072a_Folien

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```
In [1]: import numpy as np
        from scipy import stats
        from matplotlib import pyplot as plt
        %matplotlib inline
```

1 Beschreibende Statistik

2 Wahrscheinlichkeitstheorie

3 Schließende Statistik

3.1 Punktschätzungen

3.2 Tests

- Nullhypothesentest
- Binomialtest
- Gaußtest
- χ^2-Verteilung
- Bedeutung
- χ^2-Test
- Quantil-Quantil-Plot
- χ^2-Anpassungs-Test
- t-Test

3.3 Intervallschätzungen

Erinnerung:

Erwartungswert Summe i.i.d. Zufallsvariablen $X_i \sim X$

$$\mathcal{E}\big(\sum_{i=1}^{N} X_i\big) = \sum_{i=1}^{N} \mathcal{E}(X_i) = n \cdot \mathcal{E}(X)$$

Varianz-Summe *un*abhängiger i.i.d. Zufallsvariablen

$$\mathrm{Var}\big(\sum_{i=1}^{N} X_i\big) = \sum_{i=1}^{N} Var(X_i) = n \cdot \mathrm{Var}(X)$$

Standardisieren Jede Zufallsvariable kann standardisiert werden:

$$X' = \frac{X-\mu}{\sigma}$$

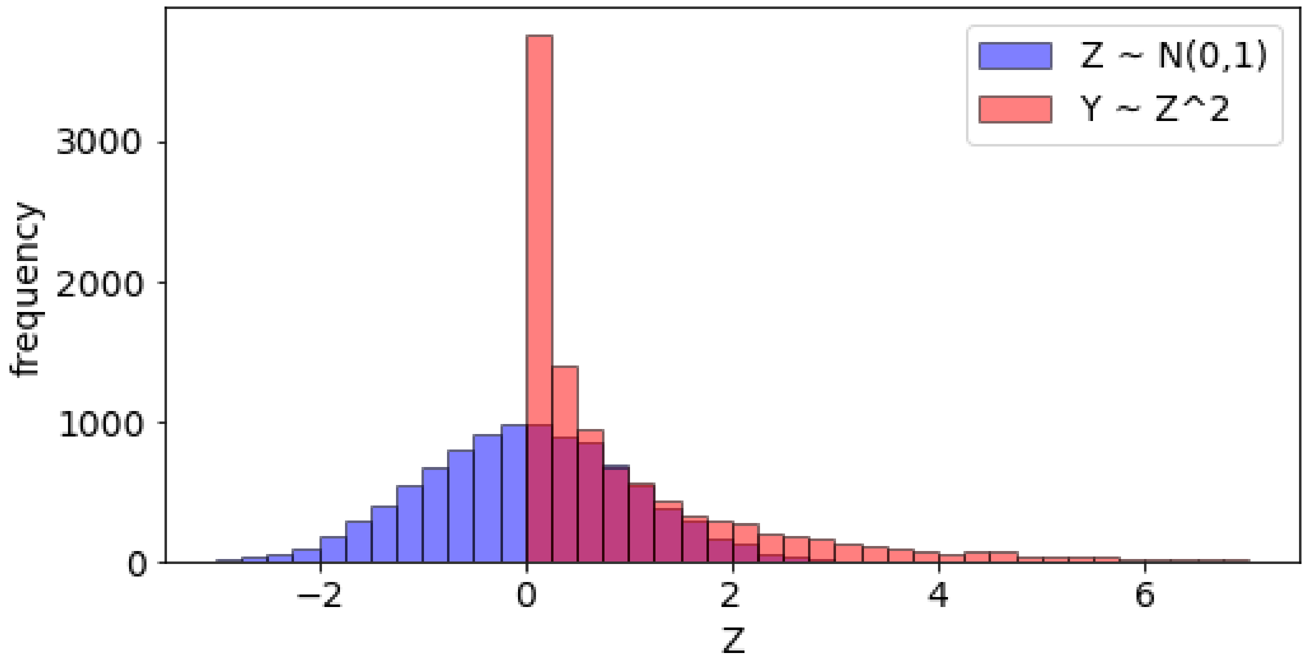
Insbesondere kann jede *normalverteilte* Zufallsvariable $X \sim \mathcal{N}(\mu, \sigma^2)$ auf die Standardnormalverteilung überführt werden:

$$Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

4 Z^2-Verteilung

Für $Z \sim \mathcal{N}(0,1)$ ist $Y = Z^2$ eine neue Zufallsvariable.

