

## 1 Summation of Series

$$\begin{aligned}\sum_{x=1}^n x &= \frac{n(n+1)}{2} \\ \sum_{x=1}^n x^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{x=1}^n x^3 &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

## 2 Matrix

### 2.1 Transformations

#### 2.1.1 Rotation

Rotation about the origin by  $\theta$  anti-clockwise:  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

## 3 Complex Numbers

1) Translation

$$w = z + a + bi : \text{translation by } \begin{pmatrix} a \\ b \end{pmatrix}$$

2) Enlargement

$$w = kz : \text{enlargement by a scale factor } k$$

3) Enlargement followed by translation

$$w = kz + a + bi : \text{enlargement by a scale factor } k \text{ followed by a translation by } \begin{pmatrix} a \\ b \end{pmatrix}$$

### 3.1 Transformations

#### 3.1.1 Example 1

Find the transformation  $w = \frac{1}{z}, z \neq 0$ , find the locus of  $w$  when  $z$  lies on the line with equation  $y = 2x + 1$

$$x + yi = \frac{1}{u + vi} = \frac{u - vi}{u^2 + v^2} = \frac{u}{u^2 + v^2} + \frac{-v}{u^2 + v^2}i$$

## 4 Differentiation

### 4.1 First order differentiation

$$f(x) \frac{dy}{dx} + f'(x)y = \frac{d(f(x)y)}{dx}$$

Integration factor:  $\boxed{e^{\int p dx}}$

$$\frac{dy}{dx} + py = Q \Rightarrow \frac{d(\boxed{e^{\int p dx}} y)}{dx} = \boxed{e^{\int p dx}} Q$$

### 4.2 Second order differentiation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

#### 4.2.1 Auxiliary equation

$$am^2 + bm + c = 0$$

If  $\Delta > 0$ , it has two distinct roots  $\alpha, \beta$ . General solution:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

If  $\Delta = 0$ , it has two repeated roots. General solution:

$$y = (A + Bx)e^{\alpha x}$$

If  $\Delta < 0$ , it has two complex roots,  $p + qi$  and  $p - qi$ . General solution:

$$y = e^{px}(A \cos qx + B \sin qx)$$

#### 4.2.2 Example for finding a general solution for Second order differentiation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$a = 1, b = 5, c = 6$$

$$m^2 + 5m + 6 = 0$$

$$m = -2 \text{ or } m = -3$$

$$y = Ae^{-2x} + Be^{-3x}$$

#### 4.2.3 Complementary functions

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Solution:  $y = \text{complementary function} + \text{particular integral}$

Particular integral is the general form of  $f(x)$ .

#### 4.2.4 Complementary functions example

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

Step 1. State CF and PI

$$\text{CF: } y = Ae^{2x} + Be^{6x}$$

$$\text{PI: } y = \lambda x + \mu$$

Step 2. Differentiate PI

Obtain:

$$\frac{dy}{dx} = \lambda$$

$$\frac{d^2y}{dx^2} = 0$$

Step 3. Substitute  $\frac{d^2y}{dx^2}$ ,  $\frac{dy}{dx}$ ,  $y$  into the differentiation equation.

Then find  $\lambda$  and  $\mu$ .

### 4.3 Appendix: Particular Integrals

$f(x)$	Particular integral
$k$	$\lambda$
$ax + b$	$\lambda x + \mu$
$ax^2 + bx + c$	$\lambda x^2 + \mu x + \gamma$
$ae^{kx}$	$\lambda e^{kx}$
$a \sin kx$ $a \sin kx$ $a \sin kx + b \cos kx$	$\lambda \sin kx + \mu \cos kx$

## 5 Maclaurin and Taylor series

### 5.1 Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

#### 5.1.1 Provided expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, -1 < x < 1$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots, -1 < x < 1$$

### 5.2 Taylor expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(r)}(a)}{r!}(x-a)^r + \dots$$

$$f(x-a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f^{(r)}(a)}{r!}x^r + \dots$$

## 6 Polar Coordinates

### 6.1 Sketching Graphs in Polar Coordinates

### 6.2 Integration in Polar Coordinates

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

### 6.3 Differentiation in Polar Coordinates

Polar function  $r = f(\theta)$  can be transformed to

$$y = r \sin \theta$$

$$x = r \cos \theta$$

Then differentiation:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

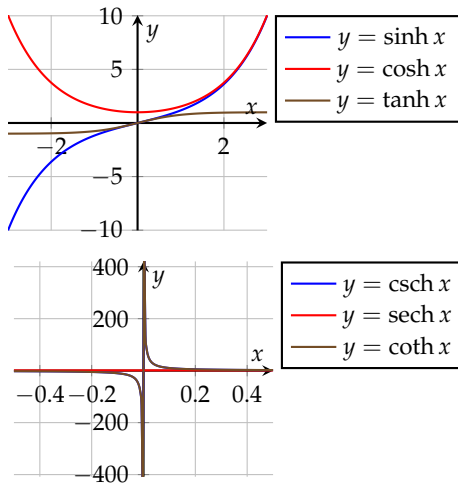
- For tangent parallel to initial line,  $\frac{dy}{dx} = 0$ , hence  $\frac{dy}{d\theta} = 0$ .
- For tangent perpendicular to initial line,  $\frac{dy}{dx}$  is undefined, hence  $\frac{dx}{d\theta} = 0$

## 7 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

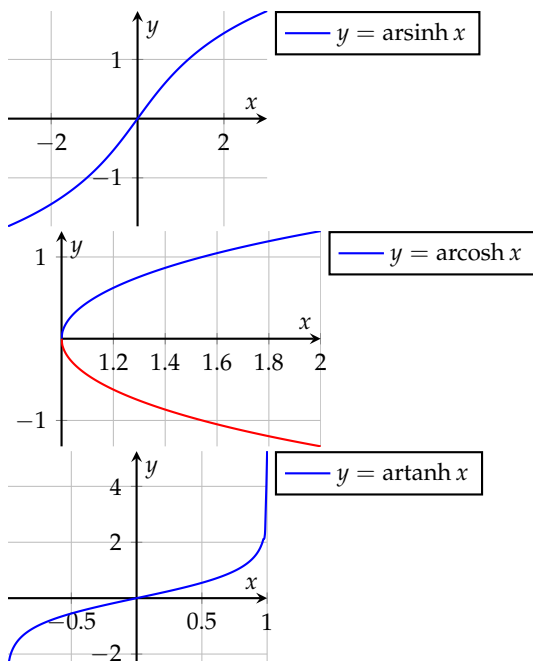
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$$

$$\operatorname{arcosh} x = \ln(x \pm \sqrt{x^2 - 1})$$

$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$



## 7.1 Osborn's rule

Replace  $\sin$  with  $\sinh$ ,  $\cos$  with  $\cosh$ ,  $\sin^2$  with  $-\sinh^2$

## 8 Further coordinates

### 8.1 Ellipses

#### 8.1.1 Standard equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

#### 8.1.2 Parametric equation

$$\begin{cases} x = a \cos \theta \\ y = b \sin \theta \end{cases}$$

#### 8.1.3 Gradient of tangent for ellipse

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \cos \theta}{-a \sin \theta}$$

## 9 Appendix: Formulas of Integration and Differentiation

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x))$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\int \sec x dx = \ln(\sec x + \tan x) + C$$

$$\int \csc x dx = -\ln(\csc x + \cot x) + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$