Bochner technique and its applications

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摘要

This article mainly introduces Bochner technique and its two applications: Hodge theory and Cheeger-Gromoll splitting theorem.

1 Introduction

In this part we introduce the main theorems ,necessary background knowledge and notations.

1.1 Introductions to the Main Theorems

The Bochner technique was, as the name indicates, invented by Bochner. However, Bernstein knew about it for harmonic functions on domains in Euclidean space. Specifically, he used

$$\Delta\frac{1}{2}\left|\nabla u\right|^2 = \left|\mathrm{Hess}u\right|^2$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $\Delta u = 0$. It was Bochner who realized that when the same trick is attempted on Riemannian manifolds, a curvature term also appears Namely, for $u : (M, g) \to \mathbb{R}$ with $\Delta_g u = 0$ one has

$$\Delta\frac{1}{2}\left|\nabla u\right|^2 = \left|\mathrm{Hess}u\right|^2 + \mathrm{Ric}\left(\nabla u, \nabla u\right).$$

With this in mind it is clear that curvature influences the behavior of harmonic functions. The next nontrivial step Bochner took was to realize that one can compute $\Delta\frac{1}{2}|\omega|^2$ for any harmonic form ω and then try to get information about the topology of the manifold. The key ingredient here is of course Hodge's theorem, which states that any cohomology class can be uniquely represented by a harmonic form.

We will prove Bochner's formulas and show some of the applications, including basic Hodge theory and Cheeger-Gromoll splitting theorem.Before that, we first explore the integral of functions on Riemann manifolds and introduce Riemann measure in the same time.

1.2 Necessary Background Knowledge

We assume basic knowledge of manifolds and Riemann Geometry. To understand the proof of Hodge theorem, some knowledge about Sobolev spaces and functional analysis such as Riesz Representation theorem and Hahn Banach theorem should also be known. Some of important definitions and theorems are listed below:

Definition 1.1. The Laplacian of a function is defined as in vector calculus by

$$\Delta f = \mathrm{div}\,\nabla f$$

Definition 1.2. The Ricci curvature Ric is a trace or contraction of R. If $e_1, \ldots, e_n \in T_pM$ is an orthonormal basis, then

$$\begin{aligned}\mathrm{Ric}(v, w) &= \mathrm{tr}\left(x \mapsto \mathrm{R}\left(x, v\right) w\right) \\ &= \sum_{i=1}^{n} g\left(R\left(e_i, v\right) w, e_i\right) \\ &= \sum_{i=1}^{n} g\left(R\left(v, e_i\right) e_i, w\right) \\ &= \sum_{i=1}^{n} g\left(R\left(e_i, w\right) v, e_i\right).\end{aligned}$$

Definition 1.3. A ray is a half-line $\gamma_+ : [0, \infty) \longrightarrow M$,which is a normal minimizing geodesic.

Definition 1.4. (Partition of Unity) A partition of unity subordinate to Σ is an indexed family $(\psi_\alpha)_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

(i) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.

(ii) $\operatorname{supp} \psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.

(iii) The family of supports $(\operatorname{supp} \psi_\alpha)_{\alpha \in A}$ is locally finite, meaning that every point has a neighborhood that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many values of α.

(iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Theorem 1.5. (Existence of Partition of Unity) Suppose M is a smooth manifold with or without boundary, and $\Sigma = (X_\alpha)_{\alpha \in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to Σ

Theorem 1.6. (Riesz Representation theorem) Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$ .Thenthere exists a σ -algebra $\mathfrak{M}$ in X which contains all Borel sets in X, and there exists a unique positive measure μ on $\mathfrak{M}$ which represents Λ in the sense that

(a) $\Lambda f = \int_X f \, d\mu$ for every $f \in C_c(X)$, and which has the following additional properties:

(b) $\mu(K) < \infty$ for every compact set $K \subset X$.

(c) For every $E \in \mathfrak{M}$, we have

$$\mu(E) = \inf\{\mu(V) : E \subset V, V \text{ open}\}.$$

(d) The relation

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$$

holds for every open set E, and for every $E \in \mathfrak{M}$ with $\mu(E) < \infty$.

(e) If $E \in \mathfrak{M}$, $A \subset E$, and $\mu(E) = 0$, then $A \in \mathfrak{M}$.

Theorem 1.7. (Rellich lemma). Let $\{\omega_i\}$ be a bounded sequence in $\Omega^*(M)$ with respect to the H^1-norm. Then $\{\omega_i\}$ contains a subsequence that is a Cauchy sequence with respect to the H^0-norm.

Theorem 1.8. (Gärding inequality).

(1) There exist constants $c_1, c_2 > 0$ so that for any $\omega \in \Omega^*(M)$,

$$(\Delta\omega, \omega) \geq c_1 \|\omega\|_1^2 - c_2 \|\omega\|_0^2.$$

(2) There exists a constant $c_0 > 0$ so that for any $\omega \in \mathcal{H}^\perp$,

$$\|\omega\|_1^2 \leq c_0 (\Delta\omega, \omega).$$

Theorem 1.9. (Weyl's lemma). If $\omega \in H^1(M)$ such that $\Delta\omega = \eta \in \Omega^*(M)$ weakly, then $\omega \in \Omega^*(M)$.

We will use these theorems without proof.

1.3 Notations

Without extra explanation, we assume the following notations:

- M refers to a Riemann manifold.
- g refers to the Riemann metric and (g_{ij}) is it's local coordinates. G refers to the determinant of g_{ij} .
- $\partial_i = \dfrac{\partial}{\partial x_i}$ refers to the basis vector in the tangent space T_pM for some point $p \in M$.
- Riemann curvature is given by $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$,while Rm is a $(0,4)$-tensor defined by $Rm(X,Y,Z,T) = g(R(X,Y)Z,T)$.
- $\Omega^k(M)$ denotes the differential k-forms on $M, \Omega^*(M) = \oplus \Omega^k(M)$.

2 Riemann measure and Differential Operators

In this part we introduce measure structure onto Riemann manifold and derive some basic results of the gradient operator and Laplacian operator for reference.

2.1 Riemann measure

2.1.1 Integrals of compactly supported continuous functions

Definition 2.1. (Integral in one chart) Let (M,g)be a Riemannian manifold. f is a continuous function with compact support, so that $supp(f)$is contained in one chart (ϕ, U, V). Define

$$\int_M f dV_g := \int_V (f\sqrt{G}) \circ \varphi^{-1} dx^1 \cdots dx^m$$

Lemma 2.2. The definition above is independent of the choices of coordinate charts containing $supp(f)$

证明. Let $(\tilde{\varphi}, \tilde{U}, \tilde{V})$ be another coordinate chart containing $\mathrm{supp}(f)$, on which the coordinates are denoted by $y^1, \cdots, y^m$. Then

$$(g_{ij}) = J^T(\tilde{g}_{kl})J,$$

where $J = \left(\dfrac{\partial y^i}{\partial x^j}\right)$ is the Jacobian of the map $\widetilde{\varphi} \circ \varphi^{-1}$.

As a consequence, we get

$$\sqrt{G(p)} = \sqrt{\widetilde{G}(p)} \cdot |\det(J(\varphi(p)))|,$$

for $p = \varphi^{-1}(x) = \tilde{\varphi}^{-1}(y)$. Thus, by the change of variables in $\mathbb{R}^m$,

$$\sqrt{\widetilde{G} \circ \widetilde{\varphi}^{-1}}\, dy^1 \cdots dy^m = \sqrt{\widetilde{G} \circ \widetilde{\varphi}^{-1}(\widetilde{\varphi} \circ \varphi^{-1})} \cdot |\det(J)|\, dx^1 \cdots dx^m = \sqrt{G \circ \varphi^{-1}}\, dx^1 \cdots dx^m.$$

$\square$

Note that by locally finiteness of U_α and compactness of $\mathrm{supp}(f)$, the sum is in fact a finite sum. Moreover, if $\{(\tilde{\varphi}_\beta, \tilde{U}_\beta, \tilde{V}_\beta)\}$ is another atlas, the by the above Lemma ,

$$\int_{\varphi_\alpha(U_\alpha\cap\tilde{U}_\beta)} (f\rho_\alpha\tilde{\rho}_\beta\sqrt{G^\alpha})\circ(\varphi_\alpha)^{-1}dx_\alpha^1\cdots dx_\alpha^m = \int_{\tilde{\varphi}_\beta(U_\alpha\cap\tilde{U}_\beta)} (f\rho_\alpha\tilde{\rho}_\beta\sqrt{G^\beta})\circ(\varphi_\beta)^{-1}dx_\beta^1\cdots dx_\beta^m$$

since both sides equal to $\int_M \rho_\alpha\tilde{\rho}_\beta f dV_g$, which implies

$$\sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (f\rho_\alpha\sqrt{G^\alpha})\circ(\varphi_\alpha)^{-1}dx_\alpha^1\cdots dx_\alpha^m = \sum_\beta \int_{\varphi_\beta(U_\beta)} (f\rho_\beta\sqrt{G^\beta})\circ(\varphi_\beta)^{-1}dx_\beta^1\cdots dx_\beta^m.$$

In other words, $\int_M f dV_g$ is well-defined for any $f \in C_c(M)$.

