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Floquet Phase-1

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1 Floquet formalism

Floquet formalism or Floquet theory is constructed within a time-periodic driven quantum system, and is partly similar to Bloch theory for a spatial periodic system. Consider a single-particle form Hamiltonian

$$\mathcal{H}(t) = \mathcal{H}(t+T), \tag{1}$$

with period T. The discrete time translation symmetry implies a time translation operator $U_F := U(T)$ (Floquet operator), whose basis of irreducible group representation are the so-called Floquet states or the **quasienergy** states $|\varepsilon\rangle$ with $U(T)|\varepsilon\rangle = e^{-i\varepsilon T}|\varepsilon\rangle$. This is an analogue to Bloch theory, where $U(T) \leftrightarrow \mathcal{T}, |\varepsilon\rangle \leftrightarrow |\mathbf{k}\rangle, \mathcal{T}|\mathbf{k}\rangle = e^{-ika}|\mathbf{k}\rangle$. We can also define the quasienergy Brillouin zone (or the Floquet-Brillouin zone): $[-\frac{\pi}{T}, \frac{\pi}{T})$, then the whole time axis can be mapped into a circle S^1. For a general Floquet quantum material with both discrete spatial and time translation symmetry, there are two good quantum numbers ε and k, so the Floquet spectrum is usually a spectrum drown in 1-st BZ $\times [-\frac{\pi}{T}, \frac{\pi}{T})$ =: FBZ or a torus $\mathbb{T}^2 = \mathbb{C}/\text{FBZ}$, with eigenstates labled as $|\varepsilon_n(\mathbf{k})\rangle$.

The Floquet operator can be written down directly in the form of a usual time evolution operator. We define that

$$U_F := U(T) = \mathrm{T}\exp\Big[-\frac{i}{\hbar}\int_0^T dt\mathcal{H}(t)\Big] =: \exp\Big[-\frac{i}{\hbar}T\mathcal{H}^{\text{eff}}_{\epsilon}\Big], \tag{2}$$

where we set a dimensionless parameter $\epsilon = \varepsilon T$ and $\mathcal{H}^{\text{eff}}_{\epsilon} = \frac{i\hbar}{T}\ln_{-\epsilon} U_F$ is the effective Hamiltonian over one period with eigenvalues $\hbar\varepsilon$. Here, it is conventional to set the branch cut of the logarithm as:

$$\ln_{-\epsilon} e^{i\phi} = i\phi,\ -\varepsilon T - 2\pi < \phi < -\varepsilon T, \tag{3}$$

namely,

$$\ln_{-\epsilon} e^{-i\varepsilon T + i0^-} = -i\varepsilon T,\ \ln_{-\epsilon} e^{-i\varepsilon T + i0^+} = -i\varepsilon T - 2\pi i. \tag{4}$$

If the Floquet operator can be decomposed into

$$U_F = \sum_n \lambda_n(\mathbf{k})\,|\varepsilon_n(\mathbf{k})\rangle\langle\varepsilon_n(\mathbf{k})|, \tag{5}$$

then the effective Hamiltonian is just

$$\mathcal{H}^{\text{eff}}_{\epsilon} = \frac{i\hbar}{T}\sum_n \ln_{-\epsilon}(\lambda_n(\mathbf{k}))\,|\varepsilon_n(\mathbf{k})\rangle\langle\varepsilon_n(\mathbf{k})|. \tag{6}$$

The Floquet formalism can provide us the stroboscopic dynamic of a system, where it behaves like a static system with effective Hamiltonian $\mathcal{H}^{\text{eff}}_{\epsilon}$ whose eigenvalues are the quasienergies.

2 Floquet SSH model

Let's consider a periodic driven SSH model. The Hamiltonian is written as:

$$\mathcal{H}(t) = \sum_i v(t) c^\dagger_{i,A} c_{i,B} + w(t) c^\dagger_{i,A} c_{i+1,B} + \text{h.c.}, \tag{7}$$

where

$$\begin{cases} v(t) = 1 - \theta(t - t_1) \\ w(t) = \theta(t - t_1) \end{cases} \quad (0 < t, t_1 < T),\ v(t+T) = v(t), w(t+T) = w(t). \tag{8}$$

Such Hamiltonian describes a periodic driven dynamic with period T in which the system hops between two phases periodically:

$$\mathcal{H}(t) = \begin{cases} \sum_i c^\dagger_{i,A} c_{i,B} + \text{h.c.}, 0 < t < t_1 \text{ (trivial)} \\ \sum_i c^\dagger_{i,A} c_{i+1,B} + \text{h.c.}, t_1 < t < T \text{ (topological)} \end{cases}. \tag{9}$$

Inside the bulk, we apply the periodic boundary condition and get the familiar result of the bulk Bloch Hamiltonian:

$$\begin{aligned} \mathcal{H}(t) &= \sum_k \psi_k^\dagger \mathcal{H}(k,t) \psi_k \\ &= \sum_k \psi_k^\dagger \begin{pmatrix} 0 & v(t) + w(t)e^{-ik} \\ v(t) + w(t)e^{ik} & 0 \end{pmatrix} \psi_k \\ &= \sum_k \psi_k^\dagger U^\dagger \begin{pmatrix} \sqrt{v^2 + 2vw\cos k + w^2} & 0 \\ 0 & -\sqrt{v^2 + 2vw\cos k + w^2} \end{pmatrix} U \psi_k, \end{aligned} \tag{10}$$

where the unitary matrix $U = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} & 1 \\ e^{i\phi} & -1 \end{pmatrix}, e^{i\phi} = \frac{v + we^{ik}}{\sqrt{v^2 + 2vw\cos k + w^2}}$. The gap or the phase are controlled by the factor $\gamma = \frac{v}{w}$ and are protected by chiral symmetry if the Hamiltonian is time-independent.

