053_Folien

November 23, 2018

```
In [1]: import numpy as np                        # mathematical methods
        from scipy import stats                   # statistical methods
        from matplotlib import pyplot as plt      # plotting methods
        %matplotlib inline
```

1 Wahrscheinlichkeitstheorie

1.0.1 Zufallsvariable und Wahrscheinlichkeitsraum

1.0.2 Diskrete Zufallsvariablen und Wahrscheinlichkeitsverteilungen

1.0.3 Kontinuierliche Zufallsvariable und Wahrscheinlichkeitsverteilungen

Kontinuierliche Verteilungen

- Rechteckverteilung, Gauß'sche Normal-Verteilung, Exponentialverteilung

Mehrdimensionale Verteilungen

Außergewöhnliche Beispiele kontinuierlicher Verteilungen

- Pareto-Verteilung
- Cauchy-Verteilung

Zusammengesetzte Verteilungen

1.0.4 Wiederholung: Gleichverteilung / Rechteckverteilung

$$x \in [a, b]$$

$$f(x) = const = \frac{1}{b-a}$$

$$\mathcal{E}(X) = \mu = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \sigma^2 = \frac{(b-a)^2}{12}$$

1.0.5 Wiederholung: Normalverteilung $\mathcal{N}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mathcal{E}(X) = \mu$$

$$\mathrm{Var}(X) = \sigma^2$$

$$\text{Schiefe} = 0$$

$$\text{Kurtosis} = 0$$

Standardnormalverteilung $\phi(z)$

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$
$$\mathcal{E}(Z) = 0$$
$$\mathrm{Var}(Z) = 1$$

(kumulierte) Wahrscheinlichkeitsfunktion

$$F(x) = \int_{-\infty}^{x} f(x')\,\mathrm{d}x'$$

68-95-99,7-Prozent-Regel

1.0.6 Wiederholung: Exponentialverteilung

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{sonst} \end{cases}$$
$$\mathcal{E}(X) = \frac{1}{\lambda}$$
$$x_{med} = \frac{\ln 2}{\lambda}$$
$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

2 Außergewöhnliche Verteilungen

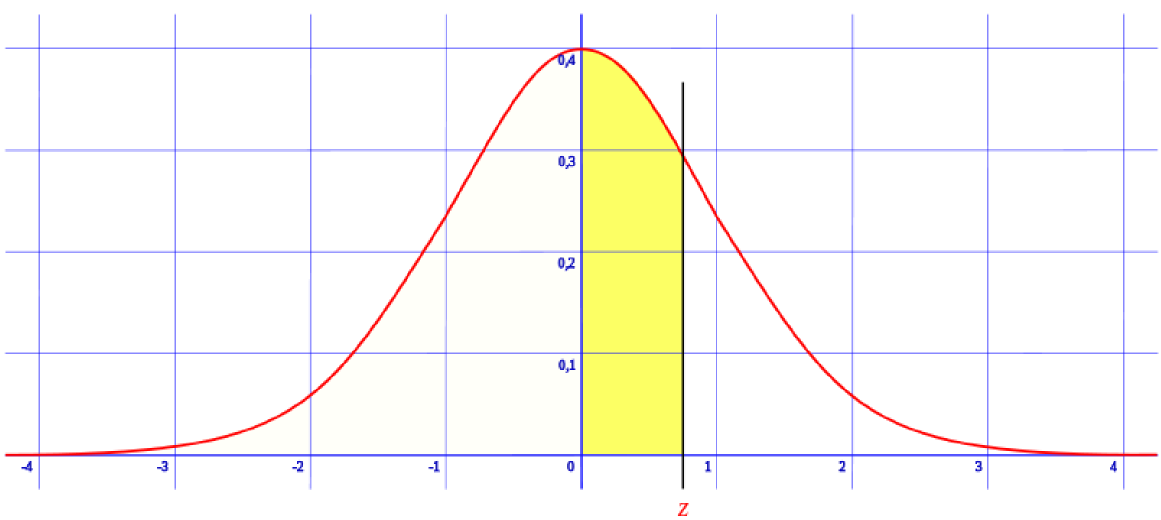
2.0.1 "Pareto-Effekt"

oder 80-zu-20-Regel:

- 80% der Ergebnisse werden mit 20% des Gesamtaufwandes erreicht
- Verbleibende 20% der Ergebnisse benötigen mit 80% die meiste Arbeit

2.1 Pareto Verteilung

$$f(x) = \begin{cases} \frac{k \cdot x_{\min}^k}{x^{(k+1)}} & \text{für } x \geq x_{\min} \\ 0 & \text{sonst} \end{cases}$$



Beispiel aus der Praxis
Quelle: Wikipedia (CC0 1.0 Verzicht auf das Copyright)

Wahrscheinlichkeitsdichte Pareto-Verteilung

$$f(x) = \begin{cases} \frac{k \cdot x_{\min}^k}{x^{(k+1)}} & \text{für } x \geq x_{\min} \\ 0 & \text{sonst} \end{cases}$$

Speziell mit Paramtern $k = 2$ und $x_{\min} = 1$:

Wahrscheinlichkeitsdichte

$$f(x) = \begin{cases} \frac{2}{x^3} & \text{für } x \geq 1 \\ 0 & \text{sonst} \end{cases}$$

Verteilungsfunktion

$$F(x) = \begin{cases} 1 - \frac{1}{x^2} & \text{für } x \geq 1 \\ 0 & \text{sonst} \end{cases}$$

2.1.1 Erwartungswert

$$\mathcal{E}(X) = \int\limits_1^\infty \frac{2}{x^2}\,\mathrm{d}x = 2$$

Wahrscheinlichkeitsdichte

$$f(x) = \begin{cases} \frac{2}{x^3} & \text{für } x \geq 1 \\ 0 & \text{sonst} \end{cases}$$

2.1.2 Besonderheit: Varianz

$$\mathcal{E}(X^2) = \int\limits_1^\infty \frac{2}{x}\,\mathrm{d}x = \infty$$

Und damit auch

$$\mathrm{Var}(X) = \mathcal{E}(X^2) - (\mathcal{E}(X))^2 = \infty$$

2.1.3 Modus:

$$x_{\min}$$

2.1.4 Median:

$$x_{\min}\sqrt[k]{2}$$

2.1.5 letztes Quintil:

$$Q\_\{0.8\} = x\_\{\min\}\sqrt[k]{5}$$

$$\mathcal{E}(X|X > Q_{0.8}) = x_{\min}\frac{k}{k-1}/5^{\frac{k-1}{k}}$$

80% ergibt sich für $k \approx 1,16$.

In [2]:
```
'''German Cities' population > 10.000'''
# source: https://www.destatis.de/DE/ZahlenFakten/LaenderRegionen/Regionales/
#         Gemeindeverzeichnis/Administrativ/Archiv/GVAuszugQ/AuszugGV3QAktuell.html
import pandas as pd                      # allows easy import from data
limit = 14815  # 10000                   # cities, not all villages, arbitrary
data = pd.read_table('data/Staedte2015_10kplus.csv', sep=',') # read destatis data
print('population of German cities:')
print(data[:8])
pop = np.asarray([r[1] for i, r in enumerate(data.values)])   # get data in array
pop = pop[pop>=limit]                                          # restrict above limit
```

```
population of German cities:
                Name        0
0             Berlin  3421829
1            Hamburg  1746342
2            München  1407836
3               Köln  1034175
4  Frankfurt am Main   701350
5          Stuttgart   604297
6         Düsseldorf   598686
7           Dortmund   575944
```

```
In [4]: '''fit empirical data to pareto distribution'''
        pfit = stats.pareto.fit(pop, floc=0)     #fit shape and scale, keep location fixed@0
        print('German cities populations can be fitted as Pareto({:.3f}, {:.0f}) distributed'
              .format(pfit[0], pfit[2]) )
```

```
German cities populations can be fitted as Pareto(1.310, 14825) distributed
```

```
In [5]: '''Quintiles of German cities' population'''
        # empirical quintiles from sorted cities' index
        quintiles = np.round(np.linspace(0, pop.shape[0]-1, 5+1)).astype(int)
        # theor. quintiles of fitted pareto distribution; be careful @ borders
        fitq = stats.pareto.ppf([.0005, .2, .4, .6, .8, .9995], b=pfit[0], scale=pfit[2])
        print('From {} real cities,'.format(pop.shape[0]))
        empqpop = np.zeros_like(fitq)    # (space for) empirical population in fitted quintiles
        for q in range(5):
            pfrom, pto = (fitq[q], fitq[q+1])
            part  = pop[(pfrom<pop) & (pop<=pto)]
            empqpop[q+1] = empqpop[q] + part.shape[0]
            print('     {:.0f} are in {}. fit-quintile from {:6.0f} to {:7.0f}'.
                  format(empqpop[q+1]-empqpop[q], q+1, pfrom, pto))
```