Definition 2.3. (Integral globally) Let $\{(\varphi_\alpha, U_\alpha, V_\alpha)\}$ be a system of locally finite coordinate charts that cover M, with local coordinates $\{x_\alpha^1, \cdots, x_\alpha^m\}$ on each U_α, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open covering $\{U_\alpha\}$. Then we define

$$\int_M f dV_g := \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (f\rho_\alpha\sqrt{G^\alpha})\circ(\varphi_\alpha)^{-1}dx_\alpha^1\cdots dx_\alpha^m$$

2.1.2 The Riemannian measure

Since manifolds are always locally compact and Hausdorff, and since the linear functional

$$\mu : C_c(M) \to \mathbb{R}, \quad f \mapsto \mu(f) = \int_M f dV_g$$

is positive (i.e. $f \geq 0$ implies $\mu(f) \geq 0$), by Riesz representation theorem, μ gives rise to a unique Radon measure on M.

Using the standard technique in real analysis, we can extend the integral to more general functions.

- First, define the (upper) integral of a lower semi-continuous positive function f to be the supremum of integrals of compactly-supported functions that are no more than f.
- Then, define the (upper) integral of a positive function f as the infimum of the (upper) integral of all lower semi-continuous positive functions that are greater than f.
- A function f is said to be integrable if there exists a sequence g_n in $C_c(M)$ such that the (upper) integrals of the sequence $|g_n - f|$ converge to 0.

For any $1 \leq p < \infty$ one can define the L^p norm on C_c^∞ via

$$\|f\|_{L^p} := \left(\int_M |f|^p dV_g\right)^{1/p},$$

and define $L^p(M,g)$ to be the completion of C_c^∞ under the L^p norm. Similarly one can define $L^\infty(M,g)$. It is not hard to extend the theory to complex-valued functions. In the special case $p=2$, one can define an inner product structure on $L^2(M,g)$ by

$$\langle f_1, f_2\rangle_{L^2} := \int_M f_1\bar{f}_2 dV_g$$

which make $L^2(M,g)$ into a Hilbert space.

One can also talk about the volume of any Borel set (or more generally, measurable subsets) A in M,which is defined to be

$$\mathrm{Vol}(A) = \int_M \chi_A dV_g$$

Remark 2.4. In the above definition, we do not assume M to be oriented or compact. What we really obtain is a volume density, which, in a local chart, can be written as

$$dV_g = \sqrt{G}\circ\varphi^{-1}\, dx^1\cdots dx^m.$$

We will refer to dVol as the Riemannian volume element (or volume density) on (M,g).

In the special case where M is oriented, then we may choose an orientation-compatible coordinate patch near each point, and define (locally on each chart)

$$\omega_g = \sqrt{G}dx^1\wedge\cdots\wedge dx^m.$$

One can check that ω_g is a well-defned global volume form on M,which is called the Riemannian volume form for the oriented Riemannian manifold (M,g).

2.2 Gradient and Divergence

By musical isomorphism , we can define gradient as follows:

Definition 2.5. The gradient vector field of f is $\nabla f = \sharp(df)$, i.e. $\langle\nabla f, v\rangle = df(v) = v(f), \forall v\in TM$.

Through the Riemannian metric, we can calculate that given $X = X^i\partial_i$,

$$g(\nabla f, X) = df(X) = Xf = X^i\partial_i f.$$

It follows

$$\nabla f = g^{ij}\partial_i f\partial_j.$$

Now let's consider the definition of divergence. Recall the volume form is $\omega_g = \sqrt{G}dx^1\wedge\cdots\wedge dx^m$ under local coordinate $(U, x^1,\cdots,x^m)$ One may choose other coordinates on U, then the corresponding volume forms are either the same, or differ by a negative sign. As a result, the following definition is independent of the choice of coordinate charts:

Definition 2.6. The divergence of X is the function $\mathrm{div}(X)$ on M such that $(\mathrm{div}X)\omega_g = d\left(\iota(X)\omega_g\right).$

Using Cantan's magic formula ,the definition above is equivalent to $\mathcal{L}_X(\omega_g) = \mathrm{div}(X)\omega_g$ This coincides with the geometric definition of divergence in the case of R m: the divergence of a vector field is the infinitesimal rate of change of the volume element along the vector field.

Under local coordinates, say $X = X^i\partial i$,we have

$$\begin{aligned}(\mathrm{div}X)\sqrt{G}dx^1\wedge\cdots\wedge dx^m &= d\left(\iota(X^i\partial_i)\sqrt{G}dx^1\wedge\cdots\wedge dx^m\right)\\ &= d\left(\sum_i X^i\sqrt{G}(-1)^{i-1}dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^m\right)\\ &= \partial_i(X^i\sqrt{G})dx^1\wedge\cdots\wedge dx^m,\end{aligned}$$

so we conclude

$$\mathrm{div}(X^i\partial_i) = \frac{1}{\sqrt{G}}\partial_i(X^i\sqrt{G}).$$

We may replace X by fX to get

$$\mathrm{div}(fX) = f\mathrm{div}X + (\partial_i f)X^i = f\mathrm{div}X + g(\nabla f, X).$$

So we get

Corollary 2.7. For any smooth vector field $X \in \Gamma^\infty(TM)$ and any smooth function $f \in C^\infty(\dot{M})$,one has

$$\mathrm{div}(fX) = f\mathrm{div}X + g(\nabla f, X).$$

As an application, we can prove

Theorem 2.8. (The Divergence theorem I). Let X be a smooth vector field with compact support on a Riemannian manifold (M, g), then

$$\int_M \mathrm{div}(X)dV_g = 0.$$

证明. First we assume that X is supported in a local chart (φ, U, V) and $X = X^i\partial_i$ with $X^i \in C_c^\infty(U)$. Then

$$\begin{aligned}\int_M \mathrm{div}(X)dV_g &= \int_U \frac{1}{\sqrt{G}}\partial_i(X^i\sqrt{G})dV_g\\ &= \int_{\varphi(U)} \partial_i(X^i\sqrt{G}\circ\varphi^{-1})dx^1\cdots dx^m = 0.\end{aligned}$$

The general case follows from partition of unity and the Corollary 2.7

$$\sum_\alpha \rho_\alpha\mathrm{div}(X) = \sum_\alpha \mathrm{div}(\rho_\alpha X) - g(\nabla(\sum_\alpha\rho_\alpha), X) = \sum_\alpha \mathrm{div}(\rho_\alpha X)$$

and thus

$$\int_M \mathrm{div}(X)dV_g = \int_M \sum \rho_\alpha\mathrm{div}(X)dV_g = \int_M \sum_\alpha \mathrm{div}(\rho_\alpha X)dV_g = 0.$$

$\square$

There's more that we can say about gradient.

First, we can define gradient of tensor fields, or , in most literature , the covariant derivative of tensors. Roughly saying, if S is a (p,q)-tensor field, we define ∇S to be a $(p,q+1)$-tensor field by

$$\begin{aligned}\nabla S(X,Y_1,\cdots,Y_q,\omega_1,\cdots,\omega_p) &= (\nabla_X S)(Y_1,\cdots,Y_q,\omega_1,\cdots,\omega_p)\\ &= \nabla_X(S(Y_1,\cdots,Y_q,\omega_1,\cdots,\omega_p))\\ &\quad -\sum_{j=1}^{q} S(\cdots,Y_{j-1},\nabla_X Y_j,Y_{j+1},\cdots)\\ &\quad -\sum_{i=1}^{p} S(\cdots,\omega_{i-1},\nabla_X\omega_i,\omega_{i+1},\cdots).\end{aligned}$$

It is an induction-type definition , we know everything as long as we know the definition of covariant derivative of vector fields.

Second, the following lemmas will be useful for further discussion:

Lemma 2.9. (1) ∇X is symmetric, i.e.$\langle\nabla_u X,v\rangle = \langle\nabla_v X,u\rangle$ for all $u,v\in T_pM$ if and only if $\omega=\flat X$ is closed

(2) ∇X is anty-symmetric i.e.$\langle\nabla_u X,v\rangle = -\langle\nabla_v X,u\rangle$ for all $u,v\in T_pM$ if and only if X is a Killing field

证明. We only prove (1) here, (2) will be proved later. if $\omega=\flat X$ is closed ,then $d\omega = d\flat X = 0$

By definition of exterior derivative, we have $\mathrm{d}\omega(Y,Z) = Y\omega(Z) - Z\omega(Y) - \omega([Y,Z])$

Recall the fact that the Levi-Civita connection is torsion-free,one has

$$\begin{aligned}\mathrm{d}\omega(Y,Z) &= Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z])\\ &= g(\nabla_Y X,Z) - g(\nabla_Z X,Y) + g(X,\nabla_Y Z - \nabla_Z Y - [Y,Z])\\ &= g(\nabla_Y X,Z) - g(\nabla_Z X,Y).\end{aligned}$$

Hence we get

$$\begin{aligned}\mathrm{d}\omega = 0 &\Longleftrightarrow \forall Y,Z,\quad g(\nabla_Y X,Z) = g(\nabla_Z X,Y)\\ &\Longleftrightarrow \nabla X \text{ is symmetric.}\end{aligned}$$

□

2.3 Laplacian

We can define the Hessian of a function on a Riemannian manifold as a $(0,2)$-tensor using Lie-derivative:

Definition 2.10. $\mathrm{Hess}f\,(X,Y) = \frac{1}{2}\left(L_{\nabla f}g\right)(X,Y)$.

