Particle connected to a spool

Mechanics Aplied to Aerospace Engeniering

Laboratory session 1

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Authors:

Sofía Marín Puche

Alejandra Menéndez Lucini

Andrés Velázquez Vela

This report addresses a mechanical problem involving a heavy particle P of mass m , which is connected to a cylindrical spool of radius a via an inelastic, massless string. The spool rotates at a constant angular rate, causing the particle to move. The objective of this assignment is to analyze the system, derive the equations, and implement a numerical solution for the motion of the particle using Matlab.

The study begins by verifying that the system has a single degree of freedom, which can be parameterized by the angle ϕ, the angle between the horizontal axis and the point of tangency between the string and the spool. Subsequently, the position, velocity, and acceleration vectors of the particle are expressed in terms of the constants of the problem, time t, and the angle ϕ and its derivatives.

Next, the forces acting on the particle are examined, and a second-order differential equation for ϕ is derived, describing the dynamics of the system without depending on the tension in the string. Additionally, an equation for the tension T is formulated as a function of ϕ and its derivatives.

The core of this assignment lies in converting the second-order differential equation into two first-order differential equations for the state vector X. These equations are implemented in a Matlab function that will be integrated numerically using the `ode45` solver. This step provides a crucial framework for analyzing the motion of the particle in subsequent cases.

1 Introduction

The purpose of this report is to study the dynamics of a heavy particle connected by a string to a rotating spool. The particle is subject to gravity, and the spool rotates at a constant angular velocity. The system is modeled with one degree of freedom, parameterized by the angle between the string and the horizontal axis. By solving the equations of motion numerically, the position, velocity, and forces acting on the particle can be determined. The aim of the study is to explore the motion and energy of the particle under different initial conditions.

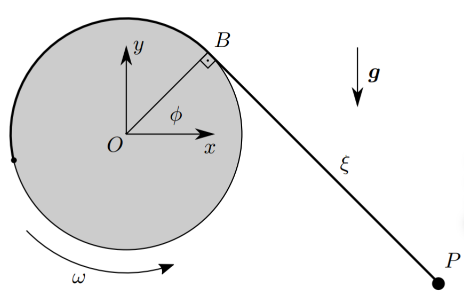


Figure 1: The exercise.

2 Methodology

The dynamics of the particle are modeled using classical mechanics. The system's configuration is defined by the angle ϕ between the spool and the string. The position, velocity, and acceleration of the particle were derived in terms of this angle and its derivatives. The forces acting on the particle include tension in the string and gravitational force.

To simulate the particle's motion, the second-order differential equations were transformed into a system of first-order equations. The `ode45` function in MATLAB was used to solve these equations numerically. Boundary conditions, such as initial velocities and angular velocities, were varied to observe the system's behavior under different scenarios.

3 Results

3.1 Verify that there is only one configuration degree of freedom.

A system's degree of freedom refers to the minimum number of independent parameters required to define its configuration at any given time. For this specific system, we aim to show that it has only one degree of freedom, which can be parametrized by the angular displacement $\phi(t)$, the angle between the spool's point of tangency and the horizontal axis.

The system consist of a spool of radius a, rotating at a constant angular velocity , fixed at the center O. And a particle P of mass m attached to the spool by a massless string of fixed length l.

Key constraints of the system:

- The string is always under tension and remains tight throughout the motion of the particle.
- As the spool rotates ccw, part of the string is wrapped around the spool, reducing the free portion of the string from point B to point P, $(\xi(t,\phi))$.
- The system's configuration depends on how much string is wrapped around the spool, which in turn depends on the angular displacement ϕ.

To verify that the system has only one degree of freedom, we observe that the position P can be described in polar coordinates as:

$$\vec{r}_O^P = a\vec{e}_r + [-\xi_0 + a(\omega t - (\phi - \phi_0))]\vec{e}_\phi \tag{1}$$

Here $\overline{e}_r$ and $\overline{e}_\phi$ are the radial and tangential unit vectors. Importantly, the distance between P and the spool depends directly on the length of the unwrapped portion of the string, which is given by:

$$\xi(t,\phi) = \xi_0 - \omega ta + (\phi - \phi_0)a \tag{2}$$

This shows that once we know $\phi(t)$, the dsitance is determined.

