Acoustic solitons in waveguides with Helmholtz resonators: transmission line approach

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We report experimental results and study theoretically soliton formation and propagation in an air-filled acoustic waveguide side loaded with Helmholtz resonators. Our theoretical approach relies on a transmission-line description of this setting, which leads to a nonlinear dynamical lattice model. The latter is treated analytically, by means of dynamical systems and multiscale expansion techniques, and leads to various soliton solutions for the pressure. These include Boussinesq-like and Korteweg-de Vries pulse-shaped solitons, as well as nonlinear Schrödinger envelope solitons. The analytical predictions are in excellent agreement with direct numerical simulations and in qualitative agreement with the experimental observations.

I. INTRODUCTION

Solitons, namely robust localized waves propagating undistorted in nonlinear dispersive media [1–3], have been studied extensively in various physical contexts. Indeed, soliton formation, stability, dynamics and interactions have been analyzed, both in theory and in experiments, in water waves [4, 5], plasma physics [5], nonlinear optics [6], atomic Bose-Einstein condensation (BEC) [7], and so on.

On the other hand, solitons have also been studied in acoustics, both in solids and fluids [8]. In particular, solitary nonlinear waves have been the subject of many studies the last years in granular chains [9] and crystalline solids (see Ref. [10] and references therein). In the latter case, solitons and solitary waves in crystals and their surfaces have been attained by nanosecond and picosecond laser ultrasonics methods. However, solitons in fluids have been studied less extensively: in fact, pertinent studies include seminal work by Sugimoto and co-workers, who studied theoretically [11–13] and demonstrated experimentally [12, 13] propagation of one-dimensional (1D) acoustic solitary waves in an air-filled waveguide, with a periodic array of Helmholtz resonators. In these works, the analysis was based on nonlinear wave equations with fractional derivative terms accounting for losses. For this model, soliton solutions were found in an implicit form, and turned out to be close to Korteweg-de Vries (KdV) solitons in some asymptotic limit; additionally, numerical studies on the model proposed in Refs. [11–13] were recently reported too [14]. Other relevant works include Refs. [15], where diffusive soliton solutions to the so-called Kuznetsov equation (which models weakly nonlinear acoustic wave propagation in viscoelastic media) were studied. Note that traveling wave solutions of a higher-order nonlinear acoustic wave equation of the Kuznetsov-type (valid for larger values of acoustic Mach number) were rigorously studied as well [16]. It is also relevant to mention the work of Ref. [17], where envelope solitons (holes) were predicted to occur in cylindrical acoustic waveguides (in this system, higher-order dispersive modes were taken into account).

In this work, motivated by studies on nonlinear effects of acoustic wave propagation in a lattice made of Helmholtz resonators side connected to a tube [18], and – more importantly – by recent observations of acoustic solitons in this setting that we report below, we revisit this theme. In particular, we propose to use a transmission line (TL) approach in order to study analytically soliton propagation in a Helmholtz resonators lattice, compare with pertinent experimental results, and also predict other types of solitons (namely envelope solitons of the bright and dark type) that can be supported in this setting.

Generally, the TL approach, is a powerful tool commonly used in electromagnetic (EM) wave applications [19], and has recently gained considerable attention due to its applicability in the analysis and design of both EM [20] and acoustic [21] metamaterials. This approach also allows for the study of nonlinear effects, and particularly soliton formation and propagation, a theme that has been studied extensively in the past in the context of electrical TLs [2], and more recently in the realm of TL metamaterials [22]. As we explain in more detail below, the TL approach can also be directly used for similar studies, related to soliton formation and propagation, in the setting under consideration, namely the 1D lattice of Helmholtz resonators.

Our analysis relies on the study of an electrical TL, as per the electro-acoustic analogy, where the voltage corresponds to the acoustic pressure, and the current to the volume velocity flowing through the waveguide's cross-sectional area [23]. In addition, nonlinear effects are taken into regard by incorporating nonlinear elements in the unit-cell circuit, accounting for the dependence of wave celerity on the pressure (note that Helmholtz resonators are assumed to have a linear response, while nonlinearity originates only for the large-amplitude wave propagation within the waveguide). This representation allows for the derivation of a nonlinear lattice model, which is studied numerically and analytically. In the numerical simulations, using initial conditions relevant to our experiments, we are able to reproduce soliton profiles and characteristics (speed, width, etc) in a good agreement with the experimental observations. Furthermore, employing the continuum approximation, we study analytically the lattice model, and show that it is intimately related (in proper temporal and spatial scales) to models that have been studied in the past in other branches of physics: these include a Boussinesq-type model and a KdV equation (originally used to describe shallow water waves [3, 4], waves in plasmas [5], solitons in
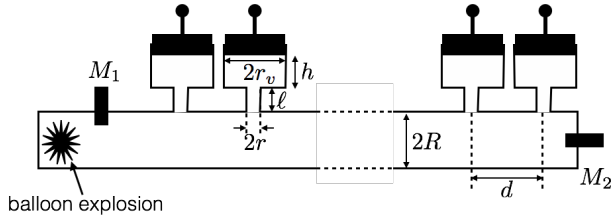
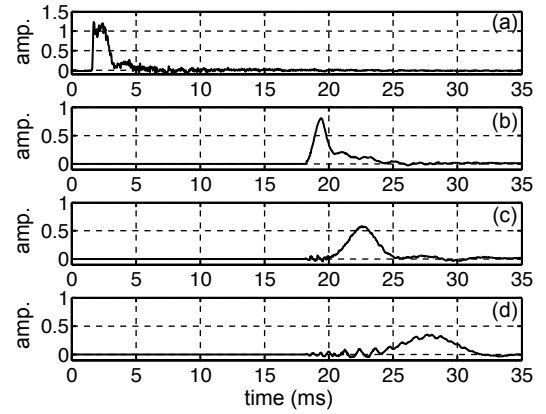FIG. 1: Schematic illustration of the experimental setup.

FIG. 2: Panel (a) shows the initial acoustic pressure, measured at $x = 0$ m. Panels (b), (c) and (d) show, respectively, the acoustic pressure measured at the end of the lattice ($x = 6.2$ m) for resonator cavity length $h = 0.02$ m, $h = 0.07$ m, and $h = 0.165$ m.

electrical TLs [2], etc), as well as a nonlinear Schrödinger (NLS) equation (describing deep water waves [3, 4], optical solitons [1–3, 6], dynamics of BEC [7], etc). This way, we derive approximate pulse-like solitons of the Boussinesq and KdV type, as well as bright and dark envelope solitons satisfying an effective NLS equation. In all cases, we identify parameter regimes where different types of solitons can be formed, and present numerical results that are found to be in excellent agreement with the analytical predictions.

The paper is structured as follows. In Section II, we describe the experimental setup and present experimental results for the formation of acoustic solitons in the 1D lattice of Helmholtz resonators. We also introduce our model and, by employing the TL approach, derive the nonlinear lattice equation and compare numerical findings for the latter with relevant experimental results. Section III is devoted to our analytical study: there, we present the various types of solitons that can be formed in our setting, identify relevant parameter regimes and spatio-temporal scales, and investigate their propagation characteristics. Finally, in Section IV we present our conclusions and discuss future research directions.

