# 1 Approximation

## 1.1 Newton-Raphson process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

## 1.2 Linear interpolation

Draw triangles, use similar triangles.

## 1.3 Interval bisection

| $a$ | $f(a)$ | $b$ | $f(b)$ | $\frac{a+b}{2}$ | $f\left(\frac{a+b}{2}\right)$ |
|---|---|---|---|---|---|
| 2 | $-1$ | 3 | 2 | 2.5 | 0.1569 |
| 2 | $-1$ | 2.5 | 0.1569 | 2.25 | $-0.493$ |

# 2 Summation of Series

## 2.1 Summation of Series

$$\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$$

$$\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{x=1}^{n} x^3 = \frac{n^2(n+1)^2}{4}$$

## 2.2 Summation of Arithmetic Progression

$$S_n = a_1 n + \frac{(n)(n-1)d}{2}$$

$$S_n = a_0 n + \frac{(n)(n+1)d}{2}$$

$$S_n = \frac{n \times (a_1 + a_n)}{2}$$

$$S_n = n \times a_{\frac{n+1}{2}}$$

## 2.3 Summation of Geometric Progression

$$S_n = \frac{a_1 \times (1 - q^n)}{1 - q}$$

$$S_\infty = \frac{a_1}{1 - q}$$

# 3 Matrices

## 3.1 Transformations

### 3.1.1 Enlargement

- Stretch in x-direction by a scale factor $k$: $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
- Stretch in y-direction by a scale factor $k$: $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
- Enlargement with centre of the origin by a scale factor $k$: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

### 3.1.2 Reflection

- Reflection in x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection in y-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection in $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Reflection in $y = -x$: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

### 3.1.3 Rotation

- Rotation about the origin by $\theta$ anti-clockwise: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

### 3.1.4 A point to a point

Point $A$ is transformed by $T$, resultant point is $T \cdot A$.

### 3.1.5 A line to a line

Equation $\begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \\ a_3 + tb_3 \end{pmatrix}$ is transformed by $T$, resultant line is $T \cdot \begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \\ a_3 + tb_3 \end{pmatrix}$.

## 3.2 Inverse matrix 2*2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det A = ad - bc$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If $\det A = 0$, A is singular, so A has no inverse.

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

## 3.3 Inverse matrix 3*3

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} A & -B & C \\ -D & E & -F \\ G & -H & I \end{pmatrix}^{T}$$

where

$$A = \begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - hf$$

$$\Delta = aA - bB + cC$$

### 3.3.1 Transpose

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{T} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

That is, rows become corresponding columns.

### 3.4 Calculating area of an triangle

$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$

$$A = \frac{1}{2}\left(x_2y_1 + x_3y_2 + x_1y_3 - x_1y_2 - x_2y_3 - x_3y_1\right)$$

## 4 Complex Numbers

1) Translation

   $w = z + a + bi$ : translation by $\begin{pmatrix} a \\ b \end{pmatrix}$

2) Enlargement

   $w = kz$ : enlargement by a scale factor k

3) Enlargement followed by translation

   $w = kz + a + bi$ : enlargement by a scale factor k followed by a translation by $\begin{pmatrix} a \\ b \end{pmatrix}$

### 4.1 Transformations

#### 4.1.1 Example 1

Find the transformation $w = \frac{1}{z}, z! = 0$, find the locus of $w$ when $z$ lies on the line with equation $y = 2x + 1$

$$x + yi = \frac{1}{u + vi} = \frac{u - vi}{u^2 + v^2} = \frac{u}{u^2 + v^2} + \frac{-v}{u^2 + v^2}i$$

## 5 Vector

### 5.1 Scalar product ($\cdot$)

$$\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 + z_1 + z_2$$

#### 5.1.1 Angle between $\vec{a}$ and $\vec{b}$

$$\cos\theta = \frac{\vec{a}\vec{b}}{|\vec{a}|\left|\vec{b}\right|}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right|\cos\theta$$

### 5.2 Vector product ($\times$)

Vector product is perpendicular to both the vectors.

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right|\sin\theta$$

$$\vec{a} \times \vec{b} = |\vec{a}|\left|\vec{b}\right|\sin\theta \cdot \vec{u}$$

### 5.3 Calculate Area of a Triangle

$$A = \frac{1}{2}|\vec{a}|\left|\vec{b}\right|\sin\theta = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

### 5.4 Calculate Area of a Tetrahedron

$$A = \frac{1}{6}\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\sin\theta$$

### 5.5 Calculate Area of a Prism

$$A = \frac{1}{6}\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) \sin\theta$$

### 5.6 Cartesian equation of a straight line

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \lambda$$

where

$$fixedpoint(a_1, a_2, a_2)$$

,

$$direction = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

### 5.7 Vector equation of a straight line

$$\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$$

$$\vec{r} = \vec{a} + t\vec{b}$$

### 5.8 Find the distance from a point to the line using vector product

$$\frac{\left|\vec{d}\right|}{\left|\vec{AP}\right|} = \sin\theta$$

$$\vec{d} = \frac{\left|\vec{AP} \times \vec{b}\right|}{\left|\vec{b}\right|}$$

### 5.9 Find the shortest distance between two lines

- Two parallel lines, choose one point from a line and calculate by

$$\vec{d} = \frac{\left|\vec{AP} \times \vec{b}\right|}{\left|\vec{b}\right|}$$

.