Furthermore, the string has a total, fixed, length l, which is divided into the part of the spool that is an ∞ wrapped segment and the remaining length $\xi(t, \phi)$. Since the total length is constant, the wrapped portion directly determines the remaining length. This means that the configuration of the system is determined by $\phi(t)$, as any change in the angle alters the amount of string wrapped and consequently the length from P to O.

In conclusion, the system's motion at any given time can be described by just one parameter, $\phi(t)$, confirming that the system has only one degree of freedom.

3.2 Express the lenght $\xi(t, \phi)$ and the vectors: $[\vec{r}_O^P, \vec{v}_O^P, \vec{a}_O^P]$.

In this section, we will express the length $\xi(t, \phi)$ as well as the position, velocity, and acceleration vectors for the particle connected to the spool. The expression for the length $\xi(t, \phi)$ is given by:

$$\xi(t, \phi) = \xi_0 - \omega t a + (\phi - \phi_0)a \tag{2}$$

Where:

- ξ_0 is the initial unspooled length of the string.
- ω is the angular velocity of the spool.
- a is the radius of the spool.
- ϕ is the current angle of the particle, and ϕ_0 is its initial angle.

Next, we define the vectors:

$$\begin{cases} \vec{r}_O^P = a\vec{e}_r + [-\xi_0 + a(\omega t - (\phi - \phi_0))]\vec{e}_\phi \\ \vec{v}_O^P = \dot{\phi}[\xi_0 - a(\omega t - (\phi - \phi_0))]\vec{e}_r + a\omega\vec{e}_\phi \\ \vec{a}_O^P = [a\dot{\phi}^2 - 2a\omega\dot{\phi} + \ddot{\phi}(\xi_0 + a((\phi - \phi_0) - t\omega))]\vec{e}_r + \dot{\phi}^2[\xi_0 - a(\omega t - (\phi - \phi_0))]\vec{e}_\phi \end{cases} \tag{3}$$

Where:

- $\vec{r}_O^P$ is the position vector of particle P relative to point O.
- $\vec{v}_O^P$ is the velocity vector of particle P in terms of its radial and tangential components.
- $\vec{a}_O^P$ is the acceleration vector of particle P, comprising both radial and tangential components.

3.3 Mathematical expression of the forces acting on P.

According to Newton's Second Law of motion, the sum of the forces acting on a particle is equal to the mass of the particle times its acceleration:

$$\sum \vec{F} = m\vec{a} \tag{4}$$

For particle P, we can express the forces acting on it in terms of the radial ($\vec{e}_r$) and tangential ($\vec{e}_\phi$) components. The equations for the forces can be written as:

$$\begin{cases} \vec{e}_r : \cancel{m}(a\dot{\phi}^2 - 2a\omega\dot{\phi} + \ddot{\phi}(\xi_0 + a((\phi - \phi_0) - t\omega))) = -\cancel{m}g \sin\phi \\ \vec{e}_\phi : m\dot{\phi}^2(\xi_0 - a(\omega t - (\phi - \phi_0))) = -mg\cos\phi + T \end{cases} \tag{5}$$

Explanation of the Equations:

- Radial Component ($\vec{e}_r$): - The first equation represents the balance of forces in the radial direction. The term on the left-hand side includes contributions from the particle's radial acceleration. The right-hand side accounts for the gravitational force component acting in the radial direction.

- Tangential Component ($\vec{e}_\phi$): - The second equation describes the forces in the tangential direction. The left-hand side represents the net force due to the tangential acceleration of the particle, while the right-hand side accounts for the gravitational force acting in the tangential direction and the tension T.

