THEOREM 6.13 *Let Ω be a bounded $C^{2,\alpha}$-domain in $\mathbb{R}^n$ and f be a C^1-function in $\bar{\Omega} \times \mathbb{R}$. Suppose $\underline{u}, \bar{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfy $\underline{u} \le \bar{u}$,*

$$\Delta \underline{u} \ge f(x, \underline{u}) \ \text{in } \Omega, \quad \underline{u} \le 0 \ \text{on } \partial\Omega,$$
$$\Delta \bar{u} \le f(x, \bar{u}) \ \text{in } \Omega, \quad \bar{u} \ge 0 \ \text{on } \partial\Omega.$$

Then there exists a solution $u \in C^{2,\alpha}(\bar{\Omega})$ of

$$\Delta u = f(x,u) \ \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \ \underline{u} \le u \le \bar{u} \text{ in } \Omega.$$

记号约定：$m = \inf_{\bar{\Omega}} \underline{u}$

$M = \sup_{\bar{\Omega}} \bar{u}$.

$\lambda > 0$ 充分大使 $f_z(x,z) \le \lambda$

$\forall (x,z) \in \bar{\Omega} \times \mathbb{R}$.

① 从 $u_0 := \underline{u}$ 开始，构造迭代序列：

假设 $u_k \in C^{2,\alpha}(\bar{\Omega})$ 已规定，则定义 u_{k+1} 为以下问题的解：

$$\begin{cases} Lu := \Delta u - \lambda u = f(x, u_k) - \lambda u_k \\ u|_{\partial\Omega} = 0 \end{cases} \Rightarrow \Delta u = f(x, u_k).$$

$u_k \longrightarrow u$.

我们断言 $\underline{u} \le u_k \le \bar{u}$，$\forall k \in \mathbb{N}$ 成立.

事实上，对 k 用归纳法：$u_0 = \underline{u}$，成立.

其次如果对 k 成立，

则 $\Delta(u_{k+1} - \underline{u}) - \lambda(u_{k+1} - \underline{u}) \le \big(f(x, u_k) - f(x, \underline{u})\big) - \lambda(u_k - \underline{u})$

$= (-\lambda + f_z)(x, \theta)(u_k - \underline{u})$

$L = \Delta - \lambda$

$Lu = \Delta u - \lambda u.$

$\le 0.$ 由归纳假设 ≥ 0.

$L(u_{k+1} - \underline{u}) \le 0$，且 $u_{k+1} - \underline{u}|_{\partial\Omega} \ge 0$

$0 \quad \le 0$

$\Rightarrow u_{k+1} \ge \underline{u}$.

完全类似地有：

$$L(\bar{u}-u_{k+1}) \le -f(x,u_k)+\lambda u_k + f(x,\bar{u})-\lambda\bar{u}$$
$$= \left(f_z(x,\theta)-\lambda\right)\underbrace{(\bar{u}-u_k)}_{\text{由归纳假设}\ \ge 0.}$$
$$\le 0.$$

即： $L(\bar{u}-u_{k+1}) \le 0$ 且 $\underbrace{\bar{u}}_{\ge 0}-\underbrace{u_{k+1}}_{0} \ge 0$ on $\partial\Omega$

$\Longrightarrow$ $\underline{u} \le u_k \le \bar{u}$, as desired.

② 对 $\{u_k\}$ 作整体估计：

THEOREM 5.26 *For some constant $\alpha \in (0,1)$, let Ω be a bounded $C^{2,\alpha}$-domain in $\mathbb{R}^n$, and a_{ij}, b_i, and c be $C^\alpha(\bar{\Omega})$-functions. Suppose $u \in C^{2,\alpha}(\bar{\Omega})$ is a solution of (5.29) for some $f \in C^\alpha(\bar{\Omega})$ and $\varphi \in C^{2,\alpha}(\bar{\Omega})$. Then*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \le C\{\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\bar{\Omega})} + \|\varphi\|_{C^{2,\alpha}(\bar{\Omega})}\},$$

where C is a positive constant depending only on n, α, λ, Ω, and the $C^\alpha(\bar{\Omega})$-norms of a_{ij}, b_i, and c.

