2. Una partícula se mueve en el potencial:

$V(r) = \frac{\alpha}{r} + \frac{b}{r^2}$.

a) Determine la órbita de la partícula $r = r(\theta)$

La posición de la partícula esta dada por:

$\vec{r} = r\hat{r} \longrightarrow \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\rightarrow$ Su lagrangiano es: $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \alpha/r - b/r^2$

Que, como el potencial no depende de las velocidades y el lagrangiano es independiente del tiempo; nos da:

$$\frac{dL}{dt} = \frac{\partial L}{\partial r}\dot{r} + \frac{\partial L}{\partial \dot{r}}\ddot{r} + \cancel{\frac{\partial L}{\partial t}} \rightarrow 0$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)\dot{r} + \frac{\partial L}{\partial \dot{r}}\ddot{r} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\dot{r}\right]$$

$$\rightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\dot{r} - L\right] = 0 \longrightarrow \frac{\partial L}{\partial \dot{r}}\dot{r} - L = cte$$

$$\rightarrow 2T - T + V = T + V = E_m = cte.$$

i.e, $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\alpha}{r} + \frac{b}{r^2}$

$$\rightarrow \frac{1}{2}m\dot{r}^2 = E - \frac{\alpha}{r} - \frac{b}{r^2} - \frac{m}{2}r^2\dot{\theta}^2 \geq 0$$

$$\rightarrow \dot{r} = \sqrt{\frac{2}{m}\left(E - \frac{\alpha}{r} - \frac{b}{r^2} - \frac{m r^2\dot{\theta}^2}{2}\right)} = \frac{dr}{dt}$$

$$\therefore \int_{t_0=0}^{t} dt = \int_{r_0}^{r} \sqrt{\frac{m}{2}} \frac{dr}{\sqrt{E - \frac{\alpha}{r} - \frac{b}{r^2} - \frac{m}{2}r^2\dot{\theta}^2}} \Rightarrow t(r)$$

$$\mu\ddot{r} = f(r) + \frac{l^2}{\mu r^3}$$

Ahora, debido a que el potencial es central, tenemos que:

$$\tau = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = 0 \leftrightarrow \frac{d}{dt}\vec{L} = 0 \leftrightarrow \vec{L} = cte$$

$\rightarrow$ Tanto el ángulo como el módulo del momento angular se conservan. Por lo cual, podemos obtener del lagrangiano que:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \cancel{\frac{\partial L}{\partial \theta}}^{0} = 0 \leftrightarrow \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = l$$

Por lo cual podemos hacer que:

$$\frac{d\theta}{dt} = \frac{d\theta}{dr}\frac{dr}{dt} = \frac{l}{mr^2}$$

$$\rightarrow \frac{d\theta}{dr} = \frac{l}{mr^2}\frac{1}{\frac{dr}{dt}} = \frac{l}{mr^2}\frac{1}{\sqrt{\frac{2}{m}\left(E - \frac{1}{2}mr^2\dot{\theta}^2 - \frac{a}{r} - \frac{b}{r^2}\right)}}$$

$-$ Pero de $mr_0^2\dot{\theta} = l$ $\quad \frac{l^2}{2mr^2}$

$$\rightarrow \int_{\theta_0}^{\theta} d\theta = \frac{l}{m}\int_{r_0}^{r}\frac{1}{r^2}\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{2}{m}\left(E - \frac{l^2}{2mr^2} - \frac{\alpha}{r} - \frac{b}{r^2}\right)}}$$

$$\therefore \theta(r) = \frac{l}{m}\int_{r_0}^{r}\frac{dr}{r^2\sqrt{\frac{2}{m}\left(E - \frac{l^2}{2mr^2} - \frac{\alpha}{r} - \frac{b}{r^2}\right)}} + \theta_0$$

¡Pero nos piden una ecuación del movimiento para $r(\theta)$! Entonces, de la ecuación de Euler-Lagrange para r:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \rightarrow m\ddot{r} - mr\dot{\theta}^2 + \frac{a}{r^2} + \frac{2b}{r^3} = 0 \quad (1)$$

Pero, haciendo lo siguiente:

$$\frac{dg}{dt} = \frac{dg}{d\theta}\frac{d\theta}{dt} = \frac{l}{mr^2}\frac{dg}{d\theta} \rightarrow \frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$$

para una función g arbitraria. Luego,

$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right)$$

$$= \frac{l}{mr^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) \quad (L)$$

