051_Folien

November 23, 2018

```
In [1]: import numpy as np                        # mathematical methods
        from scipy import stats                   # statistical methods
        from matplotlib import pyplot as plt      # plotting methods
        %matplotlib inline
```

1 Wahrscheinlichkeitstheorie

1.1 Zufallsvariable und Wahrscheinlichkeitsraum

1.2 Erwartungswert und Varianz

1.3 Diskrete Zufallsvariablen und Wahrscheinlichkeitsverteilungen

1.4 Mehrdimensionale Verteilungen

1.5 Kontinuierliche Zufallsvariable und Wahrscheinlichkeitsverteilungen

1.6 Wahrscheinlichkeitsverteilung kontinuierlicher Zufallsvariable

Unendlich viele dichte Ereignisse (meist $x \in \mathbb{R}$) möglich.

Beispiele

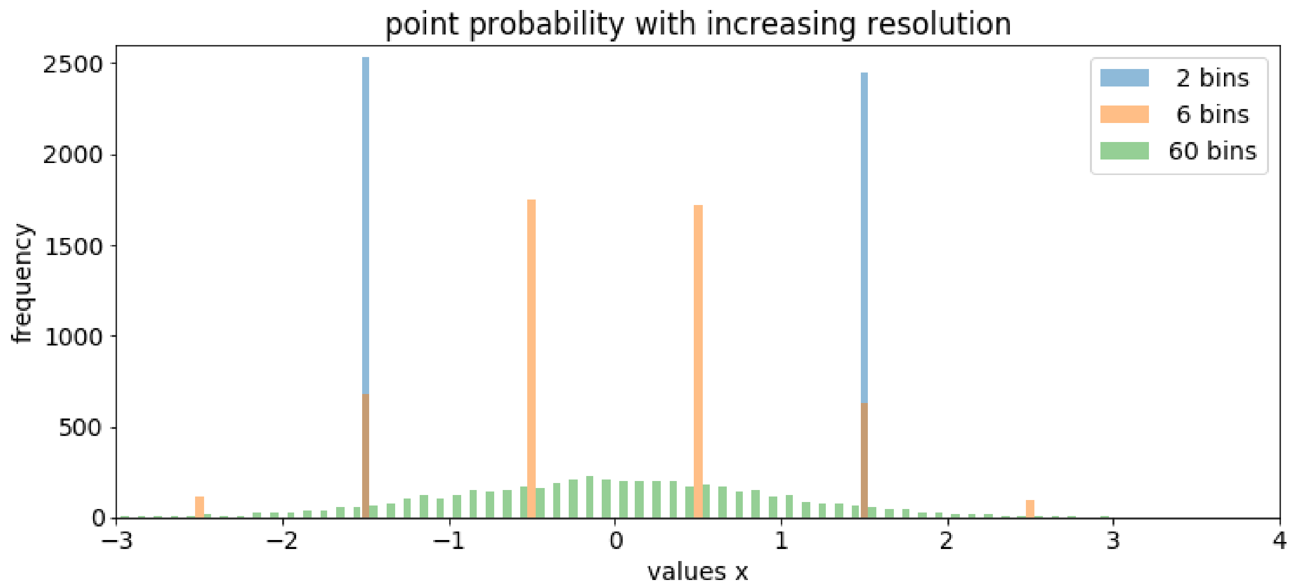
- Rauschspannung
- Aktienkurse
- Chemikalienkonzentrationen

1.6.1 Problem: Punktwahrscheinlichkeit

$$P(x = a) \to 0$$

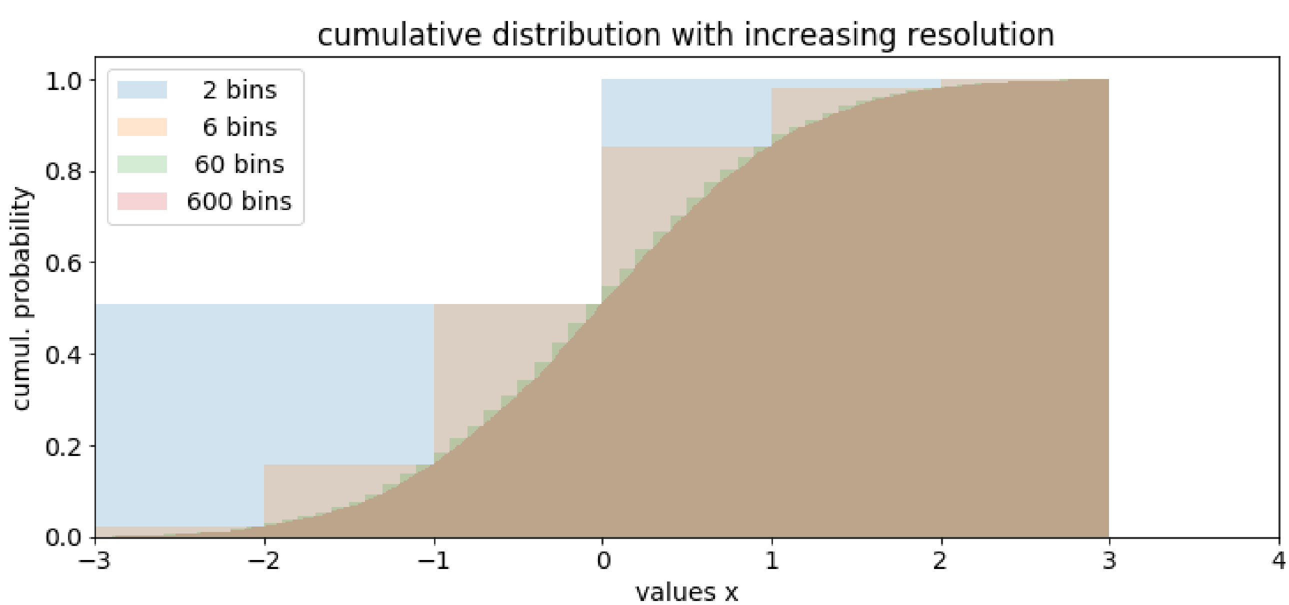
```
In [3]: '''discrete --> continuous distribution: single probability'''
        f = plt.figure(figsize=(12, 5))
        np.random.seed(9876543)
        x = stats.norm(0., 1.).rvs(size=5000)      # draw random numbers

        for n in (2, 6, 60):                       # discretisize to n bins
            h = plt.hist(x, bins=np.linspace(-3., 3., n+1), align='mid',
                         rwidth=0.008*n, alpha=0.5, label='{:2d} bins'.format(n))
            print('discrete {:3d} has center probabilities {}'
                  .format(n, h[0][max(0, int(n/2-3)):min(n, int(n/2+3))]))
        plt.axis((-3., 4., 0, 2600 ))
        plt.title('point probability with increasing resolution')
        plt.xlabel('values x')
        plt.ylabel('frequency')
        plt.legend();
```

```
discrete   2 has center probabilities [2537. 2443.]
discrete   6 has center probabilities [ 117.  675. 1745. 1716.  632.   95.]
discrete  60 has center probabilities [205. 226. 207. 198. 195. 201.]
```




```
In [4]: '''discrete --> continuous distribution: cumulative'''
        f = plt.figure(figsize=(12, 5))
        for n in (2, 6, 60, 600):                # discretisize to n bins
            h = plt.hist(x, bins=np.linspace(-3., 3., n+1), density=True,
                         cumulative=True, alpha=0.2, label='{:3d} bins'.format(n))
            print('discrete {:4d} bins have center probabilities {}'
                  .format(n, h[0][max(0, int(n/2-2)):min(n, int(n/2+2))]))
        plt.axis((-3., 4., 0, 1.05 ))
        plt.title('cumulative distribution with increasing resolution')
        plt.xlabel('values x')
        plt.ylabel('cumul. probability')
        plt.legend();
```

```
discrete    2 bins have center probabilities [0.50943775 1.        ]
discrete    6 bins have center probabilities [0.15903614 0.50943775 0.85401606 0.98092369]
discrete   60 bins have center probabilities [0.46787149 0.50943775 0.54919679 0.58835341]
discrete  600 bins have center probabilities [0.50421687 0.50943775 0.5126506  0.51767068]
```



