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Closed p-elastic curves in spheres of Lorentz-Minkowski space

Tesax A&M University, College Station, Texas.
Texas Analysis and Mathematical Physics Symposium

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February 10, 2024

Content

$p-$elastic Functionals

$p-$elastic functional defined by

$$\Theta_p(\gamma) = \int_\gamma \kappa^p \, ds \tag{1}$$

Historical Context:

During the classical era of mathematics, D. Bernoulli and L. Euler embarked on a significant endeavor, namely, the pursuit of extrema for the total squared curvature functional.

In a letter exchanged between D. Bernoulli and L. Euler, the idea of exploring extrema for p-elastic functionals, also known as generalized curves, was introduced.

Some Interesting Cases

- $p = 0$ - length functional and geodesics
- $p = 2$ - bending/elastic energy
- $p = 1/2$ - Blaschke and catenaries
- $p = 1/3$ - Blaschke, parabolas equi-affine length
- $p = 1$ - total curvature and topological invariant
- $p > 2$ for $p \in \mathbb{N}$ - generating curves of Willmore-Chen submanifolds
- Some $p \in \mathbb{Q} \cap (0, 1)$ - generating curves of biconservative hyper-surfaces.
- $p = -1$ - cycloids and brachistochrone problem.

Motivation

There are plenty of papers studying the $p-$elastic curves in several special cases. Among them, in the year 2022, Anthony Gruber, Alvaro Pampano, and Magdalena Toda authored a paper titled "Instability of Closed p-Elastic Curves in $\mathbb{S}^2$." This research delves into the analysis of p-elastic curves within the context of $\mathbb{S}^2$, with a focus on values of p falling within the open interval (0, 1).

Are there closed $p-$elastic curves in $p \in \mathbb{R} \backslash (0, 1)$?

Spaces

The Lorentz-Minkowski Space

The Lorentz-Minkowski 3-space $\mathbb{L}^3$ is $\mathbb{R}^3$ endowed with the metric $g \equiv \langle \cdot, \cdot \rangle = dx^2 + dy^2 - dz^2$

Hyperbolic Plane

The hyperbolic plane, denoted as $\mathbb{H}_0^2(\rho)$ where $\rho < 0$, is the space-like surface of $\mathbb{L}^3$ and is represented by the top part of the hyperboloid of two sheets.

$$\mathbb{H}_0^2(\rho) = \{(x, y, z) | \langle (x, y, z), (x, y, z) \rangle = x^2 + y^2 - z^2 = 1/\rho,\ z > 0\}$$

The de-Sitter 2-space

The de-Sitter space, denoted by $\mathbb{H}_1^2(\rho)$, where $\rho > 0$, is the time-like surface of $\mathbb{L}^3$ and is represented by

$$\mathbb{H}_1^2(\rho) = \{(x, y, z) \in \mathbb{R}^3 | \langle (x, y, z), (x, y, z) \rangle = 1/\rho\}$$

Spaces

For visualization purposes, we identify $\mathbb{H}^2_0$ with the disk $\mathbb{D} \subset \mathbb{R}^2$ centered at the origin and endowed with the Poincaré metric by means of the isometry

$$(x, y, z) \in \mathbb{H}^2_0 \to \frac{1}{1+z}(x, y) \in \mathbb{D}.$$

This is the Poincaré disk model for $\mathbb{H}^2_0$.

We can identify the bottom half of $\mathbb{H}^2_1$ as $\mathbb{H}^2_{10} = \mathbb{H}^2_1 \cap \{z < 0\}$ with the once-punctured unit disk $\mathring{\mathbb{D}} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \text{ and } x^2 + y^2 \neq 0\}$ via the diffeomorphism

$$(x, y, z) \in \mathbb{H}^2_{10} \to \frac{1}{x^2 + y^2}(x, y) \in \mathring{\mathbb{D}}.$$

Euler-Lagrange Equation

The critical points for Θ_p must satisfy the Euler-Lagrange equation:

$$p\frac{d^2}{ds^2}\kappa^{p-1} + \epsilon_1\epsilon_2(p-1)\kappa^{p+1} - \epsilon_2 p\kappa^{p-1} = 0,$$

where ϵ_1, ϵ_2 are the causal characters of the Frenet frame associated with the critical points. This implies that $\kappa(s)$ is a solution to the first-order ordinary differential equation:

$$p^2(p-1)^2\kappa^{2(p-2)}(\kappa')^2 + \epsilon_1\epsilon_2(p-1)^2\kappa^{2p} - \epsilon_2 p^2\kappa^{2(p-1)} = a,$$

where $a \in \mathbb{R}$ is a constant of integration.