II. THE HELMHOLTZ RESONATOR LATTICE

A. Experimental setup and observations

We start by presenting our experimental setup, which consists of a long cylindrical waveguide, of length $L = 6$ m, with an inner radius $R = 25 \times 10^{-3}$ m and a 5×10^{-3} m thick wall. This waveguide is connected to an array of 60 Helmholtz resonators, which are periodically distributed. The distance between two consecutive resonators is $d = 0.1$ m. Each resonator is composed by a neck (cylindrical tube with an inner radius $r = 10 \times 10^{-3}$ m and a length $\ell = 20 \times 10^{-3}$ m) and a variable length cavity (cylindrical tube with an inner radius $r_v = 21.5 \times 10^{-3}$ m and a maximum length $h = 165 \times 10^{-3}$ m). Notice that the end of the waveguide, located at $d/2$ from the last resonator, is rigidly closed.

The input signal is generated by the explosion of a balloon. The balloon is located at 20 cm of the lattice into a waveguide connected to the main tube and is inflated until its explosion. The produced acoustic wave is measured with 2 PCB $106B$ microphones, carefully calibrated, which are located 20 cm in front of the lattice and at the end of lattice (the microphone is embedded in the rigid end); recall that the propagation distance is $L = 6.2$ m. The experimental setup is shown in Fig. 1.

Figure 2(a) shows the temporal profiles of the normalized acoustic pressure measured at the first microphone located 20 cm before the first resonator ($x = 0$ m). The input signal, generated by the balloon explosion, can be described by a gate-signal with a large amplitude (around 30 kPa) and a width around 1.5 ms. Figures 2(b), 2(c) and 2(d) present the temporal profiles of the acoustic pressure measured after a 6 m propagation in the Helmholtz resonators lattice ($x = 6.2$ m) for the cases of $h = 0.02$ m, $h = 0.07$ m and $h = 0.165$ m respectively. Oppositely to the case of a waveguide without resonators where a shock wave is formed [13, 24], we observe the propagation of a wave with a smooth shape through the lattice. The characteristics of this wave, namely shape, amplitude and velocity, are strongly dependent on the cavity length of the resonators, which defines the dispersion characteristics of the lattice (see Sec. III.B). As it is seen, for $h = 0.07$ m and $h = 0.165$ m, the wave shape is clearly symmetrical, while for $h = 0.02$ m this is not the case. Generally, it is observed that the competition between nonlinearities (due to a cumulative effect occurring for large amplitude pulse input) and dispersion in the medium (due to the presence of Helmholtz resonators) produces waves of constant shape, with amplitude dependent velocity, which are in fact acoustic solitons (note that we use the term "soliton" in a loose sense, without implying complete integrability [3]).

B. The discrete model: transmission line approach

Next, in order to model our system and provide theoretical results for the above experimental observations of acoustic solitons, we will employ the TL approach. Our starting point relies on the consideration of an ideal fluid, and use of the fluid-dynamic equations, neglecting viscosity and other dissipative terms. If we restrict our analysis to 1D flow – as in the case of the experimental results of Fig. 2 – wave propagation
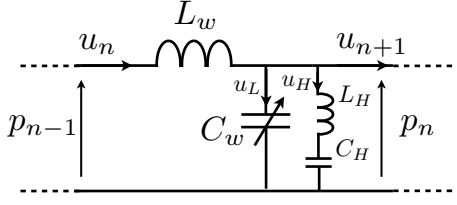
FIG. 3: The unit-cell circuit of the nonlinear Helmholtz lattice model.

is described by the following equations:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho v) = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -\frac{1}{\varrho}\frac{\partial p}{\partial x}, \tag{2}$$

where $\varrho(p, s)$ is the fluid mass density, s is the entropy, v is the acoustic fluid velocity and p is the acoustic pressure. We assume that the entropy s is constant, while the mass density ϱ and wave celerity $c \equiv (\partial \varrho / \partial p)^{1/2}$ are considered as functions of the total pressure p. Accordingly the acoustic fluid velocity v can can be written as a single-valued function of the pressure p so that $\partial v / \partial t = (dv/dp)\, \partial p / \partial t$. We wish to model the acoustic propagation along the waveguide in the low frequency regime, where only plane waves can propagate, by means of the electro-acoustic analogy [23]. Considering the long-wavelength limit, the mass conservation and Euler's equations (1)-(2) between two points separated by dx (much smaller than the acoustic wavelength) can be approximated as:

$$u_n = \frac{S dx}{\varrho c^2}\frac{\partial p_{n+1}}{\partial t} + u_{n+1} \tag{3}$$

$$p_n = \frac{\varrho dx}{S}\frac{\partial u_{n+1}}{\partial t} + p_{n+1} \tag{4}$$

where S is the waveguide cross-section, u is the acoustic volume velocity, and the subscripts n and $n + 1$ are related, respectively, to left and right side of the tube at some point dx. According to the electrical analogy, the propagation along a unit-cell with length dx can be modelled by a simple electrical circuit for the "current" u_n and the "voltage" p_n, consisting of an inductance L_w and a capacitance C_w, given by:

$$L_w = \varrho dx/S, \quad C_w = S dx/\varrho c^2. \tag{5}$$

For our analysis below, we will assume that the inductance L_w is linear while the capacitance C_w is nonlinear, depending on the pressure p; this choice, relies on the approximation that (to a first order) the density does not depend on p, while the celerity c depends on p –see below.

In order to model the experimental setup that incorporates the Helmholtz resonators, we will include an additional parallel branch in the unit-cell circuit, composed by a serial combination of an inductance L_H and a capacitance C_H, as shown in Fig. 3. We consider the response of the Helmholtz resonators to be linear. Nonlinearity originates only from the large amplitude acoustic propagation within the waveguide.

Thus, in the low frequency approximation, the relevant inductance and capacitance are given by $L_H = \varrho \ell / S_n$ and $C_H = V_H / \varrho c^2$, respectively, where ℓ, S_n and V_H are the length and the cross-sectional area of the resonator neck, and the total volume of the resonator cavity, respectively. Notice that, by including a resonator in each unit-cell, it is natural to set $dx = d$ (recall that d is the distance between two successive resonators).

Using the unit-cell circuit of Fig. 3, we can now use Kirchhoff's voltage and current laws and derive an evolution equation for the pressure p_n in the n-th cell of the lattice. Let us first consider the Kirchhoff's voltage law for two successive cells, which yields:

$$p_{n-1} - p_n = L_w \frac{d}{dt}u_n, \tag{6}$$

$$p_n - p_{n+1} = L_w \frac{d}{dt}u_{n+1}. \tag{7}$$

Subtracting the above equations, we obtain the difference equation:

$$\hat{\delta}^2 p_n = \hat{\mathcal{L}}(u_n - u_{n+1}), \tag{8}$$

where $\hat{\delta}^2 p_n \equiv p_{n+1} - 2p_n + p_{n-1}$ and $\hat{\mathcal{L}} \equiv L_w d/dt$. On the other hand, Kirchhoff's current law yields:

$$u_n - u_{n+1} = \frac{d}{dt}(C_w p_n) + \hat{P}^{-1}\frac{dp_n}{dt}, \tag{9}$$

where the first and second terms in the right-hand side denote the currents across the capacitance C_w and the Helmholtz branch, respectively, with $\hat{P}^{-1}$ being the inverse of the operator $\hat{P} \equiv L_H d^2/dt^2 + 1/C_H$.