- Two skew lines:

$$\vec{r_1} = \vec{a} + \lambda\vec{b} \quad (1)$$

$$\vec{r_2} = \vec{c} + \lambda\vec{d} \quad (2)$$

$$d = \left| \frac{(\vec{a} - \vec{c}) \cdot \left(\vec{b} \times \vec{d}\right)}{\left|\vec{b} \times \vec{d}\right|} \right|$$

### 5.10 Vector Equation of a Plane

$$\vec{n} \cdot (\vec{r} - \vec{a}) = 0$$

### 5.11 Cartesian Equation of a Plane

$$ax + by + cz = d = \vec{n} \cdot \vec{a}, \vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where $\vec{a}$ is a fixed point.

### 5.12 Determine a Plane with two lines

1. Find normal of the plane by $\vec{b_1} \times \vec{b_2}$
2. Find intersection
3. Express in cartesian form

### 5.13 Find the distance between a point and a plane

Plane: $\vec{r} \cdot \vec{n} = ax + by + cz = d$, Point: $P(x_1, y_1, z_1)$

#### 5.13.1 Method 1, use formulae directly

$$d = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

#### 5.13.2 Method 2, use perpendicular foot

$F$ is the perpendicular foot of $P$ to the plane.

Step 1. Find  for line $PF$, which is a expression of $F$

Step 2. $F$ is in the plane, so put F into the equation of the plane and find $F$.

Step 3. Calculate distance by $\left|\vec{PF}\right|$

### 5.14 Line and plane

#### 5.14.1 The line lies in the plane

Method 1. Prove two random points lie in the plane

Method 2. Prove all points are in the plane

#### 5.14.2 The line is parallel to the plane

Method 1. $\vec{b}\vec{n} = 0$,

#### 5.14.3 The line is intersecting with the plane

Method 1. Find point of intersection

Method 2. Find angle between $l$ and the plane

### 5.15 Tow planes

#### 5.15.1 Distance between two planes

$$ax + by + cz = d_1 \tag{3}$$

$$ax + by + cz = d_2 \tag{4}$$

$$d = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

## 6 Differentiation

### 6.1 First order differentiation

$$f(x)\frac{dy}{dx} + f'(x)y = \frac{d(f(x)y)}{dx}$$

Integration factor: $\boxed{e^{\int pdx}}$

$$\frac{dy}{dx} + py = Q \rightrightarrows \frac{d(\boxed{e^{\int pdx}}y)}{dx} = \boxed{e^{\int pdx}}Q$$

### 6.2 Second order differentiation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

#### 6.2.1 Auxiliary equation

$$am^2 + bm + c = 0$$

If $\Delta > 0$, it has two distinct roots $\alpha$, $\beta$. General solution:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

If $\Delta = 0$, it has two repeated roots. General solution:

$$y = (A + Bx)e^{\alpha x}$$

If $\Delta < 0$, it has two complex roots, $p + qi$ and $p - qi$. General solution:

$$y = e^{px}(A\cos qx + B\sin qx)$$

#### 6.2.2 Example for finding a general solution for Second order differentiation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$
$$a = 1, b = 5, c = 6$$
$$m^2 + 5m + 6 = 0$$
$$m = -2 or m = -3$$
$$y = Ae^{-2x} + Be^{-3x}$$

#### 6.2.3 Complementary functions

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Solution: $y = complementary\ function + particular\ integral$

Particular integral is the general form of $f(x)$.

#### 6.2.4 Complementary functions example

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

Step 1. State CF and PI
CF: $y = Ae^{2x} + Be^{6x}$
PI: $y = \lambda x + \mu$

Step 2. Differentiate PI

Obtain:

$$\frac{dy}{dx} = \lambda$$
$$\frac{d^2y}{dx^2} = 0$$

Step 3. Substitute $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, $y$ into the differentiation equation.

Then find $\lambda$ and $\mu$.

