052_Folien

November 23, 2018

```
In [1]: import numpy as np                          # mathematical methods
        from scipy import stats                     # statistical methods
        from matplotlib import pyplot as plt        # plotting methods
        %matplotlib inline
```

1 Wahrscheinlichkeitstheorie

1.0.1 Zufallsvariable und Wahrscheinlichkeitsraum

1.0.2 Diskrete Zufallsvariablen und Wahrscheinlichkeitsverteilungen

1.0.3 Kontinuierliche Zufallsvariable und Wahrscheinlichkeitsverteilungen

Wahrscheinlichkeitsdichte

- Unendlich viele dichte Ereignisse ($x \in \mathbb{R}$)

kontinuierliche Verteilungen

- Gleich- / Rechteckverteilung
- Normalverteilung
- Exponentialverteilung
- besondere Verteilungen

1.0.4 Wiederholung: diskrete binomiale Wahrscheinlichkeitsverteilungen

- Bernoulli-Experiment
- mehrere unabhängige identische Wiederholungen *i.i.d*

Binomialverteilung

- Anzahl des Eintretens

Geometrische Verteilung

- Anzahl der Versuche bis zum Eintreten

Poissonverteilung

- Anzahl seltener Ereignisse im Intervall

1.0.5 Gemeinsamkeit: Anzahl $\in \mathbb{N}$

2 Multinomiale Verteilung

Erweiterung des Bernoulli-Zufall-Experiments auf mehrere mögliche Ergebnisse (Elementarereignisse):

$$\Omega = \{a_i\} \quad i > 2$$

Experiment: ***Ein*** Objekt wird daraus gezogen/beobachtet/gemessen/...

Beispiele

- Roulette
- Objekte bei Verkehrszählung
- Eigenschaften bei Populationen
- Parteien zur Wahl

```
np.random.multinomial?
multinomial(n, pvals, size=None)
The multinomial distribution is a multivariate generalisation of the binomial distribution.
```

```
In [3]: '''multinomial example: a single die'''
        N = 6                                          # how many possible results?
        omega = 1+np.arange(6)                         # the pips of the die (starting at one)
        print('x: ', omega)                            # reference: possible outcomes
        ps = N*[1/N]                                   # Laplace: 6 equally distributed
        x = np.random.multinomial(n=1, pvals=ps)       # one experiment: draw a single sample
        print('f: ', x)                                # vector of *where* the result is!
        print('we obtained a {}'.format(omega[x==1]))  # which of the sides shows up face?
```

```
x:  [1 2 3 4 5 6]
f:  [0 0 1 0 0 0]
we obtained a [3]
```

```
In [4]: '''multinomial with two dice'''
        x2 = np.random.multinomial(n=2, pvals=ps)      # a combined experiment: throw two dice at once
        print(omega)
        print(x2)
        pasch = len(x2[x2==2]) > 0                     # 2 of same kind? vector has (one) element: a 2
        if pasch:
            print('we got a {}-pasch'.format(omega[x2==2]))     # show the only place of the 2
        else:                                                   # show 1st and 2nd place of two 1s
            print('we got a {} and a {}'.format(omega[x2==1][0], omega[x2==1][1]))
```

```
[1 2 3 4 5 6]
[0 1 0 0 1 0]
we got a 2 and a 5
```

```
In [5]: '''repeat throwing two dice five times'''
        x25 = np.random.multinomial(n=2, pvals=ps, size=5)  # five combined experiments: two dice at once
        print(' ', omega, ' omega')
        print(x25, ' results')                              # now a 2D matrix
        print(x25.shape, 'shape of results-matrix')         #             of shape (#repetitions x #elements)
        pasches = len(x25[x25==2])                          # how many 2 of same kind?
        print('we got {} of same kind'.format(pasches))
```

```
  [1 2 3 4 5 6]  omega
[[0 0 0 1 1 0]
 [0 0 1 0 0 1]
 [0 1 0 1 0 0]
 [0 0 0 1 0 1]
 [0 0 0 1 1 0]]  results
(5, 6) shape of results-matrix
we got 0 of same kind
```

```
In [6]: print(omega, 'numbers')
        print(x25.sum(axis=0), 'frequency total')
```

```
[1 2 3 4 5 6] numbers
[0 1 1 4 2 2] frequency total
```

```
In [7]: '''another variant: without replacement'''
        N = 49                                                      # how many possible results?
        p49 = N*[1/N]                                               # Laplace: 49 possible numbers
        omega49 = np.arange(1, 49+1)                                # ball's numbers
        n49 = 6                                                     # how many repetitions?
        lotto = np.random.choice(a=omega49, size=n49, replace=False) # draw 6 out of 49 without replacement
        print('Lotto today gives {}. Maybe.'.format(np.sort(lotto)))
```

```
Lotto today gives [ 3 18 31 34 43 49]. Maybe.
```

2.1 Zusammenfassung Multinomiale Verteilung

- Mehr als zwei mögliche Elementarereignisse
- i.i.d. Wiederholungen / Kombinationen möglich

2.1.1 Python

- Keine Verteilung in `scipy.stats`
- Möglichkeiten Zufallszahlen zu gewinnen
  - `np.random.choice()`
  - `np.random.multinomial()`

