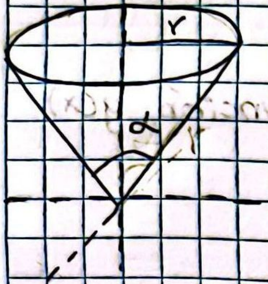
Segunda Tanda de Problemas

1) Calcule la trayectoria que da la distancia más corta entre dos puntos sobre la superficie de un cono invertido con ángulo de vértice α. Use coordenadas cilíndricas.



$$P(x,y,z) \longrightarrow P(r,\theta,z)$$

$$r = \sqrt{x^2+y^2} = z\tan\beta$$

$$\theta = \operatorname{Arctan}\left(\frac{y}{x}\right) \qquad \text{Donde } \beta = \frac{\alpha}{2}$$

$$z = z$$

La distancia más corta entre dos puntos en coordenadas cilíndricas es:

$$\operatorname{Tan}\beta = \frac{r}{z} \Rightarrow z = r\operatorname{Cot}\beta$$

$$dz = \operatorname{Cot}\beta\, dr$$

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$ds^2 = dr^2 + r^2 d\theta^2 + \operatorname{Cot}^2\beta\, dr^2$$

$$ds^2 = dr^2(1+\operatorname{Cot}^2\beta) + r^2 d\theta^2$$

$$ds^2 = \operatorname{Csc}^2\beta\, dr^2 + r^2 d\theta^2$$

$$\frac{ds^2}{d\theta^2} = \operatorname{Csc}^2\beta \frac{dr^2}{d\theta^2} + r^2\frac{d\theta^2}{d\theta^2}$$

$$ds = \left(\operatorname{Csc}^2\beta\left(\frac{dr}{d\theta}\right)^2 + r^2\right)^{1/2} d\theta$$

* Parametrizamos r en función de θ, de esta forma la longitud entre los dos puntos dependerá del ángulo que se recorra en la superficie del cono.

$$L = \int_{\theta_1}^{\theta_2}\left[\operatorname{Csc}^2\beta\left(\frac{dr}{d\theta}\right)^2 + r^2\right]^{1/2} d\theta$$

→ Definimos la función lagrangiana como:

* Aplicando la ecuación de Euler-Lagrange

$$\frac{d}{d\theta}\left(\frac{\partial \mathcal{L}}{\partial r'}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \qquad \text{donde } r' = \frac{dr}{d\theta} \ \wedge \ \mathcal{L} = \left[\operatorname{Csc}^2\beta (r')^2 + r^2\right]^{1/2}$$

- $$\frac{\partial \mathcal{L}}{\partial r'} = \frac{1}{2}\cdot\frac{2\cdot \operatorname{Csc}^2\beta\, r'}{(\operatorname{Csc}^2\beta(r')^2+r^2)^{1/2}} = \frac{\operatorname{Csc}^2\beta\, r'}{(\operatorname{Csc}^2\beta(r')^2+r^2)^{1/2}}$$

- $$\frac{d}{d\theta}\left(\frac{\partial \mathcal{L}}{\partial r'}\right) = \frac{d}{d\theta}\left(\frac{\operatorname{Csc}^2\beta\, r'}{(\operatorname{Csc}^2\beta(r')^2+r^2)^{1/2}}\right) \Rightarrow u = \operatorname{Csc}^2\beta\, r' \ \wedge \ v = (\operatorname{Csc}^2\beta(r')^2+r^2)^{1/2}$$

$$u' = \operatorname{Csc}^2\beta\, r'' \ \wedge \ v' = \frac{2(\operatorname{Csc}^2\beta\, r')}{2(\operatorname{Csc}^2\beta(r')^2+r^2)^{1/2}}$$

$$\Rightarrow \frac{(\operatorname{Csc}^2\beta\, r'')(\operatorname{Csc}^2\beta(r')^2+r^2)^{1/2} - (\operatorname{Csc}^2\beta\, r')\left(\frac{\operatorname{Csc}^2\beta\, r'}{(\operatorname{Csc}^2\beta(r')^2+r^2)^{1/2}}\right)}{\operatorname{Csc}^2\beta(r')^2+r^2}$$

$$\frac{\frac{(\csc^2\beta r'')(\csc^2\beta (r')^2 + r^2) - \csc^4\beta r'}{(\csc^2\beta (r')^2 + r^2)^{1/2}}}{\csc^2\beta (r')^2 + r^2}$$

