EBERHARD KARLS UNIVERSITÄT TÜBINGEN

FACULTY OF SCIENCE

Data Science & Analytics

Data Mining and Probabilistic Reasoning

Basics of Statistics

Main source: *All of statistics*

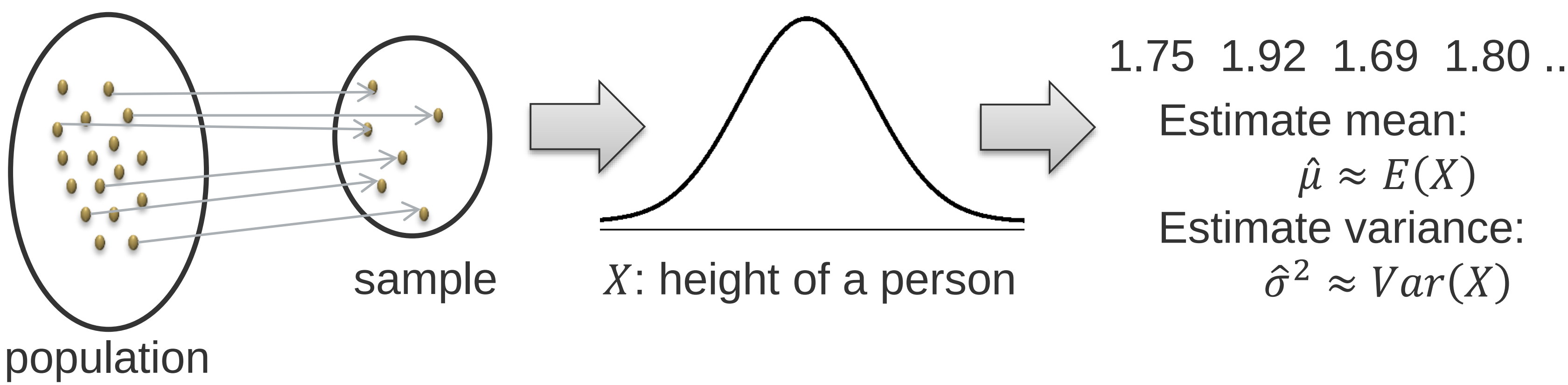
Outline

- Sampling
- Estimators
  - Bias
  - Consistency
  - Variance
  - Mean squared error
- Law of Large Numbers
- Central Limit Theorem
- Hypothesis testing

What is statistics about?

Statistics is concerned with data that are subject to random variations

- Data is typically collected through sampling
- Data is summarized and analyzed by estimating the parameters of the underlying distribution(s)



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Sampling

Population

biased

Convenience sample

Only data that is easy to accessible is sampled

unbiased

Random sample

Every element has equal probability of being selected

biased

Systematic sample

Ordered sampling: randomly select first element and then select every k'th element

. . .

unbiased

Stratified sample

When there is variance in subpopulations, it is better to sample each population Independently (sample size should reflect the proportion of the subpopulation)

Estimators: mean and variance

An **estimator** is a function that uses input from the sample space to estimate a parameter of the underlying data distribution

Examples: Let $x_1, \dots, x_n$ be the values of i.i.d. random variables X_i

- **Empirical mean and the sample mean**: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$
- **Empirical variance**: $S_{em}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2$
- **Sample variance**: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2$

Estimators: covariance and correlation

Further examples: Let $(x_1, y_1), \dots, (x_n, y_n)$ be samples of i.i.d. random variables X_i, Y_i

- **Empirical covariance**: $\hat{C}_{em} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})(y_i - \bar{Y})$
- **Sample covariance**: $\hat{C} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})(y_i - \bar{Y})$
- **Correlation**: $r = \frac{\hat{C}}{s_X s_Y}$; for linear dependency between two variables, e.g., $Y = aX + b$:

$$r = \begin{cases} 1, & a > 0 \\ -1, & a < 0 \end{cases}$$

Empirical distribution function and empirical median

Let $x_1, \dots, x_n$ be the values of i.i.d. random variables X_i

- **Empirical distribution function**: $\hat{F}_{X_{i:n}}(x) = \frac{1}{n}\sum_{i=1}^{n}[\![x_i \le x]\!]$

  where $[\![x_i \le x]\!] \coloneqq \begin{cases} 1, x_i \le x \\ 0, x_i > x \end{cases}$ is called the indicator function

- **Empirical median** $\hat{x}_{med}$ is defined as $\hat{F}_{X_{i:n}}(\hat{x}_{med}) = \frac{1}{2}$, that is, for ordered $x_{i_1} \le \cdots \le x_{i_n}$:

$$\hat{x}_{med} = \begin{cases} x_{\left(i_{(n+1)/2}\right)} & \text{for odd } n \\ \frac{\left(x_{i_{n/2}} + x_{i_{(n+2)/2}}\right)}{2} & \text{for even } n \end{cases}$$

Example

What is the expected life time of a specific electronic device (in months)?