### 6.3 Appendix: Particular Integrals

| $f(x)$ | Particular integral |
|---|---|
| $k$ | $\lambda$ |
| $ax+b$ | $\lambda x + \mu$ |
| $ax^2+bx+c$ | $\lambda x^2 + \mu x + \gamma$ |
| $ae^{kx}$ | $\lambda e^{kx}$ |
| $a \sin kx$<br>$a \sin kx$<br>$a \sin kx + b \cos kx$ | $\lambda \sin kx + \mu \cos kx$ |

## 7 Maclaurin and Taylor series

### 7.1 Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^r(0)}{r!}x^r + \dots$$

#### 7.1.1 Provided expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, -1 < x < 1$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots, -1 < x < 1$$

### 7.2 Taylor expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^r(a)}{r!}(x-a)^r + \dots$$

$$f(x-a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f^r(a)}{r!}x^r + \dots$$

## 8 Polar Coordinates

### 8.1 Sketching Graphs in Polar Coordinates

### 8.2 Integration in Polar Coordinates

$$A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

### 8.3 Differentiation in Polar Coordinates

Polar function $r = f(\theta)$ can be transformed to

$$y = r\sin\theta$$

$$x = r\cos\theta$$

Then differentiation:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

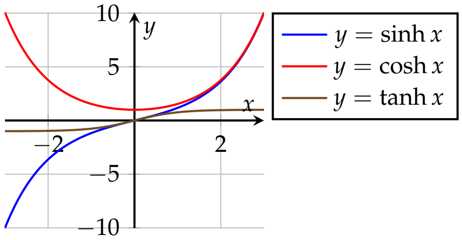
- For tangent parallel to initial line, $\frac{dy}{dx} = 0$, hence $\frac{dy}{d\theta} = 0$.
- For tangent perpendicular to initial line, $\frac{dy}{dx}$ is undefined, hence $\frac{dx}{d\theta} = 0$

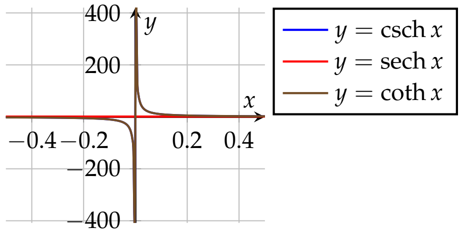
## 9 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

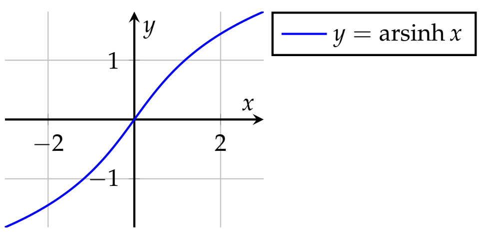


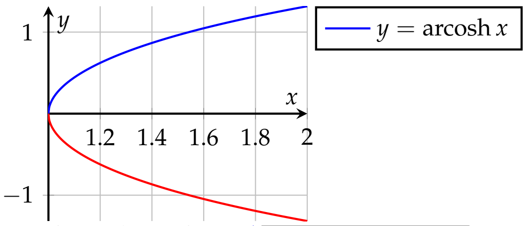



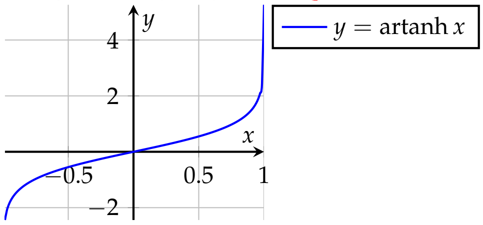

$$\operatorname{arsinh} x = \ln{(x + \sqrt{x^2+1})}$$

$$\operatorname{arcosh} x = \ln{(x \pm \sqrt{x^2-1})}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln{(\frac{1+x}{1-x})}$$






### 9.1 Osborn's rule

Replace sin with sinh, cos with cosh, $\sin^2$ with $-\sinh^2$

## 10 Further integration

### 10.1 General formulae

$$\int f'(x)f^n(x)dx = \frac{f^{n+1}(x)}{n+1}$$

### 10.2 Useful formulae

$$\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\cosh^2(x)$$
$$\cosh^2(x) = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2(x) = \frac{\cosh 2x - 1}{2}$$

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\operatorname{artanh} x) = 1 - x^2$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$
$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin(\frac{x}{a}) + C$$

### 10.3 Substitution in Integration

| $\int f(x)dx$ | Substitution |
|---|---|
| $\int \frac{1}{\sqrt{a^2-x^2}}dx$<br>$\int \sqrt{a^2-x^2}dx$ | $x = a\sin\theta$ |
| $\int \frac{1}{\sqrt{x^2-a^2}}dx$<br>$\int \sqrt{x^2-a^2}dx$ | $x = a\cosh\theta$ |
| $\int \frac{1}{\sqrt{x^2+a^2}}dx$<br>$\int \sqrt{x^2+a^2}dx$ | $x = a\sinh\theta$ |
| $\int \frac{1}{x^2+a^2}dx$ | $x = a\tan\theta$ |