In time-dependent case, we denote the Hamiltonians in two periods as $\mathcal{H}_1 \in (0, t_1)$ and $\mathcal{H}_2 \in (t_1, t_1 + t_2)$. We know that both $\mathcal{H}_1$ and $\mathcal{H}_2$ own a chiral symmetry, but it turns out that not all Floquet operator and the effective Hamiltonian it induces own it. The chiral symmetry constraints on the time evolution operator implies that $\Gamma^\dagger U(t) \Gamma = U(-t) \implies \Gamma^\dagger U(T) \Gamma = U^\dagger(T)$, since $v(t), w(t)$ are real function of time, we should choose a good initial time t_0 so that $v(t), w(t)$ are even function of time. So, in order to inherit the chiral symmetry from the non-driven system, we choose $t_0 = \frac{1}{2}t_1$ and put a translation to $v(t), w(t) \mapsto v(t + t_0), w(t + t_0)$, that is we have the Bloch Hamiltonian:

$$\mathcal{H}(k,t) = \begin{cases} \mathcal{H}_1 = v\sigma^x,\ t \in (0, \frac{1}{2}t_1) \\ \mathcal{H}_2 = w\cos k\sigma^x + w\sin k\sigma^y,\ t \in (\frac{1}{2}t_1, \frac{1}{2}t_1 + t_2) \\ \mathcal{H}_1 = v\sigma^x,\ t \in (\frac{1}{2}t_1 + t_2, t_1 + t_2) \end{cases}. \tag{11}$$

The Floquet operator in k-space can be calculated as follow:

$$
\begin{aligned}
U_F := U(T) &= U(t_1+t_2, \frac{1}{2}t_1+t_2)U(\frac{1}{2}t_1+t_2, \frac{1}{2}t_1)U(\frac{1}{2}t_1, 0) \\
&= \exp\Big[-\frac{i}{\hbar}\mathcal{H}_1\frac{t_1}{2}\Big]\exp\Big[-\frac{i}{\hbar}\mathcal{H}_2 t_2\Big]\exp\Big[-\frac{i}{\hbar}\mathcal{H}_1\frac{t_1}{2}\Big] \\
&= \exp\Big[-\frac{i}{\hbar}\begin{pmatrix}0 & v\\ v & 0\end{pmatrix}\frac{t_1}{2}\Big]\exp\Big[-\frac{i}{\hbar}\begin{pmatrix}0 & we^{-ik}\\ we^{ik} & 0\end{pmatrix}t_2\Big]\exp\Big[-\frac{i}{\hbar}\begin{pmatrix}0 & v\\ v & 0\end{pmatrix}\frac{t_1}{2}\Big] \\
&= \begin{pmatrix}\cos\frac{vt_1}{2\hbar} & -i\sin\frac{vt_1}{2\hbar}\\ -i\sin\frac{vt_1}{2\hbar} & \cos\frac{vt_1}{2\hbar}\end{pmatrix}\begin{pmatrix}\cos\frac{wt_2}{\hbar} & -ie^{-ik}\sin\frac{wt_2}{\hbar}\\ -ie^{ik}\sin\frac{wt_2}{\hbar} & \cos\frac{wt_2}{\hbar}\end{pmatrix}\begin{pmatrix}\cos\frac{vt_1}{2\hbar} & -i\sin\frac{vt_1}{2\hbar}\\ -i\sin\frac{vt_1}{2\hbar} & \cos\frac{vt_1}{2\hbar}\end{pmatrix} \\
&= \begin{pmatrix}\cos v\cos\omega - \sin v\sin\omega\cos k & -i\sin v\cos\omega + i(-\cos v\sin\omega\cos k + i\sin\omega\sin k)\\ -i\sin v\cos\omega + i(-\sin v\sin\omega\cos k - i\sin\omega\sin k) & \cos v\cos\omega - \sin v\sin\omega\cos k\end{pmatrix} \\
&= (\cos v\cos\omega - \sin v\sin\omega\cos k)\sigma^0 - i(\sin v\cos\omega + \cos v\sin\omega\cos k)\sigma^x - i(\sin\omega\sin k)\sigma^y \\
&= \cos E_{k\pm} \pm i\sin E_{k\pm}\mathcal{R}(k),
\end{aligned}
\tag{12}
$$

where we set two dimensionless parameter $v := \frac{vt_1}{\hbar}, \omega := \frac{wt_2}{\hbar}$, while it can be shortened as $U_F = \tau_0\sigma^0 + i\boldsymbol{\tau}\cdot\boldsymbol{\sigma}$, where

$$
\tau_0 = \cos v\cos\omega - \sin v\sin\omega\cos k, \boldsymbol{\tau} = (-\sin v\cos\omega - \cos v\sin\omega\cos k, -\sin\omega\sin k, 0) \tag{13}
$$

satisfying $\tau_0^2 + \boldsymbol{\tau}\cdot\boldsymbol{\tau} = 1$, and operator $\mathcal{R}(k) := \frac{1}{\sin E_{k+}}\boldsymbol{\tau}\cdot\boldsymbol{\sigma}$.

The quasienergy bands can be calculated through

$$
\exp(iE_{k\pm}) = \tau_0 \pm i\sqrt{\boldsymbol{\tau}\cdot\boldsymbol{\tau}} \Longrightarrow E_{k\pm} = -i\ln_{-\epsilon}(\tau_0 \pm i\sqrt{\boldsymbol{\tau}\cdot\boldsymbol{\tau}}), \tag{14}
$$

where ϵ is the bulk gap branch cut. Here we set $\epsilon = -\pi \sim \pi$ (i.e. we choose the branch cut $-\pi < \varphi < \pi$) to get a pair of quasienergies.

The gap-closing points are found at $k = 0, \pm\pi$:

$$
\tau_0 = \cos E_{k\pm} = \cos v\cos\omega - \sin v\sin\omega\cos k\Big|_{k=0,\pm\pi} = \pm 1 \Longrightarrow \begin{cases} k = 0 : v + \omega = m\pi, \quad m \in \mathbb{Z} \\ k = \pm\pi : v - \omega = n\pi, \quad n \in \mathbb{Z} \end{cases}. \tag{15}
$$

The gap-closing points define the phase boundaries of this system, and give birth to two gapless edge states:0-mode and π-mode. They are protected by the static chiral symmetry which is inherited by our Floquet operator (i.e. $\sigma^z U_F \sigma^z = U_F^{-1}$).

Through static chiral symmetry, we can calculate the winding number of each band. To do that, we first find the eigenstates of our Floquet operator. Note that they are just eigenstates of $\boldsymbol{\tau}\cdot\boldsymbol{\sigma}$. By mapping $\boldsymbol{\tau}$ onto the Bloch sphere, we normalize it into:

$$
\hat{\tau} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \Longrightarrow \begin{cases} \cos\theta = \frac{\tau_z}{|\tau|} = 0 \\ e^{i\phi} = \frac{\tau_x + i\tau_y}{|\tau_x + i\tau_y|} = -\frac{\sin v\cos\omega + \cos v\sin\omega\cos k + i\sin\omega\sin k}{\sqrt{\tau_x^2+\tau_y^2}} \end{cases}. \tag{16}
$$

Note that chiral symmetry demands it lie on the equator of Bloch sphere, so there is a well-defined winding number. The two eigenstates under Southern gauge are:

$$|\pm\rangle = \begin{pmatrix} \sqrt{\frac{1}{2}(1 \pm \frac{\tau_z}{|\tau|})} \\ \pm \sqrt{\frac{1}{2}(1 \mp \frac{\tau_z}{|\tau|})}\frac{\tau_x + i\tau_y}{\sqrt{\tau_x^2+\tau_y^2}} \end{pmatrix}, \tag{17}$$

specifically we have $|\pm\rangle = \frac{1}{\sqrt{2}}(1, \pm e^{i\phi})^T$.

