GAME THEORY

Game theory attempts to mathematically capture behavior in strategic situations, or *games*, in which an individual's success in making choices depends on the choices of others

Myerson, 1991

GAME THEORY

- Zero-sum / Non-zero-sum games
  - When a player gains something, another player may lose something
  - The total sum of gainings and losses is 0 / may be anything
- Cooperative / Non-cooperative games
  - Players cooperate / everyone plays for its own

MODEL SETTING

- Strategic Game:
  - A set of *players*
  - For each player, a set of *strategies,* each defining the *action* to perform in a certain situation, based on the actions of all the players
    - Ordinal preferences among the strategies (Player *1* prefers *a* to *b*)
- Time is absent from the model
  - All the players choose simultaneously
- Rationality assumption
  - Each player makes her best choice

PRISONER'S DILEMMA

- Two suspects are held in separate cells.
- There is enough evidence to convict each of them of a minor offense, but not enough to convict either of them of the major crime, unless one of them finks.
- If they both stay quiet, each will be convicted of the minor offense and spend 1 year in prison.
- If only one of them finks, he/she will be set free and used as a witness against the other, who will spend 3 years in prison.
- If they both fink, each will spend 2 years in prison.

PRISONER'S DILEMMA

- *Players*: the two suspects *S1* and *S2*
- *Actions:* Either *Quiet* or *Fink,* for both players
- *Preferences among strategies:*
  - *S1: {F,Q} (payoff 0) > {Q,Q} (-1) > {F,F} (-2) > {Q,F} (-3)*
  - *S2: {Q,F} (0) > {Q,Q} (-1) > {F,F} (-2) > {F,Q} (-3)*

	Q	**F**
Q	(-1,-1)	(-3,0)
F	(0,-3)	(-2,-2)

PRISONER'S DILEMMA

	Q	**F**
Q	(-1,-1)	(-3,0)
F	(0,-3)	(-2,-2)

- *How would you act if you were suspect S1?*
- *How would you act if you were suspect S2?*
- *How would you act if you were one of the suspects and you could speak with the other?*

NASH EQUILIBRIUM

- A state of the game (a set of *strategies*, one for each *player)*, such that no player has incentive to *unilaterally* change action
  - Each player is making the best decision, taking into account the decisions of the others.
- Nash equilibria are used to analyze the outcome of the *strategic interaction* of several *decision makers*
  - There could be zero, one, or more than one Nash equilibria
    - If it is only one, then the outcome is *certain*
- We cannot predict the result of the choices of multiple decision makers if we analyze those decisions in isolation

NASH EQUILIBRIUM – PARETO OTPIMUM

- **A Nash Equilibrium is not necessarily the best solution for everyone**

- In a Nash equilibrium no player can improve his outcome by using a different strategy, but a set of players can improve the global outcome by changing their strategies simultaneously

- Nash equilibrium <> *Pareto Optimum*
  - None of them implies the other

PRISONER'S DILEMMA

	Q	F
Q	(-1,-1)	(-3,0)
F	(0,-3)	(-2,-2)

- *How many Nash Equilibria?*
- *What is the best global solution?*
  - *Is it a Nash Equilibrium?*

MATCHING PENNIES

	H	**T**
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

- *How many Nash Equilibria?*
- *What is the best global solution?*
  - *Is it a Nash Equilibrium?*

SAY THE SAME

	A	B
A	(1,1)	(0,0)
B	(0,0)	(1,1)

- *How many Nash Equilibria?*
- *What is the best global solution?*
  - *Is it a Nash Equilibrium?*

SAY THE SAME

	A	**B**
A	(2,1)	(0,0)
B	(0,0)	(1,2)

- *How many Nash Equilibria?*
- *What is the best global solution?*
  - *Is it a Nash Equilibrium?*

SAY THE SAME

	A	B
A	(2,2)	(0,0)
B	(0,0)	(1,1)

- *How many Nash Equilibria?*
- *What is the best global solution?*
  - *Is it a Nash Equilibrium?*

MIXED STRATEGIES

- Instead of simply choosing an action, players choose *probability distributions* over the set of available actions

- Such distributions can be represented by a *function* that *assigns a real number* to each strategy
  - *von Neumann-Morgenstern utility function*

- One lottery is preferred to another if it results in a higher expected value of this utility function

NASH THEOREM

Every finite game with mixed strategies has at least one Nash equilibrium

- Finite game: finite number of players and strategies

MATCHING PENNIES – MIXED STRATEGIES

	H	**T**
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

- A mixed strategy in which both players choose A with probability ½ and B with probability ½ is a Nash Equlibrium

COMMUTING PROBLEM

- How do you choose your way to work/uni?
  - Shortest?
  - Fastest?
  - Less traffic?
  - Nicer?
  - Flatter?
  - Passing in front of a cigarette shop?
  - Taking into account how much additional traffic you generate for the others?
    - If this is not your choice, then you are *selfish*

MOTIVATION

- Large real-world systems cannot be administrated centrally
  - Internet
  - Road traffic

- Each agent is autonomous and tries to maximize its own benefit
  - The benefit of an agent depends on the behavior of the other agents

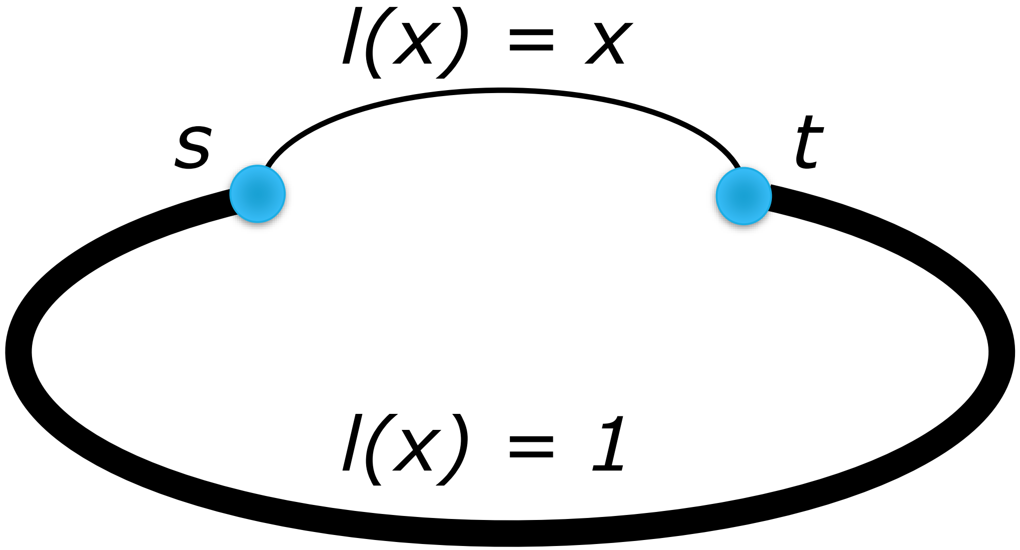
- The behavior of all agents determines a social benefit