- Random variable X:= life time in # months
- Random sample:

$$x_1 = 38,\ x_2 = 33,\ x_3 = 35,\ x_4 = 32,\ x_5 = 9,\ x_6 = 36,\ x_7 = 31,$$
$$x_8 = 37,\ x_9 = 22,\ x_{10} = 40,\ x_{11} = 30$$

  - Empirical mean: $\bar{X} = \frac{1}{11}\sum_{i=1}^{11} x_i \approx 31.2$
  - Empirical median: 33
  - Empirical variance: $S_{em}^2 = \frac{1}{11}\sum_{i=1}^{11}(x_i - \bar{X})^2 \approx 70.69$
  - Sample variance: $S^2 = \frac{1}{10}\sum_{i=1}^{11}(x_i - \bar{X})^2 \approx 77.76$

Unbiased and consistent estimators

How “good” is an estimator?

- How well does it approximate the true parameter on average?
- Can it yield the true parameter with more and more data?
- What is the variance of the estimator?

Unbiased estimator: An estimator $\hat{\gamma}$ is unbiased if its expected value $E(\hat{\gamma})$ is equal to the true value of the parameter γ it estimates, i.e., $E(\hat{\gamma}) = \gamma$, otherwise $\hat{\gamma}$ is biased with squared bias $(E(\hat{\gamma}) - \gamma)^2$

Consistent estimator: An estimator $\hat{\gamma}$ derived from n values of i.i.d. random variables X_i is consistent if $\lim_{n\to\infty} P(|\hat{\gamma} - \gamma| > \varepsilon) = 0$ for all ε>0

Biased and unbiased estimators

Let $x_1, \dots, x_n$ be the values of i.i.d. random variables X_i

- The empirical mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is an **unbiased consistent** estimator of the true mean $E(X)$
- The empirical variance $S_{em}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2$ is a **biased consistent** estimator of the true variance $Var(X)$, it can be shown that $E(S_{em}^2) = \frac{n-1}{n} Var(X)$
- The sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2$ is an **unbiased consistent** estimator of the true variance $Var(X)$
- The sample covariance $\hat{C} = \frac{1}{n-1}\sum_{i=1}^{n}(x_1 - \bar{X})(y_i - \bar{Y})$ is an **unbiased consistent** estimator of $Cov(X)$
- The empirical distribution function $\hat{F}_{X_{i:n}}(x) = \frac{1}{n}\sum_{i=1}^{n}[\![x_i \leq x]\!]$ is an **unbiased consistent** estimator of the true cumulative distribution F_X

Law of Large Numbers

Let $x_1, x_2, \dots, x_n$ be a random sample from a distr. $f_X(x)$ and $\bar{X} = \sum_i \frac{x_i}{n}$

Weak law of large numbers (weak consistency of the empirical mean)

$\lim_{n\to\infty} P(|\bar{X} - E(X)| > \varepsilon) = 0$ for all ε>0

Sample average converges in probability towards the mean of the distr. of X

Strong law of large numbers (strong consistency of the empirical mean)

$P(\lim_{n\to\infty} |\bar{X} - E(X)| > \varepsilon) = 0$ for all ε>0

Sample average converges almost surely towards the mean of the distr. of X

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Best estimators

An **unbiased estimator** $\hat{\gamma}$ is the **best estimator** of the true parameter γ if it has lowest variance among all other **unbiased estimators**, i.e., **for all unbiased estimators** $\hat{\boldsymbol{\gamma}}'$ **of** $\boldsymbol{\gamma}$**:** $\boldsymbol{Var(\hat{\gamma}) \leq Var(\hat{\gamma}')}$

Notes

- The **sample mean is the best estimator** of the true mean for many useful distributions
- The **sample variance is the best estimator** of the true variance for normally distributed data