Gemeinsamkeit *Steigung*: diskret : $\frac{\Delta F}{\Delta x} = m$ kontinuierlich : $\frac{\mathrm{d}F}{\mathrm{d}x} = f$

1.7 Wahrscheinlichkeitsdichte $f(x)$

$$f(x) \geq 0$$

$$P(a \leq x \leq b) = \int_a^b f(x)\,\mathrm{d}x \;\leq 1$$

Möglich:

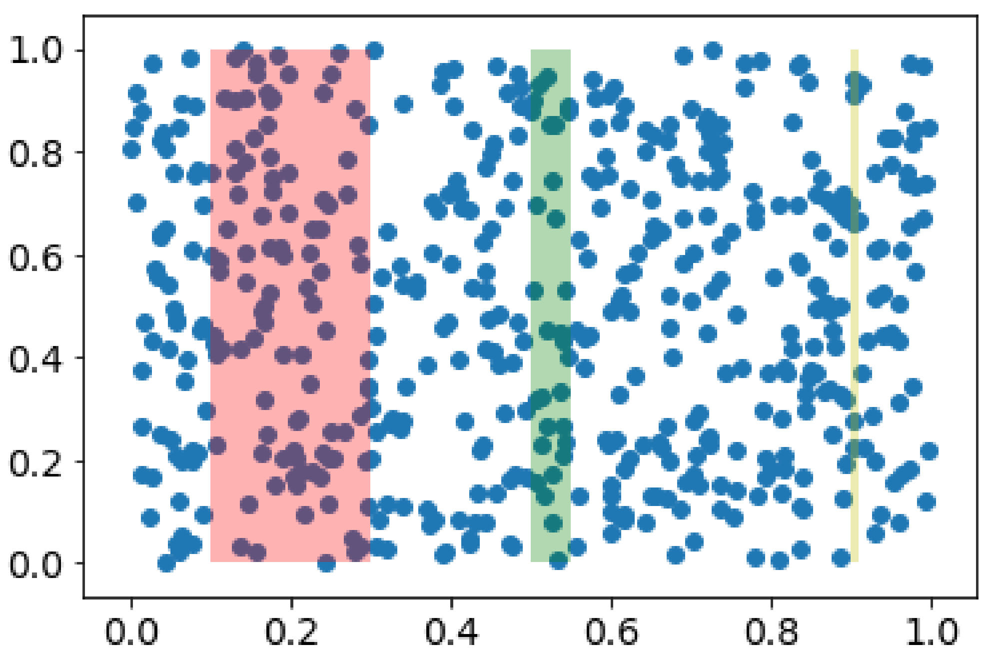
$$f(x) \not\leq 1$$

Normierung

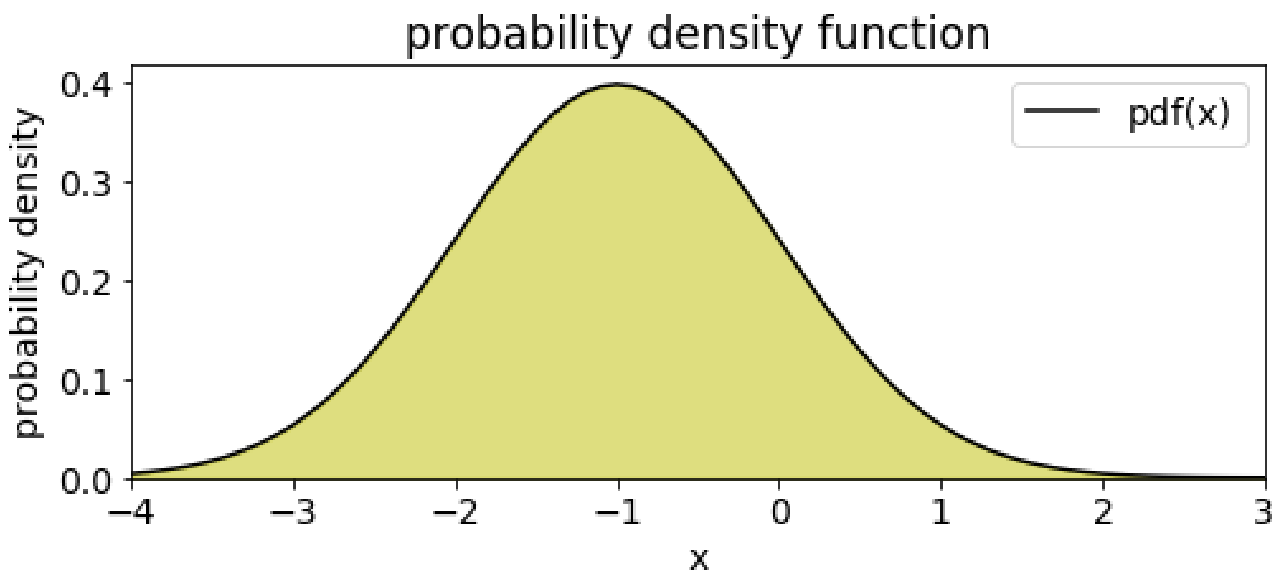
$$\int_{x=-\infty}^{\infty} f(x)\mathrm{d}x = 1$$

```
In [6]: '''raindrops are falling - show density'''
        np.random.seed(98765432)
        x = np.random.rand(500)                    # 500 raindrops x coordinate 0..1
        y = np.random.rand(500)                    #                y coordinate 0..1
        plt.scatter(x,y)                           # show the raindrops
        # define three stripes (location, width, color)
        s = ((.1, .2, 'r'), (.5, .05, 'g'), (.9, .01, 'y'))
        for si in s:                               # choose one of above
            a, w, col = si                         # separate location a and width w
            # barh(y-bottom, x-width, y-height, x-left)
            plt.barh( .5, w, 1., a, color=col, alpha=.3)
            # select x between stripe borders
            n = x[np.logical_and(a<x, x<=a+w)].shape[0]
            print('in width={:.2f} are {:3d} drops or {:4.1f}%, density={:.1f}'
                  .format(w, n, n/500*100, n/w/500))
```

```
in width=0.20 are 100 drops or 20.0%, density=1.0
in width=0.05 are  29 drops or  5.8%, density=1.2
in width=0.01 are   6 drops or  1.2%, density=1.2
```

