consider a system described by

$$\dot{x} = f(x) \tag{12.1}$$

By a solution of system (12.1) we mean a continuous function $x(\cdot) : [0, \infty) \to \mathbb{R}^n$ which identically satisfies $\dot{x}(t) = f(x(t))$.

12.1 Invariant sets

A subset $\mathcal{M}$ of the state space $\mathbb{R}^n$ is an invariant set for system (12.1) if it has the property that every solution which originates in $\mathcal{M}$ remains in $\mathcal{M}$ for all future time, that is, for every solution $x(\cdot)$,

$$x(0) \in \mathcal{M} \qquad \text{implies} \qquad x(t) \in \mathcal{M} \quad \text{for all} \quad t \geq 0$$

The simplest example of an invariant set is a set consisting of a single equilibrium point. As another example of an invariant set, consider any linear system $\dot{x} = Ax$ and let v be any real eigenvector of the matrix A. Then the line consisting of all vectors of the form cv where c is any real number is an invariant set for the linear system.

Figure 12.1: Invariant sets

Lemma 14 *Suppose*

$$DV(x)f(x) \leq 0 \qquad \textit{when} \qquad V(x) < c$$

Then

$$\mathcal{M} = \{x : V(x) < c\}$$

is an invariant set.

Figure 12.2: Invariant set lemma

Example 125

$$\dot{x}_1 = -x_1 + x_1^3$$

Considering $V(x) = x^2$ and $c = 1$, one can readily show that the interval $(-1, 1)$ is an invariant set.

Largest invariant set. It should be clear that the union and intersection of two invariant sets is also invariant. Hence given any subset $\mathcal{S}$ of $\mathbb{R}^n$, we can talk about the largest invariant set $\mathcal{M}$ contained in $\mathcal{S}$, that is $\mathcal{M}$ is an invariant set and contains every other invariant set which is contained in $\mathcal{S}$. As an example, consider the Duffing system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 \end{aligned}$$

and let $\mathcal{S}$ correspond the the set of all points of the form $(x_1, 0)$ where x_1 is arbitrary. If a solution starts at one of the equilibrium states

$$(-1, 0) \qquad (0, 0) \qquad (1, 0)$$

it remains there. Hence, the set

$$\{(-1,0),(0,0),(1,0)\}$$

is an invariant set for the system under consideration. We claim that this is the largest invariant set in $\mathcal{S}$. To see this, let $\mathcal{M}$ be the largest invariant set in $\mathcal{S}$ and consider any solution $(x_1(\cdot), x_2(\cdot))$ which lies in $\mathcal{M}$. Then this solution lies in $\mathcal{S}$ from which it follows that $x_2(t)$ is zero for all t. This implies that $\dot{x}_2(t)$ is zero for all t. From the system description, we must have $x_1(t) - x_1^3(t) = 0$. Hence $x_1(t)$ must be -1, 0 , or 1.

Figure 12.3: Largest invariant set

12.2 Limit sets

A state x^* is called a positive limit point of a solution $x(\cdot)$ if there exists a sequence $\{t_k\}_{k=0}^{\infty}$ of times such that:

$$\begin{aligned} \lim_{k\to\infty} t_k &= \infty \\ \lim_{k\to\infty} x(t_k) &= x^* \end{aligned}$$

The set of all positive limit points of a solution is called the positive limit set of the solution.

The solution $x(t) = e^{-t}$ has only zero as a positive limit point, whereas $x(t) = \cos t$ has the interval $[-1, 1]$ as its positive limit set. What about $x(t) = e^{-t}\cos t$? The positive limit set of $x(t) = e^t$ is empty. Note that the positive limit set of a periodic solution consists of all the points of the solution.

Suppose a solution converges to a single state x^*; then the positive limit set of this solution is the set consisting of that single state, that is, $\{x^*\}$. For another example, consider the Van der Pol oscillator. Except for the zero solution, the positive limit set of every solution is the set consisting of all the states in the limit cycle.

Exercise 39 What are the positive limit sets of the following solutions?

(a) $x(t) = \sin(t^2)$

(b) $x(t) = e^t \sin(t)$

• Distance of a point x from a set $\mathcal{M}$:

$$d(x, \mathcal{M}) := \inf\{||x - y|| \ : \ y \in \mathcal{M}\}$$

Figure 12.4: Distance of a point from a set

Convergence to a set: A solution $x(\cdot)$ converges to a set $\mathcal{M}$ if

$$\lim_{t \to \infty} d(x(t), \mathcal{M}) = 0$$

Figure 12.5: Convergence to a set

Lemma 15 (Positive limit set of a bounded solution.) *The positive limit set $\mathcal{L}^+$ of a bounded solution to (12.1) is nonempty and has the following properties.*

1. *It is closed and bounded.*
2. *It is an invariant set.*
3. *The solution converges to $\mathcal{L}^+$.*

Proof. See Khalil.

Example 126 (Van der Pol oscillator)

12.3 LaSalle's Theorem

In the following results, V is a scalar valued function of the state, that is, $V : \mathbb{R}^n \to \mathbb{R}$, and is continuously differentiable.

Lemma 16 (A convergence result for bounded solutions.) *Suppose $x(\cdot)$ is a bounded solution of (12.1) and there exists a function V such that*

$$DV(x(t))f(x(t)) \leq 0$$

for all $t \geq 0$. Then $x(\cdot)$ converges to the largest invariant set $\mathcal{M}$ contained in the set

$$\mathcal{S} := \{x \in \mathbb{R}^n \mid DV(x)f(x) = 0\}$$

PROOF. Since $x(\cdot)$ is a bounded solution of (12.1), it follows from the previous lemma that it has a nonempty positive limit set $\mathcal{L}^+$. Also, $\mathcal{L}^+$ is an invariant set for (12.1) and $x(\cdot)$ converges to this set.

We claim that V is constant on $\mathcal{L}^+$. To see this, we first note that

$$\frac{d}{dt}(V(x(t)) = DV(x(t))f(x(t) \leq 0\,,$$

hence, $V(x(t))$ does not increase with t. Since $x(\cdot)$ is bounded and V is continuous, it follows that $V(x(\cdot))$ is bounded below. It now follows that there is a constant c such that

$$\lim_{t\to\infty} V(x(t)) = c\,.$$

Consider now any member x^* of $\mathcal{L}^+$. Since $\mathcal{L}^+$ is the positive limit set of $x(\cdot)$, there is a sequence $\{t_k\}_{k=0}^{\infty}$ such that

$$\lim_{k\to\infty} t_k = \infty \qquad \text{and} \qquad \lim_{k\to\infty} x(t_k) = x^*\,.$$

Since V is continuous,

$$V(x^*) = \lim_{k\to\infty} V(x(t_k)) = c\,.$$

We have now shown that $V(x^*) = c$ for all x^* in $\mathcal{L}^+$. Thus V is constant on $\mathcal{L}^+$.

Since $\mathcal{L}^+$ is an invariant set for (12.1) and V is constant on $\mathcal{L}^+$, it follows that $DV(x^*)f(x^*)$ is zero for all x^* in $\mathcal{L}^+$. Hence $\mathcal{L}^+$ is contained in $\mathcal{S}$ and, since $\mathcal{L}^+$ is an invariant set, it must be contained in the largest invariant set $\mathcal{M}$ in $\mathcal{S}$.

Finally, since $x(\cdot)$ converges to $\mathcal{L}^+$ and $\mathcal{L}^+$ is contained in $\mathcal{M}$, the solution $x(\cdot)$ also converges to $\mathcal{M}$. ■

The above result tells us that the solution converges to largest invariant set which is contained in the set in which $\dot{V}$ is zero. Thus, the solution also converges to the set in which $\dot{V}$ is zero.

Remark 7 Note that the above result does not require that V have any special properties such as positive definiteness or radial unboundedness.

Example 127 Consider the damped simple pendulum described by

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2 \,.\end{aligned}$$

Suppose V is the total mechanical energy of this system, that is,

$$V(x) = 1 - \cos x_1 + \frac{1}{2}x_2^2 \,.$$

Then $\dot{V}(x) = -x_2^2 \leq 0$. Since $\dot{V}(x) = 0$ is equivalent to $x_2 = 0$, it follows that every bounded solution of this system converges to the largest invariant set $\mathcal{M}$ in

$$\mathcal{S} = \{(x_1, 0) : x_1 \text{ is an arbitrary real number }\} \,.$$

Consider any solution $x(\cdot)$ contained in $\mathcal{M}$. Since this solution is in $\mathcal{S}$, we must have $x_2(t) = 0$ for all t. Hence $\dot{x}_2(t)$ is zero for all t. It now follows from the system description that

$$\sin x_1(t) = 0 \qquad \text{for all} \quad t \,.$$

Thus, $x_1(t)$ must belong to the set $\mathcal{E}$ of equilibrium states, that is,

$$\mathcal{E} = \{(n\pi, 0) : n \text{ is an integer }\} \,.$$

Thus $\mathcal{M}$ is contained in $\mathcal{E}$. Since $\mathcal{E}$ is invariant, $\mathcal{E} = \mathcal{M}$. Thus all bounded solutions converge to $\mathcal{E}$.

Theorem 22 (LaSalle's Theorem.) *Suppose there is a radially unbounded function V such that*

$$DV(x)f(x) \leq 0$$

for all x. Then all solutions of (12.1) are bounded and converge to the largest invariant set $\mathcal{M}$ contained in the set

$$\mathcal{S} := \{x \in \mathrm{I\!R}^n \mid DV(x)f(x) = 0\}$$

PROOF. We have already seen that the hypothesis of the theorem guarantee that all solutions are bounded. The result now follows from Lemma 16. ■

Example 128 The damped mass.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2\end{aligned}$$

Consider

$$V(x) = \frac{1}{2}x_1^2 + x_1x_2 + x_2^2$$

Figure 12.6: The damped mass

Since V is a positive definite quadratic form, it is radially unbounded. Also,

$$\dot{V}(x) = -x_2^2 \leq 0$$

Hence all solutions approach the set

$$\mathcal{M} = \mathcal{S} = \{ \begin{bmatrix} x_1 & 0 \end{bmatrix}^T : x_1 \in \mathrm{I\!R} \}$$

that is, the x_1 axis.

Figure 12.7: Phase portrait of the damped mass

Example 129 *(Damped Duffing oscillator.)*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 - cx_2 \end{aligned}$$

with $c > 0$.

Exercise 40 Using LaSalle's Theorem, show that all solutions of the system

$$\begin{aligned} \dot{x}_1 &= x_2^2 \\ \dot{x}_2 &= -x_1 x_2 \end{aligned}$$

must approach the x_1 axis.

Exercise 41 Consider the scalar nonlinear mechanical system

$$\ddot{q} + c(\dot{q}) + k(q) = 0$$

If the term $-c(\dot{q})$ is due to damping forces it is reasonable to assume that $c(0) = 0$ and

$$c(\dot{q})\dot{q} > 0 \qquad \text{for all} \quad \dot{q} \neq 0$$

Suppose the term $-k(q)$ is due to conservative forces and define the potential energy by

$$P(q) = \int_0^q k(\eta)\, d\eta$$

Show that if $\lim_{q\to\infty} P(q) = \infty$, then all motions of this system must approach one of its equilibrium positions.

12.4 Asymptotic stability and LaSalle

12.4.1 Global asymptotic stability

Using LaSalle's Theorem, we can readily obtain the following result on global asymptotic stability. This result does not require $\dot{V}$ to be negative for all nonzero states.

Theorem 23 (Global asymptotic stability) *Suppose there is a positive definite function V with*

$$DV(x)f(x) \leq 0$$

for all x and the only solution for which

$$DV(x(t))f(x(t)) \equiv 0$$

is the zero solution. Then the origin is a globally asymptotically stable equilibrium state.

Proof.

Example 130 A damped nonlinear oscillator

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1^3 - cx_2
\end{aligned}$$

with $c > 0$.

Example 131 A nonlinearly damped oscillator

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - cx_2^3
\end{aligned}$$

with $c > 0$.

Exercise 42 Consider a nonlinear mechanical system described by

$$m\ddot{q} + c\dot{q} + k(q) = 0$$

where q is scalar, $m, c > 0$ and k is a continuous function which satisfies

$$\begin{aligned} k(0) &= 0 \\ k(q)q &> 0 \qquad \text{for all } q \neq 0 \\ \lim_{q\to\infty} \int_0^q k(\eta)\, d\eta &= \infty \end{aligned}$$

(a) Obtain a state space description of this system.

(b) Prove that the state space model is GUAS about the state corresponding to the equilibrium position $q = 0$.

  (i) Use a La Salle type result.

  (ii) Do not use a La Salle type result.

Figure 12.8: Single link manipulator

Exercise 43 Consider the inverted pendulum (or one link manipulator) as illustrated where u is a control torque. This system can be described by

$$\ddot{q} - a\sin q = bu$$

with $a = mgl/I$, $b = 1/I$, where m is the mass of $\mathcal{B}$, I is the moment of inertia of $\mathcal{B}$ about its axis of rotation through O, l is the distance between O and the mass center of $\mathcal{B}$, and g is the gravitational acceleration constant of YFHB. We wish to stabilize this system about the position corresponding to $q = 0$ by a linear feedback controller of the form

$$u = -k_p q - k_d \dot{q}$$

Using the results of the last problem, obtain the least restrictive conditions on the controller gains k_p, k_d which assure that the closed loop system is GUAS about the state corresponding to $q(t) \equiv 0$. Illustrate your results with numerical simulations.

12.4.2 Asymptotic stability

Theorem 24 (Asymptotic stability.) *Suppose there is a locally positive definite function V and a scalar $R > 0$ such that*

$$DV(x)f(x) \leq 0 \qquad \textit{for} \quad ||x|| \leq R$$

and the only solution for which

$$DV(x(t))f(x(t)) \equiv 0 \qquad \textit{and} \quad ||x(t)|| \leq R$$

is the zero solution. Then the origin is an asymptotically stable equilibrium state.

Example 132 Damped simple pendulum.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mgl}{I}\sin x_1 - \frac{c}{I}x_2 \end{aligned}$$

with $c > 0$.

12.5 LTI systems

$$\dot{x} = Ax$$

Theorem 25 *The following statements are equivalent.*

(a) The system $\dot{x} = Ax$ *is asymptotically stable.*

(b) There exist a positive definite hermitian matrix P *and a matrix* C *with* (C, A) *observable which satisfy the Lyapunov equation*

$$\boxed{PA + A^*P + C^*C = 0} \tag{12.2}$$

(c) For every *matrix* C *with* (C, A) *observable, the Lyapunov equation (12.2) has a unique solution for* P *and this solution is hermitian positive-definite.*

12.6 Applications

12.6.1 A simple stabilizing adaptive controller

Consider a simple scalar system described by

$$\dot{x} = ax + bu \tag{12.3}$$

where the state x and the control input u are scalars. The scalars a and b are unknown but constant. The only information we have on these parameters is that

$$b > 0\,.$$

We wish to obtain a controller which guarantees that, regardless of the values of a and b, all solutions of the resulting closed loop system satisfy

$$\lim_{t\to\infty} x(t) = 0\,.$$

To achieve the above goal, we propose the following adaptive controller

$$\begin{aligned} u &= -\hat{k}x && \text{(12.4a)}\\ \dot{\hat{k}} &= \gamma x^2 && \text{(12.4b)} \end{aligned}$$

where γ is some positive scalar.

We now show that the above controller results in the desired goal and in addition, the adaptive parameter $\hat{k}$ is bounded. Let k any scalar for which

$$\tilde{a} := a - bk < 0\,.$$

Letting $\delta k = \hat{k} - k$, the closed loop system resulting from the proposed adaptive controller can be described by

$$\begin{aligned} \dot{x} &= \tilde{a}x - b\delta k x && \text{(12.5a)}\\ \delta\dot{k} &= \gamma x^2 && \text{(12.5b)} \end{aligned}$$

As a candidate Lyapunov function, consider,

$$V(x, \delta k) = x^2 + (b/\gamma)(\delta k)^2$$

Along any solution of the closed loop system (12.5), we have

$$\begin{aligned} \dot{V} &= 2x\dot{x} + 2(b/\gamma)\delta k\delta\dot{k}\\ &= 2\tilde{a}x^2 - 2x^2 b\delta k + 2b\delta k x^2\\ &= 2\tilde{a}x^2 \le 0\,. \end{aligned}$$

Since b and γ are positive, V is radially unbounded; also $\dot{V} \le 0$. It now follows that x and δk are bounded. Also, all solutions approach the largest invariant set in the set corresponding to $x = 0$. This implies that $x(t)$ approaches zero as t goes infinity.

Exercise 44 (a) Consider an uncertain system described by

$$\dot{x} = -3x + \delta \sin x + u \tag{12.6}$$

where $t \in \mathbb{R}$ is "time", $x(t) \in \mathbb{R}$ is the state, and $u(t) \in \mathbb{R}$ is the control input. The uncertainty in the system is due to the unknown constant real scalar parameter δ. We wish to design a controller generating $u(t)$ which assures that, for all initial conditions and for all $\delta \in \mathbb{R}$, the closed-loop system satisfies

$$\lim_{t\to\infty} x(t) = 0 \tag{12.7}$$

If δ were known, one could let

$$u = -\delta \sin x$$

to obtain a GUAS closed-loop system and, hence, the desired behavior. Since δ is unknown, we propose the following *parameter adaptive controller*

$$u = -\hat{\delta} \sin x \tag{12.8a}$$

$$\dot{\hat{\delta}} = \gamma x \sin x \tag{12.8b}$$

where $\gamma > 0$ is a constant.

(i) Prove that the closed-loop system (12.6),(12.8) has property (12.7).

(ii) Numerically simulate the open loop system (system (12.6) with $u = 0$) and the closed loop system ((12.6),(12.8)) for different initial conditions, and for different values of δ and γ.

(b) Consider now an uncertain described by

$$\dot{x} = f(x) + \delta g(x) + u \tag{12.9}$$

where t, x, u, δ are as before and f and g are continuous functions with f satisfying

$$\begin{aligned} f(x) > 0 \quad &\text{if} \quad x < 0 \\ f(x) < 0 \quad &\text{if} \quad x > 0 \end{aligned}$$

Design a controller generating u which assures that, for all initial conditions and for all δ, the resulting closed-loop system has property (12.7). Prove that your design works.

12.6.2 PI controllers for nonlinear systems

Here we consider systems described by

$$\dot{x} = f(x) + g(x)(u + w) \tag{12.10}$$

where the n-vector $x(t)$ is the state, the m-vector $u(t)$ is the control input while the constant m-vector w is an unknown constant disturbance input.

We assume that there is a positive definite Lyapunov function V such that

$$DV(x)f(x) < 0 \qquad \text{for all} \quad x \neq 0\,.$$

This implies that the nominal uncontrolled undisturbed system

$$\dot{x} = f(x)$$

is GAS about zero.

We wish to design a controller for u which assures that, for every constant disturbance input, one has

$$\lim_{t\to\infty} x(t) = 0\,.$$

To this end, we propose the following PI-type controllers:

$$u(t) = -K_P y(t) - K_I \int_0^t y(\tau)\,d\tau \qquad \text{where} \quad y = g(x)'DV(x)' \tag{12.11}$$

and the gain matrices K_P and K_I are arbitrary positive definite symmetric matrices. We claim that these controllers achieve the desired behavior.

To prove our claim, note that the closed loop system can be described by

$$\begin{aligned}
\dot{x} &= f(x) - g(x)K_P y + g(x)(w - x_c) \\
\dot{x}_c &= K_I y \\
y &= g(x)'DV(x)'
\end{aligned}$$

and consider the candidate Lyapunov function

$$W(x, x_c) = V(x) + \frac{1}{2}(x_c - w)'K_I^{-1}(x_c - w)\,.$$

Exercise 45 Consider the system of exercise 43 subject to an unknown constant disturbance torque w, that is,

$$\ddot{q} - a\sin q = b(u + w)\,.$$

Design a controller which assures that, for any initial conditions, the angle q approaches some desired constant angle q^e as time goes to infinity. Consider a controller structure of the form:

$$u = -k_p(q - q^e) - k_d\dot{q} - k_I \int (q - q^e)\,.$$

Prove your design works and illustrate your results with numerical simulations.

Chapter 13

Stability of nonautonomous systems

Here, we are interested in the stability properties of systems described by

$$\dot{x} = f(t, x) \tag{13.1}$$

where the state $x(t)$ is an n-vector at each time t. By a solution of the above system we mean any continuous function $x(\cdot) : [t_0, t_1) \to \mathbb{R}^n$ (with $t_0 < t_1$) which satisfies $\dot{x}(t) = f(t, x(t))$ for $t_0 \le t < t_1$. We refer to t_0 as the initial time associated with the solution.

We will mainly focus on stability about the origin. However, suppose one is interested in stability about some nonzero solution $\bar{x}(\cdot)$. Introducing the new state $e(t) = x(t) - \bar{x}(t)$, its evolution is described by

$$\dot{e} = \tilde{f}(t, e) \tag{13.2}$$

with $f(t, e) = f(t, \bar{x}(t) + e) - \dot{\bar{x}}(t)$. Since $e(t) = 0$ corresponds to $x(t) = \bar{x}(t)$, one can study the stability of the original system (13.1) about $\bar{x}(\cdot)$ by studying the stability of the error system (13.2) about the origin.

Scalar linear systems. All solutions of

$$\dot{x} = a(t)x$$

satisfy

$$x(t) = e^{\int_{t_0}^{t} a(\tau)\, d\tau} x(t_0)\,.$$

Consider

$$\dot{x} = -e^{-t}x$$

Here $a(t) = -e^{-t}$. Since $a(t) < 0$ for all t, one might expect that all solutions converge to zero. This does not happen. Since,

$$x(t) = e^{(e^{-t} - e^{-t_0})}$$

we have

$$\lim_{t\to\infty} x(t) = e^{-e^{-t_0}} x(t_0)\,.$$

Hence, whenever $x(t_0) \neq 0$, the solution does not converge to zero.

13.1 Stability and boundedness

13.1.1 Boundedness of solutions

DEFN. Boundedness. *A solution $x(\cdot)$ is* bounded *if there exists $\beta \geq 0$ such that*

$$||x(t)|| \leq \beta \qquad \textit{for all} \quad t \geq t_0$$

A solution is unbounded if it is not bounded.

DEFN. Global uniform boundedness (GUB). *The solutions of (13.1) are* globally uniformly bounded *if for each initial state x_0 there exists a scalar $\beta(x_0)$ such that for all t_0*

$$x(t_0) = x_0 \quad \Longrightarrow \quad ||x(t)|| \leq \beta(x_0) \qquad \textit{for all} \quad t \geq t_0$$

Note that, in the above definition, the bound $\beta(x_0)$ is independent of the initial time t_0.

13.1.2 Stability

Suppose the origin is is an equilibrium state of system (13.1), that is,

$$f(t, 0) = 0$$

for all t.

DEFN. Stability (S). *The origin is* stable *if for each $\epsilon > 0$ and each t_0 there exists $\delta > 0$ such that*

$$||x(t_0)|| < \delta \quad \Longrightarrow \quad ||x(t)|| < \epsilon \quad \textit{for all} \quad t \geq t_0$$

Figure 13.1: Stability

The origin is unstable if it is not stable.

Note that, in the above definition, the scalar δ may depend on the initial time t_0. When δ can be chosen independent of t_0, we say that the stability is uniform.

DEFN. Uniform stability (US). *The origin is* uniformly stable *if for each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* t_0*,*

$$||x(t_0)|| < \delta \quad \Longrightarrow \quad ||x(t)|| < \epsilon \quad \textit{for all} \quad t \geq t_0$$

Example 133 Stable but not uniformly stable. Consider the scalar linear time-varying system:

$$\dot{x} = (6t \sin t - 2t)x$$

We will analytically show that this system is stable about the origin. We will also show that not only is it not uniformly stable about zero, the solutions are not uniformly bounded. If one simulates this system with initial condition $x(t_0) = x_0$ for different initial times t_0 while keeping x_0 the same one will obtain a sequence of solutions whose peaks go to infinity as t_0 goes to infinity. This is illustrated in Figure 13.2.

Figure 13.2: Growing peaks

All solutions $x(\cdot)$ satisfy

$$x(t) = x(t_0)e^{[\alpha(t)-\alpha(t_0)]} \qquad \text{where} \qquad \alpha(t) = 6\sin t - 6t\cos t - t^2$$

Clearly, for each initial time t_0, there is a bound $\beta(t_0)$ such that

$$e^{[\alpha(t)-\alpha(t_0)]} \leq \beta(t_0) \qquad \text{for all } t \geq t_0$$

Hence every solution $x(\cdot)$ satisfies $|x(t)| \leq \beta(t_0)|x(t_0)|$ for $t \geq t_0$ and we can demonstrate stability of the origin by choosing $\delta = \epsilon/\beta(t_0)$ for any $\epsilon > 0$.

To see that this system is not uniformly stable consider $t_0 = 2n\pi$ for any nonnegative integer n. Then $\alpha(t_0) = -12n\pi - 4n^2\pi^2$ and for $t = (2n+1)\pi$ we have $\alpha(t) = 6(2n+1)\pi - (2n+1)^2\pi^2$. Hence

$$\alpha(t) - \alpha(t_0) = (6-\pi)(4n+1)\pi$$

So for any solution $x(\cdot)$ and any nonnegative integer n, we have

$$x(2n\pi + \pi) = x(t_0)e^{(6-\pi)(4n+1)\pi}$$

So, regardless of how small a nonzero bound one places on $x(t_0)$, one cannot place a bound on $x(t)$ which is independent of t_0. Hence the solutions of this system are not uniformly bounded. Also, this system in not uniformly stable about the origin.

13.1.3 Asymptotic stability

Defn. Asymptotic stability (AS). *The origin is* asymptotically stable *if*

(a) It is stable.

(b) For each t_0, there exists $R(t_0) > 0$ such that

$$||x(t_0)|| < R(t_0) \quad \Longrightarrow \quad \lim_{t\to\infty} x(t) = 0 \tag{13.3}$$

Figure 13.3: Asymptotic stability

Note that, in the above definition, R may depend on t_0.

Defn. Uniform asymptotic stability (UAS). *The origin is* uniformly asymptotically stable *if*

(a) It is uniformly stable.

(b) There exists $R > 0$ such that the following holds for each initial state x_0 with $||x_0|| < R$.

i) There exists $\beta(x_0)$ such that for all t_0,

$$x(t_0) = x_0 \quad \Longrightarrow \quad ||x(t)|| \le \beta(x_0) \qquad \textit{for all} \quad t \ge t_0$$

ii) For each $\epsilon > 0$, there exists $T(\epsilon, x_0) \ge 0$ such that for all t_0,

$$x(t_0) = x_0 \quad \Longrightarrow \quad ||x(t)|| < \epsilon \qquad \textit{for all} \quad t > t_0 + T(\epsilon, x_0)$$

13.1.4 Global asymptotic stability

Defn. Global asymptotic stability (GAS). *The origin is* globally asymptotically stable *if*

(a) It is stable.

(b) Every solution satisfies

$$\lim_{t\to\infty} x(t) = 0$$

Defn. Global uniform asymptotic stability (GUAS). *The origin is* globally uniformly asymptotically stable *if*

(a) *It is uniformly stable.*

(b) *The solutions are globally uniformly bounded.*

(c) *For each initial state* x_0 *and each each* $\epsilon > 0$, *there exists* $T(\epsilon, x_0) \geq 0$ *such that for all* t_0,

$$x(t_0) = x_0 \quad \Longrightarrow \quad ||x(t)|| < \epsilon \qquad \text{for all} \quad t > t_0 + T(\epsilon, x_0)$$

13.1.5 Exponential stability

Defn. Uniform exponential stability (UES). *The origin is* uniformly exponentially stable *with* rate of convergence $\alpha > 0$ *if there exists* $R > 0$ *and* $\beta > 0$ *such that whenever* $||x(t_0)|| < R$ *one has*

$$||x(t)|| \quad \leq \quad \beta||x(t_0)||e^{-\alpha(t-t_0)} \qquad \text{for all} \quad t \geq t_0$$

Note that exponential stability implies asymptotic stability, but, in general, the converse is not true.

Defn. Global uniform exponential stability (GUES). The origin is globally exponentially stable with rate of convergence $\alpha > 0$ if there exists $\beta > 0$ such that every solution satisfies

$$||x(t)|| \quad \leq \quad \beta||x(t_0)||e^{-\alpha(t-t_0)} \qquad \text{for all} \quad t \geq t_0$$

Chapter 14

Lyapunov theory for nonautonomous systems

14.1 Introduction

Here, we are interested in the stability properties of the system,

$$\dot{x} = f(t, x) \tag{14.1}$$

where $x(t) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We will focus on stability about the origin. However, suppose one is interested in stability about some nonzero solution $\bar{x}(\cdot)$. Introducing the new state $e(t) = x(t) - \bar{x}(t)$, its evolution is described by

$$\dot{e} = \tilde{f}(t, e) \tag{14.2}$$

with $f(t, e) = f(t, \bar{x}(t) + e) - \dot{\bar{x}}(t)$. Since $e(t) = 0$ corresponds to $x(t) = \bar{x}(t)$, one can study the stability of the original system (14.1) about $\bar{x}(\cdot)$ by studying the stability of the error system (14.2) about the origin.

In this chapter, V is a scalar-valued function of the time and state, that is, $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Suppose V is continuously differentiable. Then, at any time t, the derivative of V along a solution $x(\cdot)$ of system (14.1) is given by

$$\begin{aligned} \frac{dV(t, x(t))}{dt} &= \frac{\partial V}{\partial t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t))\dot{x}(t) \\ &= \frac{\partial V}{\partial t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t))f(t, x(t)) \end{aligned}$$

where $\frac{\partial V}{\partial t}$ is the partial derivative of V with respect to t and $\frac{\partial V}{\partial x}$ is the partial derivative of V with respect to x and is given by

$$\frac{\partial V}{\partial x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \dots & \frac{\partial V}{\partial x_n} \end{bmatrix}$$

Note that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \ldots + \frac{\partial V}{\partial x_n} f_n$$

In what follows, if a condition involves $\frac{\partial V}{\partial t}$ or $\frac{\partial V}{\partial x}$, then it is implicitly assumed that V is continuously differentiable.

14.2 Stability

14.2.1 Locally positive definite functions

A function V is said to be locally positive definite (lpd) if there is a locally positive definite function $V_1 : \mathbb{R}^n \to \mathbb{R}$ and a scalar $R > 0$ such that

$$V_1(x) \leq V(t, x) \qquad \text{for all } t \text{ and for } ||x|| < R$$

A function V is said to be locally decresent (ld) if there is a locally positive definite function $V_2 : \mathbb{R}^n \to \mathbb{R}$ and a scalar $R > 0$ such that

$$V(t, x) \leq V_2(x) \qquad \text{for all } t \text{ and for } ||x|| < R$$

Example 134

(a)

$$V(t, x) = (2 + \cos t)x^2$$

lpd and ld

(b)

$$V(t, x) = e^{-t^2} x^2$$

ld but not lpd

(c)

$$V(t, x) = (1 + t^2)x^2$$

lpd but not ld

14.2.2 A stability theorem

Theorem 26 (Uniform stability) *Suppose there exists a function V with the following properties.*

(a) V is locally positive definite and locally decresent.

(b) There is a scalar $R > 0$ such that

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x} f(t, x) \leq 0 \qquad \textit{for all } t \textit{ and for } ||x|| < R$$

Then the origin is a uniformly stable equilibrium state for $\dot{x} = f(t, x)$.

If V satisfies the hypotheses of the above theorem, then V is said to be a Lyapunov function which guarantees the stability of origin for system (14.1).

Example 135

$$\dot{x} = a(t)x$$

where $a(t) \leq 0$ for all t.

14.3 Uniform asymptotic stability

Theorem 27 (Uniform asymptotic stability) *Suppose there exists a function V with the following properties.*

(a) V is locally positive definite and locally decresent.

(b) There is a continuous function W and a scalar $R > 0$ such that for all t and all nonzero x with $||x|| < R$,

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \leq -W(x) < 0$$

Then the origin is an uniformly asymptotically stable equilibrium state for system (14.1).

Example 136 To see that $\dot{V} < 0$ is not sufficient, consider

$$\dot{x} = -e^{-t}x$$

With $V(x) = x^2$, we have $\dot{V} = -2e^{-t}x^2 < 0$ for all nonzero x. However, with $x(0) = x_0 \neq 0$, we have

$$x(t) = e^{(e^{-t}-1)}x_0$$

Hence $\lim_{t\to\infty} x(t) = e^{-1}x_0 \neq 0$ and this system is not asymptotically stable about zero.