```
In [3]: '''distribution of Z^2'''
        Z = stats.norm.rvs(loc=0, scale=1, size=10000)    # Standard normal distribution
        Y = Z**2                                           # new random variable Z^2
        bins = np.linspace(-3., 7., 41)
        fig = plt.figure(figsize=(8, 4))
        plt.hist(Z, bins=bins, alpha=0.5, color='b', label='Z ~ N(0,1)', edgecolor='black')
        plt.hist(Y, bins=bins, alpha=0.5, color='r', label='Y ~ Z^2', edgecolor='black')
        plt.xlabel('Z')
        plt.ylabel('frequency')
        plt.legend();
```



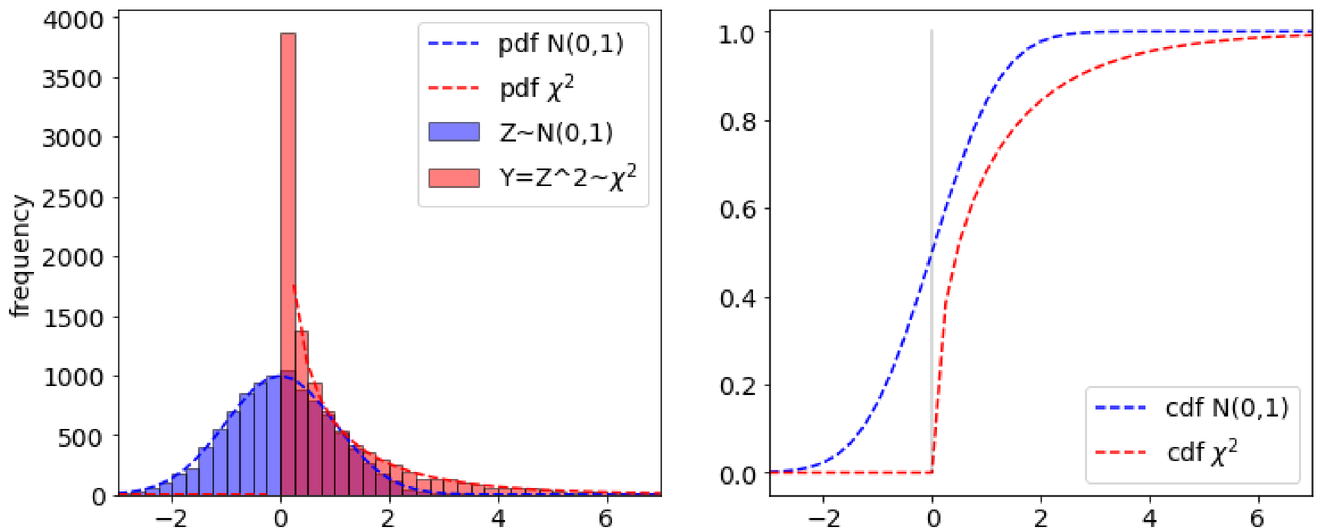
5 χ^2-Verteilung in Python scipy.stats

Zur Berechnung hat **scipy.stats** die Methoden

```
stats.chi2?
``rvs(df, loc=0, scale=1, size=1)``   Random variates.
``pdf(x, df, loc=0, scale=1)``        Probability density function.
``cdf(x, df, loc=0, scale=1)``        Cumulative density function.
``ppf(q, df, loc=0, scale=1)``        Percent point function (inverse of cdf).
...
```