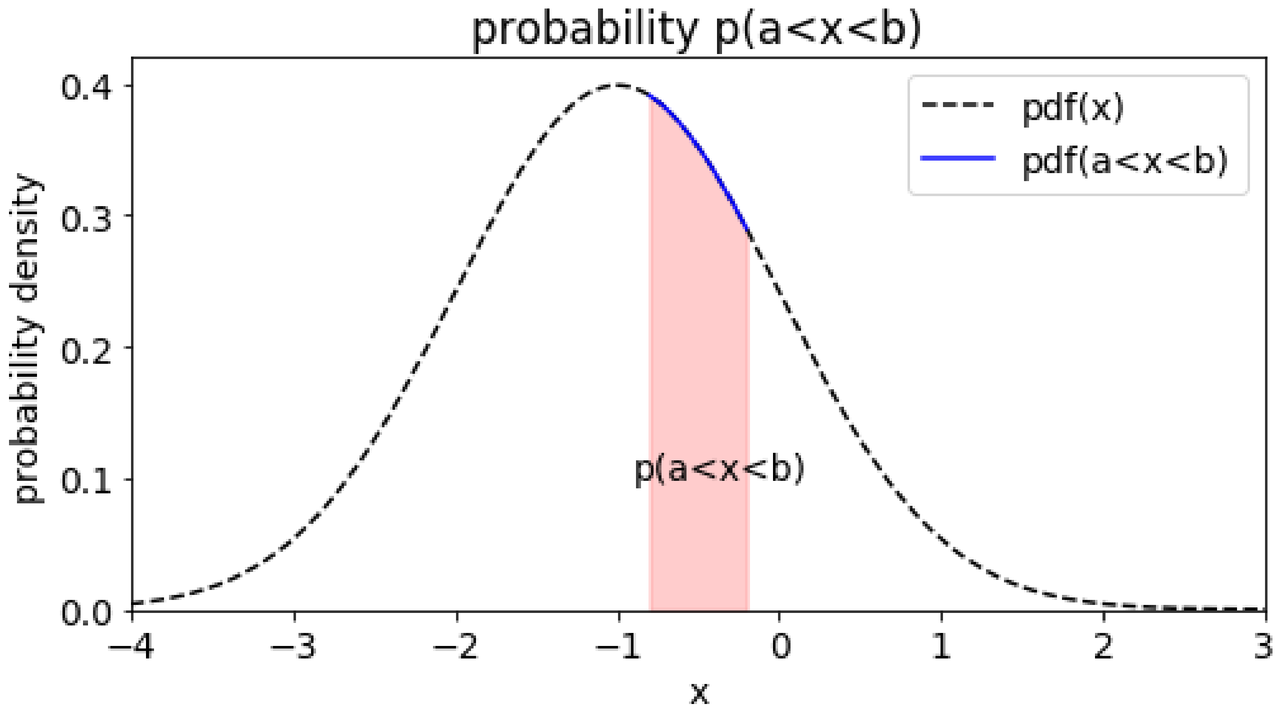


In [6]:
```python
'''probability density function of x'''
xgrid = np.linspace(-4, 3, 70+1) # just some x values
distrib = stats.norm(-1., 1)      # define a probability distribution & freeze this
f = plt.figure(figsize=(8, 3))
plt.title('probability density function')
plt.plot(x, distrib.pdf(x), 'k-', label='pdf(x)')
# make area under curve yellow
plt.fill_between(x, 0, distrib.pdf(x), color='y', alpha=0.5)
plt.axis((-4., 3., 0, 0.42))
plt.xlabel('x')
plt.ylabel('probability density')
plt.legend();
```



In [7]:
```python
'''probability of x between a and b'''
x = np.linspace(-6, 4, 1001)          # dense enough x values for smooth graphics
a = -0.8                              # chose lower
b = -0.2                              # and upper boundary of ROI

f = plt.figure(figsize=(8, 4))
plt.title('probability p(a<x<b)')
# complete distribution, frozen from above
plt.plot(x, distrib.pdf(x), 'k--', label='pdf(x)')
xab = np.linspace(a, b, 11)  # select ROI, make smooth enough
plt.plot(xab, distrib.pdf(xab), 'b-', label='pdf(a<x<b)') # highlight ROI in blue
plt.fill_between(xab, 0, distrib.pdf(xab), color='r', alpha=0.2) # and make area visible
plt.text(-0.9, 0.1, 'p(a<x<b)')
plt.axis((-4., 3., 0, 0.42))
plt.xlabel('x')
plt.ylabel('probability density')
plt.legend();

ptotal = distrib.pdf(x).mean()*(4.-(-6.))             # normalized?
xab = np.linspace(a, b, 1001)                         # dense enough x values
p = distrib.pdf(xab).mean()*(b-a)                     # pseudo integral /
print('probability of x between {:.2f} and {:.2f} is {:.3f} (of norm={:.4f})'
      .format(a, b, p, ptotal))
```

```
probability of x between -0.80 and -0.20 is 0.209 (of norm=0.9990)
```