Results

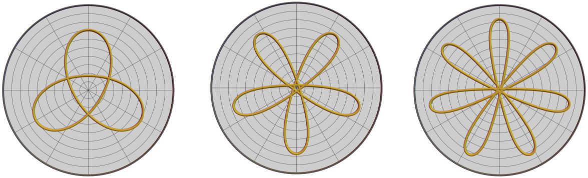
Theorem [A. Pámpano, M. Samarakkody, H. Tran (2024)]

Let γ be a closed p-elastic curve in $\mathbb{H}^2_\epsilon$ with non-constant curvature. Then γ is a space-like curve with $0 > a > a_* := -((-1)^\epsilon p)^p((-1)^\epsilon(p-1))^{1-p}$. Moreover,

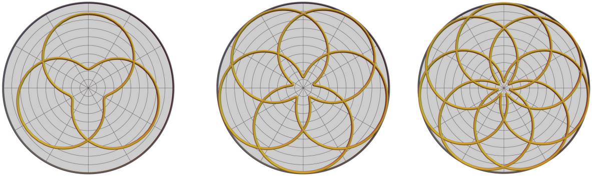
- if it is a hyperbolic curve, then $p > 1$.
- if it is a pseudo-hyperbolic curve, then $p < 0$.

Conversely, assume the above restrictions on p for each ambient space and let (n, m) relatively prime, such that $m < 2n < \sqrt{2}m$, then there exist a closed space-like p-elastic curve in $\mathbb{H}^2_\epsilon$.

Results



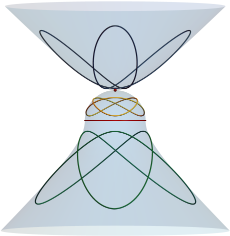
Three hyperbolic $p-$elastic curves for $p = 3/2$ in $\mathbb{H}^2_0$ corresponding to the values $q = 2/3$, $q = 3/5$, and $q = 4/7$. They are represented in the Poincare disk model.



Three pseudo-hyperbolic $p-$elastic curves for $p = -1$ in $\mathbb{H}^2_1$ corresponding to the values $q = 2/3$, $q = 3/5$, and $q = 4/7$. They are represented in the once punctured unit disk.

Results

This figure is evolution of closed $p-$elastic curve of type $\gamma_{2,3}$. In black, the $p-$elastic curve $\gamma_{2,3}$ for $p = 2$ immersed in $\mathbb{H}^2_0$; in yellow, the $p-$elastic curve $\gamma_{2,3}$ for $p = 0.2$ immersed in $\mathbb{S}^2$; and, in green the $p-$elastic curve $\gamma_{2,3}$ for $p = -1$ immersed in $\mathbb{H}^2_1$. The red point is the pole $(0, 0, 1) \in \mathbb{R}^3$ and the red circle is the equator.



Applications

Generalized elastic curves find practical applications in various fields, including:

- Structural engineering
- Deformable objects in computer graphics
- Significantly, biophysics

Applications

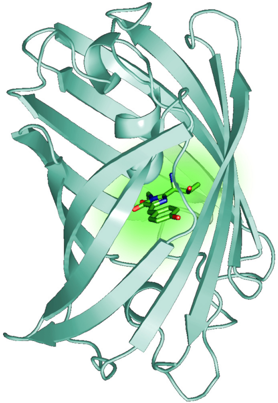


Figure: Beta Sheets of a protein

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[1] Figure Source: Researchgate.net

Applications

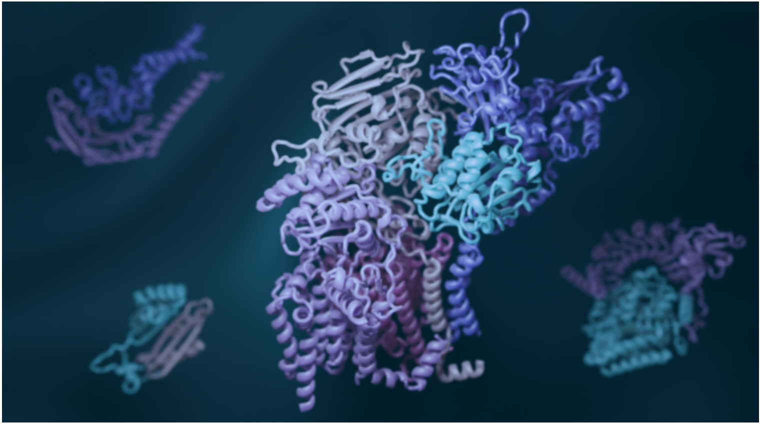

Figure: Protein Structure

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[2] Figure Source: GeekWire

References

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Figure References

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Thank you!