(Results marked (*) are in the Edexcel formula booklet)

$\int \sinh x \, dx = \cosh x + C$ (*)

$\int \cosh x \, dx = \sinh x + C$ (*)

$\int \operatorname{sech}^2 x \, dx = \tanh x + C$

$\int \operatorname{cosech}^2 x \, dx = -\coth x + C$

$\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$

$\int \operatorname{cosech} x \coth x dx = -\operatorname{cosech} x + C$

$\int \frac{1}{\sqrt{(a^2 - x^2)}} dx = \arcsin\left(\frac{x}{a}\right), \; |x| < a$ (*) $\qquad \int \frac{1}{\sqrt{(1 - x^2)}} dx = \arcsin x + C, \; |x| < 1$

$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$ (*) $\qquad \int \frac{1}{1 + x^2} dx = \arctan x + C$

$\int \frac{1}{\sqrt{(a + x^2)}} dx = \operatorname{arsinh}\left(\frac{x}{a}\right)$ (*) $\qquad \int \frac{1}{\sqrt{(1 + x^2)}} dx = \operatorname{arsinh} x + C$

$\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \operatorname{arcosh}\left(\frac{x}{a}\right), \; x > a$ (*) $\qquad \int \frac{1}{\sqrt{(x^2 - 1)}} dx = \operatorname{arcosh} x + C$

$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln\left|\frac{a + x}{a - x}\right|, \; |x| < a$ (*)

$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln\left|\frac{x - a}{x + a}\right|$ (*)

### 10.4 Arc Length

$$S = \int_{x_A}^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$S = \int_{y_A}^{y_B} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$S = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

### 10.5 Surface Area

Rotating about x-axis

$$S = 2\pi \int_{x_A}^{x_B} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$S = 2\pi \int_{x_A}^{x_B} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dx$$

Rotating about y-axis

$$S = 2\pi \int_{x_A}^{x_B} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$S = 2\pi \int_{x_A}^{x_B} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dx$$

## 11 Further coordinates

### 11.1 Ellipses

#### 11.1.1 Gradient of tangent for ellipse

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta}$$

**Conics**

| | Ellipse | Parabola | Hyperbola | Rectangular Hyperbola |
|---|---|---|---|---|
| Standard Form | $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$ | $y^2=4ax$ | $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ | $xy=c^2$ |
| Parametric Form | $(a\cos\theta, b\sin\theta)$ | $(at^2, 2at)$ | $(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$ | $\left(ct, \frac{c}{t}\right)$ |
| Eccentricity | $e<1$ $b^2=a^2(1-e^2)$ | $e=1$ | $e>1$ $b^2=a^2(e^2-1)$ | $e=\sqrt{2}$ |
| Foci | $(\pm ae, 0)$ | $(a, 0)$ | $(\pm ae, 0)$ | $(\pm\sqrt{2}c, \pm\sqrt{2}c)$ |
| Directrices | $x=\pm\frac{a}{e}$ | $x=-a$ | $x=\pm\frac{a}{e}$ | $x+y=\pm\sqrt{2}c$ |
| Asymptotes | none | none | $\frac{x}{a}=\pm\frac{y}{b}$ | $x=0, y=0$ |

## 11.2 Hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

### 11.2.1 Asymptotes

$$y=\pm\frac{b}{a}x$$

### 11.2.2 Intersections

$$x=\pm a$$

### 11.2.3 Parametric equations

$$\begin{cases} x=a\sec\theta \\ y=b\tan\theta \end{cases}$$

$$\begin{cases} x=a\cosh\theta \\ y=b\sinh\theta \end{cases}$$

### 11.2.4 Differentiation

$$\frac{dy}{dx}=\frac{b}{a}\csc\theta$$

$$\frac{dy}{dx}=\frac{b}{a}\coth\theta$$

## 11.3 Eccentricity

$$e=\frac{distance\ to\ focus}{distance\ to\ directrix}$$

- If $0<e<1$, it's an ellipse. $foci(\pm ae, 0)$. directrix: $x=\pm\frac{a}{e}$
- If $e=1$, it's an parabola.

Eccentricity for ellipse:

$$b^2=a^2(1-e^2)$$

$$e^2=1-\frac{b^2}{a^2}$$

Eccentricity for hyperbola:

$$a^2 = b^2(e^2 - 1)$$

$$e^2 = 1 + \frac{a^2}{b^2}$$

## 12 Appendix: Formulas of Integration and Differentiation

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x))$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\int \sec x dx = \ln(\sec x + \tan x) + C$$

$$\int \csc x dx = -\ln(\csc x + \cot x) + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$