2.2 Wiederholung: Mittelwert ⇔ Erwartungswert

Arithmetischer **Mittelwert** von n Ereignissen x_j

$$\overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$$

Diskrete Ereignisse $x_i, i \in \{1 \ldots N\}$ mit absoluter Häufigkeit H_i, $\sum H_i = n$, dann

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{N} H_i \cdot x_i$$

Mit relativen Häufigkeiten $h_i = \frac{H_i}{n}$ und damit $\sum h_i = 1$

$$\overline{x} = \sum_{i=1}^{N} h_i \cdot x_i$$

Übergang zur (theoretischen) Wahrscheinlichkeit p_i mit $\sum p_i = 1$: diskreter **Erwartungswert** der Zufallsvariable X

$$\mathcal{E}(X) = \mu = \sum_{i=1}^{N} p_i \cdot x_i$$

mit $p_i = p(X = x_i)$

3 Kontinuierliche Verteilungen: $x \in \mathbb{R}$

3.0.1 Wiederholung: Erwartungswert

Diskret

$$\mathcal{E}(X) = \mu = \sum_{i=1}^{N} p_i \cdot x_i$$

mit $p_i = p(X = x_i)$

Kontinuierlich

$$\mathcal{E}(X) = \mu = \int_{-\infty}^{\infty} f(x) \cdot x \, \mathrm{d}x$$

mit Wahrscheinlichkeitsdichte $f(x)$

3.0.2 Wiederholung: kontinuierliche Zufallsvariable

Kontinuierliche Zufallsvariable X mit $x \in \mathbb{R}$

Problem: weil Punktwahrscheinlichkeit $P(x = a) \to 0$

Lösung (endliche) Wahrscheinlichkeitsdichte $f(x)$

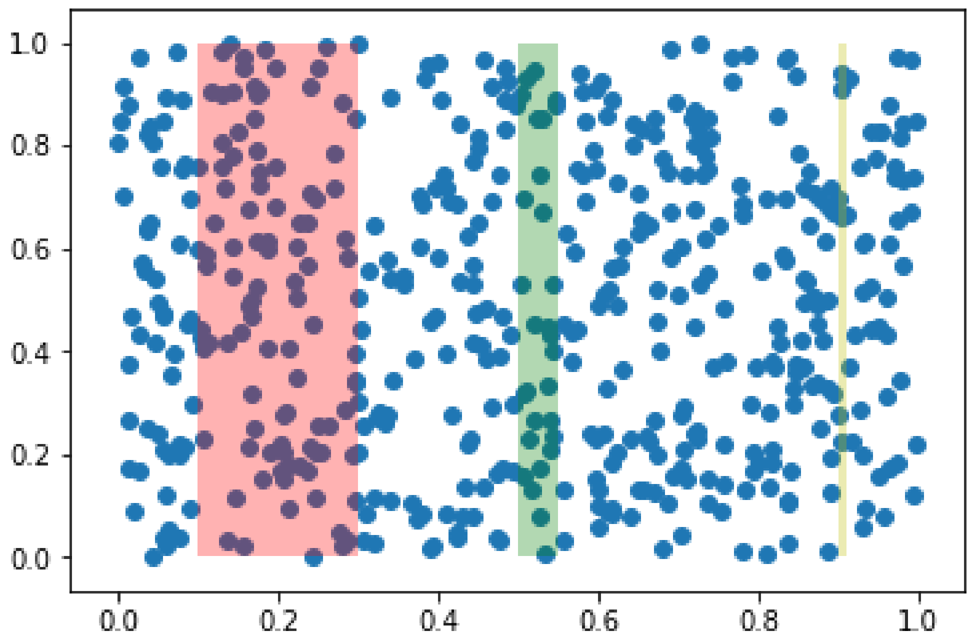
$$f(x) \geq 0$$

$$P(a \leq x \leq b) = \int_a^b f(x)\,\mathrm{d}x \ \leq 1$$

Normierung $\int_{x=-\infty}^{\infty} f(x)\mathrm{d}x = 1$

```
In [8]: ''''500 raindrops are falling - show density'''
```

```
in width=0.20 are 100 drops, density=500.0
in width=0.05 are  29 drops, density=580.0
in width=0.01 are   6 drops, density=600.0
```



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4 Kontinuierliche Verteilungen

5 Gleichverteilung / Rechteckverteilung

$$x \in [a, b]$$

$$f(x) = const$$

$$= \frac{1}{b-a}$$

5.1 Kennwerte Rechteck-/Gleichverteilung

$$\mathcal{E}(X) = \frac{a+b}{2}$$

$$Median(X) = \frac{a+b}{2}$$

$$\text{Var}(X) = \sigma^2 = \frac{(b-a)^2}{12}$$

$$\sigma = \frac{|b-a|}{2\sqrt{3}}$$

Beweis: [ÜA]

