- Notations.

Sobolev space. $\underline{\underline{W^{k,p}}}$: $\partial_\alpha f \in \underline{\underline{L^p}}$, $|\alpha| \le k$.

Hilbert space. $H^k = W^{k,2}$

Norm: $\|f\|_{W^{k,p}} = \left(\|f\|_{L^p}^p + \sum_{|\alpha|\le k} \|\partial_\alpha f\|_{L^p}^p\right)^{\frac{1}{p}}$

Inner product: $\langle f, g\rangle_{H^k} = \int fg$.

$\underline{W^{k,p}_{loc}}(U)$: $\forall V \subset\subset U$, $f \in W^{k,p}(V)$
$\bar{V} \subset U$.

$\underline{W^{k,p}_0}(U)$: $\underline{C_c(U)\text{'s closure in } W^{k,p}(U)}$ $(\approx f|_{\partial U} = 0)$

$\underline{H^k_0}$, H^k_{loc} $(p=2)$ $\qquad$ $\underline{H^k_0}$

$\boxed{f|_{\partial U} = 0}$

Δ $\underline{H^1_0\text{'s}}$ inner product: $\underline{\langle f, g\rangle_{H^1_0(\Omega)} \triangleq \int_\Omega \nabla f \cdot \nabla g \, dx}$.

Evans: $\underline{\langle f, g\rangle_{H^1_0(\Omega)} = \int_\Omega (fg + \nabla f \cdot \nabla g) dx}$.

$\|f\|_{H^1_0} = \|f\|_{W^{1,2}_0} = \left(\int_\Omega (f^2 + |\nabla f|^2)\, dx\right)^{\frac{1}{2}}$

Poincaré inequlity: (Evans page 281)

U bounded open $\subseteq \mathbb{R}^n$, $1 \le p < n$, $u \in W^{1,p}_0(U)$,

Then $\underline{\|u\|_{L^q}} \le \underset{p,n,q,U}{C} \underline{\|\nabla u\|_{L^p}}$ When $q=p=2$, $\underline{\underline{\|u\|_{L^2} \le C\|\nabla u\|_{L^2}}}$

Where $q \in [1, p^*]$, p^* is the Sobolev conjugate of p,

$p^* = \frac{np}{n-p}$ (i.e. $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, $p^* > p$) ($\Leftrightarrow$ Sobolev embadding $W^{k,p} \hookrightarrow L^q$)

$$\left(\int |\nabla f|^2\right)^{\frac{1}{2}} \le \left(\int f^2 + |\nabla f|^2\right)^{\frac{1}{2}} \le \sqrt{(1+C^2)}\left(\int |\nabla f|^2\right)^{\frac{1}{2}}$$

$$\left(\int |\nabla f|^2\right)^{\frac{1}{2}} \le \left(\int f^2 + |\nabla f|^2\right)^{\frac{1}{2}} \le \sqrt{(1+c^2)}\left(\int |\nabla f|^2\right)^{\frac{1}{2}}$$

$$\|f\|_{Qing} \qquad \|f\|_{Evans} \qquad \sqrt{1+c^2}\,\|f\|_{Qing}.$$

Banach: complete normed linear space $\sim V^n$

Hilbert: complete linear space with inner product $\sim (V^n, g)$ $\langle \cdot, \cdot \rangle$

Linear functional: $F: B \to \mathbb{R}$. $\sim$ 余切向量. 1-form. dual.

