Closed $p-$Elastic Curves in 2-Space Forms

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Texas Tech University

AMS Sectional Meeting (Meeting No: 1198)
University of Texas, San Antonio.
September 15, 2024

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- D. Bernoulli proposed to investigate extrema of the functionals

$$\Theta_p(\gamma) := \int_\gamma \kappa^p ds,$$

where
$\kappa-$ the curvature of the curve γ;
$s-$ arc length;
$p \in \mathbb{R}$.

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- $p = 2$; Bending/elastic energy.
- $p > 2$ for $p \in \mathbb{N}$; Generating curves of Willmore-Chen submanifolds.

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Are there closed p-elastic curves for $p \in \mathbb{R}$, and what are the conditions for a p-elastic curve to be closed?

Some Previous Results

For $p = 2$, J. Langer, D.A. Singer, P. Griffiths, R. Bryant and other values of p, J. Arroyo, M. Barros, O.J. Garay, R. López, J. Mencía, S. Montaldo, E. Musso, C. Oniciuc, A. Pámpano.

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Then for general value of p in sphere

Theorem [A. Gruber, A. Pámpano, M. Toda (2023)]

Let n and m be two relatively prime natural numbers satisfying $m < 2n < \sqrt{2}m$. Then, for every $p \in (0, 1)$, there exists a closed $p-$elastic curve with non-constant curvature.

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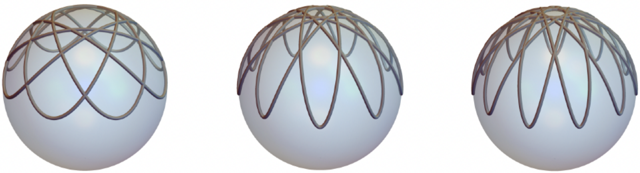

Figure: Closed $p-$elastic curves for $p = 0.3$ in $\mathbb{S}^2$ of type $\gamma_{5,8}$, $\gamma_{5,9}$ and $\gamma_{6,11}$, respectively.

Preliminaries

Let (x, y, z) be standard coordinates of $\mathbb{R}^3$. The Lorentz-Minkowski 3-space $\mathbb{L}^3$ is $\mathbb{R}^3$ endowed with the canonical metric of index one $g \equiv \langle \cdot, \cdot \rangle = dx^2 + dy^2 - dz^2$.

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The Hyperbolic Plane

The hyperbolic plane denoted as $\mathbb{H}^2_0$ is a space like surface of $\mathbb{L}^3$ and is represented by the top part of the hyperboloid of two sheets.

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$$\mathbb{H}_0^2 = \{x^2 + y^2 - z^2 = -1, z > 0\}$$

For visualization purposes, we identify $\mathbb{H}_0^2$ with the Poincaré disk model

$$(x, y, z) \in \mathbb{H}_0^2 \to \frac{1}{1+z}(x, y) \in \mathbb{D}$$

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The de Sitter Space

The de Sitter 2-space, denoted by $\mathbb{H}_1^2$, is a time like surface of $\mathbb{L}^3$ and is represented by

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The de Sitter 2-space, denoted by $\mathbb{H}^2_1$, is a time like surface of $\mathbb{L}^3$ and is represented by

$$\mathbb{H}^2_1 = \{x^2 + y^2 - z^2 = 1\}.$$

We can identify the bottom half $\mathbb{H}^2_{10} = \mathbb{H}^2_1 \cap \{z < 0\}$ with the once punctured unit disk via the diffeomorphism

$$(x, y, z) \in \mathbb{H}^2_{10} \to \frac{1}{x^2 + y^2}(x, y) \in \mathring{\mathbb{D}}$$

Notations

For $\mathbb{H}^2_\epsilon$; ($\mathbb{H}^2_0-$Hyperbolic plane, $\mathbb{H}^2_1-$de Sitter space)

- $\langle T(s), T(s) \rangle = \epsilon_1 = \pm 1$

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Frenet-Serret equations:

- $\nabla_T T(s) = \epsilon_2 \kappa(s) N(s)$
- $\nabla_T N(s) = -\epsilon_1 \kappa(s) T(s)$

Variational Problem

Let $C_*^\infty(I, \mathbb{H}_\epsilon^2)$ be the space of smooth non-null immersed convex curves $\gamma : I \subset \mathbb{R} \to \mathbb{H}_\epsilon^2$ parametrized by the arc length $s \in I$.

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For fixed real number $p \in \mathbb{R}$, the $p-$elastic functional is given by

$$\Theta_p(\gamma) := \int_\gamma \kappa^p \, ds,$$

and acts on $C_*^\infty(I, \mathbb{H}_\epsilon^2)$.

Euler-Lagrange Equation

The critical points for Θ_p must satisfy the Euler-Lagrange equation:

$$p\frac{d^2}{ds^2}\kappa^{p-1} + \epsilon_1\epsilon_2(p-1)\kappa^{p+1} - \epsilon_2 p\kappa^{p-1} = 0.$$

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Therefore $\kappa(s)$ is either constant or a solution to the first-order ordinary differential equation:

$$p^2(p-1)^2\kappa^{2(p-2)}(\kappa')^2 + \epsilon_1\epsilon_2(p-1)^2\kappa^{2p} - \epsilon_2 p^2\kappa^{2(p-1)} = a,$$

where $a \in \mathbb{R}$ is a constant of integration.

