

## 1 Approximation

### 1.1 Newton-Raphson process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

### 1.2 Linear interpolation

Draw triangles, use similar triangles.

### 1.3 Interval bisection

$a$	$f(a)$	$b$	$f(b)$	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
2	-1	3	2	2.5	0.1569
2	-1	2.5	0.1569	2.25	-0.493

## 2 Summation of Series

### 2.1 Summation of Series

$$\begin{aligned}\sum_{x=1}^n x &= \frac{n(n+1)}{2} \\ \sum_{x=1}^n x^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{x=1}^n x^3 &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

### 2.2 Summation of Arithmetic Progression

$$S_n = a_1 n + \frac{(n)(n-1)d}{2}$$

$$S_n = a_0 n + \frac{(n)(n+1)d}{2}$$

$$S_n = \frac{n \times (a_1 + a_n)}{2}$$

$$S_n = n \times a_{\frac{n+1}{2}}$$

### 2.3 Summation of Geometric Progression

$$S_n = \frac{a_1 \times (1 - q^n)}{1 - q}$$

$$S_\infty = \frac{a_1}{1 - q}$$

## 3 Matrices

### 3.1 Transformations

#### 3.1.1 Enlargement

- Stretch in x-direction by a scale factor  $k$ :  $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
- Stretch in y-direction by a scale factor  $k$ :  $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
- Enlargement with centre of the origin by a scale factor  $k$ :  $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

### 3.1.2 Reflection

- Reflection in x-axis:  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection in y-axis:  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection in  $y = x$ :  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Reflection in  $y = -x$ :  $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

### 3.1.3 Rotation

- Rotation about the origin by  $\theta$  anti-clockwise:  $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

## 3.2 Inverse matrix 2\*2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\det A = ad - bc$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If  $\det A = 0$ ,  $A$  is singular, so  $A$  has no inverse.

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

## 3.3 Inverse matrix 3\*3

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} A & -B & C \\ -D & E & -F \\ G & -H & I \end{pmatrix}^T$$

where

$$A = ei - hf$$

$$\Delta = aA - bB + cC$$

### 3.3.1 Transpose

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

## 3.4 Calculating area of an triangle

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

$$A = \frac{1}{2} (x_2y_1 + x_3y_2 + x_1y_3 - x_1y_2 - x_2y_3 - x_3y_1)$$

## 4 Complex Numbers

1) Translation

$$w = z + a + bi : \text{translation by } \begin{pmatrix} a \\ b \end{pmatrix}$$

2) Enlargement

$$w = kz : \text{enlargement by a scale factor } k$$

3) Enlargement followed by translation

$$w = kz + a + bi : \text{enlargement by a scale factor } k \text{ followed by a translation by } \begin{pmatrix} a \\ b \end{pmatrix}$$

### 4.1 Transformations

#### 4.1.1 Example 1

Find the transformation  $w = \frac{1}{z}, z \neq 0$ , find the locus of  $w$  when  $z$  lies on the line with equation  $y = 2x + 1$

$$x + yi = \frac{1}{u + vi} = \frac{u - vi}{u^2 + v^2} = \frac{u}{u^2 + v^2} + \frac{-v}{u^2 + v^2}i$$

## 5 Differentiation

### 5.1 First order differentiation

$$f(x) \frac{dy}{dx} + f'(x)y = \frac{d(f(x)y)}{dx}$$

Integration factor:  $\boxed{e^{\int p dx}}$

$$\frac{dy}{dx} + py = Q \Rightarrow \frac{d(\boxed{e^{\int p dx}} y)}{dx} = \boxed{e^{\int p dx}} Q$$

### 5.2 Second order differentiation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

#### 5.2.1 Auxiliary equation

$$am^2 + bm + c = 0$$

If  $\Delta > 0$ , it has two distinct roots  $\alpha, \beta$ . General solution:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

If  $\Delta = 0$ , it has two repeated roots. General solution:

$$y = (A + Bx)e^{\alpha x}$$

If  $\Delta < 0$ , it has two complex roots,  $p + qi$  and  $p - qi$ . General solution:

$$y = e^{px}(A \cos qx + B \sin qx)$$

#### 5.2.2 Example for finding a general solution for Second order differentiation

$$\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = 0$$

$$a = 1, b = 5, c = 6$$

$$m^2 + 5m + 6 = 0$$

$$m = -2 \text{ or } m = -3$$

$$y = Ae^{-2x} + Be^{-3x}$$

### 5.2.3 Complementary functions

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

Solution:  $y = \text{complementary function} + \text{particular integral}$

Particular integral is the general form of  $f(x)$ .