Example 137

$$\dot{x} = (-2 + \sin t)x + x^2$$

14.4 Global uniform asymptotic stability

14.4.1 Positive definite functions

A function V is said to be positive definite (pd) if there is a positive definite function $V_1 : \mathbb{R}^n \to \mathbb{R}$ such that

$$V_1(x) \leq V(t,x) \qquad \text{for all } t \text{ and } x$$

A function V is said to be decresent (d) if there is a positive definite function V_2 such that

$$V(t,x) \leq V_2(x) \qquad \text{for all } t \text{ and } x$$

Example 138 Recall Example 134.

14.4.2 A result on global uniform asymptotic stability

Theorem 28 (Global uniform asymptotic stability) *Suppose there exists a function V with the following properties.*

(a) V is positive definite and decresent.

(b) There is a continuous function W such that for all t and all nonzero x,

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x} f(t,x) \leq -W(x) < 0$$

Then the origin is a globally uniformly asymptotically stable equilibrium state for $\dot{x} = f(t,x)$.

Example 139

$$\begin{aligned}
\dot{x}_1 &= -x_1 - \frac{1}{4}(\sin t)x_2 \\
\dot{x}_2 &= x_1 - x_2
\end{aligned}$$

As a candidate Lyapunov function, consider

$$V(t,x) = x_1^2 + (1 + \frac{1}{4}\sin t)x_2^2$$

Since

$$x_1^2 + \frac{3}{4}x_2^2 \leq V(t,x) \leq x_1^2 + \frac{5}{4}x_2^2$$

it follows that V is positive definite and decresent. Along any solution of the above system, we have

$$\begin{aligned}
\dot{V} &= \frac{1}{4}\cos t\, x_2^2 + 2x_1(-x_1 - \frac{1}{4}\sin t\, x_2) + (2 + \frac{1}{2}\sin t)x_2(x_1 - x_2) \\
&= -2x_1^2 - (2 + \frac{1}{2}\sin t - \frac{1}{4}\cos t)x_2^2 + 2x_1x_2 \\
&\leq -W(x)
\end{aligned}$$

where

$$W(x) = 2x_1^2 - 2x_1x_2 + \frac{5}{4}x_2^2$$

Since $W(x)$ is positive for all nonzero x, it follows from the previous theorem that the above system is globally uniformly stable about the origin.

14.4.3 Proof of theorem 28

In what follows $x(\cdot)$ represents any solution and t_0 is its initial time.

Uniform stability. Consider any $\epsilon > 0$. We need to show that there exists $\delta > 0$ such that whenever $||x(t_0)|| < \delta$, one has $||x(t)|| < \epsilon$ for all $t \geq t_0$.

Since V is positive definite, there exists a positive definite function V_1 such that

$$V_1(x) \leq V(t,x)$$

for all t and x. Since V_1 is positive definite, there exists $c > 0$ such that

$$V_1(x) < c \qquad \Longrightarrow \qquad ||x|| < \epsilon$$

Since V is decresent, there exists a positive definite function V_2 such that

$$V(t,x) \leq V_2(x)$$

for all t and x. Since V_2 is continuous and $V(0) = 0$, there exists $\delta > 0$ such that

$$||x|| < \delta \quad \Longrightarrow \quad V_2(x) < c$$

Along any solution $x(\cdot)$, we have

$$\frac{dV(t,x(t))}{dt} \leq 0$$

for all $t \geq t_0$; hence

$$V_1(x(t)) \leq V(t,x(t)) \leq V(t_0,x(t_0)) \leq V_2(x(t_0)) \qquad \text{for} \quad t \geq t_0.$$

that is,

$$V_1(x(t)) \leq V_2(x(t_0)) \quad t \geq t_0$$

It now follows that if $||x(t_0)|| < \delta$, then for all $t \geq t_0$:

$$V_1(x(t)) \leq V_2(x(t_0)) < c$$

that is, $V_1(x(t)) < c$; hence, $||x(t)|| < \epsilon$.

Global uniform boundedness. Consider any initial state x_0. We need to show that there is a bound β such that whenever $x(t_0) = x_0$, one has $||x(t)|| \leq \beta$ for all $t \geq t_0$.

Since V_1 is a radially unbounded function, there exists $\beta > 0$ such that

$$V_1(x) \leq V_2(x_0) \quad \Longrightarrow \quad ||x|| \leq \beta$$

We have already seen that along any solution $x(\cdot)$ we have

$$V_1(x(t)) \leq V_2(x(t_0)) \quad t \geq t_0$$

It now follows that if $x(t_0) = x_0$, then for all $t \geq t_0$:

$$V_1(x(t)) \leq V_2(x_0)\,;$$

hence $||x(t)|| \leq \beta$.

Global uniform convergence. Consider any initial state x_0 and any $\epsilon > 0$. We need to show that there exists $T \geq 0$ such that whenever $x(t_0) = x_0$, one has $||x(t)|| < \epsilon$ for all $t \geq t_0 + T$.

Consider any solution with $x(t_0) = x_0$. From uniform stability, we know that there exists $\delta > 0$ such that for any $t_1 \geq t_0$,

$$||x(t_1)|| < \delta \quad \Longrightarrow \quad ||x(t)|| < \epsilon \text{ for } \quad t \geq t_1$$

We have shown that there exists $\beta \geq 0$ such that

$$||x(t)|| \leq \beta \text{ for } \quad t \geq t_0$$

We also have

$$\frac{dV(t, x(t))}{dt} \leq -W(x(t))$$

for all $t \geq t_0$. Let

$$\gamma := \min\{W(x) : \delta \leq ||x|| \leq \beta\}$$

Since W is continuous and $W(x) > 0$ for $x \neq 0$, the above minimum exists and $\gamma > 0$. Let

$$T := V_2(x_0)/\gamma$$

We now show by contradiction, there exists t_1 with $t_0 \leq t_1 \leq t_2$, $t_2 := t_0 + T$, such that $||x(t_1)|| < \delta$; from this it will follow that $||x(t)|| < \epsilon$ for all $t \geq t_1$ and hence for $t \geq t_0 + T$.

So, suppose, on the contrary, that

$$||x(t)|| \geq \delta \qquad \text{for} \quad t_0 \leq t \leq t_2$$

Then $V_1(x(t_2)) > 0$, and for $t_0 \leq t \leq t_2$,

$$\delta \leq ||x(t)|| \leq \beta\,;$$

hence

$$\frac{dV(t, x(t))}{dt} \leq -\gamma$$

So,

$$\begin{aligned} V_1(x(t_2)) &\leq V(x(t_2), t_2) \\ &= V(t_0, x(t_0)) + \int_{t_0}^{t_2} \frac{dV(t, x(t))}{dt}\, dt \\ &\leq V_2(x_0) - \gamma T \\ &\leq 0 \end{aligned}$$

that is,

$$V_1(x(t_2)) \leq 0$$

This contradicts $V_1(x(t_2)) > 0$. Hence, there must exist t_1 with $t_0 \leq t_1 \leq t_0 + T$ such that $||x(t_1)|| < \delta$. ∎

14.5 Exponential stability

14.5.1 Exponential stability

Theorem 29 (Uniform exponential stability) *Suppose there exists a function V and positive scalars α, β_1, β_2, and R such that for all t and all x with $\|x\| < R$,*

$$\beta_1 ||x||^2 \leq V(t,x) \leq \beta_2 ||x||^2$$

and

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x} f(t,x) \quad \leq \quad -2\alpha V(t,x)$$

Then, for system (14.1), the origin is a uniformly exponentially stable equilibrium state with rate of convergence α.

14.5.2 Global uniform exponential stability

Theorem 30 (Global uniform exponential stability) *Suppose there exists a function V and positive scalars α, β_1, β_2 such that for all t, x,*

$$\beta_1 ||x||^2 \leq V(t,x) \leq \beta_2 ||x||^2$$

and

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x} f(t,x) \quad \leq \quad -2\alpha V(t,x)$$

Then, for system (14.1), the origin is a globally exponentially uniformly stable equilibrium state with rate of convergence α.

PROOF.

Example 140

$$\dot{x} = -(2 + \cos t)x - x^3$$

14.6 Quadratic stability

From the above it should be clear that all the results in the section on quadratic stability also hold for time-varying systems.

14.7 Boundedness

A scalar valued function V of time and state is said to be uniformly radially unbounded if there are radially unbounded functions V_1 and V_2 of state only such that

$$V_1(x) \leq V(t,x) \leq V_2(x)$$

for all t and x.

Example 141

$$\begin{array}{rcll} V(t,x) & = & (2+\sin t)x^2 & \qquad \text{yes} \\ V(t,x) & = & (1+e^{-t^2})x^2 - x & \qquad \text{yes} \\ V(t,x) & = & e^{-t^2}x^2 & \qquad \text{no} \\ V(t,x) & = & (1+t^2)x^2 & \qquad \text{no} \end{array}$$

Theorem 31 *Suppose there exists a uniformly radially unbounded function V and a scalar $R \geq 0$ such that that for all t and x with $||x|| \geq R$,*

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x} f(t,x) \leq 0$$

Then the solutions of $\dot{x} = f(t,x)$ are globally uniformly bounded.

Note that, in the above theorem, V does not have to be positive away from the origin.

Example 142 Consider the disturbed nonlinear system,

$$\dot{x} = -x^3 + w(t)$$

where w is a bounded disturbance input. We will show that the solutions of this system are GUB.

Let β be a bound on the magnitude of w, that is, $|w(t)| \leq \beta$ for all t. Consider $V(x) = x^2$. Since V is (uniformly) radially unbounded and

$$\begin{array}{rcl} \dot{V} & = & -2x^4 + 2xw(t) \\ & \leq & -2|x|^4 + 2\beta|x| \end{array}$$

the hypotheses of the above theorem are satisfied with $R = \beta^{\frac{1}{3}}$; hence GUB.

Exercise 46 *(Forced Duffing's equation with damping.)* Show that all solutions of the system

$$\begin{array}{rcll} \dot{x}_1 & = & x_2 & \\ \dot{x}_2 & = & x_1 - x_1^3 - cx_2 + w(t) & \qquad c > 0 \\ |w(t)| & \leq & \beta & \end{array}$$

are bounded.
Hint: Consider

$$V(x) = \frac{1}{2}\lambda c^2 x_1^2 + \lambda c x_1 x_2 + \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$$

where $0 < \lambda < 1$. Letting

$$P = \frac{1}{2}\begin{bmatrix} \lambda c^2 & \lambda c \\ \lambda c & 1 \end{bmatrix}$$

note that $P > 0$ and

$$\begin{aligned} V(x) &= x^T P x - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \\ &\geq x^T P x - \frac{1}{4} \end{aligned}$$

14.8 Attractive and invariant regions

Suppose $\mathcal{E}$ is a nonempty subset of the state space $\mathbb{R}^n$. Recall that $\mathcal{E}$ is invariant for the system

$$\dot{x} = f(t, x) \tag{14.3}$$

if $\mathcal{E}$ has the property that, whenever a solution starts in $\mathcal{E}$, it remains therein thereafter, that is, if $x(t_0)$ is in $\mathcal{E}$, then $x(t)$ is in $\mathcal{E}$ for all $t \geq t_0$.

We say that the set $\mathcal{E}$ is attractive for the above system if every solution $x(\cdot)$ of (14.3) converges to $\mathcal{E}$, that is,

$$\lim_{t\to\infty} d(x(t), \mathcal{E}) = 0\,.$$

Theorem 32 *Suppose there is a continuously differentiable function V, a continuous function W and a scalar c such that the following hold.*

1) V is radially unbounded.

2) Whenever $V(x) > c$, we have

$$DV(x)f(t, x) \leq -W(x) < 0 \tag{14.4}$$

for all t.

Then, the solutions of the system $\dot{x} = f(t, x)$ are globally uniformly bounded and the set

$$\{x \in \mathbb{R}^n : V(x) \leq c\}$$

is an invariant and attractive set.

Figure 14.1: An invariant and attractive set

Example 143

$$\dot{x} = -x^3 + \cos t$$

14.9 Systems with disturbance inputs

Here we consider disturbed systems described by

$$\dot{x} = F(t, x, w) \tag{14.5}$$

where the m-vector $w(t)$ is the disturbance input at time t. As a measure of the size of a disturbance $w(\cdot)$ we shall consider its peak norm:

$$\|w(\cdot)\|_\infty = \sup_{t \geq 0} \|w(t)\| \, .$$

Figure 14.2: The peak norm of a signal

We wish to consider the following type of problem: Given the peak norm of the disturbance, obtain an ultimate bound on the norm of the state or some performance output defined by

$$z = H(t, x, w) \, . \tag{14.6}$$

We first have the following result.

Theorem 33 *Consider the disturbed system (14.5) and suppose there exists a continuously differentiable function V, a scalar $\mu_1 > 0$ and a continuous function U such that the following hold.*

1) V is radially unbounded.

2) For all t,

$$\boxed{DV(x)F(t, x, w) \leq -U(x, w) < 0 \qquad \textit{whenever} \quad V(x) > \mu_1 \|w\|^2} \tag{14.7}$$

Then, for every bounded disturbance $w(\cdot)$, the state $x(\cdot)$ is bounded and the set

$$\left\{ x \in \mathbb{R}^n : V(x) \leq \mu_1 \|w(\cdot)\|_\infty^2 \right\}$$

is invariance and attractive for system (14.5).

Example 144

$$\dot{x} = -x - x^3 + w$$

Considering $V(x) = x^2$, we have

$$\begin{aligned} DV(x)\dot{x} &= -x^2 - x^4 + xw \\ &\leq -|x|^2 - |x|^4 + |x||w| \\ &= -|x|\left(|x| + |x|^3 - |w|\right) = -U(x, w) \end{aligned}$$

where $U(x, w) = |x|\left(|x| + |x|^3 - |w|\right)$. Clearly $U(x, w) > 0$ when $|x| > |w|$, that is $V(x) > |w|^2$. Hence, taking $\mu_1 = 1$, it follows from the above theorem that the following interval is invariant and attractive for this system:

$$[-\rho \quad \rho] \qquad \text{where} \quad \rho = \|w(\cdot)\|_\infty .$$

Proof of Theorem 33 Consider the disturbed system (14.5) subject to a specific bounded disturbance $w(\cdot)$. This disturbed system can be described by $\dot{x} = f(t, x)$ with $f(t, x) = F(t, x, w(t))$. Let

$$\rho = \|w(\cdot)\|_\infty = \sup_{t \geq 0} \|w(t)\| .$$

We shall apply Theorem 32 with $c = \mu_1 \rho^2$.

Consider any x which satisfies $V(x) > c$. Then, whenever $\|w\| \leq \rho$, we have $V(x) > \mu_1 \|w\|^2$. If we define

$$W(x) = \min\{U(x, w) : \|w\| \leq \rho\}$$

then, it follows from condition (14.7) that $W(x) > 0$ whenever $V(x) > c$.

Since $\|w(t)\| \leq \rho$ for all $t \geq 0$, it follows that whenever $V(x) > c$, we have

$$\begin{aligned} DV(x)f(t, x) &= DV(x)F(t, x, w(t)) \\ &\leq -U(x, w(t)) \\ &\leq -W(x) < 0 \end{aligned}$$

It now follows from Theorem (32) that the state $x(\cdot)$ is bounded and the set

$$\left\{x \in \mathbb{R}^n : V(x) \leq \mu_1 \|w(\cdot)\|_\infty^2\right\}$$

is invariance and attractive. ■

Corollary 2 *Consider the disturbed system (14.5) equipped with a performance output specified in (14.6). Suppose the hypotheses of Theorem 33 are satisfied and there exists a scalar μ_2 such that for all t, x and w,*

$$\boxed{\|H(t,x,w)\|^2 \leq V(x) + \mu_2\|w\|^2} \tag{14.8}$$

Then for every disturbance bounded disturbance $w(\cdot)$, the output $z(\cdot)$ is bounded and satisfies

$$\limsup_{t\to\infty} \|z(t)\| \leq \gamma\|w(\cdot)\|_\infty$$

where

$$\gamma = \sqrt{\mu_1 + \mu_2}\,. \tag{14.9}$$

If in addition, $V(0) = 0$ then

$$\boxed{\|z(\cdot)\|_\infty \leq \gamma\|w(\cdot)\|_\infty}$$

when $x(0) = 0$.

14.9.1 Linear systems with bounded disturbance inputs

Consider a disturbed linear system:

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \end{aligned}$$

where all the eigenvalues of A have negative real part and w is a bounded input.

Suppose there exists a positive real scalar α such that

$$\begin{pmatrix} PA + A^TP + \alpha P & PB \\ B^TP & -\alpha\mu_1 I \end{pmatrix} < 0 \tag{14.10a}$$

$$\begin{pmatrix} C^TC - P & C^TD \\ D^TC & D^TD - \mu_2 I \end{pmatrix} \leq 0 \tag{14.10b}$$

Then

$$\boxed{\limsup_{t\to\infty} \|z(t)\| \leq \gamma\|w(\cdot)\|_\infty}$$

where

$$\gamma = \sqrt{\mu_1 + \mu_2}\,. \tag{14.11}$$

Also, when $x(0) = 0$, we have

$$\boxed{\|z(\cdot)\|_\infty \leq \gamma\|w(\cdot)\|_\infty}$$

PROOF.

For each fixed α we have LMIs in P, μ_1 and μ_2. One can minimize $\mu_1 + \mu_2$ for each α and do a line search over α.

The above results can be generalized to certain classes of nonlinear systems like polytopic systems. Above results can also be used for control design.

14.10 Regions of attraction

Suppose that the origin is an asymptotically stable equilibrium state of the system

$$\dot{x} = f(t, x) \tag{14.12}$$

where the state $x(t)$ is an n-vector. We say that a non-empty subset $\mathcal{A}$ of the state space $\mathbb{R}^n$ is a region of attraction for the origin if every solution which originates in $\mathcal{A}$ converges to the origin, that is,

$$x(t_0) \in \mathcal{A} \qquad \text{implies} \qquad \lim_{t\to\infty} x(t) = 0\,.$$

Figure 14.3: Region of attraction

The following result can be useful in computing regions of attraction.

Theorem 34 *Consider system (14.12) and suppose there is a continuously differentiable function V, a continuous function W and a scalar c with the following properties.*

1) V is positive definite.

2) Whenever $x \neq 0$ and $V(x) < c$, we have

$$DV(x)f(t,x) \leq -W(x) < 0$$

for all t.

Then, the origin is a uniformly asymptotically stable equilibrium state with

$$\mathcal{A} = \{x \in \mathbb{R}^n : V(x) < c\}$$

as a region of attraction. Also, $\mathcal{A}$ is invariant.

Figure 14.4:

Example 145

$$\dot{x} = -x(1 - x^2)$$

Example 146

$$\begin{aligned}\dot{x}_1 &= -x_1(1-x_1^2-x_2^2)+x_2 \\ \dot{x}_2 &= -x_1-x_2(1-x_1^2-x_2^2)\end{aligned}$$

Considering $V(x) = x_1^2 + x_2^2$, we have

$$\dot{V} = -2V(1-V)$$

14.10.1 A special class of nonlinear systems

We consider here nonlinear systems described by

$$\dot{x} = A(t,x)x \tag{14.13}$$

where

$$A(t,x) = A_0 + \psi(t,x)\Delta A \qquad \text{and} \qquad -a(||Cx||) \le \psi(t,x) \le b(||Cx||) \tag{14.14}$$

while a, b are non-decreasing functions and C is a matrix. We have previously considered these systems where a and b are constants and obtained sufficient conditions for global uniform exponential stability using quadratic Lyapunov functions.

Example 147 Consider

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1-(1-x_1^2)x_2\end{aligned}$$

This system can be expressed as

$$\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \left[\begin{pmatrix}0 & 1 \\ -1 & -1\end{pmatrix} + x_1^2\begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}x_1 \\ x_2\end{pmatrix} \tag{14.15}$$

Here

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad a(\mu) = 0, \qquad b(\mu) = \mu^2 .$$

Theorem 35 *Suppose there is a positive definite symmetric P and a positive scalar μ such that*

$$PA_0 + A_0^TP - a(\mu)(P\Delta A + \Delta A^TP) < 0 \tag{14.16a}$$
$$PA_0 + A_0^TP + b(\mu)(P\Delta A + \Delta A^TP) < 0 \tag{14.16b}$$
$$C^TC - P \le 0 \tag{14.16c}$$

Then the origin is a uniformly asymptotically stable equilibrium state with

$$\mathcal{A} = \{x \in \mathbb{R}^n : x^TPx < \mu^2\}$$

as a region of attraction. Also $\mathcal{A}$ is invariant. If, in addition,

$$P - \beta I \le 0 \tag{14.17}$$

for positive number β, then the ball

$$\{x \in \mathbb{R}^n : ||x|| < \beta^{-\frac{1}{2}}\mu\}$$

is a region of attraction.

Proof. First note that the inequality, $C^TC - P \leq 0$, implies that

$$\|Cx\|^2 \leq x^TPx$$

for all x. So, consider any x for which $x^TPx < \mu^2$. Then

$$\|Cx\| < \mu\,.$$

Using the nondecreasing property of b, we now obtain that

$$\psi(t,x) \leq b(\|Cx\|) \leq b(\mu)\,.$$

In a similar fashion, we also obtain that

$$\psi(t,x) \geq -a(\|Cx\|) \geq -a(\mu)\,.$$

Hence, whenever $x^TPx < \mu^2$, we have

$$-a(\mu) \leq \psi(t,x) \leq b(\mu)\,.$$

Considering $V(x) = x^TPx$ as a candidate Lyapunov function, one may use inequalities (14.16a)-(14.16b) to show that whenever $x^TPx < \mu^2$ and $x \neq 0$, one has

$$DV(x)\dot{x} \leq -W(x) < 0\,.$$

Hence the set

$$\mathcal{A} = \{x \in \mathbb{R}^n : x^TPx < \mu^2\}$$

is invariant and a region of attraction for the origin.

Now note that the inequality, $P - \beta I \leq 0$, implies that

$$x^TPx \leq \beta^{-1}\|x\|^2$$

for all x. So whenever, $\|x\| < \beta^{-\frac{1}{2}}\mu$, we must have $x^TPx < \mu^2$, that is x is in $\mathcal{A}$. Hence the set of states of norm less than $\beta^{-\frac{1}{2}}\mu$ is a region of attraction for the origin. ∎

14.11 Linear time-varying systems

Here we consider systems described by

$$\dot{x} = A(t)x \tag{14.18}$$

All solutions satisfy

$$x(t) = \Phi(t,\tau)x(\tau)$$

where Φ is the state transition matrix.

Computation of the state transition matrix. Let X be the solution of

$$\dot{X}(t) = A(t)X(t) \tag{14.19a}$$

$$X(0) = I \tag{14.19b}$$

where I is the $n \times n$ identity matrix. Then

$$\Phi(t,\tau) = X(t)X(\tau)^{-1} \tag{14.20}$$

Example 148 (Markus and Yamabe, 1960.) This is an example of an unstable second order system whose time-varying A matrix has constant eigenvalues with negative real parts. Consider

$$A(t) = \begin{bmatrix} -1 + a\cos^2 t & 1 - a\sin t\cos t \\ -1 - a\sin t\cos t & -1 + a\sin^2 t \end{bmatrix}$$

with $1 < a < 2$. Here,

$$\Phi(t,0) = \begin{bmatrix} e^{(a-1)t}\cos t & e^{-t}\sin t \\ -e^{(a-1)t}\sin t & e^{-t}\cos t \end{bmatrix}$$

Since $a > 1$, the system corresponding to this $A(t)$ matrix has unbounded solutions. However, for all t, the characteristic polynomial of $A(t)$ is given by

$$p(s) = s^2 + (2-a)s + 2$$

Since $2 - a > 0$, the eigenvalues of $A(t)$ have negative real parts.

14.11.1 Lyapunov functions

As a candidate Lyapunov function for a linear time-varying system consider a time-varying quadratic form V given by:

$$V(t,x) = x^T P(t)x$$

where $P(\cdot)$ is continuously differentiable and $P(t)$ is symmetric for all t.

To guarantee that V is positive definite and decrescent, we need to assume that there exists positive real scalars c_1 and c_2 such that for all t,

$$0 < c_1 \leq \lambda_{\min}(P(t)) \qquad \text{and} \qquad \lambda_{\max}(P(t)) \leq c_2 \tag{14.21}$$

This guarantees that

$$c_1\|x\|^2 \le V(t,x) \le c_2\|x\|^2 .$$

Considering the rate of change of V along a solution, we have

$$\begin{aligned}\dot V(t,x) &= \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)\dot x \\ &= x'\dot P(t)x + 2x'P(t)A(t)x \\ &= -x'Q(t)x\end{aligned}$$

where $Q(t)$ is given by

$$\boxed{\dot P + PA + A'P + Q = 0} \tag{14.22}$$

To satisfy the requirements on $\dot V(t,x)$ for GUAS and GUES we need

$$0 < c_3 \le \lambda_{\min}(Q(t)) \tag{14.23}$$

for all t. This guarantees that

$$\dot V(t,x) \le -c_3\|x\|^2 .$$

Hence

$$\dot V(t,x) \le -2\alpha V(t,x) \qquad \text{where} \qquad \alpha = c_3/2c_2 .$$

So, if there are matrix functions $P(\cdot)$ and $Q(\cdot)$ which satisfy (14.21)-(14.23) , we can conclude that the LTV system (14.18) is GUES. The following theorem provides the converse result.

Theorem 36 *Suppose $A(\cdot)$ is continuous and bounded and the corresponding linear time-varying system (14.18) is globally uniformly asymptotically stable. Then, for all t, the matrix*

$$P(t) = \int_t^\infty \Phi(\tau,t)^T\Phi(\tau,t)\,d\tau \tag{14.24}$$

is well defined, where Φ is the state transition matrix associated with (14.18), and

$$\dot P + PA + A'P + I = 0 . \tag{14.25}$$

Also, there exists positive real scalars c_1 and c_2 such that for all t,

$$c_1 \le \lambda_{\min}(P(t)) \qquad \textit{and} \qquad \lambda_{\max}(P(t)) \le c_2$$

Proof. See Khalil or Vidyasagar.

A consequence of the above theorem is that GUES and GUAS are equivalent for systems where $A(\cdot)$ is continuous and bounded.

Example 149 The following system has a time varying quadratic Lyapunov function but does not have a time-invariant one.

$$\dot x = (-1 + 2\sin t)x$$

Here

$$\Phi(\tau,t) = e^{\int_t^\tau(-1+2\sin s)\,ds} = e^{(t-\tau+2\cos t-2\cos\tau)} = e^{(t-\tau)}e^{2(\cos t-\cos\tau)}$$

Hence,

$$P(t) = \int_t^\infty e^{2(t-\tau)}e^{4(\cos t-\cos\tau)}\,d\tau = \int_0^\infty e^{-2\tau}e^{4(\cos t-\cos(t+\tau))}\,d\tau .$$

14.12 Linearization

$$\dot{x} = f(t, x)$$

Suppose that $\bar{x}(\cdot)$ is a specific solution.
Let

$$\delta x = x - \bar{x}$$

Linearization about $\bar{x}(\cdot)$

$$\delta\dot{x} = A(t)\delta x \tag{14.26}$$

where

$$A(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t))\,.$$

Chapter 15

The describing function method

Here we consider the problem of determining whether or not a nonlinear system has a limit cycle (periodic solution) and approximately finding this solution.

We consider systems which can be described by the negative feedback combination of a SISO LTI (linear time invariant) system and a SISO memoryless nonlinear system; see Figure 15.2. If the LTI system is represented by its transfer function $\hat{G}$ and the nonlinear

Figure 15.1: Systems under consideration

system is represented by the function ϕ, then the system can be described by

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s) \tag{15.1a}$$

$$u(t) = -\phi(y(t)) \tag{15.1b}$$

where u and y are the input and output, respectively for the LTI system and their respective Laplace transforms are $\hat{u}$ and $\hat{y}$.

As a general example, consider any system described by

$$\dot{x} = Ax - B\phi(Cx) \tag{15.2}$$

where $x(t)$ is an n-vector. In this example

$$\hat{G}(s) = C(sI - A)^{-1}B$$

A solution $y(\cdot)$ is periodic with period $T > 0$ if

$$y(t+T) = y(t) \qquad \text{for all} \quad t\,.$$

We are looking for periodic solutions to (15.1).

15.1 Periodic functions and Fourier series

Consider any piecewise continuous signal $s : \mathbb{R} \to \mathbb{R}$ which is periodic with period T. Then s has a Fourier series expansion, that is, s can be expressed as

$$s(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t)$$

where $\omega = 2\pi/T$ is called the natural frequency of s. The Fourier coefficients $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ are uniquely given by

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)\,dt \\ a_k &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)\cos(k\omega t)\,dt \qquad \text{and} \qquad b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)\sin(k\omega t)\,dt \qquad \text{for} \quad k = 1, 2, \ldots \end{aligned}$$

If s is odd, that is, $s(-t) = -s(t)$ then,

$$a_k = 0 \qquad \text{for} \quad k = 0, 1, 2, \ldots$$

and

$$b_k = \frac{4}{T}\int_0^{\frac{T}{2}} s(t)\sin(k\omega t)\,dt \qquad \text{for} \quad k = 1, 2, \ldots$$

Example 150 Sometimes it is easier to compute Fourier coefficients without using the above integrals. Consider

$$s(t) = \sin^3 t\,.$$

Clearly, this is an odd periodic signal with period $T = 2\pi$. Since

$$\sin t = \frac{e^{jt} - e^{-jt}}{2j}$$

we have

$$\begin{aligned} \sin^3(t) &= \left(\frac{e^{jt} - e^{-jt}}{2j}\right)^3 = \frac{e^{3jt} - 3e^{jt} + 3e^{-jt} - e^{-3jt}}{-8j} \\ &= \frac{3(e^{jt} - e^{-jt})}{8j} - \frac{e^{3jt} - e^{-3jt}}{8j} \\ &= \frac{3}{4}\sin t - \frac{1}{4}\sin 3t\,. \end{aligned}$$

So,

$$b_1 = \frac{3}{4}, \qquad b_3 = -\frac{1}{4}$$

and all other Fourier coefficients are zero.

Exercise 47 Obtain the Fourier series for $s(t) = \sin^5 t$

15.2 Describing functions

The describing function for a static nonlinearity is an approximate description of its frequency response. Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is a piecewise continuous function and consider the signal

$$s(t) = \phi(a \sin \omega t)$$

for $a, \omega > 0$. This signal is the output of the nonlinear system defined by ϕ and subject to input $a \sin \omega t$.

The signal s is piecewise continuous and periodic with period $T = 2\pi/\omega$. Hence it has a Fourier series expansion and its Fourier coefficients are given by

$$a_0(a) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi(a \sin \omega t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(a \sin \theta)\, d\theta \qquad (\theta = \omega t = 2\pi t/T)$$

and for $k = 1, 2, \ldots$,

$$a_k(a) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(a \sin \theta) \cos k\theta\, d\theta \qquad \text{and} \qquad b_k(a) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(a \sin \theta) \sin k\theta\, d\theta$$

Real describing functions. Suppose that the function $\phi : \mathbb{R} \to \mathbb{R}$ is piecewise continuous and odd, that is,

$$\phi(-y) = -\phi(y)$$

If ϕ is odd, then s is odd; hence

$$a_k(a) = 0 \qquad \text{for} \quad k = 0, 1, \ldots$$

and

$$b_k(a) = \frac{2}{\pi} \int_0^{\pi} \phi(a \sin \theta) \sin k\theta\, d\theta\,.$$

If we approximate s by neglecting its higher harmonics, that is,

$$s(t) \approx b_1(a) \sin(\omega t)$$

then

$$s(t) \approx N(a) a \sin \omega t$$

where N is called the describing function for ϕ and is given by

$$\boxed{N(a) = b_1(a)/a = \frac{2}{\pi a} \int_0^{\pi} \phi(a \sin \theta) \sin \theta\, d\theta}$$

Basically, $N(a)$ is an approximate system gain for the nonlinear system when subject to a sinusoidal input of amplitude a. Notice that, unlike a dynamic system, this gain is independent of the frequency of the input signal. For a linear system, the gain is independent of amplitude.