Now the winding number for conduction band can be calculated as follow:

$$\begin{aligned} W &= \frac{1}{\pi}\int_{-\pi}^{\pi} dk \, \langle +|i\partial_k|+\rangle \\ &= \frac{i}{\pi}\int_{-\pi}^{\pi} dk\partial_k \ln e^{i\phi} \\ &= \frac{i}{\pi}\int_{-\pi}^{\pi} dk\partial_k \ln -\frac{\sin\nu\cos\omega + \cos\nu\sin\omega\cos k + i\sin\omega\sin k}{\sqrt{\tau_x^2+\tau_y^2}}. \end{aligned} \tag{18}$$

Geometrically, this winding number characterizes the winding around the origin point on $\mathbb{C}$ of a loop $(\tau_x, \tau_y) : \mathrm{BZ} \to \mathbb{C}$, this loop turns out to be an ellipse:

$$\left(\frac{\tau_x + \sin\nu\cos\omega}{\cos\nu\sin\omega}\right)^2 + \left(\frac{\tau_y}{\sin\omega}\right)^2 = 1. \tag{19}$$

So the winding number is determined by the ratio factor: $\gamma = \frac{|\sin\nu\cos\omega|}{|\cos\nu\sin\omega|}$.

If $\sin\nu\cos\omega > 0, \cos\nu\sin\omega > 0$, then the ellipse is located on the negative real axis with semiminor axis $a = \cos\nu\sin\omega$, so when $\gamma = \frac{\sin\nu\cos\omega}{\cos\nu\sin\omega} > 1 \Leftrightarrow \sin(\nu-\omega) > 0 \Leftrightarrow \nu-\omega \in (2m\pi, (2m+1)\pi), m \in \mathbb{Z}$, the winding number is 0, else $\nu-\omega \in ((2m+1)\pi, (2m+2)\pi), m \in \mathbb{Z}$ is 1.

If $\sin\nu\cos\omega < 0, \cos\nu\sin\omega < 0$, then the ellipse is located on the postive real axis with semiminor axis $a = -\cos\nu\sin\omega$, so when $\gamma = \frac{-\sin\nu\cos\omega}{-\cos\nu\sin\omega} > 1 \Leftrightarrow \sin(\nu-\omega) < 0 \Leftrightarrow \nu-\omega \in ((2m+1)\pi, (2m+2)\pi), m \in \mathbb{Z}$, the winding number is 0, else $\nu-\omega \in (2m\pi, (2m+1)\pi), m \in \mathbb{Z}$ is 1.

If $\sin\nu\cos\omega > 0, \cos\nu\sin\omega < 0$, then the ellipse is located on the negative real axis with semiminor axis $a = -\cos\nu\sin\omega$, so when $\gamma = \frac{\sin\nu\cos\omega}{-\cos\nu\sin\omega} > 1 \Leftrightarrow \sin(\nu+\omega) > 0 \Leftrightarrow \nu+\omega \in (2m\pi, (2m+1)\pi), m \in \mathbb{Z}$, the winding number is 0, else $\nu+\omega \in ((2m+1)\pi, (2m+2)\pi), m \in \mathbb{Z}$ is 1.

If $\sin\nu\cos\omega < 0, \cos\nu\sin\omega > 0$, then the ellipse is located on the positive real axis with semiminor axis $a = \cos\nu\sin\omega$, so when $\gamma = \frac{-\sin\nu\cos\omega}{\cos\nu\sin\omega} > 1 \Leftrightarrow \sin(\nu+\omega) < 0 \Leftrightarrow \nu+\omega \in ((2m+1)\pi, (2m+2)\pi), m \in \mathbb{Z}$, the winding number is 0, else $\nu+\omega \in (2m\pi, (2m+1)\pi), m \in \mathbb{Z}$ is 1.

Again, the phase boundaries fit the gapless points of quasienergies we have obtained. So we can draw the phase diagram as below.

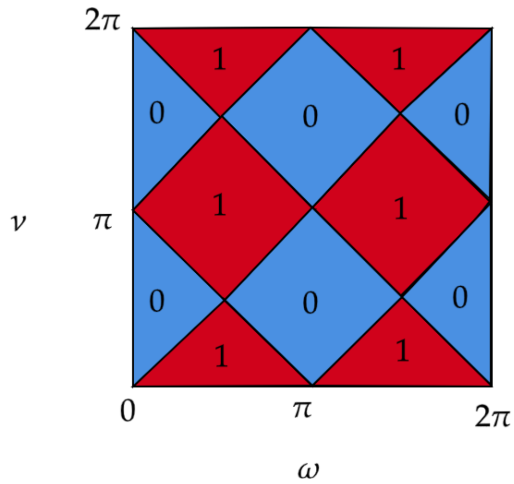
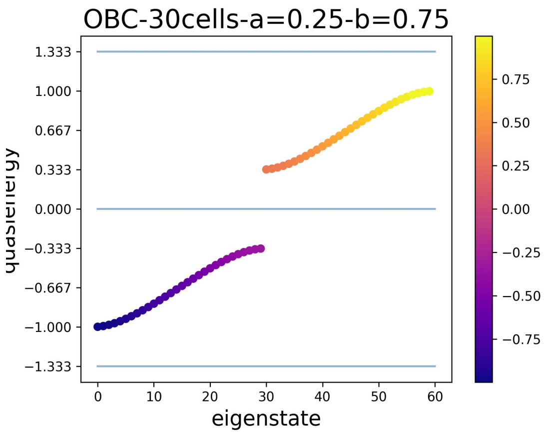


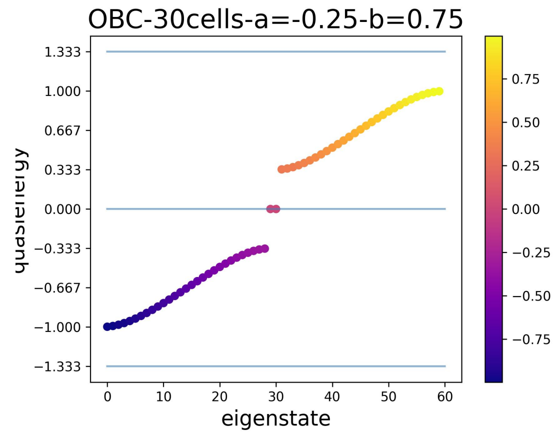
Fig. 1. Phase diagram of Floquet SSH model.