SELFISH ROUTING

- It is usually not possible to enforce a centralized strategy in a real-world network, as users make their own choices selfishly
  - We cannot act centrally to maximize the social benefit

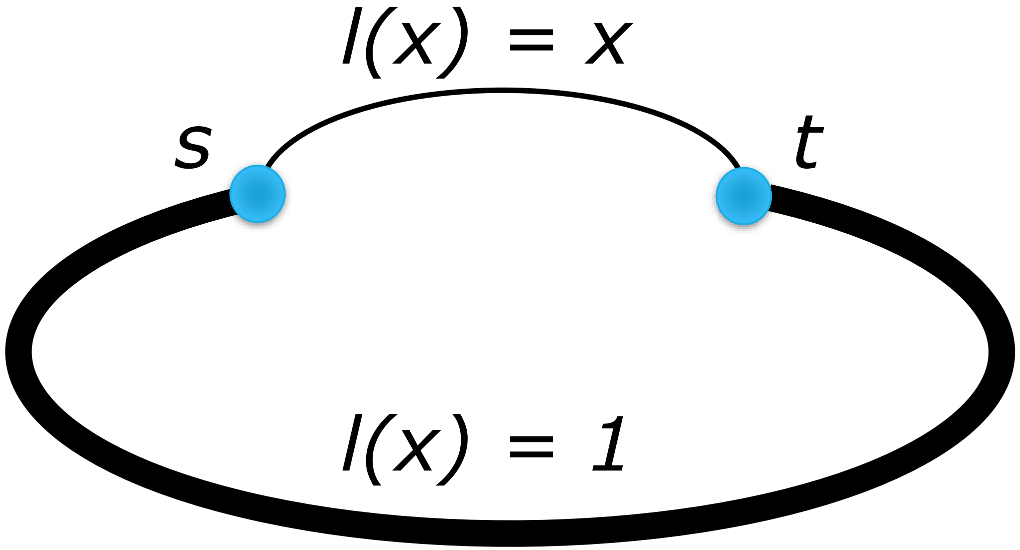
- In general, the result of a local optimization by selfish users is (significantly) worse than the global optimum that could be obtained if they cooperated

SELFISH ROUTING: PIGOU'S EXAMPLE



- One road is short but small
- One road is large but long
- x is the percentage [0,…,1] of traffic passing through a road

SELFISH ROUTING: PIGOU'S EXAMPLE



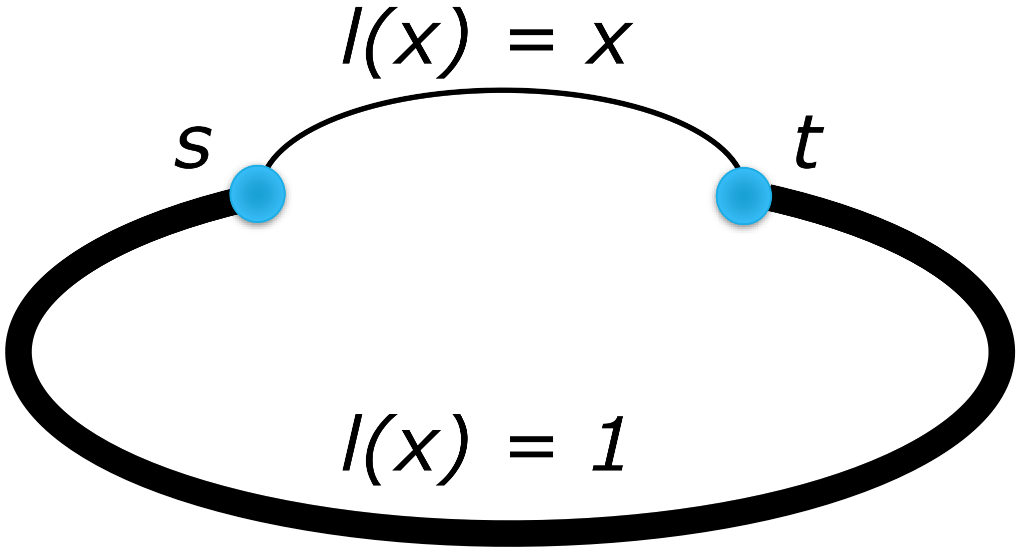
- Everyone wants to use the short road, and it takes 1 hour for all
- If they split (half-half), then it takes ½ hour for half of them and 1 for the others

