物理学院
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Floquet Phase-1

聂嘉会

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1 Floquet formalism

Floquet formalism or Floquet theory is constructed within a time-periodic driven quantum system, and is partly similar to Bloch theory for a spatial periodic system. Consider a single-particle form Hamiltonian

$$\mathcal{H}(t) = \mathcal{H}(t+T), \tag{1}$$

with period T. The discrete time translation symmetry implies a time translation operator $U_F := U(T)$ (Floquet operator), whose basis of irreducible group representation are the so-called Floquet states or the **quasienergy** states $|\varepsilon\rangle$ with $U(T)|\varepsilon\rangle = e^{-i\varepsilon T}|\varepsilon\rangle$. This is an analogue to Bloch theory, where $U(T) \leftrightarrow \mathcal{T}, |\varepsilon\rangle \leftrightarrow |\mathbf{k}\rangle, \mathcal{T}|\mathbf{k}\rangle = e^{-ika}|\mathbf{k}\rangle$. We can also define the quasienergy Brillouin zone (or the Floquet-Brillouin zone): $[-\frac{\pi}{T}, \frac{\pi}{T})$, then the whole time axis can be mapped into a circle S^1. For a general Floquet quantum material with both discrete spatial and time translation symmetry, there are two good quantum numbers ε and k, so the Floquet spectrum is usually a spectrum drown in 1-st BZ $\times [-\frac{\pi}{T}, \frac{\pi}{T}) =:$ FBZ or a torus $\mathbb{T}^2 = \mathbb{C}/\text{FBZ}$, with eigenstates labled as $|\varepsilon_n(\mathbf{k})\rangle$.

The Floquet operator can be written down directly in the form of a usual time evolution operator. We define that

$$U_F := U(T) = \mathrm{T}\exp\Big[-\frac{i}{\hbar}\int_0^T dt\mathcal{H}(t)\Big] =: \exp\Big[-\frac{i}{\hbar}T\mathcal{H}_\epsilon^{\text{eff}}\Big], \tag{2}$$

where we set a dimensionless parameter $\epsilon = \varepsilon T$ and $\mathcal{H}_\epsilon^{\text{eff}} = \frac{i\hbar}{T}\ln_{-\epsilon} U_F$ is the effective Hamiltonian over one period with eigenvalues $\hbar\varepsilon$. Here, it is conventional to set the branch cut of the logarithm as:

$$\ln_{-\epsilon} e^{i\phi} = i\phi,\ -\varepsilon T - 2\pi < \phi < -\varepsilon T, \tag{3}$$

namely,

$$\ln_{-\epsilon} e^{-i\varepsilon T + i0^-} = -i\varepsilon T,\ \ln_{-\epsilon} e^{-i\varepsilon T + i0^+} = -i\varepsilon T - 2\pi i. \tag{4}$$

If the Floquet operator can be decomposed into

$$U_F = \sum_n \lambda_n(\mathbf{k})|\varepsilon_n(\mathbf{k})\rangle\langle\varepsilon_n(\mathbf{k})|, \tag{5}$$

then the effective Hamiltonian is just

$$\mathcal{H}_\epsilon^{\text{eff}} = \frac{i\hbar}{T}\sum_n \ln_{-\epsilon}(\lambda_n(\mathbf{k}))|\varepsilon_n(\mathbf{k})\rangle\langle\varepsilon_n(\mathbf{k})|. \tag{6}$$

The Floquet formalism can provide us the stroboscopic dynamic of a system, where it behaves like a static system with effective Hamiltonian $\mathcal{H}_\epsilon^{\text{eff}}$ whose eigenvalues are the quasienergies. Such a stroboscopic picture often gives us a anomalous band structure and spectrum, where the compare of the driven energy and the bandwidth and the gap width as well becomes a crucial factor.

2 Floquet SSH model

Let's consider a periodic driven SSH model. The Hamiltonian is written as:

$$\mathcal{H}(t) = \sum_i v(t) c_{i,A}^\dagger c_{i,B} + w(t) c_{i,A}^\dagger c_{i+1,B} + \text{h.c.}, \tag{7}$$

where

$$\begin{cases} v(t) = 1 - \theta(t - t_1) \\ w(t) = \theta(t - t_1) \end{cases} \quad (0 < t, t_1 < T),\ v(t+T) = v(t), w(t+T) = w(t). \tag{8}$$

Such Hamiltonian describes a periodic driven dynamic with period T in which the system hops between two phases periodically:

$$\mathcal{H}(t) = \begin{cases} \sum_i c_{i,A}^\dagger c_{i,B} + \text{h.c.}, 0 < t < t_1 \text{ (trivial)} \\ \sum_i c_{i,A}^\dagger c_{i+1,B} + \text{h.c.}, t_1 < t < T \text{ (topological)} \end{cases}. \tag{9}$$

Inside the bulk, we apply the periodic boundary condition and get the familiar result of the bulk Bloch Hamiltonian:

$$\begin{aligned} \mathcal{H}(t) &= \sum_k \psi_k^\dagger \mathcal{H}(k,t) \psi_k \\ &= \sum_k \psi_k^\dagger \begin{pmatrix} 0 & v(t) + w(t) e^{-ik} \\ v(t) + w(t) e^{ik} & 0 \end{pmatrix} \psi_k \\ &= \sum_k \psi_k^\dagger U^\dagger \begin{pmatrix} \sqrt{v^2 + 2vw\cos k + w^2} & 0 \\ 0 & -\sqrt{v^2 + 2vw\cos k + w^2} \end{pmatrix} U \psi_k, \end{aligned} \tag{10}$$

where the unitary matrix $U = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} & 1 \\ e^{i\phi} & -1 \end{pmatrix}$, $e^{i\phi} = \frac{v + we^{ik}}{\sqrt{v^2 + 2vw\cos k + w^2}}$.