The mean squared error

The **mean squared error** between an estimator $\hat{\gamma}$ and the true parameter γ is:

$$\boldsymbol{mse(\hat{\gamma}-\gamma) = E\left((\hat{\gamma}-\gamma)^2\right) = Var(\hat{\gamma}) + Bias(\hat{\gamma})^2}$$

because:

$$Var(\hat{\gamma}) = Var(\hat{\gamma}-\gamma) = E\left((\hat{\gamma}-\gamma)^2\right) - E^2(\hat{\gamma}-\gamma)$$

$$Bias(\hat{\gamma}) = E(\hat{\gamma}) - \gamma = E(\hat{\gamma}) - E(\gamma) = E(\hat{\gamma}-\gamma)$$

Usefulness of estimators

How useful is an estimator for the understanding of the underlying distribution?
It depends on the distribution!

Example

- Random variable X:= yearly income in \$1000
- Random sample:

$$x_1 = 58;\ x_2 = 74;\ x_3 = 69;\ x_4 = 81;\ x_5 = 64;\ x_6 = 120;\ x_7 = 55;\ x_8 = 71;\ x_9 = 77;\ x_{10} = 65;\ x_{11} = 23{,}000 \quad \Rightarrow \quad \bar{X} \approx 2{,}158$$

→ **Empirical median** is more insightful in this case

Other useful estimators

Maximum Likelihood Estimator, i.e., $argmax_\theta P(x_1, \dots, x_n|\theta)$ is

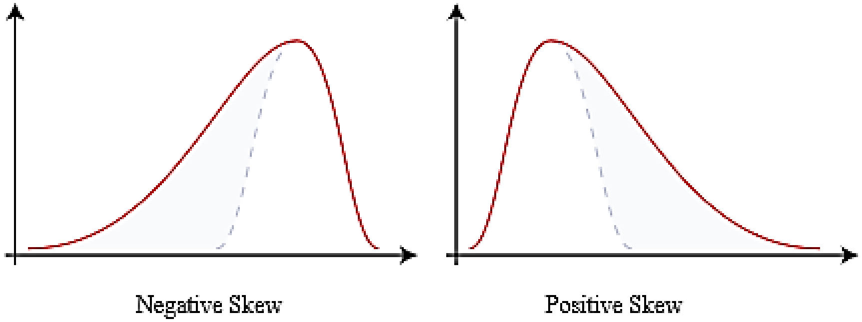
- Consistent
- Asymptotically normal
- Asymptotically optimal, i.e., with smallest variance

Minimum: $Min(x_1, \dots, x_n)$

Maximum: $Max(x_1, \dots, x_n)$

Empirical skew

$$Sk = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2\right)^{3/2}}$$



Source: Wikipedia

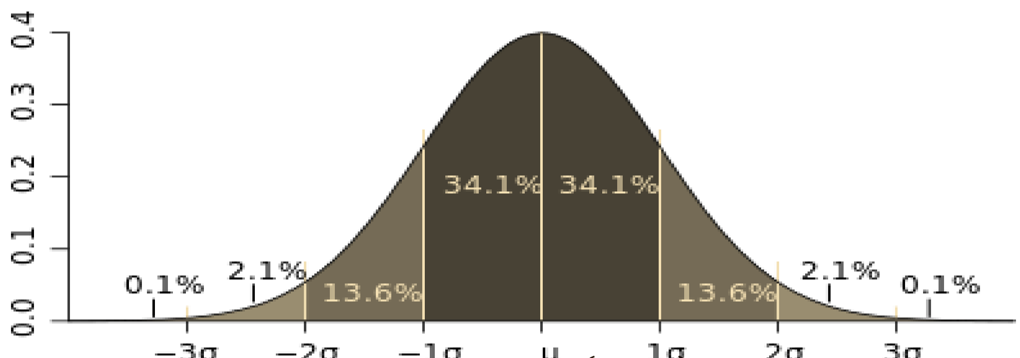
Recap of the normal distribution

X is **normally distributed** $\Leftrightarrow\ X \sim N(\mu, \sigma^2) \Leftrightarrow\ f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

μ: mean, σ: standard deviation

Standard normal distribution: $N(0,1)$

Cumulative distribution of $N(0,1)$: $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}\, dx$



Theorem: If $X \sim N(\mu, \sigma^2)$ then $Y \coloneqq \frac{(X-\mu)}{\sigma} \sim N(0,1)$