Price of Anarchy = 4/3

SELFISH ROUTING AS A GAME

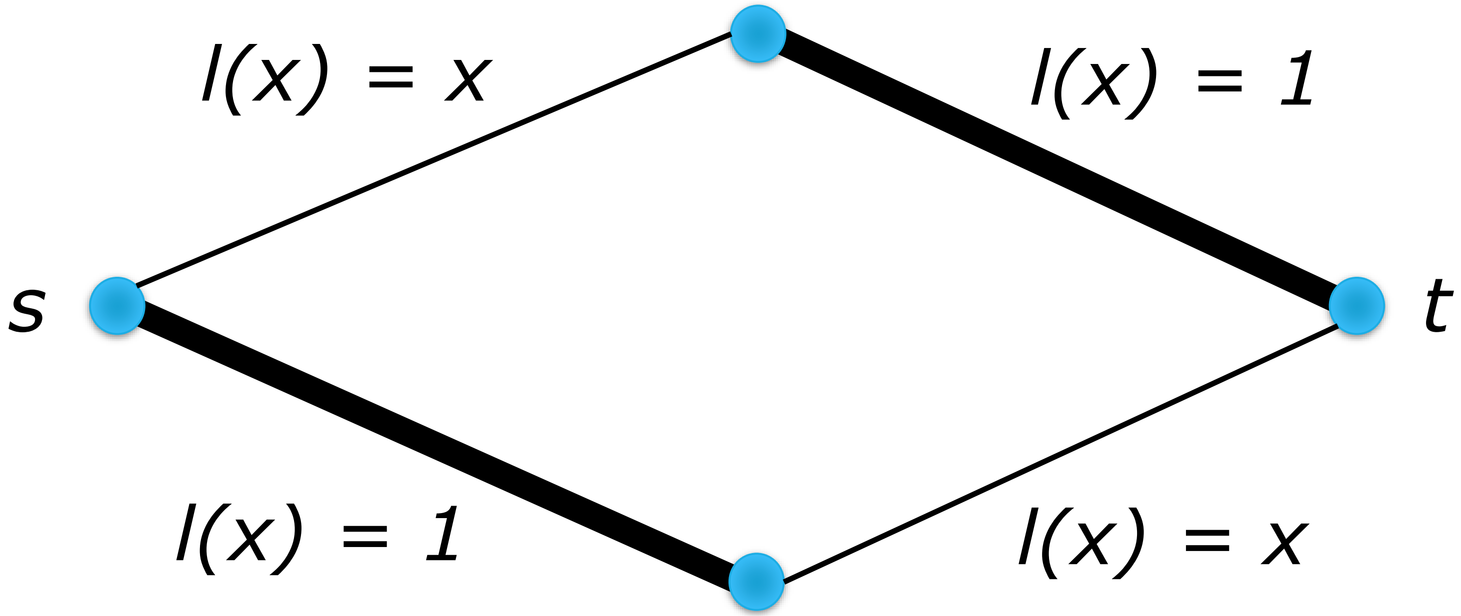
- Everyone wants to use the short road, and it takes 1 hour for all
  - Situation that is reached when everyone plays selfishly
  - Nash Equilibrium
- If they split (half-half), then it takes ½ hour for half of them and 1 for the others
  - Optimal solution when a central entity can coordinate
  - Pareto Optimum
- $Price\ of\ Anarchy = \frac{cost(NE)}{cost(PO)}$

SELFISH ROUTING AS A GAME



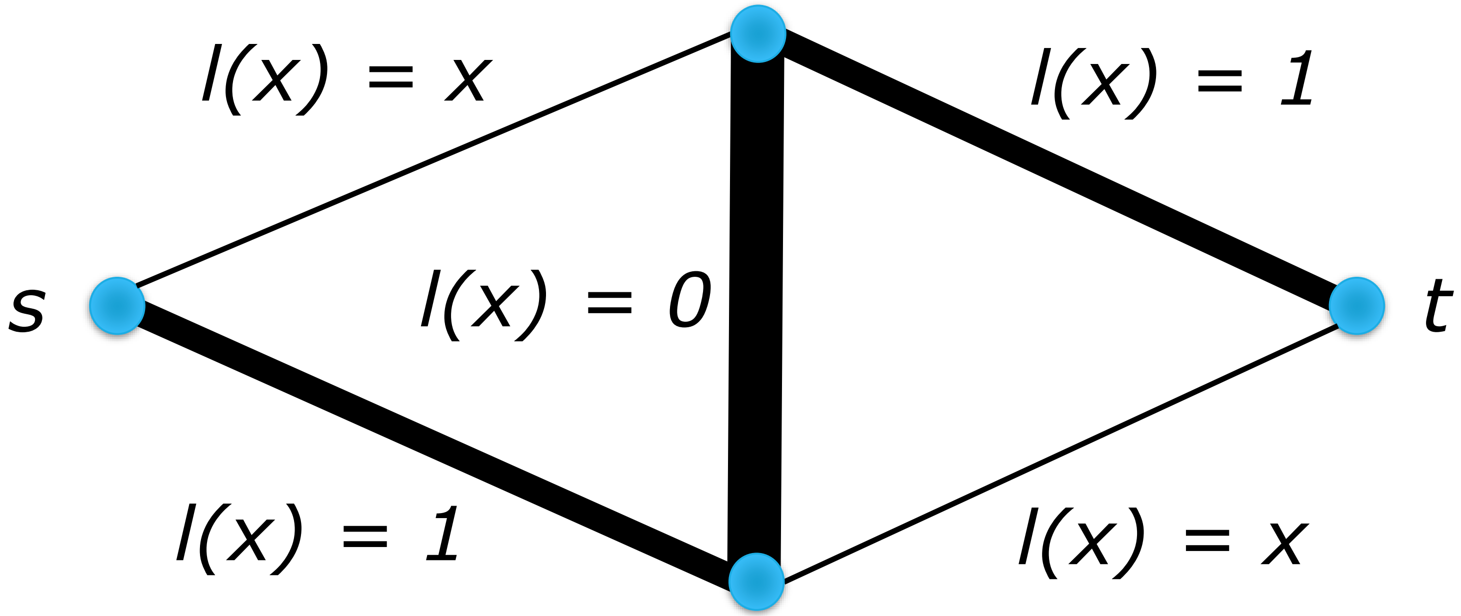
- In the Pareto Optimum (half long – half short), a single car going the long way (cost 1) would prefer to move to the short way (cost 0,5 + a small amount), so this is not a Nash Equilibrium

BRÄß PARADOX



- The traffic splits into half-half, both in the NE and in the PO
- It takes 1,5 hours for everyone, PoA = 1