### 5.2.4 Complementary functions example

$$\frac{d^2 y}{dx^2} - 8 \frac{dy}{dx} + 12y = 36x$$

Step 1. State CF and PI

$$\text{CF: } y = Ae^{2x} + Be^{6x}$$

$$\text{PI: } y = \lambda x + \mu$$

Step 2. Differentiate PI

Obtain:

$$\frac{dy}{dx} = \lambda$$

$$\frac{d^2 y}{dx^2} = 0$$

Step 3. Substitute  $\frac{d^2 y}{dx^2}$ ,  $\frac{dy}{dx}$ ,  $y$  into the differentiation equation.

Then find  $\lambda$  and  $\mu$ .

### 5.3 Appendix: Particular Integrals

$f(x)$	Particular integral
$k$	$\lambda$
$ax + b$	$\lambda x + \mu$
$ax^2 + bx + c$	$\lambda x^2 + \mu x + \gamma$
$ae^{kx}$	$\lambda e^{kx}$
$a \sin kx$ $a \sin kx$ $a \sin kx + b \cos kx$	$\lambda \sin kx + \mu \cos kx$

## 6 Maclaurin and Taylor series

### 6.1 Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

#### 6.1.1 Provided expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, -1 < x < 1$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots, -1 < x < 1$$

## 6.2 Taylor expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(r)}(a)}{r!}(x-a)^r + \dots$$

$$f(x-a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f^{(r)}(a)}{r!}x^r + \dots$$

## 7 Polar Coordinates

### 7.1 Sketching Graphs in Polar Coordinates

### 7.2 Integration in Polar Coordinates

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

### 7.3 Differentiation in Polar Coordinates

Polar function  $r = f(\theta)$  can be transformed to

$$y = r \sin \theta$$

$$x = r \cos \theta$$

Then differentiation:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

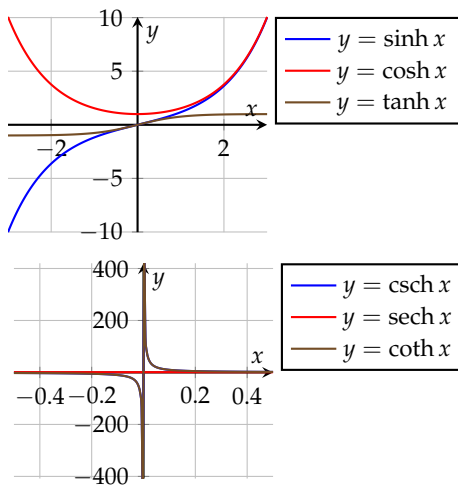
- For tangent parallel to initial line,  $\frac{dy}{dx} = 0$ , hence  $\frac{dy}{d\theta} = 0$ .
- For tangent perpendicular to initial line,  $\frac{dy}{dx}$  is undefined, hence  $\frac{dx}{d\theta} = 0$

## 8 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

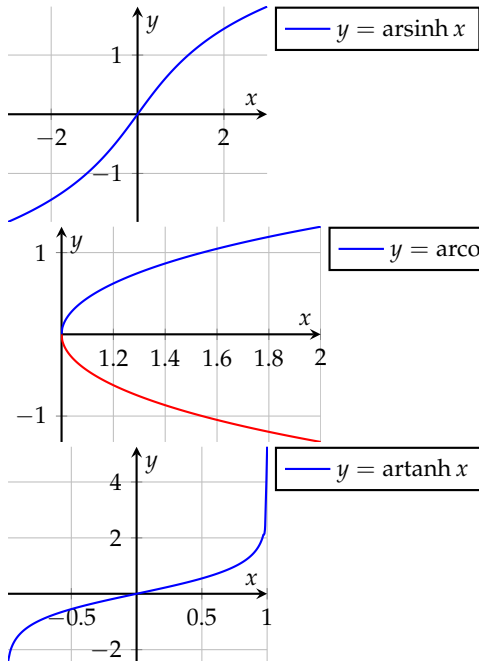
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$$

$$\operatorname{arcosh} x = \ln(x \pm \sqrt{x^2 - 1})$$

$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$



### 8.1 Osborn's rule

Replace  $\sin$  with  $\sinh$ ,  $\cos$  with  $\cosh$ ,  $\sin^2$  with  $-\sinh^2$