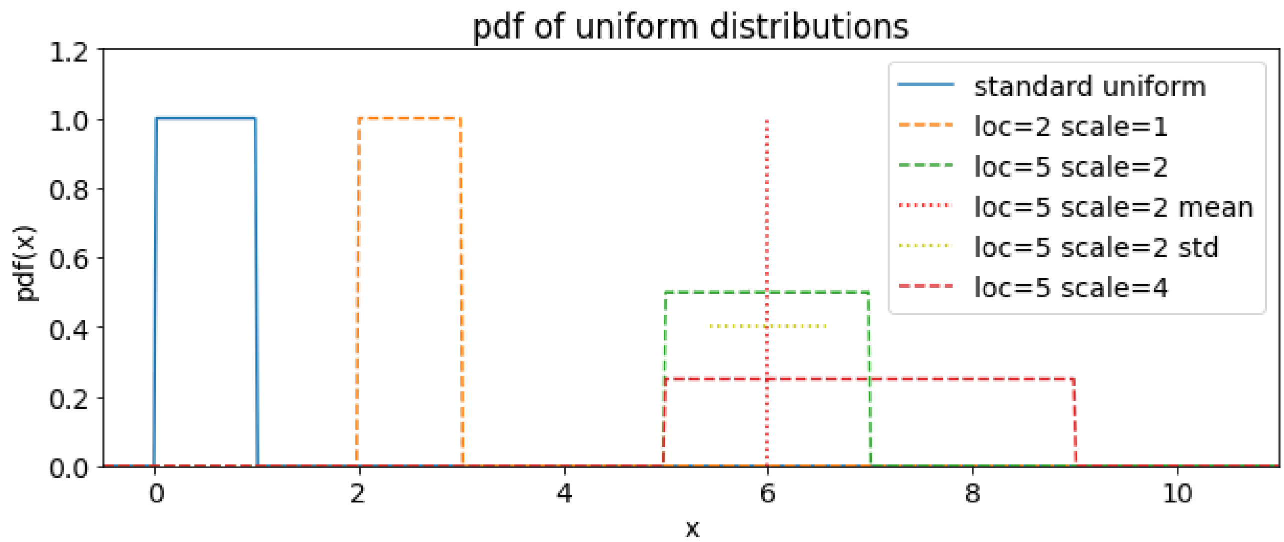
```
In [9]: '''uniform distribution % rectangle shaped
           probability density > 0 in range [a...b] = [loc...loc+range]
                               = 0 elsewhere                              '''
        # uniform without parameters is in range [0, 1]
        print('uniform()     has expectation value {:.5f} and variance {:.5f}'.
              format(stats.uniform.expect(), stats.uniform.var()))

        # uniform in range [5..7] = [5..5+2]
        unif_5_2 = stats.uniform(loc=5.0, scale=2.0)         # freeze these parameters
        m52 = unif_5_2.expect()                              # frozen distribution's expectation
        s52 = unif_5_2.std()                                 # ... and standard deviation
        print('uniform(5, 2) has expectation value {:.5f} and std-dev. {:.5f}'.format( m52, s52))
```

```
uniform()     has expectation value 0.50000 and variance 0.08333
uniform(5, 2) has expectation value 6.00000 and std-dev. 0.57735
```

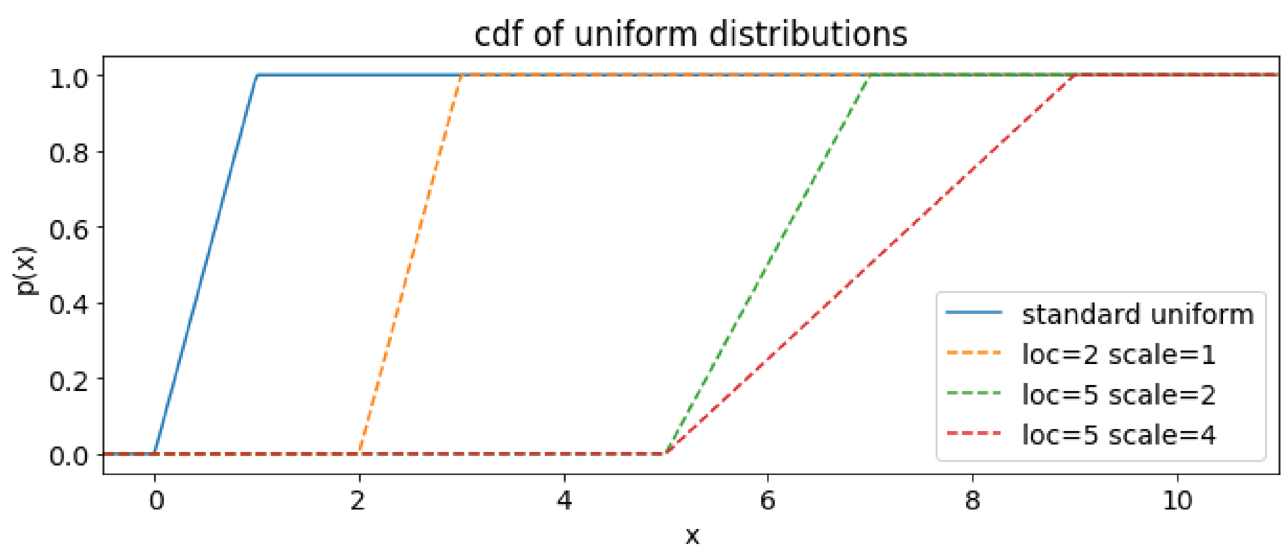
5.1.1 Wahrscheinlichkeitsdichte $f(x)$, pdf

```
In [10]: '''uniform distribution examples: probability density '''
         plt.figure(figsize=(11,4))
         x = np.linspace(-.5, 11., 465)
         plt.plot(x, stats.uniform.pdf(x), label='standard uniform')
         plt.plot(x, stats.uniform(loc=2., scale=1.).pdf(x), '--', label='loc=2 scale=1')
         plt.plot(x, unif_5_2.pdf(x), '--', label='loc=5 scale=2')
         plt.plot(2*[m52], [0, 1], 'r:', label='loc=5 scale=2 mean')
         plt.plot([m52-s52, m52+s52], 2*[.4], 'y:', label='loc=5 scale=2 std')
         plt.plot(x, stats.uniform(loc=5., scale=4).pdf(x), '--', label='loc=5 scale=4')
         plt.xlabel('x')
         plt.ylabel('pdf(x)')
         plt.xlim(-0.5, 11.)
         plt.ylim(0., 1.2)
         plt.legend(loc='upper right')
         plt.title('pdf of uniform distributions');
```