Example 151 *Cubic nonlinearity.* Consider

$$\phi(y) = y^3$$

Then, recalling the last example, we see that

$$\phi(a \sin\theta) = a^3 \sin^3\theta = \frac{3a^3}{4}\sin\theta - \frac{a^3}{4}\sin 3\theta$$

Hence $b_1(a) = 3a^3/4$ and

$$N(a) = \frac{b_1(a)}{a} = \frac{3a^2}{4}\,.$$

Example 152 *Signum function.* Recall the signum function given by

$$\phi(y) = \begin{cases} -1 & \text{if} \quad y < 0 \\ 0 & \text{if} \quad y = 0 \\ 1 & \text{if} \quad y > 0 \end{cases}$$

Its describing function is given by

$$N(a) \quad = \quad \frac{2}{\pi a}\int_0^{\pi} \phi(a\sin\theta)\sin\theta\, d\theta = \frac{2}{\pi a}\int_0^{\pi} \sin\theta\, d\theta = \frac{4}{\pi a}\,.$$

Example 153 Consider the following approximation to the signum switching function.

Figure 15.2: Approximation to signum function

$$\phi(y) = \begin{cases} -1 & \text{if} \quad y < -e \\ 0 & \text{if} \quad -e \le y \le e \\ 1 & \text{if} \quad y > e \end{cases} \tag{15.3}$$

where e is some positive real number. This is an odd function. Clearly, $N(a) = 0$ if $a \le e$. If $a > e$, let θ_e be the unique number between 0 and $\pi/2$ satisfying $\theta_e = \arcsin(e/a)$ that is, $a\sin(\theta_e) = e$. Then,

$$\int_0^{\pi} \phi(a\sin\theta)\sin\theta\ d\theta = \int_{\theta_e}^{\pi-\theta_e} \sin\theta\ d\theta = -\cos(\pi - \theta_e) + \cos\theta_e = 2\cos\theta_e$$

Since $\sin\theta_e = e/a$ and $0 \le \theta_e < \pi/2$, we have $\cos\theta_e = \sqrt{1-(e/a)^2}$. Thus

$$b_1(a) = \frac{4\sqrt{a^2-e^2}}{\pi a}$$

Hence,

$$N(a) = \begin{cases} 0 & \text{if} \quad a \le e \\ \dfrac{4\sqrt{a^2-e^2}}{\pi a^2} & \text{if} \quad a > e \end{cases} \tag{15.4}$$

One can readily show that the maximum value of $N(a)$ is $2/(\pi e)$ and this occurs at $a = \sqrt{2}e$.

Example 154 Consider a sector bounded nonlinearity satisfying

$$\alpha y^2 \le y\phi(y) \le \beta y^2$$

for some constants α and β. We will show that

$$\alpha \le N(a) \le \beta$$

Complex describing functions. If the function ϕ is not odd then, its describing function may be complex valued; see [1] for an example. Assuming $a_0(a) = 0$, in this case, we let

$$\boxed{N(a) = b_1(a)/a + \jmath a_1(a)/a} \tag{15.5}$$

where

$$a_1(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(a\sin\theta)\cos\theta\, d\theta \qquad \text{and} \qquad b_1(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(a\sin\theta)\sin\theta\, d\theta$$

Note that

$$N(a) = |N(a)|e^{j\phi(a)}$$

where

$$|N(a)| = \frac{\sqrt{a_1(a)^2 + b_1(a)^2}}{a}$$

and $\phi(a)$ is the unique angle in $[0, 2\pi)$ given by

$$\cos(\phi(a)) = \frac{b_1(a)}{\sqrt{a_1(a)^2+b_1(a)^2}} \qquad \sin(\phi(a)) = \frac{a_1(a)}{\sqrt{a_1(a)^2+b_1(a)^2}}$$

Here, the approximate output of the nonlinearity due to input $a\sin(\omega t)$ is given by

$$\begin{aligned} s(t) &\approx a_1(a)\cos(\omega t) + b_1(a)\sin(\omega t) \\ &= |N(a)|\ a\left[\sin(\phi(a))\cos(\omega t) + \cos(\phi(a))\sin(\omega t)\right] \\ &= \approx |N(a)|\ a\sin(\omega t + \phi(a)) \end{aligned}$$

that is,

$$s(t) \approx |N(a)| a\sin(\omega t + \phi(a)) \tag{15.6}$$

where $\phi(a)$ is the argument of $N(a)$. Notice the phase shift in s due to $\phi(a)$.

Exercise 48 Compute the describing function for $\phi(y) = y^5$.

Exercise 49 Consider the saturation function *sat* described by

$$sat(y) = \begin{cases} -1 & \text{if} \quad y < -1 \\ y & \text{if} \quad -1 \le y \le 1 \\ 1 & \text{if} \quad y > 1 \end{cases}$$

Show that its describing function is given by

$$N(a) = \begin{cases} 1 & \text{if} \quad 0 \le a \le 1 \\ \frac{2}{\pi}\left[\sin^{-1}(\frac{1}{a}) + \frac{\sqrt{a^2-1}}{a^2}\right] & \text{if} \quad 1 < a \end{cases}$$

15.3 The describing function method

Recall the nonlinear system described in (15.1) Suppose we are looking for a periodic solution for y which can be approximately described by

$$y(t) \approx a\sin(\omega t)$$

where $a > 0$.

Real describing functions. Suppose ϕ is odd,

$$u(t) \approx -\phi(a\sin(\omega t)) = -\sum_{k=1}^{\infty} b_k(a)\sin(k\omega t)$$

where

$$b_k(a) \quad = \quad \frac{2}{\pi}\int_0^{\pi} \phi(a\sin\theta)\sin(k\theta)\,d\theta$$

So,

$$u(t) \quad \approx \quad -b_1(a)\sin(\omega t) = -N(a)a\sin(\omega t)$$

where N is the describing function for ϕ. Since $u(t) \approx -N(a)a\sin(\omega t)$ and $y(t) \approx a\sin(\omega t)$ and $\hat{y} = \hat{G}\hat{u}$, it follows that

$$\hat{G}(\jmath\omega) \approx -1/N(a)\,.$$

So, if for some pair $a, \omega > 0$, the condition

$$\boxed{1 + \hat{G}(\jmath\omega)N(a) = 0} \tag{15.7}$$

is satisfied, it is likely that the nonlinear system will have a periodic solution with

$$y(t) \approx a\sin\omega t$$

When $N(a)$ is real the above condition can be expressed

$$\Im(\hat{G}(\jmath\omega)) \quad = \quad 0 \tag{15.8a}$$

$$1 + \Re(\hat{G}(\jmath\omega))N(a) \quad = \quad 0 \tag{15.8b}$$

The first condition in (15.8) simply states that the imaginary part of $\hat{G}(\jmath\omega)$ is zero, that is, $\hat{G}(\jmath\omega)$ is real. This condition is independent of a and can be solved for values of ω. The values of a are then determined by the second condition in (15.8).

Example 155

$$\ddot{y} + y^3 = 0$$

With $\phi(y) := y^3$, this system is described by

$$\begin{aligned} \ddot{y} &= u \\ u &= -\phi(y) \end{aligned}$$

Hence

$$\hat{G}(s) = \frac{1}{s^2}$$

and

$$N(a) = \frac{3a^2}{4}$$

Here condition (15.7) is

$$1 - \frac{3a^2}{4\omega^2} = 0$$

Hence, we predict that this system has periodic solutions of all amplitudes a and with approximate periods

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{3}a/2} = \frac{4\pi}{\sqrt{3}a}$$

Example 156 Consider the nonlinear system,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4x_1 - 5x_2 - \mathrm{sgm}\,(x_1 - x_2) \end{aligned}$$

This can be represented as the LTI system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4x_1 - 5x_2 + u \\ y &= x_1 - x_2 \end{aligned}$$

subject to negative feedback from the memoryless nonlinearity $u = -\mathrm{sgm}\,(y)$. Hence

$$\hat{G}(s) = \frac{-s+1}{s^2+5s+4}$$

and $N(a) = 4/\pi a$. The condition $1 + \hat{G}(\jmath\omega)N(a) = 0$ results in

$$1 + \frac{-\jmath\omega + 1}{-\omega^2 + 4 + 5\jmath\omega} N(a) = 0$$

that is

$$-\omega^2 + 4 + 5\jmath\omega + (-\jmath\omega + 1)N(a) = 0$$

or

$$\begin{aligned} 4 - \omega^2 + N(a) &= 0 \\ 5\omega - \omega N(a) &= 0 \end{aligned}$$

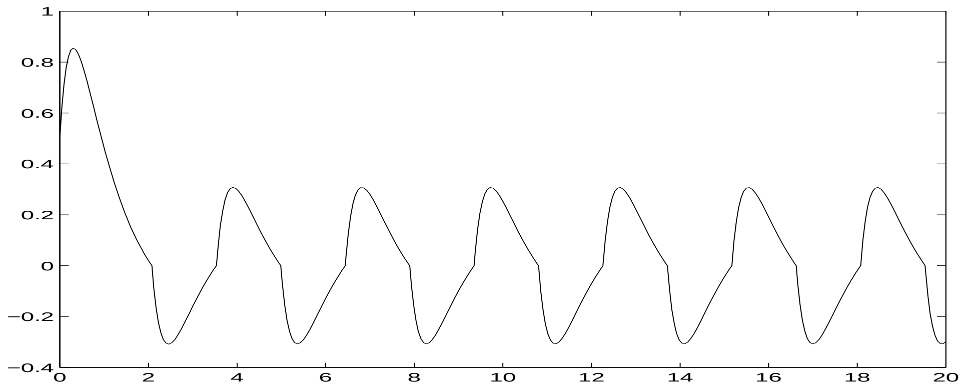

Figure 15.3: Limit cycle for system in example 156

Solving yields $\omega = 3$ and $N(a) = 5$. This results in

$$a = \frac{4}{5\pi} = 0.2546\,, \qquad T = \frac{2\pi}{\omega} = 2.0944$$

The y-response of this system subject to initial conditions $x_1(0) = 0.5$ and $x_2(0) = 0$ is illustrated in Figure 15.3

Example 157 Consider the system of the previous example with the signum function replaced with an approximation as described in Example 153, that is,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4x_1 - 5x_2 - \phi(x_1 - x_2) \end{aligned}$$

where ϕ is given by (15.3) for some $e > 0$. Proceeding as in the previous example, the describing function method still gives the following conditions for a limit cycle

$$\omega = 3 \qquad \text{and} \qquad N(a) = 5\,.$$

Recalling the expression for $N(a)$ in 15.4, we must have $a > e$ and

$$\frac{4\sqrt{a^2 - e^2}}{\pi a^2} = 5$$

Solving this for a^2 yields two solutions

$$a^2 = \frac{8 \pm 4\sqrt{4 - (5\pi e)^2}}{(5\pi)^2}$$

For these solutions to be real, we must have $4 - (5\pi e)^2 \geq 0$ that is

$$e \leq \frac{2}{5\pi} = 0.1273\,.$$

Complex describing functions. We still have the criterion (15.7) for a limit cycle, that is,

$$1 + \hat{G}(\jmath\omega)N(a) = 0$$

However, we cannot usually simplify things as in (15.8).

Example 158 (Van der Pol oscillator) Here we demonstrate a generalization of the method.

$$\ddot{y} + \mu(y^2 - 1)\dot{y} + y = 0$$

We can describe this system by

$$\begin{aligned} \ddot{y} - \mu\dot{y} + y &= \mu u \\ u &= -y^2\dot{y} \end{aligned}$$

Here

$$\hat{G}(s) = \frac{\mu}{s^2 - \mu s + 1}$$

If $y(t) = a\sin\omega t$ then

$$y^2\dot{y} = a^3\omega\sin^2\omega t\cos\omega t \approx \frac{a^3\omega}{4}\cos\omega t \approx \frac{a^3\omega}{4}\sin(\omega t + \pi/2)$$

So here

$$N(a,\omega) = \frac{a^2\omega e^{\jmath\pi/2}}{4} = \frac{\jmath a^2\omega}{4}\,.$$

Solving $1 + \hat{G}(\jmath\omega)N(a,\omega) = 0$ yields

$$\omega = 1 \qquad a = 2\,.$$

15.3.1 Rational transfer functions

When the describing function of the nonlinearity is real, the approximate frequencies ω at which a limit cycle occurs are a subset of those frequencies for which $\hat{G}(\jmath\omega)$ is real, that is, the imaginary part of $\hat{G}(\jmath\omega)$ is zero. Here we consider strictly proper rational transfer functions $\hat{G}$ and show how one can easily determine the frequencies ω for which $\hat{G}(\jmath\omega)$ is real. This involves a state space representation $(A, B, C, 0)$ of $\hat{G}$.

Suppose $\hat{G}(s) = C(sI - A)^{-1}B$ is a scalar transfer function with real A, B, C and A has no imaginary eigenvalues. Noting that

$$\begin{aligned}(\jmath\omega I - A)^{-1} &= (-\jmath\omega I - A)(-\jmath\omega I - A)^{-1}(\jmath\omega I - A)^{-1}\\ &= (-\jmath\omega I - A)(\omega^2 I + A^2)^{-1}\\ &= -A(\omega^2 I + A^2)^{-1} - \jmath\omega(\omega^2 I + A^2)^{-1},\end{aligned}$$

we obtain that

$$\hat{G}(\jmath\omega) = -CA(\omega^2 I + A^2)^{-1}B - \jmath\omega C(\omega^2 I + A^2)^{-1}B. \tag{15.9}$$

The above expression splits $\hat{G}(\jmath\omega)$ into its real and imaginary part, in particular the imaginary part of $\hat{G}(\jmath\omega)$ is given by

$$\Im\left(\hat{G}(\jmath\omega)\right) = -\omega C(\omega^2 I + A^2)^{-1}B \tag{15.10}$$

To determine the nonzero values (if any) of ω for which the imaginary part of $\hat{G}(\jmath\omega)$ is zero, we note that

$$\begin{aligned}\det\begin{pmatrix} -\omega^2 I - A^2 & B \\ C & 0 \end{pmatrix} &= \det\left[-\omega^2 I - A^2\right]\,\det\left[C(\omega^2 I + A^2)^{-1}B\right]\\ &= \det\left[-\omega^2 I - A^2\right]\,C(\omega^2 I + A^2)^{-1}B.\end{aligned}$$

Introducing the matrix pencil given by

$$\boxed{P(\lambda) = \begin{pmatrix} -A^2 - \lambda I & B \\ C & 0 \end{pmatrix}} \tag{15.11}$$

we have shown that, for $\omega \neq 0$,

$$\det P(\omega^2) = -\det\left[-\omega^2 I - A^2\right]\Im\left(\hat{G}(\jmath\omega)\right)/\omega$$

Since it is assumed that A does not have any purely imaginary eigenvalues, A^2 does not have any negative real eigenvalues. This implies that $\det(-\omega^2 I - A^2)$ is non-zero for all ω. Hence, for nonzero ω,

$$\Im\left(\hat{G}(\jmath\omega)\right) = 0 \qquad \text{if and only if} \qquad \det P(\omega^2) = 0 \tag{15.12}$$

Note that

$$P(\lambda) = \begin{pmatrix} -A^2 & B \\ C & 0 \end{pmatrix} - \lambda\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

The values of λ for which $\det P(\lambda) = 0$ are the finite generalized eigenvalues of the matrix pencil P. You can use the Matlab command `eig` to compute generalized eigenvalues. Thus we have the following conclusion:

$\hat{G}(\jmath\omega)$ is real if and only if $\omega = \sqrt{\lambda}$ and λ is a positive, real, finite, generalized eigenvalue of the matrix pencil P

Example 159 For example 156 we have

$$A = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

Hence

$$P(\lambda) = \begin{pmatrix} -A^2 - \lambda I & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 4-\lambda & 5 & 0 \\ -20 & -21-\lambda & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

and $\det P(\lambda) = -\lambda + 9$. Hence 9 is the only finite generalized eigenvalue of P and it is positive real. Thus, there is a single nonzero value of ω for which $\hat{G}(\jmath\omega)$ is real and this is given by $\omega = 3$.

15.4 Exercises

Exercise 50 Obtain the describing function of the nonlinear function

$$\phi(y) = y^5$$

Exercise 51 Determine whether or not the following Duffing system has a periodic solution. Determine the approximate amplitude and period of all periodic solutions.

$$\ddot{y} - y + y^3 = 0\,.$$

Exercise 52 Determine whether or not the following system has a periodic solution. Determine the approximate amplitude and period of all periodic solutions.

$$\ddot{y} + \mu(\dot{y}^3/3 - \dot{y}) + y = 0\,.$$

where μ is a positive real number.

Exercise 53 Use the describing function method to predict period solutions to

$$\ddot{y} + \operatorname{sgm}(y) = 0$$

Illustrate your results with numerical simulations.

Exercise 54 Use the describing function method to predict period solutions to

$$\dot{x}(t) = -x(t) - 2\operatorname{sgm}(x(t-h))$$

Illustrate your results with numerical simulations.

Exercise 55 Consider the double integrator

$$\ddot{q} = u$$

subject to a saturating PID controller

$$u = \mathrm{sat}(\tilde{u}) \qquad \text{where} \qquad \tilde{u} = -k_P q - k_I \int q - k_D \dot{q}$$

(a) For $k_P = 1$ and $k_D = 2$ determine the largest of $k_I \geq 0$ for which the closed loop system is stable about $q(t) \equiv 0$.

(b) For $k_P = 1$ and $k_D = 2$, use the describing function method to determine the smallest value $k_I \geq 0$ for which the closed loop system has a periodic solution.

Bibliography

[1] Boniolo, I., Bolzern, P., Colaneri,P., Corless, M., Shorten, R. "Limit Cycles in Switching Capacitor Systems: A Lure Approach"

Chapter 16

Aerospace/Mechanical Systems

16.1 Aerospace/Mechanical systems

Let the real scalar t represent time. At each instant of time, the configuration of the aerospace/mechanical systems under consideration can be described by a real N-vector $q(t)$. We call this the vector of generalized coordinates; each component is usually an angle, a length, or a displacement. It is assumed that there are no constraints, either holonomic or non-holonomic, on q. So, N is the number of degrees-of-freedom of the system. We let the real scalar T represent the kinetic energy of the system. It is usually given by

$$T = \frac{1}{2}\dot{q}^T M(q)\dot{q} \; = \sum_{j=1}^{N}\sum_{k=1}^{N} M_{jk}(q)\dot{q}_j\dot{q}_k$$

where the symmetric $N \times N$ matrix $M(q)$ is called the system mass matrix. Usually, it satisfies

$$M(q) > 0 \qquad \text{for all } q$$

The real N-vector Q is the sum of the generalized forces acting on the system. It includes conservative and non-conservative forces. It can depend on $t, q, \dot{q}$.

Example 160 Two cart system.

Here

$$T = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2$$

So,

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

Example 161 Single link manipulator (inverted pendulum).

$$q = \theta$$
$$T = \frac{1}{2}J\dot{\theta}^2$$

So, $M = J$. Also, $Q = Wl\sin\theta$.

Example 162 Particle undergoing central force motion.

Figure 16.1: Central force motion

Consider a particle P of mass m undergoing a central force motion. The force center is O which is fixed in inertial reference frame $\mathbf{i}$. Here,

$$q = (r \quad \theta)^T$$

and the kinetic energy is given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Hence

$$M(q) = \begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix}$$

Also, the generalized forces are given by

$$Q = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

16.2 Equations of motion

La Grange's Equation

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \qquad\qquad \text{for } i = 1, 2, \cdots, N}$$

In vector form:

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{q}}\right) - \frac{\partial T'}{\partial q} = Q$$

Example 163 *Single link manipulator (inverted pendulum).* Application of La Grange's equation yields:

$$J\ddot{\theta} - Wl\sin\theta = 0 \tag{16.1}$$

Example 164 *Particle undergoing central force motion.* Application of La Grange's equation yields

$$\begin{array}{ccccc} m\ddot{r} & - & mr\dot{\theta}^2 & -F & = 0 \\ mr^2\ddot{\theta} & + & 2mr\dot{r}\dot{\theta} & & = 0 \end{array}$$

The following result makes a statement about the structure of the differential equations obtained from the application of La Grange's equation.

Proposition 1 *Application of La Grange's equation yields*

$$\boxed{M(q)\ddot{q} + c(q,\dot{q}) - Q = 0} \tag{16.2}$$

where the i-th component of c is a quadratic function of $\dot{q}$ and is given by

$$c_i(q,\dot{q}) = \sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k$$

Proof. Since

$$T = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

it follows that

$$\frac{\partial T'}{\partial \dot{q}} = M(q)\dot{q}$$

Hence

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{q}}\right) = M\ddot{q} + \dot{M}\dot{q}$$

and application of La Grange's equation yields

$$M\ddot{q} + c - Q = 0$$

where

$$c = \dot{M}\dot{q} - \frac{\partial T'}{\partial q}\,. \tag{16.3}$$

Since

$$T = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} M_{jk}(q)\dot{q}_j\dot{q}_k\,,$$

it follows that

$$\frac{\partial T}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k \tag{16.4}$$

Noting that

$$(\dot{M}\dot{q})_i = \sum_{j=1}^{N} \dot{M}_{ij}\dot{q}_j = \sum_{j=1}^{N} \left(\sum_{k=1}^{N} \frac{\partial M_{ij}}{\partial q_k}\dot{q}_k \right) \dot{q}_j = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{\partial M_{ij}}{\partial q_k}\dot{q}_j\dot{q}_k \tag{16.5}$$

we have

$$c_i = (\dot{M}\dot{q})_i - \frac{\partial T}{\partial q_i} = \sum_{j=1}^{N}\sum_{k=1}^{N} \left(\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_i} \right) \dot{q}_j\dot{q}_k \qquad \blacksquare$$

16.3 Some fundamental properties

In this section, we present some fundamental properties of aerospace/mechanical systems which are useful in the analysis and control design for these systems.

16.3.1 An energy result

First we have the following result.

Proposition 2

$$\dot{q}'\left(c - \frac{1}{2}\dot{M}\dot{q}\right) = 0 \tag{16.6}$$

PROOF. It follows from (16.3) that

$$\dot{q}'\left(c - \frac{1}{2}\dot{M}\dot{q}\right) = \dot{q}'\left(\frac{1}{2}\dot{M}\dot{q} - \frac{\partial T}{\partial q}'\right) = \frac{1}{2}\dot{q}'\dot{M}\dot{q} - \frac{\partial T}{\partial q}\dot{q} \tag{16.7}$$

Using (16.4),

$$\frac{\partial T}{\partial q}\dot{q} = \sum_{i=1}^{N}\frac{\partial T}{\partial q_i}\dot{q}_i = \sum_{i=1}^{N}\left(\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k\right)\dot{q}_i = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_i\dot{q}_j\dot{q}_k \ . \tag{16.8}$$

Using (16.5),

$$\frac{1}{2}\dot{q}'\dot{M}\dot{q} = \frac{1}{2}\sum_{i=1}^{N}\dot{q}_i(\dot{M}\dot{q})_i = \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_j\dot{q}_k\right)\dot{q}_i = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j\dot{q}_k$$

By interchangining the indices i and k and using the symmetry of M, we obtain that

$$\frac{1}{2}\dot{q}'\dot{M}\dot{q} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_i\dot{q}_j\dot{q}_k \ . \tag{16.9}$$

Substituting (16.8) and (16.9) into (16.7) yields

$$\dot{q}'\left(c - \frac{1}{2}\dot{M}\dot{q}\right) = 0 \qquad \blacksquare$$

Now we have the main result of this section.

Theorem 37 *The time rate of change of the kinetic energy equals the power of the generalized forces, that is,*

$$\boxed{\frac{dT}{dt} = \dot{q}'Q} \tag{16.10}$$

PROOF. Since the kinetic energy is given by

$$T = \frac{1}{2}\dot{q}'M\dot{q}$$

application of (16.2) and (16.6) yields:

$$\begin{aligned}
\frac{dT}{dt} &= \dot{q}'M\ddot{q} + \frac{1}{2}\dot{q}'\dot{M}\dot{q} \\
&= \dot{q}'(-c+Q) + \frac{1}{2}\dot{q}'\dot{M}\dot{q} \;=\; \dot{q}'Q - \dot{q}'\left(c - \frac{1}{2}\dot{M}\dot{q}\right) \\
&= \dot{q}'Q\,.
\end{aligned}$$

■

16.3.2 Potential energy and total energy

A generalized force is said to be conservative if it satisfies the following two properties:

(a) It depends only on q.

(b) It can be expressed as the negative of the gradient of some scalar valued function of q. This function is called the potential energy associated with the force.

Example 165 Consider

$$Q = Wl\sin\theta \qquad \text{and} \qquad q = \theta$$

Here $Q = -\frac{\partial U}{\partial q}'$ where

$$U = Wl(\cos\theta - 1)\,.$$

Example 166 Consider

$$Q = \begin{pmatrix} F \\ 0 \end{pmatrix} \qquad q = \begin{pmatrix} r \\ \theta \end{pmatrix} \qquad F = -\frac{\mu m}{r^2}$$

Here $Q = -\frac{\partial U}{\partial q}'$ where

$$U = -\frac{\mu m}{r}\,.$$

Suppose one splits the generalized force vector Q into a conservative piece $-K$ and a 'non-conservative' piece $-D$, thus

$$Q = -K(q) - D(q, \dot{q}) \tag{16.11}$$

where $-K$ is the sum of the conservative forces and $-D$ is the sum of the 'non-conservative' forces. Let U be the total potential energy associated with the conservative forces, then

$$K = \frac{\partial U'}{\partial q} \,. \tag{16.12}$$

Introducing the Lagrangian

$$L = T - U \tag{16.13}$$

Lagrange's equation can now be written as

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) - \frac{\partial L'}{\partial q} + D = 0\,.$$

In particular, if all the forces are conservative, we have

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) - \frac{\partial L'}{\partial q} = 0\,.$$

Introducing the total mechanical energy

$$E = T + U \tag{16.14}$$

we obtain that

$$\boxed{\frac{dE}{dt} = -\dot{q}' D} \tag{16.15}$$

that is, *the time rate of change of the total energy equals the power of the non-conservative forces.* To see this:

$$\begin{aligned} \frac{dE}{dt} &= \frac{dT}{dt} + \frac{dU}{dt} \\ &= \dot{q}'Q + \frac{\partial U}{\partial q}\dot{q} \\ &= -\dot{q}'\frac{\partial U'}{\partial q} - \dot{q}'D(q,\dot{q}) + \frac{\partial U}{\partial q}\dot{q} \\ &= -\dot{q}'D(q,\dot{q})\,. \end{aligned}$$

In particular, if all the forces are conservative, we have

$$\boxed{\frac{dE}{dt} = 0} \tag{16.16}$$

thus E is constant, that is, energy is conserved.

16.3.3 A skew symmetry property

Proposition 3 *For each $q, \dot{q} \in \mathbb{R}^N$, there is a skew-symmetric matrix $S(q, \dot{q})$ such that*

$$c = \frac{1}{2}\left(\dot{M} - S\right)\dot{q} \tag{16.17}$$

Proof. Using (16.3),

$$c = \frac{1}{2}\dot{M}\dot{q} + \frac{1}{2}\dot{M}\dot{q} - \frac{\partial T'}{\partial q}$$

From (16.5),

$$(\dot{M}\dot{q})_i = \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_j\dot{q}_k = \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{ik}}{\partial q_j}\dot{q}_j\dot{q}_k$$

Thus, using (16.4),

$$\begin{aligned}\left(\frac{1}{2}\dot{M}\dot{q} - \frac{\partial T'}{\partial q}\right)_i &= \frac{1}{2}(\dot{M}\dot{q})_i - \frac{\partial T'}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k \\ &= -\sum_{j=1}^{N}\frac{1}{2}S_{ij}\dot{q}_j\end{aligned}$$

where

$$S_{ij} := \sum_{k=1}^{N}\left(\frac{\partial M_{jk}}{\partial q_i} - \frac{\partial M_{ik}}{\partial q_j}\right)\dot{q}_k \tag{16.18}$$

Hence $S_{ji} = -S_{ij}$,

$$\frac{1}{2}\dot{M}\dot{q} - \frac{\partial T'}{\partial q} = -\frac{1}{2}S\dot{q}$$

and

$$c = \frac{1}{2}\left(\dot{M} - S\right)\dot{q}$$

with S skew-symmetric. ∎

Corollary 3 *For each $q, \dot{q} \in \mathbb{R}^N$, there is a matrix $C(q, \dot{q})$ such that*

$$c = C\dot{q} \tag{16.19}$$

and $\dot{M} - 2C$ is skew-symmetric.

Proof. Let

$$C = \frac{1}{2}(\dot{M} - S) \tag{16.20}$$

where S is skew-symmetric as given by previous result. Then $c = C\dot{q}$ and

$$\dot{M} - 2C = S$$

∎

Remark 8 Substituting (16.19) into (16.2) yields

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} - Q = 0, \tag{16.21}$$

and using (16.20) to (16.18) we obtain

$$C_{ij} = \frac{1}{2}\sum_{k=1}^{N}\left(\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_k \quad . \tag{16.22}$$

16.4 Linearization of aerospace/mechanical systems

Here we consider aerospace/mechanical systems described by (16.2), that is,

$$M(q)\ddot{q} + c(q,\dot{q}) - Q(q,\dot{q}) = 0 \tag{16.23}$$

The equilibrium configurations $q(t) \equiv q^e$ of the above system are given by

$$Q(q^e, 0) = 0$$

Linearization about $q(t) \equiv q^e$ results in

$$\tilde{M}\delta\ddot{q} + \tilde{D}\delta\dot{q} + \tilde{K}\delta q = 0 \tag{16.24}$$

where

$$\tilde{M} = M(q^e), \qquad \tilde{D} = -\frac{\partial Q}{\partial \dot{q}}(q^e, 0), \qquad \tilde{K} = -\frac{\partial Q}{\partial q}(q^e, 0)$$

Note that the c term does not contribute anything to the linearization; the contribution of $M(q)\ddot{q}$ is simply $M(q^e)\delta\ddot{q}$.

We have now the following stability result.

Theorem 38 *Consider the linearization (16.24) of system (16.23) about $q(t) \equiv q^e$ and suppose that $\tilde{M}$ and $\tilde{K}$ are symmetric and positive definite. Suppose also that for all $\dot{q}$,*

$$\delta\dot{q}'\tilde{D}\delta\dot{q} \geq 0$$

and

$$\delta\dot{q}(t)'\tilde{D}\delta\dot{q}(t) \equiv 0 \Rightarrow \delta\dot{q}(t) \equiv 0$$

then the nonlinear system (16.23) is asymptotically stable about the solution $q(t) \equiv q^e$.

PROOF. As a candidate Lyapunov function for the linearized system (16.24), consider

$$V(\delta q, \delta\dot{q}) = \frac{1}{2}\delta\dot{q}'\tilde{M}\delta\dot{q} + \frac{1}{2}\delta q'\tilde{K}\delta q$$

■

Example 167

$$\begin{array}{lcccccc} m_1\ddot{q}_1 & + & d\dot{q}_1 & +(k_1+k_2)q_1 & -k_2q_2 & = & 0 \\ m_2\ddot{q}_2 & & & -\,k_2q_1 & +k_2q_2 & = & 0 \end{array}$$

This system is asymptotically stable.

Example 168

$$\begin{array}{lcccccc} m\ddot{q}_1 & + & d\dot{q}_1 - d\dot{q}_2 & +(k_1+k_2)q_1 & -k_2q_2 & = & 0 \\ m\ddot{q}_2 & - & d\dot{q}_1 + d\dot{q}_2 & -\,k_2q_1 & +(k_1+k_2)q_2 & = & 0 \end{array}$$

This system is not asymptotically stable.

Figure 16.2: An example

Figure 16.3: Another example

16.5 A class of GUAS aerospace/mechanical systems

Here we consider aerospace/mechanical systems described by

$$\boxed{M(q)\ddot{q} + c(q,\dot{q}) + D(q,\dot{q}) + K(q) = 0} \tag{16.25}$$

where $q(t)$ is a real N-vector and

$$M(q)' = M(q) > 0$$

for all q. So, here $Q = -D(q,\dot{q}) - K(q)$

Assumption 1 There is an N-vector q^e such that:

(i)

$$K(q^e) = 0 \qquad \text{and} \qquad K(q) \neq 0 \quad \text{for} \quad q \neq q^e$$

(ii)

$$D(q,0) = 0 \qquad \text{for all } q$$

This first assumption guarantees that $q(t) \equiv q^e$ is a unique equilibrium solution of the above system. The next assumption requires that the generalized force term $-K(q)$ be due to potential forces and that the total potential energy associated with these forces is a positive definite function.

Assumption 2 There is a function U, which is positive definite about q^e, such that

$$K(q) = \frac{\partial U'}{\partial q}(q)$$

for all q.