Constant Curvature Case

Proposition [A. Pámpano, M. Samarakkody, H. Tran (2024)]

Let γ be a non-geodesic $p-$elastic circle immersed in $\mathbb{H}^2_\epsilon \subset \mathbb{L}^3$. Then, γ is space like and its constant curvature is given by

$$\kappa = \sqrt{\frac{p}{p-1}}.$$

Equivalently, the radius of γ, viewed as a curve in $\mathbb{L}^3$, is $r = \sqrt{(-1)^\epsilon(p-1)}$.

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Equivalently, the radius of γ, viewed as a curve in $\mathbb{L}^3$, is $r = \sqrt{(-1)^\epsilon(p-1)}$.
Moreover:

- If $\gamma \subset \mathbb{H}^2_0$ is a hyperbolic curve, then $p > 1$ holds.
- If $\gamma \subset \mathbb{H}^2_1$ is a pseudo-hyperbolic curve, then $p < 0$ holds.

Existence of Periodic Curvatures

Theorem [A. Pámpano, M. Samarakkody, H. Tran (2024)]

Let γ be a p-elastic curve in $\mathbb{H}^2_\epsilon$ with non-constant periodic curvature. Then γ is a space-like curve with $0 > a > a_* := -((-1)^\epsilon p)^p((-1)^\epsilon(p-1))^{1-p}$.
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Conversely, there exists a space-like convex $p-$elastic curve $\gamma_a : I \subset \mathbb{R} \to \mathbb{H}^2_\epsilon$ with non-constant curvature κ_a. If, in addition, γ_a is defined on its maximal domain, then it is complete ($I = \mathbb{R}$) and its curvature κ_a is a periodic function.

Parameterization

We use an approach by Langer and Singer involving Killing vector fields along curves to find the parametrization.

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The $p-$elastic curve with periodic curvature γ in $\mathbb{H}^2_\epsilon$ can be parametrized in terms of its arc length parameter $s \in \mathbb{R}$, as

$$\gamma = \frac{1}{\sqrt{-a}}(\sqrt{\epsilon_2 a + p^2\kappa^{2(p-1)}}\cos\theta(s), \sqrt{\epsilon_2 a + p^2\kappa^{2(p-1)}}\sin\theta(s), p\kappa^{p-1}),$$

where

$$\theta(s) := \epsilon_2(p-1)\sqrt{-a}\int \frac{k^p}{\epsilon_2 a + p^2\kappa^{2(p-1)}}ds,$$

is the angular progression and $\epsilon_2 = (-1)^\epsilon$.

Geometric Properties

1. The trajectory of γ is contained between two parallels of $\mathbb{H}^2_\epsilon$. If $\gamma \subset \mathbb{H}^2_0$ is a hyperbolic curve, it never meets the pole $(0, 0, 1)$.

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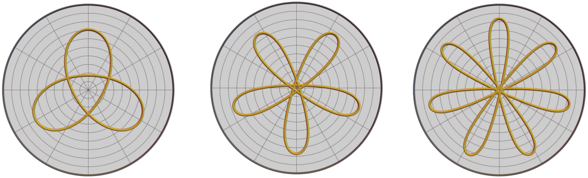
2. The curve γ meets the bounding parallels tangentially at the maximum and minimum values of its curvature.
3. The angular progression is monotonic with respect to the arc length parameter of the curve.
4. The $p-$elastic curve γ is closed if and only if the angular progression along a period of the curvature is a rational multiple of 2π.

Results

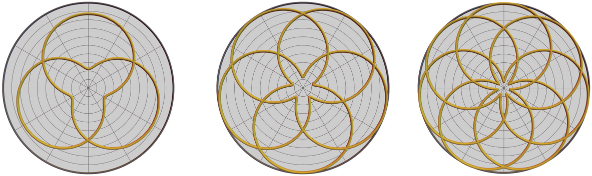
Theorem [A. Pámpano, M. Samarakkody, H. Tran (2024)]

Let (n, m) be relative prime natural numbers such that $m < 2n < \sqrt{2}m$, then there exists a closed space like $p-$elastic curve in $\mathbb{H}^2_\epsilon$, where $p > 1$ for a hyperbolic curve and $p < 0$ for a pseudo-hyperbolic curve.

Results



Three hyperbolic $p-$elastic curves for $p = 3/2$ in $\mathbb{H}^2_0$
corresponding to the values $q = 2/3$, $q = 3/5$, and $q = 4/7$.
They are represented in the Poincare disk model.



Three pseudo-hyperbolic $p-$elastic curves for $p = -1$ in $\mathbb{H}^2_1$
corresponding to the values $q = 2/3$, $q = 3/5$, and $q = 4/7$.
They are represented in the once punctured unit disk.