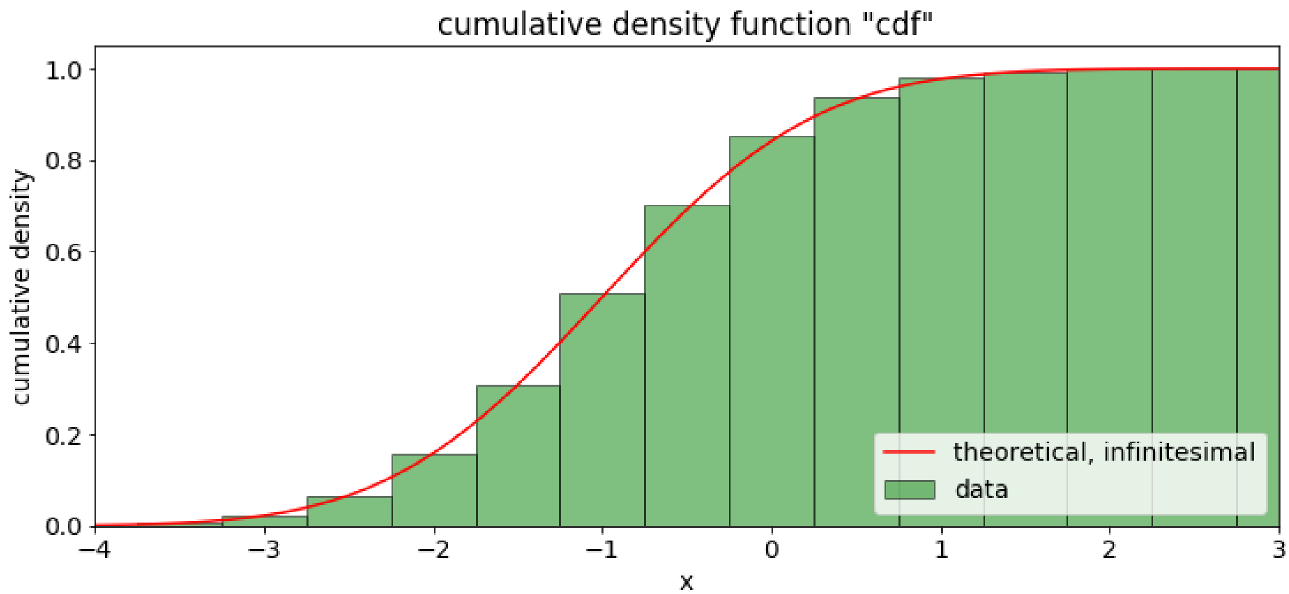
1.8 (Wahrscheinlichkeits-) Verteilungsfunktion

$$F(x) = \int_{x'=-\infty}^{x} f(x')\,\mathrm{d}x'$$

Vergleiche mit diskreter Verteilungsfunktion:

$$F(x) = \sum_{x_i <= x} p(x_i)$$

```
In [8]: ''' N=10 bins as discrete example above
            x from above: origin was standard normal distribution'''
        np.random.seed(9876543)
        x = distrib.rvs(size=5000)          # draw random numbers from chosen norm
        f = plt.figure(figsize=(12, 5))
        plt.hist(x, bins=np.linspace(-4., 3., 14+1), label='data',
                 color='green', cumulative=True, alpha=0.5, density=True, align='right',
                 edgecolor='black', linewidth=1.)
        # cdf
        xi = np.linspace(-4., 3., 70+1)     # make an x-axis with finer resolution
        y = distrib.cdf(xi)                 # use the cdf-fct (with data x) for xi
        plt.plot(xi, y, 'r-', label='theoretical, infinitesimal')
        plt.axis((-4., 3., 0, 1.05))
        plt.xlabel('x')
        plt.ylabel('cumulative density')
        plt.title('cumulative density function "cdf"')
        plt.legend(loc='lower right');
```



1.9 Eigenschaften der Verteilungsfunktion

für integrierbare Dichten $f(x)$

$$F(x) = \int_{-\infty}^{x} f(x')\, \mathrm{d}x'$$

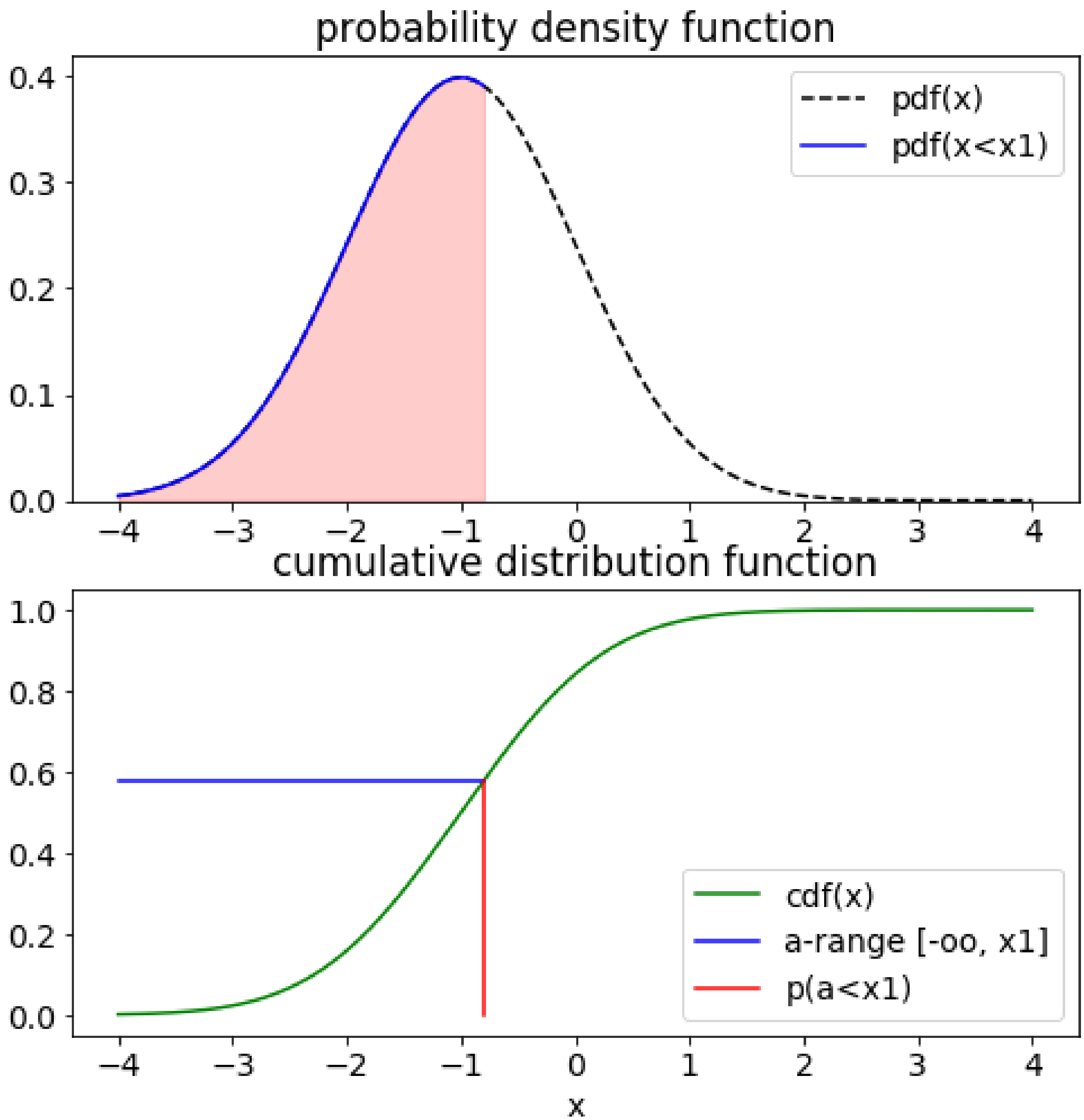
- $F(x)$ ist monoton in x
- $F(-\infty) = 0$
- $F(\infty) = 1$
- $P(a \leq x \leq b) = \int_a^b f(x)\, \mathrm{d}x = F(b) - F(a)$
- $P(X \geq c) = 1 - F(c)$
- Punktwahrscheinlichkeit: wenn $a = x = b$ dann $P(X{=}b) = F(b) - F(b) = 0$ wichtig!