```
In [4]: '''chi-square distribution: Python scipy.stats.chi2'''
        N = 10000
        Z = stats.norm.rvs(loc=0, scale=1, size=N) # standard normal distribution
        Y = Z**2                                   # (vectorized) squared as before
        bins = np.linspace(-3, 7, 41)
        f = plt.figure(figsize=(12,5))
        f.add_subplot(121)                         # --- left graph ---
        plt.xlim((-3, 7))
        plt.hist(Z, bins=bins, alpha=0.5, color='b', label='Z~N(0,1)', edgecolor='black')
        plt.hist(Y, bins=bins, alpha=0.5, color='r', label='Y=Z^2~$\chi^2$', edgecolor='black')
```

```
x = np.linspace(-3., 7., 41)
plt.plot(x, N/4*stats.norm.pdf(x), 'b--', label='pdf N(0,1)')
plt.plot(x, N/4*stats.chi2(1).pdf(x), 'r--', label='pdf $\chi^2$')
plt.ylabel('frequency')
plt.legend()
f.add_subplot(122)                          # --- right graph ---
plt.xlim((-3, 7))
plt.plot(x, stats.norm.cdf(x), 'b--', label='cdf N(0,1)')
plt.plot(x, stats.chi2(1).cdf(x), 'r--', label='cdf $\chi^2$')
plt.plot((0, 0),(0, 1),'k-', alpha=.2)
plt.legend(loc='lower right');
```



Für $Z \sim \mathcal{N}(0,1)$ ist $\qquad Y = Z^2 \sim \chi^2(1)$

6 Familie von Verteilungen: $\chi^2(N)$

Seien $Z_i \sim \mathcal{N}(0,1)$, dann ist

$$Y = \sum_{i=1}^{N} Z_i^2 \sim \chi^2(N)$$

Chi-Quadrat Verteilung eingeführt 1876 von F. R. Helmert; 1900 so benannt von K. Pearson.

Parameter "*Freiheitsgrad*"-Paramter N: **df** (*degree of freedom*)

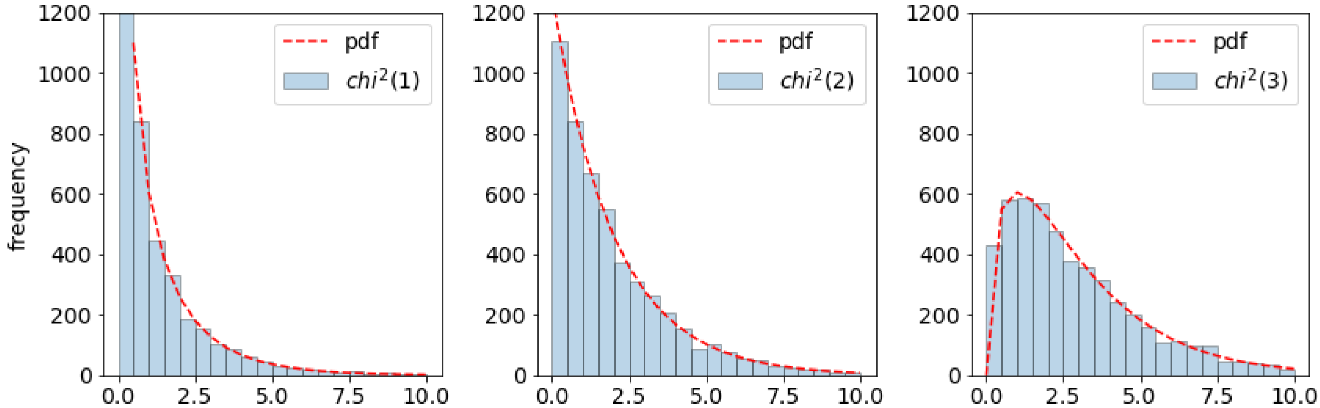
Vektor-Notation

$$Y = \sum Z_i^2 = \mathbf{z}^T\mathbf{z}$$

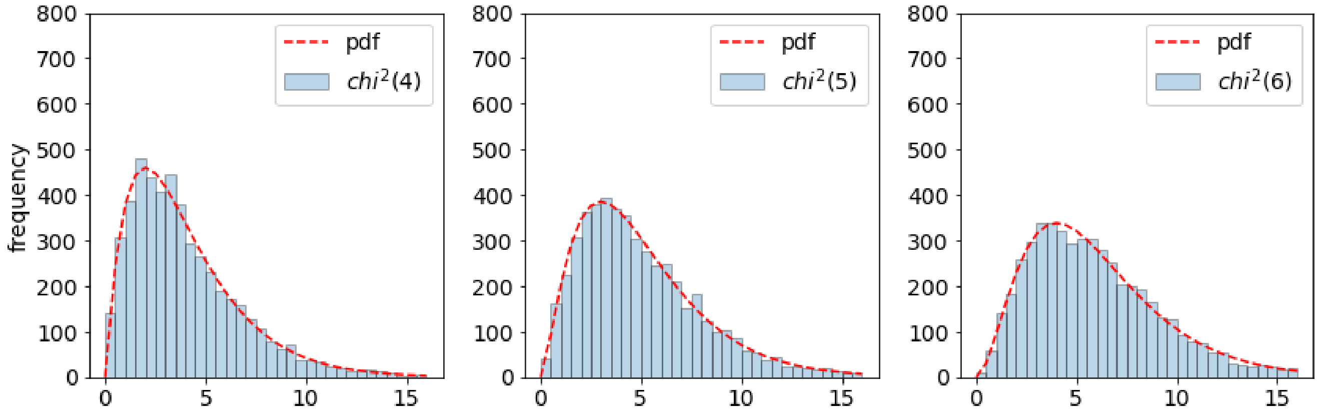
```
stats.chi2?