Por lo cual, en la ecuación (1):

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{a}{r^2} + \frac{2b}{r^3} = 0$$

$$\rightarrow m\ddot{r} - \frac{l^2}{mr^3} = -\frac{a}{r^2} - \frac{2b}{r^3} = f(r) = -\frac{\partial V}{\partial r} \quad (x)$$

Pero $\frac{\partial V}{\partial r} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial r}$

Qu, para $u = \frac{1}{r} \rightarrow \frac{\partial V}{\partial r} = \frac{\partial V}{\partial u}\left(-\frac{1}{r^2}\right) = \frac{\partial V}{\partial u}(-u^2)$

$\rightarrow$ en (x), con (L):

$$\frac{\partial u}{\partial \theta} = \left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial \theta} = -u^2\frac{\partial r}{\partial \theta}$$

$$\rightarrow \frac{l^2u^2}{m}\frac{d}{d\theta}\left(-\frac{\partial u}{\partial \theta}\right) - \frac{l^2u^3}{m} = -u^2\frac{\partial V}{\partial u}$$

$$\rightarrow \frac{l^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\partial V}{\partial u}$$

La anterior ecuación constituye la ecuación diferencial de la órbita para $r(\theta) = 1/u(\theta)$ en términos del potencial $V(r) = V(1/u)$

$$\rightarrow \frac{\partial V}{\partial u} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial u} = \left(-\frac{a}{r^2} - \frac{2b}{r^3}\right)\left(-\frac{1}{u^2}\right)$$

$$= (au^2 + 2bu^3)\left(\frac{1}{u^2}\right) = a + 2bu$$

$$\rightarrow -\frac{\partial V}{\partial u} = a + 2bu = \frac{l^2}{m}\left(\frac{\partial^2 u}{\partial \theta} + u\right)$$

$$\rightarrow a + 2bu - \frac{l^2}{m}u = \frac{l^2}{m}\frac{\partial^2 u}{\partial \theta^2}$$

$$\rightarrow a + u\left(2b - \frac{l^2}{m}\right) = \frac{l^2}{m}\frac{\partial^2 u}{\partial \theta^2}$$

$$a + u\left(2b - \frac{l^2}{m}\right) = \frac{l^2}{m}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)$$

$$\frac{l^2}{m}\frac{\partial^2 u}{\partial \theta^2} + \left(\frac{l^2}{m} - 2b\right)u = a$$

$$\rightarrow \frac{\partial^2 u}{\partial \theta^2} + \left(1 - \frac{2bm}{l^2}\right)u = \frac{ma}{l^2}$$

Sea $m^2 + \left(1 - \frac{2bm}{l^2}\right) = 0$

$$\rightarrow m^2 = \frac{2bm}{l^2} - 1$$

$$\rightarrow m_{1,2} = \pm\sqrt{\frac{2bm}{l^2} - 1} = \pm\omega_0$$

$\therefore u_c(\theta) = Ae^{\omega_0 \theta} + Be^{-\omega_0 \theta}$

Ahora, sea $\frac{\partial^2 u}{\partial \theta^2} + \left(1 - 2\frac{mb}{l^2}\right)u = \frac{ma}{l^2}$

$\longrightarrow$ Sea $y_p(\theta) = C$

$\longrightarrow \left(1 - \frac{2mb}{\ell^2}\right) C = \frac{ma}{\ell^2}$

$\longrightarrow C = \frac{ma}{\ell^2 - 2mb}$

$\therefore u(\theta) = u_c + u_p$

$= A e^{w_0 \theta} + B e^{-w_0 \theta} + \frac{ma}{\ell^2 - 2mb}$

pero, como $r = 1/u$,

$\longrightarrow r(\theta) = \frac{1}{u(\theta)} = \frac{1}{A e^{w_0 \theta} + B e^{-w_0 \theta} + \frac{ma}{\ell^2 - 2mb}}$

Un cometa de masa $M_T/8$ ... colisión ...

(b) Dibuje esquemáticamente el potencial efectivo para este sistema.

$$E = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\alpha}{r} + \frac{b}{r^2}$$

$$\rightarrow V_{ef}(r) = r^2\dot{\theta}^2 + \frac{a}{r} + \frac{b}{r^2}$$