Bounded linear functional: $|F(X)| \le C \cdot \|X\|$ $\sim V^*$ $\to B^*$ (dual space)

Operator norm on dual space: Banach space ? Hilbert space ?

$$\|F\|_{B^*} = \sup_{X \in B} \frac{|F(X)|}{\|X\|_B} = \sup \left\{ F(X) \,\middle|\, \|X\|_B = 1 \right\}$$

Rieze Representation: For a Hilbert space H, H^* and H 可自然等同, i.e. $F: H \to \mathbb{R}$ bound, $\exists$ unique $Y \in H$ s.t.

$$F(X) = \langle X, Y \rangle, \ \forall X \in H.$$

Moreover, the map $F \mapsto Y$ is isometry. $(\|F\|_{H^*} = \|Y\|_H)$

g 1-form $\sim$ vector $\omega(X) = \langle \omega \lrcorner g, X \rangle$

$\omega = \omega^i dx_i$, $g = \delta_{ij}\, dx_i \otimes dx_j$. $\|\omega\|_{T_p^* M} = \left(\sum (\omega^i)^2\right)^{\frac{1}{2}} = \|\omega \lrcorner g\|_g$

$(\omega^1, \omega^2, \dots, \omega^n)$

- Theorem (Lax-Milgram) H: Hilbert space. $\langle \cdot, \cdot \rangle_H$. $H \xrightarrow{\text{linear}} H^*$ $B(u, \cdot) + B(v, \cdot) = B(u+v, \cdot)$

  $B: H \times H \to \mathbb{R}$ is a bilinear form on H. satisfying:

  inner product. (a) $|B(u, v)| \le \alpha \|u\| \|v\|$ ($\alpha = 1$, by Cauchy)

  (b) $\beta \|u\|^2 \le B(u, u)$ ($\beta = 1$)

  where $\alpha, \beta > 0$.

  $\boxed{B(u, \cdot) = B(v, \cdot) \Rightarrow u = v.}$

where $\alpha, \beta > 0$. $\boxed{B(u,\cdot) = B(v,\cdot) \Rightarrow u = v.}$

Then for any F bounded linear functional in H^*, $\exists$ unique u s.t. $\boxed{B(u,\cdot)} = F$. $\left(\Leftrightarrow \sigma(B(u,\cdot)) = \sigma F \Leftrightarrow Au = \boxed{\sigma F} \right)$

$A: H \to H$ is injective onto H

Proof:

$$\begin{tikzcd} & H^* \arrow[dr, "\sigma"] & \\ H \arrow[ur, "B"] \arrow[rr, "A"'] & & H \end{tikzcd}$$

- $B(u) = B(u,\cdot)$
- By Riesz, $H^* \xrightarrow[\text{iso}]{\sigma} H$
- $\langle A(X), Y \rangle_H = B(X,Y)$.

The conditions reinterpreted by A:

- $|B(u,v)| \le \alpha \|u\| \|v\| \Leftrightarrow \|B(u,\cdot)\|_{H^*} \le \alpha \|u\|_H$

  $\Leftrightarrow \|Au\|_H \le \alpha \|u\|_H$.

- $\beta \|u\|^2 \le B(u,u) \Leftrightarrow \langle Au, u \rangle_H \ge \beta \|u\|_H^2$

  $\left(\Rightarrow \|Au\|_H \ge \beta \|u\|_H \right)$

In conclusion: $\beta\|u\|_H \le \|Au\|_H \le \alpha \|u\|_H$, $\langle Au, u \rangle_H \ge \beta \|u\|_H^2$

(1) injective: by $\|Au\|_H \ge \beta \|u\|_H$

(2) surjective: $(\mathrm{Im} A)^\perp = \{0\}$.

$W \subseteq H$ subspace.

$W^\perp \triangleq \{ x \in H \mid \langle x, v \rangle = 0, \forall v \in W \}$

(1) $W^\perp$ is closed subspace.

(2) If W closed, $H = W \oplus W^\perp$

i.e. $\forall x \in H$, $x = \alpha + \beta$, $\alpha \in W$, $\beta \in W^\perp$

$\mathrm{Im} A$ closed: by $\|Au\|_H \le \alpha \|u\|_H$

If $x \in (\mathrm{Im} A)^\perp$,

$0 = \langle Ax, x \rangle_H \ge \beta \|x\|_H^2 \Rightarrow x = 0$.