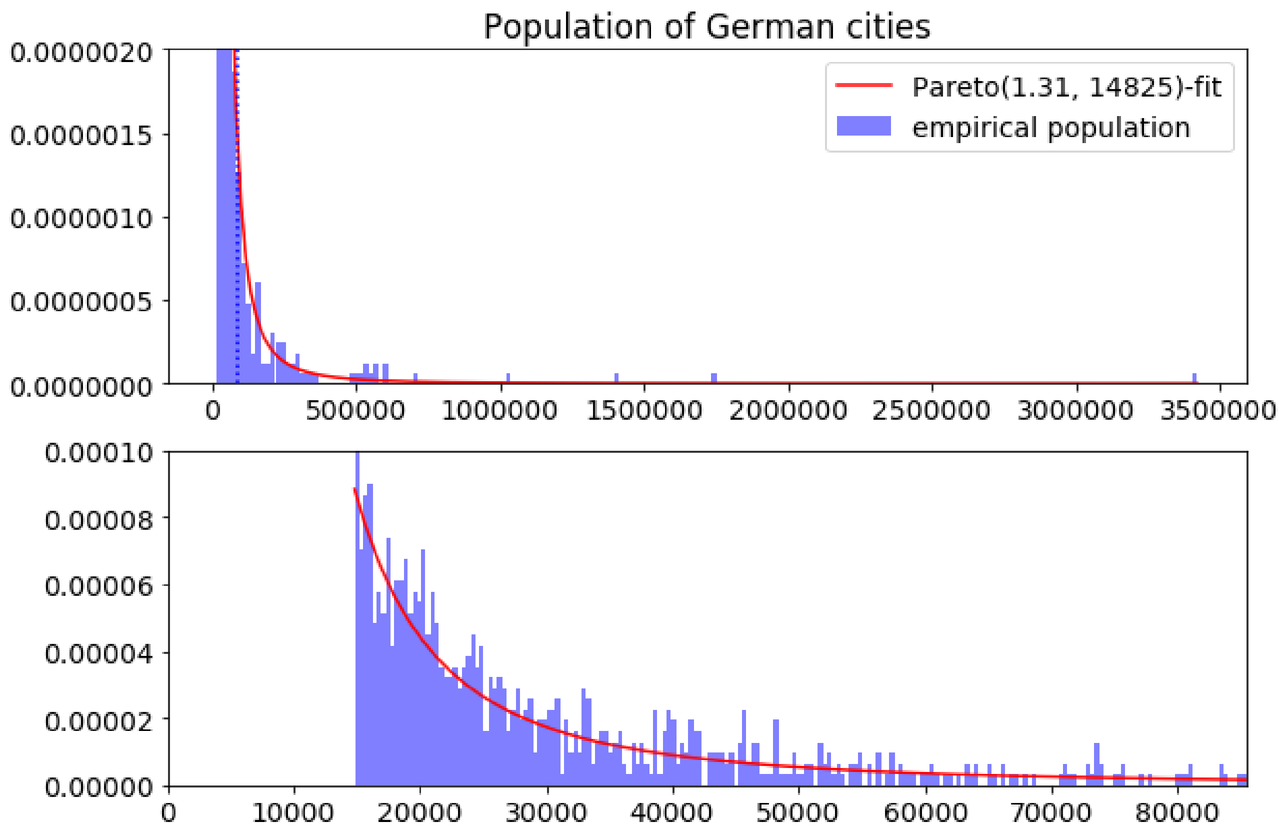
```
From 974 real cities,
     170 are in 1. fit-quintile from  14831 to   17579
     213 are in 2. fit-quintile from  17579 to   21898
     198 are in 3. fit-quintile from  21898 to   29844
     211 are in 4. fit-quintile from  29844 to   50668
     181 are in 5. fit-quintile from  50668 to 4916705
```

```
In [6]: '''Test of pareto principle 80%20-20%80'''
        n1 = np.asarray([n for n in pop[:182]]).sum()
        n2 = np.asarray([n for n in pop[182:]]).sum()
        print('{} habitants in 20% biggest cities'.format(n1))
        print('{} habitants in 80% smaller cities'.format(n2))
```

```
32302386 habitants in 20% biggest cities
20010417 habitants in 80% smaller cities
```

```
In [7]: '''German cities - Pareto distribution - histogram'''
        f = plt.figure(figsize=(10,7))
        f.add_subplot(211)
        plt.title('Population of German cities')
        # plot empirical data
        pthreas = 85400    # Tuebingen
        bins = np.linspace(pop.min(), pop.max(), 201)
        plt.hist(pop, color='b', normed=True, bins=bins, alpha=.5,
                 label='empirical population')
        # plot the fit
```

```
popi = np.linspace(pop.min(), pop.max(), 201)
plt.plot(popi, stats.pareto.pdf(popi, b=pfit[0], scale=pfit[2]), 'r-',
         label='Pareto({:.2f}, {:.0f})-fit'.format(pfit[0], pfit[2]))
plt.ylim(0, 0.000002);
plt.legend()
plt.plot(2*[pthreas], [0, 0.000002], 'b:')
f.add_subplot(212)
# plot empirical data up to threashold
bins = np.linspace(pop.min(), pthreas, 201)
plt.hist(pop, color='b', normed=True, bins=bins, alpha=.5)
# plot fit
popi = np.linspace(pop.min(), pthreas, 201)
plt.plot(popi, stats.pareto.pdf(popi, b=pfit[0], scale=pfit[2]), 'r-')
plt.axis((0, pthreas, 0, 0.0001));
```