2) Calcule el valor mínimo de la integral donde la función $y(x)$ satisface $y(0)=0$ y $y(1)=1$

$$I = \int_0^1 \left[(y')^2 + 12xy \right] dx$$

→ Definimos la cantidad que queremos extremar:

$$\mathcal{L} = \left(\frac{dy}{dx}\right)^2 + 12xy \qquad \text{→ Aplicando Euler-Lagrange} \qquad \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial \mathcal{L}}{\partial y}$$

- $\frac{\partial \mathcal{L}}{\partial y'} = 2y'$ • $\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = 2y''$

- $\frac{\partial \mathcal{L}}{\partial y} = 12x \Rightarrow \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial \mathcal{L}}{\partial y} = 2y'' - 12x = 0$

→ Despejando el valor de y'' se obtiene:

- $y'' = \frac{12x}{2} = 6x$ • $y' = \int 6x\,dx = \frac{6x^2}{2} + C_1 = 3x^2 + C_1$

- $y = \int 3x^2 + C_1\,dx = x^3 + C_1 x + C_2$ → Reemplazamos las condiciones de frontera para hallar C_1 y C_2.

* $y(0) = 0$

$y(0) = (0)^3 + C_1(0) + C_2 = 0$

$C_2 = 0$

* $y(1) = 1$

$y(1) = (1)^3 + C_1(1) + 0 = 1$

$C_1 = 1 - 1 = 0$

→ La función $y(x)$ que extrema la integral y cumple las condiciones de frontera es $y(x) = x^3$ ∧ $y'(x) = 3x^2$

$$I = \int_0^1 \left[(3x^2)^2 + 12x(x^3)\right] dx = \int_0^1 9x^4 + 12x^4\,dx = \int_0^1 21x^4\,dx$$

$$I = 21\frac{x^5}{5}\Big|_0^1 = 21/5$$ → Es el valor mínimo de la integral

3) Encuentre la geodésica (i.e. la trayectoria de menor distancia) entre los puntos $P_1=(a,0,0)$ y $P_2=(-a,0,\pi)$ sobre la superficie $x^2+y^2-a^2=0$. Use coordenadas cilíndricas.

- $P_1=(a,0,0)$
- $P_2=(-a,0,\pi)$

Sobre una superficie:

$$x^2+y^2-a^2=0$$

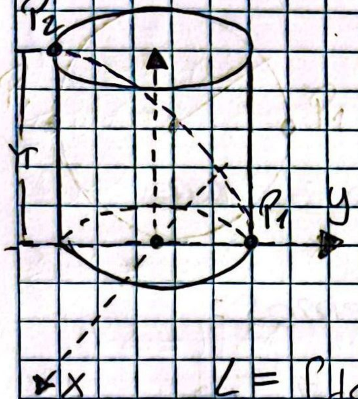
$$x^2+y^2=a^2$$

$$\theta=\tan^{-1}\left(\frac{x}{y}\right)$$

$$r=\sqrt{x^2+y^2}$$

$$z=z$$

$$r=a=\sqrt{x^2+y^2}$$



→ El radio del cilindro es a.

$$ds^2=r^2\dot\theta^2+\dot z^2+\dot r^2 \qquad r^2=a^2 \quad \dot r=0$$

$$ds^2=r^2\dot\theta^2+\dot z^2\left[\frac{1}{d\theta}\right] \qquad r=a$$

$$ds=\sqrt{r^2+\left(\frac{dz}{d\theta}\right)^2}\,d\theta$$

$$L=\int ds=\int\left(r^2+(z')^2\right)^{1/2}d\theta$$

- Mi lagrangiano será:

$$\mathcal{L}(z,z',\theta)=\left(r^2+(z')^2\right)^{1/2}d\theta$$

- Dado que el lagrangiano no depende explícitamente de θ podemos aplicar la identidad de Beltrami.