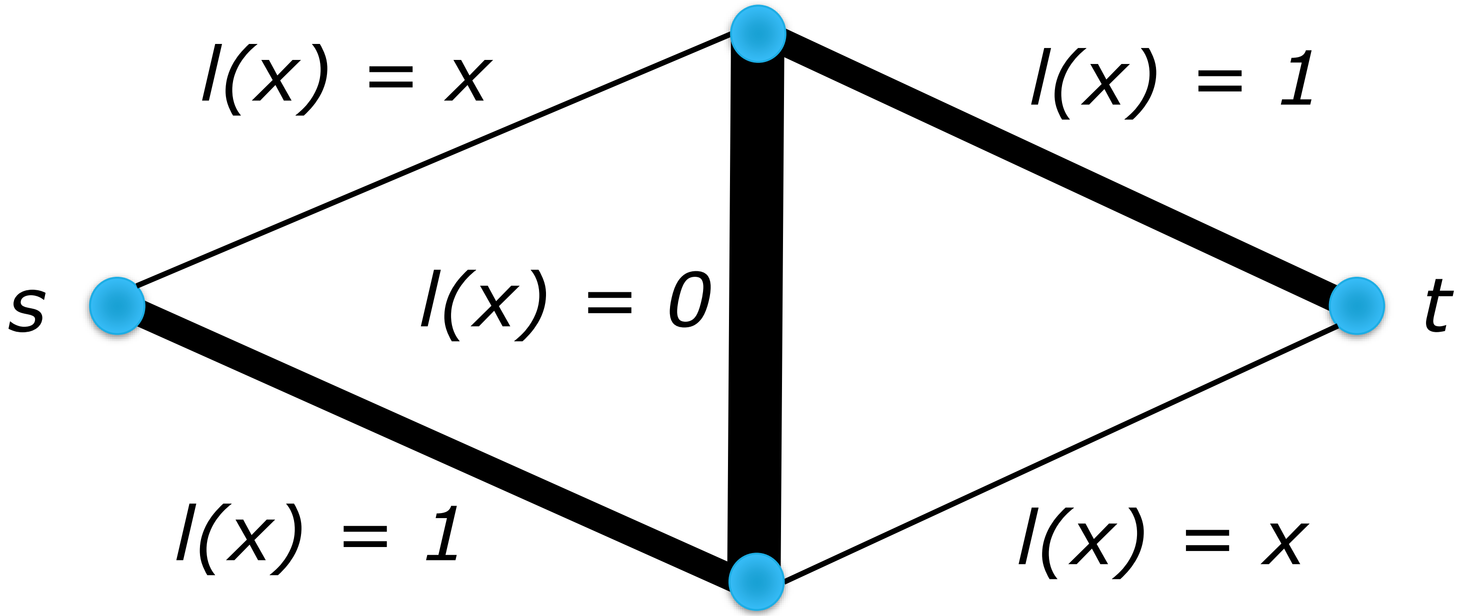
BRÄß PARADOX



- To improve the traffic, the government builds a very large and very short road

What is it going to happen?

BRÄß PARADOX



Selfish users want to use the brand new road…

It now takes 2 hours for everyone!

Price of Anarchy = 4/3

BRÄß PARADOX

- The optimal centralized solution is better than the selfish one *for everyone*
  - In the Pigou's example only half of the users were improving in the centralized solution

- It may be worse to have more alternatives
  - Should we close some roads?

SELFISH ROUTING: MODEL

- A *directed graph* with many *(source-target) pairs*, each exchanging a certain amount of (*splittable*) *flow*
  - *Multi-commodity flow*

- Each edge is equipped with a *latency function l(x)*, where *x* is the total traffic passing through it
  - Continuous and non-decreasing, usually linear or at least convex

- For each pair, we want a path between source and target with minimal total latency

HOW BAD IS SELFISH ROUTING?

- What is the (worst-case) ratio between the total latency of a flow in a Nash equilibrium and in an optimal centralized solution?
  - The Price of Anarchy

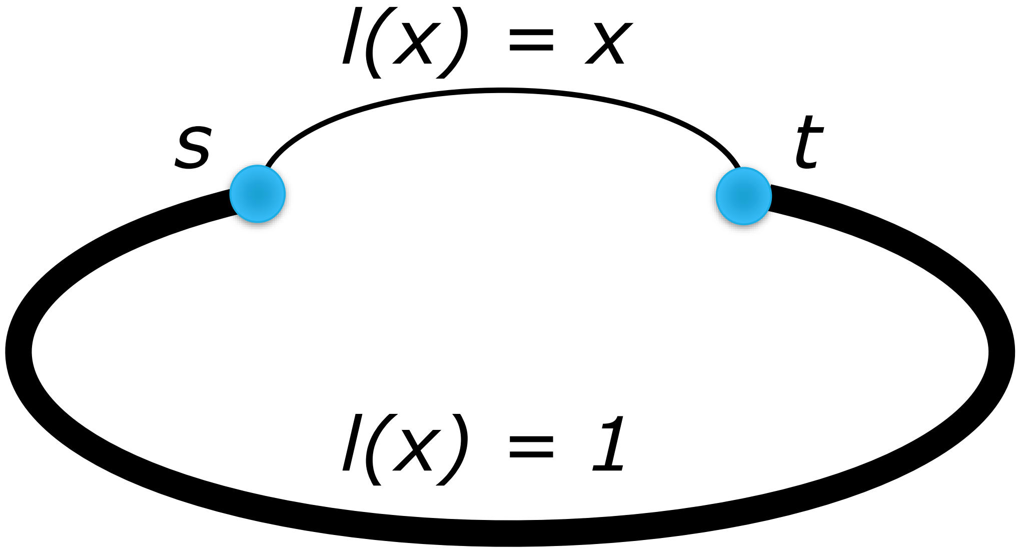
- The first question is whether a Nash Equilibrium exists at all
  - In this case: yes, and it is unique (they can be more, but cost the same)

COMPUTING THE NASH EQUILIBRIUM

- In a flow at Nash equilibrium, all the paths between a pair source-destination with a positive amount of flow have equal latency
  - Otherwise, one single car would prefer to change route

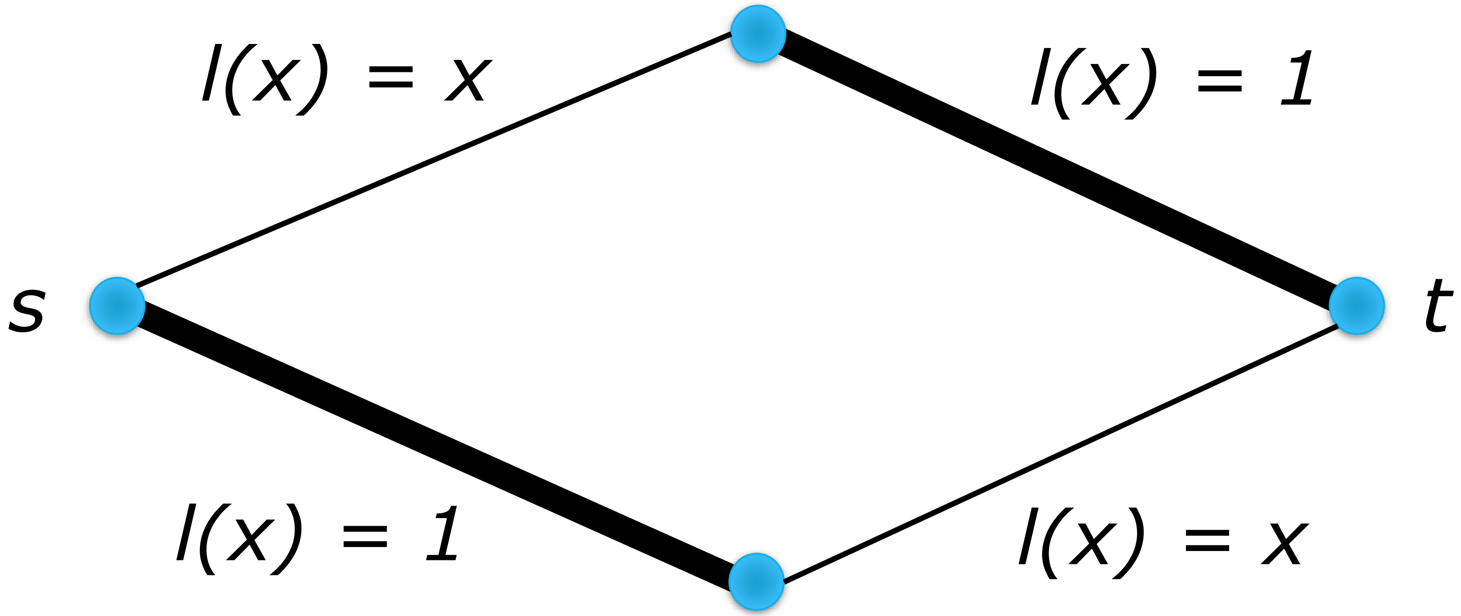
- Hence, to compute the Nash Equilibrium, it is enough to consider all the possible source-destination paths and solve a linear equation system
  - They can be many!