- Energy estimate:

$a(u,\cdot) = \langle f, \cdot \rangle$

weak solution: $\int a_{ij} \partial_i u \partial_j \varphi + b_i \partial_i u \varphi + c u \varphi = \int f \varphi$, $\forall \varphi \in H_0^1$

bilinear form: $a(u,v) \triangleq \int a_{ij}\partial_i u \partial_j v + b_i \partial_i u v + cuv$

uniformly elliptic

on Ω. a_{ij}, b_i, c, bounded

a_{ij}: $\lambda\|\xi\|^2 \le \xi^i a_{ij}^{(x)} \xi^j \le \Lambda\|\xi\|^2$

for any ξ, x.

$L'u = Lu + \mu u$, $\mu \ge \gamma$.

1° $|a(u,v)| \le \alpha \|u\|_{H_0^1} \|v\|_{H_0^1}$

2° $\beta \|u\|_{H_0^1}^2 \le a(u,u) + \gamma \|u\|_{L^2}^2$ → coercive

where $\alpha, \beta > 0$, $\gamma \ge 0$.

Proof: 1° $|a(u,v)| \le \int \underset{①}{|a_{ij}\partial_i u \partial_j v|} + \underset{②}{|b_i \partial_i u v|} + \underset{③}{|cuv|}$

① suppose $A = \{a_{ij}\}$. $A \overset{\text{definite}}{=} P^T P$.

$$\int |a_{ij}\partial_i u \partial_j v| = \int |\nabla u^T A \nabla v| = \int |(P\nabla u)^T \cdot (P\nabla v)|$$

$$= \int \|P\nabla u\| \|P\nabla v\| \le \boxed{\sup_\Omega \|P\|^2} \int |\nabla u||\nabla v|$$

$$\le C\left(\int |\nabla u|^2\right)^{\frac{1}{2}} \left(\int |\nabla v|^2\right)^{\frac{1}{2}} = C\|u\|_{H_0^1} \|v\|_{H_0^1}$$

② $$\int |b_i \partial_i u v| = \int |b^T \nabla u||v| \le \sup_\Omega \|b\| \int |\nabla u||v|$$

$$\le C\left(\int |\nabla u|^2\right)^{\frac{1}{2}} \left(\int |v|^2\right)^{\frac{1}{2}} \overset{\text{Poincaré}}{\le} C'\left(\int |\nabla u|^2\right)^{\frac{1}{2}} \left(\int |\nabla v|^2\right)^{\frac{1}{2}}$$

2° $$\lambda \int |\nabla u|^2 \le \int a_{ij}\partial_i u \partial_j u = a(u,u) - \left(\int b_i \partial_i u u + cu^2\right)$$

$$\le a(u,u) + D\int \varepsilon|\nabla u| \cdot \frac{1}{\varepsilon}|u| - \int cu^2$$

$$\le a(u,u) + D\left(\int \varepsilon^2 |\nabla u|^2 + \int \frac{1}{\varepsilon^2}|u|^2\right) - \int cu^2$$

$$\underset{>0}{(\lambda - D\varepsilon^2)} \int |\nabla u|^2 \le a(u,u) + \underset{>0}{\left(\frac{D}{\varepsilon^2} - c\right)} \int u^2$$

取 $\varepsilon \to 0$.

If $\gamma = 0$. i.e. $|a(u,u) \ge \beta \|u\|_{...}|$ → 强制 → ... 存在唯一

If $\gamma=0$. i.e. $\boxed{a(u,u) \geq \beta \|u\|_{H_0^1}}$, 强制 $\Rightarrow$ Dirichlet 存在唯一

If $\gamma>0$, $\exists \mu$, s.t. $\forall \mu' \geq \mu$, Dirichlet solution for $Lu+\mu' u$ 存在且唯一、、

$H_0^1 =$ C_c's closure in $H^1 = W^{1,2}$

$\hookrightarrow$ $f|_{\partial\Omega}=0$.

$H_0^1 = \{f \in H^1 \mid Tf = 0\}$ $f|_{\partial\Omega}=0$.

$f \in W^{1,2}_{(\Omega)} \cap C(\overline{\Omega})$