Parameters
----------
df : array_like                        shape parameters
loc : array_like, optional             location parameter (default=0)
scale : array_like, optional           scale parameter (default=1)
size : int or tuple of ints, optional  shape (default computed from input arguments )
```

```
In [5]: '''by hand: sum up to six squared random variables from standard normal distribution'''
        Nmax = 6
        np.random.seed(345678)
        Zij = stats.norm.rvs(size=(Nmax,5000))  # draw 6 times 5000 rvs of standardnormal
```

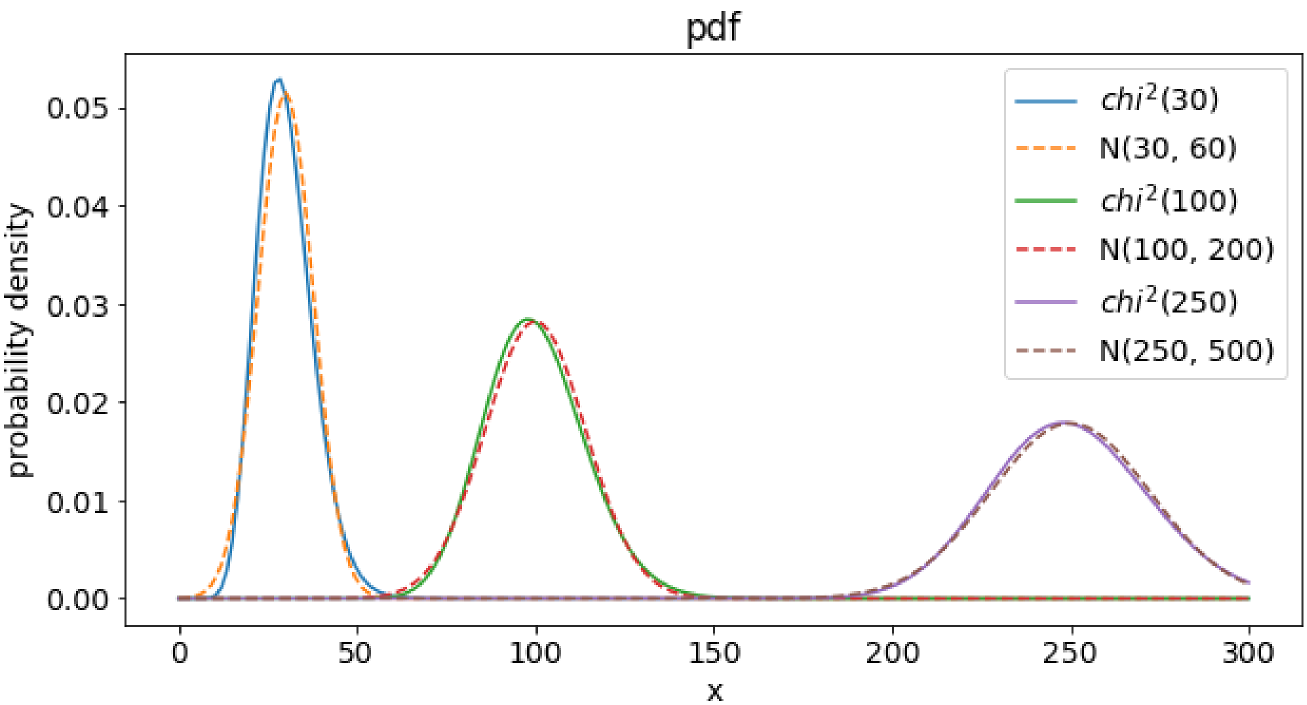
```
f = plt.figure(figsize=(12,4))
bins = np.linspace(0., 10., 21)
chisq = np.zeros_like( Zij[0] )           # initialize chisquare's sum of Z to zero
for n in range(1, 4):                     # range = {1..3}  inclusive
    chisq += Zij[n-1][:]**2               # add the n-th distribution squared
    f.add_subplot(1, 3, n)
    plt.hist(chisq, bins=bins, alpha=.3, label='$chi^2$({})'.format(n), edgecolor='black')
    plt.ylim((0, 1200))
    plt.plot(bins, 2500*stats.chi2(n).pdf(bins),'r--', label='pdf')
    if n==1: # first, left most, only
        plt.ylabel('frequency')
    plt.legend(loc='upper right')
plt.tight_layout();
```



In [6]:
```
'''continue above 4..6, keep sum of {1..3}'''
f = plt.figure(figsize=(12,4))
bins = np.linspace(0., 16., 33)
for n in range(4, Nmax+1):                # range = {4..N}  inclusive
    chisq += Zij[n-1][:]**2               # add the n-th distribution squared
    f.add_subplot(1, 3, n-3)
    plt.hist(chisq, bins=bins, alpha=.3, label='$chi^2$({})'.format(n), edgecolor='black')
    plt.ylim((0, 800))
    plt.plot(bins, 2500*stats.chi2(n).pdf(bins),'r--', label='pdf')
    if n==4: # first, left most, only
        plt.ylabel('frequency')
    plt.legend(loc='upper right')
plt.tight_layout();
```