In fact , there are several ways to define the Hessian of a function , for example, using the following proposition:

Proposition 2.11. For $f: M\to\mathbb{R}$,

$$\left(\nabla_X df\right)(Y) = g\left(\nabla_X\nabla f,Y\right) = \mathrm{Hess}f\,(X,Y).$$

证明. First observe that

$$\begin{aligned}(\nabla df)(X,Y) &= (\nabla_X df)(Y) \\ &= \nabla_X \nabla_Y f - df(\nabla_X Y) \\ &= \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f.\end{aligned}$$

This shows

$$(\nabla_X df)(Y) - (\nabla_Y df)(X) = [\nabla_X, \nabla_Y] f - \nabla_{[X,Y]} f = 0.$$

Thus $(\nabla_X df)(Y)$is symmetric. This can be used to establish the formulas

$$\begin{aligned}(\nabla df)(X,Y) &= (\nabla_X df)(Y) \\ &= D_X g(\nabla f, Y) - g(\nabla f, \nabla_X Y) \\ &= g(\nabla_X \nabla f, Y) \\ &= \frac{1}{2} g(\nabla_X \nabla f, Y) + \frac{1}{2} g(X, \nabla_Y \nabla f) \\ &= \frac{1}{2}(\nabla_{\nabla f} g)(X,Y) + \frac{1}{2} g(\nabla_X \nabla f, Y) + \frac{1}{2} g(X, \nabla_Y \nabla f) \\ &= \frac{1}{2} D_{\nabla f} g(X,Y) - \frac{1}{2} g([\nabla f, X], Y) - \frac{1}{2} g(X, [\nabla f, Y]) \\ &= \frac{1}{2}(L_{\nabla f} g)(X,Y).\end{aligned}$$

□

Also ,using covariant derivative, we get

$$\nabla^2 f(X,Y) = \nabla(\nabla f(X,Y)) = (\nabla_X df)(Y) = XY(f) - (\nabla_X Y)(f) = g(\nabla_X \nabla f, Y) = \mathrm{Hess} f(X,Y).$$

Definition 2.12. Laplacian of a function $f : M \to \mathbb{R}$ is defined as $\Delta f = \operatorname{div} \nabla f$

Locally, Δf is given by

$$\Delta f = -\mathrm{div}(g^{ij}\partial_i f \partial_j) = -\frac{1}{\sqrt{G}}\partial_i(\sqrt{G} g^{ij}\partial_j f)$$

and we omit the proof of this formula.

3 Bochner Technique

3.1 Bochner's formula for vector fields

Theorem 3.1. Let (M,g) be a Riemannian manifold, and $X \in \Gamma^\infty(TM)$.

If ∇X is symmetric, i.e., $(\nabla_u X, v) = \langle \nabla_v X, u\rangle$ for all $u, v \in T_x M$, then $\frac{1}{2}\Delta(|X|^2) = |\nabla X|^2 + \langle X, \nabla(\operatorname{div} X)\rangle + Ric(X,X)$.

If ∇X is anti-symmetric, i.e., $(\nabla_u X, v) = -\langle \nabla_v X, u\rangle$ for all $u, v \in T_x M$, then $\frac{1}{2}\Delta(|X|^2) = |\nabla X|^2 - Ric(X,X)$.

证明. (1) With Riemannian normal coordinates centered at x ,we have

$$\nabla_{\partial_i}\partial_j(x) = 0, \quad \forall i, j.$$

We know from the last section that

$$(\nabla^2 f)_x(\partial_i, \partial_j) = (\partial_i\partial_j f)(x) \qquad (\Delta f)(x) = \sum(\partial_i\partial_i f)(x).$$

Hence we have

$$\begin{aligned}
\frac{1}{2}\Delta(|X|^2) = \frac{1}{2}\sum_i \partial_i\partial_i\langle X, X\rangle &= \sum_i \partial_i\langle \nabla_{\partial_i}X, X\rangle \\
&\overset{\star}{=} \sum_i \partial_i\langle \nabla_X X, \partial_i\rangle \\
&= \sum_i \langle \nabla_{\partial_i}\nabla_X X, \partial_i\rangle \\
&= \sum_i \big(\langle \nabla_X\nabla_{\partial_i}X, \partial_i\rangle - \langle \nabla_{[X,\partial_i]}X, \partial_i\rangle - \langle R(X,\partial_i)X, \partial_i\rangle\big) \\
&= \sum_i \big(\langle \nabla_X\nabla_{\partial_i}X, \partial_i\rangle - \langle \nabla_{[X,\partial_i]}X, \partial_i\rangle + Rm(X,\partial_i, X, \partial_i)\big) \\
&= \sum_i \big(\langle \nabla_X\nabla_{\partial_i}X, \partial_i\rangle - \langle \nabla_{[X,\partial_i]}X, \partial_i\rangle\big) + Ric(X, X).
\end{aligned}$$

Note that if we write $X = X^i\partial_i$, then

$$\mathrm{Tr}(\nabla X) = \sum\langle \nabla_{\partial_i}X, \partial_i\rangle \overset{\bullet}{=} \sum_i \partial_i\langle X, \partial_i\rangle = \sum_i \partial_i X^i = \mathrm{div}X.$$

It follows that

$$\sum_i\langle \nabla_X\nabla_{\partial_i}X, \partial_i\rangle = \sum_i X\langle \nabla_{\partial_i}X, \partial_i\rangle = X(\mathrm{div}X) = \langle X, \nabla\mathrm{div}X\rangle.$$

On the other hand, since $[\partial_i, X] = \nabla_{\partial_i}X - \nabla_X\partial_i = \nabla_{\partial_i}X$,

$$-\sum_i\langle \nabla_{[X,\partial_i]}X, \partial_i\rangle = \sum_i\langle \nabla_{\nabla_{\partial_i}X}X, \partial_i\rangle \overset{\star}{=} \sum_i\langle \nabla_{\partial_i}X, \nabla_{\partial_i}X\rangle = |\nabla X|^2.$$

So the conclusion follows.

(2) If ∇X is anti-symmetric, then there will be a negative sign at the right hand side of the two $\overset{\star}{=}$, and we will get 0 after the$\overset{\bullet}{=}$. So the conclusion follows.

□

In particular, if $u \in C^\infty(M)$, then $X = \nabla u$ is smooth and ∇X is symmetric(by 2.9). Moreover,div$X =$ div$\nabla u = \Delta u$, hence it follows

Theorem 3.2. For any $u \in C^\infty(M)$

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\nabla^2 u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + Ric(\nabla u, \nabla u).$$

3.2 Bochner's formula for closed 1-forms

Given any smooth 1-form the musical isomorphism produce a smooth vector field $X = \sharp\omega$

Through dual Riemannian metric on the cotangent space, we can define Riemannian metric for tensors.Thus we can define the norm of 1-form $|\omega|$

Definition 3.3. Let $U, x_1, \cdots, x_m$ be a local coordinate near $p \in M$ For any 1-forms $\omega = \omega_i dx^i$and$\eta = \eta_i dx^i$, we define $g^*(\omega, \eta) = \langle \omega, \eta \rangle_p^* := g^{ij}(p)\omega_i(p)\eta_j(p)$.

It's easy to check that this definition is independent of the choices of coordinates. Moreover, for tensors

$$T = T^{i_1 \cdots i_k}_{j_1 \cdots j_l} \partial_{i_1} \otimes \cdots \otimes \partial_{i_k} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}, S = S^{a_1 \cdots a_k}_{b_1 \cdots b_l} \partial_{a_1} \otimes \cdots \otimes \partial_{a_k} \otimes dx^{b_1} \otimes \cdots \otimes dx^{b_l}$$

We can define

$$\langle T, S \rangle = g^{j_1 b_1} \cdots g^{j_l b_l} g_{i_1 a_1} \cdots g_{i_k a_k} T^{i_1 \cdots i_k}_{j_1 \cdots j_l} S^{a_1 \cdots a_k}_{b_1 \cdots b_l}$$

With the definitions above, it's not hard to check

$$|X| = |\omega|, \quad |\nabla X|^2 = |\nabla \omega|^2.$$

And we can prove the following theorem:

Theorem 3.4. Let (M, g) be a compact Riemannian manifold. Then, for any closed 1-form $\omega \in \Omega^1(M)$, the following holds: $\frac{1}{2}\Delta(|\omega|^2) = |\nabla\omega|^2 - \langle \omega, \Delta\omega \rangle + Ric(\sharp\omega, \sharp\omega)$.