## 9 Further integration

### 9.1 General formulae

$$\int f'(x)f^n(x)dx = \frac{f^{n+1}(x)}{n+1}$$

### 9.2 Useful formulae

$$\cosh 2x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2(x)$$

$$\cosh^2(x) = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2(x) = \frac{\cosh 2x - 1}{2}$$

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{artanh} x) = 1 - x^2$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C$$

(Results marked (\*) are in the Edexcel formula booklet)

$$\int \sinh x \, dx = \cosh x + C \quad (*)$$

$$\int \cosh x \, dx = \sinh x + C \quad (*)$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\int \operatorname{cosech}^2 x \, dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + C$$

$$\int \frac{1}{\sqrt{(a^2 - x^2)}} dx = \arcsin\left(\frac{x}{a}\right), \quad |x| < a \quad (*) \qquad \int \frac{1}{\sqrt{(1 - x^2)}} dx = \arcsin x + C, \quad |x| < 1$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) \quad (*) \qquad \int \frac{1}{1 + x^2} dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{(a^2 + x^2)}} dx = \operatorname{arsinh}\left(\frac{x}{a}\right) \quad (*) \qquad \int \frac{1}{\sqrt{(1 + x^2)}} dx = \operatorname{arsinh} x + C$$

$$\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \operatorname{arcosh}\left(\frac{x}{a}\right), \quad x > a \quad (*) \qquad \int \frac{1}{\sqrt{(x^2 - 1)}} dx = \operatorname{arcosh} x + C$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right|, \quad |x| < a \quad (*)$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| \quad (*)$$

### 9.3 Substitution in Integration

$\int f(x) dx$	Substitution
$\int \frac{1}{\sqrt{a^2 - x^2}} dx$ $\int \sqrt{a^2 - x^2} dx$	$x = a \sin \theta$
$\int \frac{1}{\sqrt{x^2 - a^2}} dx$ $\int \sqrt{x^2 - a^2} dx$	$x = a \cosh \theta$
$\int \frac{1}{\sqrt{x^2 + a^2}} dx$ $\int \sqrt{x^2 + a^2} dx$	$x = a \sinh \theta$
$\int \frac{1}{x^2 + a^2} dx$	$x = a \tan \theta$

### 9.4 Arc Length

$$S = \int_{x_A}^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$S = \int_{y_A}^{y_B} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$S = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### 9.5 Surface Area

Rotating about x-axis

$$S = 2\pi \int_{x_A}^{x_B} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$S = 2\pi \int_{x_A}^{x_B} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dx$$

Rotating about y-axis

$$S = 2\pi \int_{x_A}^{x_B} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$S = 2\pi \int_{x_A}^{x_B} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dx$$

## 10 Further coordinates

### Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a \cos \theta, b \sin \theta)$	$(at^2, 2at)$	$(a \sec \theta, b \tan \theta)$ $(\pm a \cosh \theta, b \sinh \theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm \sqrt{2}c, \pm \sqrt{2}c)$
Directrices	$x = \pm \frac{a}{e}$	$x = -a$	$x = \pm \frac{a}{e}$	$x + y = \pm \sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm \frac{y}{b}$	$x = 0, y = 0$

### 10.1 Ellipses

#### 10.1.1 Gradient of tangent for ellipse

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \cos \theta}{-a \sin \theta}$$

### 10.2 Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

#### 10.2.1 Asymptotes

$$y = \pm \frac{b}{a}x$$

#### 10.2.2 Intersections

$$x = \pm a$$

#### 10.2.3 Parametric equations

$$\begin{cases} x = a \sec \theta \\ y = b \tan \theta \end{cases}$$

$$\begin{cases} x = a \cosh \theta \\ y = b \sinh \theta \end{cases}$$



### 10.2.4 Differentiation

$$\frac{dy}{dx} = \frac{b}{a} \csc \theta$$

$$\frac{dy}{dx} = \frac{b}{a} \coth \theta$$

### 10.3 Eccentricity

$$e = \frac{\text{distance to focus}}{\text{distance to directrix}}$$

- If  $0 < e < 1$ , it's an ellipse. *foci*( $\pm ae, 0$ ). *directrix*:  $x = \pm \frac{a}{e}$
- If  $e = 1$ , it's an parabola.