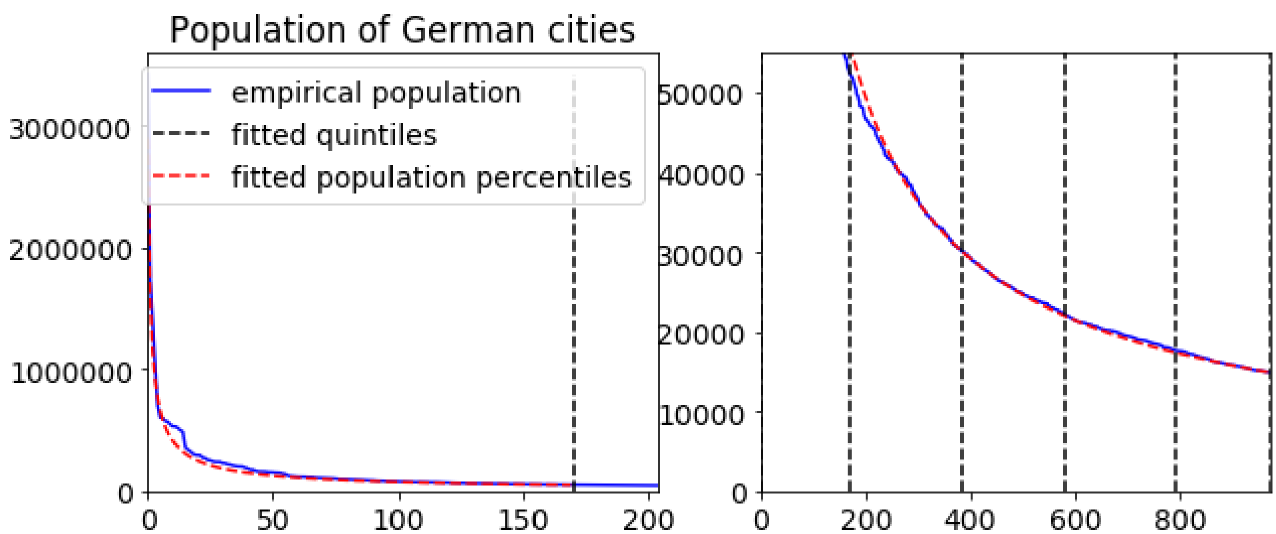



```
In [8]: '''German cities - Pareto distribution - x-axis: index of city'''
        f = plt.figure(figsize=(10,4))
        f.add_subplot(121)                       # --- plot first quantile ---
        plt.title('Population of German cities')
        # plot empirical data
        plt.plot(pop, 'b-', label='empirical population')
        plt.xlim(0, 1.2*empqpop[1])              # show only 1st 20% of cities
        plt.ylim(0, 1.05*pop.max())              # max range including biggest city
        plt.plot(2*[empqpop[1]], (0, pop.max()), 'k--',
                 label='fitted quintiles')       # mark the border between 1st and 2nd
        # plot fitted data compressed to city after city
        n_q1 = np.int(empqpop[1])                # number of cities in highest quintile
        a_q1 = pop[0]                            #  between  3,5Mio and
        b_q1 = pop[n_q1-1]                       #  and      ~50K
        print('{} biggest cities have populations range {} to {}.'.format(n_q1, b_q1, a_q1))
        x = np.linspace(0.999, 0.8, n_q1)        # probabilities for biggest quintile
```

```
popf = stats.pareto.ppf(x, b=pfit[0], scale=pfit[2])   # city population at percentile
plt.plot(popf, 'r--', label='fitted population percentiles')
plt.legend(loc='upper right')

f.add_subplot(122)                      # --- same plot for whole range ---
plt.plot(pop, 'b-')
plt.xlim(0, empqpop[-1])                # show complete range to last quintile border
ymax = 1.05*pop[n_q1]
plt.ylim(0, ymax)                       # max range starting 2nd quintile
# plot fitted data compressed to city after city
n_q1 = pop.shape[0]                     # number of cities total
x = np.linspace(0.999, 0.001, n_q1)     # probabilitie percentiles for number of cities
popf = stats.pareto.ppf(x, b=pfit[0], scale=pfit[2])  # city population at percentile
plt.plot(popf, 'r--')
for q in empqpop:
    plt.plot((q, q), (0, ymax), 'k--'); # show quintiles of fitted distribution
```

```
170 biggest cities have populations range 52400 to 3421829.
```