证明. $\omega \in \Omega^1(M)$is a closed 1-form. Then locally ω is exact, i.e. locally of the form $\omega = du$. As a result ,$X = \sharp\omega = \nabla u$, from theorem 3.1 we know that ∇X is symmetric.So we may apply part (1) of Theorem 3.1 to get

$$\frac{1}{2}\Delta(|\omega|^2) = |\nabla\omega|^2 + \langle \sharp\omega, \nabla \text{div}\sharp\omega \rangle + Ric(\sharp\omega, \sharp\omega)$$

Suppose$\omega = \omega_i dx^i$,then

$$\sharp\omega = \sum_i \omega_i \partial_i, \quad \text{div}\sharp\omega = \sum_i \partial_i \omega_i, \quad \nabla\text{div}\sharp\omega = \sum_i \partial_j \partial_i \omega_i \partial_j.$$

and thus

$$\langle \sharp\omega, \nabla\text{div}\sharp\omega \rangle = \sum_{i,j} \omega_j \partial_j \partial_i \omega_i.$$

On the other hand, apply $X = \sharp\omega$ to theorem 2.7,we get

$$\text{div}(f\sharp\omega) = f\text{div}\sharp\omega + \langle \nabla f, \sharp\omega \rangle.$$

Since M is compact , by theorem 2.8, we get

$$0 = \int_M \text{div}(f\sharp\omega) dV_g = \int_M (f \,\text{div}\, \sharp\omega + \langle \nabla f, \sharp\omega \rangle) dV_g = \int_M (f \,\text{div}\, \sharp\omega + \langle df, \omega \rangle) dV_g.$$

It follows that

$$\langle df, \omega \rangle_{L^2} = \int_M \langle df, \omega \rangle dV_g = \int_M f(-\text{div}\sharp\omega) dV_g = \langle f, -\text{div}\sharp\omega \rangle_{L^2}.$$

If we define $\delta\omega = -\mathrm{div}\sharp\omega$, then $\delta : \Omega^1(M) \to C^\infty(M)$ is the L^2 dual of d.

$$\langle df, \omega\rangle_{L^2} = \langle f, \delta\omega\rangle_{L^2}, \quad \forall f \in C^\infty(M), \omega \in \Omega^1(M).$$

For any closed 1-form ω we define $\Delta\omega = d\delta\omega$. Then locally

$$\Delta\omega = d(-\sum_i \partial_i\omega_i) = -\sum_{i,j} \partial_j\partial_i\omega_i$$

and thus

$$\langle\omega, \Delta\omega\rangle = -\sum \omega_j\partial_j\partial_i\omega_i = -\langle\sharp\omega, \nabla\mathrm{div}\sharp\omega\rangle.$$

、 Thus the conclusion follows □

In analysis, we say a function f is harmonic iff $\Delta f = 0$, similarly, a 1-form ω is harmonic iff $\Delta\omega = 0$,combining the theorem we just proved, we get:If (M, g) is compact and has $Ric \geq 0$, then every harmonic 1-formω is parallel,i.e.$\nabla\omega = 0$

Moreover, we can define harmonic k-form and get similar results,which will be presented later.

3.3 Bochner's formula for Killing Forms

We first present the definition of Killing forms

Definition 3.5. Any vector field X defines a local family of diffeomorphisms

$$\phi_t^X : U \subset M \to \phi(U) \subset M, \quad p \mapsto \phi_t^X(p) = \gamma_{p,X_p}(t)$$

for $-\varepsilon < t < \varepsilon$,which satisfes $\phi_t^X \circ \phi_s^X = \phi_{t+s}^X$ for $t, s, t+s \in (-\varepsilon, \varepsilon)$. Now suppose (M, g) is a Riemannian manifold. We say X is a Killing vector freld if these ϕ_t^X's are isometries.

Proposition 3.6. Let $X, Y, Z \in \Gamma^\infty(TM)$. The following statements are equivalent:

(i) X is a Killing vector field.

(ii) $\mathcal{L}_X g = 0$.

(iii) $X\langle Y, Z\rangle = \langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle$.

(iv) $\langle\nabla_Y X, Z\rangle + \langle\nabla_Z X, Y\rangle = 0$.

(v) The $(1, 1)$-tensor field ∇X defined by $\nabla X : V \mapsto \nabla_V X$ is skew-symmetric.

证明. By definition of Lie derivative, (i) $\iff$ $\phi_t^* g = g \quad \text{for all } t \in I.$ $\iff$ $\dfrac{d}{dt}(\phi_t^* g)_{|t=0} = 0.$ $\iff$ (ii)

According to the formula $d(\flat X)(V, W) + (L_X g)(V, W) = 2g(\nabla_V X, W)$ and the fact that differential forms are skew-symmetric, we have (ii) $\iff$ (iv) $\iff$ (v)

Recall the fact that $\nabla_X Y - \nabla_Y X = [X, Y]$ and $X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$, we get (iii) $\iff$ (iv)

□

Theorem 3.7. For any Killing vector field on M,

$$\frac{1}{2}\Delta(|X|^2) = |\nabla X|^2 - Ric(X, X).$$

证明. Combining 3.1(2) and 3.6, we immediately get the desired result. □

As a result, we get

Theorem 3.8. (Bochner, 1946)Any Killing vector field of a compact Riemannian manifold with negative Ricci curvature must be zero.

4 Applications

4.1 Basic Hodge Theory

4.1.1 Hodge Laplacian

Recall that the Riemannian volume form (which is independent of the choice of coordinates) is given by

$$\omega_g = \sqrt{G} dx^1 \wedge \cdots \wedge dx^m$$

Now let $p \in M$. Then the Riemannian metric g induces a dual inner product structure on $T_p^* M$ via $\langle \omega_i dx^i, \eta_j dx^j \rangle = g^{ij} \omega_i \eta_j$.

As in the case of functions, the pointwise inner product induces an L^2 inner product structure on $\Omega_c^k(M)$ via

$$(\omega, \eta) := \int_M \langle \omega, \eta \rangle \omega_g.$$

We also let differential forms with different order to be orthogonal, thus the inner product above becomes an inner product in the space of differential form$\Omega^*(M)$

With these preparations, we can define **Hodge star operator** (or $*$ operator), it maps k-form to $(n-k)$-form. For simplicity, we take a normal orthogonal basis on T_pM:$\{e_1, \cdots, e_n\}$,and denote $\{\omega^1, \cdots, \omega^n\}$ as its dual basis, in this case, $(g_{ij}) = (\delta_{ij})$.

Definition 4.1. On a basis of k-form $\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_k}\}$, operator $*$ is defined as:

$$*(\omega^{i_1} \wedge \cdots \wedge \omega^{i_k}) = \varepsilon_{i_1 \cdots i_k} \cdot \omega^1 \wedge \cdots \wedge \widehat{\omega^{i_1}} \wedge \cdots \wedge \widehat{\omega^{i_k}} \wedge \cdots \wedge \omega^n$$

where $\varepsilon_{i_1 \cdots i_k} = \pm 1$ s.t.

$$\omega^{i_1} \wedge \cdots \wedge \omega^{i_k} \wedge *(\omega^{i_1} \wedge \cdots \wedge \omega^{i_k}) = \omega = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$$

Since a minus sign will appear if we switch $\omega_i \wedge \omega_j$, we can show by direct calculation that

$$\varepsilon_{i_1 \cdots i_q} = (-1)^{i_1 + \cdots + i_q + 1 + \cdots + q}.$$

Generally, if

$$\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k}\omega^{i_1}\wedge\cdots\wedge\omega^{i_k}$$

then

$$*\omega = \sum_{i_1<\cdots<i_k} \varepsilon_{i_1\cdots i_k}\omega_{i_1\cdots i_k}\omega^1\wedge\cdots\wedge\widehat{\omega^{i_1}}\wedge\cdots\wedge\widehat{\omega^{i_k}}\wedge\cdots\wedge\omega^n.$$

Proposition 4.2. Let

$$\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1\cdots i_k}\omega^{i_1}\wedge\cdots\wedge\omega^{i_k}, \eta = \sum_{i_1<\cdots<i_k} \eta_{i_1\cdots i_k}\omega^{i_1}\wedge\cdots\wedge\omega^{i_k}$$

,then

(1)$*1 = \omega_g, *\omega_g = 1, **\omega = (-1)^{k(n-k)}\omega$

(2)$\omega\wedge*\eta = \langle\omega,\eta\rangle\omega_g, \langle*\omega,*\eta\rangle = \langle\omega,\eta\rangle$.

证明. (1) $*1 = \omega_g, *\omega_g = 1$ is derived directly from definition.And we have

$$**\omega = \varepsilon_{i_1\cdots i_k}\varepsilon_{1\cdots\widehat{i_1}\cdots\widehat{i_k}\cdots n}\omega = (-1)^{k(n-k)}\omega.$$

It remains to check the following elementary identity:

$$\frac{n(n+1)}{2}+\frac{k(k+1)}{2}+\frac{(n-k)(n-k+1)}{2}\equiv k(n-k) \pmod 2.$$

(2) From definition we have

$$\begin{aligned}\omega\wedge*\eta &= \sum_{i_1<\cdots<i_k}\omega_{i_1\cdots i_k}\eta_{j_1\cdots,j_k}\omega^{i_1}\wedge\cdots\wedge\omega^{i_k}\wedge*(\omega^{j_1}\wedge\cdots\wedge\omega^{j_k})\\ &= \sum_{i_1<\cdots<i_k}\omega_{i_1\cdots i_k}\eta_{i_1\cdots i_k}\omega_g = \langle\omega,\eta\rangle\omega_g.\end{aligned}$$

Thus we have

$$\langle*\omega,*\eta\rangle\omega_g = (*\omega)\wedge*(*\eta) = (-1)^{k(n-k)}(*\omega)\wedge\eta = \eta\wedge*\omega = \langle\eta,\omega\rangle\omega_g$$

□

We know that the exterior derivative operator d maps $(k-1)$-forms to k-forms. In the inner product space $(\Omega^*(M),(\cdot,\cdot))$, d has an adjoint operator δ, which maps k-forms to $(k-1)$-forms. Using the Hodge star operator $*$, we can express δ. Specifically, let ω be a $(k-1)$-form and η a k-form. Then

$$\begin{aligned}d(\omega\wedge*\eta) &= d\omega\wedge*\eta+(-1)^{k-1}\omega\wedge d*\eta\\ &= d\omega\wedge*\eta+(-1)^{n(k-1)}\omega\wedge*(*d*\eta).\end{aligned}$$

If we define $\delta = (-1)^{n(k-1)+1}*d*:\Omega^k(M)\to\Omega^{k-1}(M)$, the equation can be written as

$$d(\omega\wedge*\eta) = d\omega\wedge*\eta-\omega\wedge*\delta\eta.$$

By Stokes' theorem, we have

$$\int_M d\omega \wedge *\eta = \int_M \omega \wedge *\delta\eta,$$

which implies

$$(d\omega, \eta) = (\omega, \delta\eta),$$

showing that δ is indeed the adjoint operator of d under the inner product $(\cdot, \cdot)$.