COMPUTING THE NASH EQUILIBRIUM



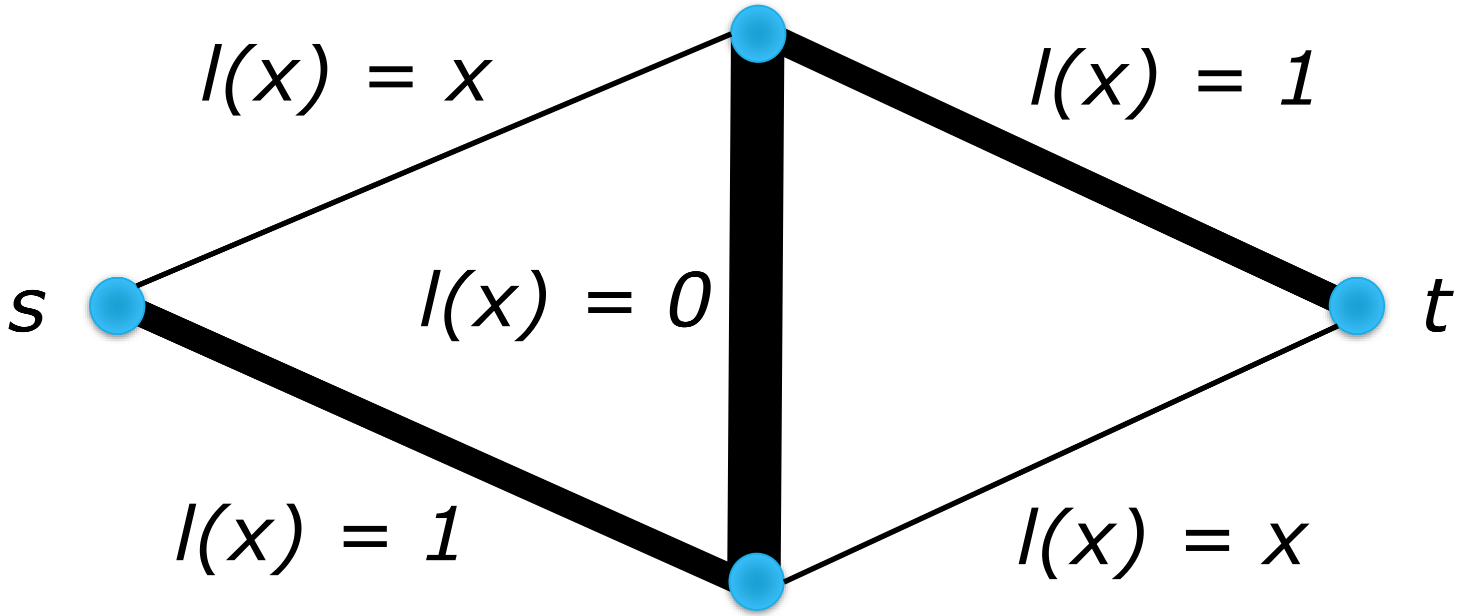
- x_1 cars use the short way, and pay x_1 each
- x_2 cars use the long way, and pay 1 each
- Equations are x_1=1 and $x_1 + x_2 = 1$
  - Solution is x_1=1, so in the NE all the traffic is through the short way

COMPUTING THE NASH EQUILIBRIUM



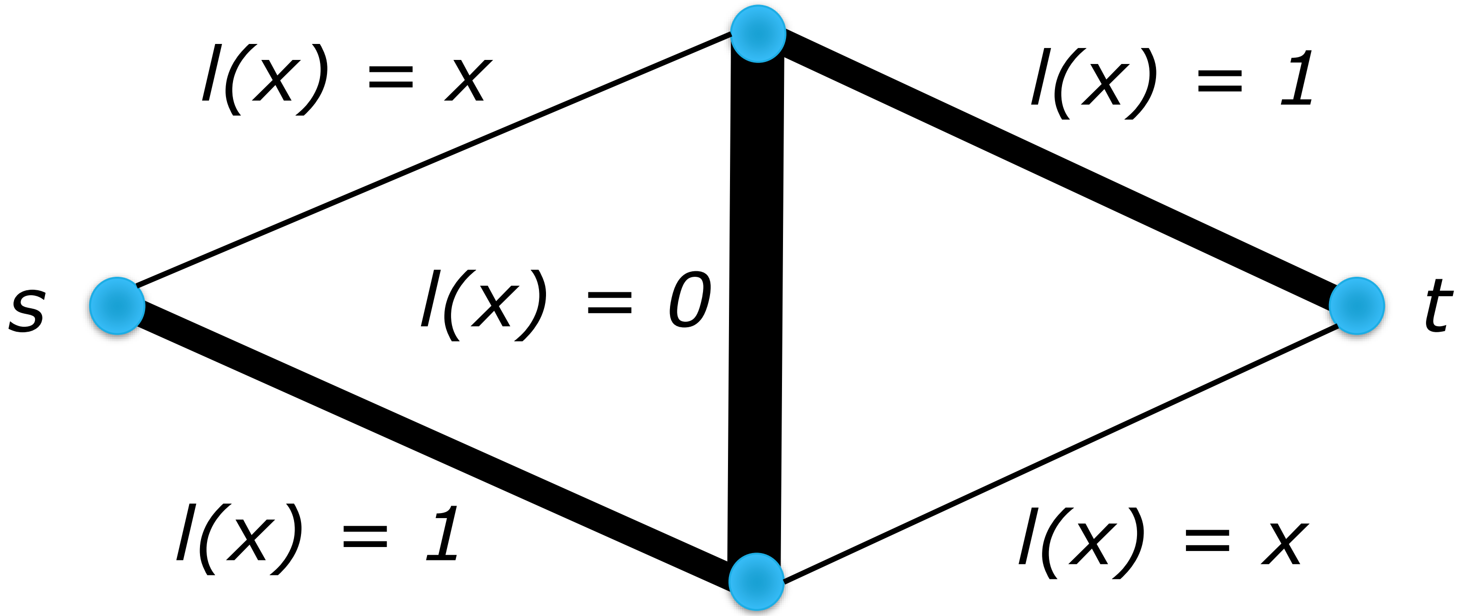
- x_1 cars use the upper way, and pay $1 + x_1$ each
- x_2 cars use the lower way, and pay $1 + x_2$ each
- Equations are $1 + x_1 = 1 + x_2$ and $x_1 + x_2 = 1$
  - Solution is $x_1 = x_2 = 0{,}5$, so in the NE the traffic splits