The next assumption requires that the generalized force term $-D(q, \dot{q})$ be due to damping forces and that these forces dissipate energy along every solution except constant solutions.

Assumption 3 For all $q, \dot{q}$,

$$\dot{q}' D(q, \dot{q}) \geq 0$$

If $q(\cdot)$ is any solution for which

$$\dot{q}(t)' D(q(t), \dot{q}(t)) \equiv 0$$

then

$$\dot{q}(t) \equiv 0$$

Theorem 39 *Suppose the mechanical system (16.25) satisfies assumptions 1-3. Then the system is GUAS about the equilibrium solution $q(t) \equiv q^e$.*

Proof Consider the total mechanical energy (kinetic + potential)

$$V = \frac{1}{2}\dot{q}' M(q)\dot{q} + U(q)$$

as a candidate Lyapunov function and use a LaSalle type result. ■

Example 169

$$\begin{array}{lcllllcl} I_1\ddot{q}_1 & + & d\dot{q}_1 & +(k_1+k_2)q_1 & -k_2 q_2 & & = & 0 \\ I_2\ddot{q}_2 & & & -\,k_2 q_1 & +k_2 q_2 & -Wl\sin(q_2) & = & 0 \end{array}$$

where all parameters are positive and

$$Wl < \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

Exercise 56 Consider the sprung pendulum subject to aerodynamic drag described by

$$I\ddot{\theta} + d|\dot{\theta}|\dot{\theta} + Wl\sin\theta + k\theta = 0$$

where $k > Wl$ and $d > 0$.

(i) Linearize this system about the zero solution. What are the stability properties of the linearized system? Based on this linearization, what can you say about the stability of the nonlinear system about the zero solution?

(ii) Use Theorem 39 to show that the nonlinear system is GUAS about the zero solution.

16.6 Control of aerospace/mechanical systems

Here we consider aerospace/mechanical systems with control inputs described by

$$\boxed{M(q)\ddot{q} + c(\dot{q}, q) + F(q, \dot{q}) = u} \tag{16.26}$$

where $q(t)$ and the control input $u(t)$ are real N-vectors. So, $Q = -F + u$. The salient feature of the systems considerd here is that the number of scalar control inputs is the same as the number of degrees of freedom of the system.

16.6.1 Computed torque method

This is a special case of the more general technique of feedback linearization. Suppose we let

$$u = c(q, \dot{q}) + F(q, \dot{q}) + M(q)v \tag{16.27}$$

where v is to be regarded as a new control input. Then, $M(q)\ddot{q} = M(q)v$ and since $M(q)$ is nonsingular, we obtain

$$\ddot{q} = v \tag{16.28}$$

This can also be expressed as a bunch of N decoupled double integrators:

$$\ddot{q}_i = v_i \qquad \text{for} \quad i = 1, 2, \cdots, N$$

Now use YFLTI (your favorite linear time invariant) control design method to design controllers for (16.28).

Disadvantages: Since this method requires exact cancellation of F and exact knowledge of M, it may not be robust with respect to uncertainty in M and F.

16.6.2 Linearization

Consider any desired controlled equilibrium configuration

$$q(t) \equiv q^e$$

for the system

$$M(q)\ddot{q} + c(\dot{q}, q) + F(q, \dot{q}) = u$$

This can be achieved with the constant input

$$u(t) \equiv u^e := F(q^e, 0)$$

If we linearize the above system about $(q(t), u(t)) \equiv (q^e, u^e)$, we obtain

$$\tilde{M}\delta\ddot{q} + \tilde{D}\delta\dot{q} + \tilde{K}\delta q = \delta u \tag{16.29}$$

where

$$\tilde{M} = M(q^e), \qquad \tilde{D} = \frac{\partial F}{\partial \dot{q}}(q^e, 0), \qquad \tilde{K} = \frac{\partial F}{\partial q}(q^e, 0)$$

Now use YFLTI control design method to design controllers for δu and let

$$u = u^e + \delta u \tag{16.30}$$

16.6.3 Globally stabilizing PD controllers

Here we consider mechanical systems described by

$$\boxed{M(q)\ddot{q} + c(\dot{q}, q) + D(q, \dot{q}) + K(q) = u} \tag{16.31}$$

So, $F(q, \dot{q}) = D(q, \dot{q}) + K(q)$.

Assumption 4 There is an N-vector q^e uch that

$$K(q^e) = 0 \qquad \text{and} \qquad D(q, 0) = 0 \quad \text{for all } q$$

This first assumption guarantees that $q(t) \equiv q^e$ is an equilibrium solution of the uncontrolled ($u = 0$) system. It does not rule out the possibility of other equilibrium solutions.

Assumption 5 There is a function U and a scalar β such that for all q

$$K(q) = \frac{\partial U'}{\partial q}(q)$$

and

$$\frac{\partial K}{\partial q}(q) \geq -\beta I$$

Assumption 6 For all q and $\dot{q}$,

$$\dot{q}' D(q, \dot{q}) \geq 0$$

PD controllers

$$\boxed{u = -K_p(q - q^e) - K_d\dot{q}} \tag{16.32a}$$

with

$$\boxed{K_p' = K_p > \beta I\,, \qquad K_d + K_d' > 0} \tag{16.32b}$$

Theorem 40 *Consider a system described by (16.31) which satisies assumptions 4-6 and is subject to PD controller (16.32). Then the resulting closed system is GUAS about $q(t) \equiv q^e$.*

Proof. We show that the closed loop system

$$M(q)\ddot{q} + c(q, \dot{q}) + D(q, \dot{q}) + K_d\dot{q} + K(q) + K_p(q - q^e) = 0 \tag{16.33}$$

satisfies the hypotheses of Theorem 39. ■

Example 170 *Single link manipulator with drag.*

$$I\ddot{\theta} + d\dot{\theta}|\dot{\theta}| - Wl\sin\theta = u$$

with $d \geq 0$. Here $q = \theta$,

$$K(\theta) = -Wl\sin\theta \qquad \text{and} \qquad D(\theta, \dot{\theta}) = d\dot{\theta}|\dot{\theta}|$$

Hence

$$\frac{\partial K}{\partial q}(q) = -Wl\cos q \geq -Wl$$

So $\beta = Wl$ and a globally stabilizing controller is given by

$$u = -k_p q - k_d \dot{q}$$

$$k_p > Wl\,, \qquad k_d > 0$$

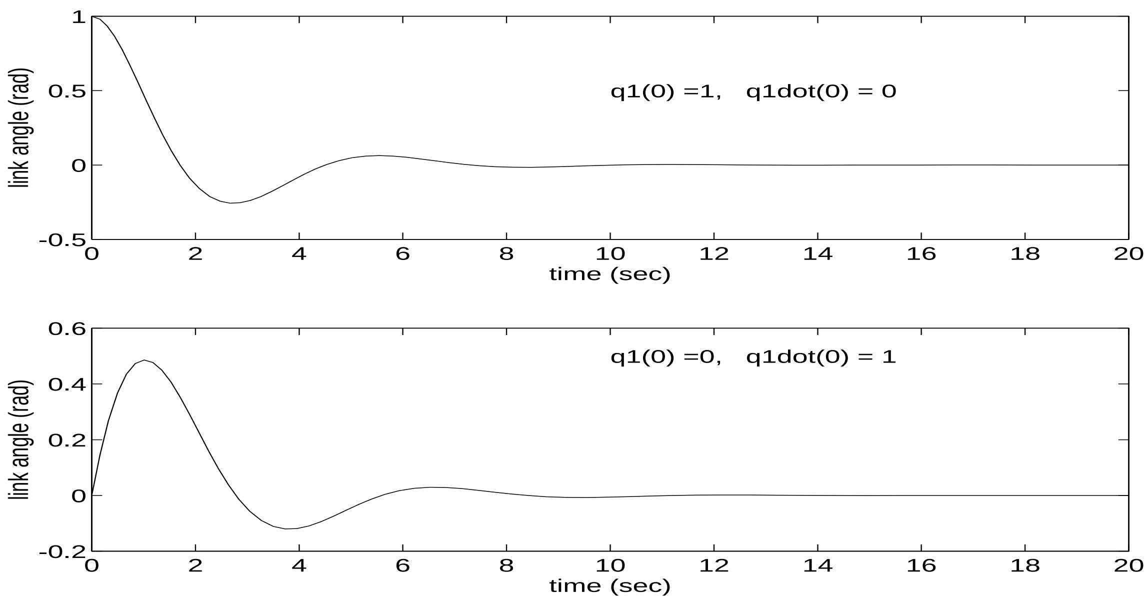

Figure 16.4: Response of closed-loop, rigid, single-link manipulator

Exercise 57 Consider the two link manipulator with two control input torques, u_1 and u_2:

$$[m_1 lc_1^2 + m_2 l_1^2 + I_1]\ddot{q}_1 \quad +[m_2 l_1 lc_2 \cos(q_1 - q_2)]\ddot{q}_2 \quad +m_2 l_1 lc_2 \sin(q_1 - q_2)\dot{q}_2^2 - [m_1 lc_1 + m_2 l_1] g \sin(q_1) \quad = u_1$$

$$[m_2 l_1 lc_2 \cos(q_1 - q_2)]\ddot{q}_1 \quad +[m_2 lc_2^2 + I_2]\ddot{q}_2 \quad -m_2 l_1 lc_2 \sin(q_1 - q_2)\dot{q}_1^2 - m_2 g lc_2 \sin(q_2) \quad = u_2$$

Using each of the 3 control design methods of this section, obtain controllers which asymptotically stabilize this system about the upward vertical configuration. Design your controllers using the following data:

m_1	l_1	lc_1	I_1	m_2	l_2	lc_2	I_2
kg	m	m	$kg \cdot m^2$	kg	m	m	$kg \cdot m^2$
10	1	0.5	10/12	10	1	0.5	5/12

Table 16.1: Parameters of the two-link manipulator

Simulate the closed loop system with each of your three controller and for the following values of m_2: 5 kg and 10 kg.

Chapter 17

Quadratic stabilizability

17.1 Preliminaries

Here we consider the problem of designing stabilizing state feedback controllers for specific classes of nonlinear systems. Controller design is based on quadratic Lyapunov functions.

First recall that a system

$$\dot{x} = f(t, x) \tag{17.1}$$

is quadratically stable if there exist positive definite symmetric matrices P and Q which satisfy

$$2x'Pf(t, x) \leq -2x'Qx \tag{17.2}$$

for all t and x. When this is the case, the system is globally uniformly exponentially stable (GUES) about the origin and the scalar $\alpha = \lambda_{\min}(P^{-1}Q)$ is a rate of exponential convergence. Also, we call P a Lyapunov matrix for (17.1).

Consider now a system with control input u where $u(t)$ is an m-vector:

$$\dot{x} = F(t, x, u)\,. \tag{17.3}$$

Suppose u is given by a static linear state feedback controller

$$u = Kx\,; \tag{17.4}$$

then the resulting closed loop system is described by

$$\dot{x} = F(t, x, Kx)\,. \tag{17.5}$$

We refer to K as a state feedback gain matrix.

We say that system (17.3) is quadratically stabilizable via linear state feedback if there exists a matrix K such that the closed loop system (17.5) is quadratically stable, that is, exist a matrix K and positive definite symmetric matrices P and Q which satisfy

$$2x'PF(t, x, Kx) \leq -2x'Qx \tag{17.6}$$

for all t and x.

If we let $\tilde{x} = Px$, $S = P^{-1}$ and $L = KP^{-1}$ and $\tilde{Q} = P^{-1}QP^{-1}$, the above requirement can we expressed as:

$$2\tilde{x}'F(t, S\tilde{x}, L\tilde{x}) \leq -2\tilde{x}'\tilde{Q}\tilde{x} \tag{17.7}$$

for all t and $\tilde{x}$. Then

$$P = S^{-1} \qquad K = LS^{-1}\,. \tag{17.8}$$

17.2 Some special classes of nonlinear systems

Here we consider nonlinear systems with control inputs described by

$$\boxed{\dot{x} = A(t, x)x + B(t, x)u} \tag{17.9}$$

where the real n-vector $x(t)$ is the state and the real m-vector $u(t)$ is the control input. The following observation is sometimes useful in reducing control design problems to the solution of linear matrix inequalities (LMIs).

• *Suppose there exist a matrix L and positive definite symmetric matrices S and $\tilde{Q}$ which satisfy*

$$\boxed{A(t, x)S + B(t, x)L + SA(t, x)' + L'B(t, x)' \leq -\tilde{Q}} \tag{17.10}$$

for all t and x. Then the linear state feedback controller

$$\boxed{u = LS^{-1}x} \tag{17.11}$$

renders system (17.9) GUES about the origin with Lyapunov matrix $P = S^{-1}$.

To see this, suppose (17.10) holds and let $P = S^{-1}$ and $K = LS^{-1}$. Pre- and post-multiply inequality (17.10) by S^{-1} to obtain

$$P[A(t, x) + B(t, x)K] + [A(t, x) + B(t, x)K]'P \leq -P\tilde{Q}P \tag{17.12}$$

Now notice that the closed loop system resulting from controller (17.11) is described by

$$\dot{x} = [A(t, x) + B(t, x)K]x \tag{17.13}$$

that is, it is described by (17.1) with $f(t, x) = [A(t, x) + B(t, x)K]x$. Thus, using inequality (17.12), the closed loop system satisfies

$$\begin{aligned}
2x'Pf(t, x) &= 2x'P[A(t, x) + B(t, x)K]x \\
&= x'P[A(t, x) + B(t, x)K] + [A(t, x) + B(t, x)K]'Px \\
&\leq -x'P\tilde{Q}Px\,.
\end{aligned}$$

Thus, the closed loop system satisfies inequality (17.6) with $P\tilde{Q}P$ replacing Q. Therefore, the closed loop system (17.13) is GUES with Lyapunov matrix $P = S^{-1}$.

17.2.1 A special class of nonlinear systems

We first consider systems described by (17.9) where the time/state dependent matrices $A(t,x)$ and $B(t,x)$ have the following structure:

$$\boxed{A(t,x) = A_0 + \psi(t,x)\Delta A\,, \qquad B(t,x) = B_0 + \psi(t,x)\Delta B} \tag{17.14}$$

The matrices A_0, B_0 and $\Delta A, \Delta B$ are constant and ψ is a scalar valued function of time t the state x which is bounded above and below, that is,

$$\boxed{a \leq \psi(t,x) \leq b} \tag{17.15}$$

for some constants a and b. Examples of such functions are given by $\sin x, \cos x, e^{-x^2}$ and $\sin t$.

Example 171 (Single-link manipulator) A single-link manipulator (or inverted pendulum) subject to a control torque u is described by

$$J\ddot{\theta} - Wl\sin\theta = u$$

with $J,\, Wl > 0$. Letting $x_1 = \theta$ and $x_2 = \dot{\theta}$ this system has the following state space description:

$$\dot{x}_1 = x_2 \tag{17.16a}$$
$$J\dot{x}_2 = Wl\sin x_1 + u \tag{17.16b}$$

This description is given by (17.9) and (17.14) with

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 \\ 1/J \end{pmatrix}, \qquad \Delta A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \Delta B = 0$$

and

$$\psi(t,x) = \begin{cases} (Wl/J)\sin x_1/x_1 & \text{if} \quad x_1 \neq 0 \\ (Wl/J) & \text{if} \quad x_1 = 0 \end{cases}$$

Since $|\sin x_1| \leq |x_1|$, we have

$$-Wl/J \leq \psi(t,x) \leq Wl/J;$$

hence ψ satisfies inequalities (17.15) with $a = -Wl/J$ and $b = Wl/J$.

The following theorem yields controllers for the global exponential stabilization of systems described by (17.9) and (17.14) and satisfying (17.15).

Theorem 41 *Consider a system described by (17.9) and (17.14) and satisfying (17.15). Suppose there exist a matrix L and a positive-definite symmetric matrix S which satisfy the following linear matrix inequalities:*

$$A_1S + B_1L + SA_1' + L'B_1 < 0 \tag{17.17a}$$
$$A_2S + B_2L + SA_2' + L'B_2 < 0 \tag{17.17b}$$

with

$$A_1 := A_0 + a\Delta A\,, \qquad B_1 := B_0 + a\Delta B \tag{17.18a}$$
$$A_2 := A_0 + b\Delta A\,, \qquad B_2 := B_0 + b\Delta B\,. \tag{17.18b}$$

Then the controller given by (17.11) renders system (17.9),(17.14) globally uniformly exponentially stable about the origin with Lyapunov matrix S^{-1}.

PROOF. Suppose S and L satisfy inequalities (17.17) and let $\tilde{Q}$ be any positive definite matrix which satisfies

$$A_1S + B_1L + SA_1' + L'B_1 \le -\tilde{Q} \tag{17.19a}$$
$$A_2S + B_2L + SA_2' + L'B_2 \le -\tilde{Q}\,. \tag{17.19b}$$

For each t and x,

$$\begin{aligned} N(t,x) &:= A(t,x)S + B(t,x)L + SA(t,x)' + L'B(t,x)' \\ &= N_0 + \psi(t,x)\Delta N \end{aligned}$$

where

$$N_0 = A_0S + B_0L + SA_0' + L'B_0' \qquad \text{and} \qquad \Delta N = \Delta AS + \Delta BL + S\Delta A' + L'\Delta B'$$

Since $N(t,x) = N_0 + \psi(t,x)\Delta N$ and $a \le \psi(t,x) \le b$, it follows that

$$N(t,x) \le -\tilde{Q}$$

for all t and x if

$$N_0 + a\Delta N \le -\tilde{Q} \qquad \text{and} \qquad N_0 + b\Delta N \le -\tilde{Q}$$

These last two inequalities are precisely inequalities (17.19). ■

Exercise 58 Prove the following result: Consider a system described by (17.9),(17.14) and satisfying (17.15). Suppose there exists a positive-definite symmetric matrix S, a matrix L and a positive scalar α which satisfy

$$A_1S + B_1L + SA_1' + L'B_1 + 2\alpha S \le 0 \tag{17.20a}$$
$$A_2S + B_2L + SA_2' + L'B_2 + 2\alpha S \le 0 \tag{17.20b}$$

where A_1, B_1, A_2, B_2 are given by (17.18). Then controller (17.11) renders system (17.9), (17.14), GUES about the origin with rate of convergence α and Lyapunov matrix $S = P^{-1}$.

Note that the existence of a positive definite symmetric S and a matrix L satisfying inequalities (17.17) or (17.20) is equivalent to the existence of another symmetric matrix S and another matrix L satisfying $S \ge I$ and (17.17) or (17.20), respectively.

Suppose $B_1 = B_2$ and $(S,\, L)$ is a solution to (17.17) or (17.20), then $(S,\, L-\gamma B_1')$ is also a solution for all $\gamma \ge 0$. Since the feedback gain matrix corresponding to the latter pair is

$K = L_0S^{-1} - \gamma B_1'S^{-1}$, this results in large gain matrices for large γ. So how do we restrict the size of K? One approach is to minimize β subject to

$$KSK' \leq \beta I \tag{17.21}$$

If we also require $I \leq S$, we obtain that $KK' \leq KSK' \leq \beta I$. Thus, β is an upper bound on $||K||^2$. If we minimize β subject to inequalities (17.17), the resulting controller may yield an unacceptably small rate of convergence. To avoid this, we can choose a specific desired rate of convergence $\alpha > 0$ and minimize β subject to (17.20). Finally, note that inequality (17.21) can be written as $LS^{-1}L' \leq \beta I$, or,

$$\begin{pmatrix} \beta I & L \\ L' & S \end{pmatrix} \geq 0$$

To summarize, we can globally stabilize system (17.9),(17.14),(17.15) with rate of convergence α by solving the following optimization problem:

$$\boxed{\begin{array}{l} \text{Minimize } \beta \text{ subject to} \\ \\ \begin{array}{rcl} A_1S + B_1L + SA_1' + L'B_1 + 2\alpha S & \leq & 0 \\ A_2S + B_2L + SA_2' + L'B_2 + 2\alpha S & \leq & 0 \\ I & \leq & S \\ \begin{pmatrix} \beta I & L' \\ L & S \end{pmatrix} & \geq & 0 \end{array} \end{array}} \tag{17.22}$$

and letting $u = LS^{-1}x$.

Example 172 Recall the single-link manipulator of Example 171. For $J = Wl = 1$, the following Matlab program uses the results of this section and the LMI toolbox to obtain stabilizing controllers.

```
% Quadratic stabilization of  single-link manipulator
%
clear all
% Data and specs
alpha = 0.1;                        %The specified rate of convergence
J = 1;
Wl = 1;
%
% Description of manipulator
A0 = [0 1;  0 0];
DelA = [0 0; 1 0];
B0 = [0; 1/J];
DelB = 0;
a= -Wl/J
b= Wl/J
```

```
A1  = A0 + a*DelA;
A2  = A0 + b*DelA;
B1  = B0 + a*DelB;
B2  = B0 + b*DelB;
%
% Form the system of LMIs
setlmis([])
%
S=lmivar(1, [2,1]);
L=lmivar(2, [1,2]);
beta=lmivar(1, [1,1]);
%
lmi1=newlmi;
lmiterm([lmi1,1,1,S], A1,1,'s')         %A1*S + S*A1'
lmiterm([lmi1,1,1,L], B1,1,'s')         %B1*L + L'*B1'
lmiterm([lmi1,1,1,S], 2*alpha, 1)       %2*alpha*S
%
lmi2=newlmi;
lmiterm([lmi2,1,1,S], A2,1,'s')         %A2*S + S*A2'
lmiterm([lmi2,1,1,L], B2,1,'s')         %B2*L + L'*B2'
lmiterm([lmi2,1,1,S], 2*alpha, 1)       %2*alpha*S
%
Slmi= newlmi;
lmiterm([-Slmi,1,1,S],1,1)
lmiterm([Slmi,1,1,0],1)
%
lmi4=newlmi;
lmiterm([-lmi4,1,1,beta],1,1)           %beta*I
lmiterm([-lmi4,1,2,L],1,1)              %L
lmiterm([-lmi4,2,2,S],1,1)              %S
lmis = getlmis;
%
c=mat2dec(lmis,0,0,1);                   %specify weighting
options=[1e-5 0 0 0 0];                  %Minimize
[copt,xopt]=mincx(lmis,c,options)
%
S = dec2mat(lmis,xopt,S)
L = dec2mat(lmis,xopt,L)
%
K =L*inv(S)                              %Controller gain matrix
```

Running this program yields:

$$
\begin{aligned}
K &= \begin{pmatrix} -1.3036 & -1.2479 \end{pmatrix} \qquad \text{for} \qquad \alpha = 0.1 \\
K &= \begin{pmatrix} -3.6578 & -2.8144 \end{pmatrix} \qquad \text{for} \qquad \alpha = 1
\end{aligned}
$$

$$K = \begin{pmatrix} -152.9481 & -20.1673 \end{pmatrix} \qquad \text{for} \qquad \alpha = 10$$

For initial conditions

$$\theta(0) = \pi \qquad\qquad \dot{\theta}(0) = 0\,,$$

Figure 17.1 illustrates the closed loop behavior of θ corresponding to $\alpha = 1$ and $\alpha = 10$.

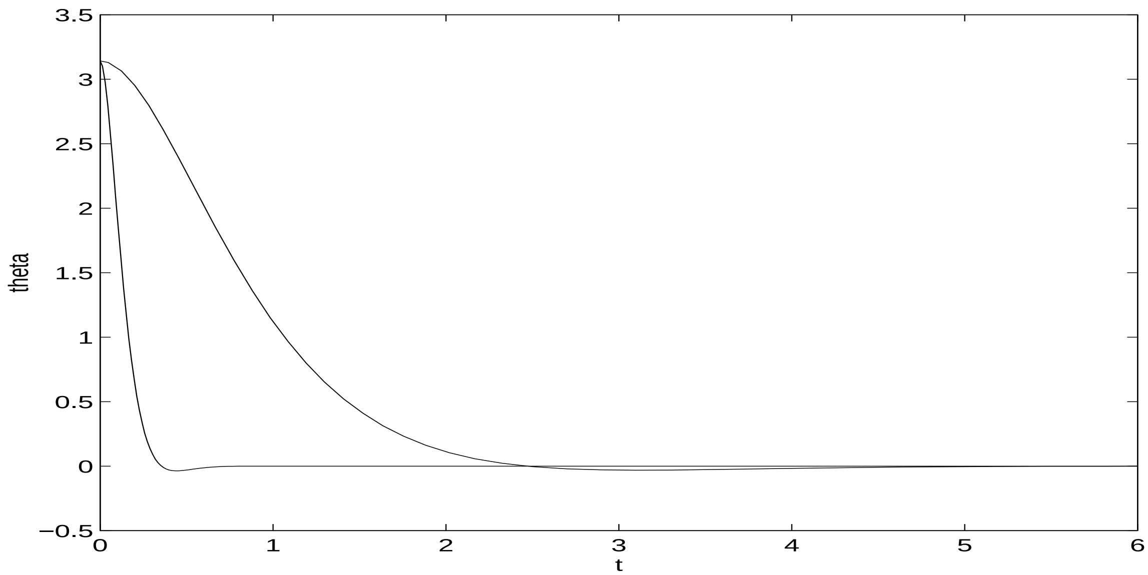

Figure 17.1: Closed loop single-link manipulator

Exercise 59 (PID regulation of single-link manipulator) Consider a single-link manipulator subject to a unknown constant disturbance torque w:

$$J\ddot{\theta} - Wl\sin\theta = w + u\,.$$

Suppose we wish to stabilize this system about some desired constant motion $\theta(t) \equiv \theta_d$. The corresponding steady state value of u is unknown and is given by $u_d = -Wl\sin\theta_d - w$. Consider now the following PID controller:

$$u = -k_p(\theta - \theta_d) - k_I \int_0^t (\theta - \theta_d)\,d\tau - k_d\dot{\theta} \tag{17.23}$$

The resulting closed loop system is described by

$$\begin{aligned} J\ddot{\theta} + k_d\dot{\theta} + k_p(\theta - \theta_d) - Wl\sin\theta - w + k_I\eta &= 0 \\ \dot{\eta} &= \theta - \theta_d \end{aligned}$$

The equilibrium solutions $\theta(t) \equiv \theta_d, \eta(t) \equiv \eta_d$ of this system satisfy

$$\begin{aligned} \theta &= \theta_d \\ u_d + k_I\eta &= 0 \end{aligned}$$

So, when $u_d \neq 0$, one needs $k_I \neq 0$ and the equilibrium η_d value of η is given by $\eta_d = -u_d/k_I$. Introduce states

$$\begin{aligned} x_1 &= \theta - \theta_d \\ x_2 &= \dot{\theta} \\ x_3 &= \eta - \eta_d \end{aligned}$$

to obtain the following state space description:

$$\dot{x}_1 = x_2 \tag{17.24a}$$
$$J\dot{x}_2 = -k_p x_1 - k_d x_2 - k_I x_3 + Wl[\sin(\theta_d + x_1) - \sin(\theta_d)] \tag{17.24b}$$
$$\dot{x}_3 = x_1 \tag{17.24c}$$

If one now obtains a matrix

$$K = \begin{pmatrix} -k_p & -k_d & -k_I \end{pmatrix}$$

so that system (17.24) is GUES about the origin, then the PID controller (17.23) will guarantee that the angle θ of the manipulator will exponentially approach the desired value θ_d and all other variables will be bounded. Note that a stabilizing K will have $k_I \neq 0$.

Using Theorem 41 and the LMI toolbox, obtain a stabilizing K for $J = Wl = 1$. Simulate the resulting closed loop system for several initial conditions and values of θ_d and w.

17.2.2 Generalization

One can readily generalize the results of this section to systems described by (17.9) where the time/state dependent matrices $A(t,x)$ and $B(t,x)$ have the following structure

$$A(t,x) = A_0 + \psi_1(t,x)\Delta A_1 + \cdots + \psi_l(t,x)\Delta A_l \tag{17.25a}$$
$$B(t,x) = B_0 + \psi_1(t,x)\Delta B_1 + \cdots + \psi_l(t,x)\Delta B_l \tag{17.25b}$$

Here, $A_0, \Delta A_1, \cdots, \Delta A_l$ are constant $n \times n$ matrices and $B_0, \Delta B_1, \cdots, \Delta B_l$ are constant $n \times m$ matrices. Also, each ψ_i is a scalar valued function of the time t and state x which satisfies

$$a_i \leq \psi_i(t,x) \leq b_i \tag{17.26}$$

for some known constants a_i and b_i.

Example 173 (Robust control of single-link manipulator) Recall from Example 171 that the motion of a single-link manipulator (or inverted pendulum) subject to a control torque u can be described by

$$\dot{x}_1 = x_2 \tag{17.27a}$$
$$\dot{x}_2 = (Wl/J)\sin x_1 + (1/J)u \tag{17.27b}$$

with J, $Wl > 0$. Suppose that we do not know the parameters J and Wl exactly and only have knowledge on their upper and lower bounds, that is,

$$0 < \underline{J} \leq J \leq \overline{J} \qquad \text{and} \qquad \underline{Wl} \leq Wl \leq \overline{Wl}$$

where the bounds $\underline{J}$, $\overline{J}$, $\underline{Wl}$, $\overline{Wl}$ are known. Introducing

$$\psi_1(t,x) = \begin{cases} (Wl/J)\sin x_1/x_1 & \text{if} \quad x_1 \neq 0 \\ (Wl/J) & \text{if} \quad x_1 = 0 \end{cases} \qquad \text{and} \qquad \psi_2(t,x) = 1/J\,,$$

the above state space description is given by (17.9) and (17.25) with $l = 2$ and

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Delta A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Delta B_1 = 0\,, \quad \Delta A_2 = 0\,, \quad \Delta B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $|\sin x_1| \leq |x_1|$, we have

$$\begin{aligned} -\overline{Wl}/\underline{J} \leq \quad & \psi_1(t,x) & \leq \overline{Wl}/\underline{J} \\ 1/\overline{J} \leq \quad & \psi_2(t,x) & \leq 1/\underline{J}\,; \end{aligned}$$

hence ψ_1 and ψ_2 satisfy inequalities (17.26) with $a_1 = -\overline{Wl}/\underline{J}$, $b_1 = \overline{Wl}/\underline{J}$ and $a_2 = 1/\overline{J}$, $b_2 = 1/\underline{J}$.

To obtain a sufficient condition for a system given by (17.9), (17.25) and (17.26), we introduce the following set of 2^l "extreme matrix pairs":

$$\mathcal{AB} = \{\; (A_0 + \delta_1\Delta A_1 + \cdots + \delta_l \Delta A_l, \quad B_0 + \delta_1 \Delta B_1 + \cdots + \delta_l \Delta B_l) \;:\; \delta_i = a_i \text{ or } b_i \text{ for } i = 1, \cdots, l\}$$

For example, with $l = 2$, $\mathcal{AB}$ consists of the following four matrix pairs:

$$\begin{array}{c} (A_0 + a_1\Delta A_1 + a_2\Delta A_2, \quad B_0 + a_1\Delta B_1 + a_2\Delta B_2) \\ (A_0 + a_1\Delta A_1 + b_2\Delta A_2, \quad B_0 + a_1\Delta B_1 + b_2\Delta B_2) \\ (A_0 + b_1\Delta A_1 + a_2\Delta A_2, \quad B_0 + b_1\Delta B_1 + a_2\Delta B_2) \\ (A_0 + b_1\Delta A_1 + b_2\Delta A_2, \quad B_0 + b_1\Delta B_1 + b_2\Delta B_2) \end{array}$$

We have now the following result.

Theorem 42 *Consider a system described by (17.9),(17.25) and satisfying (17.26). Suppose there exist a matrix L and a positive-definite symmetric matrix S which satisfy*

$$AS + BL + SA' + L'B' < 0 \tag{17.28}$$

for every pair (A, B) in $\mathcal{AB}$. Then the controller given by (17.11) renders system (17.9),(17.25) globally uniformly exponentially stable about the origin with Lyapunov matrix S^{-1}.

Recalling the discussion in the previous section, we can globally stabilize system (17.9),(17.14),(17.26) with rate of convergence α by solving the following optimization problem:

$$\boxed{\begin{array}{l} \text{Minimize } \beta \text{ subject to} \\ \begin{array}{rcll} AS + BL + SA' + L'B' + 2\alpha S & \leq & 0 & \quad \text{for all } (A,B) \text{ in } \mathcal{AB} \\ S & \geq & I & \\ \begin{pmatrix} \beta I & L' \\ L & S \end{pmatrix} & \geq & 0 & \end{array} \end{array}} \tag{17.29}$$

and letting $u = LS^{-1}x$.

