1 Approximation

1.1 Newton-Raphson process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1.2 Linear interpolation

Draw triangles, use similar triangles.

2 Summation of Series

$$\begin{aligned}\sum_{x=1}^{n} x &= \frac{n(n+1)}{2} \\ \sum_{x=1}^{n} x^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{x=1}^{n} x^3 &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

3 Matrix

3.1 Transformations

3.1.1 Enlargement

- Stretch in x-direction by a scale factor k: $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
- Stretch in y-direction by a scale factor k: $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
- Enlargement with centre of the origin by a scale factor k: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

3.1.2 Reflection

- Reflection in x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection in y-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection in $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Reflection in $y = -x$: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

3.1.3 Rotation

- Rotation about the origin by θ anti-clockwise: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

3.2 Inverse matrix 2*2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det A = ad - bc$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If $\det A = 0$, A is singular, so A has no inverse.

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

3.3 Inverse matrix 3*3

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} A & -B & C \\ -D & E & -F \\ G & -H & I \end{pmatrix}^{T}$$

where

$$A = ei - hf$$

$$\Delta = aA - bB + cC$$

3.3.1 Transpose

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{T} = \begin{pmatrix} a & d & c \\ b & e & h \\ g & f & i \end{pmatrix}$$

4 Complex Numbers

1) Translation

   $w = z + a + bi :$ translation by $\begin{pmatrix} a \\ b \end{pmatrix}$

2) Enlargement

   $w = kz :$ enlargement by a scale factor k

3) Enlargement followed by translation

   $w = kz + a + bi :$ enlargement by a scale factor k followed by a translation by $\begin{pmatrix} a \\ b \end{pmatrix}$

4.1 Transformations

4.1.1 Example 1

Find the transformation $w = \frac{1}{z}, z! = 0$, find the locus of w when z lies on the line with equation $y = 2x + 1$

$$x + yi = \frac{1}{u + vi} = \frac{u - vi}{u^2 + v^2} = \frac{u}{u^2 + v^2} + \frac{-v}{u^2 + v^2} i$$

5 Differentiation

5.1 First order differentiation

$$f(x)\frac{dy}{dx} + f'(x)y = \frac{d(f(x)y)}{dx}$$

Integration factor: $\boxed{e^{\int pdx}}$

$$\frac{dy}{dx} + py = Q \rightrightarrows \frac{d(\boxed{e^{\int pdx}}y)}{dx} = \boxed{e^{\int pdx}}Q$$

5.2 Second order differentiation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

5.2.1 Auxiliary equation

$$am^2 + bm + c = 0$$

If $\Delta > 0$, it has two distinct roots α, β. General solution:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

If $\Delta = 0$, it has two repeated roots. General solution:

$$y = (A + Bx)e^{\alpha x}$$

If $\Delta < 0$, it has two complex roots, $p + qi$ and $p - qi$. General solution:

$$y = e^{px}(A\cos qx + B\sin qx)$$

5.2.2 Example for finding a general solution for Second order differentiation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$
$$a = 1, b = 5, c = 6$$
$$m^2 + 5m + 6 = 0$$
$$m = -2orm = -3$$
$$y = Ae^{-2x} + Be^{-3x}$$

5.2.3 Complementary functions

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Solution: $y = complementary\ function + particular\ integral$

Particular integral is the general form of $f(x)$.

5.2.4 Complementary functions example

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

Step 1. State CF and PI
CF: $y = Ae^{2x} + Be^{6x}$
PI: $y = \lambda x + \mu$

Step 2. Differentiate PI
Obtain:

$$\frac{dy}{dx} = \lambda$$
$$\frac{d^2y}{dx^2} = 0$$

Step 3. Substitute $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, y into the differentiation equation.
Then find λ and μ.

5.3 Appendix: Particular Integrals

$f(x)$	Particular integral
k	λ
$ax+b$	$\lambda x+\mu$
ax^2+bx+c	$\lambda x^2+\mu x+\gamma$
ae^{kx}	λe^{kx}
$a\sin kx$ $a\sin kx$ $a\sin kx + b\cos kx$	$\lambda\sin kx+\mu\cos kx$

6 Maclaurin and Taylor series

6.1 Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^r(0)}{r!}x^r + \dots$$

6.1.1 Provided expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, -1 < x < 1$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots, -1 < x < 1$$

6.2 Taylor expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^r(a)}{r!}(x-a)^r + \dots$$

$$f(x-a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f^r(a)}{r!}x^r + \dots$$

