$B[u,v] = \langle Au, v\rangle$, AX is closed set (try to prove)

$|Ax| \geq b|x|$. $\{Au_n\}$ is a cauchy sequence, $\{u_n\} \to u$, $Au_n \xrightarrow{?} Au$

$|Ax| \leq c|x|$

1) $B[u,v] \leq d|u||v|$, $A: X \to X$, $B[u,v] = \langle Au, v\rangle$, $AX = X$

2) $B[u,u] \geq \beta|u|^2$

$Lu = -(a_{ij} u_{x_i})_{x_j} + b^i u_{x_i} + cu$, $Lu + \mu u = \underline{L_\mu u}$

$$(1) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{in } \partial\Omega. \end{cases}$$

The solution of (1) satisfies that $I[u] = \inf\limits_{w \in H_1} \{I[w]\}$

$I[w] = \int_\Omega \frac{1}{2}|Dw|^2 - wf$, $w|_{\partial\Omega} = g$, $H_1 = \{f : f|_{\partial\Omega} = g\}$

$J(u) = \frac{1}{2}\int_\Omega (a_{ij} D_i u D_j u + \underline{cu^2})\,dx - \underline{\int_\Omega uf\,dx}$, $H_0^1(\Omega)$

$\underline{i(\tau) = J(u + \tau v)}$. $u, \tau \in H_0^1(\Omega)$, $\underline{i'(0) = \int_\Omega a^{ij} D_i u D_j v + cuv\,dx - \int_\Omega vf\,dx = 0}$

$J(u)$ exists a lower bound.

$\underline{\frac{1}{\lambda}|\xi|^2} \leq \underline{a_{ij}\xi_i\xi_j} \leq \Lambda|\xi|^2$, $a_{ij} = a_{ji}$, $c > 0$

$\int_\Omega u^2\,dx \leq c\int_\Omega |Du|^2\,dx$,

$$\int_\Omega |uf|\,dx \le \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega f^2 dx\right)^{\frac{1}{2}}$$

$$\le \sqrt{C}\left(\int |\nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega f^2 dx\right)^{\frac{1}{2}}$$

$$\le \frac{1}{4\lambda}\int_\Omega |\nabla u|^2 dx + C\lambda \int_\Omega f^2 dx$$

$$J(u) = \frac{1}{2}\int_\Omega a_{ij} D_i u D_j u + cu^2\,dx - \int_\Omega uf\,dx$$

$$\ge \frac{1}{2\lambda}|\nabla u|^2 - \int_\Omega |uf|\,dx \ge \frac{1}{4\lambda}\int_\Omega |\nabla u|^2 dx - C\lambda \int_\Omega f^2 dx$$

$$\ge -C\lambda \int_\Omega f^2 dx$$

$J_0 = \inf\{J(u) : u \in H_0^1(\Omega)\}$,

$\{u_k\}$, $J(u_k) \longrightarrow J_0$, $k \to \infty$,

$$\int_\Omega |\nabla u_k|^2 dx \le 4\lambda J(u_k) + 4C\lambda^2 \int_\Omega f^2 dx$$

$\{u_k\}$ is bounded $H_0^1(\Omega)$

By Rellich's theorem, H_0^1 is compacted set in $L^2(\Omega)$

$\{u_{k_i}\} \longrightarrow u$, $\quad J(u) \le \liminf_{k_i \to \infty} J(u_{k_i})$

open mapping theorem: let E, F be two Banach spaces,

and let T be continous linear operator from E to F is surjective,

Then there exists a constant c such that

$$T(B_E(0,1)) \supset B_F(0,c)$$

$$U = B_E(x,r), \quad T(U) = T(x) + T(B_E(0,r))$$

Corollary 1: T is also injective, T^{-1} is continous (bounded) operator,

$$T(B_E(0,1)) \supset B_F(0,c),$$

$\forall x \in E$, if $\|T(x)\| < c$, $\|x\| < 1$, $\quad \|x\| \leq \frac{1}{c}\|Tx\|$

$\exists x' \in B_E(0,1)$, such that $Tx' = Tx$, $x' = x$

$\forall x \in E$, $x \neq 0$, $\|T(kx)\| = c$, $\|kx\| \leq 1 \quad k\|x\| \leq \frac{k}{c}\|Tx\|$

Corollary 2: let E be a vector space with two norms, $\|\cdot\|_1$, $\|\cdot\|_2$,

Assume E be a Banach space, and there exists a constant C,

continous

$$\left(\|x\|_2 \leq C\|x\|_1 \right) \ \forall x \in E,$$

$$T: (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$$

conclusion

$$T^{-1}: (E, \|\cdot\|_2) \to (E, \|\cdot\|_1), \quad \boxed{\|x\|_1 \leq c\|x\|_2}$$

$$\|u\|_{H_0^1} = \int_\Omega |Du|^2 dx, \quad \|u\|'_{H_0^1} = \int_\Omega u^2 + |Du|^2 dx, \quad \text{in } H_0^1(\Omega)$$

Step 1, $E = \bigcup_{k=1}^{\infty} \overline{T(B_E(0,k))}$, $\mathrm{Int}\left(\overline{T(B_E(0,1))}\right) \neq \phi$

$\exists\, y \in \mathrm{Int}\left(\overline{T(B_E(0,1))}\right)$, $\overset{A_1}{B(y,4c)} \subset \overset{A_2}{\overline{T(B(0,1))}}$, $\qquad A_1 + B_1 = \{x+y : x \in A_1, y \in B_1\}$

$\overset{B_1}{-y} \in \overline{T(B(0,1))}\ B_2$

$$B(0,4c) \subset \overline{T(B(0,1))} + \overline{T(B(0,1))} \subseteq \overline{T(B(0,2))}$$

$$B(0,2c) \subset \overline{T(B(0,1))}$$

Step 2 : Try to prove $T(B(0,1)) \supset B(0,c)$

$B(0,c) \subset \overline{TB(0,\frac{1}{2})}$, $\quad B(0,\frac{c}{2^k}) \subset \overline{TB(0,\frac{1}{2^{k+1}})}$

$\forall \|y\| < c$, $\exists\, x_1 \in B(0,\frac{1}{2})$, $\|y - Tx_1\| < \frac{c}{2}$

$y - Tx_1 \subset B(0,\frac{c}{2})$, $\exists\, x_2 \in B(0,\frac{1}{4})$,

$\|(y - Tx_1) - Tx_2\| \leq \frac{c}{4}$,

$\|y - T(x_1 + \cdots x_n)\| \longrightarrow 0$, $n \longrightarrow \infty$, $T(x_1 + \cdots x_n) \longrightarrow y$,

$\|x_k\| \leq \frac{1}{2^k}$, $\quad x_1 + x_2 + \cdots \longrightarrow x$, $\|x\| < 1$