The Floquet operator in k-space can be calculated as follow:

$$
\begin{aligned}
U_F &= \exp\Big[-\frac{i}{\hbar}\int_0^T dt\mathcal{H}(k,t)\Big] \\
&= \exp\Big[-\frac{i}{\hbar}\Big(\int_0^{t_1} dt\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+\int_{t_1}^{t_1+t_2} dt\begin{pmatrix}0 & e^{-ik}\\ e^{ik} & 0\end{pmatrix}\Big)\Big] \\
&= \exp\Big[-\frac{i}{\hbar}\begin{pmatrix}0 & t_1+t_2e^{-ik}\\ t_1+t_2e^{ik} & 0\end{pmatrix}\Big] \\
&= \exp\Big[-\frac{i}{\hbar}V^\dagger\begin{pmatrix}\sqrt{t_1^2+2t_1t_2\cos k+t_2^2} & 0\\ 0 & -\sqrt{t_1^2+2t_1t_2\cos k+t_2^2}\end{pmatrix}V\Big] \\
&= V^\dagger\exp\Big[-\frac{i}{\hbar}\begin{pmatrix}\sqrt{t_1^2+2t_1t_2\cos k+t_2^2} & 0\\ 0 & -\sqrt{t_1^2+2t_1t_2\cos k+t_2^2}\end{pmatrix}\Big]V \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\varphi} & e^{-i\varphi}\\ 1 & -1\end{pmatrix}\begin{pmatrix}e^{-\frac{i}{\hbar}\varepsilon T} & 0\\ 0 & e^{\frac{i}{\hbar}\varepsilon T}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\varphi} & 1\\ e^{i\varphi} & -1\end{pmatrix} \\
&= \begin{pmatrix}\cos\frac{\varepsilon T}{\hbar} & -ie^{-i\varphi}\sin\frac{\varepsilon T}{\hbar}\\ -ie^{i\varphi}\sin\frac{\varepsilon T}{\hbar} & \cos\frac{\varepsilon T}{\hbar}\end{pmatrix}.
\end{aligned}
\tag{11}
$$

Here $\varepsilon = \varepsilon(k) = \frac{1}{T}\sqrt{t_1^2+2t_1t_2\cos k+t_2^2}$ is the quasienergy, $e^{i\varphi} = \frac{t_1+t_2e^{ik}}{\sqrt{t_1^2+2t_1t_2\cos k+t_2^2}}$. Such Floquet operator permits a spectrum decomposition:

$$
U_F = e^{-\frac{i}{\hbar}\varepsilon T}\left|\varepsilon_+(k)\right\rangle\left\langle\varepsilon_+(k)\right| + e^{\frac{i}{\hbar}\varepsilon T}\left|\varepsilon_-(k)\right\rangle\left\langle\varepsilon_-(k)\right|, \tag{12}
$$

where $\left|\varepsilon_\pm(k)\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\varphi}\\ \pm 1\end{pmatrix}$ are two quasienergy Floquet band states. Set a dimensionless parameter $\epsilon = \varepsilon T/\hbar$. The effective Hamiltonian can also be computed:

$$
\begin{aligned}
\mathcal{H}_\epsilon^{\text{eff}}(k) &= \frac{i\hbar}{T}\ln_{-\epsilon}e^{-i\epsilon}\left|\epsilon_+(k)\right\rangle\left\langle\epsilon_+(k)\right| + \frac{i\hbar}{T}\ln_{-\epsilon}e^{i\epsilon}\left|\epsilon_-(k)\right\rangle\left\langle\epsilon_-(k)\right| \\
&= \begin{pmatrix}0 & \frac{i\hbar}{T}\ln_{-\epsilon}e^{-i\epsilon}\cdot e^{-i\varphi}\\ \frac{i\hbar}{T}\ln_{-\epsilon}e^{-i\epsilon}\cdot e^{i\varphi} & 0\end{pmatrix}.
\end{aligned}
\tag{13}
$$

We choose the branch cut $\epsilon = -\pi$, so the phase interval we focus is $-\pi < \varphi < \pi$. The effective Hamiltonian living on this branch can give us the correct Floquet band structure, which may be a mixture of conduction and valence band due to the time driven dynamic. Geometrically, the Floquet bands are actually drawn on a torus $\mathbb{T}^2 = \mathcal{S}^1\times\mathcal{S}^1$. So there are two gaps: 0-gap and π-gap, or more generally speaking the original 0-gap in static system module π. This can also be seen by chiral symmetry, for the states $|0\pm m\pi\rangle, m\in\mathbb{Z}$ are identical, which contribute to the edge states which appear periodically. The two gapless points identify the phase boundaries of our system: $|t_1-t_2| \equiv -|t_1-t_2| \mod \hbar\pi \Longrightarrow |t_1-t_2| \equiv 0 \mod \hbar\pi$ and $t_1+t_2 \equiv -(t_1+t_2) \mod \hbar\pi \Longrightarrow t_1+t_2 \equiv 0 \mod \hbar\pi$.

We can have a look of the Floquet band structure. If, for instance, luckily we have $|t_1-t_2| < \hbar\pi, t_1+t_2 < \hbar\pi$, that is both the bandwidth and 0-gap are smaller than the energy of the "driven photon" $\hbar\omega = \frac{2\hbar\pi}{T}$, then

the band structure is just like the static band structure, where the only gap can be closed is the 0-gap at gap-closing point $t_1 = t_2$. It can be plotted like Fig. 1 (the code is in Appendix A):

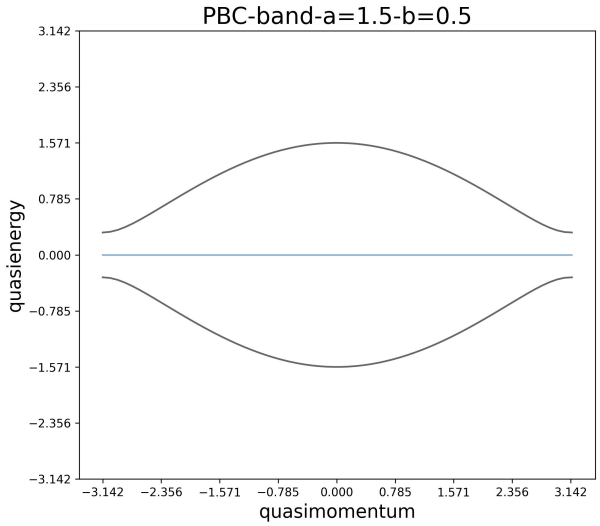


Fig. 1. The quasienergy bands for bandwidth $0.5\hbar\omega$, $vt_1/wt_2 = 1.5$.

The effective Hamiltonian is just the usual static one $\frac{1}{T}(t_1 + t_2\cos k, t_2 \sin k, 0)\cdot\boldsymbol{\sigma}$, which defined a circle in the Pauli space, the winding number can be calculated as usual:

$$\begin{aligned} C &= \frac{1}{\pi}\int_{-\pi}^{\pi} dk\,\langle\varepsilon_-(k)|i\partial_k|\varepsilon_-(k)\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dk e^{i\varphi} i\partial_k e^{-i\varphi} \\ &= \frac{i}{2\pi}\int_{-\pi}^{\pi} dk\partial_k \ln e^{-i\varphi} \\ &= -\frac{i}{2\pi}\ln e^{-i\varphi}\Big|_{-\pi}^{\pi} \\ &= \theta(t_2 - t_1). \end{aligned} \tag{14}$$

That is, if the 0-gap has once closed, then it contributes to a winding number 1, else zero.