3 Anomalous spectrum

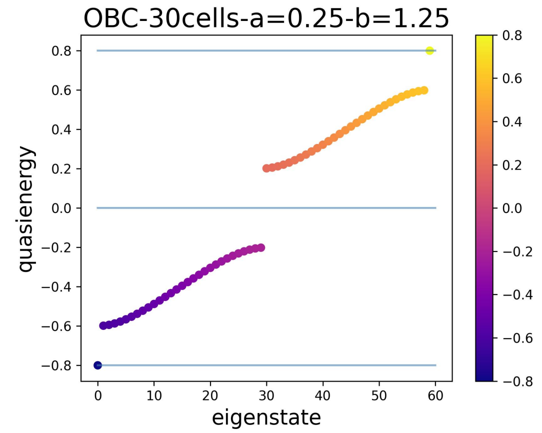
We can plot the spectrum in OBC. The code is in Appendix A.



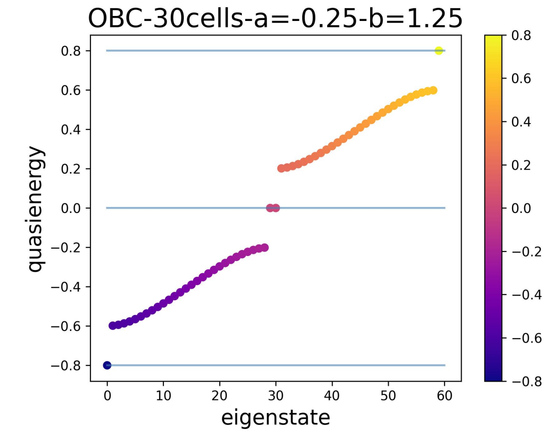
(a) $\alpha = 0.25, \beta = 0.75$



(b) $\alpha = -0.25, \beta = 0.75$



(c) $\alpha = 0.25, \beta = 1.25$



(d) $\alpha = -0.25, \beta = 1.25$

Fig. 2. The spectrums of four different phases of Floquet SSH model. The range of the ordinate is from $-\frac{\pi}{T}$ to $\frac{\pi}{T}$. Here $\alpha = \frac{\nu-\omega}{\pi}, \beta = \frac{\nu+\omega}{\pi}$ are two phase factor which divide the phase diagram. Numerical simulation shows that (a) is a normal trivial phase with winding number 0 and no edge states, (b) is a topological phase with winding number 1 and one 0-mode edge state, (c) is a topological phase with winding number 1 and one π-mode edge state, (d) is a anomalous "trivial" phase with winding number 0 but both 0-mode and π-mode edge states.

4 Floquet winding number

In order to calculate the right winding number that reveals the number of edge states correctly, we need to build a new topological invariant. This topological invariant is determined by the symmetry class the system belongs to and the opening and closing of the gaps of the system. For our Floquet SSH model, it owns a chiral symmetry and has two gaps. In a Floquet system, we know that energy bands are quasienery bands which are the eigenvalues of the Floquet operator $U_F(\mathbf{k}) = \mathrm{T}\exp\left[-\frac{i}{\hbar}\int_0^t dt \mathcal{H}(\mathbf{k}, t)\right]$. However in order to involve the periodic dynamic into the topological invariant, we need to do such decomposition of the time evolution operator:

$$U(\mathbf{k},t) = U_\epsilon(t)[U_F(\mathbf{k})]_\epsilon^{\frac{t}{T}} = U_\epsilon(t)e^{-\frac{i}{\hbar}\mathcal{H}_\epsilon^{\text{eff}}(\mathbf{k})t}. \tag{20}$$

Here, ϵ again is the branch cut of taking root $\frac{t}{T}$ and logarithm as well. This decomposition separates the normal part $[U_F(\mathbf{k})]_\epsilon^{\frac{t}{T}}$ and the anomalous part $U_\epsilon(t)$ of the time evolution operator. For the original time evolution operator is not periodic, but the anomalous part $U_\epsilon(t) = U(\mathbf{k},t)e^{\frac{i}{\hbar}\mathcal{H}_\epsilon^{\text{eff}}(\mathbf{k})t}$ is:

$$U_\epsilon(t+T) = U(\mathbf{k},t+T)e^{\frac{i}{\hbar}\mathcal{H}_\epsilon^{\text{eff}}(\mathbf{k})(t+T)} = U(\mathbf{k},t)U_F U_F^{-1}e^{\frac{i}{\hbar}\mathcal{H}_\epsilon^{\text{eff}}(\mathbf{k})t} = U_\epsilon(t). \tag{21}$$

Physically, the anomalous time evolution operator characterizes the evolution of polarization inside the time-driven crystal [6].

It is known that for such a class which is denoted as class AIII, for a d-spatial dimension Floquet system with a D-dimension surrounding system and a chiral symmetry, the topological invariant for $d-D$ is odd is [3]

$$W[U_\epsilon(\mathbf{k},\mathbf{r},\frac{T}{2})] = e^{i\epsilon}K_{d+D}\int_{\mathbb{T}^d\times S^D} d^dkd^Dr \times \mathrm{Tr}\Big\{\varepsilon^{\alpha_1\cdots\alpha_{d+D}}\frac{I-\Gamma}{2}[U_\epsilon^{-1}\partial_{\alpha_1}U_\epsilon)\cdots[U_\epsilon^{-1}\partial_{\alpha_{d+D}}U_\epsilon]\Big\}, \tag{22}$$

where

$$K_{d+D} = \frac{(-1)^{\frac{d+D-1}{2}}(\frac{d+D-1}{2})!}{(d+D)!}\Big(\frac{i}{2\pi}\Big)^{\frac{d+D+1}{2}}. \tag{23}$$