Remark 4.3. A more general (coordinate free) definition for Hodge star operator is through its proposition: $\omega \wedge *\eta = \langle \omega, \eta \rangle \omega_g$

The exterior differential operator d and its adjoint operator can both be expressed using a connection.

Proposition 4.4. Let ∇ be the Levi-Civita connection, then

$$(1) d = \omega^i \wedge \nabla_{e_i}.$$
$$(2) \delta = -\sum_j \iota(e_j) \nabla_{e_j}.$$

证明. One can check that the right hand side is independent of choice of basis. So at each point p that is fixed, with out loss of generality one may take $e_i = \partial_i$ to be the coordinate vector field for a normal coordinate system centered at p. The dual basis is then dx^i . Recall that by definition, at the point p one has, for any i, j, k,

$$(\nabla_{\partial_i} dx^j)(\partial_k) = \nabla_{\partial_i}(dx^j(\partial_k)) - dx^j(\nabla_{\partial_i}\partial_j) = 0.$$

At p, one has $\nabla_{\partial_i} dx^j = 0$ for any i, j.

Now we denote

$$\bar{d} = \omega^i \wedge \nabla_{e_i} = dx^i \wedge \nabla_{\partial_i}.$$

Consider $\eta = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Then at p,

$$\bar{d}\eta = \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = d\eta.$$

This implies $d = \bar{d}$.

(2) The proof is similar. We denote

$$\bar{\delta} = -\sum_j \iota_{e_j} \nabla_{e_j} = -\sum_j \iota_{\partial_j} \nabla_{\partial_j}.$$

Then at p,

$$\bar{\delta}\eta = -\sum_j (-1)^{j-1} (\partial_{i_j} f) dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_k}.$$

On the other hand, by the definition of δ one can calculate $\delta\eta$ and prove that at p,

$$\delta\eta = \sum_i (-1)^j (\partial_{i_j} f) dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_k}.$$

□

Now we can present the definition of Hodge Laplacian

Definition 4.5. The Hodge-Laplace operator on k-forms is

$$\Delta = d\delta + \delta d : \Omega^k(M) \to \Omega^k(M).$$

It has the following properties:

Proposition 4.6. (1) If f is a smooth function on M, then $\Delta f = -\mathrm{tr}\nabla^2 f$;

(2) Δ is a self-adjoint operator, i.e., $(\Delta\omega, \eta) = (\omega, \Delta\eta)$ for any differential forms ω and η;

(3) Δ is a positive operator, i.e., $(\Delta\omega, \omega) \geq 0$, and equality holds if and only if $\Delta\omega = 0$;

(4) $*\Delta = \Delta *$.

证明. (1) When f is a function, $\delta f = 0$, by 4.4 we have

$$\Delta f = \delta df + d\delta f = \delta df = -\iota(e_j)\nabla_{e_j} df = -\mathrm{tr}\nabla^2 f.$$

(2)We only need to consider when the orders of ω and η are the same, by definition

$$(\Delta\omega, \eta) = (\delta d\omega, \eta) + (d\delta\omega, \eta) = (d\omega, d\eta) + (\delta\omega, \delta\eta).$$

Hence the conclusion follows

(3)Let $\eta = \omega$ in

$$(\Delta\omega, \eta) = (\delta d\omega, \eta) + (d\delta\omega, \eta) = (d\omega, d\eta) + (\delta\omega, \delta\eta).$$

, we get

$$(\Delta\omega, \omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega) \geq 0$$

The conclusion follows

(4)Let ω be a k-form, we know that

$$\delta\omega = (-1)^{n(k-1)+1} * d * \omega, * * \omega = (-1)^{k(n-k)}\omega.$$

Therefore

$$*\delta\omega = (-1)^{n(k-1)+1}(-1)^{(n-k+1)(k-1)} d * \omega = (-1)^k d * \omega.$$

Similarly we get $\delta * \omega = (-1)^{k+1} * d\omega$ and thus

$$*d\delta\omega = (-1)^k \delta * \delta\omega = \delta d * \omega.$$

We get $*\delta d = d\delta *$, which implies

$$*\Delta = *d\delta + *\delta d = \delta d * + d\delta * = \Delta *$$

□

Remark 4.7. Notice from (1) that the Hodge Laplacian is different from the usual Laplacian by a minus sign.

Definition 4.8. We say $\omega \in \Omega(M)$ is a harmonic form if $\Delta\omega$=0. We denote the set of harmonic k-form on M as $\mathcal{H}^k(M)$

From 4.6 we know that ω is harmonic iff $d\omega = 0, \delta\omega = 0$

Now we come back to the Bochner formula. Now suppose ω is a harmonic 1-form on compact Riemannian manifold (M, g). Then ω is closed, by 3.4 , we get

$$0 = \int_M \frac{1}{2}\Delta(|\omega|^2) = \int_M |\nabla\omega|^2 + \int_M Rc(\sharp\omega, \sharp\omega).$$

Thus we get

Theorem 4.9. (The vanishing theorem, Bochner, 1948) Let (M, g) be a compact Riemannian manifold, then

(1) Suppose Ric ≥ 0. If $\Delta\omega = 0$, then $\nabla\omega = 0$.

(2) Suppose Ric 0, and Ric > 0 at one point. If $\Delta\omega = 0$, then $\omega = 0$.

In fact 3.4 can be generalized to harmonic k-form ω, define

$$\mathrm{tr}\nabla^2\omega = \nabla^2\omega(\cdot, e_i, e_i)$$

where $\{e_i\}$ are orthogonal normal basis. Direct calculation shows

$$\mathrm{tr}\nabla^2\omega = \nabla_{e_i}\nabla_{e_i}\omega - \nabla_{\nabla_{e_i}e_i}\omega.$$

Lemma 4.10. (Weitzenböck formula) ω is k-form, then

$$\Delta\omega = -\mathrm{tr}\nabla^2\omega + \omega^i \wedge i(e_j)R(e_i, e_j)\omega,$$

where R denotes the curvature tensor.

证明. By 4.4, we have

$$\begin{aligned}
\Delta\omega &= d\delta\omega + \delta d\omega \\
&= \omega^i \wedge \nabla_{e_i}(-\iota(e_j)\nabla_{e_j}\omega) - \iota(e_j)\nabla_{e_j}(\omega^i \wedge \nabla_{e_i}\omega) \\
&= \omega^i \wedge (-\iota(\nabla_{e_i}e_j)\nabla_{e_j}\omega - \iota(e_j)\nabla_{e_i}\nabla_{e_j}\omega) \\
&- \iota(e_j)(\nabla_{e_j}\omega^i) \wedge \nabla_{e_i}\omega - \iota(e_j)\omega^i \wedge \nabla_{e_j}\nabla_{e_i}\omega \\
&= -\omega^i \wedge \iota(e_j)\nabla_{e_i}\nabla_{e_j}\omega - \nabla_{e_i}\nabla_{e_i}\omega + \omega^i \wedge \iota(e_j)\nabla_{e_j}\nabla_{e_i}\omega \\
&= -\mathrm{tr}\nabla^2\omega + \omega^i\iota \wedge \iota(e_j)R(e_i, e_j)\omega,
\end{aligned}$$

□

Corollary 4.11. (Bochner formula for k-form) Let ω be a k-form, then

$$-\frac{1}{2}\Delta\langle\omega, \omega\rangle = -\langle\Delta\omega, \omega\rangle + |\nabla\omega|^2 + F(\omega)$$

where

$$F(\omega) = \langle\omega^i \wedge i(e_j)R(e_i, e_j)\omega, \omega\rangle.$$

证明.

$$
\begin{aligned}
\langle \mathrm{tr}\nabla^2\omega, \omega\rangle &= \langle \nabla_{e_i}\nabla_{e_i}\omega - \nabla_{\nabla_{e_i}e_i}\omega, \omega\rangle \\
&= \nabla_{e_i}\langle \nabla_{e_i}\omega, \omega\rangle - \langle \nabla_{e_i}\omega, \nabla_{e_i}\omega\rangle - \frac{1}{2}\nabla_{e_i}e_i\langle\omega,\omega\rangle \\
&= \frac{1}{2}\nabla_{e_i}\nabla_{e_i}\langle\omega,\omega\rangle - \frac{1}{2}\nabla_{e_i}e_i\langle\omega,\omega\rangle - |\nabla\omega|^2 \\
&= -\frac{1}{2}\Delta\langle\omega,\omega\rangle - |\nabla\omega|^2.
\end{aligned}
$$

Combing 4.10 ,the conclusion follows □

4.1.2 Duality

Let M be a smooth compact manifold. Then from basic manifold theory we know that the differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfies $d^2 = 0$. As a consequence, one can define the k^{th} de Rham cohomology group

$$
H^k_{dR}(M) := \frac{\ker\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\mathrm{im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right)}.
$$

The famous **de Rham theorem** says that the de Rham cohomology group is isomorphic to the singular cohomology group $H^k(M, \mathbb{R})$. Note that by definition, $H^k_{dR}(M)$ depends on the smooth structure on M, while $H^k(M, \mathbb{R})$ depends only on the topological structure of M. So the de Rham theorem is a bridge between topological invariants and smooth invariants.