If, however, we keep $|t_1 - t_2| < \hbar\pi$ but $\hbar\pi < t_1 + t_2 < 2\hbar\pi$, which means the bandwidth is now larger than $\hbar\omega = \frac{2\hbar\pi}{T}$, then the Floqet bands can be plotted in Fig. 2:

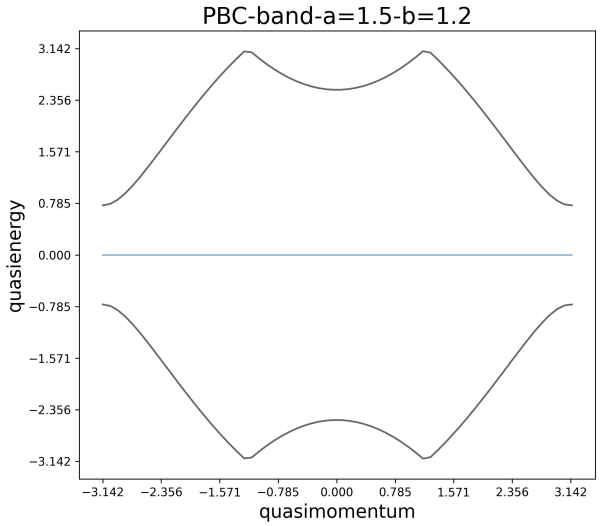
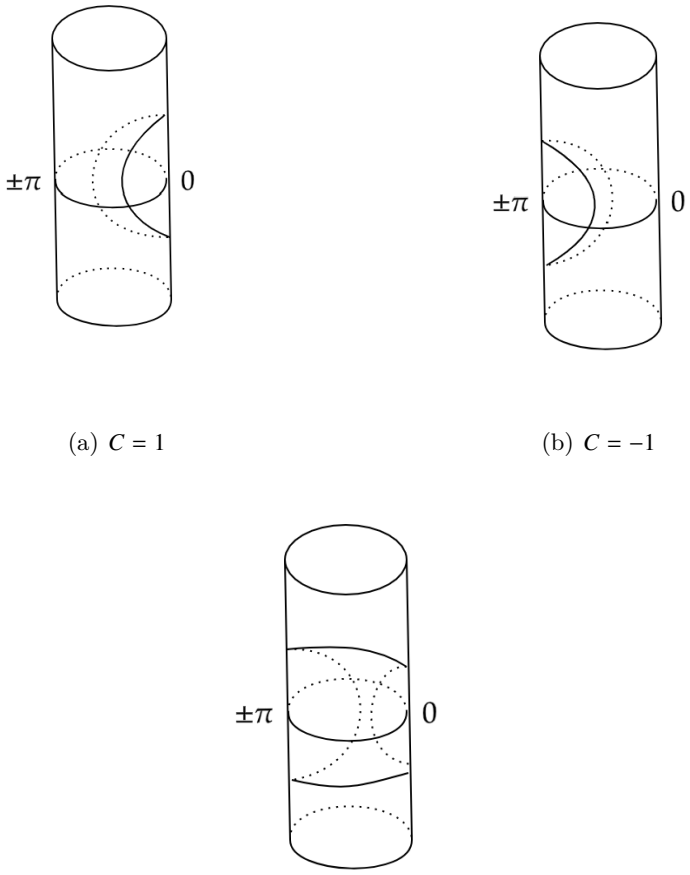


Fig. 2. The quasienergy bands for bandwidth $1.2\hbar\omega$, $vt_1/wt_2 = 1.5$.

We notice that now for quasienergies larger (smaller) than $\frac{\hbar\pi}{T}$ ($-\frac{\hbar\pi}{T}$), the branch cut of the logarithm give an additional -2π ($+2\pi$) phase, making the excess part jump to the lower (upper) part of the Floquet-Brillouin zone, or let's say the conduction band and the valence band overlap. Note that the crossings of two bands are actually gapped due to quantum perturbation. Then the effective Hamiltonian vector is deformed into $\boldsymbol{h} = (h_x, h_y, 0)$:

$$h_x = \frac{1}{T}\Big[(t_1 + t_2\cos k)\theta(\hbar\pi - \sqrt{t_1^2 + 2t_1t_2\cos k + t_2^2}) + \frac{\sqrt{t_1^2 + 2t_1t_2\cos k + t_2^2} - 2\hbar\pi}{\sqrt{t_1^2 + 2t_1t_2\cos k + t_2^2}}(t_1 + t_2\cos k)\theta(\sqrt{t_1^2 + 2t_1t_2\cos k + t_2^2} - \hbar\pi)\Big],$$
$$h_y = \frac{1}{T}\Big[(t_2\sin k)\theta(\hbar\pi - \sqrt{t_1^2 + 2t_1t_2\cos k + t_2^2}) + \frac{\sqrt{t_1^2 + 2t_1t_2\cos k + t_2^2} - 2\hbar\pi}{\sqrt{t_1^2 + 2t_1t_2\cos k + t_2^2}}(t_2\sin k)\theta(\sqrt{t_1^2 + 2t_1t_2\cos k + t_2^2} - \hbar\pi)\Big]. \tag{15}$$

Here we define the Theta function like $\theta(x) = \begin{cases} 1, x \geq 0 \\ 0, x < 0 \end{cases}$. The Hamiltonian curve turns out to be not connected! How to solve the winding number of this Hamiltonian curve? Here I think we can use an alternative way: to deform the complex plane itself rather than deform the Hamiltonian. We can cut $\mathbb{C}$ with the lines $x = \pi$ and $x = -\pi$, then stick them together, this operation deforms the 2-D plane $\mathbb{C}$ into a infinity long 2-D cylindrical surface with radius 1: $\mathbb{C} \to \ln_\pi \mathbb{C}$. Now the Hamiltonian curve is just a circle drawn on the surface of this cylinder, and the winding number can be counted obviously as usual. However, this time, we need to be careful about the direction of our loop about the origin point, for if the diameter of the circle is very large and contains the half perimeter π of the cylinder for many times, the circle will wind around the origin point for several times but each time with minus direction to the former one. In simpler words, every time the circle winds around the origin point contributes a winding number 1, while every time it winds around the point $(\pm\pi, 0)$ contributes a winding number -1. Some examples are showed in Fig. 3.



(a) $C = 1$ (b) $C = -1$

(c) $C = 0$

Fig. 3. The windings of effective Hamiltonian circle on deformed complex cylindrical surface $\ln_\pi \mathbb{C}$.