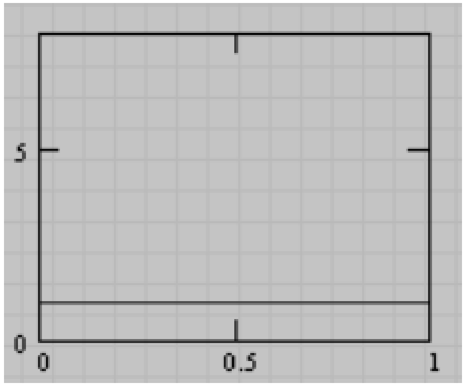
Central Limit Theorem

Central Limit Theorem: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a distr. with mean μ and finite non-zero variance σ^2. The cdf of the random variable $Z := \sum_i X_i$ converges to the cdf of the normal distribution $N(n\mu, n\sigma^2)$. That is:

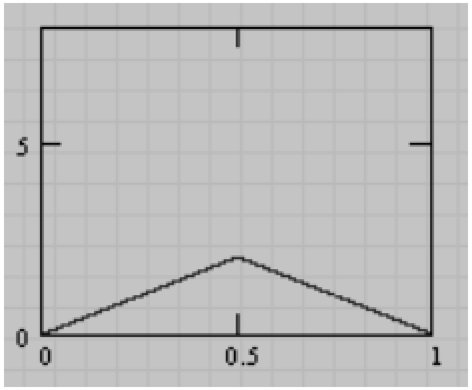
$$\lim_{n\to\infty} P\left(a \leq \frac{Z - n\mu}{\sqrt{n}\sigma} \leq b\right) = \Phi(b) - \Phi(a)$$

Corollary: The cdf of $Z := \frac{1}{n}\sum_i X_i$ converges to the cdf of $N\left(\mu, \frac{\sigma^2}{n}\right)$

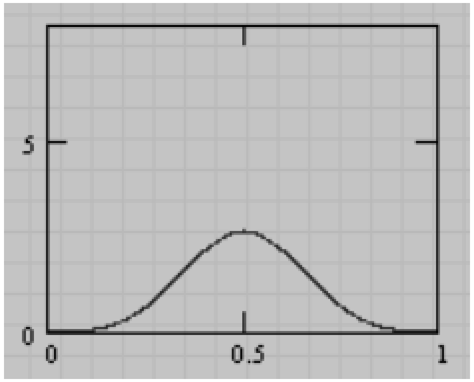
Central Limit Theorem: Example



$f_X(x)$ is uniform

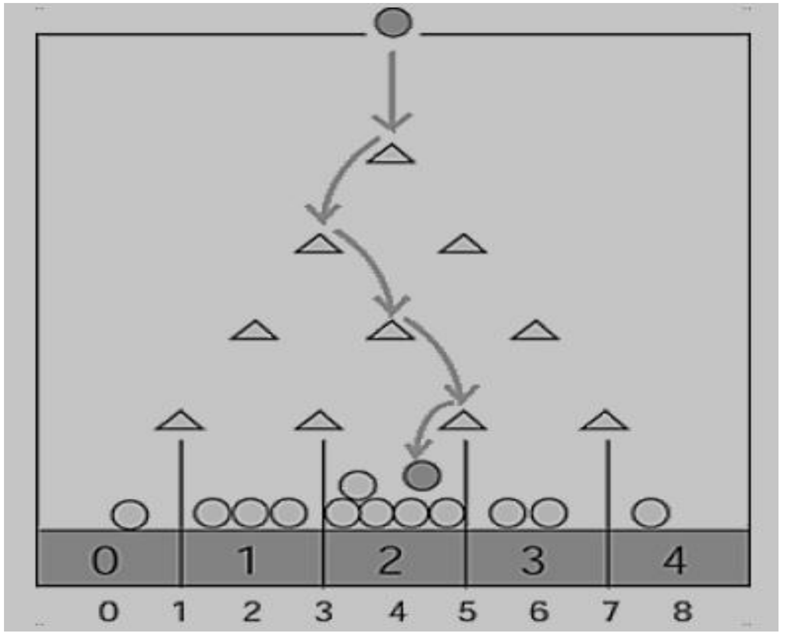


Avg. of X_1, X_2 sampled repeatedly from $f_X(x)$



Avg. of X_1, X_2, X_3, X_4 sampled repeatedly from $f_X(x)$

Galton Machine



Empirical evidence for the Central Limit Theorem (by considering sequences of i.i.d. Bernoulli variables) and for the Law of Large Numbers (by considering random samples from a Binomial distribution)