Substituting Eq. (9) into Eq. (8), we obtain the following equation for the pressure p_n:

$$L_w C_H \frac{d^2 p_n}{dt^2} - \left(1 + L_H C_H \frac{d^2}{dt^2}\right)\hat{\delta}^2 p_n$$
$$+ L_w \frac{d^2}{dt^2}\left(1 + L_H C_H \frac{d^2}{dt^2}\right)(C_w p_n) = 0, \tag{10}$$

where it is reminded that the capacitance C_w depends on the pressure. In order to quantify this dependence, we first note that one may use Eqs. (1)-(2) and, for sufficiently small pressures, approximate the wave celerity $c = c(p)$ as [23]:

$$c \approx c_0(1 + \beta_0 p/\varrho_0 c_0^2), \tag{11}$$

where $c_0 = 343.26$ m/s is the speed of sound at room temperature, ϱ_0 is the density evaluated at the equilibrium state, and $\beta_0 = 1.2$ for the case of air. Then, for density $\varrho = \varrho_0$, the second of Eqs. (5) leads to the following pressure-dependent capacitance C_w:

$$C_w(p_n) \approx C_{w0} + C_w' p_n, \tag{12}$$

where $C_{w0} = Sd/\varrho_0 c_0^2$ is a constant capacitance (relevant to the linear case) and $C_w' = -2\frac{\beta_0}{\varrho_0 c_0^2}C_{w0}$. Substituting Eq. (12)

into Eq. (10), we obtain the equation:

$$\frac{d^2 p_n}{dt^2} - \frac{c_0^2}{\kappa d^2}\left(1 + \frac{1}{\omega_0^2}\frac{d^2}{dt^2}\right)\hat{\delta}^2 p_n$$
$$+ \frac{1}{\kappa}\frac{d^2}{dt^2}\left(1 + \frac{1}{\omega_0^2}\frac{d^2}{dt^2}\right)\left[p_n\left(1 - 2\frac{\beta_0}{\varrho_0 c_0^2}p_n\right)\right] = 0, \quad (13)$$

where $\omega_0 = c_0\sqrt{S_n/S_v h\ell}$ is the Helmholtz resonance frequency ($S_n = \pi r^2$ and $S_v = \pi r_v^2$ are the cross-sectional areas of the resonator neck and cavity, respectively), $\kappa = V_H/V$ is a geometrical factor (ratio of the volume of the Helmholtz resonator V_H over the tube volume V in a unit cell of length d, and we have used the following equations connecting the transmission line parameters with the acoustic waveguide characteristics:

$$L_w C_H = \frac{\kappa}{d^2 c_0^2}, \quad L_H C_H = \frac{1}{\omega_0^2}, \quad L_w C_{w0} = \frac{d^2}{c_0^2}, \quad (14)$$

where L_w is also evaluated at $\varrho = \varrho_0$. The above nonlinear dynamical lattice equation is one of the main results of the present work: it describes the propagation of acoustic waves in a tube with an array of Helmholtz resonators. This simplified model will be used below in order to derive analytical solitary wave solutions that are supported in this setting –as is also evident from the experimental results shown in Fig. 2.

C. Comparison with the experiment

Let us now proceed by comparing results that can be derived in the framework of the lattice model (13) with the experimental results presented above.

We numerically integrate Eq. (13) by means of a 4th-order Runge-Kutta method, using an initial condition similar to the experiments, as shown in the top panel of Fig. 2. In particular, we use a super-Gaussian pulse of the form

$$p_{n=0}(t) = A\exp\left[-\left((t - t_0)/w_0\right)^{16}\right], \quad (15)$$

of amplitude $A = 30$ kPa and width $w_0/2 = 400$ Hz. The values of the coefficients of the various terms of Eq. (13) depend actually only on the cavity length h, since all other parameters are fixed.

The results of the direct numerical simulations, corresponding to the three different cavity lengths used in the experiment ($h = 0.02, 0.07, 0.165$ m), are shown in Fig. 4. In all cases shown in panels (b)-(d) of Fig. 4, the profile of each pulse is shown for the lattice cite $n = 60$, corresponding to a distance 6 m from the first resonator. The profiles in panels (b)-(d) are time shifted by $\Delta t \approx 0.9$ ms corresponding to the propagating time needed for the initial pulse to reach the first resonator; this is done to facilitate direct comparison with the experimental results of Fig. 2.

Comparing corresponding panels of Figs. 2 and 4 for each of the three different values of h, it is seen that the solitary waves obtained numerically via Eq. (13) have approximately the same width as those observed in the experiment. Notice

that quantitative differences between numerical and experimental soliton amplitudes, as well as the presence of "tails" attached to the solitons (which are absent in the experimental data), may be qualitatively understood by (i) the presence of losses in the experiment [which are not included in the simplified model of Eq. (13)], and (ii) the fact that the initial conditions used in the experiment and simulations are different.

In any case, the above comparison shows that Eq. (13) can be used to describe, in a fairly good agreement with the experiment, the formation of acoustic solitary waves. Below we will show that, using this simplified model, we can obtain analytically different types of acoustic solitons in different experimentally relevant regimes.

III. ACOUSTIC SOLITONS

A. The continuum approximation

For our analytical considerations, we will focus on the continuum limit of Eq. (13), corresponding to $n \to \infty$ and $d \to 0$ (but with nd being finite); in such a case, the pressure becomes $p_n(t) \to p(x, t)$, where $x = nd$ is a continuous variable. Then, the difference operator $\hat{\delta}^2$ is approximated by $\hat{\delta}^2 p_n \approx d^2 p_{xx}$, where terms of the order $\mathcal{O}(d^4)$ and higher are neglected, and subscripts denote partial derivatives. It is also convenient to express our model in dimensionless form; this can be done upon introducing the normalized variables χ and τ and normalized pressure P [of order $\mathcal{O}(1)$], which are defined as follows:

$$\tau = \tilde{\omega}_0 t, \qquad \chi = \frac{\tilde{\omega}_0}{c_0\sqrt{\alpha}}x, \qquad \frac{p}{p_0} = \epsilon P, \quad (16)$$

where $\tilde{\omega}_0$ is a characteristic spectral width or inverse temporal width (which is set by the initial condition), $p_0 = \varrho_0 c_0^2/2\beta_0$, $\alpha = 1/(1 + \kappa)$, and ϵ is a dimensionless small parameter ($\epsilon \ll 1$), defining the strength of the nonlinearity. In these variables, the continuum limit of Eq. (13) reads:

$$P_{\tau\tau} - P_{\chi\chi} - \Omega^2(P_{\chi\chi\tau\tau} - \alpha P_{\tau\tau\tau\tau})$$
$$- \epsilon\alpha\left[(P^2)_{\tau\tau} + \Omega^2(P^2)_{\tau\tau\tau\tau}\right] = 0, \quad (17)$$

where $\Omega = \tilde{\omega}_0/\omega_0$. Equation (17) is a Boussinesq-like model, which has been originally proposed for studies of solitons in shallow water [3, 4], but later was used in studies of solitons in different contexts, including electrical TLs [2]. In our case, the dispersion terms of Eq. (17) are due to discreteness and the presence of Helmholtz resonators, and their strength is measured by the dimensionless parameter Ω. The strength of the nonlinear terms, on the other hand, is set by the parameter ϵ. Notice that in the absence of the Helmholtz resonators, i.e., for $\omega_0 \to \infty$ and $\kappa = 0$ (i.e., $\Omega = 0$ and $\alpha = 1$), Eq. (17) is reduced to the well-known Westervelt equation, which is a common nonlinear model describing 1D acoustic wave propagation [25].
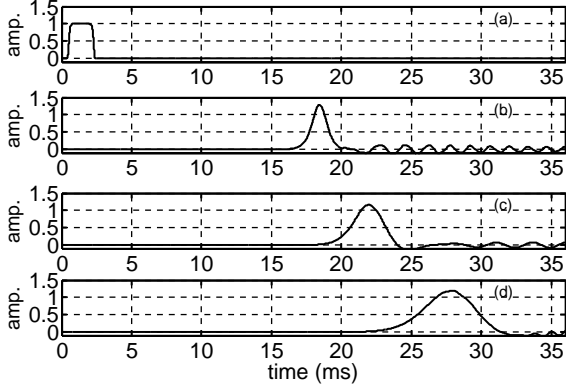
FIG. 4: Top panel: Initial condition p_0 used for the numerical integration of Eq. (13). Rest of the panels (time shifted by 0.9 ms –see text) show the pressure at $n = 60$ as a function of time, for the same value of the cavity length as in the respective experimental data (cf. Fig. 2).

B. Linear theory

We start by considering the linear limit of Eq. (17) and the respective dispersion relation. Note that in the limit of $\epsilon \to 0$, Eq. (17) is reduced to the linear wave equation (in the lossless case) studied in Ref. [26] (see Eq. (61) of this work).

Assuming propagation of plane waves in the lattice, of the form $P \propto \exp[i(k\chi - \omega\tau)]$, we obtain the following dispersion relation connecting the wavenumber k and frequency ω:

$$D(\omega, k) \equiv k^2 - \omega^2 - \Omega^2(k^2\omega^2 - \alpha\omega^4) = 0. \quad (18)$$

In Fig. 5 the frequency $f = \omega/2\pi$ is plotted as a function of the normalized wavenumber kd for the three different values of the Helmholtz resonator cavity length h used in the experiment; the result is depicted by the dashed (black) line. On the other hand, the solid (green) line in the same figure shows the respective dispersion relation obtained using Bloch theory and the transfer matrix method [26] :

$$\cos(kd) = \cos\left(\frac{\omega}{c_0}d\right) + i\frac{Z_0}{2Z_b}\sin\left(\frac{\omega}{c_0}d\right), \quad (19)$$

where Z_b is the input impedance of the Helmholtz resonator branch, and $Z_0 = \varrho_0 c_0/S$ the acoustic characteristic impedance of the waveguide; for the lossless case $Z_b = i(\omega L_H - 1/\omega C_H)$. It is clear that the model of Eq. (17) is in very good agreement with the numerical results, especially in the regime of low frequencies, where the transmission line approach is expected to be more accurate. From left to right the three panels of Fig. 5 show the dispersion relation for $h = 0.02$ m, $h = 0.07$ m and $h = 0.165$ m. It is readily seen that increasing the Helmholtz cavity length, the resonance frequency decreases, and additionally the dispersion in the low frequency regime increases.

Thus, it is reasonable to expect that different types of solitons may be obtained in different regimes of the dispersion relation, by exploiting the relative strength between dispersion and nonlinearity. A relevant study is appended in the following sections.
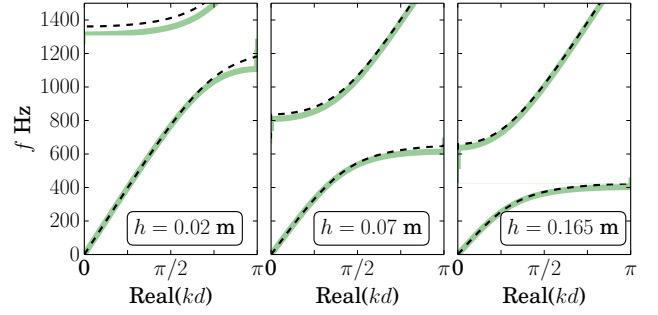

FIG. 5: (Color online) The dispersion relation as obtained via Eq. (19) [solid (green) line] compared to the result of Eq. (18) [dashed (black) line] for three different values of the Helmholtz resonator cavity length h.

C. Boussinesq and KdV pulse-like solitons

First we focus on the regime where the dispersion and nonlinearity terms of Eq. (17) are of the same order, i.e., $\epsilon \sim \Omega^2$. Given that we have already assumed a weak nonlinearity, it is obvious that the last term in the left-hand side of Eq. (17), which is $\propto \epsilon\Omega^2$ can be neglected. In such a case, Eq. (17) is reduced to the following equation:

$$P_{\tau\tau} - P_{\chi\chi} - \Omega^2(P_{\chi\chi\tau\tau} - \alpha P_{\tau\tau\tau\tau}) - \epsilon\alpha(P^2)_{\tau\tau} = 0, \quad (20)$$

which is actually a combination of the so-called bad and improved Boussinesq equation (see, e.g., Ref. [27] for the definition and discussion of these models). Travelling wave solutions of the above equation can readily be obtained by introducing the ansatz $P(\chi, \tau) = \Phi(\xi)$, where $\xi = \delta(\tau - \chi/v)$, while v and δ denote the velocity and inverse width of the wave. Then, assuming vanishing boundary conditions for Φ, namely $\Phi \to 0$ as $|\xi| \to \infty$, we derive from Eq. (20) the following ordinary differential equation (ODE) for $\Phi(\xi)$:

$$A\Phi'' + B\Phi - \epsilon\alpha\Phi^2 = 0, \quad (21)$$

where primes denote differentiation with respect to ξ, while $A = \Omega^2\left(\alpha - 1/v^2\right)$ and $B = 1 - 1/v^2$. Equation (21) can be seen as an equation of motion of a particle in the presence of the potential $V(\Phi) = (B/2A)\Phi^2 - (\epsilon\alpha/3A)\Phi^3$. A straightforward analysis shows that the only physically relevant solution, with the correct (vanishing) boundary conditions, corresponds to a homoclinic orbit, for $A < 0$, $B > 0$, relevant to the hyperbolic fixed point $\Phi = 3B/2\epsilon\alpha$. This solution reads:

$$P(\chi, \tau) = \left(\frac{\Omega^2}{\epsilon}\right)\left(\frac{6\kappa\delta^2}{1 + 4\delta^2\Omega^2}\right)\text{sech}^2\left[\delta\left(\tau - \frac{\chi}{v}\right)\right], (22)$$

where the velocity is given by $v = [(1 + 4\delta^2\Omega^2)/(1 + 4\alpha\delta^2\Omega^2)]^{1/2}$. Obviously, the above solution is characterized by one free parameter, the inverse width δ. Note that since $\Omega^2/\epsilon \sim 1$ (as per our assumption above), the free parameter δ is also ~ 1. Thus, the normalized pressure P, along with its spectral width, are of the order of unity as well. Using
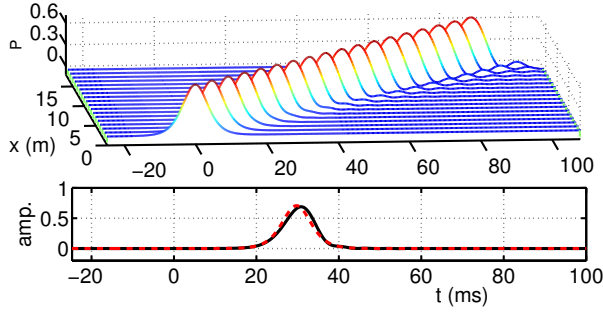
FIG. 6: Top panel: 3D plot depicting the evolution of a soliton of the form of Eq. (22), obtained by numerically integrating Eq. (13) for a distance corresponding to 200 sites (physical distance $x = 20$ m. The bottom panel shows the temporal profile of the normalized pressure, p/p_0, at the site $n = 60$. Parameter values correspond to the experimental ones, for a Helmholtz resonator with cavity length $h = 0.07$ m. The dashed (red) line in the bottom panel depicts the analytical result of Eq. (22), while the solid (black) line the result of the simulation.

Eq. (22), we can express –for the sake of clarity– the corresponding approximate solution of Eq. (13) in terms of the original space and time coordinates as follows:

$$\frac{p(x,t)}{p_0} \approx \frac{3\kappa\delta^2(\tilde{\omega}_0/\omega_0)^2}{1 + 4\delta^2(\tilde{\omega}_0/\omega_0)^2} \operatorname{sech}^2\left[\delta\tilde{\omega}_0\left(t - \frac{x}{v}\right)\right]. \quad (23)$$

Notice that, in physical units, the velocity of the soliton reads:

$$v = c_0\sqrt{\alpha}\sqrt{\frac{\omega_0^2 + 4\delta^2\tilde{\omega}_0^2}{\omega_0^2 + 4\alpha\delta^2\tilde{\omega}_0^2}}, \quad (24)$$

and is bounded (as follows from the requirements $A < 0$ and $B > 0$ mentioned above) according to:

$$c_0\sqrt{\alpha} < v < c_0. \quad (25)$$

This shows that the velocity of the Boussinesq-type soliton of Eq. (23) is lower than the speed of sound (i.e., the soliton is subsonic), in accordance with the analysis of Ref. [13] for small geometrical factor κ [see Eq. (2.14) of this work].

We have numerically integrated the nonlinear lattice model of Eq. (13), using as an initial condition, p_1 (i.e., the pressure at the first site of the lattice), the functional form of the soliton of Eq. (23) at $x = 0$; we have used the parameter values $\delta\tilde{\omega}_0 = 0.1$, and a cavity length $h = 0.07$ m. The results of our simulations are shown in Fig. 6. The top panel shows a three-dimensional (3D) plot depicting the evolution of the pressure p, while the bottom panel shows the temporal profile of the pressure at the lattice site $n = 60$, corresponding to a physical distance $x = 6$ m. It is observed that the soliton propagates for about 20 m with almost no distortion. In fact, the only noticeable effect is a small amount of radiation emitted by the soliton during its evolution (cf. the structure formed at the leading edge of the pulse); this effect can naturally be attributed to the fact that Eq. (23) is nothing but an approximate solution –derived in the continuum limit– of the lattice model of Eq. (13). Nevertheless, as is also shown in the bottom panel
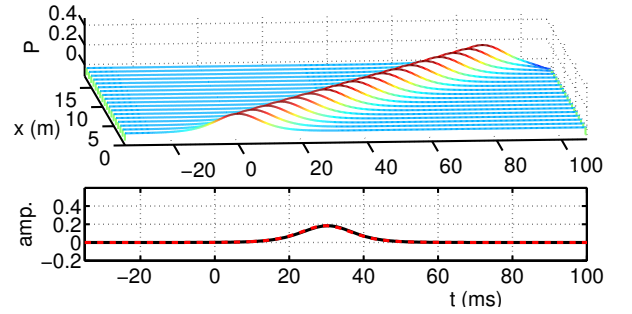

FIG. 7: Same as in Fig. 6, but for an initial condition corresponding to a KdV soliton of the form of Eq. (29).

of Fig. 6, our analytical approximation is very good –at least for propagation distances up to 40 m: indeed, the analytical result [dashed (red) line] for the soliton profile (at $x = 6$ m) in the bottom panel of the figure, almost coincides with the corresponding numerical result [solid (black) line].

For longer propagation distances ($x \gtrsim 40$ m), however, the continuous emission of radiation of the Boussinesq-type solitons eventually lead to their disintegration. More robust soliton solutions –in the same parametric region– can be obtained upon considering the long-wavelength, far-field limit of the Boussinesq-type Eq. (17), which is the KdV equation. Indeed, using a formal multiscale expansion method, we can reduce Eq. (17) to a KdV equation, and use the latter to derive approximate solutions of Eq. (13). We thus proceed upon using the slow variables:

$$T = \epsilon^{1/2}(\tau - \chi), \qquad X = \epsilon^{3/2}\chi, \quad (26)$$

and express Eq. (17) as follows:

$$2\epsilon^2 P_{XT} - \epsilon^3 P_{XX} - \Omega^2\left(\epsilon^2 P_{TTTT} - 2\epsilon^3 P_{XX} + \epsilon^4 P_{XX}\right.$$
$$\left. -\alpha\epsilon^2 P_{TTTT}\right) - \epsilon^2\alpha\left[(P^2)_{TT} + \epsilon(P^2)_{TTTT}\right] = 0. \quad (27)$$

Next, introducing the expansion $P = P_1 + \epsilon P_2 + \cdots$, and integrating Eq. (27) once in T, at order $\mathcal{O}(\epsilon^2)$ we obtain the following KdV equation for P_1:

$$P_{1X} - \frac{\Omega^2}{2}(1 - \alpha)P_{1TTT} - \alpha P_1 P_{1T} = 0. \quad (28)$$

To this end, using the soliton solution of Eq. (28) for P_1, namely $P_1 = 6\kappa\Omega^2\operatorname{sech}^2(T - X/V)$ (where $V^{-1} = 2\Omega^2\kappa\alpha$), we can write the approximate KdV soliton solution for $p(x,t)$ as follows:

$$\frac{p(x,t)}{p_0} \approx 3\epsilon\kappa(\tilde{\omega}_0/\omega_0)^2 \operatorname{sech}^2\left[\sqrt{\epsilon}\tilde{\omega}_0\left(t - \frac{x}{v}\right)\right], \quad (29)$$

where the velocity of the KdV soliton is given by

$$v \approx c_0\sqrt{a}(1 + 2\epsilon\Omega^2\kappa\alpha). \quad (30)$$

It is observed that the amplitude of the normalized pressure p/p_0 is now of order $\epsilon\Omega^2$ and, thus, KdV solitons are of smaller amplitude than the Boussinesq-type solitons [cf.