COROLLARY 5.28 *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, a_{ij} be continuous functions in Ω, and b_i, c be bounded functions in Ω. For some constant $p > n$, suppose $u \in W^{2,p}(\Omega)$ is a solution of (5.29) for some $f \in L^p(\Omega)$ and $\varphi \in W^{2,p}(\Omega)$. Then*

$$\|u\|_{C^{1,1-\frac{n}{p}}(\Omega)} \le C\{\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|\varphi\|_{W^{2,p}(\Omega)}\},$$

where C is a positive constant depending only on n, p, λ, Ω, the moduli of continuity of a_{ij} and the $L^\infty(\Omega)$-norms of a_{ij}, b_i, and c.

在方程 $\begin{cases} \Delta u_{k+1} - \lambda u_{k+1} = f(x,u_k) - \lambda u_k \\ u_{k+1}\big|_{\partial\Omega} = 0 \end{cases}$ 中，

因为 $u_k \in [m, M]$，故 $f(u_k, x) - \lambda u_k$ 一致有界，

利用整体 $C^{1,\alpha}$ 估计：

$$\|u_k\|_{C^{1,\alpha}(\Omega)} \le C_0 \{\underbrace{\|u_k\|_{L^p(\Omega)}}_{\text{被 } m(\Omega)^{1/p}\cdot(M+m) \text{ 控制}} + \underbrace{\|f(u_k,x)-\lambda u_k\|_{L^p(\Omega)}}_{\text{同理.}}\}$$
$$=: C$$

$\Longrightarrow$ $\|u_k\|_{C^{1,\alpha}(\Omega)}$ 是一致有界的.

于是又可以利用 $C^{2,\alpha}$ 估计：

$$\|u_k\|_{C^{2,\alpha}} \le C'$$

即 u_k 在 $C^{2,\alpha}$ 范数下也是一致有界的。

$\Longrightarrow$ u_k，D^1u_k，D^2u_k 是等度连续的。

定理 1.1 (Arzelà–Ascoli). 对闭区间 $X=[a,b]$ 上的一族连续函数 $\mathscr{F}\subset C(X,\mathbb{R})$ 而言，以下两个条件等价：

- $\mathscr{F}$ 中每个序列都有一致收敛的子列.
- $\mathscr{F}$ 等度连续并且一致有界.

$\Longrightarrow$ 存在 u_k 子列，不妨仍记为 u_k，使：

$$u_k \xrightarrow{C^0} u,\quad Du_k \xrightarrow{C^0} Du,\quad D^2u_k \xrightarrow{C^0} D^2u. \qquad C^2(\bar{\Omega}).$$

$\Longrightarrow$ 可对方程 $\Delta u_k - \lambda u_k = f(x,u_k) - \lambda u_k$ 两端取极限

得：$\Delta u - \lambda u = f(x,u) - \lambda u$

$\Longrightarrow$ $\Delta u = f(x,u)$. #.

COROLLARY 6.14 *Let Ω be a bounded $C^{2,\alpha}$-domain in $\mathbb{R}^n$ and f be a bounded C^1-function in $\bar{\Omega}\times\mathbb{R}$. Then there exists a solution $u\in C^{2,\alpha}(\bar{\Omega})$ of*

$$\begin{aligned} \Delta u &= f(x,u) \quad \text{in } \Omega,\\ u &= 0 \qquad\quad\ \text{on } \partial\Omega. \end{aligned}$$

PROOF: Set

$$M = \sup_{\Omega\times\mathbb{R}} |f|.$$

Let $\underline{u},\bar{u}\in C^{2,\alpha}(\bar{\Omega})$ satisfy

$$\begin{aligned} \Delta\underline{u} &= M \quad \text{in } \Omega, \quad \underline{u}=0 \text{ on } \partial\Omega,\\ \Delta\bar{u} &= -M \quad \text{in } \Omega, \quad \bar{u}=0 \text{ on } \partial\Omega; \end{aligned}$$

then

$$\Delta\underline{u} \ge \Delta\bar{u} \quad \text{in } \Omega, \quad \underline{u}=\bar{u} \text{ on } \partial\Omega.$$

By the maximum principle, we have $\underline{u}\le\bar{u}$ in Ω. It is obvious that

$$\Delta\underline{u} \ge f(x,\underline{u}), \quad \Delta\bar{u} \le f(x,\bar{u}) \ \text{ in } \Omega.$$

Hence $\underline{u}$ and $\bar{u}$ satisfy the conditions in Theorem 6.13. We obtain the desired result by Theorem 6.13. □