Theorem 4.12.

$$
H^k_{dR}(M) \cong \mathcal{H}^k(M)
$$

In fact,each de Rham cohomology class contains a unique harmonic representative, and the harmonic representative is the one that minimize the L^2norm in that cohomology class, which is the following lemma:

Lemma 4.13. Let ω be a harmonic k-form on (M, g).Then for any $d\eta \neq 0$,

$$
(\omega + d\eta, \omega + d\eta) > (\omega, \omega).
$$

证明.

$$
\text{Recall } \Delta\omega = 0 \Longleftrightarrow d\omega = 0,\ \delta\omega = 0.
$$

So

$$
\begin{aligned}
(\omega + d\eta, \omega + d\eta) &= (\omega, \omega) + 2(\omega, d\eta) + (d\eta, d\eta) \\
&= (\omega, \omega) + 2(\delta\omega, \eta) + (d\eta, d\eta) \\
&= (\omega, \omega) + (d\eta, d\eta) \\
&> (\omega, \omega).
\end{aligned}
$$

□

Consider the linear map

$$i : \mathcal{H}^k(M) \to H^k_{dR}(M), \omega \mapsto [\omega].$$

any harmonic k-form ω must be closed. So it defines a cohomology group $[\omega]$ in $H^q_{dR}(M)$

from the above lemma ,i is injective.To prove that it's surjective ,we need the following fact known as Hodge decomposition:

$$\Omega^k(M) = \mathcal{H}^k \oplus \mathrm{im}(d) \oplus \mathrm{im}(\delta).$$

Assuming this fact, let $[\omega]$in$H^k_{dR}(M)$, then by the decomposition:$\omega = \omega_0 + d\eta_1 + \delta\eta_2$.

However, since ω is closed and the above decomposition is orthogonal,

$$(\delta\eta_2, \delta\eta_2) = (\omega, \delta\eta_2) = (d\omega, \eta_2) = 0$$

. So the decomposition for closed formω is simplified to $\omega = \omega_0 + d\eta_1$. This implies that ω_0 is a harmonic form inside the cohomology class $[\omega]$.

Remark 4.14. The three spaces $\mathcal{H}^k, \mathrm{im}(d), \mathrm{im}(\delta)$ are orthogonal to each other. In fact, For any harmonic k-form ω and any $(k-1)$-form η_1, any $(k+1)$-form η_2,

$$(\omega, d\eta_1) = (\delta\omega, \eta_1) = 0, \quad (\omega, \delta\eta_2) = (d\omega, \eta_2) = 0.$$

and

$$(d\eta_1, \delta\eta_2) = (dd\eta_1, \eta_2) = 0.$$

Remark 4.15. The Hodge theorem identifies the smooth (and in fact topological) invariant $H^k_{dR}(M)$ with the space of harmonic k- forms $\mathcal{H}^k(M)$, which depends on the Riemannian metric. As a consequence, one can use analysis to study topology. This turns out to be very useful. For example, by the standard elliptic theory of Δ one concludes that $H^k_{dR}(M)$ is always finite dimensional.

Another duality theorem is the famous **Poincaré** duality, saying that $H^k_{dR}(M) \cong H^{n-k}_{dR}(M)$, which can be derived from the Hodge theorem.All we need is a lemma:

Lemma 4.16. Let ω be a harmonic k-form on (M, g).Then $*\omega$ is a harmonic $(n-k)$-form, moreover, this implies $\mathcal{H}^k(M) \cong \mathcal{H}^{n-k}(M)$

证明. From 4.6(4), $\Delta * \omega = *\Delta\omega = 0$. We also have $** = (-1)^{k(n-k)}$,thus $* : \mathcal{H}^k(M) \to \mathcal{H}^{n-k}(M)$ is an isomorphism □

Combining the above lemma with Hodge theorem, we get the **Poincaré** duality.

4.1.3 Decomposition

As we have seen in the previous section, to prove the duality theorem $H^k_{dR}(M) \cong \mathcal{H}^k(M)$. We have to prove the following decomposition theorem:

Theorem 4.17. (Hodge decomposition theorem). Let (M, g) be a closed oriented Riemannian manifold. Then for each k, $\mathcal{H}^k(M)$ is finite dimensional. Moreover, there exists a bounded linear operator (called a parametrix of Δ)

$$G : \Omega^k(M) \to \Omega^k(M)$$

such that:

(a) $\mathrm{Ker}(G) = \mathcal{H}^k(M)$.

(b) G commutes with $d, \delta, *$.

(c) G is compact (with respect to the L^2-inner product).

(d) $\mathrm{Id} = \pi + \Delta \circ G$, where π is the orthogonal projection from $\Omega^k(M)$ onto $\mathcal{H}^k(M)$. [Since $\mathcal{H}^k(M)$ is finite dimensional, it is a closed subspace.]

Note that according to (d), any ω can be written as

$$\omega = \omega_0 + \Delta G\omega = \omega_0 + d\delta G\omega + \delta dG\omega.$$

Together with 4.14, this implies $\Omega^k(M) = \mathcal{H}^k \oplus \mathrm{im}(d) \oplus \mathrm{im}(\delta)$. Moreover, this gives us an explicit way to write down the harmonic representative in a cohomology class $[\omega]$:

$$\omega_0 = \omega - Gd\delta\omega.$$

Remark 4.18. The existence of harmonic forms implies that Δ is not invertible. The Hodge decomposition theorem above claims that Δ is "almost invertible" : $\Delta \circ G = G \circ \Delta = I - \pi$ differs from the identity only by a fnite rank operator, which is very "small". So G can be viewed as an "almost inverse" of Δ.

Let's do some preparations first.

Again, we denote

$$\Omega^*(M) = \oplus\Omega^k(M), \quad H^*_{dR}(M) = \oplus H^k_{dR}(M), \quad \mathcal{H} = \oplus\mathcal{H}^k(M).$$

Of course one can extend d, δ, L^2-inner products etc to the graded space $\Omega^*(M)$ through the natural gradings. For example, one just defne the inner product between differential forms of different degrees to be zero.

Now let $\omega, \eta \in \Omega^*(M)$ be two differential forms on M. For any non-negative integer s we can define their H^s-inner product to be

$$(\omega, \eta)_s := \sum_{j=0}^{s} (\nabla^j\omega, \nabla^j\eta),$$

where the L^2 inner product $(\nabla^j\omega, \nabla^j\eta)$ is defined as

$$(\nabla^j\omega, \nabla^j\eta) = \int_M \langle\nabla^j\omega, \nabla^j\eta\rangle\omega_g.$$

Note $\nabla^j\omega$ is no longer a differential form but a tensor ,but we can extend the inner product on T_pM to $\otimes^{r,s} T_pM$ by the same way we did earlier.

The completion of $\Omega^*(M)$ under the H^s-inner product is called the Sobolev space, and is denoted by $H^s(M)$. Note

• $H^0(M)$ is exactly the L^2 space of differential forms on M.

• In general it makes no sense to take derivative of elements in $H^s(M)$. However, one can define the *weak derivative.*

Definition 4.19. Let P be a differential operator (e.g. the operators $d, \delta, d+\delta$ or Δ), and $\omega \in H^s(M)$. We say $P\omega = \eta \in H^t(M)$ *weakly if*

$$(\omega, P\xi) = (\eta, \xi), \quad \forall \xi \in \Omega^*(M).$$

We now start the proof of Hodge decomposition theorem.

证明. •*step 1:* dim $\mathcal{H}$ is finite.

Suppose dim$\mathcal{H} = \infty$. Then one can find an orthonormal set $\{\omega_1, \omega_2, \cdots\}$ in $\mathcal{H}$ (with respect to the L^2-inner product). By Gärding inequality 1.8, there exists positive constants c_1, c_2 so that

$$\|\omega_i\|_1^2 \le \frac{1}{c_1}\left[(\Delta\omega_i, \omega_i) + c_2\|\omega_i\|_0^2\right] = \frac{c_2}{c_1}.$$

So by Rellich lemma 1.7, the sequence $\{\omega_i\}$contains a subsequence that is Cauchy with respect to the H^0norm. This is impossible since for any $i \ne j$, $\|\omega_i - \omega_j\|_0^2 = 2$.