```
In [7]: x = np.linspace(0., 300, 201)           # x values
        plt.figure(figsize=(9,5))
        for n in [30, 100, 250]:                # three examples with df=30, 100, 250
            plt.plot(x, stats.chi2(n).pdf(x), '-', label='$chi^2$({})'.format(n))
            plt.plot(x, stats.norm(loc=n, scale=np.sqrt(2*n)).pdf(x), '--',
                     label='N({}, {})'.format(n, 2*n))
        plt.title('pdf')
        plt.xlabel('x')
        plt.ylabel('probability density')
        plt.legend(loc='upper right')
        plt.tight_layout();
```



6.0.1 Form der Verteilung

$$f(x) = cx^{\frac{N}{2}-1}e^{-\frac{x}{2}} \qquad fr\ x \geq 0$$

mit der Konstanten $c = ...\Gamma...$

6.1 Erwartungswert

Der Erwartungswert von $Y = Z^2$ mit $Z \sim \mathcal{N}(0,1)$ ist

$$\mathcal{E}(Y) = 1$$

Beweis: $\mathcal{E}(Y) = \mathcal{E}(Z^2) = \mathrm{Var}(Z) = \sigma^2 = 1$

6.1.1 $N \geq 1$

Seien $Z_i \sim \mathcal{N}(0,1)$ unabhängige Zufallsvariablen und $Y = \sum_{i=1}^{N} Z_i^2$, dann gilt für den Erwartungswert eines $\chi^2(N)$-verteilten Y

$$\mathcal{E}(Y) = N$$

Beweis: $\mathcal{E}(Y) = \mathcal{E}(\sum_{i=1}^{N} Z_i^2) = \sum_{i=1}^{N} \mathcal{E}(Z_i^2) = N\mathcal{E}(Z^2) = N \cdot 1$

6.2 Varianz

Die $\chi^2(N)$-Verteilung mit $Y = \sum_{i=1}^N Z_i^2$ mit $Z_i \sim \mathcal{N}(0,1)$ und damit Erwartungswert N hat die Varianz