Eq. (22)]. In fact, the KdV soliton (29) can be obtained as the small-amplitude limit of Eq. (22), corresponding to $\delta = \sqrt{\epsilon} \ll 1$ (and, accordingly, the velocity (24) is reduced to (30) in the same limit).

The evolution of the small-amplitude KdV soliton was also studied numerically: in Fig. 7 we show the result of a direct numerical simulation, for the same parameters as in Fig. (6), where the initial condition (at the first site as before) for Eq. (13) was the KdV soliton (29) at $x = 0$, with an amplitude $\epsilon \Omega^2 = 0.05$. It is observed that the KdV soliton is much more robust, and no noticeable emission of radiation occurs; this is natural as, in this case, the KdV Eq. (28) is the long-wavelength far-field limit of Eq. (13) as mentioned above. The analytical result for the temporal soliton profile (cf. bottom panel of the figure) is found to be in excellent agreement with the numerically obtained solution. Notice that the KdV solitons were found to be robust for propagation distances of the order of 60 m (which was the distance used in the simulations).

It is interesting to compare the above approximate KdV soliton solution with the corresponding solution discussed in Refs. [11–13]. In both cases, the soliton amplitude is analogous to the square root of the soliton inverse width, and also analogous to the geometrical factor κ. Additionally, both in our case and in Refs. [11–13], the KdV solitons were obtained in the same asymptotic limit of small amplitude and large width.

We complete this subsection by noting the following: if the initial condition for Eq. (13) is fixed (i.e., the spectral width $\tilde{\omega}_0$ and amplitude are fixed) then the soliton amplitude will also be fixed. Nevertheless, if the cavity length h is increased then the soliton width $w = (\delta \tilde{\omega}_0)^{-1}$ [cf. Eq. (23)] is also increased (this occurs for both the Boussinesq-like and KdV solitons). This theoretical prediction – which is based on our analytical approximations – is in accordance with the numerical and experimental results shown in Figs. 4 and 2, respectively; for the latter, however, the presence of dissipation results – additionally – in unequal soliton amplitudes.

D. NLS envelope solitons

Our analytical approach allows us to predict still another type of soliton solutions, namely *envelope solitons* of the bright and dark type [6], that can be supported in the acoustic waveguide structure under consideration. In particular, in this Section we will show that such solitons can be found as approximate solutions of the nonlinear evolution equation (17). Our methodology relies on the use of the multiple scales perturbation method [28], by means of which Eq. (17) is reduced to an effective NLS equation; then, employing the latter, we identify parameter regimes envelope bright or dark acoustic solitons can be formed in our setting.

We start our analysis by introducing the slow variables

$$\chi_n = \epsilon^n \chi, \quad \tau_n = \epsilon^n \tau, \quad n = 0, 1, 2, \ldots \quad (31)$$

where parameter ϵ is the one appearing in Eq. (17), and will again be treated as a formal small parameter; furthermore, we express P as an asymptotic series in ϵ:

$$P = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \ldots, \quad (32)$$

where the unknown real functions P_n ($n = 0, 1, 2, \ldots$) depend on the variables (31). Then, substituting Eq. (32) into Eq. (17), and using Eq. (31), we obtain a hierarchy of equations at various orders in ϵ (see Appendix A).

In particular, at the leading order, i.e., at $\mathcal{O}(1)$, the resulting equation [cf. Eq. (A1) in Appendix A] corresponds to the linear limit of Eq. (17); this equation possesses plane wave solutions of the form:

$$\begin{aligned} P_0(\tau_0, \chi_0, \tau_1, \chi_1, \ldots) &= \Phi(\tau_1, \chi_1, \tau_2, \chi_2, \ldots) \\ &\times \exp\left[i\theta(\tau_0, \chi_0)\right] + \text{c.c.}, \end{aligned} \quad (33)$$

where Φ is the unknown envelope function of P_0, the phase $\theta(\tau_0, \chi_0)$ is given by $\theta(\tau_0, \chi_0) = k\chi_0 - \omega\tau_0$, while k and ω satisfy the linear dispersion relation –cf. Eq. (18).

Next, at the order $\mathcal{O}(\epsilon)$, the solvability condition for the corresponding equation [cf. Eq. (A2) in Appendix A] is $\tilde{L}_1 P_0 = 0$; this condition is nothing but the requirement that the secular part (which is in resonance with $\tilde{L}_0 P_1$) vanishes. This condition yields the following equation:

$$\left(k' \frac{\partial}{\partial \tau_1} - \frac{\partial}{\partial \chi_1} \right) \Phi(\chi_1, \tau_1, \ldots) = 0, \quad (34)$$

where $k' \equiv \partial k / \partial \omega$ is the inverse group velocity. Equation (34) is satisfied as long as Φ depends on the variables χ_1 and τ_1 through the traveling-wave coordinate $\tilde{\tau}_1 = \tau_1 + k'\chi_1$ (i.e., Φ travels with the group velocity), namely $\Phi(\chi_1, \tau_1, \ldots) = \Phi(\tilde{\tau}_1, \chi_2, \tau_2, \ldots)$. Additionally, at the same order, we obtain the form of the field P_1, namely :

$$P_1 = -\frac{4\alpha\omega^2(1 - 4\Omega^2\omega^2)}{D(2\omega, 2k)} \Phi^2(\tilde{\tau}_1)e^{2i\theta} + Be^{i\theta} + \text{c.c.}, \quad (35)$$

where B is an unknown function that can in principle be found at a higher-order approximation.