As a consequence, we see $\mathcal{H}$ is a closed subspace of $H^0(M)$. In particular, its orthogonal complement is well-defned. By restricting to smooth forms, it is not hard to see

$$\Omega^*(M) = \mathcal{H} \oplus \mathcal{H}^\perp,$$

where

$$\mathcal{H}^\perp = \{\omega \in \Omega^*(M) \mid (\omega, \eta) = 0, \forall \eta \in \mathcal{H}\}.$$

Note that for any $\omega \in \Omega^*(M)$ and any $\eta \in \mathcal{H}$,

$$(\Delta\omega, \eta) = (\omega, \Delta\eta) = 0.$$

So Δ maps $\Omega^*(M)$, and thus maps $\mathcal{H}^\perp$, into $\mathcal{H}^\perp$.

•*step 2:* $\Delta : \mathcal{H}^\perp \to \mathcal{H}^\perp$ *is bijective.*

$\boxed{\text{Injectivity}}$:Suppose$\omega \in \mathcal{H}^\perp$ and $\Delta\omega = 0$. Then $\omega \in \mathcal{H} \cap \mathcal{H}^\perp$. So $\omega = 0$.

$\boxed{\text{Surjectivity}}$:Fix any $\eta \in \mathcal{H}^\perp$. We need to find $\omega \in \mathcal{H}^\perp$ so that $\Delta\omega = \eta$. We let $\overline{\mathcal{H}^\perp}$ be the closure of $\mathcal{H}^\perp$ in $H^1(M)$. According to Weyl's lemma, it is enough to find $\omega \in \overline{\mathcal{H}^\perp}$ so that $\Delta\omega = \eta$ weakly, i.e. for any $\xi \in \Omega^*(M)$,

$$(\omega, \Delta\xi) = (\eta, \xi).$$

Since this holds trivially for $\xi \in \mathcal{H}$, it is enough to find ω so that it holds for $\xi \in \mathcal{H}^\perp$.

We define a bilinear function on $\mathcal{H}^\perp$ by

$$[\cdot,\cdot] : \mathcal{H}^\perp \times \mathcal{H}^\perp \to \mathbb{R}, \quad [\omega_1,\omega_2] := (\omega_1, \Delta\omega_2).$$

It is symmetric since Δ is symmetric. It is positive since by Gärding inequality 1.8(2),

$$[\omega_1,\omega_1] = (\omega_1, \Delta\omega_1) \geq \frac{1}{c_0}\|\omega_1\|_1^2.$$

So $[\cdot,\cdot]$ is an inner product on $\mathcal{H}^\perp$. On the other hand, one can prove that there exists $c > 0$ so that

$$[\omega_1,\omega_1] \leq c\|\omega_1\|_1^1, \forall \omega_1 \in \mathcal{H}^\perp.$$

By Hahn-Banach, one can extend $[\cdot,\cdot]$ uniquely to an inner product on $\overline{\mathcal{H}^\perp}$, so that

$$[\omega,\omega] \geq \frac{1}{c_0}\|\omega\|_1^2, \quad \forall \omega \in \overline{\mathcal{H}^\perp}.$$

Finally we define a linear functional ι on $\overline{\mathcal{H}^\perp}$ by

$$l(\xi) = (\eta,\xi), \quad \forall \xi \in \overline{\mathcal{H}^\perp}.$$

Then l is continuous with respect to $[\cdot,\cdot]$, since

$$|l(\xi)| = |(\eta,\xi)| \leq \|\eta\|_0\|\xi\|_0 \leq \|\eta\|_0\|\xi\|_1 \leq \sqrt{c_0}\|\eta\|_0\sqrt{[\xi,\xi]}.$$

So by Riesz representation theorem, there exists $\omega \in \mathcal{H}^\perp$ so that $l(\xi) = [\omega,\xi]$. This is exactly what we need.

•*step 3*: Construction of the paramatrix G.

We define the linear map $G : \Omega^*(M) \to \Omega^*(M)$ by requiring $G|\mathcal{H} = 0, \quad G|\mathcal{H}^\perp = (\Delta|_{\mathcal{H}^\perp})^{-1}$. G is bounded (with respect to L^2 norm) because for any $\omega \in \mathcal{H}^\perp$, $|G\omega|_0^2 \leq |G\omega|_1^2 \leq c_0(\Delta G\omega,\omega) = c_0|\omega|_0^2$. It remains to check (a)-(d) of the Hodge decomposition theorem.

Part(a) follows trivially from the definition of G

Part(b) We will prove the following even stronger statement: For any linear map $L : \Omega^*(M) \to \Omega^*(M)$ such that $L\Delta = \Delta L$, one has $LG = GL$. Obviously one only need to check this on $\mathcal{H}$ and $\mathcal{H}^\perp$.

Since $L\Delta = \Delta L$ and since $\mathcal{H} = \ker\Delta$, we immediately see $L(\mathcal{H}) \subset \mathcal{H}$. It follows that $GL|_{\mathcal{H}} = 0 = LG|_{\mathcal{H}}$.

On $\mathcal{H}^\perp$, we know $\Delta|_{\mathcal{H}^\perp}$ is invertible, and the inverse if $G|_{\mathcal{H}^\perp}$. We also have $L(\mathcal{H}^\perp) \subset \mathcal{H}^\perp$, since

$$(\mathcal{H}, L\mathcal{H}^\perp) = (\mathcal{H}, L\Delta G\mathcal{H}^\perp) = (\mathcal{H}, \Delta LG\mathcal{H}^\perp) = (\Delta\mathcal{H}, LG\mathcal{H}^\perp) = 0.$$

So from the fact $L\Delta|_{\mathcal{H}^\perp} = \Delta L|_{\mathcal{H}^\perp}$ one immediately gets $LG|_{\mathcal{H}^\perp} = GL|_{\mathcal{H}^\perp}$.

Part(c) According to Gärding inequality 1.8(2), for any $\omega \in \mathcal{H}^\perp$,

$$\|\omega\|_1^2 \leq c_0(\Delta\omega,\omega) \leq c_0\|\Delta\omega\|_0\|\omega\|_0 \leq c_0\|\Delta\omega\|_0\|\omega\|_1.$$

So $\|\omega\|_1 \leq c_0\|\Delta\omega\|_0$ for any $\omega \in \mathcal{H}^\perp$.

Now let $\{\omega_i\}$ be any sequence such that $\|\omega_i\|_0 \leq 1$. Apply the above result to $G\omega_i$'s, we get

$$\|G\omega_i\|_1 \leq c_0\|\Delta G\omega_i\|_0 \leq c_0\|\omega_i\|_0 \leq c_0,$$

where the second inequality follows from the fact that $\Delta G\omega_i = \omega_i - \pi\omega_i$ is the projection of ω_i onto $\mathcal{H}^\perp$. Now apply Rellich lemma 1.7, we conclude that $\{G\omega_i\}$ contains a convergent subsequence (with respect to H^0-norm). This proves G is compact.

Part(d) Recall that $\pi : \Omega^*(M) \to \mathcal{H}$ is the orthogonal projection. Any $\omega \in \Omega^*(M)$ can be written as $\omega = \pi(\omega) + (\omega - \pi(\omega))$, where $\omega - \pi(\omega) \in \mathcal{H}^\perp$. By definition of G,

$$\omega - \pi(\omega) = \Delta \circ G(\omega - \pi(\omega)) = \Delta(G\omega).$$

So $Id = \pi + \Delta \circ G$. □

4.2 Cheeger-Gromoll splitting Theorem

One of the applications of Bochner formula is Cheeger-Gromoll splitting Theorem, stated as follows:

Theorem 4.20. (Cheeger-Gromoll, 1971). *Let* (M, g) be a complete non-compact Riemannian manifold with $Ric \geq 0$. Suppose there exists a line in M. Then (M, g) *is* isometric to $\mathbb{R}\times N$, where N is an $(m-1)$-*dimensional* complete Riemannian manifold with $Ric \geq 0$.

Before the proof of the theorem, we need to do two preparations: Calabi-Hopf maximum principle and Buseman function.

4.2.1 Calabi-Hopf maximum principle

We first introduce a "weak" notion :

Definition 4.21. Let f be a continuous function defined on (M, g).

1. If $g \in C^2(U)$ is defined in a neighborhood U of p, and

$$f(p) = g(p), \quad \text{and} \quad f(q) \leq g(q), \forall q \in U,$$

then we call g an upper barrier function of f at p.

2. If for any $\varepsilon > 0$, there exists an upper barrier function g_ε of f at p, such that

$$\Delta g_\varepsilon(p) \leq c + \varepsilon,$$

then we say

$$\Delta f(p) \leq c \quad \text{in the barrier sense.}$$

3. If for any normal geodesic σ with $\sigma(0) = q$, one has $(f \circ \sigma)''(0) \leq c$ in the barrier sense, then we say

$$(\nabla^2 f)(q) \leq c \cdot \mathrm{Id} \quad \text{in the barrier sense.}$$

We also mention the following theorem extending the corresponding one in PDE regularity theory:

Theorem 4.22. (Hopf-Calabi strong maximum principle). Let $\Omega \subset M$ be a connected open set. Suppose $\Delta f \leq 0$ in Ω in the barrier sense, and f has an interior minimum, then f is constant on Ω.

4.2.2 Buseman function

To let a Riemann manifold split(in one direction), we need a function that is linear in some sense, which is the Buseman function.