Mathematically, the winding number comes from the winding times around origin point on $\ln_\pi \mathbb{C}$, while physically, it comes from the opening and closing of the gaps. Then it can be concluded that, if we remain 0-gap open and continue to turn off and on π-gap, then if $\left\lfloor \frac{t_1+t_2}{\hbar\pi} \right\rfloor$ is even, it gives a winding number of 0, and if $\left\lfloor \frac{t_1+t_2}{\hbar\pi} \right\rfloor$ is odd, it gives a winding number of -1. That is, start from the usual static band structure in Fig. 1, turn off and on π-gap for odd (even) times, we have the winding number -1 (0). While if we remain π-gap open (this actually cannot be held true permanently, because in order to turn off 0-gap, we need to turn off π-gap before that, but we can focus on every period that π-gap is closed and reopened while 0-gap is open), if $\left\lfloor \frac{t_1-t_2}{\hbar\pi} \right\rfloor$ is even, it gives a winding number of 0, if $\left\lfloor \frac{t_1-t_2}{\hbar\pi} \right\rfloor$ is odd, it gives a winding number of 1. Like the curves plotted in Fig. 5. That is, start from the usual static band structure in Fig .1, turn off and on 0-gap for odd (even) times, we have the winding number 1 (0).

In general, if we initially set $v(t \in (0, t_1)) = v, w(t \in (t_1, T)) = w$ to satisfy a formal dimension, then we can generate two dimensionless parameters $\mathfrak{v} = vt_1/\hbar, \mathfrak{w} = wt_2/\hbar$, and the winding number may be expressed

like:

$$C = \sin\left[\frac{1}{2}\left\|\left\lfloor\frac{\mathfrak{v}-\mathfrak{w}}{\pi}\right\rfloor\right\|\pi\right] - \sin\left[\frac{1}{2}\left\lfloor\frac{\mathfrak{v}+\mathfrak{w}}{\pi}\right\rfloor\pi\right]. \tag{16}$$

Question: Analytically, we can get system's winding number by counting the winding times around origin point of the following effective Hamiltonian curve living on $\mathbb{C}$:

$$\begin{aligned} h_x &= \frac{1}{T}\sum_{m=0}^{\infty}\frac{\sqrt{\mathfrak{v}^2+2\mathfrak{v}\mathfrak{w}\cos k+\mathfrak{w}^2}-2m\pi}{\sqrt{\mathfrak{v}^2+2\mathfrak{v}\mathfrak{w}\cos k+\mathfrak{w}^2}}(\mathfrak{v}+\mathfrak{w}\cos k)\theta(\sqrt{\mathfrak{v}^2+2\mathfrak{v}\mathfrak{w}\cos k+\mathfrak{w}^2}-m\pi)\theta((m+1)\pi-\sqrt{\mathfrak{v}^2+2\mathfrak{v}\mathfrak{w}\cos k+\mathfrak{w}^2}), \\ h_y &= \frac{1}{T}\sum_{m=0}^{\infty}\frac{\sqrt{\mathfrak{v}^2+2\mathfrak{v}\mathfrak{w}\cos k+\mathfrak{w}^2}-2m\pi}{\sqrt{\mathfrak{v}^2+2\mathfrak{v}\mathfrak{w}\cos k+\mathfrak{w}^2}}(\mathfrak{w}\sin k)\theta(\sqrt{\mathfrak{v}^2+2\mathfrak{v}\mathfrak{w}\cos k+\mathfrak{w}^2}-m\pi)\theta((m+1)\pi-\sqrt{\mathfrak{v}^2+2\mathfrak{v}\mathfrak{w}\cos k+\mathfrak{w}^2}). \end{aligned} \tag{17}$$

Are there any ways to calculate directly through the normal Hamiltonian curve to get the right result instead of a geometric intuitive way?

Then, such static winding numbers relied on these two parameters can give as a phase diagram located in the first region:

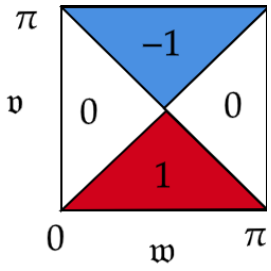


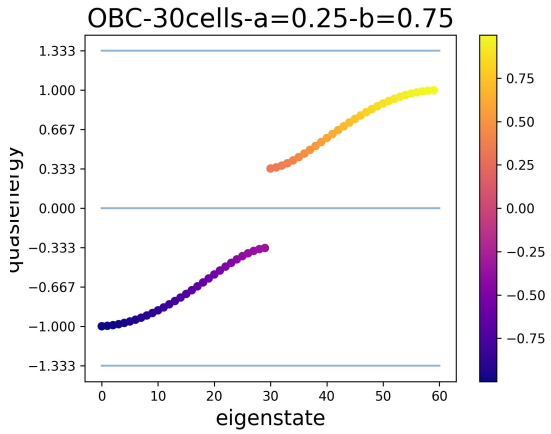
Fig. 4. A first part of phase diagram of Floquet SSH model $(\mathfrak{v},\mathfrak{w}) \in [0,\pi)\times[0,\pi)$. The red part denotes the phase with winding number $C = 1$, the blue part denotes the phase with winding number $C = -1$, and the white part denotes the phase with winding number $C = 0$.

If we define two phase index $\lfloor\alpha\rfloor = \left\lfloor\frac{\mathfrak{v}-\mathfrak{w}}{\pi}\right\rfloor$ and $\lfloor\beta\rfloor = \left\lfloor\frac{\mathfrak{v}+\mathfrak{w}}{\pi}\right\rfloor$, We thus can summarize the phases of our system from static topological band theory:

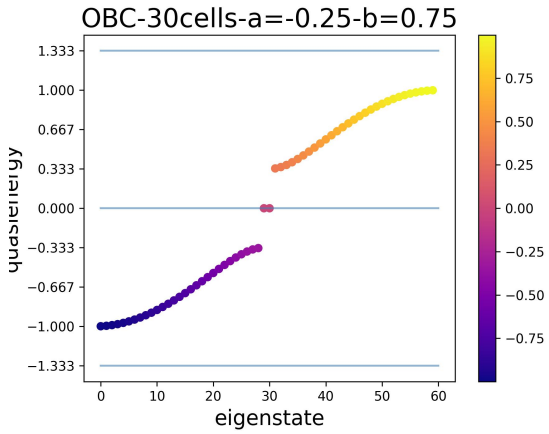
- α is odd $\wedge$ β is odd: trivial with winding number 0
- α is odd $\wedge$ β is even: topological with winding number 1
- α is even $\wedge$ β is odd: topological with winding number -1
- α is even $\wedge$ β is even: trivial with winding number 0

3 Anomalous spectrum

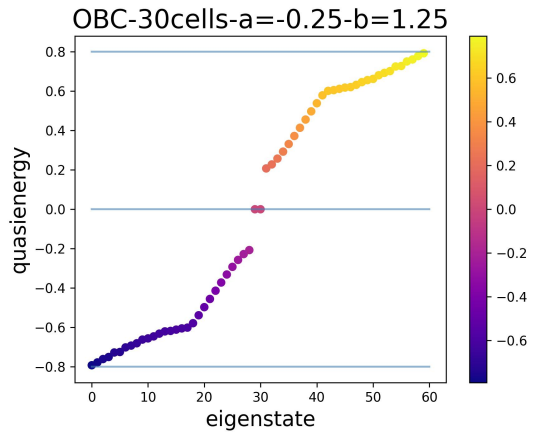
We should do a numerical simulation to see the Floquet spectrum with OBC. The code is in Appendix B. We use the reduced parameters $\alpha = \frac{\mathfrak{v}-\mathfrak{w}}{\pi}$ and $\beta = \frac{\mathfrak{v}+\mathfrak{w}}{\pi}$ to draw the spectrums in different phases. The spectrums are shown in Fig. 4.