Hypothesis testing

Example hypotheses:

- Sample originates from normal distribution
- Two random variables are independent
- Sample is Bernoulli distributed with p=0.5

Goal: Falsification of hypothesis by lack of statistical evidence

- Hypothesis to be falsified: H_0 (**null hypothesis**)
- Counter hypothesis: H_1
- Test region R from cdf of test variable X

  $X \in R \Rightarrow$ reject H_0

  $X \notin R \Rightarrow$ retain H_0

	Retain H_0	Reject H_0
H_0 true	ok	Type I error
H_1 true	Type II error	ok

Hypothesis testing: Example

Assume average IQ
of students is 100
H_0: $\mu = 100$

Run IQ test
on sample

IQ_1 IQ_2 IQ_3 IQ_4 IQ_5 IQ_6 IQ_7 IQ_8

$\overline{IQ} = 115$

Is this likely given $\mu = 100$?

If yes retain H_0 else reject

Confidence interval and confidence level

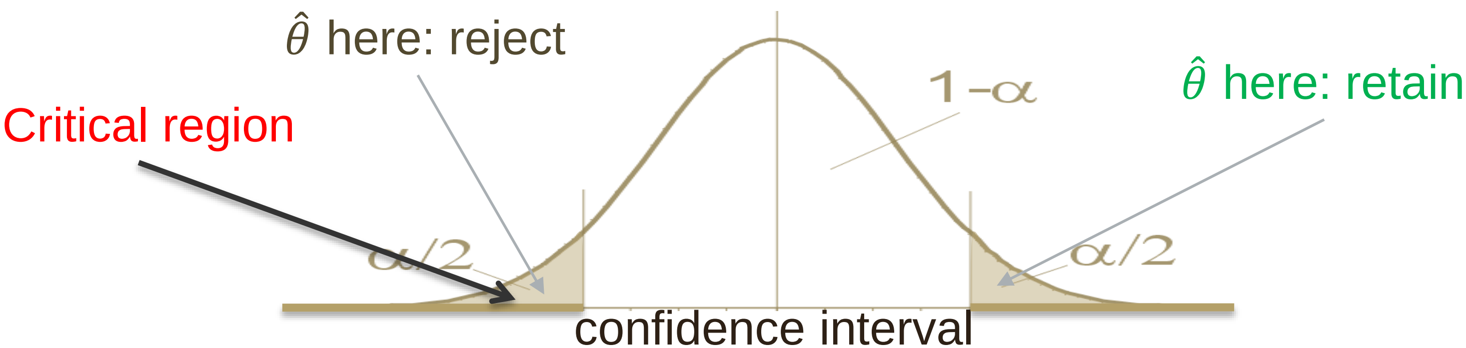
How well is a parameter estimated?

- Consider estimator $\hat{\theta}$ for parameter θ
- How well does $\hat{\theta}$ represent θ?

$$P\left(\hat{\theta} - c \leq \theta \leq \hat{\theta} + c\right) = 1 - \alpha$$

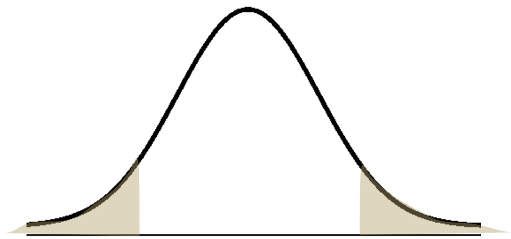
Definitions

- The interval $[\hat{\theta} - c, \hat{\theta} + c]$ is the **confidence interval**
- The value $1 - \alpha$ is the **confidence level**
- α is the **significance level** (typically: 0.01, 0.05, 0.1)



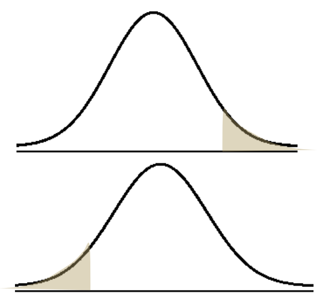
One sided and two-sided tests

- A test of the form $H_0: \theta = \theta_0 \ vs. H_1: \theta \neq \theta_0$ is called a **two-sided test**



- A test of either of these forms
  - $H_0: \theta \leq \theta_0 \ vs. H_1: \theta > \theta_0$
  - $H_0: \theta \geq \theta_0 \ vs. H_1: \theta < \theta_0$

  is called a **one-sided test**



Unknown mean and known variance

Consider i.i.d. random variables $X_1, \dots, X_n,\ n \gg 1$, from a distribution with **unknown, non-zero mean** μ and **known finite variance** σ^2.