Since (M, g) is complete and non-compact, for any ray $\gamma : [0, +\infty) \to M$, let

$$b_\gamma^t : M \to \mathbb{R}, \quad b_\gamma^t(x) = t - d(x, \gamma(t)).$$

By the triangle inequality, it is easy to see:

- $b_\gamma^t(x) \le d(\gamma(0), x)$,
- for any $t < s$, one has $b_\gamma^s(x) - b_\gamma^t(x) = (s - t) + d(x, \gamma(t)) - d(x, \gamma(s)) \ge 0$,
- $|b_\gamma^t(x) - b_\gamma^t(y)| \le d(x, y)$.

As a result, the limit

$$b_\gamma(x) := \lim_{t \to +\infty} (t - d(x, \gamma(t)))$$

is well-defined and is Lipschitz with Lipschitz constant 1. We call the function $b_\gamma : M \to \mathbb{R}$ the Buseman function associated with γ.

Proposition 4.23. Let b_γ be the Busemann function associated with a ray γ, then $\Delta b_\gamma \ge 0$ in the barrier sense.

To prove the proposition, we present the Laplacian comparison theorem without proof:

Theorem 4.24. (The Laplacian Comparison Theorem) Let M be a complete m-dimensional Riemannian manifold with

Ricci curvature bounded from below by

$$Ric \ge (m - 1)K$$

for some constant K. Then the Laplacian of the distance function satisftes

$$\Delta r(x) \le \begin{cases} (m-1)\sqrt{K}\cot(\sqrt{K}r) & for \quad K > 0, \\ (m-1)r^{-1} & for \quad K = 0, \\ (m-1)\sqrt{-K}\coth(\sqrt{-K}r) & for \quad K < 0 \end{cases}$$

in the sense of distribution. where $r(x)$ is the distance function from x to a fixed point $p \in M$

Proof of the proposition 4.23: We need to construct barrier functions of buseman function.

Given our ray γ, as before, and $p \in M$, consider a family of unit speed segments $\sigma_t : [0, L_t] \to (M, g)$ from p to $\gamma(t)$. As in the construction of rays this family subconverges to a ray $\tilde{\gamma} : [0, \infty) \to M$, with $\tilde{\gamma}(0) = p$. Such $\tilde{\gamma}$ are called asymptotes for γ from p and need not be unique.

We need the following lemma:

Lemma 4.25. The Busemann functions are related by:

(1)$b_\gamma(x) \ge b_\gamma(p) + b_{\tilde{\gamma}}(x)$.

(2)$b_\gamma(\tilde{\gamma}(t)) = b_\gamma(p) + b_{\tilde{\gamma}}(\tilde{\gamma}(t)) = t - b_\gamma(p)$.

证明. Let $\sigma_i : [0, L_i] \to (M, g)$ be the segments converging to $\tilde{\gamma}$. To check (1), observe that

$$\begin{aligned} d(x\gamma(s)) - s &\leq d(x, \tilde{\gamma}(t)) + d(\tilde{\gamma}(t), \gamma(s)) - s \\ &= t - d(x, \tilde{\gamma}(t)) + d(p, \tilde{\gamma}(t)) + s - d(\tilde{\gamma}(t), \gamma(s)) \\ &\to t - d(x, \tilde{\gamma}(t)) + d(p, \tilde{\gamma}(t)) + b_\gamma(\tilde{\gamma}(t)) \quad \text{as } s \to \infty. \end{aligned}$$

Thus, we see that (1) is true provided that (2) is true. To establish (2), note that

$$d(p, \gamma(t_i)) = d(p, \sigma_i(s)) + d(\sigma_i(s), \gamma(t_i))$$

for some sequence $t_i \to \infty$. Then $\sigma_i(s) \to \tilde{\gamma}(s)$ and

$$\begin{aligned} b_\gamma(p) &= \lim\left(t_i - d(p, \gamma(t_i))\right) \\ &= \lim\left(d(p, \tilde{\gamma}(s)) + (t_i - d(\tilde{\gamma}(s), \gamma(t_i)))\right) \\ &= d(p, \tilde{\gamma}(s)) + \lim\left(t_i - d(\tilde{\gamma}(s), \gamma(t_i))\right) \\ &= s + b_\gamma(\tilde{\gamma}(s)) \\ &= -b_{\tilde{\gamma}}(\tilde{\gamma}(s)) + b_\gamma(\tilde{\gamma}(s)). \end{aligned}$$

□

Since $b_\gamma(p) + b_{\tilde{\gamma}}$ is a support function from below at p, we only need to check that $\Delta b_{\tilde{\gamma}} \leq 0$ at p. To see this, observe that the functions

$$b_t(x) = t - d(x, \tilde{\gamma}(t))$$

are barrier functions from below for $b_{\tilde{\gamma}}$ at p. Furthermore, these functions are smooth at p , and by 4.24,

$$\Delta b_t(p) \geq -\frac{n-1}{t} \to 0 \text{ as } t \to \infty.$$

Hence 4.23 follows.

4.2.3 The proof of Cheeger-Gromoll splitting theorem

Now we apply the Bochner formula to the function b_{γ_+} to get

$$\frac{1}{2}\Delta(|\nabla b_{\gamma_+}|^2) = |\nabla^2 b_{\gamma_+}|^2 + \mathrm{Ric}(\nabla b_{\gamma_+}, \nabla b_{\gamma_+}) \geq 0.$$

So, $|\nabla b_{\gamma_+}|^2$ is a subharmonic function that achieves its interior maximum. By the Calabi-Hopf strong maximum principle 4.22,

$$|\nabla b_{\gamma_+}|^2 = 1.$$

It follows that $|\nabla b_{\gamma_+}| = 1$, and in particular, ∇b_{γ_+} is a complete vector field. Moreover, it follows that

$$\Delta(|\nabla b_{\gamma_+}|) = 0,$$

and thus $|\nabla^2 b_{\gamma_+}|^2 = 0$, i.e.,

$$\nabla^2 b_{\gamma_+} = 0.$$

Finally, we construct the splitting. Let $M_t = b_{\gamma_+}^{-1}(t)$. Since $|\nabla b_{\gamma_+}| = 1$, any $t \in \mathbb{R}$ is a regular value of b_{γ_+}. Thus, M_t is a smooth submanifold of M of dimension $m-1$. Denote $N = M_0$. Let $\varphi_s : M \to M$ be the flow of the vector field ∇b_{γ_+}. Then φ_s is a diffeomorphism. Moreover, for any $s \in \mathbb{R}$ and any $x \in N = M_0$, we have $\varphi_s(x) \in M_s$.

Thus, we define a smooth map

$$\Phi : \mathbb{R} \times N \to M, \quad \Phi(s,p) := \varphi_s(p),$$

which is bijective, and whose inverse

$$\Phi^{-1} : M \to \mathbb{R} \times N, \quad x \mapsto (b_{\gamma_+}(x), \varphi_{-b_{\gamma_+}(x)}(x))$$

is smooth. Hence, Φ is a diffeomorphism.

It remains to prove that Φ is an isometry. Note that if we let $\gamma_p(s) = \varphi_s(p)$ be the integral curve passing through p, then

$$\dot{\gamma}_p = \nabla b_{\gamma_+} \Longrightarrow \nabla \dot{\gamma}_p = \nabla^2 b_{\gamma_+} = 0,$$

which implies that γ_p is the geodesic $\gamma_p(s) = \exp_p(sX_p)$, where $X_p = \nabla b_{\gamma_+}(p)$. As a result, we have:

- Φ is a radial isometry: We have $|\partial_s| = 1$, and

$$|d\Phi_{(s,p)}(\partial_s)| = |\dot{\gamma}_p| = |\nabla b_{\gamma_+}| = 1.$$

- Φ maps vectors orthogonal to the radial direction ∂_s to vectors orthogonal to the radial direction $d\Phi_{(s,p)}(\partial_s)$: For any $X_0 \in T_pN = T_pM_0$, we have

$$(d\Phi)_{(s,p)}(0, X_0) = (d\varphi_s)_p(X_0) \in T_{\varphi_s(p)}M_s \perp \nabla b_{\gamma_+}(\varphi_s(p)) = d\Phi_{(s,p)}(\partial_s).$$

- Φ preserves the length (and thus the inner product by polarization) of all vectors orthogonal to ∂_s: For any $X_0 \in T_pN$, we may extend X_0 to a local coordinate vector field $\widetilde{X}_0$ on TN such that $[\partial_s, \widetilde{X}_0] = 0$. Then

$$\nabla_{\dot{\gamma}_p(s)}((d\varphi_s)_p(X_0)) = \nabla_{d\varphi_s(\widetilde{X}_0)}(\nabla b_{\gamma_+}) - [(\nabla b_{\gamma_+})(\varphi_s(p)), (d\varphi_s)_p \widetilde{X}_0].$$

  The first term vanishes since $\nabla(\nabla b_{\gamma_+}) = 0$, while the second term vanishes since it equals $d\varphi_s([\partial_s, \widetilde{X}_0]) = 0$.

  Thus, $(d\varphi_s)_p(X_0)$ is parallel along $\gamma_p(s)$, and

$$|d\Phi_{(s,p)}(0, X_0)| = |d\varphi_s(X_0)| = |X_0|.$$

Therefore, we conclude that (M,g) is isometric to $\mathbb{R} \times N$. Finally, since (M,g) has nonnegative Ricci curvature, and N is a Riemannian submanifold with $K(\partial_s, X_0) = 0$, we conclude that N also has nonnegative Ricci curvature.

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