Finally, following a similar procedure as above, and using the functional forms of Φ and P_1, the non-secularity condition of the equation at the order $\mathcal{O}(\epsilon^2)$ [cf. Eq. (A3) in Appendix A], yields a NLS equation for the envelope function Φ:

$$i\frac{\partial \Phi}{\partial \chi_2} - \frac{1}{2}k''\frac{\partial^2 \Phi}{\partial \tilde{\tau}_1^2} + q|\Phi|^2\Phi = 0, \quad (36)$$

where the dispersion and nonlinearity coefficients are respectively given by:

$$\begin{aligned} k'' &\equiv \frac{\partial^2 k}{\partial \omega^2} \\ &= \frac{1 - k'^2(1 - \Omega^2\omega^2) + \Omega^2(k^2 - 6\Omega^2\omega^2 - 4\Omega^2\omega kk')}{k(1 - \Omega^2\omega^2)}, \end{aligned} \quad (37)$$

$$q(\omega, k) = \frac{\alpha^2(1 - \Omega^2\omega^2)(1 - 4\Omega^2\omega^2)}{3k\Omega^2(1 - \alpha)}. \quad (38)$$
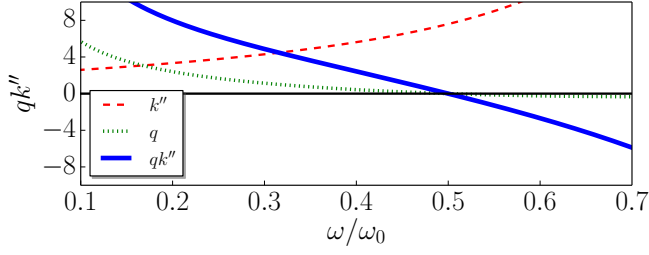
FIG. 8: (Color online) The solid line shows the frequency dependence of the product qk'' of the dispersion and nonlinearity coefficients of the NLS Eq. (36). Dashed and dotted lines shows the frequency dependence of k'' and q respectively. Parameter values correspond to the experimental ones, for a Helmholtz resonator with cavity length $h = 0.07$ m.

Importantly, the sign of the product $\sigma \equiv \mathrm{sgn}(qk'')$, determines the nature of the NLS equation, focusing ($\sigma = +1$) or defocusing ($\sigma = -1$) and, hence, the type of the soliton – bright soliton and dark soliton, respectively [6]. In Fig. 8 we show an example of the dependence of the product qk'' with respect to the frequency ω, corresponding to a Helmholtz resonator of a cavity length $h = 0.07$ m. As seen in the figure, there exist two different regimes: the low (high) frequency regime where $\sigma = +1$ ($\sigma = -1$) where bright (dark) solitons can be formed.

First we consider the low frequency regime of Fig. 8, where the NLS Eq. (36) is focusing and supports an exact analytical bright soliton solution of the form $\Phi = (\eta/\sqrt{q})\mathrm{sech}(\eta/\sqrt{|k''|}\tilde{\tau}_1)\exp[i(\eta^2/2)\chi_2]$. This expression leads to an approximate bright soliton solution of Eq. (17), which is written in terms of coordinates χ and τ as follows:

$$
\begin{aligned}
P \approx\ & \frac{2\eta}{\sqrt{q}}\mathrm{sech}\left[\frac{\epsilon\eta}{\sqrt{|k''|}}(\tau + k'\chi)\right] \\
& \times\ \cos\left[\omega\tau - \left(k - \frac{\epsilon^2\eta^2}{2}\right)\chi\right].
\end{aligned} \tag{39}
$$

In terms of the original space and time coordinates, the approximate envelope soliton solution for the pressure p is the following:

$$
\begin{aligned}
\frac{p(x,t)}{p_0} \approx\ & \frac{2\epsilon\eta}{\sqrt{q}}\mathrm{sech}\left[\frac{\epsilon\eta}{\sqrt{|k''|}}(t + \frac{k'}{c_0\sqrt{\alpha}}\chi)\right] \\
& \times\ \cos\left[\tilde{\omega}_0 t - \left(\frac{k - (\epsilon^2\eta^2)/2}{c_0\sqrt{\alpha}}\right)x\right]
\end{aligned} \tag{40}
$$

where parameters q, k', and k'', for a given frequency $\tilde{\omega}_0$, are found by using the dispersion relation in the original coordinates.

Next, we consider the high frequency regime of Fig. 8, where the NLS Eq. (36) is defocusing and admits a dark soliton solution of the form $\Phi = \sqrt{\Phi_0}\tanh[\sqrt{\Phi_0/|k''|}\tilde{\tau}_1]\exp(-i\Phi_0\chi_2)$. In this case, the cor-
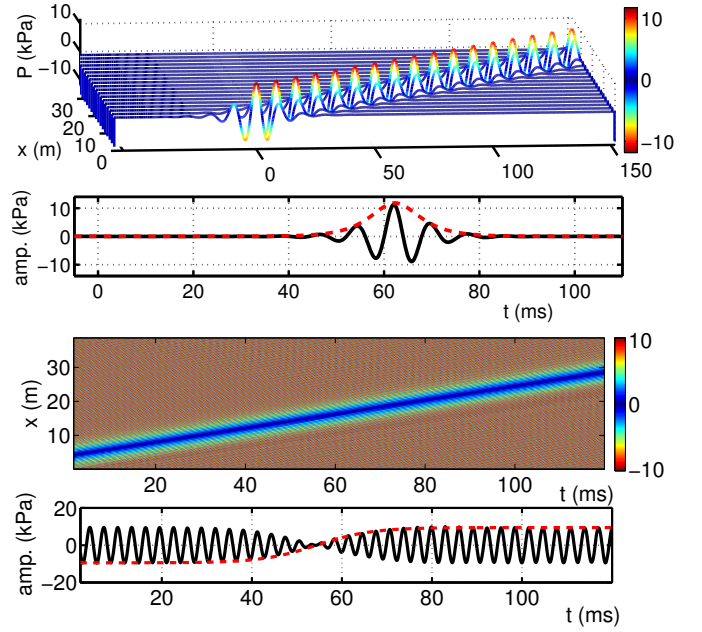

FIG. 9: (Color online) Top panels (bottom panels): Same as in Fig. 6 but for an initial condition corresponding to a bright (dark) NLS soliton of the form of Eq. (40) [Eq. (42)]. In the second (fourth) panel, the red dashed line depicts the analytical result for the sech- (tanh-)shaped envelope of Eq. (40) [Eq. (42)]. Parameter values correspond to the experimental ones, for a Helmholtz resonator with cavity length $h = 0.07$ m.

responding approximate solution of Eq. (17) reads:

$$
\begin{aligned}
P \approx\ & 2\sqrt{\frac{\Phi_0}{q}}\tanh\left[\sqrt{\frac{\Phi_0}{|k''|}}\epsilon(\tau + k'\chi)\right] \\
& \times\ \cos\left[\omega\tau - (k + \epsilon^2\Phi_0)\chi\right].
\end{aligned} \tag{41}
$$

Accordingly, the approximate dark envelope soliton solution for the pressure p in the original coordinates is given by:

$$
\begin{aligned}
\frac{p(x,t)}{p_0} \approx\ & 2\epsilon\sqrt{\frac{\Phi_0}{q}}\tanh\left[\sqrt{\frac{\Phi_0}{|k''|}}\epsilon\tilde{\omega}_0\left(t + \frac{k'}{c_0\sqrt{\alpha}}x\right)\right] \\
& \times\ \cos\left[\tilde{\omega}_0 t - \left(\frac{k + \epsilon^2\Phi_0}{c_0\sqrt{\alpha}}\right)x\right].
\end{aligned} \tag{42}
$$

Note that both the bright and the dark solitons travel with the group velocity $1/k'$ (evaluated at the frequency $\tilde{\omega}_0$).