- We know $\bar{X} = \frac{1}{n}\sum_i X_i$ is **approximately normally distributed** with $N(\mu, \frac{\sigma^2}{n})$
- We also know that $Y = \frac{(\bar{X}-\mu)\sqrt{n}}{\sigma} \sim N(0,1)$

$$P\left(-z \leq \frac{(\bar{X}-\mu)\sqrt{n}}{\sigma} \leq z\right) = \Phi(z) - \Phi(-z) = P\left(\bar{X} - \frac{z\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + \frac{z\sigma}{\sqrt{n}}\right)$$

$\Rightarrow$ For confidence interval $[\bar{X} - c, \bar{X} + c]$ set $z \coloneqq \frac{c\sqrt{n}}{\sigma}$ and look up $\Phi(z)$

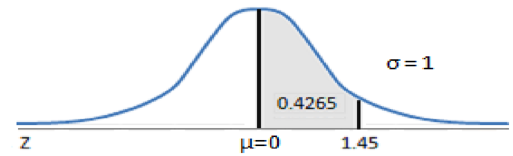
$\Rightarrow$ For confidence level $1-\alpha$, and a proposed value for μ, reject null hypothesis if $|Y| > \Phi^{-1}(1-\alpha/2)$

The **p-value** is **minimal significance level** at which H_0 can be rejected

Z-score table

Areas Under the One-Tailed Standard Normal Curve

This table provides the area between the mean and some Z score.
For example, when Z score = 1.45 the area = 0.4265.



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995

Wald Test

For a parameter $\hat{\theta}$ derived from a sample and a proposed parameter θ, we can test

$$H_0: \hat{\theta} = \theta \text{ vs. } H_1: \hat{\theta} \neq \theta$$

$s = \sqrt{Var(\hat{\theta})}$ is called the **standard error** and $Var(\hat{\theta})$ is the **sample variance**.

The test variable $W := \frac{\hat{\theta}-\theta}{s}$ is approximately $N(0,1)$-distributed (i.e., distribution of W converges to $N(0,1)$ for growing sample size).

→ Reject H_0 at level α when $|W| > \Phi^{-1}(1-\alpha/2)$.

Example

$\hat{\theta}$: Average increase of height of men compared to height of women over a 30-year period. Proposed parameter: $\theta = 0$.

Example: Life-time expectancy

What is the expected life time of a specific electronic device (in months)?

Random variable X:= life time in # months

Random sample: $x_1 = 38,\ x_2 = 33,\ x_3 = 35,\ x_4 = 32,\ x_5 = 9,\ x_6 = 36,$
$x_7 = 31,\ x_8 = 37,\ x_9 = 22,\ x_{10} = 40,\ x_{11} = 30$

- Empirical mean: $\bar{X} = \frac{1}{11}\sum_{i=1}^{11} x_i \approx 31.2$
- Sample variance: $S^2 = \frac{1}{10}\sum_{i=1}^{11}(x_i - \bar{X})^2 \approx 77.76$

Hypothesis I: Devices have a life time of around 2 years

$$W \coloneqq \frac{\hat{\theta}-\theta}{\sqrt{Var(\hat{\theta})}} \approx 0{,}82 < 1.96 \text{ (for significance level 0.05)}$$

Hypothesis II: Devices have a life time of around 1 year

$$W \coloneqq \frac{\hat{\theta}-\theta}{\sqrt{Var(\hat{\theta})}} \approx 2{,}177 > 1.96 \text{ (for significance level 0.05)}$$