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3 Cauchy/Lorentz-Verteilung

Familie von Verteilungen mit

$$f(x) = \frac{1}{\pi} \frac{s}{s^2 + (x-t)^2}$$

Speziell mit dem Zentrum $t = 0$ und der Breite $s = 1$ *Standard-Cauchy-Verteilung*:

$$f(x) = \frac{1}{\pi} \frac{1}{1 + x^2}$$

Kumulative Verteilungsfunktion

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \cdot \arctan x$$

3.1 Anwendung

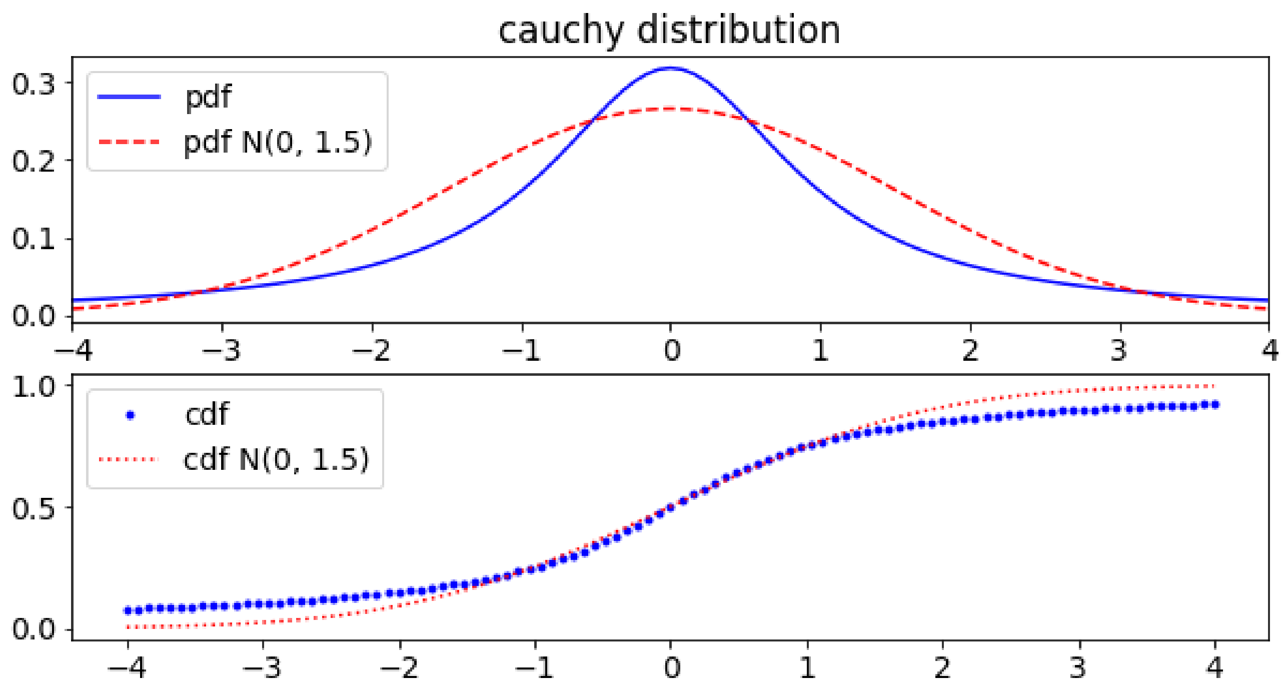
Physik:

- Resonante Schwingung
- Form von Spektrallinien

Beziehung zur Normalverteilung Der Quotient aus zwei unabhängigen standardnormalverteilten Zufallsvariablen ist Standard-Cauchy-verteilt.

Beziehung zu Student'schen t-Verteilung Die Standard-Cauchy-Verteilung ist der Spezialfall der studentschen t-Verteilung mit einem Freiheitsgrad.

```
In [10]: '''cauchy distribution - compare to Normal distribution'''
         X = 4.                                  # x = {-X..X}
         x = np.linspace(-X, X, 101)
         f = plt.figure(figsize=(10,5))
         f.add_subplot(211)
         plt.xlim((-X, X))
         plt.title('cauchy distribution')
         plt.plot(x, stats.cauchy.pdf(x), 'b-', label='pdf')
         plt.plot(x, stats.norm(scale=1.5).pdf(x), 'r--', label='pdf N(0, 1.5)')
         plt.legend(loc='upper left');
         f.add_subplot(212)
         plt.plot(x, stats.cauchy.cdf(x), 'b.', label='cdf')
         plt.plot(x, stats.norm(scale=1.5).cdf(x), 'r:', label='cdf N(0, 1.5)')
         plt.legend(loc='upper left');
```



3.1.1 Besonderheit:

Keines der Momente ist definiert,
die Integrale für Erwartungswert, Varianz, ... konvergieren alle **nicht**!
Moment-Integrale berechnen über Residuensatz. Oder Fouriertransformation....