Exercise 60 Consider the double inverted pendulum described by

$$\begin{aligned}\ddot{\theta}_1 + k\theta_1 - k\theta_2 - \sin\theta_1 &= u \\ \ddot{\theta}_2 - k\theta_1 + k\theta_2 - \sin\theta_2 &= 0\end{aligned}$$

with $k = 1$. Using the above results, obtain a state feedback controller which renders this system globally exponentially stable about the zero solution.

17.3 Another special class of nonlinear systems

Here we consider nonlinear systems which have the following structure:

$$\boxed{\dot{x} = Ax + B_1\phi(t, Cx + Du) + B_2 u} \tag{17.30}$$

where the real n-vector $x(t)$ is the state and the real m_2-vector $u(t)$ is the control input. The matrices B_1, B_2 and C are constant with dimensions $n \times m_1$, $n \times m_2$, and $p \times n$, respectively. The function $\phi : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^{m_1}$ satisfies

$$\boxed{||\phi(t, z)|| \leq \gamma ||z||} \tag{17.31}$$

Note that if we introduce a fictitious output $z = Cx + Du$ and a fictitious input $w = \phi(t, Cx + Du)$, the nonlinear system (17.30) can be described as a feedback combination of a linear time invariant system

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u \\ z &= Cx + Du\end{aligned}$$

and a memoryless nonlinearity

$$w = \phi(t, z)$$

This is illustrated in Figure 17.2.

Figure 17.2: A feedback system

Example 174 *Single link manipulator.* Recall the single-link of Example 171 described by (17.16). This system can be described by the above general description where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 0 \\ Wl/J \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 \\ 1/J \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad D = 0$$

and $\phi(t, z) = \sin z$. Also,

$$||\phi(t, z)|| = |\sin z| \leq |z| = ||z||$$

The following theorem yields controllers for for the global exponential stabilization of system (17.30).

Theorem 43 *Consider a system described by (17.30) and satisfying (17.31). Suppose there exist a matrix L, a positive-definite symmetric matrix S and a positive scalar μ which satisfy the following matrix inequality :*

$$AS + B_2L + SA' + L'B_2' + \mu\gamma^2 B_1B_1' + \mu^{-1}(CS + DL)'(CS + DL) < 0 \tag{17.32}$$

Then the controller given by (17.11) renders system (17.30) globally uniformly exponentially stable about the origin with Lyapunov matrix S^{-1}.

PROOF. Consider any K and $u = Kx$. Then

$$\dot{x} = (A + B_2K)x + B_1\phi(t, (C + DK)x)$$

Recalling quadratic stability results, this system is GUES with Lyapunov matrix P if there exists $\mu > 0$ such that

$$P(A + B_2K) + (A + B_2K)'P + \mu\gamma^2 PB_1B_1'P + \mu^{-1}(C + DK)'(C + DK) < 0 = 0$$

Letting $S = P^{-1}$ and $L = KS$, this is equivalent to

$$AS + B_2L + SA' + L'B_2' + \mu\gamma^2 B_1B_1' + \mu^{-1}(CS + DL)'(CS + DL) < 0$$

■

Exercise 61 Consider a system described by (17.30) and satisfying (17.31). Suppose there exist a matrix L, a positive-definite symmetric matrix S and a positive scalar μ which satisfy

$$AS + B_2L + SA' + L'B_2' + \mu\gamma^2 B_1B_1' + 2\alpha S + \mu^{-1}(CS + DL)'(CS + DL) \leq 0 \tag{17.33}$$

Then the controller given by (17.11) renders system (17.30) globally uniformly exponentially stable about the origin with rate of convergence α and Lyapunov matrix S^{-1}

Using a Schur complement result, one can show that satisfaction of quadratic matrix inequality (17.32) is equivalent to satisfaction of the following matrix inequality :

$$\begin{pmatrix} AS + B_2L + SA' + L'B_2' + \mu\gamma^2 B_1B_1' & (CS + DL)' \\ CS + DL & -\mu I \end{pmatrix} < 0 \tag{17.34}$$

Note that this inequality is linear in S, L, and μ. Recalling the discussion in Section 17.2, we can globally stabilize system (17.30) with rate of convergence α by solving the following optimization problem:

$$\boxed{\begin{array}{c} \text{Minimize } \beta \text{ subject to} \\[1em] \begin{array}{rcl} \begin{pmatrix} AS + B_2L + SA' + L'B_2' + \mu\gamma^2 B_1B_1' + 2\alpha S & (CS + DL)' \\ CS + DL & -\mu I \end{pmatrix} & \leq & 0 \\[1em] S & \geq & I \\[1em] \begin{pmatrix} \beta I & L' \\ L & S \end{pmatrix} & \geq & 0 \end{array} \end{array}} \tag{17.35}$$

and letting $u = LS^{-1}x$.

17.3.1 Generalization

Here we consider systems whose nonlinearity is characterized by several nonlinear functions, $\phi_i : \mathbb{R}^{p_i} \to \mathbb{R}^{m_i}, \ i = 1, 2, \ldots, l$:

$$\dot{x} = Ax + B_{11}\phi_1(t, C_1x + D_1u) + \ldots + B_{1l}\phi_l(t, C_lx + D_1u) + B_2u \tag{17.36}$$

where A, B_2 and B_{1i}, C_i, D_i $(i = 1, 2, \cdots, l)$ are constant matrices of appropriate dimensions. Each ϕ_i satisfies

$$||\phi_i(t, z_i)|| \leq \gamma ||z_i|| \tag{17.37}$$

If we introduce fictitious outputs $z_i = C_ix + D_iu$ and fictitious inputs $w_i = \phi_i(t, C_ix + D_iu)$, the above nonlinear system can be described as a feedback combination of a linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + B_{11}w_1 + \ldots + B_{1l}w_l + B_2u \\ z_1 &= C_1x + D_1u \\ &\vdots \\ z_l &= C_lx + D_lu \end{aligned}$$

and multiple memoryless nonlinearities:

$$w_i = \phi_i(t, z_i) \qquad \text{for} \quad i = 1, 2, \cdots, l$$

This is illustrated in Figure 17.3.

Figure 17.3: A multi-loop feedback system

One can readily obtain the following result:

Theorem 44 *Consider a system described by (17.36) and satisfying (17.37). Suppose there exist a matrix L, a positive-definite symmetric matrix S and l positive scalars $\mu_1, \mu_2, \ldots, \mu_l$ which satisfy the following matrix inequality :*

$$AS + B_2L + SA' + L'B_2' + \sum_{i=1}^{l} \left(\mu_i \gamma^2 B_{1i} {B_{1i}}' + \mu_i^{-1} (C_iS + D_iL)'(C_iS + D_iL) \right) < 0 \tag{17.38}$$

Then the controller given by (17.11) renders system (17.36) globally uniformly exponentially stable about the origin with Lyapunov matrix S^{-1}.

PROOF. Similar to proof of last theorem. ∎

Note that, using a Schur complement result, inequality (17.38) is equivalent to the following inequality which is linear in L, P and the scaling parameters $\mu_1, \cdots, \mu_l$:

$$\begin{pmatrix} AS + B_2L + SA' + L'B_2' + \sum_{i=1}^{l} \mu_i \gamma^2 B_{1i} B_{1i}' & (C_1S + D_1L)' & (C_2S + D_2L)' & \cdots & (C_lS + D_lL)' \\ C_1S + D_1L & -\mu_1 I & 0 & \cdots & 0 \\ C_2S + D_2L & 0 & -\mu_2 I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_lS + D_lL & 0 & 0 & \cdots & -\mu_l I \end{pmatrix} < 0$$

Recalling previous discussions, we can globally stabilize system (17.36) with rate of convergence α by solving the following optimization problem:

$$\boxed{\begin{array}{c} \text{Minimize } \beta \text{ subject to} \\ \begin{pmatrix} AS + B_2L + SA' + L'B_2' + \sum_{i=1}^{l} \mu_i \gamma^2 B_{1i} B_{1i}' + 2\alpha S & (C_1S + D_1L)' & (C_2S + D_2L)' & \cdots & (C_lS + D_lL)' \\ C_1S + D_1L & -\mu_1 I & 0 & \cdots & 0 \\ C_2S + D_2L & 0 & -\mu_2 I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_lS + D_lL & 0 & 0 & \cdots & -\mu_l I \end{pmatrix} \leq 0 \\ S \geq I \\ \begin{pmatrix} \beta I & L' \\ L & S \end{pmatrix} \geq 0 \end{array}} \tag{17.39}$$

and letting $u = LS^{-1}x$.

17.4 Yet another special class

$$\dot{x} = Ax + B_1\phi(t, Cx + Du) + B_2 u \tag{17.40}$$

where

$$z'\phi(t, z) \geq 0 \tag{17.41}$$

for all z.

Figure 17.4: A first and third quadrant nonlinearity

Theorem 45 *Consider a system described by (17.40) and satisfying (17.41). Suppose there exist a matrix L, a positive-definite symmetric matrix S and a positive scalar μ which satisfy*

$$AS + B_2L + SA' + L'B_2' \;<\; 0 \tag{17.42a}$$

$$\mu B_1' \;=\; CS + DL \tag{17.42b}$$

Then controller (17.11) renders system (17.40) globally exponentially stable about the origin with Lyapunov matrix S^{-1}.

Proof. Apply corresponding analysis result to closed loop system. ■

• Note that (17.42) has a solution with $S, \mu > 0$ if and only if the following optimization problem has a minimum of zero:

Minimize β subject to

$$\begin{array}{rcl} AS + B_2L + SA' + L'B_2' & < & 0 \\ \\ S & \geq & I \\ \\ \mu & > 0 & \\ \\ \begin{pmatrix} \beta I & \mu B_1' - CS - DL \\ \mu B_1 - SC' - L'D' & \beta I \end{pmatrix} & \geq & 0 \end{array}$$

Exercise 62 Prove the following result: Consider a system described by (17.40) and satisfying (17.41). Suppose there exist a matrix L, a positive-definite symmetric matrix S, and positive scalars μ, α which satisfy

$$\begin{aligned} AS + B_2L + SA' + L'B_2' + 2\alpha S &\leq 0 && \text{(17.43a)} \\ \mu B_1' &= CS + DL && \text{(17.43b)} \end{aligned}$$

Then controller (17.11) renders system (17.40) globally uniformly exponentially stable about the origin with rate α and with Lyapunov matrix S^{-1}.

17.4.1 Generalization

Here we consider systems described by

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u \\ z &= Cx + D_1w + D_2u \\ w &= -\phi(t,z) \end{aligned} \tag{17.44}$$

where

$$z'\phi(t,z) \geq 0 \tag{17.45}$$

for all z.

Theorem 46 *Consider a system described by (17.44) and satisfying (17.45). Suppose there exist a matrix L, a positive-definite symmetric matrix S and positive scalars α and μ which satisfy*

$$\begin{pmatrix} AS + SA' + B_2L + L'B_2' + 2\alpha S & (CS + D_2L)' - \mu B_1 \\ CS + D_2L - \mu B_1' & -\mu(D_1 + D_1') \end{pmatrix} \leq 0 \tag{17.46}$$

Then controller (17.11) renders system (17.40) globally uniformly exponentially stable about the origin with rate α and with Lyapunov matrix S^{-1}.

PROOF. ∎

- Note that (17.46) is linear in L, S, μ.
  If $D + D' > 0$, then (17.46) is equivalent to

$$AS + SA' + B_2L + L'B_2' + 2\alpha S + \mu^{-1}(\mu B_1 - SC' - L'D_2')(D_1 + D_1')^{-1}(\mu B_1' - CS - D_2L) \leq 0$$

Chapter 18

Singularly perturbed systems

18.1 Introduction and examples

Roughly speaking, a singularly perturbed system is a system which depends on a scalar parameter in such a fashion that for a particular value of the parameter (usually taken to be zero) the system order reduces.

18.1.1 High gain output feedback

Consider a scalar-input scalar-output system described by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\\ y &= 2x_1 + x_2\end{aligned}$$

This system has the following transfer function

$$\hat{G}(s) = \frac{s+2}{s^2}$$

Suppose we subject this system to high-gain static output feedback, that is, we let

$$u = -ky$$

where k is large. Letting $\mu = k^{-1}$, the closed loop system is described by

$$\dot{x}_1 = x_2 \tag{18.1}$$
$$\mu\dot{x}_2 = -2x_1 - x_2 \tag{18.2}$$

We are interested in the behavior of this system for $\mu > 0$ small.

The eigenvalues of the closed loop system matrix

$$\begin{bmatrix} 0 & 1 \\ -2\mu^{-1} & -\mu^{-1} \end{bmatrix}$$

are $\lambda^s(\mu)$ and $\mu^{-1}\lambda^f(\mu)$ where

$$\lambda^s(\mu) = \frac{-4}{1+\sqrt{1-8\mu}}\,, \qquad \lambda^f(\mu) = \frac{1+\sqrt{1-8\mu}}{-2}$$

and

$$\lim_{\mu\to 0}\lambda^s(\mu) = -2\,, \qquad \lim_{\mu\to 0}\lambda^f(\mu) = -1$$

Thus, for small $\mu > 0$, system responses are characterized by a "slow" mode $e^{\lambda^s(\mu)t}$ and a "fast" mode $e^{\lambda^f(\mu)t/\mu}$.

For $\mu = 0$, the differential equation (18.2) reduces to the algebraic equation

$$-2x_1 - x_2 = 0$$

Substitution for x_2 into (18.1) yields

$$\dot{x}_1 = -2x_1$$

Note that the dynamics of this system are described by the limiting slow mode.

18.1.2 Single-link manipulator with flexible joint

Consider the single-link manipulator illustrated in Figure 18.1. There is a flexible joint or shaft between the link and the rotor of the actuating motor; this motor exerts a control torque u. The equations of motion for this system are:

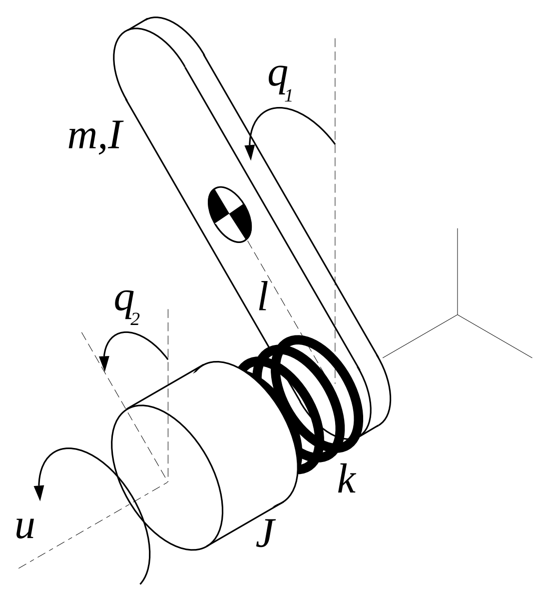

Figure 18.1: Single link manipulator with flexible joint

$$\begin{aligned} I\ddot{q}_1 &= Wl\sin q_1 &-& \quad c(\dot{q}_1-\dot{q}_2)-k(q_1-q_2) \\ J\ddot{q}_2 &= u &+& \quad c(\dot{q}_1-\dot{q}_2)+k(q_1-q_2) \end{aligned}$$

where W is the weight of the link and $k>0$ and $c>0$ represent the stiffness and damping coefficients, respectively, of the shaft. We want to design a feedback controller for u which guarantees that the resulting closed loop system is globally asymptotically stable about $q_1(t)\equiv q_2(t)\equiv 0$. Suppose for the purpose of controller design, we simplify the problem and model the shaft as rigid. This yields the following rigid model:

$$(I+J)\ddot{q}_1 = Wl\sin q_1 + u$$

A controller which globally exponentially stabilizes this model about $q_1(t)\equiv 0$ is given by

$$u = p(q_1,\dot{q}_1) = -k_p q_1 - k_d\dot{q}_1$$

with

$$k_p > Wl, \quad k_d > 0$$

Will this controller also stabilize the flexible model provided, the flexible elements are sufficiently stiff? To answer this question we let

$$k=\mu^{-2}k_0\,, \qquad c=\mu^{-1}c_0$$

where $k_0, c_0>0$ and consider system behavior as $\mu\to 0$. Introducing state variables

$$x_1=q_1\,, \qquad x_2=\dot{q}_1\,, \qquad y_1=\mu^{-2}(q_2-q_1)\,, \qquad y_2=\mu^{-1}(\dot{q}_2-\dot{q}_1)$$

the closed loop flexible model is described by

$$\begin{aligned} \dot{x}_1 &= x_2 && \\ \dot{x}_2 &= I^{-1}Wl\sin x_1 &+& \quad I^{-1}(k_0y_1+c_0y_2) \\ \mu\dot{y}_1 &= && \quad y_2 \\ \mu\dot{y}_2 &= -I^{-1}Wl\sin x_1 + J^{-1}p(x) &-& \quad \tilde{I}^{-1}(k_0y_1+c_0y_2) \end{aligned}$$

where $\tilde{I}^{-1} := I^{-1}+J^{-1}$.

Suppose we let $\mu=0$ in this description. Then the last two equations are no longer differential equations; they are equivalent to

$$\begin{aligned} y_2 &= 0 \\ y_1 &= k_0^{-1}\tilde{I}(-I^{-1}Wl\sin x_1 + J^{-1}p(x)) \end{aligned}$$

Substituting these two equations into the first two equations of the above state space description yields

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (I+J)^{-1}(Wl\sin x_1 + p(x)) \end{aligned}$$

Note that this is a state space description of the closed loop rigid model.

18.2 Singularly perturbed systems

The state space description of the above two examples are specific examples of singularly perturbed systems. In general a singularly perturbed system is described by

$$\begin{aligned} \dot{x} &= f(x, y, \mu) && \text{(18.3a)} \\ \mu\dot{y} &= g(x, y, \mu) && \text{(18.3b)} \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$ describe the state of the system at time $t \in \mathbb{R}$ and $\mu > 0$ is the singular perturbation parameter. We suppose $0 \leq \mu < \bar{\mu}$.

Reduced order system. Letting $\mu = 0$ results in

$$\begin{aligned} \dot{x} &= f(x, y, 0) \\ 0 &= g(x, y, 0) \end{aligned}$$

If we assume that there is a continuously differentiable function h_0 such that

$$g(x, h_0(x), 0) \equiv 0\,,$$

then $y = h_0(x)$. This yields the reduced order system:

$$\boxed{\dot{x} = f(x, h_0(x), 0)}$$

A standard question is whether the behavior of the original full order system (18.3) can be predicted by looking at the behavior of the reduced order system. To answer this question one must also look at another system called the boundary layer system.

Boundary layer system. First introduce "fast time" described by

$$\tau = t/\mu$$

and let

$$\begin{aligned} \xi(\tau) &:= x(\mu\tau) = x(t) \\ \eta(\tau) &:= y(\mu\tau) - h_0(x(\mu\tau)) = y(t) - h_0(x(t)) \end{aligned}$$

to obtain a regularly perturbed system:

$$\begin{aligned} \xi' &= \mu f(\xi, h_0(\xi) + \eta, \mu) \\ \eta' &= g(\xi, h_0(\xi) + \eta, \mu) - \mu \frac{\partial h_0}{\partial x}(\xi) f(\xi, h_0(\xi) + \eta, \mu) \end{aligned}$$

Setting $\mu = 0$ results in $\dot{\xi}(\tau) \equiv 0$; hence $\xi(\tau) = \xi(0) = x(0) =: x_0$ and

$$\boxed{\eta' = g(x_0, h_0(x_0) + \eta, 0)}$$

This is a dynamical system parameterized by x_0. It has a unique equilibrium state $\eta = 0$. We will refer to it as the boundary layer system.

A standard result is the following: Suppose the boundary layer system is exponentially stable and this stability is uniform with respect to x_0; then on any compact time interval not including 0, the behavior of x for the full order system approaches that of x for the reduced order system as $\mu \to 0$.

Another result is the following: Suppose the reduced order system is exponentially stable about 0 and the boundary layer is exponentially stable for all x_0. Then, under certain regularity conditions, the full order system is exponentially stable for μ sufficiently small.

Recalling the first example of the last section, the boundary layer system is given by the exponentially stable system

$$\eta' = -\eta$$

Since the reduced order system is exponentially stable, the full order system is exponentially stable for $\mu > 0$ small, that is, for $k > 0$ large.

The boundary layer system for the second example is given by

$$\begin{aligned} \eta_1' &= \eta_2 \\ \eta_2' &= -\tilde{I}^{-1}(k_0\eta_1 + c_0\eta_2) \end{aligned}$$

Clearly this system is exponentially stable. Since the reduced order system is exponentially stable, the closed loop flexible system is exponentially stable for $\mu > 0$ sufficiently small, that is, for k and c sufficiently large. This is illustrated by numerical simulations in Figure 18.2.

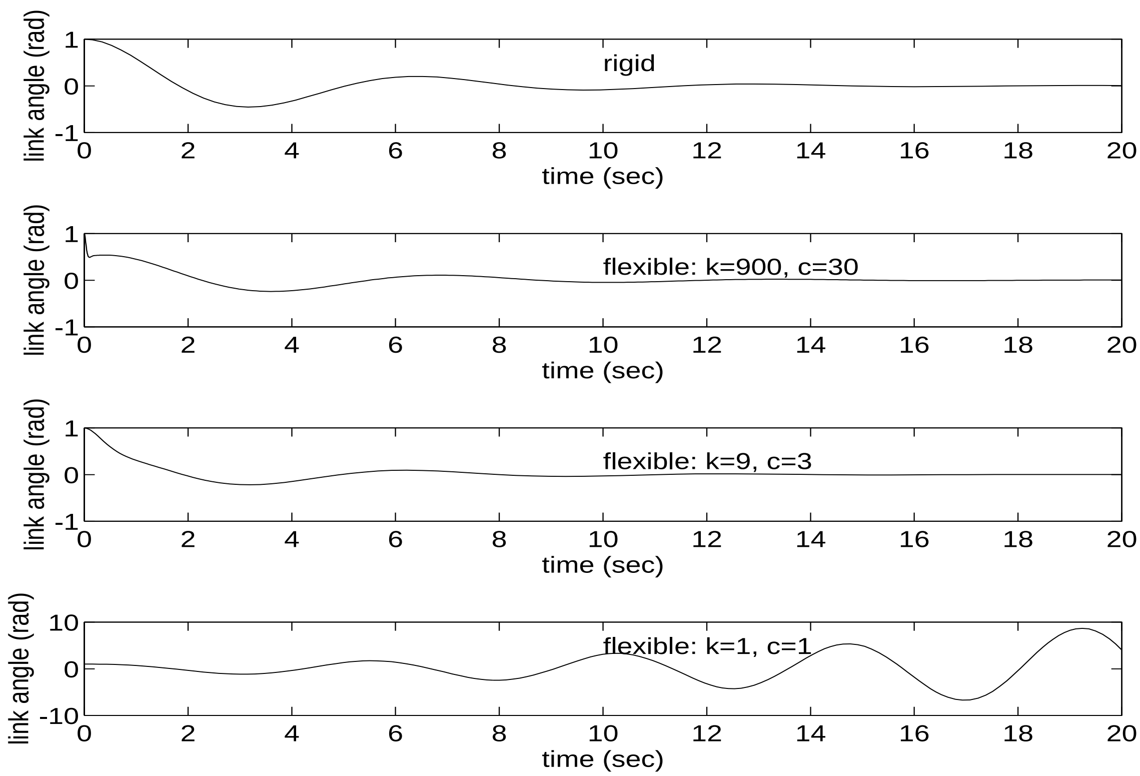

Figure 18.2: Comparison of rigid case and flexible case with large and small stiffnesses

18.3 Linear systems

A linear time invariant singularly perturbed system is described by

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)y \\ \mu\dot{y} &= C(\mu)x + D(\mu)y \end{aligned} \tag{18.4}$$

where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ are differentiable matrix valued functions defined on some interval $[0, \bar{\mu})$. For $\mu > 0$, this is a LTI system with "A-matrix"

$$\mathcal{A}(\mu) = \begin{bmatrix} A(\mu) & B(\mu) \\ \mu^{-1}C(\mu) & \mu^{-1}D(\mu) \end{bmatrix}. \tag{18.5}$$

We assume that

$$D(0) \quad \textit{is} \quad \textit{invertible}$$

Letting $\mu = 0$ results in the reduced order system:

$$\dot{x} = \bar{A}x$$

where

$$\bar{A} := A(0) - B(0)D(0)^{-1}C(0)$$

The boundary layer system is described by:

$$\eta' = D(0)\eta$$

Let

$$\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r$$

and

$$\lambda_1^{bl}, \lambda_2^{bl}, \ldots, \lambda_m^{bl}$$

be the eigenvalues of $\bar{A}$ and $D(0)$, respectively, where an eigenvalue is included p times if its algebraic multiplicity is p. Then we have the following result.

Theorem 47 *If $D(0)$ is invertible, then for each $\mu > 0$ sufficiently small, $\mathcal{A}(\mu)$ has n eigenvalues*

$$\lambda_1^s(\mu), \quad \lambda_2^s(\mu), \quad \ldots, \quad \lambda_n^s(\mu)$$

with

$$\lim_{\mu \to 0} \lambda_i^s(\mu) = \lambda_i^r\,, \qquad i = 1, 2, \ldots, n$$

and m eigenvalues

$$\lambda_1^f(\mu)/\mu, \quad \lambda_2^f(\mu)/\mu, \quad \ldots, \quad \lambda_m^f(\mu)/\mu$$

with

$$\lim_{\mu \to 0} \lambda_i^f(\mu) = \lambda_i^{bl}\,, \qquad i = 1, 2, \ldots, m$$

Recalling the high gain feedback example we have

$$\lambda_1^r = -2\,, \qquad \lambda_1^{bl} = -1$$

Hence the closed loop system has two eigenvalues $\lambda_1^s(\mu)$ and $\lambda_1^f(\mu)/\mu$ with the following properties

$$\lim_{\mu\to 0}\lambda_1^s(\mu) = -2\,, \qquad \lim_{\mu\to 0}\lambda_1^f(\mu) = -1$$

The above theorem has the following corollary: If all the eigenvalues of $\bar{A}$ and $D(0)$ have negative real parts then, for μ sufficiently small, all the eigenvalues of $\mathcal{A}(\mu)$ have negative real parts. If either $\bar{A}$ or $D(0)$ has an eigenvalue with a positive real part, then for μ sufficiently small, $\mathcal{A}(\mu)$ has an eigenvalue with positive real part.

18.4 Exact slow and fast subsystems

Suppose a singularly perturbed system has a boundary layer system which is just marginally stable, that is, just stable but not exponentially stable. Then we cannot apply our previous results. Further analysis is necessary.

The slow system. Consider a singularly perturbed system described by (18.3). A subset M of $\mathbb{R}^n \times \mathbb{R}^m$ is an invariant set for (18.3) if every solution originating in M remains there. A parameterized set M_μ, is a slow manifold for (18.3), if for each $\mu \in [0, \bar{\mu})$, M_μ is an invariant set for (18.3) and there is a continuous function $h : \mathbb{R}^n \times [0, \bar{\mu}) \to \mathbb{R}^m$ such that

$$M_\mu = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \;:\; y = h(x, \mu)\}\,. \tag{18.6}$$

When system (18.3) has a slow manifold described by (18.6), then the behavior of the system on this manifold is governed by

$$\begin{aligned} \dot{x} &= f(x, h(x, \mu), \mu) && \text{(18.7a)}\\ y &= h(x, \mu)\,. && \text{(18.7b)} \end{aligned}$$

We refer to system (18.7a) as the slow system associated with singularly perturbed system (18.3). Thus the slow system describes the motion of (18.3) restricted to its slow invariant manifold.

The slow manifold condition. Now note that if M_μ, given by (18.6), is an invariant manifold for (18.3), then h must satisfy the following slow manifold condition:

$$g(x, h(x, \mu), \mu) - \mu\frac{\partial h}{\partial x}(x, \mu) f(x, h(x, \mu), \mu) = 0 \tag{18.8}$$

For each $\mu > 0$, this condition is a partial differential equation for the function $h(\cdot, \mu)$; for $\mu = 0$, it reduces to a condition we have seen before:

$$g(x, h(x, 0), 0) = 0\,.$$

If we introduce the boundary layer state $z(t)$ given by

$$z := y - h(x,\mu)\,, \tag{18.9}$$

then the original system (18.3) is equivalently described by

$$\begin{aligned} \dot{x} &= \tilde{f}(x,z,\mu) && (18.10a)\\ \mu\dot{z} &= \tilde{g}(x,z,\mu) && (18.10b) \end{aligned}$$

with

$$\begin{aligned} \tilde{f}(x,z,\mu) &:= f(x,h(x,\mu)+z,\mu) && (18.11a)\\ \tilde{g}(x,z,\mu) &:= g(x,h(x,\mu)+z,\mu) - \mu\frac{\partial h}{\partial x}(x,\mu)f(x,h(x,\mu)+z,\mu) && (18.11b) \end{aligned}$$

Note that M_μ is an invariant manifold for system (18.3) if and only if the manifold

$$\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m \;\;:\;\; z = 0\} \tag{18.12}$$

is invariant for (18.10). Also, the slow manifold condition is equivalent to

$$\tilde{g}(x,0,\mu) = 0\,. \tag{18.13}$$

From (18.10) and (18.13), it should be clear that if the slow manifold condition holds then, for any $x(0) = x_0$ and $z(0) = 0$, the resulting system behavior satisfies (assuming uniqueness of solutions), $z(t) \equiv 0$. Hence the manifold described by (18.12) is an invariant manifold. So, the slow manifold condition is necessary and sufficient for the existence of a slow invariant manifold.

The fast system. Introducing the fast time variable

$$\tau \;:= t/\mu$$

and letting

$$\begin{aligned} \eta(\tau) &:= x(t) = x(\mu\tau)\\ \xi(\tau) &:= z(t) = z(\mu\tau) \end{aligned}$$

system (18.10) is described by the following regularly perturbed system:

$$\begin{aligned} \eta' &= \mu\tilde{f}(\eta,\xi,\mu)\\ \xi' &= \tilde{g}(\eta,\xi,\mu) \end{aligned}$$

We define

$$\dot{\xi} = \tilde{g}(\eta,\xi,\mu)$$

to be the fast system.

18.4.1 Approximations of the slow and fast systems

Suppose

$$f(x, h(x, \mu), \mu) = f_0(x) + \mu f_1(x) + o(\mu; x)$$

where

$$\lim_{\mu \to 0} o(\mu; x)/\mu = 0\,.$$

Then

$$\dot{x} = f_0(x)$$

and

$$\dot{x} = f_0(x) + \mu f_1(x)$$

are called the zero and first order slow systems, respectively. Note that the zero order slow system is the reduced order system.

To compute these approximations, we suppose that

$$h(x, \mu) = h_0(x) + \mu h_1(x) + o(\mu;\; x)$$

Then,

$$\begin{aligned} f_0(x) &= f(x, h_0(x), 0) \\ f_1(x) &= \frac{\partial f}{\partial y}(x, h_0(x), 0) h_1(x) + \frac{\partial f}{\partial \mu}(x, h_0(x), 0) \end{aligned}$$

The following expressions for h_0 and h_1 can be obtained from the slow manifold condition

$$g(x, h_0(x), 0) = 0 \tag{18.14a}$$

$$\frac{\partial g}{\partial y}(x, h_0(x), 0) h_1(x) - \frac{\partial h_0}{\partial x}(x) f_0(x) + \frac{\partial g}{\partial \mu}(x, h_0(x), 0) = 0 \tag{18.14b}$$

To approximate the fast system, suppose that

$$\tilde{g}(\eta, \xi, \mu) = g_0(\eta, \xi) + \mu g_1(\eta, \xi) + o(\mu; \eta, \xi)$$

Then

$$\xi' = g_0(\eta, \xi)$$

and

$$\boxed{\xi' = g_0(\eta, \xi) + \mu g_1(\eta, \xi)} \tag{18.15}$$

are called the zero and first order fast systems, respectively. Note that $\xi = 0$ is an equilibrium state for both. Using (18.11b) one can obtain

$$\begin{aligned} g_0(\eta, \xi) &= g(\eta, h_0(\eta) + \xi, 0) \\ g_1(\eta, \xi) &= \frac{\partial g}{\partial y}(\eta, h_0(\eta) + \xi, 0) h_1(\eta) - \frac{\partial h_0}{\partial x}(\eta) f(\eta, h_0(\eta) + \xi, 0) + \frac{\partial g}{\partial \mu}(\eta, h_0(\eta) + \xi, 0) \end{aligned}$$

Note that the zero order fast system is the boundary layer system.