1.9.1 Anschaulich

In [10]:
```
'''cumulated probability of x up to a'''
r0 = -4.
r1 = 4.
x = np.linspace(r0, r1, 801)       # dense enough a values
x1 = -0.8                          # choose an x: x1
p1 = distrib.cdf(x1)               # calculate x1' probability
print('cumulated probability up to x1={:.2f} is {:.2f}'
      .format(x1, p1))

f = plt.figure(figsize=(8, 8))
f.add_subplot(211)                 # 2 subplots, this 1st column, 1st row
plt.title('probability density function')
plt.plot(x, distrib.pdf(x), 'k--', label='pdf(x)')
xa = np.linspace(r0, x1, 81)       # x values up to x1
# plot blue pdf-line and fill space below (from 0 upwards to pdf)
plt.plot(xa, distrib.pdf(xa), 'b-', label='pdf(x<x1)')
plt.fill_between(xa, 0, distrib.pdf(xa), color='r', alpha=0.2)
plt.ylim(0, 0.42)                  # include max p ~0.4
plt.legend()

f.add_subplot(212)             # from the 2 subplots: this 1st columns, 2nd row
plt.title('cumulative distribution function')
```

```
    plt.xlabel('x')
    x = np.linspace(r0, r1, 801)        # dense enough x values
    # cdf of complete distribution over all x
    plt.plot(x, distrib.cdf(x), 'g-', label='cdf(x)')
    cc = distrib.cdf(x1)                # cdf(x=x1)
    # blue line for range of x from left to x1 in height cc
    plt.plot([r0, x1], 2*[cc], 'b-', label='a-range [-oo, x1]')
    # red line for p=0 to cc at x=x1
    plt.plot(2*[x1], [0, cc], 'r-', label='p(a<x1)')
    plt.legend(loc='lower right');
```

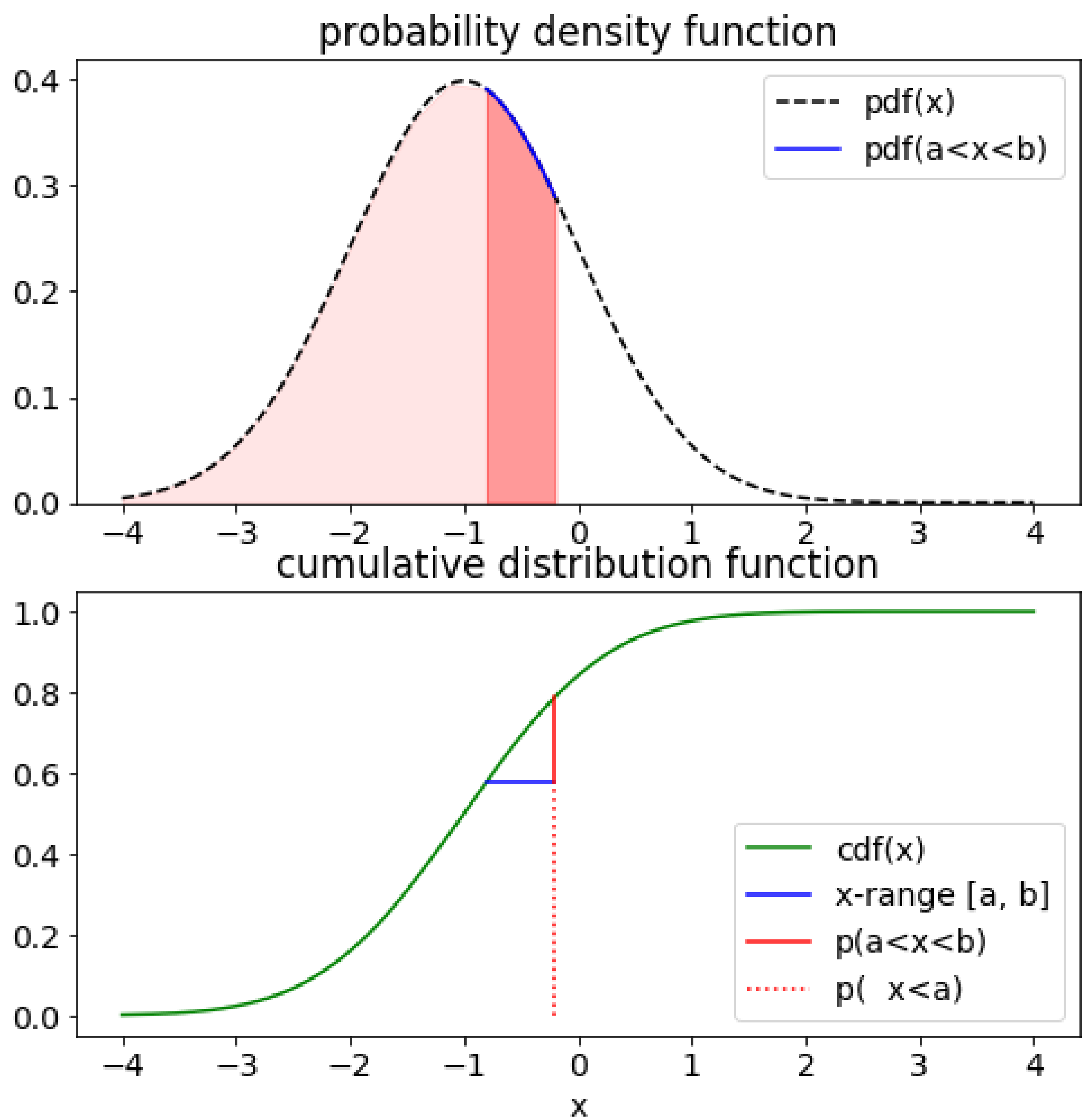
```
cumulated probability up to x1=-0.80 is 0.58
```




```
In [11]: '''probability of x between a and b'''
         x = np.linspace(-4, 4, 801)        # dense enough x values
```

```python
a = -0.8                             # chose lower
b = -0.2                             # and upper boundary
p = distrib.cdf(b)-distrib.cdf(a) # calculate probability
print('probability of x between {:.2f} and {:.2f} is {:.3f}'
      .format(a, b, p))

f = plt.figure(figsize=(8, 8))
f.add_subplot(211)                   # 2 subplots, this 1st column, 1st row
plt.title('probability density function')
plt.plot(x, distrib.pdf(x), 'k--', label='pdf(x)')
xab = np.linspace(-4, a, 11)         # x values from "-infty" to a
plt.fill_between(xab, 0, distrib.pdf(xab), color='r', alpha=0.1)
xab = np.linspace(a, b, 11)          # x values from a to b
# plot blue pdf-line from a to b and fill space below (from 0 upwards to pdf)
plt.plot(xab, distrib.pdf(xab), 'b-', label='pdf(a<x<b)')
plt.fill_between(xab, 0, distrib.pdf(xab), color='r', alpha=0.4)
plt.ylim(0, 0.42)
plt.legend()

f.add_subplot(212)
plt.title('cumulative distribution function')
plt.xlabel('x')
# green cdf of complete distribution over all x
plt.plot(x, distrib.cdf(x), 'g-', label='cdf(x)')
cc = distrib.cdf([a, b])             # cdf of left a and right b
# blue line for range of x from a to b in height cdf(a)
plt.plot([a, b], 2*[cc[0]], 'b-', label='x-range [a, b]')
# red line for p=cdf(a) to cdf(b) at x=b
plt.plot(2*[b], cc, 'r-', label='p(a<x<b)')
plt.plot(2*[b], [0, cc[0]], 'r:', label='p(  x<a)')
plt.legend(loc='lower right');
```

```
probability of x between -0.80 and -0.20 is 0.209
```