5.1.2 Verteilungsfunktion $F(x)$, kumulative Verteilung, `cdf`

```
In [11]: '''uniform cumulative distribution function examples'''
         plt.figure(figsize=(11,4))
         x = np.linspace(-.5, 11., 465)
         plt.plot(x, stats.uniform.cdf(x), label='standard uniform')
         plt.plot(x, stats.uniform(loc=2., scale=1.).cdf(x), '--', label='loc=2 scale=1')
         plt.plot(x, unif_5_2.cdf(x), '--', label='loc=5 scale=2')
         plt.plot(x, stats.uniform(loc=5., scale=4).cdf(x), '--', label='loc=5 scale=4')
         plt.xlabel('x')
         plt.ylabel('p(x)')
         plt.xlim(-0.5, 11.)
         plt.ylim(-0.05, 1.05)
         plt.legend(loc='lower right')
         plt.title('cdf of uniform distributions');
```



6 Normalverteilung

$x \in \mathbb{R}$ Wahrscheinlichkeitsdichte $f(x)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \mathcal{N}(\mu, \sigma^2)$$

6.0.1 Standardnormalverteilung $\phi(x)$

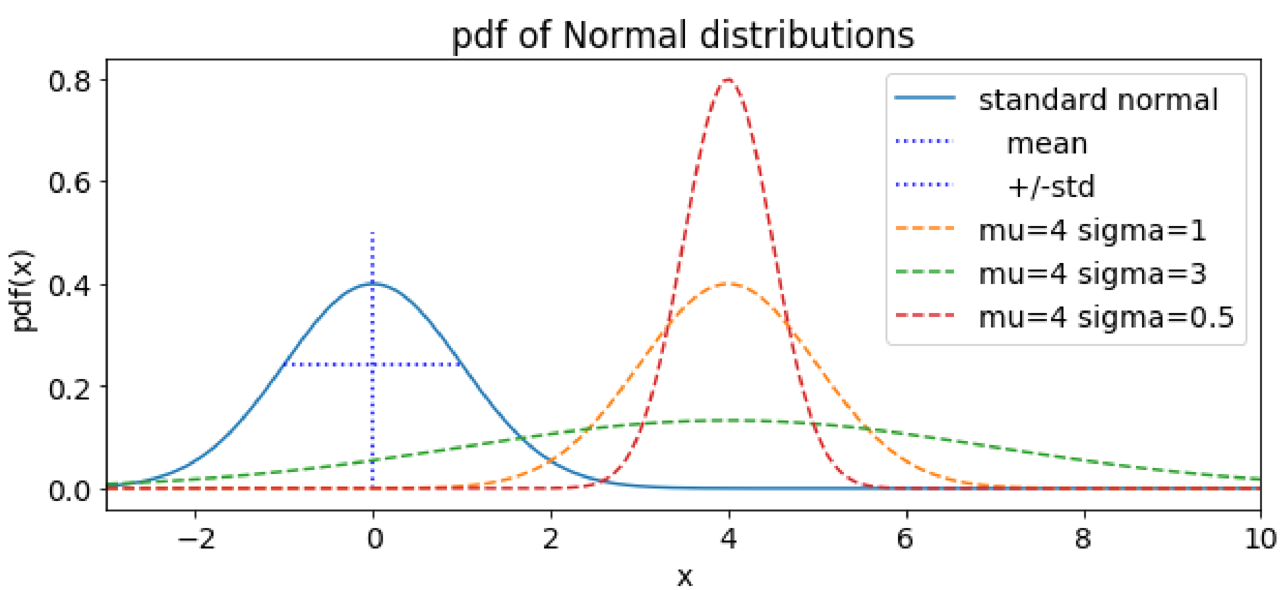
Spezialfall $\mu = 0$ und $\sigma^2 = 1$:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \mathcal{N}(0,1)$$

6.1 Kennwerte der GauSS′schen Normalverteilung

$$\mathcal{E}(X) = \mu$$
$$\mathrm{Var}(X) = \sigma^2$$
$$\text{Schiefe} = 0$$
$$\text{Kurtosis} = 0$$

