1 Summation of Series

$$\sum_{x=0}^{n} x = \frac{n(n+1)}{2}$$
$$\sum_{x=0}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{x=0}^{n} x^3 = \frac{n^2(n+1)^2}{4}$$

2 Matrix

2.1 Transformations

2.1.1 Rotation

Rotation about the origin by θ anti-clockwise: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3 Complex Numbers

1) Translation

   $w = z + a + bi$: translation by $\begin{pmatrix} a \\ b \end{pmatrix}$

2) Enlargement

   $w = kz$: enlargement by a scale factor k

3) Enlargement followed by translation

   $w = kz + a + bi$: enlargement by a scale factor k followed by a translation by $\begin{pmatrix} a \\ b \end{pmatrix}$

3.1 Transformations

3.1.1 Example 1

Find the transformation $w = \frac{1}{z}, z! = 0$, find the locus of w when z lies on the line with equation $y = 2x + 1$

$$x + yi = \frac{1}{u + vi} = \frac{u - vi}{u^2 + v^2} = \frac{u}{u^2 + v^2} + \frac{-v}{u^2 + v^2}i$$

4 Differentiation

4.1 First order differentiation

$$f(x)\frac{dy}{dx} + f'(x)y = \frac{d(f(x)y)}{dx}$$

Integration factor: $\boxed{e^{\int p dx}}$

$$\frac{dy}{dx} + py = Q \rightrightarrows \frac{d(\boxed{e^{\int p dx}}y)}{dx} = \boxed{e^{\int p dx}}Q$$

4.2 Second order differentiation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

4.2.1 Auxiliary equation

$$am^2 + bm + c = 0$$

If $\Delta > 0$, it has two distinct roots α, β. General solution:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

If $\Delta = 0$, it has two repeated roots. General solution:

$$y = (A + Bx)e^{\alpha x}$$

If $\Delta < 0$, it has two complex roots, $p + qi$ and $p - qi$. General solution:

$$y = e^{px}(A \cos qx + B \sin qx)$$

4.2.2 Example for finding a general solution for Second order differentiation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$a = 1, b = 5, c = 6$$

$$m^2 + 5m + 6 = 0$$

$$m = -2 or m = -3$$

$$y = Ae^{-2x} + Be^{-3x}$$

4.2.3 Complementary functions

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Solution: $y = complementary\ function + particular\ integral$

Particular integral is the general form of $f(x)$.

4.2.4 Complementary functions example

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

Step 1. State CF and PI
CF: $y = Ae^{2x} + Be^{6x}$
PI: $y = \lambda x + \mu$

Step 2. Differentiate PI
Obtain:

$$\frac{dy}{dx} = \lambda$$

$$\frac{d^2y}{dx^2} = 0$$

Step 3. Substitute $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, y into the differentiation equation.
Then find λ and μ.

4.3 Appendix: Particular Integrals

$f(x)$	Particular integral
k	λ
$ax+b$	$\lambda x+\mu$
ax^2+bx+c	$\lambda x^2+\mu x+\gamma$
ae^{kx}	λe^{kx}
$a\sin kx$ $a\sin kx$ $a\sin kx+b\cos kx$	$\lambda\sin kx+\mu\cos kx$

5 Maclaurin and Taylor series

5.1 Maclaurin expansion

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^r(0)}{r!}x^r+\dots$$

5.1.1 Provided expansions

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots,-1<x<1$$

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\dots,-1<x<1$$

5.2 Taylor expansion

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^r(a)}{r!}(x-a)^r+\dots$$

$$f(x+a)=f(a)+f'(a)x+\frac{f''(a)}{2!}x^2+\frac{f^r(a)}{r!}x^r+\dots$$

6 Polar Coordinates

6.1 Sketching Graphs in Polar Coordinates

6.2 Integration in Polar Coordinates

$$A=\frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

6.3 Differentiation in Polar Coordinates

Polar function $r=f(\theta)$ can be transformed to

$$y=r\sin\theta$$

$$x=r\cos\theta$$

Then differentiation:

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

- For tangent parallel to initial line, $\frac{dy}{dx} = 0$, hence $\frac{dy}{d\theta} = 0$.
- For tangent perpendicular to initial line, $\frac{dy}{dx} is undefined$, hence $\frac{dx}{d\theta} = 0$

6.4 Appendix: Formulas of Integration and Differentiation

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x))$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\int \tan x \sin x dx = \sec x + C$$

$$\int \cot x \csc x dx = -\csc x + C$$

$$\int \sec x dx = \ln\left(\sec x + \tan x\right) + C$$

$$\int \csc x dx = -(\ln \csc x + \cot x) + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$