Our analytical predictions for the existence of bright and dark solitons in the acoustic waveguide structure at hand were also compared to direct numerical simulations. As in the case of the previous soliton types, we numerically integrated the nonlinear lattice model of Eq. (13) using as initial conditions (at the first lattice site, $n = 0$) the functional forms of the envelope solitons (40) and (42) at $x = 0$. The results are shown in Fig. 9, where the two top (bottom) panels correspond to the bright (dark) soliton, respectively. We have used the following parameter values: $\tilde{\omega}_0 = 0.2\omega_0$ and amplitude $\epsilon\eta = 0.2$ for the bright soliton, $\tilde{\omega}_0 = 0.55\omega_0$ and $\epsilon\sqrt{\Phi_0} = 0.2$

for the dark soliton. In the first and third panels, we show a 3D and a contour plot showing the evolution of these two envelope soliton types, while in the second and fourth panels we show the temporal profiles of the bright and dark soliton at the site $n = 200$ (or $x = 20$ m in physical units). It is observed that the agreement between the numerical results [solid (black) line] obtained in the framework of Eq. (13) and the analytical results [dashed (red) lines depicting the envelopes of the two solitons] is excellent.

IV. CONCLUSIONS AND DISCUSSION

In conclusion, we presented experimental results showing the formation of acoustic pulse-like solitons in an air-filled quasi-1D tube with Helmholtz resonators. Additionally, we proposed a transmission line (TL) approach to theoretically study our observations. Our model, which relied on the electro-acoustic analogy, was a nonlinear dynamical lattice; the latter was analyzed by both numerical and analytical techniques.

Our numerical simulations produced results that were in qualitative agreement with the experimental findings. On the analytical side, we considered the continuum limit of the lattice model, and showed – by means of dynamical systems and multiscale expansion methods – that it can be reduced to celebrated soliton equations, namely a Boussinesq-type model, a KdV and a NLS equation. Such reductions allowed us to: (i) identify parameter regimes and appropriate spatial and temporal scales where different types of solitons can be formed, and (ii) derive various soliton solutions in a closed analytical form. In all cases, the analytical predictions were in excellent agreement with direct simulations and in qualitative agreement with the experimental observations.

In this study, our analytical approximation was simplified, due to the fact that our model did not take into account inherent losses in the system. This simplification, however, allowed us to: (a) provide analytical forms of acoustic solitons in the Helmholtz resonator lattice that were not available before (recall that soliton solutions of Refs. [11–13] were presented in an implicit form, and in an explicit form only in some asymptotic limits for the lossless case), and (b) predict envelope solitons in the setting under consideration (only dark envelope solitons were previously predicted to occur in cylindrical acoustic waveguide structures [17]). Furthermore, our analytical approximation provides a clear physical picture for the properties of solitons in various parameter regimes and can, in principle, be used for other studies (thanks to the flexibility of our experimental setting) – such as soliton collisions, soliton-defect interactions, soliton propagation in disordered lattices, and so on.

There are many future research directions that may follow this work. On the experimental side, the realization of envelope solitons and a systematic study of their properties is a particularly interesting theme. On the theoretical side, one could incorporate nonlinear elements in the parallel branch of the resonators and losses in the model and then use various asymptotic and perturbative techniques to capture the propagation properties of solitons, also quantitatively. Another interesting direction is the study of soliton formation and propagation in other waveguide structures, proposed or used in the context of acoustic metamaterials. In the same spirit, it would also be particularly challenging to extend our methodology to higher-dimensional settings. Pertinent studies are currently in progress and results will be reported in future publications.

Appendix A: Perturbation equations

Here we present the hierarchy of equations in ϵ, resulting from the substitution of Eq. (32) into Eq. (17). More specifically, at the orders $\mathcal{O}(1)$, $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$, we respectively obtain the following equatios:

$$\tilde{L}_0 P_0 = 0, \tag{A1}$$

$$\tilde{L}_0 P_1 + \tilde{L}_1 P_0 = \tilde{N}_0[P_0^2], \tag{A2}$$

$$\tilde{L}_0 P_2 + \tilde{L}_1 P_1 + \tilde{L}_2 P_0 = \tilde{N}_0[2P_0 P_1] + \tilde{N}_1[P_0^2]. \tag{A3}$$

The linear operators $\tilde{L}_0$, $\tilde{L}_1$ and $\tilde{L}_2$, as well as the nonlinear operators $\tilde{N}_0[P]$, $\tilde{N}_1[P]$ are given by:

$$\tilde{L}_0 = \frac{\partial^2}{\partial \tau_0^2} - \frac{\partial^2}{\partial \chi_0^2} - \Omega^2 \left(\frac{\partial^4}{\partial \tau_0^2 \partial \chi_0^2} - \alpha \frac{\partial^4}{\partial \tau_0^4} \right), \tag{A4}$$

$$\tilde{L}_1 = 2\frac{\partial^2}{\partial \tau_0 \partial \tau_1} - 2\frac{\partial^2}{\partial \chi_0 \partial \chi_1} - \Omega^2 \Big(2\frac{\partial^4}{\partial \tau_0^2 \partial \chi_0 \partial \chi_1} + 2\frac{\partial^4}{\partial \chi_0^2 \partial \tau_0 \partial \tau_1} - 4\alpha \frac{\partial^4}{\partial \tau_0^3 \partial \tau_1} \Big), \tag{A5}$$

$$\tilde{L}_2 = \frac{\partial^2}{\partial \tau_1^2} - \frac{\partial^2}{\partial \chi_1^2} + 2\frac{\partial^2}{\partial \tau_0 \partial \tau_2} - 2\frac{\partial^2}{\partial \chi_0 \partial \chi_2}$$
$$- \Omega^2 \left[\frac{\partial^4}{\partial \tau_0^2 \partial \chi_1^2} + \frac{\partial^4}{\partial \tau_1^2 \partial \chi_0^2} + 4\frac{\partial^4}{\partial \tau_0 \partial \chi_0 \partial \tau_1 \partial \chi_1} \right.$$
$$+ \frac{2\partial^4}{\partial \tau_0^2 \partial \chi_0 \partial \chi_2} + 2\frac{\partial^4}{\partial \chi_0^2 \partial \tau_0 \partial \tau_2}$$
$$\left. - \alpha \left(6\frac{\partial^4}{\partial \tau_0^2 \partial \tau_1^2} + 4\frac{\partial^4}{\partial \tau_0^3 \partial \tau_2} \right) \right], \tag{A6}$$

$$\tilde{N}_0[P] = \alpha \left[\frac{\partial^2 (P)}{\partial \tau_0^2} + \Omega^2 \frac{\partial^4 (P)}{\partial \tau_0^4} \right], \tag{A7}$$

$$\tilde{N}_1[P] = \alpha \left[2\frac{\partial^2 (P)}{\partial \tau_0 \partial \tau_1} + 4\frac{\partial^4 (P)}{\partial \tau_0^3 \partial \tau_1} \right]. \tag{A8}$$

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