In [12]:
```
'''a bunch of normal "Gaussian" distributions'''
plt.figure(figsize=(10,4))
x = np.linspace(-3., 10., 201)
plt.plot(x, stats.norm.pdf(x), label='standard normal')
m = stats.norm.mean()              # mean (expectation value) of distribution
s = stats.norm.std()               # standard deviation of distribution
h = stats.norm.pdf(1)              # pdf (x=sigma=1) for height
plt.plot(2*[m], [0, 0.5], 'b:', label='   mean')       # vertical line at mean
plt.plot([m-s, m+s], 2*[h], 'b:', label='   +/-std' )  # horizontal line +/-std
# plot three other distributions at mu=4, different sigma (=std, =sqrt(var))
plt.plot(x, stats.norm(loc=4., scale=1.).pdf(x), '--', label='mu=4 sigma=1')
plt.plot(x, stats.norm(loc=4., scale=3.).pdf(x), '--', label='mu=4 sigma=3')
plt.plot(x, stats.norm(loc=4., scale=.5).pdf(x), '--', label='mu=4 sigma=0.5')
plt.xlim((-3, 10))
plt.xlabel('x')
plt.ylabel('pdf(x)')
plt.title('pdf of Normal distributions')
plt.legend();
```



7 (kumulierte) Wahrscheinlichkeitsfunktion

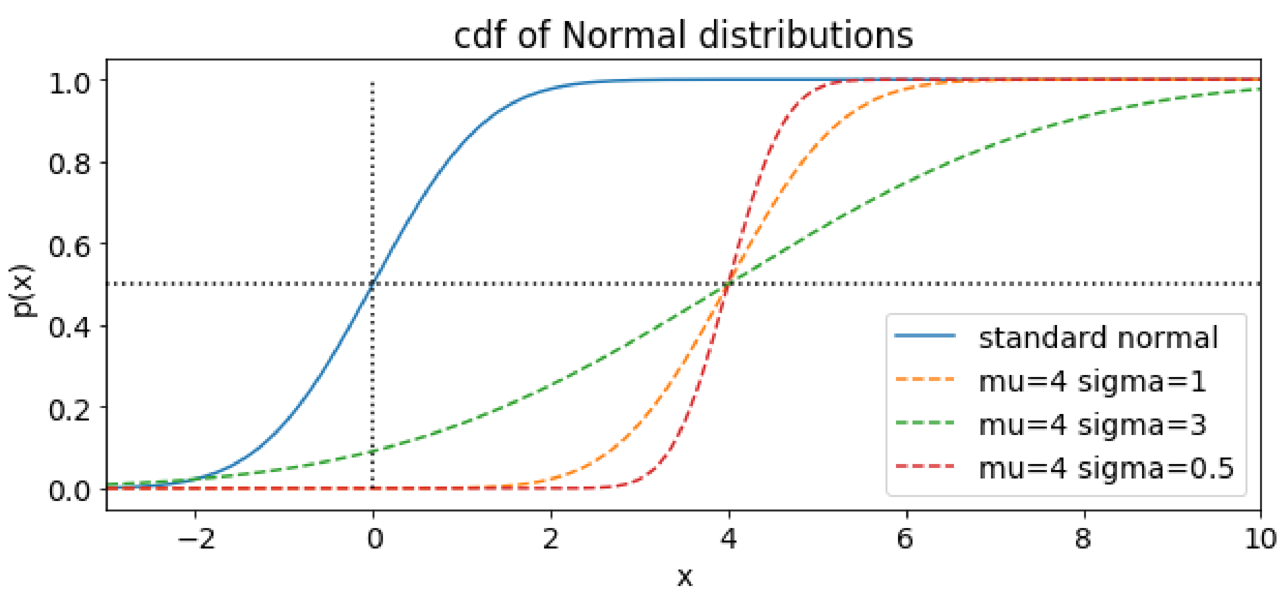
$$F(x) = \int_{-\infty}^{x} f(a)\ \mathrm{d}a$$

7.1 Standardnormalverteilungsfunktion

Für die Standardnormalverteilung $\mu = 0$ und $\sigma = 1$

$$\Phi(x) = \int_{-\infty}^{x} \phi(a)\ \mathrm{d}a$$

```
In [13]: '''a bunch of normal Gaussian cdfs'''
         plt.figure(figsize=(10,4))
         x = np.linspace(-3., 10., 201)
         # the same four distributions as before:
         plt.plot(x, stats.norm.cdf(x), label='standard normal')
         plt.plot(x, stats.norm(loc=4., scale=1.).cdf(x), '--', label='mu=4 sigma=1')
         plt.plot(x, stats.norm(loc=4., scale=3.).cdf(x), '--', label='mu=4 sigma=3')
         plt.plot(x, stats.norm(loc=4., scale=.5).cdf(x), '--', label='mu=4 sigma=0.5')
         plt.plot([-3., 10.], 2*[.5], 'k:')
         plt.plot(2*[0.], [0., 1.], 'k:')
         plt.xlabel('x')
         plt.ylabel('p(x)')
         plt.xlim((-3, 10))
         plt.title('cdf of Normal distributions')
         plt.legend(loc='lower right');
```



7.1.1 Quantile

```
In [14]: '''interesting quantiles'''
         p = np.array([.50, .75, .90, .95, .975, .99])    # interesting percentiles
         z = stats.norm.ppf(p)                             # and their probability
         print('cumulative probability p = {}'.format(np.round(p, decimals=3)))
         print('located at               z = {}'.format(np.round(z, decimals=3)))

cumulative probability p = [ 0.5    0.75   0.9    0.95   0.975  0.99 ]
located at             z = [ 0.     0.674  1.282  1.645  1.96   2.326]
```

7.1.2 Tabelle der Standardnormalverteilung

Historisch Quelle: https://en.wikipedia.org/wiki/Standard_normal_table oder jedes Statistiklehrbuch.