COMPUTING THE NASH EQUILIBRIUM



- x_1 pay $(x_1 + x_2) + 1$
- x_2 pay $(x_1 + x_2) + 0 + (x_2 + x_4)$
- x_3 pay $1 + 0 + 1$
- x_4 pay $1 + (x_2 + x_4)$
- $x_1 + x_2 + x_3 + x_4 = 1$

COMPUTING THE NASH EQUILIBRIUM



- x_1 pay $(x_1 + x_2) + 1$
- x_2 pay $(x_1 + x_2) + 0 + (x_2 + x_4)$
- x_3 pay $1 + 0 + 1$
- x_4 pay $1 + (x_2 + x_4)$
- $x_1 + x_2 + x_3 + x_4 = 1$

- Solution $x_2 = 1$, $x_1, x_3, x_4 = 0$

COMPUTING THE PARETO OPTIMUM

- The *marginal cost* of an edge describes the additional latency that is due to letting an additional amount of traffic through it
- $m_e(x) = l_e(x) + x \dfrac{dl_e(x)}{dx}$
- In a Nash equilibrium, the marginal benefit of decreasing traffic through the edges of an s,t-path is at most the same as the marginal cost of increasing the traffic through the edges of any other s,t-path

COMPUTING THE PARETO OPTIMUM

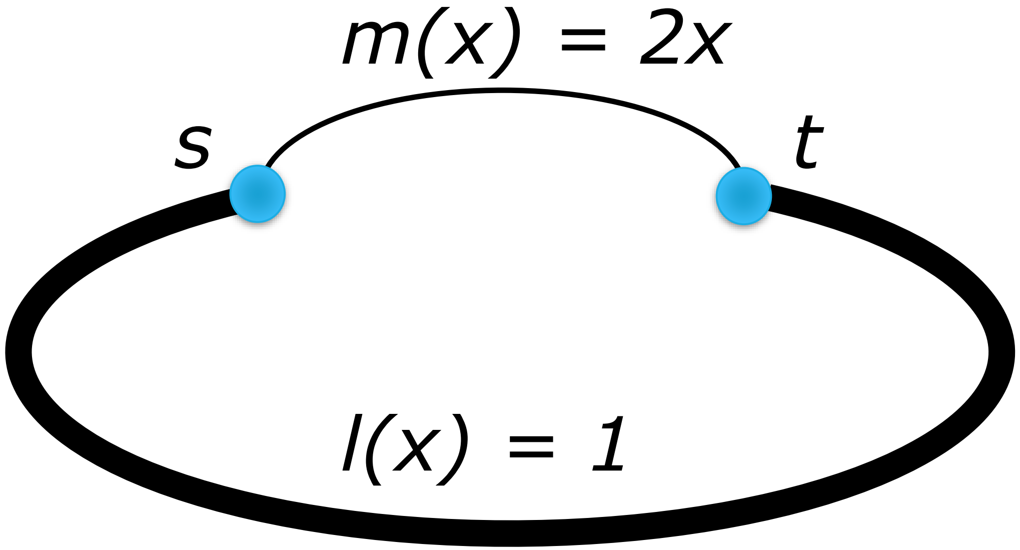
Theorem:

Let G be a network in which all the latency functions are convex. Let G* be the network obtained by replacing each latency function with its corresponding marginal cost.

Then, a flow is optimal for G if and only if it corresponds to a Nash equilibrium for G*

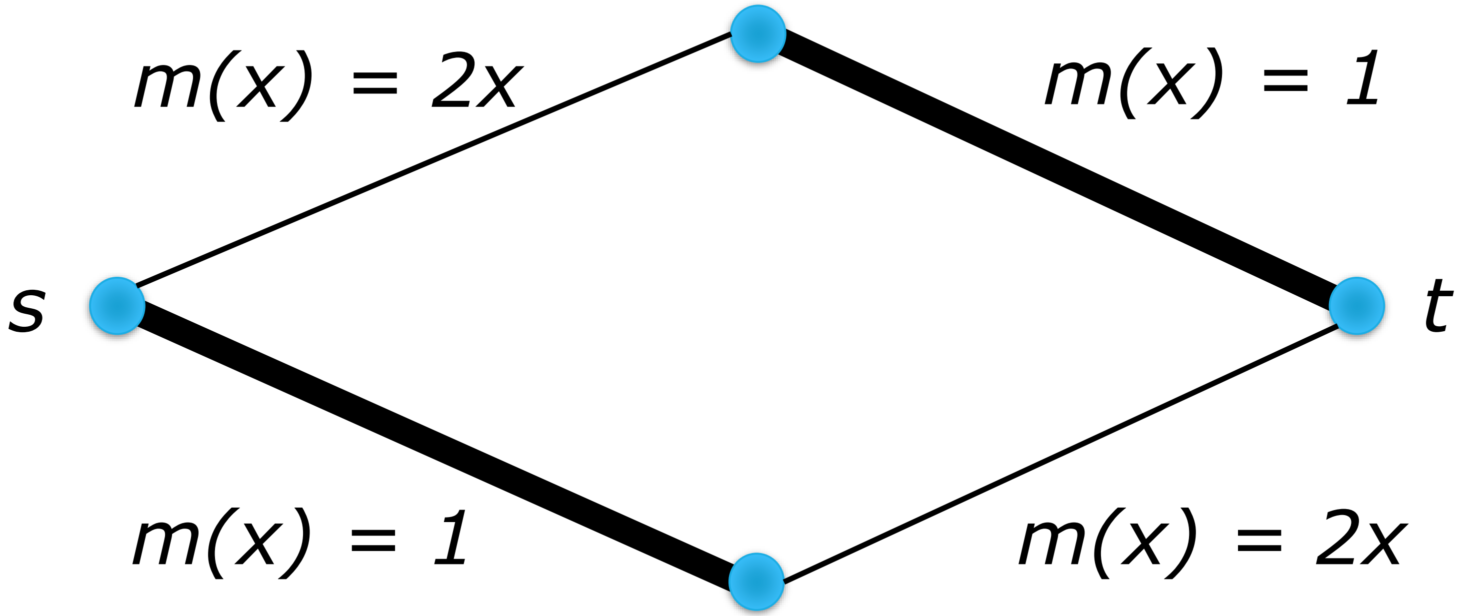
(you charge the burden that a car causes to the others as part of its own cost, and then you let it minimize its cost. Since the function is convex, a local optimum is also global)