Example: Probability of heads

H_0: coin has head probability $p = p_0$
X: test variable representing #heads in n tosses
We know that approximately $X \sim N(pn, p(1-p)n)$

$$Y := \frac{(X - pn)}{\sqrt{p(1-p)n}} \sim N(0,1)$$

→ reject H_0 at level α $(= 0.05)$ if $Y > \Phi^{-1}(1 - \alpha/2)$ or $Y < \Phi^{-1}(\alpha/2)$
$\Leftrightarrow$ reject H_0 at level α $(= 0.05)$ if $|Y| > \Phi^{-1}(1 - \alpha/2)$

t-Test for unknown mean and unknown variance

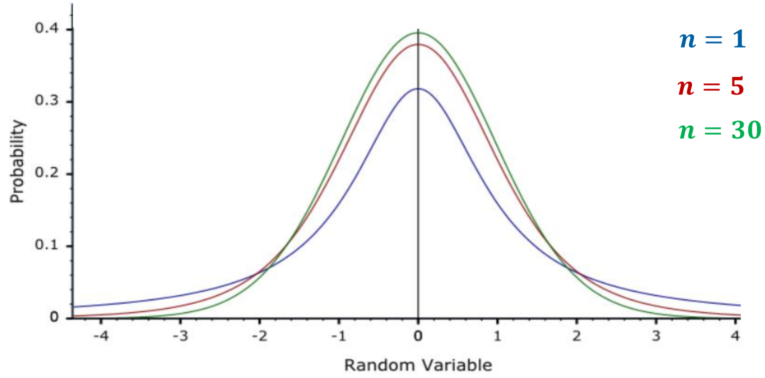
Consider i.i.d. random variables $X_1, \ldots, X_n,\ n \gg 1$, from a distribution with **unknown, non-zero mean** μ and **unknown variance.**

Let s^2 be the sample variance. $Y := \frac{(\bar{X}-\mu)\sqrt{n}}{s}$ has a Student's *t* distribution with $n-1$ degrees of freedom.

With analogous derivation as before:

$$P\left(\bar{X} - \frac{t_{n-1,1-\alpha/2}s}{\sqrt{n}} \leq \mu \leq \bar{X} + \frac{t_{n-1,1-\alpha/2}s}{\sqrt{n}}\right) = 1 - \alpha$$

⇒ For proposed μ and significance level α, reject null hypothesis if $|Y| > t_{n-1,1-\alpha/2}$



t-Test in practice

Comparison of two prediction algorithms A and A’ based on performance on k labeled datasets:

- Let $e_1, \dots, e_k$ and $e_1', \dots, e_k'$ be the error values (or any performance values), respectively.
- Are the error means any different?
- Fact: $\bar{e}$ and $\overline{e'}$ are approximately normally distributed, but we neither know the means nor the variances.
- Since σ_e and σ_e' are unknown, we need to use *t*-distribution with k-1 degrees of freedom to estimate how close μ_e and μ_e' are (H_0: $\mu_e = \mu_e'$).
- $\bar{d} = \bar{e} - \overline{e'}$ is also *t*-distributed, with k-1 degrees of freedom

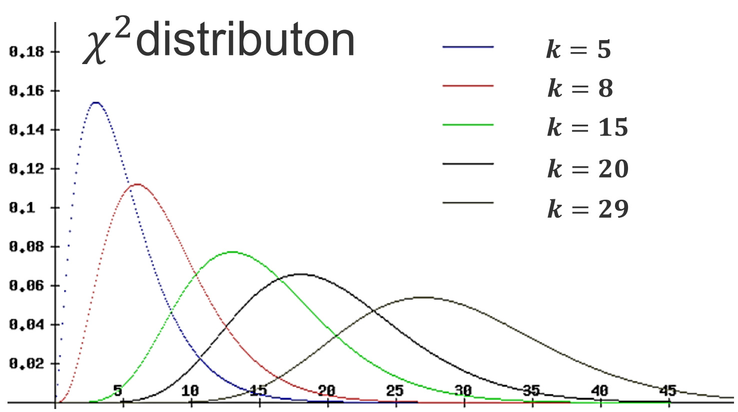
  $\Rightarrow H_0$: $\bar{d} = 0$ and $Y := \frac{(\bar{d}-0)\sqrt{k}}{s_d}$ is the t-statistics
- Use *t*-distribution table to determine the $t_{k-1,1-\alpha/2}$ score

  →If $t_{k-1,1-\alpha/2} < |Y|$ reject H_0 otherwise retain it.

Chi-Square Goodness-of-Fit-Test

Given sample $x_1, \dots, x_n$ of i.i.d. random variables X_i and absolute frequencies $h_1, \dots, h_k$ of class $c_j, 1 \le j \le k$, we can test the null hypothesis "X_i follow a proposed discrete distribution".