1.9.2 Vergleich mit diskreten Verteilungen

Variable	diskret	kontinuierlich . . . .
Wert x	$x_i \ \ i \in \mathbb{N}$	$x \in \mathbb{R}$
Wahrscheinlichkeit p .	$p(X{=}x_i) = p_i$ `pmf()`	$p(a \leq x \leq b) = \int_a^b f(x)\mathrm{d}x$ `pdf()`
Verteilungsfunktion F .	Schritte $\Delta F = p_x$ `cdf()`	kontinuierlich $dF = f\mathrm{d}x$ `cdf()`

1.10 Definition Erwartungswert

Sei X eine Zufallsvariable auf $\mathbb{R}$ mit Wahrscheinlichkeitsdichte $f(x)$, dann ist der Erwartungswert von X

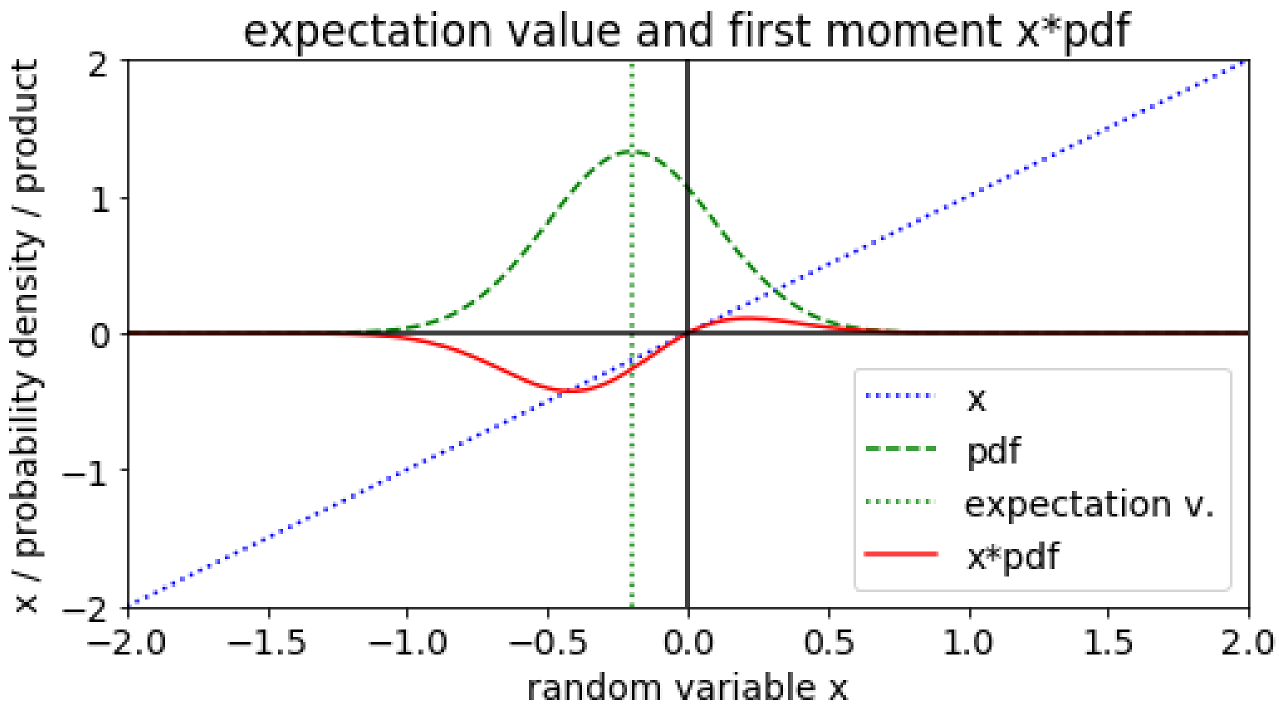
$$\mathcal{E}(X) = \mu = \int_{-\infty}^{\infty} f(x) \cdot x \, \mathrm{d}x$$

Vergleich diskreter Erwartungswert

$$\mathcal{E}(X) = \mu = \sum_{i=1}^{N} p(x_i) \cdot x_i$$

In [12]:
```
'''meaning of expectation value of x as weighted integral'''
r0, r1 = (-2.0, 2.0)                          # borders
ndistrib = stats.norm(-0.2, 0.3)              # define and freeze a (normal) probability distribution
x = np.linspace(r0, r1, 40*round(r1-r0)+1)    # dense enough values
px = ndistrib.pdf(x)                          # get x's probability

f = plt.figure(figsize=(8, 4))                # space for graphics
plt.title('expectation value and first moment x*pdf')
plt.xlabel('random variable x')
plt.ylabel('x / probability density / product')
plt.plot(x, x,    'b:',  label='x')           # x (just repeated as y value)
plt.plot(x, px,   'g--', label='pdf')         # probability density of x - as a weight

# calculate and show expectation value
print('expectation value of x under pdf is {:.3f}'.format(ndistrib.expect()))
plt.plot(2*[ndistrib.expect()], [-2, 2], 'g:', label='expectation v.')

# expectation value is integral of local x*px function
# meaning: x weighted with its probability density
# integral is then area under red, here negative
# approximation: sum over the 160 small rectangles of size dx
dx = (r1-r0)/(len(x)-1)                       # length of approximation rectangle
expectation_approx = dx*(x*px).sum()          # product is point-wise
print('approximated expectation value   is {:.3f}'.format(expectation_approx))

plt.plot(x, x*px, 'r-',  label='x*pdf')       # local product x with pdf
plt.plot([r0, r1], 2*[0], 'k-')               # coordinate system
plt.plot(2*[0], [r0, r1], 'k-')
plt.axis((r0, r1, r0, r1))                    # set borders
plt.legend(loc='lower right');                # who is who
```

```
expectation value of x under pdf is -0.200
approximated expectation value   is -0.200
```