A reasonable conjecture is the following: if the reduced order system is exponentially stable and the first order fast system is exponentially stable for $\mu > 0$ sufficiently small, then the full order system is exponentially stable for small $\mu > 0$.

18.4.2 A special class of systems

Here we consider a special class of singularly perturbed systems described by

$$\begin{aligned} \dot{x} &= \hat{f}(x,\mu) + B(\mu)y \\ \mu\dot{y} &= \hat{g}(x,\mu) + D(\mu)y \end{aligned} \tag{18.16}$$

with $D(0)$ invertible.

Utilizing (18.14a) the function h_0 is given by

$$\hat{g}(x,0) + D(0)h_0(x) = 0\,;$$

hence

$$h_0(x) = -D(0)^{-1}\hat{g}(x,0)$$

and

$$f_0(x) = \hat{f}(x,0) - B(0)D(0)^{-1}\hat{g}(x,0)\,.$$

Utilizing (18.14b) the function h_1 is given by

$$D(0)h_1(x) - \frac{\partial h_0}{\partial x}(x)f_0(x) + \frac{\partial \hat{g}}{\partial \mu}(x,0) + \frac{\partial D}{\partial \mu}(0)h_0(x) = 0$$

The function g_0 is given by

$$g_0(\eta,\xi) = D(0)\xi$$

and the function g_1 is given by

$$\begin{aligned} g_1(\eta,\xi) &= D(0)h_1(\eta) - \frac{\partial h_0}{\partial x}(\eta)[f_0(\eta) + B(0)\xi] + \frac{\partial \hat{g}}{\partial \mu}(x,0) + \frac{\partial D}{\partial \mu}(0)[h_0(\eta)+\xi] \\ &= D(0)h_1(\eta) - \frac{\partial h_0}{\partial x}(\eta)f_0(\eta) + \frac{\partial \hat{g}}{\partial \mu}(x,0) + \frac{\partial D}{\partial \mu}(0)h_0(\eta) \quad + \quad [-\frac{\partial h_0}{\partial x}(\eta)B(0) + \frac{\partial D}{\partial \mu}(0)]\xi \\ &= D_1(\eta)\xi \end{aligned}$$

where

$$\boxed{D_1(\eta) = D(0)^{-1}\frac{\partial \hat{g}}{\partial x}(\eta,0)B(0) + \frac{\partial D}{\partial \mu}(0)}$$

So the first order fast system is described by

$$\boxed{\xi' = [D(0) + \mu D_1(\eta)]\xi}$$

Example 175 Consider now the closed loop flexible one link manipulator with

$$k = \mu^{-2}k_0\,, \qquad c = c_0$$

This is described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= I^{-1}Wl\sin x_1 && + \quad I^{-1}(k_0y_1 + \mu c_0 y_2) \\ \mu\dot{y}_1 &= y_2 \\ \mu\dot{y}_2 &= -I^{-1}Wl\sin x_1 + J^{-1}p(x) && - \quad \tilde{I}^{-1}(k_0y_1 + \mu c_0 y_2) \end{aligned}$$

where $\tilde{I}^{-1} = I^{-1} + J^{-1}$. This is in the form of (18.16) with

$$B(\mu) = \begin{bmatrix} 0 & 0 \\ I^{-1}k_0 & I^{-1}\mu c_0 \end{bmatrix}, \qquad D(\mu) = \begin{bmatrix} 0 & 1 \\ -\tilde{I}^{-1}k_0 & -\tilde{I}^{-1}\mu c_0 \end{bmatrix}$$

and

$$\hat{g}(x,\mu) = \begin{bmatrix} 0 \\ -I^{-1}Wl\sin x_1 + J^{-1}p(x) \end{bmatrix}$$

Hence

$$D_1 = \begin{bmatrix} (I+J)^{-1}k_d & 0 \\ 0 & -\tilde{I}^{-1}c_0 \end{bmatrix}$$

and

$$D(0) + \mu D_1 = \begin{bmatrix} \mu(I+J)^{-1}k_d & 1 \\ -\tilde{I}^{-1}k_0 & -\mu\tilde{I}^{-1}c_0 \end{bmatrix}$$

A necessary and sufficient condition for this sytem to be exponentially for small $\mu > 0$ is

$$k_d < (I+J)\tilde{I}^{-1}c_0$$

Figures 18.3 and 18.4 illustrate numerical simulations with the above condition unsatisfied and satisfied, respectively.

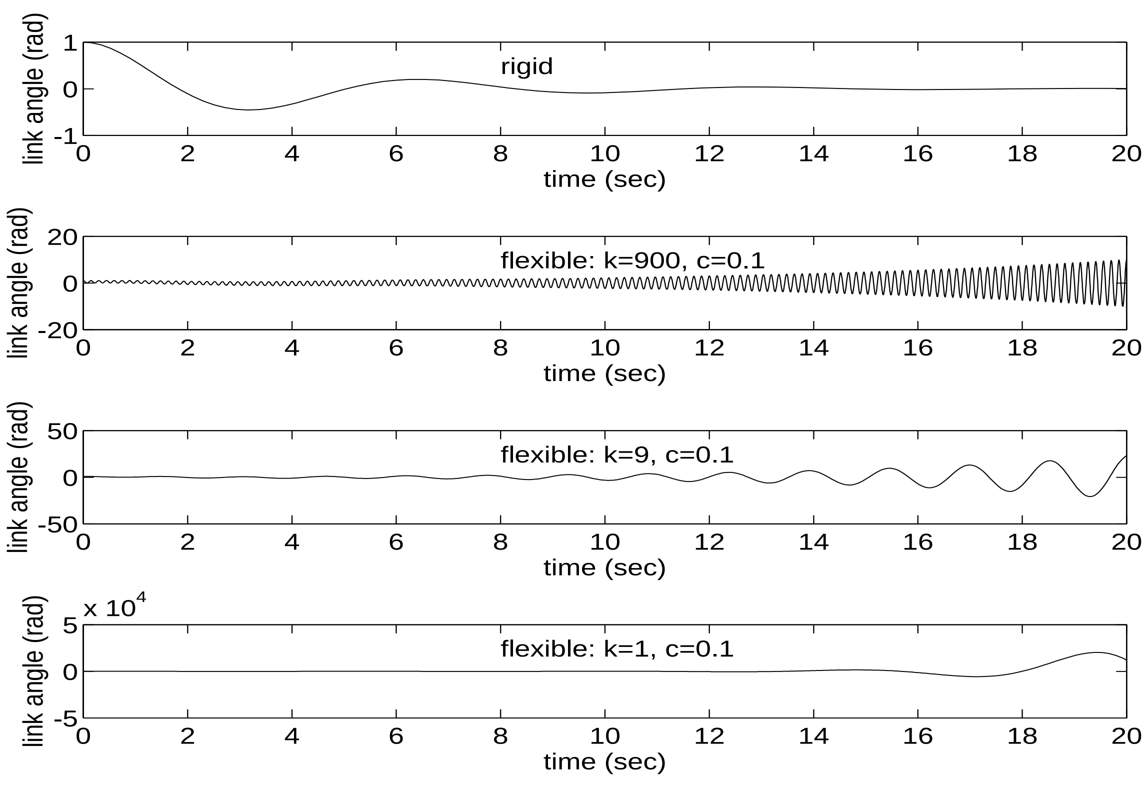

Figure 18.3: Flexible case with low damping

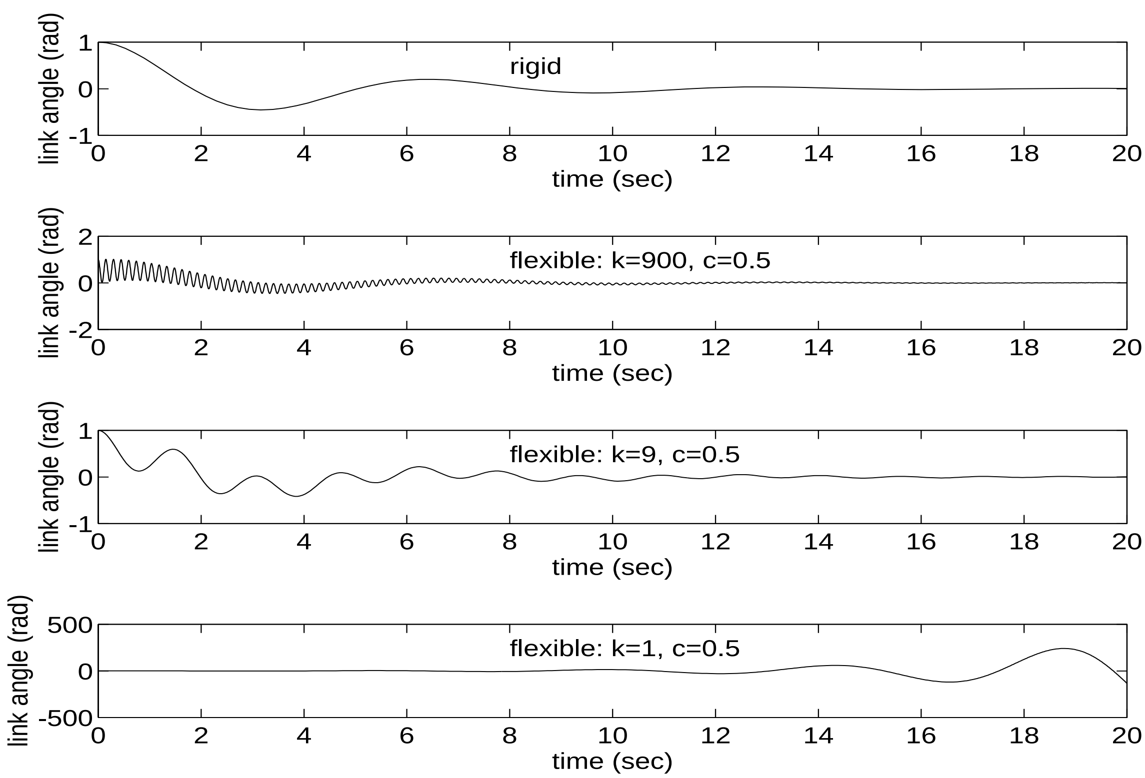

Figure 18.4: Flexible case with “medium” damping

Suppose we colocate the rate sensor with the actuator. Then

$$\begin{aligned} u &= p(q_1, \dot{q}_2) \\ &= -k_p q_1 - k_d \dot{q}_2 \\ &= -k_p x_1 - k_d x_2 - \mu k_d y_2 \\ &= p(x) - \mu k_d y_2 \end{aligned}$$

and

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= I^{-1} W l \sin x_1 && + \quad I^{-1}(k_0 y_1 + \mu c_0 y_2) \\ \mu \dot{y}_1 &= y_2 \\ \mu \dot{y}_2 &= -I^{-1} W l \sin x_1 + J^{-1} p(x) && - \quad \tilde{I}^{-1} k_0 y_1 - \mu(\tilde{I}^{-1} c_0 + J^{-1} k_d) y_2 \end{aligned}$$

Now

$$D(\mu) = \begin{bmatrix} 0 & 1 \\ -\tilde{I}^{-1} k_0 & -\mu(\tilde{I}^{-1} c_0 + J^{-1} k_d) \end{bmatrix}$$

This yields

$$D(0) + \mu D_1 = \begin{bmatrix} \mu(I+J)^{-1} k_d & 1 \\ -\tilde{I}^{-1} k_0 & -\mu(\tilde{I}^{-1} c_0 + J^{-1} k_d) \end{bmatrix}$$

This is exponentially stable for $\mu > 0$ small.

Figure 18.5 illustrates numerical simulations with $c_0 = 0$.

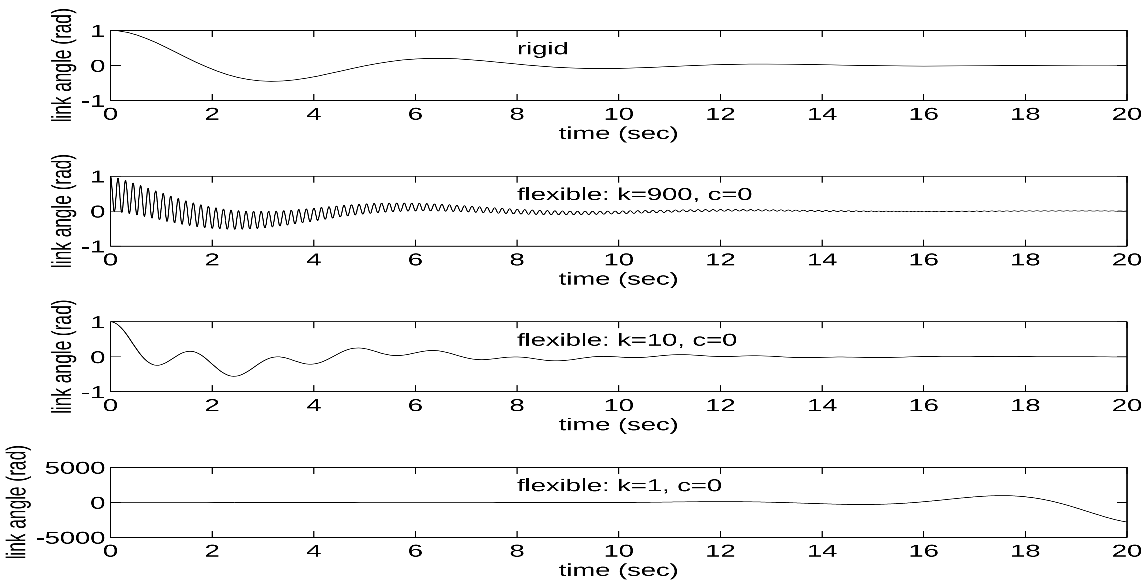

Figure 18.5: Rigid and flexible case with colocated rate feedback

Chapter 19

Input output systems

19.1 Input-output systems

To describe an input-output system, we need several ingredients. First we have the time set $\mathcal{T}$. Usually

$$\mathcal{T} = [0, \infty)$$

for continuous-time systems or

$$\mathcal{T} = \{\, 0, 1, 2, \ldots, \,\}$$

for discrete-time systems. We have the set $\mathcal{U}$ of input signals $u : \mathcal{T} \to U$ where U is the set of input values. Common examples are $U = \mathbb{R}^m$ and $\mathcal{U}$ is the set of piecewise continuous functions $u : [0, \infty) \to \mathbb{R}^m$. We also have the set $\mathcal{Y}$ of output signals $y : \mathcal{T} \to Y$ where U is the set of output values. Common examples are $Y = \mathbb{R}^o$ and $\mathcal{Y}$ is the set of piecewise continuous functions $y : [0, \infty) \to \mathbb{R}^o$.

By saying that a function $s : [0, \infty) \to \mathbb{R}^n$ is piecewise continuous we will mean that on every finite interval $[0, T]$, the function is continuous except (possibly) at isolated points.

Example 176 Two examples of piecewise continuous functions.

(i) Square wave function.

$$s(t) = \begin{cases} 1 & \text{if} \quad N \le t < N + \frac{1}{2} \\ \\ -1 & \text{if} \quad N + \frac{1}{2} \le t \le N + 1 \end{cases} \qquad N = 0, 1, 2, \ldots$$

(ii)

$$s(t) = \begin{cases} (1-t)^{-1} & \text{if} \quad 0 \le t < 1 \\ 0 & \text{if} \quad 1 \le t \end{cases}$$

An input-output system is a map

$$G : \mathcal{U} \to \mathcal{Y}$$

Figure 19.1: Input-output system

that is, for each input signal $u \in \mathcal{U}$ the system produces an output signal $y = G(u)$ with $y \in \mathcal{Y}$.

Example 177 Consider a linear time-invariant state space system described by

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\\ x(0) &= x_0\end{aligned}$$

where $x_0 \in \mathbb{R}^n$ is fixed and $t \in [0, \infty)$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^o$. For each piecewise continuous input signal u, this system produces a unique piecewise continuous output signal y, specifically

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\, d\tau + Du(t) \qquad \text{for} \quad t \geq 0\,.$$

Example 178 Consider a discrete-time linear time-invariant system described by

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k)\\ y(k) &= Cx(k) + Du(k)\\ x(0) &= x_0\end{aligned}$$

where $k \in \{\,0, 1, 2, \ldots,\,\}$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^o$, and $x_0 \in \mathbb{R}^n$ is fixed.

$$y(k) = CA^k x_0 + \sum_{j=0}^{k-1} CA^{(k-1-j)}Bu(j) + Du(k)$$

Example 179 Consider a linear time-invariant system described by a delay-differential equation:

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + A_1x(t-h) + Bu(t)\\ y &= Cx(t) + Du(t)\\ x(t) &= x_0(t) \qquad \text{for} \qquad 0 \leq t \leq h\end{aligned}$$

where the continuous function $x_0(\cdot)$ is fixed and $t \in [0, \infty)$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^o$. For each piecewise continuous input signal u, this system produces a unique piecewise continuous output signal y,

19.1.1 Causality

Basically, an input-output system is causal if the current value of the output depends only on the previous values of the input. Mathematically, we express this as follows. A system G is causal, if

$$u_1(t) = u_2(t) \qquad \text{for} \quad t \leq T$$

implies

$$G(u_1)(t) = G(u_2)(t) \qquad \text{for} \quad t \leq T\,.$$

The systems in the above examples are causal. As examples of noncausal systems, consider

$$\begin{aligned} y(t) &= u(t+1) \\ y(t) &= \int_t^\infty u(\tau)\, d\tau \end{aligned}$$

19.2 Signal norms

To talk about the size of signals, we introduce signal norms. Consider any real scalar $p \geq 1$. We say that a piecewise continuous signal $s : [0, \infty) \to \mathbb{R}^n$ is an $\mathcal{L}^p$ signal if

$$\int_0^\infty ||s(t)||^p\, dt < \infty$$

where $||s(t)||$ is the Euclidean norm of the n-vector $s(t)$. If $\mathcal{S}$ is any linear space of $\mathcal{L}^p$ signals, then the scalar valued function $||\cdot||_p$ defined by

$$\boxed{||s||_p := \left(\int_0^\infty ||s(t)||^p\, dt \right)^{\frac{1}{p}}}$$

is a norm on this space. We call this the p-norm of s. Common choices of p are one and two. For $p = 2$ we have

$$||s||_2 = \left(\int_0^\infty ||s(t)||^2\, dt \right)^{\frac{1}{2}} .$$

This is sometimes called the rms (root mean square) value of the signal. For $p = 1$,

$$||s||_1 = \int_0^\infty \|s(t)\|\, dt\,.$$

We say that s is an $\mathcal{L}^\infty$ signal if

$$\operatorname*{ess\,sup}_{t \geq 0} ||s(t)|| < \infty\,.$$

By ess sup we mean that supremum is taken over the set of times t where s is continuous. If $\mathcal{S}$ is a any linear space of $\mathcal{L}^\infty$ signals, then the scalar valued function $||\cdot||_\infty$ defined by

$$\boxed{||s||_\infty := \operatorname*{ess\,sup}_{t \geq 0} ||s(t)||}$$

is a norm on this space. We call this the ∞-norm of s. Also $||s(t)|| \leq ||s||_\infty$ a.e (almost everywhere) that is, everywhere except(possibly) where s is not continuous.

Example 180 $\mathcal{L}^p$ or not $\mathcal{L}^p$

(i) $\mathcal{L}^p$ for all p.

$$s(t) = e^{-\alpha t} \qquad \text{with} \qquad \alpha > 0\,.$$

Since $|e^{-\alpha t}| \leq 1$ for all $t \geq 0$, this signal is an $\mathcal{L}^\infty$ signal; also $\|s\|_\infty = 1$. For $1 \leq p < \infty$, we have $\int_0^\infty |e^{-\alpha t}|^p\,dt = 1/\alpha p$. Hence, this is an $\mathcal{L}^p$ signal with $\|s\|_p = (1/\alpha p)^{1/p}$.

(ii) Not $\mathcal{L}^p$ for all p.

$$s(t) = e^t$$

(iii) $\mathcal{L}^\infty$ but not $\mathcal{L}^p$ for any other p.

$$s(t) \equiv 1$$

(iv) Not $\mathcal{L}^\infty$ but $\mathcal{L}^p$ for any other p

$$s(t) = \begin{cases} (1-t)^{-\frac{1}{2}} & \text{for} \quad 0 \leq t < 1 \\ 0 & \text{for} \quad 1 \leq t \end{cases}$$

(v) Not $\mathcal{L}^1$ but $\mathcal{L}^p$ for any $p > 1$

$$s(t) = (1+t)^{-1}$$

For $1 \leq p < \infty$, we say that a piecewise continuous signal s is an extended $\mathcal{L}^p$ signal or an $\mathcal{L}^p_e$ signal if

$$\int_0^T ||s(t)||^p\,dt < \infty$$

for all $T \geq 0$. A piecewise continuous signal s is an extended $\mathcal{L}^\infty$ signal or an $\mathcal{L}^\infty_e$ signal if

$$\operatorname*{ess\ sup}_{0 \leq t \leq T} ||s(t)|| < \infty$$

for all $T \geq 0$. Clearly, if s is $\mathcal{L}^p$, then it is $\mathcal{L}^p_e$.

Example 181 $\mathcal{L}^p_e$ or not $\mathcal{L}^p_e$

(i)

$$s(t) = e^t$$

$\mathcal{L}^p_e$ for all p.

(iv)

$$s(t) = \begin{cases} (1-t)^{-\frac{1}{2}} & \text{for} \quad 0 \leq t < 1 \\ 0 & \text{for} \quad 1 \leq t \end{cases}$$

Not $\mathcal{L}^\infty_e$ but $\mathcal{L}^p$ for any other p

Note that if a signal is $\mathcal{L}^\infty_e$ then it is $\mathcal{L}^p_e$ for all p.

19.3 Input-output stability

Suppose $\mathcal{U}$ is a set of $\mathcal{L}_e^p$ signals $u : [0, \infty) \to \mathbb{R}^m$ and $\mathcal{Y}$ is a set of $\mathcal{L}_e^p$ signals $y : [0, \infty) \to \mathbb{R}^o$. Consider an input-output system

$$G : \mathcal{U} \to \mathcal{Y}\,.$$

Defn. *The system G is $\mathcal{L}^p$ stable if it has the following properties.*

(i) Whenever u is an $\mathcal{L}^p$ signal, the output $y = G(u)$ is an $\mathcal{L}^p$ signal.

(ii) There are scalars β and γ such that, for every $\mathcal{L}^p$ signal $u \in \mathcal{U}$, we have

$$\boxed{||G(u)||_p \leq \beta + \gamma ||u||_p} \tag{19.1}$$

The $\mathcal{L}^p$ gain of a system is the infimum of all γ such that there is a β which guarantees (19.1) for all $\mathcal{L}^p$ signals $u \in \mathcal{U}$.

Example 182 Consider the memoryless nonlinear SISO system defined by

$$y(t) = \sin(u(t)) + 1$$

Since

$$|y(t)| = 1 + |\sin(u(t))| \quad \leq \quad 1 + |u(t)|$$

it follows that for any $1 \leq p \leq \infty$ and any $\mathcal{L}^p$ signal u we have

$$||y||_p \leq 1 + ||u||_p$$

Hence this system is $\mathcal{L}^p$ stable with gain 1 for $1 \leq p \leq \infty$.

Example 183 Consider the simple delay system with time delay $h \geq 0$:

$$y(t) = \begin{cases} 0 & \text{for } \; 0 \leq t \leq h \\ u(t-h) & \text{for } \quad h \leq t \end{cases}$$

One can readily show that for any $1 \leq p \leq \infty$ and any $\mathcal{L}^p$ signal u we have

$$||y||_p = ||u||_p$$

Hence this system is $\mathcal{L}^p$ stable for $1 \leq p \leq \infty$.

Example 184 (Simple integrator is not io stable.)

$$\begin{aligned} \dot{x} &= u \\ y &= x \\ x(0) &= 0 \end{aligned}$$

For $1 \leq p < \infty$ consider

$$u(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t \leq 1 \\ 0 & \text{for} \quad 1 \leq t \end{cases}$$

Then

$$y(t) = \begin{cases} t & \text{for} \quad 0 \leq t \leq 1 \\ 1 & \text{for} \quad 1 \leq t \end{cases}$$

Since u is an $\mathcal{L}^p$ signal and y is not, this system is not $\mathcal{L}^p$ stable.

For $p = \infty$ consider

$$u(t) \equiv 1$$

Then

$$y(t) = t$$

Since u is an $\mathcal{L}^\infty$ signal and y is not, this system is not $\mathcal{L}^\infty$ stable.

19.4 General convolution systems

Suppose $\mathcal{U}$ is a set of $\mathcal{L}_e^p$ signals $u : [0, \infty) \to \mathbb{R}^m$ and $\mathcal{Y}$ is a set of $\mathcal{L}_e^p$ signals $y : [0, \infty) \to \mathbb{R}^o$. Consider the input-output system defined by the convolution integral

$$\boxed{y(t) = \int_0^t H(t-\tau)u(\tau)\, d\tau} \tag{19.2}$$

where $H : [0, \infty) \to \mathbb{R}^{o \times m}$ is a piecewise continuous matrix valued function of time. H is called the impulse response of the system.

As a specific example, consider a finite-dimensional linear system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= 0 \end{aligned}$$

Then $y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)\, d\tau$. Hence this is a convolution system with

$$H(t) = Ce^{At}B\,.$$

19.4.1 Systems with $\mathcal{L}^1$ impulse responses

Consider a convolution system with impulse response H. We say that H is $\mathcal{L}^1$ if

$$\int_0^\infty ||H(t)||\, dt < \infty$$

where

$$||H(t)|| = \sigma_{\max}[H(t)]$$

and we define

$$\boxed{||H||_1 := \int_0^\infty ||H(t)||\, dt}\,.$$

Example 185 Consider the SISO system

$$\begin{aligned} \dot{x} &= -\alpha x + u \\ y &= x \\ x(0) &= 0 \end{aligned}$$

with $\alpha > 0$. Here

$$y(t) = \int_0^t H(t-\tau)u(\tau)\, d\tau$$

with

$$H(t) = e^{-\alpha t}$$

For any $T > 0$,

$$\int_0^T |H(t)|\, dt = \frac{1}{\alpha}(1 - e^{-\alpha T})$$

Hence H is $\mathcal{L}^1$ and

$$||H||_1 = \frac{1}{\alpha}$$

Remark 9 Suppose there are constants $\alpha > 0$ and $\beta \geq 0$ such that

$$\|H(t)\| \leq \beta e^{-\alpha t} \tag{19.3}$$

for all $t \geq 0$. Then H is $\mathcal{L}^1$.

Remark 10 Consider

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

where A is Hurwitz. Then $H(t) = Ce^{At}B$ is $\mathcal{L}^1$. To see this, first choose any $\alpha > 0$ satisfying

$$\alpha < \bar{\alpha} := -\max\{\Re(\lambda) : \lambda \text{ is an eigenvalue of } A\}\ . \tag{19.4}$$

Then there exists $\beta > 0$ such that $\|H(t)\| \le \beta e^{-\alpha t}$.

Theorem 48 *Consider a convolution system (19.2) where the impulse response H is $\mathcal{L}^1$. For every $p \in [1, \infty]$, this system is $\mathcal{L}^p$ stable and for every $\mathcal{L}^p$ signal $u \in \mathcal{U}$,*

$$\boxed{||y||_p \le ||H||_1 ||u||_p}$$

Remark 11 The above result is a nice general result. However, except for $p = \infty$, it usually yields a conservative estimate for the $\mathcal{L}^p$ gain of a convolution system. Below we present an exact characterization of the $\mathcal{L}^2$ norm of a convolution system.

Remark 12 The main problem in applying the above result is in calculating $\|H\|_1$.

Before proving Theorem 48, we need the following result.

Holders inequality. *Suppose $f, g : [0, \infty) \to \mathbb{R}$ are piecewise continuous and consider any positive real scalars p and q which satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then, for all $T > 0$,

$$\int_0^T |f(t)g(t)|\,dt \le \left(\int_0^T |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_0^T |g(t)|^q\,dt\right)^{\frac{1}{q}} .$$

Proof of Theorem 48. First note that, for any input signal u and any time t, we have

$$\begin{aligned} ||y(t)|| &= \left\|\int_0^t H(t-\tau)u(\tau)\,dt\right\| \\ &\leq \int_0^t ||H(t-\tau)u(\tau)||\,dt \\ &\leq \int_0^t ||H(t-\tau)||\,||u(\tau)||\,dt\,. \end{aligned}$$

Case : $p=\infty$. Suppose u is $\mathcal{L}^\infty$, then $||u(\tau)|| \leq ||u||_\infty$ a.e. and

$$\begin{aligned} ||y(t)|| &\leq \int_0^t ||H(t-\tau)||\,||u||_\infty\,dt \\ &\leq \int_0^t ||H(t-\tau)||\,d\tau\,||u||_\infty \\ &= \int_0^t ||H(\tilde{\tau})||\,d\tilde{\tau}\,||u||_\infty \\ &\leq \int_0^\infty ||H(\tilde{\tau})||\,d\tilde{\tau}\,||u||_\infty \\ &= ||H||_1||u||_\infty\,. \end{aligned}$$

Hence y is $\mathcal{L}^\infty$ and

$$||y||_\infty \leq ||H||_1||u||_\infty\,.$$

Case : $p<\infty$. Suppose u is $\mathcal{L}^p$. Choose q such that

$$\frac{1}{p}+\frac{1}{q}=1\,.$$

Then, using Holders inequality, we obtain

$$\begin{aligned} ||y(t)|| &\leq \int_0^t ||H(t-\tau)||^{1/q}||H(t-\tau)||^{1/p}||u(\tau)||\,dt \\ &\leq \left(\int_0^t ||H(t-\tau)||d\tau\right)^{1/q}\left(\int_0^t ||H(t-\tau)||||u(\tau)||^p\,dt\right)^{1/p} \\ &\leq ||H||_1^{1/q}\left(\int_0^t ||H(t-\tau)||||u(\tau)||^p\,dt\right)^{1/p}\,. \end{aligned}$$

Hence, for any $T>0$, we have

$$\int_0^T ||y(t)||^p\,dt \;\leq\; ||H||_1^{p/q}\int_0^T\int_0^t ||H(t-\tau)||||u(\tau)||^p\,d\tau dt\,.$$

By changing the order of integration, we can write

$$\begin{aligned}\int_0^T \int_0^t ||H(t-\tau)||||u(\tau)||^p \, d\tau dt &= \int_0^T \int_\tau^T ||H(t-\tau)||||u(\tau)||^p \, dt d\tau \\ &= \int_0^T \int_\tau^T ||H(t-\tau)|| \, dt \, ||u(\tau)||^p \, d\tau \\ &\leq \int_0^T ||H||_1 ||u(\tau)||^p \, d\tau \\ &= ||H||_1 \int_0^T ||u(\tau)||^p \, d\tau \, .\end{aligned}$$

Since $1 + p/q = p$, we now obtain

$$\int_0^T ||y(t)||^p \leq ||H||_1^p \int_0^T ||u(\tau)||^p \, d\tau \, .$$

If u is $\mathcal{L}^p$ then $\int_0^\infty ||u(\tau)||^p$ is finite and

$$\int_0^T ||y(t)||^p \, d\tau \leq ||H||_1^p \int_0^\infty ||u(\tau)||^p \, d\tau \, ;$$

hence y is $\mathcal{L}^p$ and

$$||y||_p \leq ||H||_1 ||u||_p \, .$$

■

Remark 13 Here we provide a way of obtaining an upper bound on $\|H\|_1$ for single output systems using a Lyapunov equation. Consider a scalar output system described by

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

where A is Hurwitz, that is, all eigenvalues of A have negative real parts. Consider any α satisfying (19.4) and let S be the solution to

$$AS + SA' + 2\alpha S + BB' = 0 \, . \tag{19.5}$$

Then

$$\boxed{\|H\|_1 \leq \left(\frac{1}{2\alpha} CSC' \right)^{\frac{1}{2}}} \tag{19.6}$$

To see this first note that the above Lyapunov equation can be written as

$$(A + \alpha I)S + S(A + \alpha I)' + BB' = 0$$

Since $A + \alpha I$ is Hurwitz, S is uniquely given by

$$S = \int_0^\infty e^{(A+\alpha I)t} BB' e^{(A+\alpha I)' t} \, dt = \int_0^\infty e^{2\alpha t} e^{At} BB' e^{At} \, dt$$

Hence

$$CSC' = \int_0^\infty e^{2\alpha t} C e^{At} B B' e^{At} C \, dt = \int_0^\infty e^{2\alpha t} H(t) H(t)' \, dt = \int_0^\infty e^{2\alpha t} \|H(t)\|^2 \, dt \,.$$

We now use Holders inequality to obtain:

$$\begin{aligned}
\int_0^\infty \|H(t)\| \, dt &= \int_0^\infty e^{-\alpha t} e^{\alpha t} \|H(t)\| dt \\
&\leq \left(\int_0^\infty (e^{-\alpha t})^2 dt \right)^{\frac{1}{2}} \left(\int_0^\infty (e^{\alpha t} \|H(t)\|)^2 dt \right)^{\frac{1}{2}} \\
&= \left(\int_0^\infty e^{-2\alpha t} dt \right)^{\frac{1}{2}} \left(\int_0^\infty e^{2\alpha t} \|H(t)\|^2 dt \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{2\alpha} CSC' \right)^{\frac{1}{2}} .
\end{aligned}$$

This implies that

$$\|H\|_1 \leq \left(\frac{1}{2\alpha} CSC' \right)^{\frac{1}{2}} .$$

19.4.2 Systems with impulses in impulse response

$$H(t) = H_0(t) + \delta(t - \tau_1) H_1 + \delta(t - \tau_2) H_2 + \cdots \tag{19.7}$$

where H_0 is piecewise continuous, the matrices $H_1, H_2, \ldots$ are constant matrices, δ is the unit impulse function and $\tau_1, \tau_2, \ldots \geq 0$ are constants.