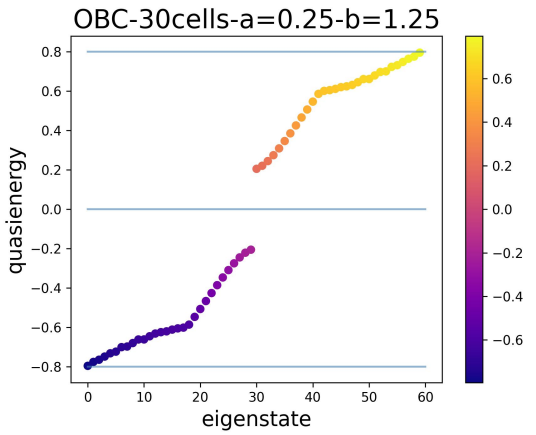
(a) $\alpha = 0.25 - \beta = 0.75$



(b) $\alpha = -0.25 - \beta = 0.75$



(c) $\alpha = -0.25 - \beta = 1.25$



(d) $\alpha = 0.25 - \beta = 1.25$

Fig. 5. Numerical simulation for the first phase region $(\alpha,\beta) \in [0,1) \times [0,1)$ with $N = 30$ cells. (a) is the normal trivial phase with $\alpha = 0.25, \beta = 0.75$ (winding number 0, no edge states); (b) is the normal topological phase with $\alpha = -0.25, \beta = 0.75$ (winding number 1, one 0-mode edge state); (c) is the anomalous "trivial" phase with $\alpha = -0.25, \beta = 1.25$ (winding number 0, but one π-mode edge state and one 0-mode edge state); (d) is the topological phase with $\alpha = 0.25, \beta = 1.25$ (winding number -1, one π-mode edge state).

We find that phase (c) is anomalous for the winding number we calculated is 0 wheras there exist two edge states.

4 Floquet winding number

In order to calculate the right winding number that reveals the number of edge states correctly, we need to build a new topological invariant. This topological invariant is determined by the symmetry class the system belongs to and the opening and closing of the gaps of the system. For our Floquet SSH model, it

owns a chiral symmetry and has two gaps. In a Floquet system, we know that energy bands are quasienery bands which are the eigenvalues of the Floquet operator $U_F(\mathbf{k}) = \mathrm{T}\exp\left[-\frac{i}{\hbar}\int_0^t dt\mathcal{H}(\mathbf{k},t)\right]$. However in order to involve the periodic dynamic into the topological invariant, we need to do such decomposition of the time evolution operator:

$$U(\mathbf{k},t) = U_\epsilon(t)[U_F(\mathbf{k})]_\epsilon^{\frac{t}{T}} = U_\epsilon(t)e^{-\frac{i}{\hbar}\mathcal{H}_\epsilon^{\mathrm{eff}}(\mathbf{k})t}. \tag{18}$$

Here, ϵ again is the branch cut of taking root $\frac{t}{T}$ and logarithm as well. This decomposition separates the normal part $[U_F(\mathbf{k})]_\epsilon^{\frac{t}{T}}$ and the anomalous part $U_\epsilon(t)$ of the time evolution operator. For the original time evolution operator is not periodic, but the anomalous part $U_\epsilon(t) = U(\mathbf{k},t)e^{\frac{i}{\hbar}\mathcal{H}_\epsilon^{\mathrm{eff}}(\mathbf{k})t}$ is:

$$U_\epsilon(t+T) = U(\mathbf{k},t+T)e^{\frac{i}{\hbar}\mathcal{H}_\epsilon^{\mathrm{eff}}(\mathbf{k})(t+T)} = U(\mathbf{k},t)U_F U_F^{-1}e^{\frac{i}{\hbar}\mathcal{H}_\epsilon^{\mathrm{eff}}(\mathbf{k})t} = U_\epsilon(t). \tag{19}$$

Physically, the anomalous time evolution operator characterizes the evolution of polarization inside the time-driven crystal [6].

It is known that for such a class which is denoted as class AIII, for a d spatial dimension Floquet system with a D dimension surrounding system and a chiral symmetry, the topological invariant is [3]

$$W[U_\epsilon(\mathbf{k},\mathbf{r},\frac{T}{2})] = e^{i\epsilon}K_{d+D}\int_{\mathbb{T}^d\times S^D} d^dkd^Dr \times \mathrm{Tr}\Big\{\varepsilon^{\alpha_1\cdots\alpha_{d+D}}\frac{I-\Gamma}{2}[U_\epsilon^{-1}\partial_{\alpha_1}U_\epsilon)\cdots[U_\epsilon^{-1}\partial_{\alpha_{d+D}}U_\epsilon]\Big\}, \tag{20}$$

where

$$K_{d+D} = \frac{(-1)^{\frac{d+D-1}{2}}(\frac{d+D-1}{2})!}{(d+D)!}\Big(\frac{i}{2\pi}\Big)^{\frac{d+D+1}{2}}. \tag{21}$$

Here, the reason why set $t = \frac{T}{2}$ is to preserve the chiral system, for the anomalous time evolution operator transforms under chiral transformation like:

$$\Gamma^\dagger U_\epsilon(\mathbf{k},\mathbf{r},t)\Gamma = U_{-\epsilon}(\mathbf{k},\mathbf{r},-t)e^{i\frac{2\pi t}{T}}, \tag{22}$$

so only when $t = \frac{T}{2}$ we have $U_\epsilon(\mathbf{k},\mathbf{r},\frac{T}{2}) = U_\epsilon(\mathbf{k},\mathbf{r},-\frac{T}{2})$, and the topological is well-defined only at the gaps $\epsilon = 0,\pi$, then we can have a good chiral symmetry/asymmetry:

$$\Gamma^\dagger U_0(\mathbf{k},\mathbf{r},\frac{T}{2})\Gamma = -U_0(\mathbf{k},\mathbf{r},\frac{T}{2}),\ \Gamma^\dagger U_\pi(\mathbf{k},\mathbf{r},\frac{T}{2})\Gamma = U_\pi(\mathbf{k},\mathbf{r},\frac{T}{2}). \tag{23}$$