$$\mathrm{Var}(Y) = \mathcal{E}((Y-N)^2) = 2N$$

Beweis (Kann über die Form der Gammafunktion bewiesen werden)
Hier als Demonstration:

```
In [8]: chisq = np.zeros_like( Zij[0] )      # initialize chisquare's sum of Z to zero
        for n in range(Nmax):                # range = {1..3}  inclusive
            chisq += Zij[n][:]**2            # add the n-th distribution squared
            print('z^2[{}].sum mean = {:.2f}  variance = {:5.2f}, theor. = {:5.2f}'
                  .format(n+1, chisq.mean(), chisq.std()**2, stats.chi2(n+1).var()))
```

```
z^2[1].sum mean = 1.01  variance =  2.14, theor. =  2.00
z^2[2].sum mean = 2.04  variance =  4.34, theor. =  4.00
z^2[3].sum mean = 3.02  variance =  6.43, theor. =  6.00
z^2[4].sum mean = 4.05  variance =  8.47, theor. =  8.00
z^2[5].sum mean = 5.04  variance = 10.29, theor. = 10.00
z^2[6].sum mean = 6.03  variance = 12.16, theor. = 12.00
```

6.3 Reproduktions-Eigenschaft

Seien Y_k unabhängig $\chi^2(n_k)$-verteilt und sei $Y = \sum_{k=1}^m Y_k$, dann ist

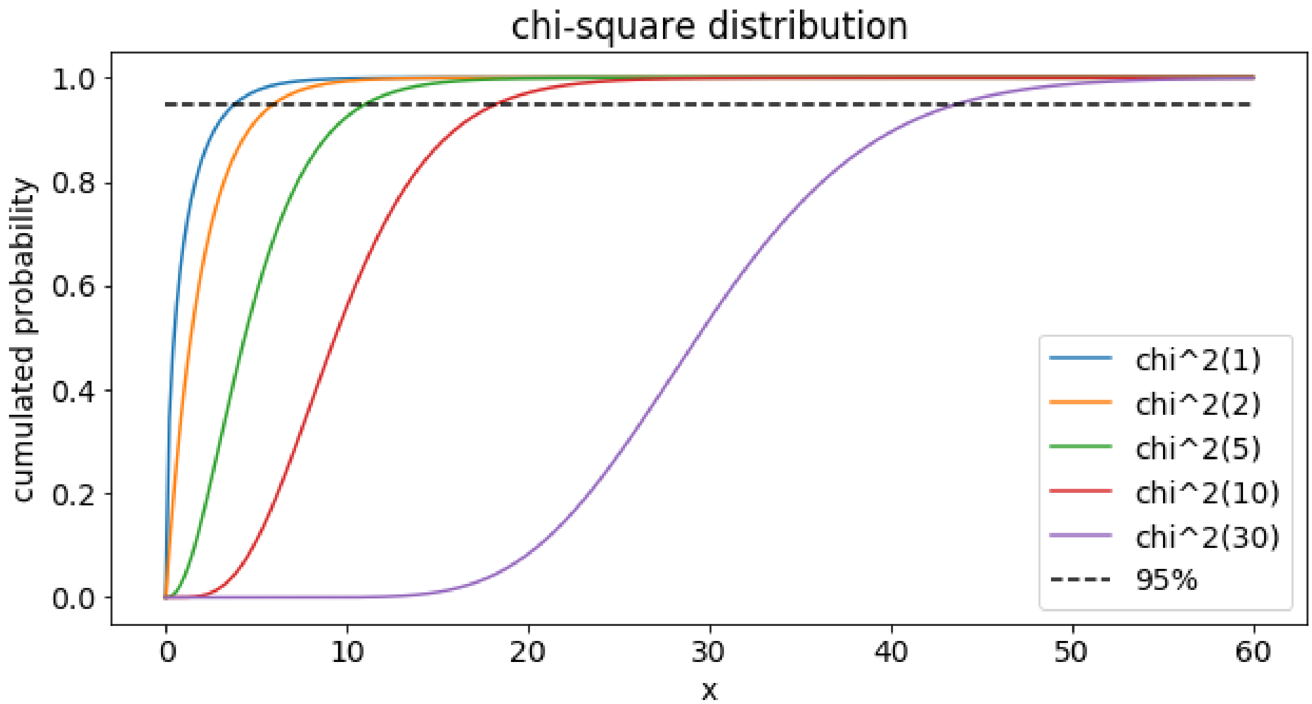
$$Y \sim \chi^2(\textstyle\sum_{k=1}^m n_k)$$

wieder χ^2 verteilt.

Beweis $Y = \sum_k Y_k = \sum_k \sum_i Z_{ik}^2 = \sum_j Z_j^2$

6.4 Wahrscheinlichkeitsverteilungsfunktion