7.1.3 68-95-99,7-Prozent-Abschätzungsregel

- 68% der Werte liegen in $\mu \pm \quad \sigma$
- 95% der Werte liegen in $\mu \pm 2 \cdot \sigma$
- 99.7% der Werte liegen in $\mu \pm 3 \cdot \sigma$

```
In [15]: '''1-2-3-sigma = 68, 95, 99.7'''
         z = np.array([1., 2., 3.])            # range 1, 2, 3 (sigma)
         p = stats.norm.cdf(z)                 # has cumulated probability
         pr = 2*p-1.                           # probability range excluding both tails
         original = np.get_printoptions()                        # save print options
         np.set_printoptions(formatter={'float': '{: 6.2f}'.format}) # print 2 digits precision
         print('range of +/- sigma:  z = {}'.format(z))
         print('probability inside:  p = {} %'.format(100*pr))
         np.set_printoptions(**original)                         # restore print options
```

```
range of +/- sigma:  z = [  1.00   2.00   3.00]
probability inside:  p = [ 68.27  95.45  99.73] %
```

7.2 Eigenschaften

(*Wichtig* für später)

7.2.1 lineare Transformation

Ist $X \sim \mathcal{N}(\mu, \sigma^2)$, dann ist $Y = aX + b$ ebenfalls normalverteilt mit:

$$Y \sim \mathcal{N}(a \cdot \mu + b, a^2 \cdot \sigma^2)$$

7.2.2 Addition

Sind $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ und $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ beide unabhängig, dann ist $Y = X_1 + X_2$

$$Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

7.2.3 Standardisierung

Die *spezielle* lineare Transformation $Z = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ ergibt

$$Z \sim \phi = \mathcal{N}(0, 1)$$

weil darunter (siehe oben)

$$\mathcal{E}(Z) = 0$$

$$\mathrm{Var}(Z) = 1$$

Umgekehrt kann man aus der Standardnormalverteilung ϕ jede Normalverteilung $\mathcal{N}(\mu, \sigma^2)$ erhalten durch

$$X = \sigma Z + \mu$$

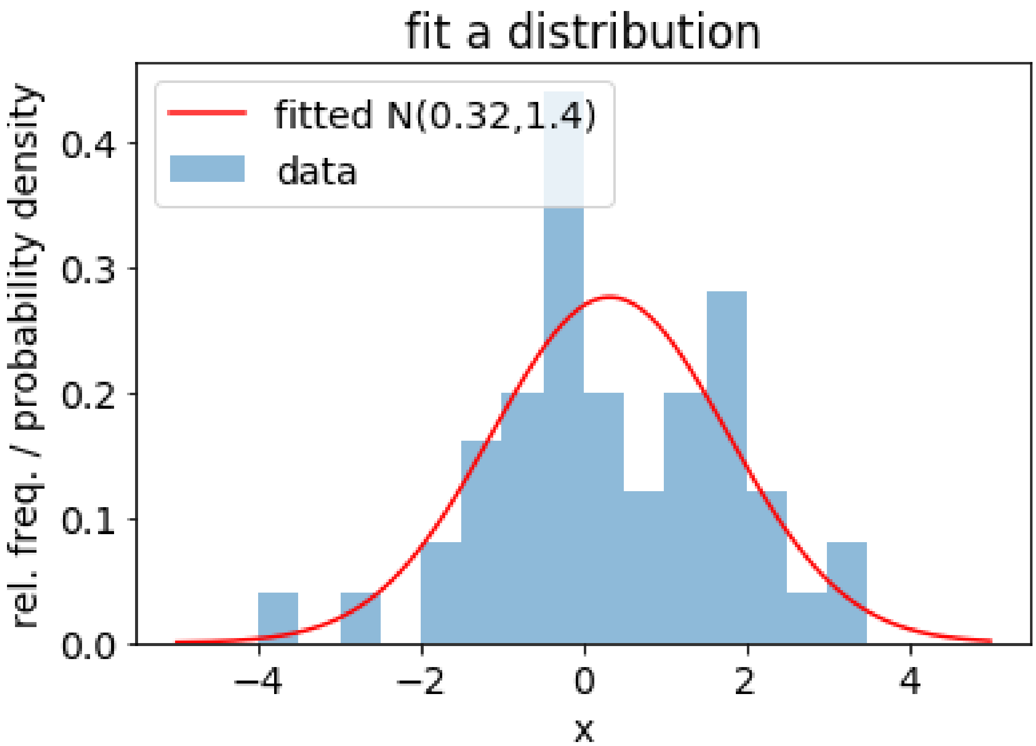
7.3 Anpassen, *fit()*

Eine Verteilung an vorhandene Daten anfitten

```
In [16]: '''fit distribution to data'''
         np.random.seed(98765432)
         data = stats.norm(0.7, 1.5).rvs(size=50)          # fake data
         (mu, sigma) = stats.norm.fit(data)                 # find best fitting Gauss for data
         plt.hist(data, bins=np.linspace(-5, 5, 21), label='data',
```

```
            normed=True, alpha=.5)                    # use relative frequencies
print('fitted normal distribution has parameters mu={:.3}, sigma={:.3}'
      .format(mu, sigma))
xgrid = np.linspace(-5, 5, 201)
plt.plot(xgrid, stats.norm(mu, sigma).pdf(xgrid), 'r-', label='fitted N({:.2},{:.2})'.format(mu, sig
plt.title('fit a distribution')
plt.xlabel('x')
plt.ylabel('rel. freq. / probability density')
plt.legend(loc='upper left');
```

```
fitted normal distribution has parameters mu=0.325, sigma=1.45
```