1.11 Definition Varianz

Sei X eine Zufallsvariable auf $\mathbb{R}$ mit Wahrscheinlichkeitsdichte $f(X)$ und Erwartungswert μ, dann

$$\begin{aligned}\mathrm{Var}(X) = \sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\mathrm{d}x \\ &= \mathcal{E}((x-\mu)^2)\end{aligned}$$

1.11.1 Verschiebungssatz:

$$\mathrm{Var}(X) = \mathcal{E}(x^2) - (\mathcal{E}(x))^2$$

Beweis: [ÜA]

1.12 Rechenregeln

Sei g(x) eine reelle Funktion. Dann gilt für $Y = g(X)$

$$\mathcal{E}(Y) = \mathcal{E}(g(X)) = \int_{-\infty}^{\infty} f(x) \cdot g(x)\ \mathrm{d}x$$

1.12.1 lineare Transformation Erwartungswert

Für $Y = aX + b$ ergibt sich

$$\mathcal{E}(Y) = \mathcal{E}(aX+b) = a\mathcal{E}(x) + b$$

Beweis:

$$\begin{aligned}\int_{-\infty}^{\infty} (ax+b)f(x)dx &= \int_{-\infty}^{\infty} axf(x)dx + \int_{-\infty}^{\infty} bf(x)dx \\ &= aE(X) + b \cdot 1\end{aligned}$$

1.12.2 lineare Transformation Varianz

Für die Varianz unter der linearen Transformation $Y = aX + b$ ergibt sich

$$\mathcal{V}\dashv\nabla(Y) = \mathcal{V}\dashv\nabla(aX + b) = a^2 \cdot \mathcal{V}\dashv\nabla(X)$$

Beweis: Wie bei der (→) diskreten Definition

$$\begin{aligned}
\mathcal{V}\dashv\nabla(aX + b) &= \mathcal{E}([aX + b - \mathcal{E}(aX + b)]^2) \\
&= \mathcal{E}([aX + b - a\mathcal{E}(X) - b]^2) \\
&= a^2\mathcal{E}([X - \mathcal{E}(X)]^2) \\
&= a^2\mathcal{V}\dashv\nabla(X)
\end{aligned}$$

1.13 Anwendung der linearen Transformation: Standardisieren

Mittels der speziellen linearen Transformation $Z = \frac{1}{\sigma}(X - \mu)$ ergibt sich

$$\mathcal{E}(Z) = 0$$

$$\mathrm{Var}(Z) = 1$$

Bitte merken für später

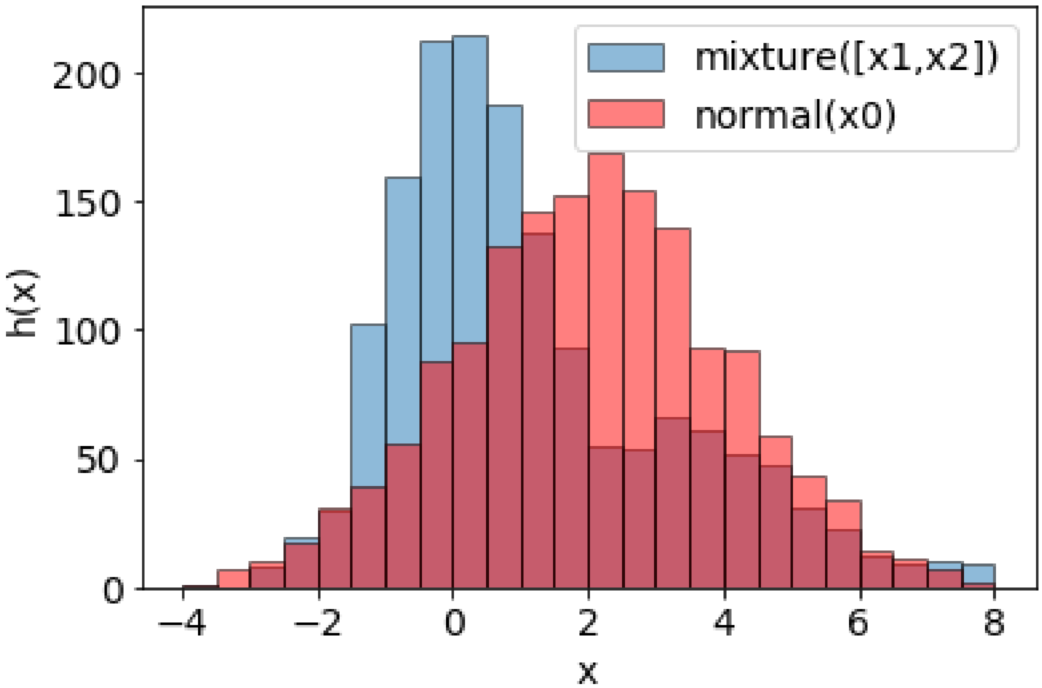
1.14 Definition Schiefe

$$\begin{aligned}
\mathrm{Schiefe}(X) &= \frac{1}{\sigma^3}\int_{-\infty}^{\infty}(x-\mu)^3 \cdot f(x)\mathrm{d}x \\
&= \mathcal{E}(z^3)
\end{aligned}$$

- $= 0$ symmetrische Verteilung
- > 0 linkssteile Verteilung
- < 0 rechtssteile Verteilung

In [13]:
```
'''Show skewness of distributions'''
np.random.seed(234567)
x0 = stats.norm(2, 2).rvs(size=1600)        # to compare: normal distribution
x1 = stats.norm(0, 1).rvs(size=1000)        # first standard normal distribution
x2 = stats.norm(3, 2).rvs(size=600)         # second normal: shiftet and broadened to "right"
bins=np.linspace(-4, 8, 25)
x3 = np.concatenate((x1, x2))               # mixture of x1 and x2
plt.hist(x3, alpha=.5, bins=bins, label='mixture([x1,x2])', edgecolor='black')
plt.hist(x0, alpha=.5, bins=bins, label='normal(x0)', color='r', edgecolor='black')
print('skewness(Normal distribution) = {:.3f}'.format(stats.norm(2, 2).stats(moments = 's')))
print('skewness(Normal data x0)      = {:.3f}'.format(stats.skew(x0)))
print('skewness(Mix data[x1,x2])     = {:.3f}'.format(stats.skew(x3)))
plt.xlabel('x')
plt.ylabel('h(x)')
plt.legend();
```

```
skewness(Normal x0)    = 0.009
skewness(Mix [x1,x2]) = 0.926
```