$$\mathcal{L}-z'\frac{\partial\mathcal{L}}{\partial z'}=C \quad \to \frac{\partial\mathcal{L}}{\partial z'}=\frac{1}{2}\frac{2z'}{(r^2+(z')^2)^{1/2}}=\frac{z'}{(r^2+(z')^2)^{1/2}}$$

Reemplazando en la identidad:

$$\left(r^2+(z')^2\right)^{1/2}-z'\left(\frac{z'}{(r^2+(z')^2)^{1/2}}\right)=\frac{r^2+(z')^2-(z')^2}{(r^2+(z')^2)^{1/2}}$$

$$=\frac{r^2}{(r^2+(z')^2)^{1/2}}=C$$

- Elevando al cuadrado ambos lados de la igualdad.

$$z'=\pm\left(\left(\frac{r^2}{c}\right)^2-r^2\right)^{1/2}=Cte \Rightarrow \frac{\partial\mathcal{L}}{\partial z'}=\frac{z'}{(r^2+(z')^2)^{1/2}}=K$$

$$\Rightarrow \frac{(z')^2}{r^2+(z')^2}=K^2 \Rightarrow (z')^2=\frac{K^2r^2}{(1-K^2)}=\pm\frac{Kr}{\sqrt{1-K^2}}=Cte$$

FABRIFOLDER S.A.S

Dado que z' es una constante, la solución que minimiza la longitud es una función lineal de la forma:

$$z(\theta) = A\theta + B$$

Aplicando las condiciones de frontera:

$P_1 \rightarrow z(0) = 0$ $\qquad$ $P_2 \rightarrow z(\pi) = \pi$

$$z(0) = A(0) + B$$
$$0 = 0 + B$$
$$B = 0$$

$$z(\pi) = A\pi$$
$$\pi = A\pi$$
$$A = 1$$

$$z(\theta) = \theta$$

Esta función corresponde a la trayectoria de una línea recta con pendiente 1.

Calculando la longitud de la geodecica obtenemos:

$$L = \int_0^{\pi} (r^2 + (z')^2)^{1/2} d\theta \qquad \text{donde} \quad \frac{dz}{d\theta} = \frac{d}{d\theta}(\theta) = 1$$

$$L = \pi\sqrt{r^2+1}$$

Desarrollo del Primer Punto usando la identedad de Beltrami:

$$\mathcal{L} = [\csc^2\beta (r')^2 + r^2]^{1/2} \qquad \mathcal{L}(\theta, r, r')$$

Dado que el lagrangiano no depente explicitamente de θ, usamos la identidad de Beltrami:

$$\mathcal{L} - r'\frac{\partial \mathcal{L}}{\partial r'} = C \qquad \cdot \frac{\partial \mathcal{L}}{\partial r'} = \frac{1}{2}\frac{2r'\csc\beta}{(\csc^2\beta(r')^2 + r^2)^{1/2}}$$

$$[\csc^2\beta(r')^2 + r^2]^{1/2} - r'\left(\frac{r'\csc\beta}{(\csc^2\beta(r')^2+r^2)^{1/2}}\right) = C$$

$$= \frac{\csc^2\beta(r')^2 + r^2 - (r')^2\csc\beta}{(\csc^2\beta(r')^2 + r^2)^{1/2}} = C$$

$$= \frac{r^2}{(\csc^2\beta(r')^2 + r^2)^{1/2}} = C$$

$$= \frac{(r^2)^2}{\csc^2\beta(r')^2 + r^2} = C$$

$$\Rightarrow (r^2)^2 = C(\csc^2\beta(r')^2 + r^2)$$

$$\frac{r^4 - r^2 C}{\csc^2\beta C} = (r')^2$$

FABRIFOLDER S.A.S

4. La acción está definida como

$$S = \int_0^T \mathcal{L}\, dt \rightarrow \mathcal{L} = T - U$$

$$T = \frac{1}{2} m \dot{y}^2 \quad \wedge \quad U = \frac{1}{2} m g y$$

• Si $y = h - g_1 t$

$$\dot{y} = -g_1$$

$$\rightarrow \mathcal{L} = \frac{1}{2} m g_1^2 - mg(h - g_1 t)$$

Por lo tanto la acción es

$$S_1 = \int_0^T \left(\frac{1}{2} m g_1^2 - mg(h - g_1 t) \right) dt$$

$$S_1 = \left[\frac{1}{2} m g_1^2 t - mg\left(ht - \frac{1}{2} g_1 t^2\right) \right]_0^T$$