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8 Exponentialverteilung

Stetige Erweiterung der *Geometrischen* Verteilung für $x \in \mathbb{R}$: Dauer bis zum Eintreten eines Ereignisses.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{sonst} \end{cases}$$

KenngröSSen

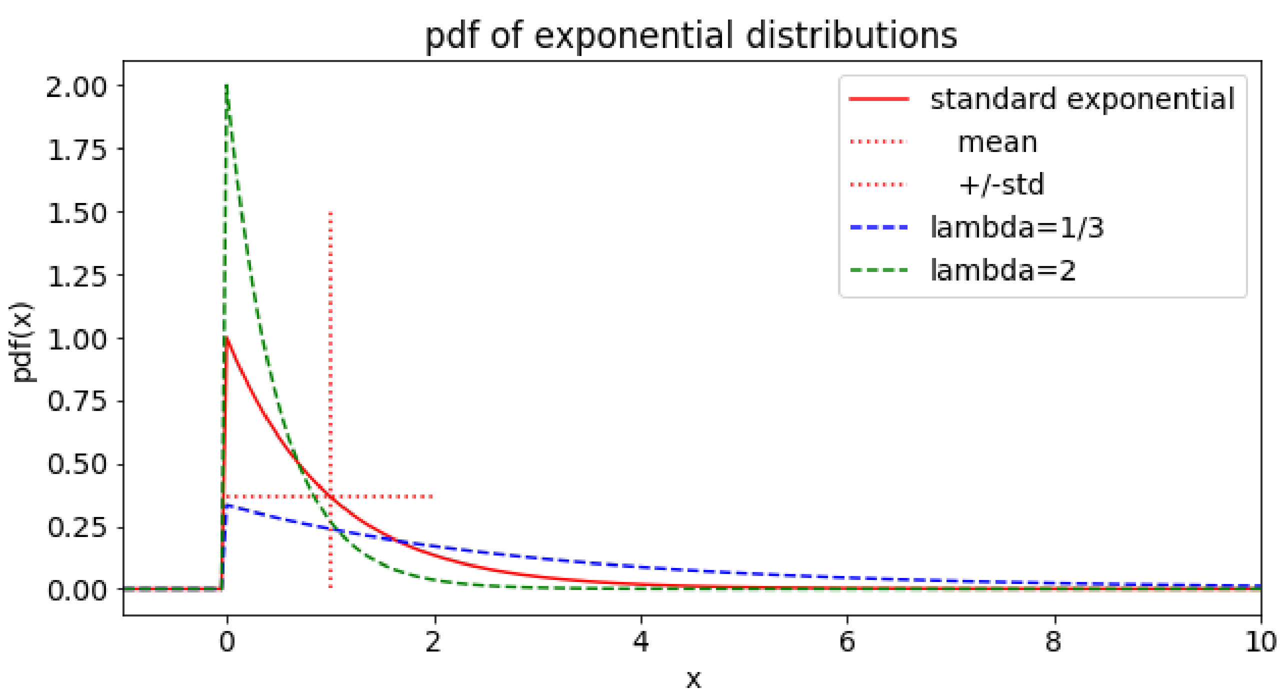
$$\mathcal{E}(X) = \frac{1}{\lambda}$$
$$x_{med} = \frac{\ln 2}{\lambda}$$
$$\text{Var}(X) = \frac{1}{\lambda^2}$$

8.0.1 Realisierung in scipy.stats

```
stats.expon?
An exponential continuous random variable.
The probability density function for `expon` is::
    expon.pdf(x) = exp(-x)     for ``x >= 0``.
The probability density above is defined in the "standardized" form.
To shift and/or scale the distribution use the ``loc`` and ``scale`` parameters.
Specifically, ``expon.pdf(x, loc, scale)`` is identically equivalent to ``expon.pdf(y) / scale``
with ``y = (x - loc) / scale``. A common parameterization for `expon` is in terms of the
rate parameter ``lambda``, such that ``pdf = lambda * exp(-lambda * x)``. This
parameterization corresponds to using ``scale = 1 / lambda``.
```