COMPUTING THE PARETO OPTIMUM



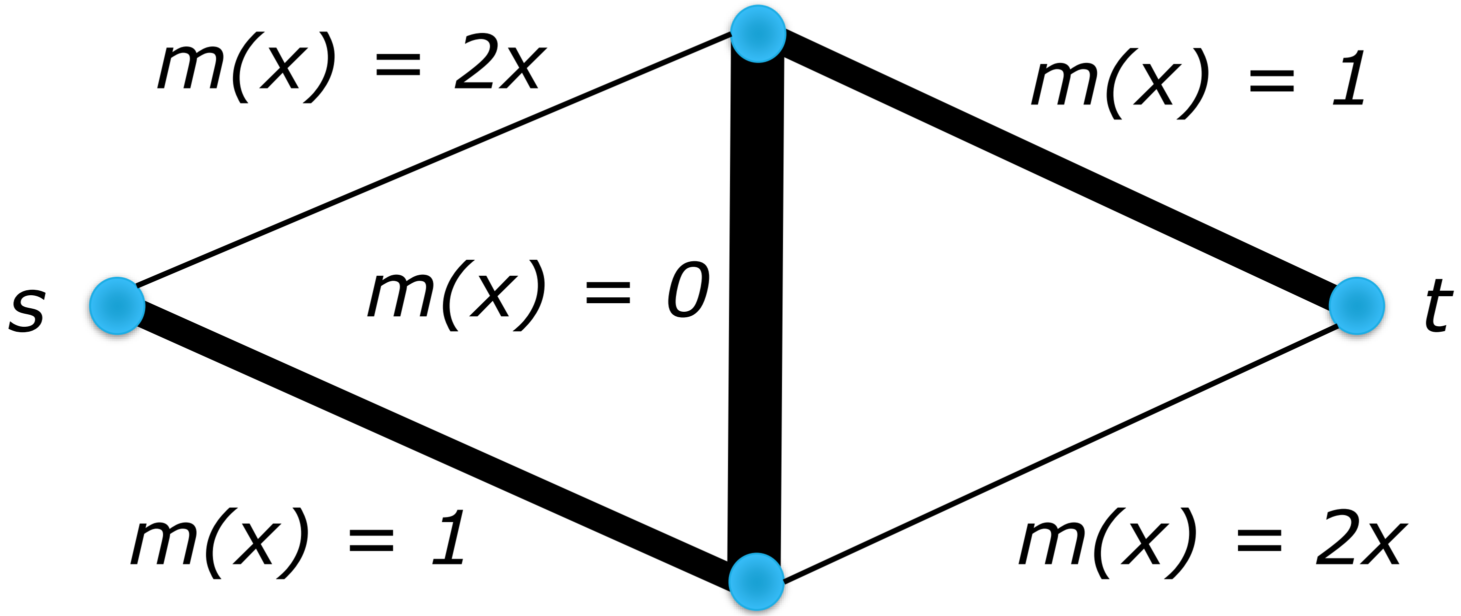
- In the Nash Equilibrium of the marginal-cost network, we have $x_1 = x_2 = ½$ as a solution to the equation system
- The traffic splits half and half

COMPUTING THE NASH EQUILIBRIUM



- x_1 cars use the upper way, and pay $1 + 2x_1$ each
- x_2 cars use the lower way, and pay $1 + 2x_2$ each
- Equations are $1 + 2x_1 = 1 + 2x_2$ and $x_1 + x_2 = 1$
  - Solution is $x_1 = x_2 = 0{,}5$, so also in the PO the traffic splits

COMPUTING THE NASH EQUILIBRIUM



- x_1 pay $2(x_1 + x_2) + 1$
- x_2 pay $2(x_1 + x_2) + 0 + 2(x_2 + x_4)$
- x_3 pay $1 + 0 + 1$
- x_4 pay $1 + 2(x_2 + x_4)$
- $x_1 + x_2 + x_3 + x_4 = 1$

- Solution $x_1 = x_4 = ½$, $x_2, x_3 = 0$

COMPUTING THE PRICE OF ANARCHY

1. Compute the flow in the Nash Equilibrium of the network
2. Compute the total cost of this flow

3. Compute the flow in the Nash Equilibrium of the marginal-cost network
4. Compute the total cost of this flow in the original network

5. Divide the two costs

HOW BAD IS SELFISH ROUTING?