$Z_k := \frac{\sum_{j=1}^{k}\left(h_j - E(h_j)\right)^2}{E(h_j)}$, with $E(h_j)$ being the expected frequency of class c_j according to the proposed distribution, is χ^2-distributed with *k-1* degrees of freedom → Reject H_0 at test level α (e.g. 0.05) if $Z_k > \chi^2_{k-1,1-\alpha}$



Chi square distribution table

d.f.	$\chi^2_{.25}$	$\chi^2_{.10}$	$\chi^2_{.05}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$	$\chi^2_{.001}$
1	1.32	2.71	3.84	5.02	6.63	7.88	10.8
2	2.77	4.61	5.99	7.38	9.21	10.6	13.8
3	4.11	6.25	7.81	9.35	11.3	12.8	16.3
4	5.39	7.78	9.49	11.1	13.3	14.9	18.5
5	6.63	9.24	11.1	12.8	15.1	16.7	20.5
6	7.84	10.6	12.6	14.4	16.8	18.5	22.5
7	9.04	12.0	14.1	16.0	18.5	20.3	24.3
8	10.2	13.4	15.5	17.5	20.1	22.0	26.1
9	11.4	14.7	16.9	19.0	21.7	23.6	27.9
10	12.5	16.0	18.3	20.5	23.2	25.2	29.6
11	13.7	17.3	19.7	21.9	24.7	26.8	31.3
12	14.8	18.5	21.0	23.3	26.2	28.3	32.9
13	16.0	19.8	22.4	24.7	27.7	29.8	34.5
14	17.1	21.1	23.7	26.1	29.1	31.3	36.1
15	18.2	22.3	25.0	27.5	30.6	32.8	37.7
16	19.4	23.5	26.3	28.8	32.0	34.3	39.3
17	20.5	24.8	27.6	30.2	33.4	35.7	40.8
18	21.6	26.0	28.9	31.5	34.8	37.2	42.3
19	22.7	27.2	30.1	32.9	36.2	38.6	32.8
20	23.8	28.4	31.4	34.2	37.6	40.0	45.3
21	24.9	29.6	32.7	35.5	38.9	41.4	46.8
22	26.0	30.8	33.9	36.8	40.3	42.8	48.3
23	27.1	32.0	35.2	38.1	41.6	44.2	49.7
24	28.2	33.2	36.4	39.4	32.0	45.6	51.2
25	29.3	34.4	37.7	40.6	44.3	46.9	52.6
26	30.4	35.6	38.9	41.9	45.6	48.3	54.1
27	31.5	36.7	40.1	43.2	47.0	49.6	55.5
28	32.6	37.9	41.3	44.5	48.3	51.0	56.9
29	33.7	39.1	42.6	45.7	49.6	52.3	58.3
30	34.8	40.3	43.8	47.0	50.9	53.7	59.7
40	45.6	51.8	55.8	59.3	63.7	66.8	73.4

Chi-Square independence test

- r = number of columns
- m = number of rows
- n_{ij} = Actual number in cell$_{ij}$
- n^*_{ij}= Expected number in cell$_{ij}$
- $(r-1)(m-1)$ = degrees of freedom

	Feature Y						Sum Σ
Feature X	1	2	…	k	…	r	$n_{j.}$
1	n_{11}	n_{12}	…	n_{1k}	…	n_{1r}	$n_{1.}$
2	n_{21}	n_{22}	…	n_{2k}	…	n_{2r}	$n_{2.}$
…	…	…	…	…	…	…	…
j	…	…	…	n_{jk}	…	…	$n_{j.}$
…	…	…	…	…	…	…	…
m	n_{m1}	n_{m2}	…	n_{mk}	…	n_{mr}	$n_{m.}$
Sum Σ	$n_{.1}$	$n_{.2}$	…	$n_{.k}$	…	$n_{.r}$	n

$$n^*_{jk} = \frac{n_{j.} \cdot n_{.k}}{n}$$

$$\chi^2 = \sum_{j=1}^{m} \sum_{k=1}^{r} \frac{(n_{jk} - n^*_{jk})^2}{n^*_{jk}}$$

→ Reject H_0 at test level α (e.g. 0.05) if $\chi^2 > \chi^2_{(r-1)(m-1),1-\alpha}$

General recipe for hypothesis testing

- Formulate null hypothesis (simplest hypothesis)
- Define corresponding random variable for the test
- Turn the variable into a $N(0,1)$-distributed variable, or a t-statistics, or a χ^2-statistics, …
- Test whether the new statistics lies in the critical region of the underlying distribution