For our model, we have $d = 1, D = 0, T = t_1 + t_2$ and $U_F = e^{-\frac{i}{\hbar}\varepsilon T}|\varepsilon_+(k)\rangle\langle\varepsilon_+(k)| + e^{\frac{i}{\hbar}\varepsilon T}|\varepsilon_-(k)\rangle\langle\varepsilon_-(k)|$ where $\varepsilon = \varepsilon(k) = \frac{1}{T}\sqrt{t_1^2 + 2t_1t_2\cos k + t_2^2}$, so the topological invariant is a one-dimensional loop integration:

$$W[U_\epsilon(\mathbf{k},\frac{T}{2})] = e^{i\epsilon}\frac{i}{2\pi}\int_{\mathbb{T}} dk \times \mathrm{Tr}\Big[\frac{I-\Gamma}{2}U_\epsilon^{-1}\partial_k U_\epsilon\Big]. \tag{24}$$

In a chiral basis ($\Gamma = \sigma_z$), the anomalous time evolution operator can be simplifed into:

$$U_0 = \begin{pmatrix} 0 & U_0^+ \\ U_0^- & 0 \end{pmatrix}, U_\pi = \begin{pmatrix} U_\pi^+ & 0 \\ 0 & U_\pi^- \end{pmatrix}. \tag{25}$$

where $U_\epsilon^\pm$ are unitary matrices. And the topological invariant can be calculated as

$$W[U_\epsilon(\mathbf{k},\frac{T}{2})] = e^{i\epsilon}\frac{i}{2\pi}\int_{\mathbb{T}} dk \times \mathrm{Tr}\Big[U_\epsilon^{+-1}\partial_k U_\epsilon^+\Big]. \tag{26}$$

In a general basis, the chiral matrix can be gained through:

$$\Lambda_{0,\pi} = VU_{0,\pi}V^\dagger \Longrightarrow \begin{cases} \Gamma_0 = V^\dagger(\cos\frac{\pi}{4}\sigma_x + \sin\frac{\pi}{4}\sigma_z)^\dagger\sigma_z(\cos\frac{\pi}{4}\sigma_x + \sin\frac{\pi}{4}\sigma_z)V \\ \Gamma_\pi = V^\dagger\sigma_z V \end{cases} . \tag{27}$$

We first try to analyse this problem analytically. The time evolution operator during a half period is

$$U(k, \frac{T}{2}) = \begin{pmatrix} \cos\tilde{\epsilon} & -ie^{-i\tilde{\varphi}}\sin\tilde{\epsilon} \\ -ie^{i\tilde{\varphi}}\sin\tilde{\epsilon} & \cos\tilde{\epsilon} \end{pmatrix}, \tag{28}$$

where

$$\tilde{\epsilon} = \begin{cases} \frac{vT}{2\hbar}, \ t_1 \geq t_2 \\ \frac{1}{\hbar}\sqrt{(vt_1)^2 + (w(\frac{T}{2}-t_1))^2 + 2vt_1 w(\frac{T}{2}-t_1)\cos k}, \ t_1 < t_2 \end{cases},$$
$$e^{i\tilde{\varphi}} = \begin{cases} 1, \ t_1 \geq t_2 \\ \frac{vt_1 + w(\frac{T}{2}-t_1)e^{ik}}{\sqrt{(vt_1)^2+(w(\frac{T}{2}-t_1))^2+2vt_1 w(\frac{T}{2}-t_1)\cos k}}, \ t_1 < t_2 \end{cases}.$$

So the anomalous time evolution operator at half period is

$$U_\epsilon(k, \frac{T}{2}) = U(k, \frac{T}{2})e^{-\frac{1}{2}(\ln_{-\epsilon} e^{-i\epsilon}|\epsilon_+(k)\rangle\langle\epsilon_+(k)| + \ln_{-\epsilon} e^{i\epsilon}|\epsilon_-(k)\rangle\langle\epsilon_-(k)|)}. \tag{29}$$

For $\epsilon = 0$, we need to change the basis to chiral basis by first diagonalizing $U_0(k, \frac{T}{2})$ and sandwiching the result with $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, then the equivalent anomalous time operator becomes anti-diagonal; for $\epsilon = \pi$, we need to change the basis to chiral basis by just diagonalizing $U_\pi(k, \frac{T}{2})$ and it becomes diagonal.

We denote $e^{-\frac{1}{2}\ln_{-\epsilon} e^{-i\epsilon}} = a, e^{-\frac{1}{2}\ln_{-\epsilon} e^{i\epsilon}} = b$, then we have

$$\begin{aligned} U_\epsilon(k, \frac{T}{2}) &= \begin{pmatrix} \cos\tilde{\epsilon} & -ie^{-i\tilde{\varphi}}\sin\tilde{\epsilon} \\ -ie^{i\tilde{\varphi}}\sin\tilde{\epsilon} & \cos\tilde{\epsilon} \end{pmatrix}(a\,|\epsilon_+(k)\rangle\langle\epsilon_+(k)| + b\,|\epsilon_-(k)\rangle\langle\epsilon_-(k)|) \\ &= \begin{pmatrix} \cos\tilde{\epsilon} & -ie^{-i\tilde{\varphi}}\sin\tilde{\epsilon} \\ -ie^{i\tilde{\varphi}}\sin\tilde{\epsilon} & \cos\tilde{\epsilon} \end{pmatrix}\frac{1}{2}\begin{pmatrix} a+b & e^{-i\varphi}(a-b) \\ e^{i\varphi} & a+b \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (a+b)\cos\tilde{\epsilon} - i(a-b)e^{i(\varphi-\tilde{\varphi})}\sin\tilde{\epsilon} & (a-b)e^{-i\varphi}\cos\tilde{\epsilon} - i(a+b)e^{-i\tilde{\varphi}}\sin\tilde{\epsilon} \\ (a-b)e^{i\varphi}\cos\tilde{\epsilon} - i(a+b)e^{i\tilde{\varphi}}\sin\tilde{\epsilon} & (a+b)\cos\tilde{\epsilon} - i(a-b)e^{-i(\varphi-\tilde{\varphi})}\sin\tilde{\epsilon} \end{pmatrix} \\ &\cong \begin{pmatrix} V_\epsilon^+ & 0 \\ 0 & V_\epsilon^- \end{pmatrix}. \end{aligned} \tag{30}$$

Here $2V_\epsilon^\pm$ is the roots of $x^2 - 2[(a+b)\cos\tilde{\epsilon} - i(a-b)\cos(\varphi-\tilde{\varphi})\sin\tilde{\epsilon}]x + 4ab = 0$. By symmetry demanding, $V_0^+ + V_0^- = 0$, so we have:

$$U_\epsilon^+ = V_\epsilon^+, \ \epsilon = 0, \pi. \tag{31}$$

Integrate this through Eq. (26) turns out to be a messy thing, and I didn't figure it out. Here I put two thoughts of mine.