Sustituyendo $T = \frac{h}{g_1} \rightarrow h - g_1 T = 0$

$$S_1 = \frac{1}{2} m g_1 h - mg\left(h \frac{h}{g_1} - \frac{1}{2} g_1 \frac{h^2}{g_1^2} \right)$$

$$S_1 = \frac{1}{2} m g_1 h - mg\left(\frac{1}{2} \frac{h^2}{g_1} \right)$$

• Si $y = h - \frac{1}{2} g_2 t^2$

$\dot{y} = -g_2 t$

$\mathcal{L} = \frac{1}{2} m g_2^2 t^2 - mg\left(h - \frac{1}{2} g_2 t^2\right)$

la acción es

$S_2 = \int_0^T \frac{1}{2} m g_2^2 t^2 - mg\left(h - \frac{1}{2} g_2 t^2\right) dt$

$S_2 = \left[\frac{1}{2} m g_2^2 \frac{t^3}{3} - mg\left(ht - \frac{1}{2} g_2 \frac{t^3}{3}\right)\right]_0^T$

Reemplazamos $T = \sqrt{2h/g_2}$

$S_2 = \frac{1}{2} m g_2^2 \frac{(\sqrt{2h/g_2})^3}{3} - mg\left(h\left(\frac{2h}{g^2}\right)^{\frac{1}{2}} - \frac{1}{2} g_2 \frac{(\sqrt{2h/g_2})^3}{3}\right.$

$S_2 = \frac{1}{6} m g_2^{1/2} (2h)^{3/2} - mgh \frac{(2h)^{1/2}}{g_2^{1/2}} + \frac{1}{6} m g g_2^{1/2} (2h)^{3/2}$

$\boxed{S_2 = -mgh \frac{(2h)^{1/2}}{g_2^{1/2}}}$

• Si $y = h - \frac{1}{4} g_3 t^3$

$\dot{y} = -\frac{3}{4} g_3 t^2$

$\mathcal{L} = \frac{1}{2} m\left(\frac{9}{16} g_3^2 t^4\right) - mg\left(h - \frac{1}{4} g_3 t^3\right)$

$$S_3 = \left[\frac{9}{32} m g_3^2 \frac{t^5}{5} - mg\left(ht - \frac{1}{4} g_3 \frac{t^4}{4}\right)\right]_0^T$$

Reemplazamos $T = (4h/g_3)^{1/3}$

$$S_3 = \frac{9}{32} m g_3^2 \frac{\left(\frac{4h}{g_3}\right)^{5/3}}{5} - mg\left(h\left(\frac{4h}{g_3}\right)^{\frac{1}{3}} - \frac{1}{4} g_3 \frac{\left(\frac{4h}{g_3}\right)^{\frac{4}{3}}}{4}\right)$$

$$S_3 = \frac{9}{160} m g_3^{1/3} (4h)^{5/3} - mgh \frac{(4h)^{\frac{1}{3}}}{g_3^{1/3}} + \frac{1}{16} mg g_3^{1/3} (4h)^{4/3}$$

$$\boxed{S_3 = mg\left(\frac{9}{160} g_3^{1/3} (4h)^{5/3} + \frac{1}{16} g_3^{1/3} (4h)^{4/3} - \frac{h(4h)^{\frac{1}{3}}}{g_3^{1/3}}\right)}$$

Punto 5

$$\mathcal{L} = \frac{m^2 \dot{x}^4}{12} + m\dot{x}^2 f(x) - f^2(x)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \rightarrow \text{Ecu. Euler-Lagrange}$$