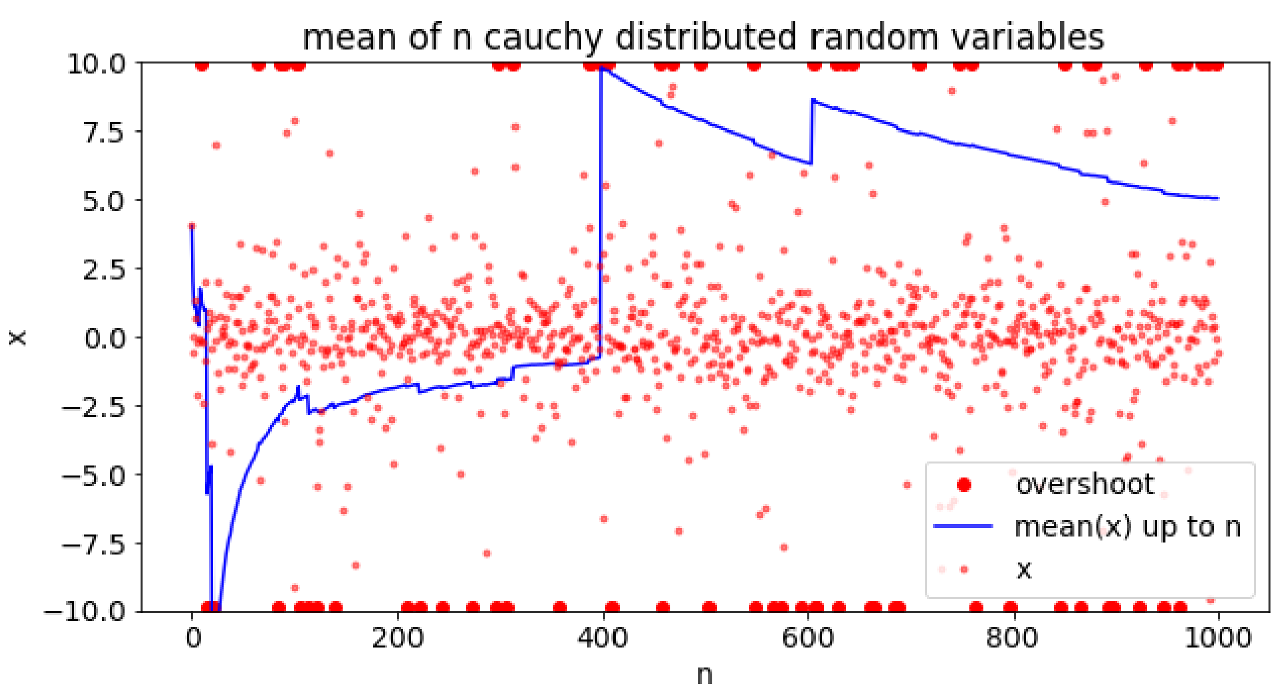
Varianz ist berechenbar:

$$
\begin{aligned}
\mathrm{E}[X^2] \propto \int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\,\mathrm{d}x \\
= \int_{-\infty}^{\infty} (1 - \frac{1}{1+x^2})\,\mathrm{d}x \\
= \int_{-\infty}^{\infty} \mathrm{d}x - \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x \\
= \int_{-\infty}^{\infty} \mathrm{d}x - \pi \\
= \infty
\end{aligned}
$$

Erwartungswert:

- undefiniert
- Warum nicht Null aufgrund von Symmetrie?

In [11]:
```
'''simulate expectation value of cauchy distributiion'''
N = 1000                                      # samples to draw
A = 10.                                       # plot x = {-A..A}
np.random.seed(987624)                        # fix random (have a nice example)
x = np.random.standard_cauchy(N)              # draw according cauchy distribution
n = np.arange(N)                              # i for x-axis; means of x until i
m = np.asarray([x[:i+1].mean() for i in n])      # mean for 0..i each
f = plt.figure(figsize=(10,5))
plt.ylim(-A, A)
q = np.ones_like(x)*2.*A                      # q out of plot region
q[x>A] = A-.1                                 # if x out of plot region, set q to border
plt.plot(n, q, 'ro', label='overshoot')       # and plot as "overshoot"
q = np.ones_like(x)*2.*A                      # and
q[x<-A] = -A+.1                               # ...
plt.plot(n, q, 'ro')                          # same for "undershoot"
plt.plot(n, m, 'b-', label='mean(x) up to n')  # the means up to i
plt.plot(n, x, 'r.', alpha=.5, label='x')     # values of drawn samples
plt.legend(loc='lower right')
plt.xlabel('n')
plt.ylabel('x')
plt.title('mean of n cauchy distributed random variables');
```



4 Zusammenfassung

Außergewöhnliche Verteilungen

- Pareto-Verteilung
- Cauchy/Lorentz-Verteilung

Kennwerte

- ?!

5 Fragen?