Example 186

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}$$

Here

$$H(t) = Ce^{At} B + \delta(t) D \,.$$

Note that when H is given by (19.7),

$$y(t) = \int_0^t H_0(t - \tau) u(\tau) \, d\tau + H_1 u(t - \tau_1) + H_2 u(t - \tau_2) + \cdots$$

Here we generalize the $\mathcal{L}_1$ norm to

$$\|H\|_{\mathcal{A}} = \|H_0\|_1 + \|H_1\| + \|H_2\| + \cdots$$

Also, whenever $\|H\|_{\mathcal{A}}$ is bounded, we have

$$\|y\|_p \leq \|H\|_{\mathcal{A}} \|u\|_p \,. \tag{19.8}$$

19.4.3 $\mathcal{L}^2$ gain and H_∞ norm

Consider a convolution system described by (19.2) and suppose the impulse response H is piecewise continuous. The Laplace Transform of H is a matrix valued function of a complex variable s and is given by

$$\hat{H}(s) = \int_0^\infty H(t)e^{-st}dt\,.$$

for all complex s where the integral is defined. If $\hat{u}$ and $\hat{y}$ are the Laplace transforms of u and y, respectively, then the convolution system can be described by

$$\hat{y}(s) = \hat{H}(s)\hat{u}(s) \tag{19.9}$$

and $\hat{H}$ is called the transfer function of the system.

Suppose H has no poles in the closed right half of the complex plane. If H is $\mathcal{L}^1$, we have

$$||\hat{H}(s)|| \leq ||H||_1$$

for all s with $\Re(s) \geq 0$.

H_∞ norm of a transfer function. Consider a transfer function $\hat{H}: \mathbb{C} \to \mathbb{C}^{m\times o}$. Suppose $\hat{H}$ is analytic in the open right half complex plane. Then, the H_∞-norm of $\hat{H}$ is defined by

$$||\hat{H}||_\infty = \sup_{\omega\in\mathbb{R}} ||\hat{H}(\jmath\omega)||$$

where

$$||\hat{H}(\jmath\omega)|| = \sigma_{\max}[\hat{H}(\jmath\omega)]\,.$$

Note that if H is $\mathcal{L}^1$ then

$$||\hat{H}||_\infty \leq ||H||_1$$

Theorem 49 *Consider a convolution system (19.2) where the impulse response H is piecewise continuous and let $\hat{H}$ be the Laplace transform of H. Then, this system is $\mathcal{L}^2$ stable if and only if $\hat{H}$ is defined for all s with $\Re(s) > 0$ and $||\hat{H}||_\infty$ is finite. In this case, whenever u is $\mathcal{L}^2$,*

$$\boxed{||y||_2 \leq ||\hat{H}||_\infty ||u||_2} \tag{19.10}$$

Actually, one can show that $||\hat{H}||_\infty$ is the $\mathcal{L}_2$ gain of the system in the sense that

$$\|\hat{H}\|_\infty = \sup_{u\in\mathcal{L}_2,\, u\neq 0} \frac{\|y\|_2}{\|u\|_2}$$

Example 187 Consider the SISO system,

$$\begin{aligned} \dot{x} &= -\alpha x + u \\ y &= x \\ x(0) &= 0 \end{aligned}$$

with $\alpha > 0$. Here

$$\hat{H}(s) = \frac{1}{s+\alpha}$$

Since $\hat{H}(\jmath\omega) = 1/(\jmath\omega + \alpha)$, it follows that $|\hat{H}(\jmath\omega)| = 1/(\omega^2 + \alpha^2)^{\frac{1}{2}}$. Hence

$$\|\hat{H}\|_\infty = \frac{1}{\alpha}\,.$$

Thus, the $\mathcal{L}_2$ gain of this system is $1/\alpha$.

19.4.4 Finite dimensional linear systems

$$\dot{x} = Ax + Bu \tag{19.11a}$$
$$y = Cx + Du \tag{19.11b}$$
$$x(0) = x_0 \tag{19.11c}$$

Here,

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)\,.$$

The transfer function of this system is given by

$$\hat{H} = C(sI - A)^{-1}B + D$$

Theorem 50 *Suppose (A, B) is stabilizable and (C, A) is detectable. Then the following statements are equivalent.*

(i) All the eigenvalues of the matrix A have negative real parts.

(ii) All the poles of the transfer function $\hat{H}$ have negative real parts.

(iii) The input-output system (19.11) is $\mathcal{L}^p$ stable for some $p \in [1, \infty]$.

(iv) The input-output system (19.11) is $\mathcal{L}^p$ stable for all $p \in [1, \infty]$.

LMI characterization of $\mathcal{L}^2$ gain

Here we provide an upper bound on the $\mathcal{L}^2$ gain of a finite-dimensional linear time-invariant system described by (19.11).

Suppose there is a symmetric matrix $P \geq 0$ and a scalar $\gamma \geq 0$ such that

$$\begin{pmatrix} PA + A'P + C'C & PB + C'D \\ B'P + D'C & -\gamma^2 I + D'D \end{pmatrix} \leq 0\,. \tag{19.12}$$

Then,

$$\int_0^T \|y(t)\|^2\,dt \leq x_0'Px_0 + \gamma^2 \int_0^T \|u(t)\|^2\,dt$$

for all $T \geq 0$. Hence, whenever u is $\mathcal{L}^2$ then so is y and

$$\|y\|_2 \leq \beta + \gamma\|u\|_2 \qquad \text{where} \qquad \beta = \sqrt{x_0'Px_0}$$

Remark 14 Inequality (19.12) is an LMI in the variables P and γ^2. The scalar γ is an upper bound on the $\mathcal{L}^2$ gain of the system.

Proof. Consider any solution $x(\cdot)$ of (19.11) and let $v(t) = x(t)'Px(t)$. Then $v_0 := v(0) = x_0'Px_0$ and

$$\dot{v} = 2x'P\dot{x} = 2x'PAx + 2x'PBu\,.$$

It now follows from the matrix inequality (19.12) that

$$x'(PA + A'P + CC')x + x'(PB + C'D)u + u'(B'P + D'C)x + u'(-\gamma^2 I + DD')u \le 0\,,$$

that is,

$$2x'PAx + 2x'PBu + \|y\|^2 - \gamma^2\|u\|^2 \le 0\,.$$

Recalling the definition of v and the expression for $\dot{v}$, we now obtain that

$$\dot{v}(t) + \|y\|^2 - \gamma^2\|u\|^2 \le 0$$

for all $t \ge 0$. Hence, for any $T \ge 0$,

$$v(t) - v(0) + \int_0^T \|y(t)\|^2\,dt - \gamma^2 \int_0^T \|u(t)\|\,dt \le 0\,.$$

Since $v(t) \ge 0$ and $v(0) = x_0'Px_0$, we obtain that

$$\int_0^T \|y(t)\|^2\,dt \le x_0'Px_0 + \gamma^2 \int_0^T \|u(t)\|\,dt$$

∎

LMI characterization of $\mathcal{L}^\infty$ gain

Here we provide an upper bound on the $\mathcal{L}^\infty$ gain of a finite-dimensional linear time-invariant system described by (19.11).

Suppose there is a symmetric matrix $P \geq 0$ and scalars $\alpha > 0$, $\mu_1, \mu_2 \geq 0$ such that

$$\begin{pmatrix} PA + A'P + 2\alpha P & PB \\ B'P & -2\alpha\mu_1 I \end{pmatrix} \leq 0 \tag{19.13}$$

and

$$\begin{pmatrix} -P + C'C & C'D \\ D'C & -\mu_2 I + D'D \end{pmatrix} \leq 0 \tag{19.14}$$

and let

$$\gamma = (\mu_1 + \mu_2)^{\frac{1}{2}}.$$

Then, for all $t \geq 0$,

$$\boxed{\|y(t)\| \leq \beta e^{-\alpha t} + \gamma \|u\|_\infty} \tag{19.15}$$

where $\beta = (x_0' P x_0)^{\frac{1}{2}}$. Hence, whenever u is $\mathcal{L}^\infty$ then so is y and

$$\boxed{\|y\|_\infty \leq \beta + \gamma \|u\|_\infty} \tag{19.16}$$

Remark 15 For a fixed α, inequalities (19.13) and (19.14) are LMIs in the variables P, μ_1 and μ_2. The scalar γ is an upper bound on the $\mathcal{L}^\infty$ gain of the system.

Proof. Consider any solution $x(\cdot)$ of (19.11) and let $v(t) = x(t)'Px(t)$. Then $v_0 := v(0) = x_0'Px_0$ and

$$\dot{v} = 2x'P\dot{x} = 2x'PAx + 2x'PBu.$$

It now follows from the matrix inequality (19.13) that

$$x'(PA + A'P + 2\alpha P)x + x'PBu + u'B'Px - 2\alpha\mu_1 u'u \leq 0,$$

that is,

$$2x'PAx + 2x'PBu \leq -2\alpha x'Px + 2\alpha\mu_1\|u\|^2.$$

Recalling the definition of v and the expression for $\dot{v}$, we now obtain that

$$\dot{v}(t) \leq -2\alpha v(t) + 2\alpha\mu_1\|u(t)\|^2$$

for all $t \geq 0$. Suppose now $u(\cdot)$ is $\mathcal{L}^\infty$. Then $\|u(t)\| \leq \|u\|_\infty$ for all $t \geq 0$ and

$$\dot{v}(t) \leq -2\alpha v(t) + 2\alpha\mu_1\|u\|_\infty^2.$$

for all $t \geq 0$. It now follows that

$$\begin{aligned} v(t) &\leq v(0)e^{-2\alpha t} + \int_0^t e^{-2\alpha(t-\tau)} 2\alpha\mu_1\|u\|_\infty^2 \, d\tau \\ &= v_0 e^{-2\alpha t} + \mu_1\|u\|_\infty^2(1 - e^{-2\alpha t}) \\ &\leq v_0 e^{-2\alpha t} + \mu_1\|u\|_\infty^2. \end{aligned}$$

Thus,

$$v(t) \le v_0 e^{-2\alpha t} + \mu_1 \|u\|_\infty^2 \,. \tag{19.17}$$

Recalling the matrix inequality (19.14) we obtain that

$$x'(-P + C'C)x + x'CDu + u'D'Cx + u'(-\mu_2 I + D'D)u \le 0\,,$$

that is,

$$\|y\|^2 = x'CCx + 2x'CDu + u'DDu \le x'Px + \mu_2 \|u\|^2 \,.$$

Hence, for all $t \ge 0$,

$$\|y(t)\|^2 \le v(t) + \mu_2 \|u(t)\|^2 \le v_0 \, e^{-2\alpha t} + \mu_1 \|u\|_\infty^2 + \mu_2 \|u\|_\infty^2 \,,$$

that is,

$$\|y(t)\|^2 \le v_0 \, e^{-2\alpha t} + \gamma^2 \|u\|_\infty^2 \,.$$

Taking the square root of both sides on this inequality yields

$$\|y(t)\| \le \beta \, e^{-\alpha t} + \gamma \|u\|_\infty$$

where $\beta = v_0^{\frac{1}{2}} = (x_0' P x_0)^{\frac{1}{2}}$. This is the desired result. ■

19.4.5 Linear differential delay systems

Consider a system with a delay described by

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + Bu(t) && (19.18a) \\ y(t) &= Cx(t) && (19.18b) \\ x(t) &= 0 \qquad -h \le t \le 0 && (19.18c) \end{aligned}$$

Suppose Φ is the unique solution to

$$\begin{aligned} \dot{\Phi}(t) &= A_0\Phi(t) + A_1\Phi(t-h) \\ \Phi(0) &= I \\ \Phi(t) &= 0 \text{ for } -h \le t < 0 \,. \end{aligned}$$

Then,

$$y(t) = \int_0^t H(t-\tau)u(\tau)\,d\tau \qquad \text{where} \qquad H(t) = C\Phi(t)B\,. \tag{19.19}$$

Proof: Consider

$$x(t) = \int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau$$

for $t \ge -h$. We will show that this is the unique solution to

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + Bu(t) && (19.20) \\ x(t) &= 0 \qquad -h \le t \le 0 && (19.21) \end{aligned}$$

Clearly $x(t) = 0$ for $-h \le t \le 0$. Now note that for $t \ge 0$,

$$x(t-h) = \int_0^{t-h} \Phi(t-h-\tau)Bu(\tau)\,d\tau = \int_0^t \Phi(t-h-\tau)Bu(\tau)\,d\tau$$

Now differentiate x to obtain

$$\begin{aligned} \dot{x}(t) &= \Phi(0)Bu(t) + \int_0^t \dot{\Phi}(t-\tau)Bu(\tau)\,d\tau \\ &= Bu(t) + \int_0^t (A_0\Phi(t-\tau) + A_1\Phi(t-\tau-h))Bu(\tau)\,d\tau \\ &= Bu(t) + A_0\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau + A_1\int_0^t \Phi(t-\tau-h)Bu(\tau)\,d\tau \\ &= A_0x(t) + A_1x(t-h) + Bu(t)\,, \end{aligned}$$

that is, x satisfies (19.20). Thus

$$y(t) = Cx(t) = C\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau = \int_0^t H(t-\tau)Bu(\tau)\,d\tau$$

where $H(t) = C\Phi(t)B$. ■

Example 188 Consider a system with delay h described by

$$\begin{aligned} \dot{x}(t) &= -ax(t) - bx(t-h) + u \\ y(t) &= x(t) \\ x(t) &= 0 \qquad \text{for} \quad -h \le t \le 0 \end{aligned}$$

with

$$|b| < a\,, \qquad h \ge 0$$

The transfer function of this system is given by

$$\hat{H}(s) = \frac{1}{d(s)} \qquad \text{where} \qquad d(s) = s + a + be^{-hs}$$

We first show that $\hat{H}$ is analytic for $\Re(s) \ge 0$. Suppose $\lambda = \alpha + \jmath\omega$ is a pole of $\hat{H}$; then λ is a zero of d, that is,

$$\alpha + \jmath\omega + a + be^{-h(\alpha+\jmath\omega)} = 0$$

Hence,

$$\alpha = -a - be^{-\alpha h}\cos(\omega h) \le -a + |b|e^{-\alpha h} \le -a + |b| < 0\,.$$

We now obtain an upper bound on the H_∞ norm of $\hat{H}$.

19.5 A class of $\mathcal{L}^p$ stable nonlinear systems

Consider an input-output system described by

$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= H(x,u) \\ x(0) &= x_0 \end{aligned}$$

We make the following assumptions.

There is a continuously differentiable function V and a positive scalars β_1, β_2, β_3 and α such that

$$\beta_1\|x\|^2 \leq V(x) \leq \beta_2||x||^2$$

$$DV(x)F(x,0) \leq -2\alpha V(x)$$

$$||DV(x)|| \leq \beta_3||x||$$

There are non-negative scalars β_4, β_5, and β_6 such that

$$||F(x,u) - F(x,0)|| \leq \beta_4||u||$$

$$||H(x,u)|| \leq \beta_5||x|| + \beta_6||u||\,.$$

Claim: *The above system is $\mathcal{L}^p$ stable for all $p \in [1,\infty]$.*

Example 189

$$\begin{aligned} \dot{x} &= -x - x^3 + u \\ y &= \sin x \end{aligned}$$

Proof of claim:

19.6 Small gain theorem

19.6.1 Stability of feedback systems

Figure 19.2: A Feedback system

Figure 19.3: Feedback system for stability definition

19.6.2 Truncations

Suppose $s : \mathcal{T} \to \mathbb{R}^n$ is a signal and consider any time T. The corresponding truncation of s is defined by

$$s_T(t) := \begin{cases} s(t) & \text{for} \quad 0 \leq t < T \\ 0 & \text{for} \quad T \leq t \end{cases}$$

Note that a system $\mathcal{G}$ is causal if and only if for every input u and every time T, we have

$$\mathcal{G}(u)_T = \mathcal{G}(u_T)_T \, .$$

For any $\mathcal{L}^p$ signal s, we have $\|s_T\|_p \le \|s\|_p$. A signal s is $\mathcal{L}^p_e$ if and only if s_T is $\mathcal{L}^p$ for all T. Also, an $\mathcal{L}^p_e$ signal s is an $\mathcal{L}^p$ signal if there exists a constant β such that $\|s_T\|_p \le \beta$ for all T.

19.6.3 Small gain theorem

Suppose $\mathcal{U}_1$ and $\mathcal{U}_2$ are vector spaces of $\mathcal{L}^p_e$ signals with the property that if they contain a signal, they also contain every truncation of the signal. Suppose $\mathcal{G}_1 : \mathcal{U}_1 \to \mathcal{U}_2$ and $\mathcal{G}_2 : \mathcal{U}_2 \to \mathcal{U}_1$ are causal $\mathcal{L}^p$-stable systems which satisfy

$$\begin{array}{rcl} ||\mathcal{G}_1(u_1)||_p & \le & \beta_1 + \gamma_1 ||u_1||_p \\ ||\mathcal{G}_2(u_2)||_p & \le & \beta_2 + \gamma_2 ||u_2||_p \end{array} \tag{19.22}$$

Consider the feedback system described by

$$\begin{array}{rcl} y_1 & = & \mathcal{G}_1(u_1) \\ y_2 & = & \mathcal{G}_2(u_2) \\ u_1 & = & r_1 + y_2 \\ u_2 & = & r_2 + y_1 \end{array} \tag{19.23}$$

We assume that for each pair (r_1, r_2) of signals with $r_1 \in \mathcal{U}_1$ and $r_2 \in \mathcal{U}_2$, the above relations uniquely define a pair (y_1, y_2) of outputs with $y_1 \in \mathcal{U}_2$ and $y_2 \in \mathcal{U}_1$.

Theorem 51 (Small gain theorem) *Consider a feedback system satisfying the above conditions and suppose*

$$\boxed{\gamma_1\gamma_2 < 1}$$

Then, whenever r_1 and r_2 are $\mathcal{L}^p$ signals, the signals u_1 u_2, y_1, y_2 are $\mathcal{L}^p$ and

$$\begin{array}{rcl} ||u_1||_p & \le & (1-\gamma_1\gamma_2)^{-1}(\beta_2 + \gamma_2\beta_1 + ||r_1||_p + \gamma_2 ||r_2||_p) \\ ||u_2||_p & \le & (1-\gamma_1\gamma_2)^{-1}(\beta_1 + \gamma_1\beta_2 + ||r_2||_p + \gamma_1 ||r_1||_p) \end{array}$$

Hence,

$$\begin{array}{rcl} ||y_1||_p & \le & \beta_1 + \gamma_1(1-\gamma_1\gamma_2)^{-1}(\beta_2 + \gamma_2\beta_1 + ||r_1||_p + \gamma_2 ||r_2||_p) \\ ||y_2||_p & \le & \beta_2 + \gamma_2(1-\gamma_1\gamma_2)^{-1}(\beta_1 + \gamma_1\beta_2 + ||r_2||_p + \gamma_1 ||r_1||_p) \end{array}$$

and the feedback system is $\mathcal{L}^p$ stable.

Proof. Suppose r_1 and r_2 are $\mathcal{L}^p$ signals. Since r_1, r_2 and y_1, y_2 are $\mathcal{L}^p_e$ signals, it follows that u_1 and u_2 are $\mathcal{L}^p_e$ signals. Consider any time $T > 0$. Since all of the signals r_1, r_2, u_1, u_2, y_1, y_2 are $\mathcal{L}^p_e$, it follows that their respective truncations r_{1T}, r_{2T}, u_{1T}, u_{2T}, y_{1T}, y_{2T} are $\mathcal{L}^p$ signals. Since $\mathcal{G}_1$ is causal,

$$y_{1T} = \mathcal{G}_1(u_1)_T = \mathcal{G}_1(u_{1T})_T$$

Hence,

$$\|y_{1T}\|_p = \|\mathcal{G}_1(u_{1T})_T\|_p \le \|\mathcal{G}_1(u_{1T})\|_p \le \beta_1 + \gamma_1\|u_{1T}\|_p$$

Since $u_{2T} = r_{2T} + y_{1T}$, it now follows that

$$\begin{aligned}\|u_{2T}\|_p &\leq \|r_{2T}\|_p + \|y_{1T}\|_p \\ &\leq \beta_1 + \|r_{2T}\|_p + \gamma_1\|u_{1T}\|_p\end{aligned}$$

In a similar fashion we can show that

$$\|u_{1T}\|_p \leq \beta_2 + \|r_{1T}\|_p + \gamma_2\|u_{2T}\|_p$$

Hence,

$$\|u_{1T}\|_p \leq \beta_2 + \|r_{1T}\|_p + \gamma_2\left(\beta_1 + \|r_{2T}\|_p + \gamma_1\|u_{1T}\|_p\right)$$

Rearranging yields

$$\begin{aligned}\|u_{1T}\|_p &\leq (1-\gamma_1\gamma_2)^{-1}\left(\beta_2 + \gamma_2\beta_1 + \|r_{1T}\|_p + \gamma_2\|r_{2T}\|_p\right) \\ &\leq (1-\gamma_1\gamma_2)^{-1}\left(\beta_2 + \gamma_2\beta_1 + \|r_1\|_p + \gamma_2\|r_2\|_p\right)\end{aligned}$$

Since the above inequality holds for all $T > 0$, it follows that u_1 is an $\mathcal{L}^p$ signal and

$$\|u_1\|_p \leq (1-\gamma_1\gamma_2)^{-1}\left(\beta_2 + \gamma_2\beta_1 + \|r_1\|_p + \gamma_2\|r_2\|_p\right)$$

In a similar fashion we can show that

$$\|u_2\|_p \leq (1-\gamma_1\gamma_2)^{-1}\left(\beta_1 + \gamma_1\beta_2 + \|r_2\|_p + \gamma_1\|r_1\|_p\right)$$

■

19.7 Application of the small gain theorem: circle criterion

Figure 19.4: A Feedback system

Consider a nonlinear system which is a feedback combination of a convolution system

$$y(t) = \int_0^t H(t-\tau)u(\tau)\,d\tau$$

and a memoryless nonlinear system

$$u(t) = \phi(t, y(t))\,.$$

Suppose that the impulse response H of the convolution system is $\mathcal{L}^1$ and, for some $\gamma \geq 0$, the nonlinearity satisfies

$$\|\phi(t,y)\| \leq \gamma\|y\| \tag{19.24}$$

for all t and y.

Since the impulse response H of the convolution system is $\mathcal{L}^1$, the $\mathcal{L}^2$ gain of the convolution system is $\|\hat{H}\|_\infty$ where $\hat{H}$ is the system transfer function. We now claim that the nonlinear system is $\mathcal{L}^2$ stable with "gain" γ. To see this, consider $u(t) = \phi(t, y(t))$ and note that, for any $T > 0$,

$$\int_0^T \|u(t)\|^2\,dt = \int_0^T \|\phi(t,y(t))\|^2\,dt \leq \int_0^T (\gamma\|y(t)\|)^2\,dt = \gamma^2\int_0^T \|y(t)\|^2\,dt\,.$$

If y is $\mathcal{L}^2$ then

$$\int_0^T \|u(t)\|^2\,dt \leq \gamma^2\int_0^\infty \|y(t)\|^2\,dt$$

for all $T \geq 0$; hence

$$\int_0^\infty \|u(t)\|^2\,dt \leq \gamma^2\int_0^\infty \|y(t)\|^2\,dt$$

This implies that u is $\mathcal{L}_2$ and $\|u\|_2 \leq \gamma\|y\|_2$. Hence the nonlinear system is $\mathcal{L}^2$ stable with "gain" γ. From the small gain theorem we obtain that the feedback system is $\mathcal{L}^2$ stable if

$$\boxed{\gamma\|\hat{H}\|_\infty < 1} \tag{19.25}$$

For a SISO system, the above condition is equivalent to the requirement that the Nyquist plot of the system transfer function lie within the circle of radius $1/\gamma$ centered at the origin. As a consequence, the above condition is sometimes referred to as the circle criterion.

Example 190 Consider

$$\dot{x} = -ax(t) + b\sin(x(t-h))$$

where $a > |b|$. Letting $y(t) = x(t-h)$ this system can be described by

$$\begin{aligned} \dot{x}(t) &= -ax(t) + u(t) \\ y(t) &= x(t-h) \\ u(t) &= b\sin(y(t)) \end{aligned}$$

Here

$$\hat{H}(s) = e^{-sh}/(s+a) \qquad \text{and} \qquad \phi(t,y) = b\sin(y)\,.$$

Thus,

$$|\hat{H}(\jmath\omega)|^2 = |e^{\jmath\omega}|^2/(\omega^2+a^2) = 1/(\omega^2+a^2)\,;$$

so, $\|H\|_\infty = 1/a$. Also,

$$|b\sin(y)| \leq |b||y|\,;$$

hence the condition (19.24) is satisfied with $\gamma = |b|$. Since $a > |b|$, we obtain that

$$\gamma\|H\|_\infty = |b|/a < 1.$$

Figure 19.5: Another feedback system

19.7.1 A more general circle criterion

Consider a nonlinear system which is a feedback combination of a SISO convolution system

$$y(t) = \int_0^t H(t-\tau)u(\tau)\,d\tau$$

and a SISO memoryless nonlinear system

$$u(t) = -\phi(t, y(t))\,.$$

Suppose that the impulse response H of the convolution system is $\mathcal{L}^1$ and, for some scalars a and b, the nonlinearity satisfies the sector condition:

$$ay^2 \le y\phi(t,y) \le by^2 \tag{19.26}$$

for all t and y.

We will convert this system into one considered in the previous section. To this end, introduce a new input

$$w = u + ky \qquad \text{where} \quad k = \frac{a+b}{2}\,.$$

Then

$$w = \tilde{\phi}(t,y) \tag{19.27}$$

where $\tilde{\phi}(t,y) := -\phi(t,y) + ky$ and, it follows from (19.26) that

$$-\gamma y^2 \le y\tilde{\phi}(t,y) \le \gamma y^2 \qquad \text{where} \quad \gamma = \frac{b-a}{2}$$

Thus,

$$|\tilde{\phi}(t,y)| \le \gamma |y| \tag{19.28}$$

Since $\hat{y} = \hat{H}\hat{u}$ and $\hat{w} = \hat{u} + k\hat{y}$, we obtain that

$$\hat{y} = \hat{H}(\hat{w} - k\hat{y})$$

Hence

$$\hat{y} = \hat{G}\hat{w} \tag{19.29}$$

where

$$\hat{G} = (1 + k\hat{H})^{-1}\hat{H}\,.$$

Thus the systems under consideration here can be described by (19.29) and (19.27) where $\tilde{\phi}$ satisfies (19.28). Using the circle criterion of the last section, this system is $\mathcal{L}_2$ stable provided all the poles of $\hat{G}$ have negative real parts and

$$\gamma\|\hat{G}\|_\infty < 1 \tag{19.30}$$

This last condition can be expressed as

$$\gamma^2 \hat{G}(\jmath\omega)'\hat{G}(\jmath\omega) < 1\,.$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$, that is,

$$\gamma^2 \left[[1 + k\hat{H}(\jmath\omega)]^{-1}\hat{H}(\jmath\omega)\right]' \left[[1 + k\hat{H}(\jmath\omega)]^{-1}\hat{H}(\jmath\omega)\right] < 1$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$, Recalling the definitions of γ and k, the above inequality can be rewritten as

$$-ab\hat{H}(\jmath\omega)'\hat{H}(\jmath\omega) - k[\hat{H}(\jmath\omega) + \hat{H}(\jmath\omega)'] < 1 \tag{19.31}$$

We have three cases to consider:

Case 1: $ab > 0$ In this case, inequality (19.31) can be rewritten as

$$[\hat{H}(\jmath\omega) + c]'[\hat{H}(\jmath\omega) + c] > R^2$$

where

$$c = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)\,, \qquad R = \frac{1}{2}\left|\frac{1}{a} - \frac{1}{b}\right|$$

This is equivalent to

$$|H(\jmath\omega) + c| > R\,.$$

that is, the Nyquist plot of $\hat{H}$ must lie outside the circle of radius R centered at $-c$. Note that this is the circle which intersects the real axis at $-1/a$ and $-1/b$.

Figure 19.6: Circle criterion for $0 < a < b$

Case 2: $ab < 0$: In this case (19.31) can be rewritten as

$$[\hat{H}(\jmath\omega) + c]'[\hat{H}(\jmath\omega) + c] < R^2$$

This is equivalent to

$$|H(\jmath\omega) + c| < R\,.$$

that is, the Nyquist plot of $\hat{H}$ must lie inside the circle of radius R centered at $-c$. Again, this is the circle which intersects the real axis at $-1/a$ and $-1/b$.

Figure 19.7: Circle criterion for $a < 0 < b$

Case 3: $ab = 0$: Suppose $a = 0$ and $b > 0$. In this case $k = b/2$ and (19.31) can be rewritten as

$$-(b/2)(\hat{H}(\jmath\omega) + \hat{H}(\jmath\omega)') < 1$$

that is,

$$\Re\left(\hat{H}(\jmath\omega)\right) > -\frac{1}{b} \tag{19.32}$$

This means that the Nyquist plot of $\hat{H}$ must lie to the right of the vertical line which intersects the real axis at $-1/b$.

Figure 19.8: "Circle" criterion for $a = 0$ and $0 < b$

References

1. Desoer, C.A. and Vidyasagar, M., "Feedback Systems: Input-Output Properties," Academic Press, 1975.

Chapter 20

Nonlinear H_∞

20.1 Analysis

Consider a system,

$$\begin{aligned}\dot{x} &= F(x) + G(x)w \\ z &= H(x)\end{aligned}$$

with initial condition

$$x(0) = x_0$$

where $w : [0, \infty) \to \mathbb{R}^m$ is a *disturbance input* and z is a *performance output* with $z(t) \in \mathbb{R}^p$.