7 Polar Coordinates

7.1 Sketching Graphs in Polar Coordinates

7.2 Integration in Polar Coordinates

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

7.3 Differentiation in Polar Coordinates

Polar function $r = f(\theta)$ can be transformed to

$$y = r\sin\theta$$

$$x = r\cos\theta$$

Then differentiation:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

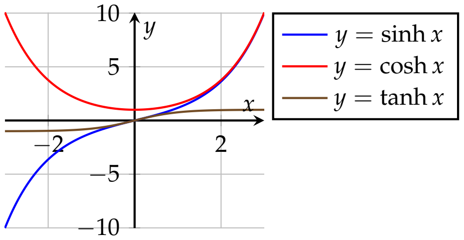
- For tangent parallel to initial line, $\frac{dy}{dx} = 0$, hence $\frac{dy}{d\theta} = 0$.
- For tangent perpendicular to initial line, $\frac{dy}{dx}$ is undefined, hence $\frac{dx}{d\theta} = 0$

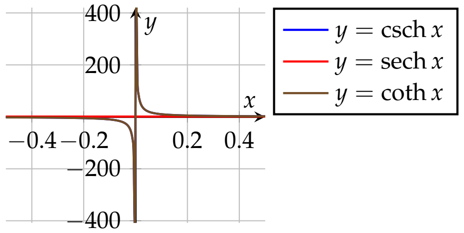
8 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

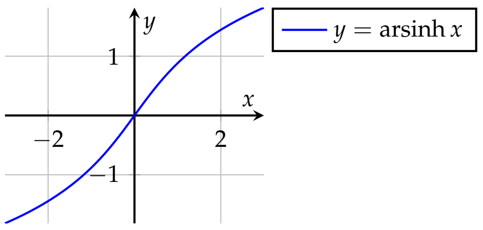


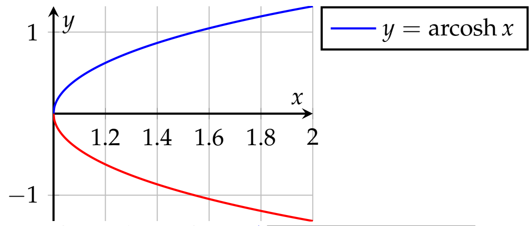


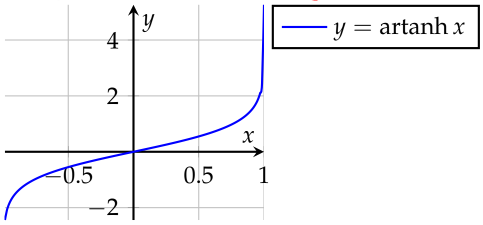
$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$$

$$\operatorname{arcosh} x = \ln(x \pm \sqrt{x^2-1})$$

$$\operatorname{artanh} x = \frac{1}{2}\ln(\frac{1+x}{1-x})$$







8.1 Osborn's rule

Replace sin with sinh, cos with cosh, $\sin^2$ with $-\sinh^2$

9 Further coordinates

9.1 Ellipses

9.1.1 Standard equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

9.1.2 Parametric equation

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$$

9.1.3 Gradient of tangent for ellipse

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta}$$

9.2 Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

9.2.1 Asymptotes

$$y = \pm\frac{b}{a}x$$

9.2.2 Intersections

$$x = \pm a$$

9.2.3 Parametric equations

$$\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$$

$$\begin{cases} x = a\cosh\theta \\ y = b\sinh\theta \end{cases}$$

9.2.4 Differentiation

$$\frac{dy}{dx} = \frac{b}{a}\csc\theta$$

$$\frac{dy}{dx} = \frac{b}{a}\coth\theta$$

9.3 Eccentricity

$$e = \frac{distance\ to\ focus}{distance\ to\ directrix}$$

- If $0 < e < 1$, it's an ellipse. $foci(\pm ae, 0)$. directrix: $x = \pm\frac{a}{e}$
- If $e = 1$, it's an parabola.

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

Eccentricity for ellipse:

$$b^2 = a^2(1 - e^2)$$

$$e^2 = 1 - \frac{b^2}{a^2}$$

Eccentricity for hyperbola:

$$a^2 = b^2(1 - e^2)$$

$$e^2 = 1 + \frac{b^2}{a^2}$$

10 Appendix: Formulas of Integration and Differentiation

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x))$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\int \sec x dx = \ln(\sec x + \tan x) + C$$

$$\int \csc x dx = -\ln(\csc x + \cot x) + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$