一、前置

Perbutation. $a_{ij} \in C$	Non-perbutation. $a_{ij} \in L$
Schauder Theory	
Ω $a_{ij}, b_i, c, f \in C^\alpha(\Omega)$	
$\Rightarrow Lu=f$ 有 $C^{2,\alpha}$ 解	
$\|u\|_{C^{2,\alpha}(\Omega')} \le C(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)})$	
$\Omega' \subset\subset \Omega$ interior	
$W^{2,p}$	
H^k	

① 书上 Ch3. Thm 3.1

Growth of Local Integrals $\Rightarrow u \in C^\alpha$

② 先验估计 + 连续性方法

前提 Ω: $\boxed{C^{2,\alpha}}$ - domains C^2

L: uniformly elliptic. $\boxed{c \le 0}$. $a_{ij}, b_i, c \in C^\alpha(\bar\Omega)$

Step 1: $f \in C^\alpha(\bar\Omega)$. $\varphi \in C^{2,\alpha}(\bar\Omega)$. $\begin{cases} \Delta u = f \\ u = \varphi \end{cases}$ 有 $C^{2,\alpha}$ 解.

See G-T: $\boxed{\text{Newton potential}}$

Step 2: 先验估计.

Thm 6.6. Ω $C^{2,\alpha}$. $u\in C^{2,\alpha}(\bar\Omega)$ 满足 $\begin{cases} Lu=f & \Omega \\ u=\varphi & \partial\Omega \end{cases}$

$$\Rightarrow |u|_{C^{2,\alpha}(\bar\Omega)} \le C\left(|u|_{C^0(\bar\Omega)} + |\varphi|_{C^{2,\alpha}(\bar\Omega)} + |f|_{C^\alpha(\bar\Omega)}\right)$$

Step 3: 连续性方法: $\begin{cases} \Delta u=f & \text{in } \Omega \\ u=\varphi & \text{on } \partial\Omega \end{cases} \Rightarrow \begin{cases} Lu=f & \text{in } \Omega \\ u=\varphi & \text{on } \partial\Omega \end{cases}$

引理: $T: V\to V$. $\|Tx-Ty\|\le\theta\|x-y\|$. $\theta\in(0,1)$

Banach V. $\Rightarrow \exists!\, x\in V$. s.t. $Tx=x$.

WLOG, $\varphi=0$.

$L_t = tL+(1-t)\Delta \qquad \Delta \rightsquigarrow L \qquad t\in[0,1]$.

$L_0=\Delta$, $L_1=L$. $L_t u = \underline{a_{ij}^t D_{ij}u + b_i^t D_i u + c^t u}$

$a_{ij}^t \xi_i\xi_j \ge \min(1,\lambda)|\xi|^2$

$|a_{ij}^t|_{C^\alpha(\bar\Omega)}, |b_i^t|_{C^\alpha(\bar\Omega)}, |c^t|_{C^\alpha(\bar\Omega)} \le \max(1,\Lambda)$.

$$|L_t u|_{C^\alpha(\bar\Omega)} \le C|u|_{C^{2,\alpha}(\bar\Omega)}$$

$\Rightarrow L_t: \mathcal{X} \longrightarrow C^\alpha(\bar\Omega)$ is bounded linear op.

$\mathcal{X} = \{u\in C^{2,\alpha}(\bar\Omega) \mid u=0 \text{ on } \partial\Omega\}$ $\boxed{\text{Banach}}$. $|\cdot|_{C^{2,\alpha}(\bar\Omega)}$

$\begin{cases} L_s u=f & \Omega \\ u=0 & \partial\Omega \end{cases}$ solvable in $C^{2,\alpha}(\bar\Omega)$. 的 s 之集 记作 $\underline{I}$

$\forall f\in C^\alpha(\bar\Omega)$.

$s\in I$. $u=L_s^{-1}f \Rightarrow |L_s^{-1}f|_{C^{2,\alpha}(\bar\Omega)} \le C\left(|u|_{C^0(\bar\Omega)} + |f|_{C^\alpha(\bar\Omega)}\right)$

$\boxed{c \le 0} \xrightarrow{\text{max prin}} \|u\|_{L^\infty(\Omega)} \le \max_{\partial\Omega}|\varphi| + C\max_{\Omega}|f|.$

$\le C|f|_{C^\alpha(\bar\Omega)}$

$$\begin{cases} L_t u = f & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega \end{cases} \Leftrightarrow L_s u = f + (L_s - L_t)u = f + (t-s)(\Delta u - Lu)$$

$$\Updownarrow$$

$$\boxed{u = L_s^{-1}(f + (t-s)(\Delta u - Lu)) \triangleq Tu. \quad u \in \mathcal{X}}$$

$T: \mathcal{X} \to \mathcal{X}$

$$\underline{|Tu - Tv|_{C^{2,\alpha}(\bar\Omega)}} = |(t-s) L_s^{-1}((\Delta - L)(u-v))|_{C^{2,\alpha}(\bar\Omega)}$$

$$\lesssim C|t-s|\,|(\Delta - L)(u-v)|_{\underline{C^\alpha(\bar\Omega)}}$$

$$\le \underline{C|t-s|\,|u-v|_{C^{2,\alpha}(\bar\Omega)}}$$

在 $|t-s| < \delta = C^{-1}$ 时 T 压缩映射.

分割区间即可.

$\square$