Here, the reason why set $t=\frac{T}{2}$ is to preserve the chiral system, for the anomalous time evolution operator transforms under chiral transformation like:

$$\Gamma^\dagger U_\epsilon(\mathbf{k},\mathbf{r},t)\Gamma = U_{-\epsilon}(\mathbf{k},\mathbf{r},-t)e^{i\frac{2\pi t}{T}}, \tag{24}$$

so only when $t=\frac{T}{2}$ we have $U_\epsilon(\mathbf{k},\mathbf{r},\frac{T}{2}) = U_\epsilon(\mathbf{k},\mathbf{r},-\frac{T}{2})$, and the topological is well-defined only at the gaps $\epsilon = 0,\pi$, then we can have a good chiral symmetry/asymmetry:

$$\Gamma^\dagger U_0(\mathbf{k},\mathbf{r},\frac{T}{2})\Gamma = -U_0(\mathbf{k},\mathbf{r},\frac{T}{2}),\ \Gamma^\dagger U_\pi(\mathbf{k},\mathbf{r},\frac{T}{2})\Gamma = U_\pi(\mathbf{k},\mathbf{r},\frac{T}{2}). \tag{25}$$

For our model, we have $d=1, D=0, T=t_1+t_2$, so the topological invariant is a one-dimensional loop integration:

$$W[U_\epsilon(\mathbf{k},\frac{T}{2})] = e^{i\epsilon}\frac{i}{2\pi}\int_{\mathbb{T}} dk \times \mathrm{Tr}\Big[\frac{I-\Gamma}{2}U_\epsilon^{-1}\partial_k U_\epsilon\Big]. \tag{26}$$

In a chiral basis ($\Gamma=\sigma_z$), the anomalous time evolution operator can be simplifed into:

$$U_0 = \begin{pmatrix} 0 & U_0^+ \\ U_0^- & 0 \end{pmatrix}, U_\pi = \begin{pmatrix} U_\pi^+ & 0 \\ 0 & U_\pi^- \end{pmatrix}. \tag{27}$$

where $U_\epsilon^\pm$ are unitary matrices. And the topological invariant can be calculated as

$$W[U_\epsilon(\mathbf{k},\frac{T}{2})] = e^{i\epsilon}\frac{i}{2\pi}\int_{\mathbb{T}} dk \times \mathrm{Tr}\Big[U_\epsilon^{+-1}\partial_k U_\epsilon^+\Big]. \tag{28}$$

For the anamolous time evolution operator, we first compute $U(t)$ at any moment.

During one period, the time evolution operator is divided into three forms:

$$
\begin{aligned}
&t \in (0, \frac{t_1}{2}): \ U(t) = \exp\Big[-\frac{i}{\hbar}\mathcal{H}_1 t\Big] = \begin{pmatrix} \cos\frac{vt}{\hbar} & -i\sin\frac{vt}{\hbar} \\ -i\sin\frac{vt}{\hbar} & \cos\frac{vt}{\hbar} \end{pmatrix}, \\
&t \in (\frac{t_1}{2}, \frac{t_1}{2}+t_2): \ U(t) = \exp\Big[-\frac{i}{\hbar}\mathcal{H}_2(t-\frac{t_1}{2})\Big]\exp\Big[-\frac{i}{\hbar}\mathcal{H}_1\frac{t_1}{2}\Big] = \begin{pmatrix} \cos\frac{w(2t-t_1)}{2\hbar} & -ie^{-ik}\sin\frac{w(2t-t_1)}{2\hbar} \\ -ie^{ik}\sin\frac{w(2t-t_1)}{2\hbar} & \cos\frac{w(2t-t_1)}{2\hbar} \end{pmatrix}\begin{pmatrix} \cos\frac{vt_1}{2\hbar} & -i\sin\frac{vt_1}{2\hbar} \\ -i\sin\frac{vt_1}{2\hbar} & \cos\frac{vt_1}{2\hbar} \end{pmatrix}, \\
&t \in (\frac{t_1}{2}+t_2, t_1+t_2): \ U(t) = \exp\Big[-\frac{i}{\hbar}\mathcal{H}_1(t-t_2-\frac{t_1}{2})\Big]\exp\Big[-\frac{i}{\hbar}\mathcal{H}_2 t_2\Big]\exp\Big[-\frac{i}{\hbar}\mathcal{H}_1\frac{t_1}{2}\Big] \\
&\qquad = \begin{pmatrix} \cos\frac{v(2t-2t_2-t_1)}{2\hbar} & -i\sin\frac{v(2t-2t_2-t_1)}{2\hbar} \\ -i\sin\frac{v(2t-2t_2-t_1)}{2\hbar} & \cos\frac{v(2t-2t_2-t_1)}{2\hbar} \end{pmatrix}\begin{pmatrix} \cos\frac{wt_2}{\hbar} & -ie^{-ik}\sin\frac{wt_2}{\hbar} \\ -ie^{ik}\sin\frac{wt_2}{\hbar} & \cos\frac{wt_2}{\hbar} \end{pmatrix}\begin{pmatrix} \cos\frac{vt_1}{2\hbar} & -i\sin\frac{vt_1}{2\hbar} \\ -i\sin\frac{vt_1}{2\hbar} & \cos\frac{vt_1}{2\hbar} \end{pmatrix}.
\end{aligned} \tag{29}
$$

For the aim to calculate Floquet winding number, we only focus on half period $t = \frac{T}{2}$, that is

$$
\begin{aligned}
U(\frac{T}{2}) &= \begin{pmatrix} \cos\frac{\omega}{2}\cos\frac{\nu}{2} - e^{-ik}\sin\frac{\omega}{2}\sin\frac{\nu}{2} & -i\cos\frac{\omega}{2}\sin\frac{\nu}{2} - ie^{-ik}\sin\frac{\omega}{2}\cos\frac{\nu}{2} \\ -i\cos\frac{\omega}{2}\sin\frac{\nu}{2} - ie^{ik}\sin\frac{\omega}{2}\cos\frac{\nu}{2} & \cos\frac{\omega}{2}\cos\frac{\nu}{2} - e^{ik}\sin\frac{\omega}{2}\sin\frac{\nu}{2} \end{pmatrix} \\
&= (\cos\frac{\omega}{2}\cos\frac{\nu}{2} - \sin\frac{\omega}{2}\sin\frac{\nu}{2}\cos k)\sigma^0 + i(-\cos\frac{\omega}{2}\sin\frac{\nu}{2} - \sin\frac{\omega}{2}\cos\frac{\nu}{2}\cos k)\sigma^x \\
&\quad + i(-\sin\frac{\omega}{2}\cos\frac{\nu}{2}\sin k)\sigma^y + i(\sin\frac{\omega}{2}\sin\frac{\nu}{2}\sin k)\sigma^z.
\end{aligned} \tag{30}
$$