1.15 Definition Wölbung, Exzeß, Kurtosis

Mit dem *vierten Moment* $m_4 = \int_{-\infty}^{\infty} (x-\mu)^4 \cdot f(x) \mathrm{d}x$ ist

$$\mathrm{Kurtosis}(X) = \frac{m_4}{(\sigma^2)^2} - 3$$

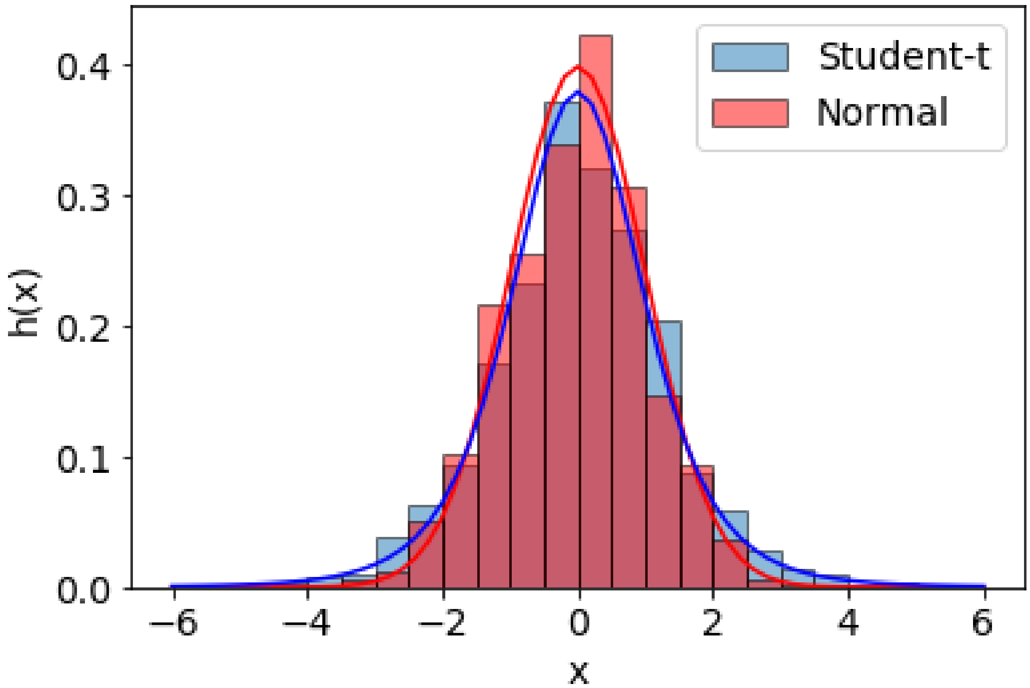
Mit der Verschiebung um 3 ist gewährleistet, daß die Normalverteilung die Kurtosis 0 hat.

- $ = 0 wie Normalverteilung $
- $ > 0 spitzer, langschwänziger$
- $ < 0 stumpfer, tailliert$

```
In [17]: '''Show kurtosis of distributions'''
         np.random.seed(123456)
         xgrid = np.linspace(-6, 6, 60+1)
         distrib1 = stats.norm(0, 1)
         distrib2 = stats.t(5)
         x1 = distrib1.rvs(size=1000)
         x2 = distrib2.rvs(size=1000)
         bins=np.linspace(-6, 6, 25)
         plt.hist(x2, alpha=.5, bins=bins, density=True, label='Student-t', edgecolor='black')
         plt.hist(x1, alpha=.5, bins=bins, density=True, label='Normal', color='r', edgecolor='black')
         plt.plot(xgrid, distrib1.pdf(xgrid), 'r-')
         plt.plot(xgrid, distrib2.pdf(xgrid), 'b-')
         print('skewness(Normal)          = {:6.3f}'.format(distrib1.stats(moments='k')))
         print('kurtosis(Normal data)     = {:6.3f}'.format(stats.kurtosis(x1)))
         print('kurtosis(Student-t)       = {:6.3f}'.format(distrib2.stats(moments='k')))
         print('kurtosis(Student-t data) = {:6.3f}'.format(stats.kurtosis(x2)))

         plt.xlabel('x')
         plt.ylabel('h(x)')
         plt.legend();
```

```
skewness(Normal)          =  0.000
kurtosis(Normal data)     = -0.040
```

```
kurtosis(Student-t)      =  6.000
kurtosis(Student-t data) =  6.449
```



1.16 Ausblick

- verschiedene wichtige kontinuierliche Verteilungen
- Woher kommt die Wahrscheinlichkeitsdichte $f(x)$ / die Verteilungsfunktion $F(x)$?

2 Zusammenfassung kontinuierliche Zufallsvariablen

- Zufallsexperiment
- Zufallsvariable
  - $X: \quad x \in \mathbb{R}$
- Wahrscheinlichkeitsdichte $f(x)$
  - Grenzwert zur objektiven Häufigkeitsverteilung $\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\Delta h}{\Delta x}$
  - subjektive (Theorie, Interpretation)
  - Normierung $[0, 1]$
- Wahrscheinlichkeitsdichte-Verteilungsfunktion
  - $F(x) = \int_{-\infty}^{x} f(x')\, \mathrm{d}x'$
- Wahrscheinlichkeit
  - $p(a \leq x \leq b) = \int_a^b f(x)\, \mathrm{d}x = F(b) - F(a)$
- Kennzahlen
  - Erwartungswert $\quad \mathcal{E}(X) = \mu = \int_{-\infty}^{\infty} f(x) \cdot x\, \mathrm{d}x$
  - Varianz $\quad \mathrm{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, \mathrm{d}x$
  - Schiefe und Kurtosis

3 Zusammenfassung Python

Statsitik-Bibliothek scipy.stats enthält kontinuierliche Verteilungen

Funktionen und Methoden

```
.expect()       # Erwartungswert
.pdf(x)         # Wahrscheinlichkeitsdichteverteilung "probability density function"
.cdf(x)         # Verteilungsfunktion "cumulative density function"
.rvs()          # Zufallserergebnis "random variables"
                # (optional Anzahl der Werte)- Python: `scipy.stats`
.mean()
.var()
.std()
.kurtosis()
...
```

In [15]:
```python
from scipy import stats

distrib = stats.norm(2, 3)   # "freeze" a normal distribution around mu=2 with sigma=3
print('mean =               {}'.format(distrib.mean()))
print('variance =           {}'.format(distrib.var()))
print('standard deviation = {}'.format(distrib.std()))
print('norm of p =          {}'.format(distrib.moment(0)))
m, v, s, k = distrib.stats(moments = 'mvsk')
print('expectation value =  {}'.format(m))
print('variance =           {}'.format(v))
print('skew =               {}'.format(s))
print('kurtosis =           {}'.format(k))
```

```
mean =               2.0
variance =           9.0
standard deviation = 3.0
norm of p =          1.0
expectation value =  2.0
variance =           9.0
skew =               0.0
kurtosis =           0.0
```

4 Fragen?