The first thought is through geometry or topology, we focus the winding number as the winding process on Bloch sphere.

For 0-gap, we see now U_0 is protected by chiral symmetry so it lives on the equator of Bloch sphere after normalized, we know the singularity of $\frac{V_0^+}{|V_0^+|}$ or the zero point of it implies the closing and opening of the gap. So we assume that during one period in BZ, if $|V_0^+|$ has even number of zero points, then $W_0 = 0$; if $|V_0^+|$ has odd number of zero points, then $W_0 = 1$. That is, two edge states can fusion together to be gapped out in the Floquet formalism. I think the number of zero points relates to the value $\mathfrak{v}$ and $\mathfrak{w}$, and branch cut here plays a crucial role.

For π-gap, chiral anti-symmetry demands U_π live on the two poles of Bloch sphere. If the radius of the sphere is equal to π, then we assume the gap is closed, and during one period in BZ, if $|V_\pi^+|$ has even number of π points, then $W_\pi = 0$, if $|V_\pi^+|$ has odd number of π points, then $W_\pi = 1$ (So both W_0 and W_π are $\mathbb{Z}_2$-valued indices). But I don't know how to geometrically visualize this as a "winding" .

The second one is through computation. The code used to compute the Floquet winding number is in Appendix C. Unfortunately, I spent lots of time adjusting and debugging my code and tried several ways to accomplish the computation but also failed, only to get a completely wrong answer every time. the result is obviously wrong, with following questions: 1. the result has nonzero imaginary part which don't change with the differentiation step; 2. the real part of the result sometimes changes with the differentiation step with the same order, like if I change the differentiation step from 10^{-5} to 10^{-3}, the result may shrink by 10^2 order; 3. the result never gives the right answer expected in theory if it is correct. For example, for W_π at $\alpha = 0.75$ and $\beta = 0.25$ (that is a trivial phase, the theoretical value is 0), I get

```
1.5707963267948966 0.7853981633974483
Winding number =  (7.0337268169738e-06-0.4514269846237881j)
```

and at $\alpha = 1.25$ and $\beta = 0.25$ (that is a topological π-phase, the theoretical value is 1), I get

```
2.356194490192345 1.5707963267948966
Winding number =  (0.0001125395359598502-0.5166317330265293j)
```

My guess is: just like [1], the winding number for $\epsilon = 0$ or $\epsilon = \pi$ yields the number of edge states at energy gap 0 or π. The static number is given by $C = W_0 - W_\pi$.

5 Questions

[1] Actually, the key part with which I am very confused is how to count the winding number analytically in the Floquet case. It turns out to have many complicated phases in which two gaps being closed and reopened and reclosed for many times. Without visualizing it to a geometric configuration with winding dynamic, I don't know how to compute it. In the static case I guess it can compute by drawing a circle on a cylindrical surface, but this is just a geometric visualization. In the Floquet case, the intergration

seems to be very complicated and I don't find a good visualization. Is there a systematic way to compute it?

[2] The second question is just raised above, I don't know what's going wrong with my code.....

A Appendix

The code to plot PBC Floquet bands in the first two phases.

```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import math
import cmath
import seaborn
import scipy
import functools

#parameter settings
v=1 #intracell hopping in time period 1
w=1 #intercell hopping in time period 2
alpha=1.5 #phase index 1 = vt1/wt2
beta=0.5 #phase index 2 = (v*t1+w*t2)/(pi)
t1=(alpha*beta*np.pi)/(v*(alpha+1)) #time period 1
t2=(beta*np.pi)/(w*(alpha+1)) #time period 2

k = np.arange(-np.pi,np.pi+0.11,0.1) #The first BZ
E = np.sqrt(t1*t1+2*t1*t2*np.cos(k)+t2*t2) #eigen-energy

interval0 = [1 if (i<=np.pi) else 0 for i in E]
interval1 = [1 if (i>=np.pi) else 0 for i in E]

#conduction band
if 0<= beta <= 1:
    E1 = np.sqrt(t1*t1+2*t1*t2*np.cos(k)+t2*t2)
if 1 < beta:
    E1 = np.sqrt(t1*t1+2*t1*t2*np.cos(k)+t2*t2)*interval0+(2*np.pi-np.sqrt(t1*t1+2*t1*t2*np.
    cos(k)+t2*t2))*interval1
#valence band
if 0<= beta <= 1:
    E2 = -np.sqrt(t1*t1+2*t1*t2*np.cos(k)+t2*t2)
if 1 < beta:
    E2 = -np.sqrt(t1*t1+2*t1*t2*np.cos(k)+t2*t2)*interval0+(-2*np.pi+np.sqrt(t1*t1+2*t1*t2*
    np.cos(k)+t2*t2))*interval1

E3 = 0*k

```

```
#plot the bands
plt.figure(figsize=(8,7))
plt.plot(k, E1, color="black", alpha=0.6)
plt.plot(k, E2, color="black", alpha=0.6)
plt.plot(k, E3, color="steelblue", alpha=0.6)
my_x_ticks = np.arange(-np.pi, np.pi+0.1, np.pi/4)
my_y_ticks = np.arange(-np.pi, np.pi+0.1, np.pi/4)
plt.xticks(my_x_ticks)
plt.yticks(my_y_ticks)
plt.xlabel("quasimomentum", fontdict={'size': 16})
plt.ylabel("quasienergy", fontdict={'size':16})
plt.title('PBC-band-'+"a="+str(round(alpha,2))+"-b="+str(round(beta,2)), fontdict={'size':
    20})
plt.savefig('PBC-band-'+"a="+str(round(alpha,2))+"-b="+str(round(beta,2))+'.jpg', dpi=300)
plt.show()
```