For operator $[{U_F}^{-t/T}]_\epsilon$, we know the eigenstates of Floquet operator is obtained already (16), noticing that $E_k = -i\ln_{-\pi}(\tau_0 + i\sqrt{\boldsymbol{\tau}\cdot\boldsymbol{\tau}}) \in [0, \pi]$, then we get

$$
\begin{aligned}
[{U_F}^{-t/T}]_0 &= e^{-iE_k\frac{t}{T}}\left|+\right\rangle\left\langle+\right| + e^{-i(2\pi-E_k)\frac{t}{T}}\left|-\right\rangle\left\langle-\right|, \\
[{U_F}^{-t/T}]_\pi &= e^{-iE_k\frac{t}{T}}\left|+\right\rangle\left\langle+\right| + e^{iE_k\frac{t}{T}}\left|-\right\rangle\left\langle-\right|.
\end{aligned} \tag{31}
$$

We know that $\left|\pm\right\rangle\left\langle\pm\right| = \frac{1}{2}\begin{pmatrix} 1 & \pm e^{-i\phi} \\ \pm e^{i\phi} & 1 \end{pmatrix}$, so we have

$$
\begin{aligned}
[{U_F}^{-t/T}]_0 &= e^{-i\pi\frac{t}{T}}\begin{pmatrix} \cos(\pi - E_k)\frac{t}{T} & ie^{-i\phi}\sin(\pi - E_k)\frac{t}{T} \\ ie^{i\phi}\sin(\pi - E_k)\frac{t}{T} & \cos(\pi - E_k)\frac{t}{T} \end{pmatrix}, \\
[{U_F}^{-t/T}]_\pi &= \begin{pmatrix} \cos E_k\frac{t}{T} & -ie^{-i\phi}\sin E_k\frac{t}{T} \\ -ie^{i\phi}\sin E_k\frac{t}{T} & \cos E_k\frac{t}{T} \end{pmatrix}.
\end{aligned} \tag{32}
$$

At half period $\frac{T}{2}$, it turns out to be

$$
\begin{aligned}
[{U_F}^{-1/2}]_0 &= -i\begin{pmatrix} \sin\frac{E_k}{2} & ie^{-i\phi}\cos\frac{E_k}{2} \\ ie^{i\phi}\cos\frac{E_k}{2} & \sin\frac{E_k}{2} \end{pmatrix} = -i\sin\frac{E_k}{2}\sigma^0 + \cos\frac{E_k}{2}\cos\phi\sigma^x + \sin\frac{E_k}{2}\sin\phi\sigma^y, \\
[{U_F}^{-1/2}]_\pi &= \begin{pmatrix} \cos\frac{E_k}{2} & -ie^{-i\phi}\sin\frac{E_k}{2} \\ -ie^{i\phi}\sin\frac{E_k}{2} & \cos\frac{E_k}{2} \end{pmatrix} = \cos\frac{E_k}{2}\sigma^0 + i(-\sin\frac{E_k}{2}\cos\phi)\sigma^x + i(-\sin\frac{E_k}{2}\sin\phi)\sigma^y.
\end{aligned} \tag{33}
$$

So, for 0-gap, we have

$$
U_0(\frac{T}{2}) = g_0^0\sigma^0 + \boldsymbol{g^0}\cdot\boldsymbol{\sigma}, \tag{34}
$$

where

$$
\begin{aligned}
g_0^0 &= (\cos\frac{\omega}{2}\cos\frac{\nu}{2} - \sin\frac{\omega}{2}\sin\frac{\nu}{2}\cos k)(-i\sin\frac{E_k}{2}) + i(-\cos\frac{\omega}{2}\sin\frac{\nu}{2} - \sin\frac{\omega}{2}\cos\frac{\nu}{2}\cos k)\cos\frac{E_k}{2}\cos\phi,\\
&\quad + i(-\sin\frac{\omega}{2}\cos\frac{\nu}{2}\sin k)\sin\frac{E_k}{2}\sin\phi\\
g_x^0 &= (\cos\frac{\omega}{2}\cos\frac{\nu}{2} - \sin\frac{\omega}{2}\sin\frac{\nu}{2}\cos k)\cos\frac{E_k}{2}\cos\phi + (-\cos\frac{\omega}{2}\sin\frac{\nu}{2} - \sin\frac{\omega}{2}\cos\frac{\nu}{2}\cos k)\sin\frac{E_k}{2},\\
&\quad + (\sin\frac{\omega}{2}\sin\frac{\nu}{2}\sin k)\sin\frac{E_k}{2}\sin\phi\\
g_y^0 &= (\cos\frac{\omega}{2}\cos\frac{\nu}{2} - \sin\frac{\omega}{2}\sin\frac{\nu}{2}\cos k)\sin\frac{E_k}{2}\sin\phi + (-\sin\frac{\omega}{2}\cos\frac{\nu}{2}\sin k)\sin\frac{E_k}{2} - (\sin\frac{\omega}{2}\sin\frac{\nu}{2}\sin k)\cos\frac{E_k}{2}\cos\phi,\\
g_z^0 &= (\sin\frac{\omega}{2}\sin\frac{\nu}{2}\sin k)\sin\frac{E_k}{2} - (-\cos\frac{\omega}{2}\sin\frac{\nu}{2} - \sin\frac{\omega}{2}\cos\frac{\nu}{2}\cos k)\sin\frac{E_k}{2}\sin\phi + (-\sin\frac{\omega}{2}\cos\frac{\nu}{2}\sin k)\cos\frac{E_k}{2}\cos\phi.
\end{aligned}
\tag{35}
$$

For π-gap, we have

$$
U_\pi(\frac{T}{2}) = g_0^\pi\sigma^0 + \boldsymbol{g^\pi}\cdot\boldsymbol{\sigma}, \tag{36}
$$