```
In [9]: '''cumulative distribution function of chi-square'''
        fig = plt.figure(figsize=(9, 5))
        ns = [1, 2, 5, 10, 30]
        x = np.linspace(0., 60., 301)
        for n in ns:
            plt.plot(x, stats.chi2(df=n).cdf(x), label='chi^2({})'.format(n))
        plt.plot([x[0], x[-1]], 2*[.95], 'k--', label='95%')
        plt.title('chi-square distribution')
        plt.xlabel('x')
        plt.ylabel('cumulated probability')
        plt.legend(loc='lower right')
        plt.tight_layout();
```




```
In [10]: '''cumulative distribution function of chi-square'''
         ns = [1, 2, 5, 10, 30]
         x = np.linspace(0., 60., 301)
         for n in ns:
             print('95% of chi²({:2d}) at {:6.3f}'.format(n, stats.chi2(df=n).ppf(.95)))
```

```
95% of chi²( 1) at  3.841
95% of chi²( 2) at  5.991
95% of chi²( 5) at 11.070
95% of chi²(10) at 18.307
95% of chi²(30) at 43.773
```

7 Zusammenfassung $\chi^2(N)$-Verteilung

- Ausgangspunkt: Standardnormalverteilungen $Z_i \sim \mathcal{N}(0,1)$
- Neue Zufallsvariable $Y_i = Z_i^2$
- Dann ist $Y = \sum_{i=1}^{N} Y_i$ $\chi^2(N)$-verteilt
- Paramter der Verteilung: N
  - Anzahl der Freiheitsgrade
  - degree of freedom: df
  - bestimmt die Form:
    * Um Null herum
    * Schiefe
    * ab N=30 ähnlich Normalverteilt
- Erwartungswert: $\mathcal{E}(\chi^2(N)) = N$
- Varianz: $\mathrm{Var}(\chi^2(N)) = 2N$
- Reproduzierbarkeit
  - $\sum \chi^2(n_i) \sim \chi^2(\sum n_i)$

8 Fragen?