Uniqueness

According to the numerical experiments, the angular progression is monotonic with respect to a. This would imply that for every $\frac{n}{m} \in \mathbb{Q}$ satisfying $m < 2n < \sqrt{2}m$, there exist a unique closed $p-$elastic curve for every $p \in (-\infty, 0) \cup (1, \infty)$ in $\mathbb{H}^2_\epsilon$ with non-constant curvature, up to isometries. In addition, none of them would be embedded.

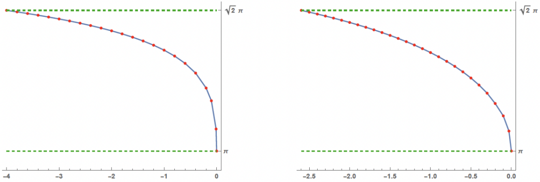


Figure: Angular progression for $p = -1$(Left) and $p = 3/2$(Right).

Uniqueness

Define $\Lambda_p(a) : (a_*, 0) \subset \mathbb{R} \to \mathbb{R}$ given by

$$\Lambda_p(a) := 2p(p-1)^2\sqrt{-a}U,$$

where

$$U = \int_\beta^\alpha \frac{\kappa^{2(p-1)}}{(a+\epsilon_2 p^2\kappa^{2(p-1)})\sqrt{a-\epsilon_2(p-1)^2\kappa^{2p}+\epsilon_2 p^2\kappa^{2(p-1)}}}\,d\kappa,$$

$\epsilon_2 = (-1)^\epsilon$ and $0 < \beta < \alpha$ are the only positive solutions of

$$f_{p,a}(\kappa) := a - \epsilon_2(p-1)^2\kappa^{2p} + \epsilon_2 p^2\kappa^{2(p-1)} = 0.$$

Uniqueness

Theorem

Let $p = 3/2$ and so $\epsilon_2 = 1$. Then, the function $\Lambda_{3/2} : (a_* = -3\sqrt{3}/2, 0) \to \mathbb{R}$ is given by

$$\Lambda_{3/2}(a) = \frac{2\sqrt{\alpha\beta(\alpha+\beta)}}{3\sqrt{2\alpha+\beta}} K(\zeta) + \pi\hat{\Lambda}\left(\arcsin\sqrt{\frac{\chi-\zeta^2}{\chi(1-\zeta^2)}}, \zeta\right),$$

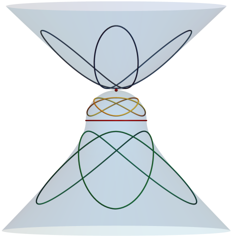
where $\alpha > \beta > 0$ are the only two positive roots of polynomial $\tilde{Q}_{3/2,a}(\kappa) = (-\kappa^3 + 9\kappa + 4a)/4$ and

$$\zeta := \sqrt{\frac{\alpha-\beta}{2\alpha+\beta}}, \quad \chi := \frac{9(\alpha-\beta)}{9\alpha - \alpha\beta(\alpha+\beta)}.$$

Moreover, the function $\Lambda_{3/2}$ decreases monotonically form $\sqrt{2}\pi$ to π.

Results

This figure is evolution of closed $p-$elastic curve of type $\gamma_{2,3}$. In black, the $p-$elastic curve $\gamma_{2,3}$ for $p = 2$ immersed in $\mathbb{H}^2_0$; in yellow, the $p-$elastic curve $\gamma_{2,3}$ for $p = 0.2$ immersed in $\mathbb{S}^2$; and, in green the $p-$elastic curve $\gamma_{2,3}$ for $p = -1$ immersed in $\mathbb{H}^2_1$. The red point is the pole $(0, 0, 1) \in \mathbb{R}^3$ and the red circle is the equator.



Kinematics

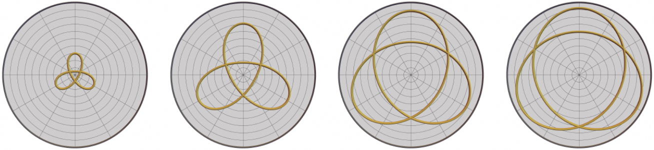


Figure: Closed hyperbolic $p-$elastic curves in $\mathbb{H}_0^2$ of type $\gamma_{2,3}$ for different values of $p > 1$. From left to the right: $p = 1.1$, $p = 2$, $p = 7$ and $p = 15$.

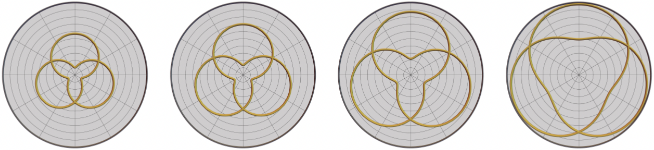


Figure: Closed pseudo-hyperbolic $p-$elastic curves in $\mathbb{H}_1^2$ of type $\gamma_{2,3}$ for different values of $p < 0$. From left to right: $p = -9$, $p = -5$, $p = -2$ and $p = -1/2$.

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