where

$$
\begin{aligned}
g_0^\pi &= (\cos\frac{\omega}{2}\cos\frac{\nu}{2} - \sin\frac{\omega}{2}\sin\frac{\nu}{2}\cos k)(\cos\frac{E_k}{2}) + (-\cos\frac{\omega}{2}\sin\frac{\nu}{2} - \sin\frac{\omega}{2}\cos\frac{\nu}{2}\cos k)\sin\frac{E_k}{2}\cos\phi,\\
&\quad + (-\sin\frac{\omega}{2}\cos\frac{\nu}{2}\sin k)\sin\frac{E_k}{2}\sin\phi\\
g_x^\pi &= i(\cos\frac{\omega}{2}\cos\frac{\nu}{2} - \sin\frac{\omega}{2}\sin\frac{\nu}{2}\cos k)(-\sin\frac{E_k}{2}\cos\phi) + i(-\cos\frac{\omega}{2}\sin\frac{\nu}{2} - \sin\frac{\omega}{2}\cos\frac{\nu}{2}\cos k)\cos\frac{E_k}{2},\\
&\quad - i(\sin\frac{\omega}{2}\sin\frac{\nu}{2}\sin k)\sin\frac{E_k}{2}\sin\phi\\
g_y^\pi &= -i(\cos\frac{\omega}{2}\cos\frac{\nu}{2} - \sin\frac{\omega}{2}\sin\frac{\nu}{2}\cos k)\sin\frac{E_k}{2}\sin\phi + i(-\sin\frac{\omega}{2}\cos\frac{\nu}{2}\sin k)\cos\frac{E_k}{2} + i(\sin\frac{\omega}{2}\sin\frac{\nu}{2}\sin k)\sin\frac{E_k}{2}\cos\phi,\\
g_z^\pi &= i(\sin\frac{\omega}{2}\sin\frac{\nu}{2}\sin k)\cos\frac{E_k}{2} + i(-\cos\frac{\omega}{2}\sin\frac{\nu}{2} - \sin\frac{\omega}{2}\cos\frac{\nu}{2}\cos k)\sin\frac{E_k}{2}\sin\phi - i(-\sin\frac{\omega}{2}\cos\frac{\nu}{2}\sin k)\sin\frac{E_k}{2}\cos\phi.
\end{aligned}
\tag{37}
$$

I don't know what is going wrong here. In chiral basis, it should be that $U_0(\frac{T}{2})$ has only anti-diagonal parts and $U_\pi(\frac{T}{2})$ has only diagonal parts. And it tunrs out they are not from my computation. So here I don't know how to move forward.

However, I find that in Appendix B of [7] mentioned in [2], the authors used another simpler method to calculate the Floquet winding number in one-dimensional chiral system. I tried their method and successfully get two winding numbers in all phases and they match the previous static result. I guess the method is only suitable for one-dimensional system.

The method is that we only need to get the half period time evolution operator $U(\frac{T}{2})$, and denote its general form as

$$
F := U(\frac{T}{2}) = \begin{pmatrix} \cos f - i\sin f\cos\theta & -i\sin f\sin\theta e^{-i\phi} \\ -i\sin f\sin\theta e^{i\phi} & \cos f + i\sin f\cos\theta \end{pmatrix} = \cos f\sigma^0 - i\sin f\sin\theta\cos\phi\sigma^x - i\sin f\sin\theta\sin\phi\sigma^y - i\sin f\cos\theta\sigma^z, \tag{38}
$$

and the winding number for each gap can be calculated as

$$\begin{aligned} W_0 &= \frac{1}{2\pi}\int dk\partial_k\phi = \frac{1}{2\pi i}\int dk\partial_k \ln\frac{\sin f\sin\theta\cos\phi + i\sin f\sin\theta\sin\phi}{|\sin f\sin\theta\cos\phi + i\sin f\sin\theta\sin\phi|}, \\ W_\pi &= \frac{1}{2\pi}\int dk\partial_k \arctan\frac{\cos f}{\sin f\cos\theta} = \frac{1}{2\pi i}\int dk\partial_k \ln\frac{\sin f\cos\theta + i\cos f}{|\sin f\cos\theta + i\cos f|}. \end{aligned} \tag{39}$$

The authors argued that the geometric visualization of these two winding numbers is: for W_0, it characterizes the winding of a circle constrained on the equator around the origin point of Bloch sphere (of radius π); for W_π, it characterizes the winding of a curve constrained on the surface of Bloch sphere around a circle of radius $\frac{\pi}{2}$ lies inside the Bloch sphere on the $z = 0$ plane.

If we substitute our result into the fomula, then we find, for 0-gap,

$$W_0 = \frac{1}{2\pi i}\int dk\partial_k \ln\frac{\cos\frac{\omega}{2}\sin\frac{\nu}{2} + \sin\frac{\omega}{2}\cos\frac{\nu}{2}\cos k + i\sin\frac{\omega}{2}\cos\frac{\nu}{2}\sin k}{|\cos\frac{\omega}{2}\sin\frac{\nu}{2} + \sin\frac{\omega}{2}\cos\frac{\nu}{2}\cos k + i\sin\frac{\omega}{2}\cos\frac{\nu}{2}\sin k|}, \tag{40}$$

and the winding number of this is just the winding number of a circle $(\tau_x^0, \tau_y^0) : \mathrm{BZ} \to \mathbb{C}$:

$$\Big(\frac{\tau_x^0 - \sin\frac{\nu}{2}\cos\frac{\omega}{2}}{\cos\frac{\nu}{2}\sin\frac{\omega}{2}}\Big)^2 + \Big(\frac{\tau_y^0}{\cos\frac{\nu}{2}\sin\frac{\omega}{2}}\Big)^2 = 1. \tag{41}$$

Just as what we did in static case, the ratio factor which controls the winding number is $\gamma_0 = |\frac{\sin\frac{\nu}{2}\cos\frac{\omega}{2}}{\cos\frac{\nu}{2}\sin\frac{\omega}{2}}|$, and we get the phase diagram:

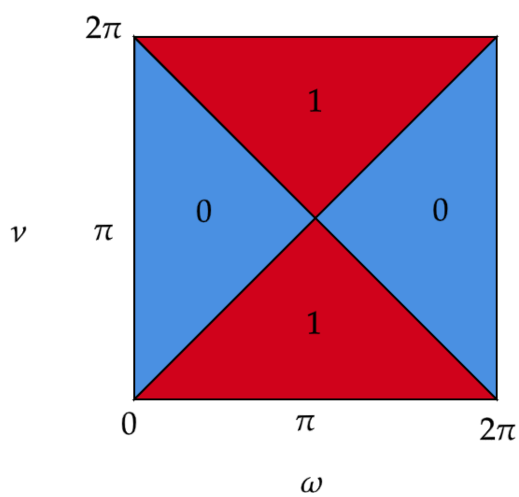


Fig. 3. Phase diagram for 0-mode edge state of Floquet SSH model.