Desired performance. We wish that the system has the following property for some $\gamma > 0$. For every initial state x_0 there is a real number β_0 such that for every disturbance input w one has

$$\boxed{\int_0^\infty ||z(t)||^2\, dt \quad \leq \quad \gamma^2 \int_0^\infty ||w||^2\, dt + \beta_0}$$

20.1.1 The HJ Inequality

Theorem 52 *Suppose there is a function V which for all x, satisfies*

$$\boxed{DV(x)F(x) + \frac{1}{4\gamma^2} DV(x)G(x)G(x)^T DV^T(x) + H(x)^T H(x) \leq 0} \tag{20.1}$$

$$V(x) \geq 0$$

for some $\gamma > 0$. Then for every initial state x_0 and for every disturbance input w one has

$$\int_0^\infty ||z(t)||^2\, dt \quad \leq \quad \gamma^2 \int_0^\infty ||w||^2\, dt + V(x_0) \tag{20.2}$$

PROOF. Consider any initial state x_0 and any disturbance input w. Along any resulting solution $x(\cdot)$, we have

$$\frac{dV(x(t))}{dt} = L(x(t), w(t))$$

where

$$\begin{aligned} L(x,w) &= DV(x)F(x) + DV(x)G(x)w \\ &= DV(x)F(x) + \frac{1}{4\gamma^2}||G(x)^T DV(x)^T||^2 + \gamma^2||w||^2 - ||\gamma w - \frac{1}{2\gamma}G(x)^T DV(x)^T||^2 \\ &\leq DV(x)F(x) + \frac{1}{4\gamma^2}||G(x)^T DV(x)^T||^2 + \gamma^2||w||^2 \end{aligned}$$

Using inequality (20.1) we have:

$$DV(x)F(x) + \frac{1}{4\gamma^2}||G(x)^T DV(x)^T||^2 \leq -||H(x)||^2$$

Hence

$$L(x,w) \leq -||H(x)||^2 + \gamma^2||w||^2$$

i.e.,

$$\frac{dV(x(t))}{dt} \leq -||z(t)||^2 + \gamma^2||w(t)||^2$$

Integrating from 0 to T yields:

$$V(x(T)) - V(x_0) \leq -\int_0^T ||z(t)||^2 dt + \gamma^2 \int_0^T ||w(t)||^2\, dt \tag{20.3}$$

Rearranging terms and recalling that $V(x(T)) \geq 0$, we obtain

$$\begin{aligned} \int_0^T ||z(t)||^2 dt &\leq \gamma^2 \int_0^T ||w(t)||^2\, dt + V(x_0) - V(x(T)) \\ &\leq \gamma^2 \int_0^T ||w(t)||^2\, dt + V(x_0) \end{aligned}$$

If w is not $\mathcal{L}_2$, then $\int_0^\infty ||w(t)||^2\, dt = \infty$ and (20.2) trivially holds. If w is $\mathcal{L}_2$, the integral $\int_0^\infty ||w(t)||^2\, dt$ is finite, hence $\int_0^\infty ||z(t)||^2 dt$ is finite and satisfies (20.2). ■

Hamilton Jacobi equation

$$\boxed{DV(x)F(x) + \frac{1}{4\gamma^2}DV(x)G(x)G(x)^T DV^T(x) + H(x)^T H(x) = 0} \tag{20.4}$$

The "worst case" w is given in feedback form as

$$w = w_*(x) = \frac{1}{2\gamma^2}G(x)^T DV(x)^T$$

Quadratic V. Suppose

$$V(x) = x^T P x$$

where P is a symmetric real matrix. Then $DV(x) = 2x^T P$ and the HJ inequality is

$$2x^T PF(x) + \gamma^{-2} x^T PG(x)G(x)^T Px + H(x)^T H(x) \leq 0$$

LTI system. Suppose

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx \end{aligned}$$

Then the HJ is

$$DV(x)Ax + \frac{1}{4\gamma^2} DV(x)BB^T DV(x) + x^T C^T Cx \leq 0$$

If $V(x) = x^T Px$, this results in

$$2x^T PAx + \gamma^{-2} x^T PBB^T Px + x^T C^T Cx \leq 0$$

This is satisfied for all x iff P satisfies

$$\boxed{PA + A^T P + \gamma^{-2} PBB^T P + C^T C \leq 0}$$

20.2 Control

$$
\begin{aligned}
\dot{x} &= F(x) + G_1(x)w + G_2(x)u \\
z &= \begin{bmatrix} H(x) \\ u \end{bmatrix}
\end{aligned}
$$

with initial condition

$$x(0) = x_0$$

Desired performance. We wish to choose a state feedback controller

$$u = k(x)$$

such that the resulting closed loop system

$$
\begin{aligned}
\dot{x} &= F(x) + G_2(x)k(x) + G_1(x)w \\
z &= \begin{bmatrix} H(x) \\ k(x) \end{bmatrix}
\end{aligned}
$$

has the following property for some $\gamma > 0$. For every initial state x_0 there is a real number β_0 such that for every disturbance input w one has

$$\int_0^\infty ||z(t)||^2\,dt \quad \leq \quad \gamma^2 \int_0^\infty ||w||^2\,dt + \beta_0$$

- We can guarantee the desired performance if there is a function $V \geq 0$ satisfying

$$DV(F + G_2 k) + \frac{1}{4\gamma^2} DV G_1 G_1^T DV^T + C^T C + k^T k \leq 0$$

Suppose $V \geq 0$ satisfies The Hamilton-Jacobi Issacs (HJI) inequality

$$\boxed{DVF + \frac{1}{4} DV(\,\gamma^{-2} G_1 G_1^T - G_2 G_2^T\,) DV^T + C^T C \leq 0} \tag{20.5}$$

Then letting

$$\boxed{k = -\tfrac{1}{2} G_2^T DV^T}$$

the desired inequality holds.

20.3 Series solution to control problem

We wish to solve the HJI equation,

$$\boxed{DVF + \tfrac{1}{4}DV(\,\gamma^{-2}G_1G_1^T - G_2G_2^T\,)DV^T + C^TC = 0} \tag{20.6}$$

for $V \geq 0$. Letting

$$G(x) := [G_1(x)\ G_2(x)] \qquad Q(x) := H(x)^T H(x), \qquad R := \begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix}$$

the HJI equation (20.6) can be rewritten as

$$DVF - \frac{1}{4}DVGR^{-1}B^TDV^T + Q = 0 \tag{20.7}$$

Also letting

$$v_* := \begin{bmatrix} u_* \\ w_* \end{bmatrix} \qquad\qquad w_* = \frac{1}{2\gamma^2}G_1^TDV^T$$

we have

$$v_* = -\frac{1}{2}R^{-1}G^TDV^T \tag{20.8}$$

20.3.1 Linearized problem

$$\dot{x} \;=\; Ax + Bv \tag{20.9a}$$

$$z \;=\; \begin{bmatrix} Cx \\ u \end{bmatrix} \tag{20.9b}$$

with

$$A = DF(0)\,, \quad B = G(0), \qquad C = DH(0)$$

Here the HJI equation (20.6) becomes

$$DV(x)Ax - \frac{1}{4}DV(x)BR^{-1}B^TDV(x) + x^TC^TCx = 0$$

Considering a quadratic form

$$V(x) = x^TPx$$

with P symmetric, as a candidate solution to the above HJE we obtain

$$x^T[PA + A^TP - PBR^{-1}B^TP + C^TC]x = 0$$

This is satisified for all x iff the matrix P solves the following Algebraic Riccati Equation (ARE)

$$PA + A^TP - PBR^{-1}B^TP + C^TC = 0 \tag{20.10}$$

From standard linear $\mathcal{H}_\infty$ theory, a necessary and sufficient condition for the linear system to have $\mathcal{L}_2$-gain less than γ is that the above Algebraic Riccati Equation (ARE) has a solution P with $A - BR^{-1}BP$ Hurwitz. When (C, A) is observable, this solution P is positive definite.

In this case v_* is given by

$$v_*(x) = -R^{-1}B^TPx \tag{20.11}$$

20.3.2 Nonlinear problem

First note that the HJI equation (20.7) can be written as

$$DVF - v_*^T R v_* + Q = 0 \tag{20.12a}$$

$$v_* + \frac{1}{2}R^{-1}G^T DV^T = 0 \tag{20.12b}$$

Suppose

$$\begin{aligned} F &= F^{[1]} + F^{[2]} + F^{[3]} + \cdots \\ G &= G^{[0]} + G^{[1]} + G^{[2]} + \cdots \\ H &= H^{[1]} + H^{[2]} + H^{[3]} + \cdots \end{aligned}$$

where $F^{[k]}$, $G^{[k]}$ and $H^{[k]}$ are homogeneous functions of order k. Note that

$$F^{[1]}(x) = Ax, \qquad G^{[0]}(x) = B, \qquad H^{[1]}(x) = Cx$$

We consider a series expansion for V of the form

$$V = V^{[2]} + V^{[3]} + V^{[4]} + \cdots \tag{20.13}$$

where $V^{[k]}$ is a homogeneous function of order k. Substituting (20.13) in equation (20.12b) one obtains that

$$v_* = v_*^{[1]} + v_*^{[2]} + v_*^{[3]} + \cdots \tag{20.14}$$

where $v_*^{[k]}$ is the homogeneous function of order k given by

$$v_*^{[k]} = -\frac{1}{2}R^{-1}\sum_{j=0}^{k-1} G^{[j]\,T} DV^{[k+1-j]\,T} \tag{20.15}$$

Substitution into (20.12a) and equating terms of order $m \geq 2$ to zero yields

$$\sum_{k=0}^{m-2} DV^{[m-k]}F^{[k+1]} \quad - \quad \sum_{k=1}^{m-1} v_*^{[m-k]\,T} R v_*^{[k]} \quad + \quad Q^{[m]} = 0 \tag{20.16}$$

For $m = 2$ equation (20.16) simplifies to

$$DV^{[2]}F^{[1]} \quad - \quad v_*^{[1]} R v_*^{[1]} \quad + \quad Q^{[2]} = 0$$

Since $F^{[1]}(x) = Ax$,

$$\begin{aligned} v_*^{[1]} &= -\frac{1}{2}R^{-1}G^{[0]\,T}DV^{[2]\,T} \\ &= -\frac{1}{2}R^{-1}B^T DV^{[2]\,T} \end{aligned}$$

and

$$Q^{[2]}(x) = x^T C^T C x$$

we obtain

$$DV^{[2]}(x)Ax - \frac{1}{4}DV^{[2]\,T}(x)BR^{-1}B^TDV^{[2]}(x) + x^TC^TCx = 0$$

which is the HJI equation for the linearized problem. Hence

$$V^{[2]}(x) = x^TPx$$

where $P^T = P > 0$ solves the ARE with $A_* := A - BR^{-1}B^TP$ Hurwitz. Hence

$$v_*^{[1]}(x) = -R^{-1}B^TPx$$

Consider now any $m \geq 3$. Using

$$v_*^{[m-1]} = -\frac{1}{2}R^{-1}\sum_{k=0}^{m-2} G^{[k]\,T}DV^{[m-k]\,T}$$

the sum of the terms in (20.16) depending on $V^{[m]}$ is given by:

$$\begin{aligned}
& DV^{[m]}F^{[1]} - v_*^{[m-1]\,T}Rv_*^{[1]} - v_*^{[1]\,T}Rv_*^{[m-1]} \;=\; DV^{[m]}F^{[1]} - 2v_*^{[m-1]\,T}Rv_*^{[1]} \\
= \;& DV^{[m]}F^{[1]} + DV^{[m]}G^{[0]}v_*^{[1]} + \sum_{k=1}^{m-2} DV^{[m-k]}G^{[k]}v_*^{[1]} \\
= \;& DV^{[m]}f^{[1]} + \sum_{k=1}^{m-2} DV^{[m-k]}G^{[k]}v_*^{[1]}
\end{aligned}$$

where

$$f = F + Gv_*^{[1]} \tag{20.17}$$

and

$$f^{[1]} = A_*x \tag{20.18}$$

The sum of the terms on the lefthandside of (20.16) can now be written as:

$$\begin{aligned}
& DV^{[m]}f^{[1]} + \sum_{k=1}^{m-2} DV^{[m-k]}F^{[k+1]} + \sum_{k=1}^{m-2} DV^{[m-k]}G^{[k]}v_*^{[1]} - \sum_{k=2}^{m-2} v_*^{[m-k]\,T}Rv_*^{[k]} + Q^{[m]} \\
= \;& DV^{[m]}f^{[1]} + \sum_{k=1}^{m-2} DV^{[m-k]}f^{[k+1]} - \sum_{k=2}^{m-2} v_*^{[m-k]\,T}Rv_*^{[k]} + Q^{[m]}
\end{aligned}$$

Hence, using

$$f^{[k+1]} = F^{[k+1]} + G^{[k]}v_*[1]$$

(20.16) can be written as

$$\boxed{DV^{[m]}f^{[1]} \quad = \quad -\sum_{k=1}^{m-2} DV^{[m-k]}f^{[k+1]} \quad + \quad \sum_{k=2}^{m-2} v_*^{[m-k]\,T}Rv_*^{[k]} \quad - \quad Q^{[m]}}$$

For example, with $Q^{[m]} = 0$,

$$\begin{aligned} DV^{[3]} f^{[1]} &= -DV^{[2]} f^{[2]} \\ &= -DV^{[2]} (F^{[2]} + G^{[1]} v_*^{[1]}) \end{aligned}$$

hence

$$v_*^{[2]} = -\frac{1}{2} R^{-1} (B^T DV^{[3]} + G^{[1]\,T} DV^{[2]})$$

For $m = 4$:

$$DV^{[4]} f^{[1]} = -DV^{[3]} f^{[2]} - DV^{[2]} f^{[3]} + v_*^{[2]\,T} R v_*^{[2]}$$

and

$$v_*^{[3]} = -\frac{1}{2} R^{-1} (B^T DV^{[4]} + G^{[1]\,T} DV^{[3]} + G^{[2]\,T} DV^{[2]})$$

If

$$F^{[2]}, G^{[1]} = 0$$

Then

$$V^{[3]}, v_*^{[2]} = 0$$

and

$$DV^{[4]} f^{[1]} = -DV^{[2]} f^{[3]}$$

$$v_*^{[3]} = -\frac{1}{2} R^{-1} (B^T DV^{[4]} + G^{[2]\,T} DV^{[2]})$$

Example 191

$$\begin{aligned} \dot{x} &= x - x^3 + w + u \\ z &= u \end{aligned}$$

HJI equation

Chapter 21

Performance

21.1 Analysis

Consider a system with output described by

$$\dot{x} = f(t,x) \tag{21.1a}$$
$$z = h(t,x) \tag{21.1b}$$

where z is a performance output.

Lemma 17 *Suppose that there exists a differentiable function V such that $V(x) \geq 0$ and*

$$DV(x)f(t,x) + h(t,x)'h(t,x) \leq 0 \tag{21.2}$$

for all t and x. Then every solution of system (21.1) satisfies

$$\int_{t_0}^{\infty} \|z(t)\|^2 \, dt \leq V(x_0) \tag{21.3}$$

where $x_0 = x(t_0)$.

Example 192 Consider

$$\begin{aligned}\dot{x} &= -\alpha x\\ z &= x\end{aligned}$$

where $\alpha > 0$. Let $V(x) = x^2/2\alpha$.

Example 193

$$\begin{aligned}\dot{x} &= -x^3\\ z &= x^2\end{aligned}$$

Let $V(x) = x^2/2$.

Example 194

$$\begin{aligned}\dot{x} &= -2x + \sin x\\ z &= x\end{aligned}$$

Let $V(x) = x^2/2$.

Linear time-invariant systems. Consider a LTI system described by

$$\dot{x} = Ax \tag{21.4}$$

$$z = Cx \tag{21.5}$$

where all the eigenvalues of A have negative real part. The Lyapunov equation

$$PA + A'P + C'C = 0 \tag{21.6}$$

has a unique solution for P; moreover P is symmetric and positive semi-definite. Post- and pre-multiplying this equation by x and its transpose yields

$$2x'PAx + (Cx)'Cx = 0\,;$$

thus, (21.2) holds with $V(x) = x'Px$. Actually, in this case one can show that

$$\int_{t_0}^{\infty} \|z(t)\|^2\, dt = V(x_0)\,.$$

21.2 Polytopic systems

21.2.1 Polytopic models

Here $\mathcal{V}$ is a vector space; so $\mathcal{V}$ could be $\mathbb{R}^n$ or the set of real $n \times m$ matrices or some more general space. Suppose $v_1, \ldots, v_l$ is a finite number of elements of $\mathcal{V}$. A vector v is a convex combination of $v_1, \ldots, v_l$ if it can be expressed as

$$v = \lambda_1 v_1 + \cdots + \lambda_l v_N$$

where the real scalars $\lambda_1, \ldots, \lambda_N$ satisfy $\lambda_j \geq 0$ for $j = 1, \ldots, N$ and

$$\lambda_1 + \cdots + \lambda_l = 1\,.$$

The set of all convex combinations of $v_1, \ldots, v_N$ is called a polytope. The vectors $v_1, \ldots, v_N$ are called the vertices of the polytope.

Polytopes and LMIs. Suppose that

$$L(v) \leq 0$$

is an LMI in v. Then this LMI holds for all v in a polytope if and only if it holds for the vertices $v_1, \ldots, v_N$ of the polytope, that is,

$$L(v_j) \leq 0 \qquad \text{for} \quad j = 1, \ldots, N\,. \tag{21.7}$$

To see this, suppose that (21.7) holds. Since $L(v)$ is affine in v, we can express it as

$$L(v) = L_0 + L_1(v)$$

where $L_1(v)$ is linear in v. Suppose v is in the polytope; then $v = \sum_{j=1}^{\infty} \lambda_j v_j$ where $\lambda_j \geq 0$ and $\sum_{j=1}^{\infty} \lambda_j = 1$. Hence

$$L(v) = L_0 + L_1(\Sigma_{j=1}^{\infty}\lambda_j v_j) = \Sigma_{j=1}^{\infty}\lambda_j L_0 + \Sigma_{j=1}^{\infty}\lambda_j L_1(v_j) = \Sigma_{j=1}^{\infty} L(v_j) \leq 0\,.$$

Obtaining polytopic models. Consider a parameter δ which satisfies

$$\underline{\delta} \leq \delta \leq \overline{\delta}\,.$$

Then, δ can be expressed as

$$\delta = \underline{\delta} + \mu(\overline{\delta} - \underline{\delta})$$

where $\mu = (\delta - \underline{\delta})/(\overline{\delta} - \underline{\delta})$. Clearly $0 \leq \mu \leq 1$. If we let $\lambda_1 = 1 - \mu$ and $\lambda_2 = \mu$ then, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$ and

$$\delta = \lambda_1 \underline{\delta} + \lambda_2 \overline{\delta}\,,$$

that is δ can be written as a convex combination of its two extreme values $\underline{\delta}$ and $\overline{\delta}$. This is a special case of the general result contained in the next remark.

Remark 16 This remark is useful in obtaining polytopic models. Suppose G is a matrix valued function of some variable ξ and for all ξ,

$$G(\xi) = \tilde{G}(\delta(\xi))$$

where δ is some parameter vector (which can depend on ξ). Suppose that $\tilde{G}$ depends in a multi-affine fashion on the components of the l-vector δ and each component of δ is bounded, that is,

$$\underline{\delta}_k \leq \delta_k(\xi) \leq \overline{\delta}_k \qquad \text{for} \qquad k = 1, \ldots, l$$

where $\underline{\delta}_1, \ldots, \underline{\delta}_l$ and $\overline{\delta}_1, \ldots, \overline{\delta}_l$ are known constants. Then, for all ξ, the vector $G(\xi)$ can be expressed as a convex combination of the $N = 2^l$ matrices $G_1, \ldots, G_N$ corresponding to the extreme values of the components of δ; these vertices G_j are given by

$$\tilde{G}(\delta) \quad \text{where} \quad \delta_k = \underline{\delta}_k \text{ or } \overline{\delta}_k \text{ for } k = 1, \ldots, l\,. \tag{21.8}$$

In applying to show satisfaction of Condition 1 below, let $\xi = (t, x)$, $G(\xi) = (A(t,x)\ C(t,x))$. Then $(A_j\ C_j) = G_j$ for $j = 1, \ldots, N$.

Example 195 Consider

$$G(\xi) = \left(\begin{array}{cc} \cos\xi_1 \sin\xi_2 + \cos\xi_1 & 1 + \cos\xi_1 + \sin\xi_2 \end{array} \right)$$

For all ξ, the matrix $G(\xi)$ can be expressed as a convex combination of the following four matrices:

$$\left(\begin{array}{cc} 0 & -1 \end{array} \right), \quad \left(\begin{array}{cc} -2 & 1 \end{array} \right), \quad \left(\begin{array}{cc} 0 & 1 \end{array} \right), \quad \left(\begin{array}{cc} 2 & 3 \end{array} \right).$$

Remark 17 The following remark is also useful for obtaining polytopic models. Consider an n-vector-valued function g of two variables t and ξ which is differential with respect to its second variable and satisfies the following two conditions:
(a)

$$g(t, 0) \equiv 0\,.$$

(b) There are matrices, $G_1, \ldots, G_N$ such that, for all t and ξ, the derivative matrix

$$\frac{\partial g}{\partial \xi}(t, \xi)$$

can be expressed as a convex combination of $G_1, \ldots, G_N$.

Then, for each t and ξ,

$$g(t, \xi) = G\xi$$

where G is some convex combination of $G_1, \ldots, G_N$.

In applying this to show satisfaction of Condition 1, let $\xi = x$, $g(t, \xi) = (f(t, x) \quad h(t, x))$. Then

$$\frac{\partial g}{\partial \xi}(t, \xi) = \left[\begin{array}{cc} \frac{\partial f}{\partial x}(t, x) & \frac{\partial h}{\partial x}(t, x) \end{array} \right]$$

and $(A_j \; C_j) = G_j$ for $j = 1, \ldots, N$.

Example 196 Consider

$$g(t, \xi) = \left(\begin{array}{c} \sin\xi_1 \\ \frac{\xi_2}{1+|\xi_2|} \end{array} \right).$$

Here $g(t, 0) = 0$ and

$$\frac{\partial g}{\partial \xi} = \left(\begin{array}{cc} \cos\xi_1 & 0 \\ 0 & \frac{1}{(1+|\xi_2|)^2} \end{array} \right).$$

Hence, $g(t, \xi) = G\xi$ where G can be expressed as a convex combination of the following four matrices:

$$\left(\begin{array}{cc} -1 & 0 \\ 0 & 0 \end{array} \right), \quad \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), \quad \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).$$

21.2.2 Performance analysis of polytopic systems

In this section, we consider a specific type of uncertain/nonlinear system described by (21.1). Specifically, we consider systems which satisfies the following condition.

Condition 1 There are matrices, A_j, C_j, $j = 1, \ldots, N$ so that for each t and x,

$$\begin{aligned} f(t,x) &= Ax && (21.9)\\ h(t,x) &= Cx && (21.10) \end{aligned}$$

where the matrix $(A\ C)$ (which can depend on t, x) is a convex combination of $(A_1\ C_1), \ldots, (A_N\ C_N)$.

So, when Condition 1 is satisfied, system 21.1 can be described by

$$\begin{aligned} \dot{x} &= A(t,x)x && (21.11a)\\ z &= C(t,x)x && (21.11b) \end{aligned}$$

where $(A(t,x)\ C(t,x))$ is contained in the polytope whose vertices are $(A_1\ C_1), \ldots, (A_N\ C_N)$. Hence, we sometimes refer to a system satisfying Condition 1 as a polytopic uncertain/nonlinear system.

Suppose that

$$PA_j + A_j'P + C_j'C_j \leq 0 \qquad \text{for} \quad j = 1, \cdots, N\,. \tag{21.12}$$

Using a Schur complement result, the above inequalities can be expressed as

$$\begin{pmatrix} PA_j + A_j'P & C_j' \\ C_j & -I \end{pmatrix} \leq 0 \qquad \text{for} \quad j = 1, \cdots, N\,. \tag{21.13}$$

Since the above inequalities are affine in $(A_j\ C_j)$, and $(A(t,x)\ C(t,x))$ is a convex combination of $(A_1\ C_1), \ldots, (A_N\ \ C_N)$ it now follows that

$$\begin{pmatrix} PA(t,x) + A(t,x)'P & C(t,x)' \\ C(t,x) & -I \end{pmatrix} \leq 0$$

for all t and x. Reusing the Schur complement now results in

$$PA(t,x) + A(t,x)'P + C(t,x)'C(t,x) \leq 0$$

for all t and x. Post- and pre-multiplying this equation by x and its transpose yields

$$2x'PA(t,x)x + (C(t,x)x)'C(t,x)x \leq 0$$

that is,

$$2x'Pf(t,x) + h(t,x)'h(t,x) \leq 0$$

for all t and x; thus, (21.2) holds with $V(x) = x'Px$. Hence every solution of the system satisfies

$$\int_{t_0}^{\infty} \|z(t)\|^2\, dt \leq x_0'Px_0 \tag{21.14}$$

where $x_0 = x(t_0)$.

To obtain a performance estimate for a fixed initial state one could minimize β subject to LMIs (21.13) and

$$x_0' P x_0 - \beta \le 0\,. \tag{21.15}$$

Note that, if we let $S = P^{-1}$ then, (21.13) can be expressed as

$$\begin{pmatrix} A_j S + A_j' S & S C_j' \\ C_j S & -I \end{pmatrix} \le 0 \qquad \text{for} \quad j = 1, \cdots, N \tag{21.16}$$

Also, using Schur complements, inequality (21.15) can be expressed as

$$\begin{pmatrix} -S & x_0 \\ x_0' & -\beta \end{pmatrix} \le 0 \qquad \text{for} \quad j = 1, \cdots, N \tag{21.17}$$

Suppose x_0 lies in a polytope with vertices $x_1, \ldots, x_M$. Then we minimize β subject to (21.16) and

$$\begin{pmatrix} -S & x_k \\ x_k' & -\beta \end{pmatrix} \le 0 \qquad \text{for} \quad k = 1, \cdots, M \tag{21.18}$$

21.3 Control for performance

21.3.1 Linear time-invariant systems

Consider

$$\begin{aligned} \dot{x} &= Ax + Bu & (21.19a) \\ z &= Cx + Du & (21.19b) \end{aligned}$$

In this case

$$\int_0^\infty \|z(t)\|^2\,dt = \int_0^\infty x(t)'C'Cx(t) + 2x(t)'C'Du(t) + u(t)'D'Du(t)\,dt\,.$$

When $C'D = 0$, this reduces to the usual performance encountered in LQR control, that is,

$$\int_0^\infty \|z(t)\|^2\,dt = \int_0^\infty x(t)'Qx(t) + u(t)'Ru(t)\,dt\,;$$

here

$$Q = C'C \qquad \text{and} \qquad R = D'D\,.$$

Suppose

$$u = Kx \tag{21.20}$$

The resulting closed loop system is described by

$$\begin{aligned} \dot{x} &= (A + BK)x & (21.21a) \\ z &= (C + DK)x & (21.21b) \end{aligned}$$

and inequality (21.2) holds for all x if and only if

$$P(A+BK)+(A+BK)'P+(C+DK)'(C+DK)\leq 0\,,$$

that is,

$$PA+A'P+C'C+(PB+C'D)K+K'(B'P+D'C)+K'D'DK\leq 0\,.$$

Assuming that $D'D$ is invertible, this can be expressed as

$$\tilde{P}A+A'\tilde{P}+C'C-(\tilde{P}B+C'D)(D'D)^{-1}(B'\tilde{P}+D'C)+\tilde{Q}\leq 0 \tag{21.22}$$

where $\tilde{P}=P$ and

$$\tilde{Q}=[K+(D'D)^{-1}(B'P+D'C)]'D'D[K+(D'D)^{-1}(B'P+D'C)]\geq 0\,.$$

Suppose P is a stabilizing solution to the following LQR Riccati equation:

$$PA+A'P+C'C-(PB+C'D)(D'D)^{-1}(B'P+D'C)=0\,. \tag{21.23}$$

Then it can be shown that $P\leq\tilde{P}$ where $\tilde{P}$ is any matrix satisfying inequality (21.22) with a stabilizing K. In this case

$$K=-(D'D)^{-1}(B'P+D'C)\,. \tag{21.24}$$

This is the LQR control gain matrix. Note that P is a stabilizing solution if $A-B(D'D)^{-1}(B'P+D'C)$ is asymptotically stable.

21.3.2 Control of polytopic systems

In this section, we consider a specific type of uncertain/nonlinear system described by

$$\dot{x} = F(t,x,u) \tag{21.25a}$$
$$z = H(t,x,u) \tag{21.25b}$$

Specifically, we consider systems which satisfies the following condition.

Condition 2 There are matrices, $A_j,B_j,C_j,D_j,\ j=1,\dots,N$ so that for each t, x and u,

$$F(t,x) = Ax+Bu \tag{21.26}$$
$$H(t,x) = Cx+Du \tag{21.27}$$

where the matrix $(A\ B\ C\ D)$ (which can depend on t,x,u) is a convex combination of $(A_1\ B_1\ C_1\ D_1)\,,\dots,(A_N\ B_N\ C_N\ D_N)$.

So, when Condition 3 is satisfied, a system under consideration can be described by

$$\dot{x} = A(t,x)x+B(t,x)u \tag{21.28a}$$
$$z = C(t,x)x+D(t,x)u \tag{21.28b}$$

where $(A(t,x)\ \ B(t,x)\ \ C(t,x)\ \ D(t,x))$ is contained in the polytope whose vertices are $(A_1\ B_1\ C_1\ D_1),\dots,(A_N\ B_N\ C_N\ D_N)$. Hence, we sometimes refer to a system satisfying Condition 3 as a polytopic uncertain/nonlinear system.

Linear state feedback controllers. We consider

$$u = Kx \tag{21.29}$$

This results in the following closed loop system

$$\dot{x} = [A(t,x) + B(t,x)K]x \tag{21.30a}$$
$$z = [C(t,x) + D(t,x)K]x \tag{21.30b}$$

Note that this system satisfies Condition 1 with vertex matrices $(A_1{+}B_1K,\ C_1{+}D_1K), \ldots, (A_N{+}B_NK)$. Hence

$$\int_{t_0}^{\infty} \|z(t)\|^2 \, dt \le x_0' P x_0$$

where $S = P^{-1}$,

$$\begin{pmatrix} A_jS + B_jL + SA_j' + L'B_j' & SC_j' + L'D_j' \\ C_jS + D_jL & -I \end{pmatrix} \le 0 \qquad \text{for} \qquad j = 1, \ldots, N \tag{21.31}$$

and

$$K = LS^{-1} \tag{21.32}$$

21.3.3 Multiple performance outputs

$$\dot{x} = F(t,x,u) \tag{21.33a}$$
$$z_i = H_i(t,x,u) \quad \text{for} \quad i = 1, \ldots, p \tag{21.33b}$$

Condition 3 There are matrices, $A_j, B_j, C_{1j}, D_{1j} \ldots, C_{pj}, D_{pj}$, $j = 1, \ldots, N$ so that for each t, x and u,

$$F(t,x) = Ax + Bu \tag{21.34}$$
$$H_i(t,x) = C_ix + D_iu \quad \text{for} \quad i = 1, \ldots, p \tag{21.35}$$

where the matrix $(A\ B\ C_1\ D_1\ , \ldots,\ C_p\ D_p)$ (which can depend on t, x, u) is a convex combination of $(A_1\ B_1\ C_{i1}\ D_{i1}\ , \ldots,\ C_{p1}\ D_{p1})\,, (A_N\ B_N\ C_{iN}\ D_{iN}\ , \ldots,\ C_{pN}\ D_{pN})\,.$

$$\int_{t_0}^{\infty} \|z_i(t)\|^2 \, dt \le x_0' P x_0 \qquad \text{for} \qquad i = 1, \ldots, p$$

where $S = P^{-1}$,

$$\begin{pmatrix} A_jS + B_jL + SA_j' + L'B_j' & SC_{ij}' + L'D_{ij}' \\ C_{ij}S + D_{ij}L & -I \end{pmatrix} \le 0 \qquad \text{for} \qquad i = 1, \ldots, p \quad \text{and} \quad j = 1, \ldots, N \tag{21.36}$$

and

$$K = LS^{-1} \tag{21.37}$$

Chapter 22

Appendix

22.1 The Euclidean norm

What is the size or magnitude of a vector? Meet *norm*.

- Consider any complex n vector x.
  The *Euclidean norm* or 2-*norm* of x is the nonegative real number given by

$$\boxed{||x|| := (|x_1|^2 + \ldots + |x_n|^2)^{\frac{1}{2}}}$$

Note that

$$\begin{aligned} ||x|| &= (\bar{x}_1 x_1 + \ldots + \bar{x}_n x_n)^{\frac{1}{2}} \\ &= (x^* x)^{\frac{1}{2}} \end{aligned}$$

If x is *real*, these expressions become

$$\begin{aligned} ||x|| &= (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} \\ &= (x^T x)^{\frac{1}{2}} \end{aligned}$$

```
>>  norm([3; 4])

ans = 5

>>  norm([1; j])

ans = 1.4142
```

Note that in the last example $x^T x = 0$, but $x^* x = 2$.

Properties of $||\cdot||$. The Euclidean norm has the following properties.

(i) For every vector x,

$$||x|| \geq 0$$

and

$$||x|| = 0 \qquad \text{iff} \qquad x = 0$$

(ii) (*Triangle Inequality.*) For every pair of fectors x, y,

$$||x + y|| \leq ||x|| + ||y||$$

(iii) For every vector x and every scalar λ.

$$||\lambda x|| = |\lambda| ||x||$$

• Any real valued function on a vector space which has the above three properties is defined to be a norm.