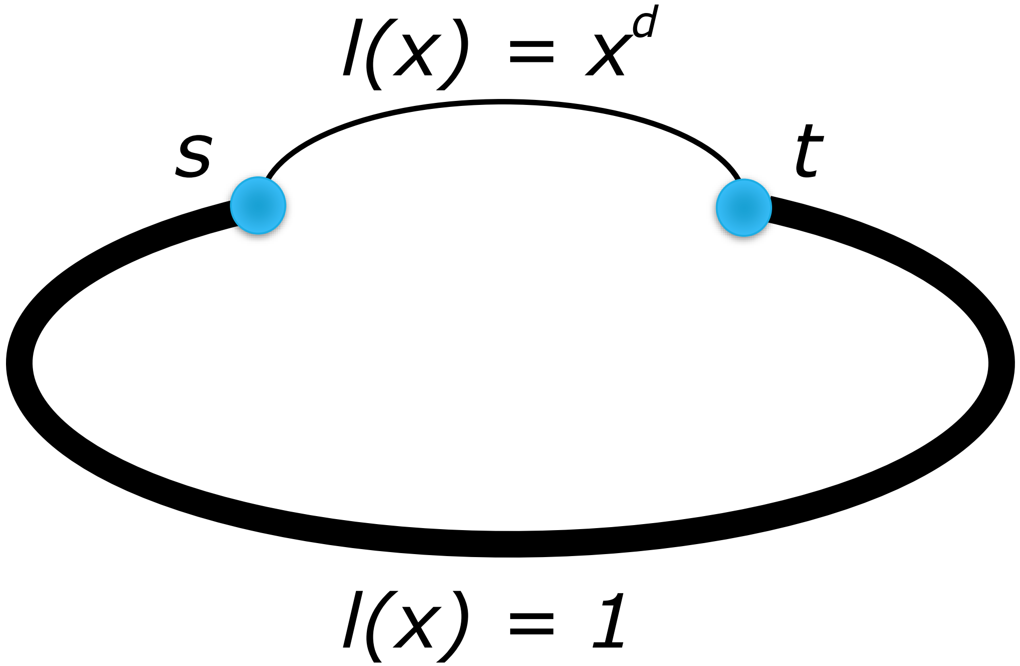
- **Theorem**: if each latency function is linear, then the price of anarchy is at most 4/3
  - Pigou and Bräß examples match the worst case
  - Latency $l(e)= a_e x + b_e$ and marginal cost $m(e) = 2a_e x + b_e$

- **Theorem**: if each latency function is a polynomius of degree d, then the price of anarchy is at most

$$[1 - d \cdot (d+1)^{-(d+1)/d}]^{-1}$$

  - $d \to \infty$ *implies PoA* $\to \infty$

HOW BAD IS SELFISH ROUTING?

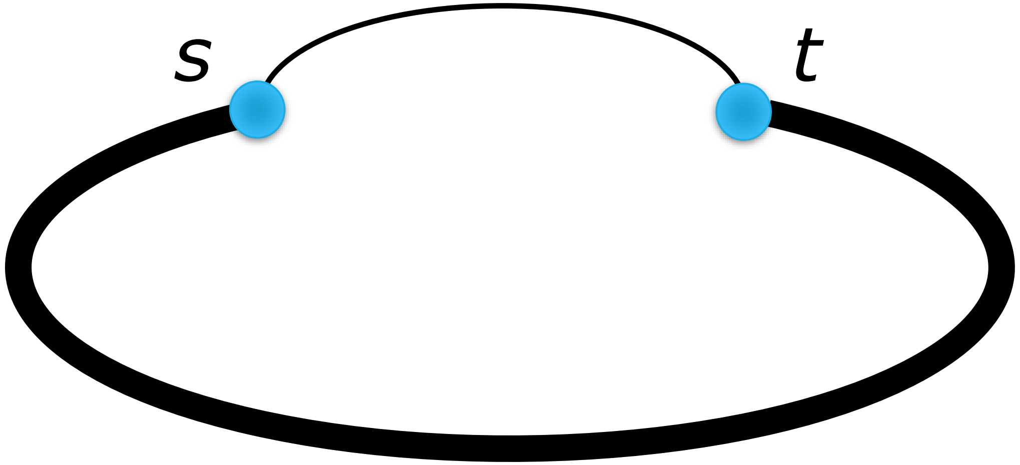


In a Nash Equilibrium, all cars use the upper path

The total cost of NE is 1

HOW BAD IS SELFISH ROUTING?

$m(x) = x^d + x\,d\,x^{d-1} = (d+1)\,x^d$



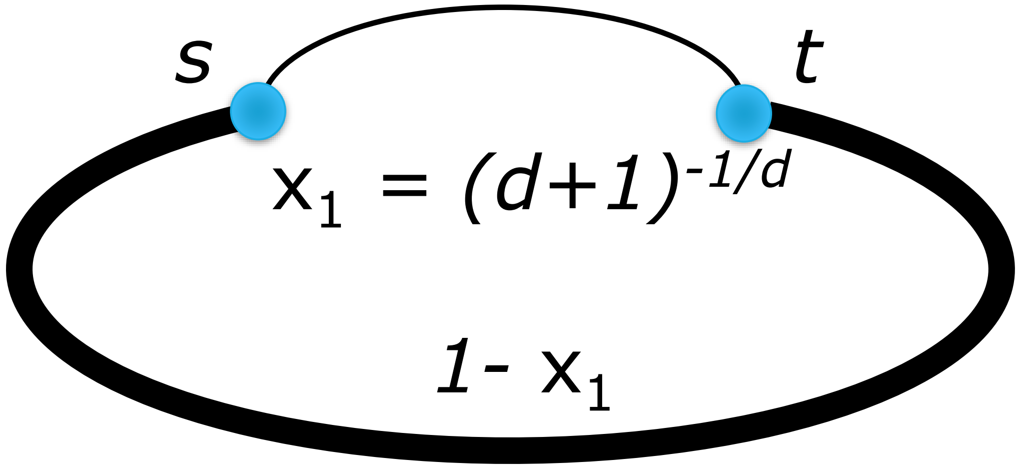
$m(x) = 1$

In a Nash Equilibrium of the marginal-cost network, $x_1 = (d+1)^{-1/d}$ cars use the upper path and

$x_2 = 1 - x_1$ use the lower

HOW BAD IS SELFISH ROUTING?

$m(x) = x^d + x\, d\, x^{d-1} = (d+1)\, x^d$



$x_1 = (d+1)^{-1/d}$

$1 - x_1$

$m(x) = 1$

Total cost: $x_1 * x_1^d + (1 - x_1) * 1 = 1 - d(d+1)^{-(d+1)/d}$

d -> ∞ implies cost -> 0 and thus PoA -> ∞

HOW BAD IS SELFISH ROUTING?

- **Theorem**: if each latency function is continuous and non-decreasing, then the total latency in a Nash equilibrium is at most the same as the optimal latency in the setting in which the flow between every pair (source, target) is doubled
  - To solve traffic issues, just make all roads larger!

LITERATURE

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