For π-gap,

$$W_\pi = \frac{1}{2\pi i}\int dk\partial_k \ln\frac{-\sin\frac{\omega}{2}\sin\frac{\nu}{2}\sin k + i(\cos\frac{\omega}{2}\cos\frac{\nu}{2} - \sin\frac{\omega}{2}\sin\frac{\nu}{2}\cos k)}{|-\sin\frac{\omega}{2}\sin\frac{\nu}{2}\sin k + i(\cos\frac{\omega}{2}\cos\frac{\nu}{2} - \sin\frac{\omega}{2}\sin\frac{\nu}{2}\cos k)|}, \tag{42}$$

and the winding number of this is just the winding number of another circle $(\tau_x^\pi, \tau_y^\pi) : \mathrm{BZ} \to \mathbb{C}$:

$$\Big(\frac{\tau_x^\pi}{\sin\frac{\nu}{2}\sin\frac{\omega}{2}}\Big)^2 + \Big(\frac{\tau_y^\pi - \cos\frac{\nu}{2}\cos\frac{\omega}{2}}{\sin\frac{\nu}{2}\sin\frac{\omega}{2}}\Big)^2 = 1. \tag{43}$$

The ratio factor is $\gamma_\pi = |\frac{\cos\frac{\nu}{2}\cos\frac{\omega}{2}}{\sin\frac{\nu}{2}\sin\frac{\omega}{2}}|$, and we get the phase diagram:

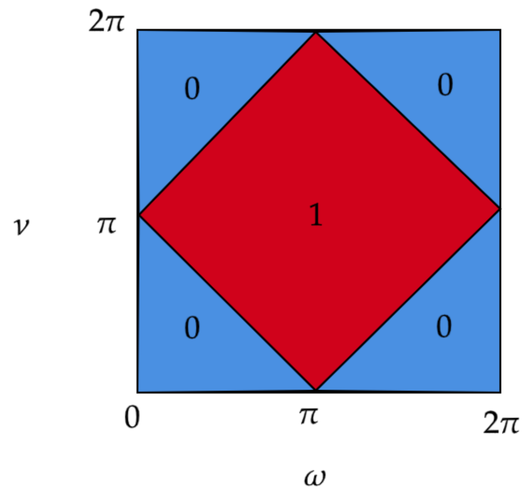


Fig. 4. Phase diagram for π-mode edge state of Floquet SSH model.

If, we define the additive group of the winding numbers as $\mathbb{Z}_2$, we discover that

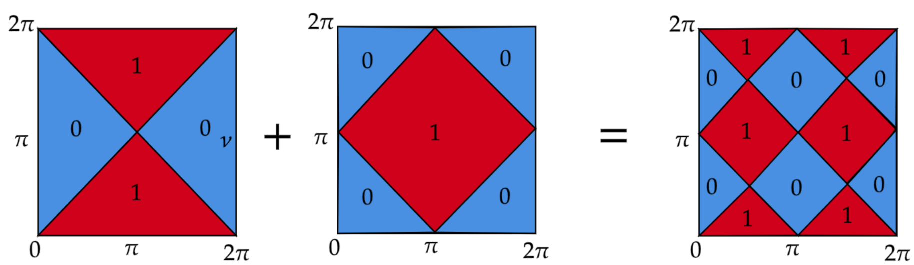


Fig. 5. The combination of two dynamic phase diagram for each gap mode edge state is the static phase diagram.

We find that the two dynamic phase diagram characterized by Floquet winding numbers give the right number of two edge states respectively. But the question is, I don't really know the correctness of this method and its relationship with the normal method.

A Appendix

The code to plot OBC Floquet spectrum.

```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

```
import math
import cmath
import seaborn
import scipy
import functools

N=30 #number of unit cells
v=1 #intracell hopping in time period 1
w=1 #intercell hopping in time period 2
aalpha=0.25
alpha= math.floor(aalpha) #phase index 1
bbeta=1.25
beta = math.floor(bbeta) #phase index 2
t1=(np.pi*(aalpha+bbeta))/(2*v) #time period 1
t2=(np.pi*(bbeta-aalpha))/(2*w) #time period 2
print(t1,t2)

dt=0.001 #step
t_array=np.arange(0,t1+t2,dt) #set time interval
length = len(t_array) #total steps

cm = plt.cm.get_cmap('plasma') #get colorbar

#get interaction parameters during one period
V = [0 for index in range(length)] #intercell hopping array
for i in range(0,length,1):
    t = t_array[i]
    if 0 <= t <= t1:
        V[i]=v
    if t1 < t <= t1+t2:
        V[i]=0
W = [0 for index in range(length)] #intracell hopping array
for i in range(0,length,1):
    t = t_array[i]
    if 0 <= t <= t1:
        W[i]=0
    if t1 < t <= t1+t2:
        W[i]=w

#calculate the evolution operator
evol = [0 for index in range(length)]
for j in range(0,length,1):
    h = np.zeros((2*N,2*N)) #hamiltonian in one step
    for i in range(0,2*N,2):
        h[i,i+1] = V[j]*dt
        h[i+1,i] = V[j]*dt
    for i in range(1,2*N-1,2):
```

```
        h[i,i+1] = W[j]*dt
        h[i+1,i] = W[j]*dt
    evol[j] = scipy.linalg.expm(-1*1j*h)
evol.reverse()

#total Floquet operator
Floquetian = functools.reduce(lambda x,y:np.dot(x,y),evol)
# print(np.round(Floquetian,2))

#solve the eigen-problem
eigenvalue, eigenvector = np.linalg.eig(Floquetian)
quasienergy = [0 for index in range(2*N)]
for i in range(0,2*N,1):
    quasienergy[i] = (cmath.phase(eigenvalue[i])/(t1+t2))

quasienergy.sort()

#number the eigenvalues
k = np.arange(0,2*N)
z = quasienergy
x=np.arange(0,2*N+0.1,0.1)
y0=0*x
y1=0*x+(np.pi)/(t1+t2)
y2=0*x-(np.pi)/(t1+t2)

#draw the spectrum
plt.scatter(k, quasienergy, c=z, s=30, cmap=cm)
plt.plot(x, y0, color="steelblue", alpha=0.6)
plt.plot(x, y1, color="steelblue", alpha=0.6)
plt.plot(x, y2, color="steelblue", alpha=0.6)
plt.colorbar()
plt.xlabel("eigenstate", fontdict={'size': 16})
plt.ylabel("quasienergy", fontdict={'size':16})
my_y_ticks = np.arange(-(np.pi)/(t1+t2), (np.pi)/(t1+t2)+(np.pi)/(4*(t1+t2)), (np.pi)/(4*(t1
    +t2)))
plt.yticks(my_y_ticks)
plt.title("OBC-"+str(N)+"cells-"+"a="+str(round(aalpha,2))+"-b="+str(round(bbeta,2)),
    fontdict={'size': 20})
plt.savefig('OBC-'+str(N)+"-"+str(round(aalpha,2))+"-"+str(round(bbeta,2))+'.jpg', dpi=300)
plt.show()
```

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