- $$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{m^2 \dot{x}^3}{3} + 2m\dot{x} f(x)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m^2 \dot{x}^2 \ddot{x} + 2m\ddot{x} f(x) + 2m\dot{x}^2 f'(x)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial x} = m\dot{x}^2 f'(x) - 2f(x) f'(x)$$

Reescribimos la ecuación de euler-Lagrange y simplificamos

$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + 2m\dot{x}f'(x) - m\dot{x}^2 f'(x) + 2f(x)f'(x) = 0$$

$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + (2m-m)\dot{x}^2 f'(x) + 2f(x)f'(x) = 0$$

$$m^2\dot{x}^2\ddot{x} + 2m\ddot{x}f(x) + m\dot{x}^2 f'(x) + 2f(x)f'(x) = 0$$

$$\ddot{x}(m^2\dot{x}^2 + 2mf(x)) + \frac{f'(x)}{m}(m^2x^2 + 2mf(x)) = 0$$

$$(m^2\dot{x}^2 + 2mf(x))(m\ddot{x} + f'(x)) = 0$$

$$\rightarrow m^2\dot{x}^2 + 2mf(x) = 0$$

$$\rightarrow m\ddot{x} + f'(x) = 0$$

6. $$\mathcal{L} = \frac{1}{2} g_{ab}(q_c)\, \dot{q}^a \dot{q}^b$$ → Lagrangiano puramente cinético

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\frac{\partial g_{bd}}{\partial q^c} + \frac{\partial g_{cd}}{\partial q^b} - \frac{\partial g_{bc}}{\partial q^d}\right)$$

↳ def. del simbolo de Christoffel

∘ Usamos la ecuación de euler - Lagrange

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) - \frac{\partial \mathcal{L}}{\partial q^a} = 0$$

$$\rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{1}{2} \frac{\partial}{\partial \dot{q}^a}\left(g_{bc}\, \dot{q}^b \dot{q}^c\right)$$

g_{bc} es la metrica del espacio de configuración que puede depender de las coordenadas q^a pero no de las velocidades $\dot{q}^a$.

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{1}{2} g_{bc} \frac{\partial}{\partial \dot{q}^a}\left(\dot{q}^b \dot{q}^c\right)$$

$$\therefore \frac{\partial}{\partial \dot{q}^a}\left(\dot{q}^a \dot{q}^c\right) = \delta^b_a \dot{q}^c + \dot{q}^b \delta^c_a = \dot{q}^c \delta^b_a + \dot{q}^b \delta^c_a$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{1}{2} g_{bc}\left(\dot{q}^c \delta^b_a + \dot{q}^b \delta^c_a\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = g_{ab}\, \dot{q}^b \quad (1)$$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \left(\frac{\partial g_{ab}}{\partial q^c}\dot{q}^c\right)\dot{q}^b + g_{ab}\ddot{q}^b \quad (2)$$

$$\rightarrow \frac{\partial \mathcal{L}}{\partial q^a} = \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\dot{q}^b\dot{q}^c \quad (3)$$

Sustituimos todos los terminos (2) y (3) en la ecuación de euler-lagrange.

$$\left(\frac{\partial g_{ab}}{\partial q^c}\dot{q}^c\right)\dot{q}^b + g_{ab}\ddot{q}^b - \frac{1}{2}\frac{\partial g_{bc}}{\partial q^a}\dot{q}^b\dot{q}^c = 0$$

Multiplicamos por la metrica inversa g^{ad}

$$\ddot{q}^d + \underbrace{\frac{1}{2}g^{ad}\left(\frac{\partial g_{bc}}{\partial q^a} + \frac{\partial g_{ac}}{\partial q^b} - \frac{\partial g_{ab}}{\partial q^c}\right)}_{\text{Simbolo de Christoffel } \Gamma^d_{bc}}\dot{q}^b\dot{q}^c = 0$$

Ec. de movimiento

$$\boxed{\ddot{q}^a + \Gamma^a_{bc}\dot{q}^b\dot{q}^c = 0}$$