B Appendix

The code to plot OBC Floquet spectrum.

```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import math
import cmath
import seaborn
import scipy
import functools

#parameter settings
N=30 #number of unit cells
v=1 #intracell hopping in time period 1
w=1 #intercell hopping in time period 2
aalpha=2.25
alpha= math.floor(aalpha) #phase index 1
bbeta=3.25
beta = math.floor(bbeta) #phase index 2
t1=(np.pi*(aalpha+bbeta))/(2*v) #time period 1
t2=(np.pi*(bbeta-aalpha))/(2*w) #time period 2
print(t1,t2)

dt=0.0001 #step
t_array=np.arange(0,t1+t2,dt) #set time interval
length = len(t_array) #total steps

cm = plt.cm.get_cmap('plasma') #get colorbar
```

```
27
28 #get interaction parameters during one period
29 V = [0 for index in range(length)] #intercell hopping array
30 for i in range(0,length,1):
31     t = t_array[i]
32     if 0 <= t <= t1:
33         V[i]=v
34     if t1 < t <= t1+t2:
35         V[i]=0
36 W = [0 for index in range(length)] #intracell hopping array
37 for i in range(0,length,1):
38     t = t_array[i]
39     if 0 <= t <= t1:
40         W[i]=0
41     if t1 < t <= t1+t2:
42         W[i]=w
43
44 #calculate the evolution operator
45 evol = [0 for index in range(length)]
46 for j in range(0,length,1):
47     h = np.zeros((2*N,2*N)) #hamiltonian in one step
48     for i in range(0,2*N,2):
49         h[i,i+1] = V[j]*dt
50         h[i+1,i] = V[j]*dt
51     for i in range(1,2*N-1,2):
52         h[i,i+1] = W[j]*dt
53         h[i+1,i] = W[j]*dt
54     evol[j] = h
55 evol.reverse()
56
57 #total Floquet operator
58 Floquetian = scipy.linalg.expm(-1*1j*functools.reduce(lambda x,y:x+y,evol))
59 # print(np.round(Floquetian,2))
60
61 #solve the eigen-problem
62 eigenvalue, eigenvector = np.linalg.eig(Floquetian)
63 quasienergy = [0 for index in range(2*N)]
64 for i in range(0,2*N,1):
65     quasienergy[i] = (cmath.phase(eigenvalue[i])/(t1+t2))
66
67 quasienergy.sort()
68
69 #number the eigenvalues
70 k = np.arange(0,2*N)
71 z = quasienergy
72 x=np.arange(0,2*N+0.1,0.1)
73 y0=0*x
```

```
y1=0*x+(np.pi)/(t1+t2)
y2=0*x-(np.pi)/(t1+t2)

#plot the spectrum
plt.scatter(k, quasienergy, c=z, s=30, cmap=cm)
plt.plot(x, y0, color="steelblue", alpha=0.6)
plt.plot(x, y1, color="steelblue", alpha=0.6)
plt.plot(x, y2, color="steelblue", alpha=0.6)
plt.colorbar()
plt.xlabel("eigenstate", fontdict={'size': 16})
plt.ylabel("quasienergy", fontdict={'size':16})
my_y_ticks = np.arange(-(np.pi)/(t1+t2), (np.pi)/(t1+t2)+(np.pi)/(4*(t1+t2)), (np.pi)/(4*(t1+t2)))
plt.yticks(my_y_ticks)
plt.title("OBC-"+str(N)+"cells-"+"a="+str(round(aalpha,2))+"-b="+str(round(bbeta,2)), fontdict={'size': 20})
plt.savefig('OBC-'+str(N)+"-"+str(round(aalpha,2))+"-"+str(round(bbeta,2))+'.jpg', dpi=300)
plt.show()
```

C Appendix

The code to compute the Floquet winding number.

```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import math
import cmath
import seaborn
import scipy
import functools
import time

#parameter settings
v=1 #intracell hopping in time period 1
w=1 #intercell hopping in time period 2
aalpha=-0.25 #phase index 1
bbeta=1.25 #phase index 2
t1=(np.pi*(aalpha+bbeta))/(2*v) #time period 1
t2=(np.pi*(bbeta-aalpha))/(2*w) #time period 2
print(t1,t2)
T=t1+t2 #total time period
Theta=0 #gap

#logarithm function with gap branch cut
def llog(z, theta):
```

```
    modulus = np.abs(z)
    argument = np.angle(z)
    if theta-2*np.pi <= argument < theta:
        argument = argument
    else:
        argument = theta-2*np.pi+np.mod(argument-theta, 2*np.pi)
    return np.log(modulus) + 1j * argument

#time evolution operator at half period
def U(k):
    matrix = np.zeros((2, 2), dtype=complex)
    if t1 >= t2:
        matrix[0,1] = v*(T/2)
        matrix[1,0] = v*(T/2)
    if t1 < t2:
        matrix[0,1] = v*t1+w*(T/2-t1)*cmath.exp(-1j*k)
        matrix[1,0] = v*t1+w*(T/2-t1)*cmath.exp(1j*k)
    return scipy.linalg.expm(-1j*matrix)

#Floquet operator cut at half period
def UF(k):
    E=np.sqrt(math.pow(v*t1,2)+math.pow(w*t2,2)+2*(v*t1)*(w*t2)*np.cos(k))
    eiphi=((v*t1)+(w*t2)*np.exp(1j*k))/(E)
    eiphic=np.conjugate(eiphi)
    TT=(1/np.sqrt(2))*np.matrix([[eiphic,eiphic],[1,-1]])
    Tc=(1/np.sqrt(2))*np.matrix([[eiphi,1],[eiphi,-1]])
    EIG=np.exp(-1j*E)
    EIGc=np.exp(1j*E)
    hamiltonian=np.matrix([[(1j/T)*llog(EIG,Theta),0],[0,(1j/T)*llog(EIGc,Theta)]])
    Hamiltonian = TT*hamiltonian*Tc #effective Hamiltonian
    return scipy.linalg.expm(1j*(T/2)*Hamiltonian)

#chiral matrix
def S(k):
    diag,P=np.linalg.eig(U(k)*UF(k)) #eigen-problem of anamolous time operator at half
    period
    Trans0=np.matrix([[1,0],[0,1]])
    Trans1=np.matrix([[1,0],[0,-1]])
    Trans2=(1/np.sqrt(2))*np.matrix([[1,1],[1,-1]])
    if Theta == np.pi:
        S=P*Trans1*scipy.linalg.inv(P)
    if Theta == 0:
        S=P*Trans2*Trans1*Trans2*scipy.linalg.inv(P)
    return np.round((1/2)*(Trans0-S),2)

#calculate the winding number
def main():
```

```
    start_clock = time.perf_counter()
    delta_1 = 1e-5 #derivation step
    delta_2 = 1e-3 #integration step
    W = 0
    for k in np.arange(-np.pi, np.pi, delta_2):
        H0 = U(k)*UF(k)
        H1 = U(k+delta_1)*UF(k+delta_1)
        W = W + np.trace(S(k)*scipy.linalg.inv(H0)*((H1-H0)/delta_1))*delta_2 # Winding
    number
    print('Winding number = ', (1j*W)/(2*np.pi))
    end_clock = time.perf_counter()

if __name__ == '__main__':
    main()
```

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