Eccentricity for ellipse:

$$b^2 = a^2(1 - e^2)$$

$$e^2 = 1 - \frac{b^2}{a^2}$$

Eccentricity for hyperbola:

$$a^2 = b^2(e^2 - 1)$$

$$e^2 = 1 + \frac{a^2}{b^2}$$

## 11 Vector

### 11.1 Scalar product ( $\cdot$ )

$$\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 + z_1 + z_2$$

#### 11.1.1 Angle between $\vec{a}$ and $\vec{b}$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

### 11.2 Vector product ( $\times$ )

Vector product is perpendicular to both the vectors.

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \cdot \vec{u}$$

### 11.3 Calculate Area of a Triangle

$$A = \frac{1}{2} |\vec{a}| |\vec{b}| \sin \theta = \frac{1}{2} |\vec{a} \times \vec{b}|$$

### 11.4 Calculate Area of a Tetrahedron

$$A = \frac{1}{6} \vec{a} \cdot (\vec{b} \times \vec{c}) \sin \theta$$

### 11.5 Calculate Area of a Prism

$$A = \frac{1}{6} \vec{a} \cdot (\vec{b} \times \vec{c}) \sin \theta$$

### 11.6 Cartesian equation of a straight line

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \lambda$$

where

$$\text{fixedpoint}(a_1, a_2, a_3)$$

,

$$\text{direction} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

### 11.7 Vector equation of a straight line

$$\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$$

$$\vec{r} = \vec{a} + t\vec{b}$$

### 11.8 Find the distance from a point to the line using vector product

$$\frac{|\vec{d}|}{|\vec{AP}|} = \sin \theta$$
$$\vec{d} = \frac{|\vec{AP} \times \vec{b}|}{|\vec{b}|}$$

### 11.9 Find the shortest distance between two lines

- Two parallel lines, choose one point from a line and calculate by

$$\vec{d} = \frac{|\vec{AP} \times \vec{b}|}{|\vec{b}|}$$

.

- Two skew lines:

$$\vec{r}_1 = \vec{a} + \lambda \vec{b} \tag{1}$$

$$\vec{r}_2 = \vec{c} + \lambda \vec{d} \tag{2}$$

$$d = \left| \frac{(\vec{a} - \vec{c}) \cdot (\vec{b} \times \vec{d})}{|\vec{b} \times \vec{d}|} \right|$$

### 11.10 Vector Equation of a Plane

$$\vec{n} \cdot (\vec{r} - \vec{a}) = 0$$

### 11.11 Cartesian Equation of a Plane

$$ax + by + cz = d = \vec{n} \cdot \vec{a}, \vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where  $\vec{a}$  is a fixed point.

### 11.12 Determine a Plane with two lines

1. Find normal of the plane by  $\vec{b}_1 \times \vec{b}_2$
2. Find intersection
3. Express in cartesian form

### 11.13 Find the distance between a point and a plane

Plane:  $\vec{r} \cdot \vec{n} = ax + by + cz = d$ , Point:  $P(x_1, y_1, z_1)$

#### 11.13.1 Method 1, use formulae directly

$$d = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

#### 11.13.2 Method 2, use perpendicular foot

$F$  is the perpendicular foot of  $P$  to the plane.

Step 1. Find for line  $PF$ , which is a expression of  $F$

Step 2.  $F$  is in the plane, so put  $F$  into the equation of the plane and find  $F$ .

Step 3. Calculate distance by  $|\vec{PF}|$

### 11.14 Line and plane

#### 11.14.1 The line lies in the plane

Method 1. Prove two random points lie in the plane

Method 2. Prove all points are in the plane

#### 11.14.2 The line is parallel to the plane

Method 1.  $\vec{b}\vec{n} = 0$ ,

#### 11.14.3 The line is intersecting with the plane

Method 1. Find point of intersection

Method 2. Find angle between  $l$  and the plane

### 11.15 Tow planes

#### 11.15.1 Distance between two planes

$$ax + by + cz = d_1 \tag{3}$$

$$ax + by + cz = d_2 \tag{4}$$

$$d = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

## 12 Appendix: Formulas of Integration and Differentiation

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x))$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\int \sec x dx = \ln(\sec x + \tan x) + C$$

$$\int \csc x dx = -\ln(\csc x + \cot x) + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$