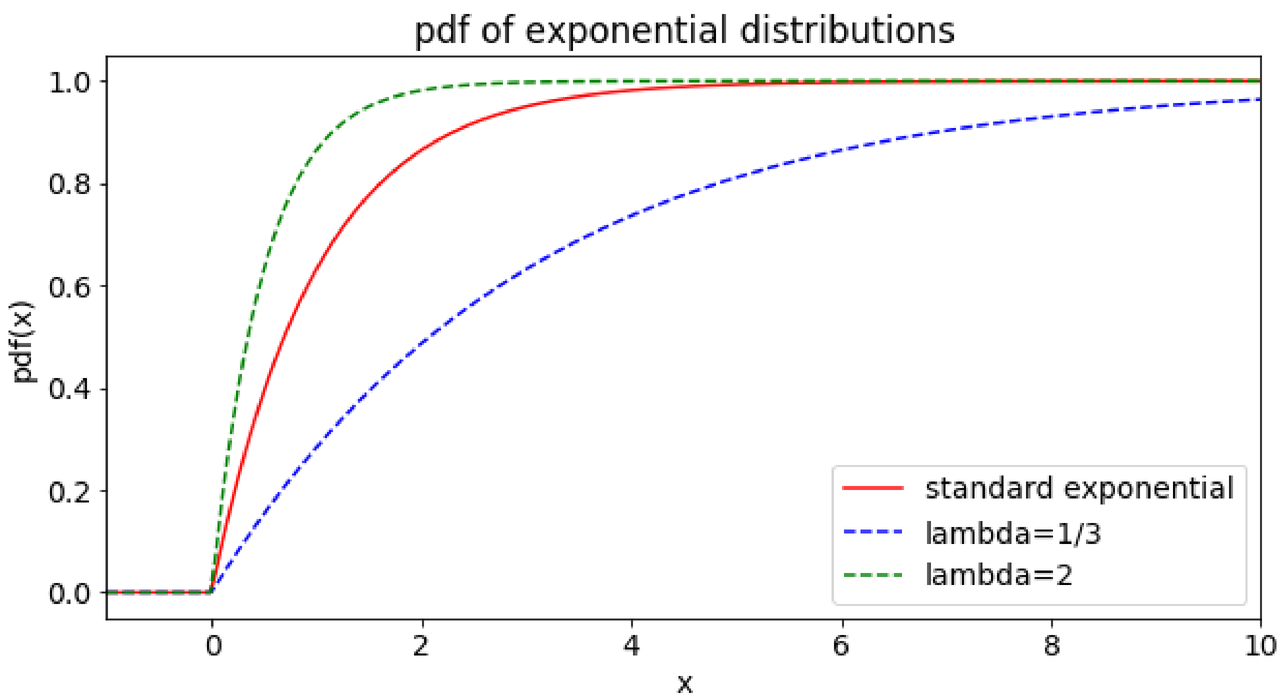
8.0.2 Wahrscheinlichkeitsdichte $f(x)$, pdf

In [17]:
```python
'''exponential distribution'''
plt.figure(figsize=(10,5))
x = np.linspace(-1., 10., 221)        # x values (negatives do not matter)
plt.plot(x, stats.expon.pdf(x), 'r-', label='standard exponential')
m = stats.expon.mean()                # mean (expectation value) of distribution
s = stats.expon.std()                 # and standard deviation
h = stats.expon.pdf(m)                # just for height at pdf(m):
plt.plot(2*[m], [0, 1.5], 'r:', label='   mean')      # vertical line for m
plt.plot([m-s, m+s], 2*[h], 'r:', label='   +/-std' ) # horiz. line for m +/- std
# two other exponentials with lambda=1/3 and 2
plt.plot(x, stats.expon(scale=3.).pdf(x), 'b--', label='lambda=1/3')
plt.plot(x, stats.expon(scale=.5).pdf(x), 'g--', label='lambda=2')
plt.xlim((-1, 10))
plt.xlabel('x')
plt.ylabel('pdf(x)')
plt.title('pdf of exponential distributions')
plt.legend();
```



8.0.3 Wahrscheinlichkeitsverteilung $F(x)$, cdf

```
In [18]: '''cdf of exponential distributions from before'''
         plt.figure(figsize=(10,5))
         x = np.linspace(-1., 10., 201)
         plt.plot(x, stats.expon.cdf(x),           'r-',  label='standard exponential')
         plt.plot(x, stats.expon(scale=3.).cdf(x), 'b--', label='lambda=1/3')
         plt.plot(x, stats.expon(scale=.5).cdf(x), 'g--', label='lambda=2')
         plt.xlim((-1, 10))
         plt.xlabel('x')
         plt.ylabel('pdf(x)')
         plt.title('pdf of exponential distributions')
         plt.legend(loc='lower right');
```



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9 Ausblick

9.0.1 AuSSergewöhnliche Verteilungen

10 Zusammenfassung

Kontinuierliche Verteilungen $x \in \mathbb{R}$

Rechteck-/Gleichverteilung

$$f(x) = \begin{cases} const. & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$

Normalverteilung, GauSSverteilung

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \mathcal{N}(\mu, \sigma^2)$$

Standardnormalverteilung $\phi(x)$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \mathcal{N}(0, 1)$$

Exponentialverteilung

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{sonst} \end{cases}$$

11 Zusammenfassung kontinuierliche V. in Python `scipy.stats`

Funktionen und Methoden

```
.uniform(loc, scale)    # borders   loc ... loc+scale
.expon(scale)           # exponent  1/scale
.norm(loc, scale)       # mu, sigma

.expect()       # Erwartungswert
.pdf(x)         # Wahrscheinlichkeitsdichteverteilung "probability density function"
.cdf(x)         # Verteilungsfunktion "cumulative density function"
.ppf(p)         # Umkehrung davon: percentile
.rvs()          # Zufallserergebnis "random variables"
                # (optional Anzahl der Werte)
.fit()          # Anfitten einer Verteilung an Ereignisse/Daten

.expect()       # Erwartungswert
.var()          # Varianz
.std()          # Standardabweichung
.kurtosis()     # Kurtosis
...
```

12 Fragen?