3.4 Second-order differential equation for ϕ that does not depend on T. Tension T as a function of ϕ, $\dot{\phi}$.

In the equation of the forces acting on P we clear the tension in the $\vec{e}_\phi$ component:

$$\vec{T} = m(g\cos\phi + \dot{\phi}^2(\xi_0 - a(\omega t - (\phi - \phi_0)))\overline{e}_\phi \tag{6}$$

From the $\vec{e}_r$ component the derivative of $\dot{\phi}$:

$$\begin{cases} \dfrac{d\phi}{dt} = \dot{\phi} \\ \dfrac{d\dot{\phi}}{dt} = \ddot{\phi} = \dfrac{2a\omega\dot{\phi} - a\dot{\phi}^2 - g\sin\phi}{\xi_0 + a((\phi - \phi_0) - t\omega)} \end{cases} \tag{7}$$

3.5 Two first-order differential equations for the state vector: $\boldsymbol{X} = [\phi, \dot{\phi}]^{\boldsymbol{T}}$.

In this section, we derive the first-order differential equations for the state vector $\mathbf{X}$ defined as:

$$X = \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} \qquad\qquad \frac{d}{dt}X = f(t, \mathbf{X}) \tag{8 and 9}$$

The system can be described by the following equations:

$$\begin{cases} \dfrac{d}{dt}X_1 = X_2 \\ \dfrac{d}{dt}X_2 = \dfrac{2a\omega\dot{\phi} - a\dot{\phi}^2 - g\sin\phi}{\xi_0 + a((\phi - \phi_0) - t\omega)} \end{cases} \tag{10}$$

Where:

- X_1 represents the angle ϕ.
- X_2 represents the angular velocity of the point B ($\dot{\phi}$).
- a is the radius of the spool.
- ω is the angular velocity of the spool.
- g is the acceleration due to gravity.
- ξ_0 is the initial unspooled length of the string.
- ϕ_0 is the initial angle.

Furthermore, the velocity vector of the particle can be expressed as:

$$\dot{x}_O^P = \vec{v}_O^P(\textit{only the x-axis component}) \tag{11}$$

As $\phi_0 = 0$ *rad*, the angular velocity $\dot{\phi}$ can be determined by the $\vec{e}_r$ component of the velocity:

$$\dot{\phi} = \frac{\dot{x}_O^P}{[\xi_0 - a(\omega t - (\phi - \phi_0))]} \tag{12}$$

3.6 Create a stop function that will detect when the particle hits the spool or the tension in the string vanishes.

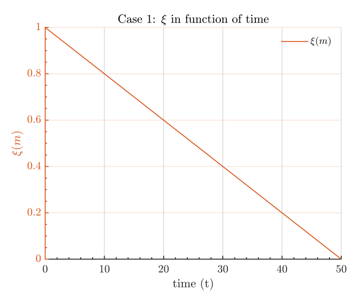
In order to create a function that detects when rather the point P touch the spool or the tension T vanishes, we have to set some parameters.

```
function [value,terminate,direction] = stopfun(t,X, xi0, m, g, phi0, omega, a, index)
    T = m*(g*cos(X(1))+ X(2)^2 * (xi0 - a*(omega(index)*t - (X(1)- phi0)) ));
    xi = xi0-omega(index)*t*a + (X(1)- phi0)*a;
    value = [xi, T];
    terminate =[1,1];
    direction = [-1, -1];
end
```

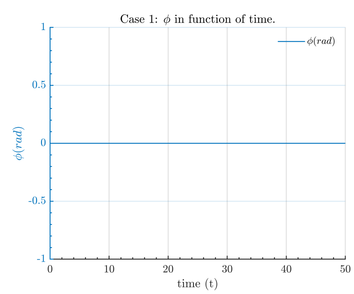
The expression `terminate = [1, 1]` means that when either condition in value crosses zero, the integration should stop. Also, `direction = [-1, -1]` means the function will detect only downward crossings (when the value goes from positive to negative).

3.7 Evolution of ϕ, the amount of unspooled string ξ and the tension T as a function of time.

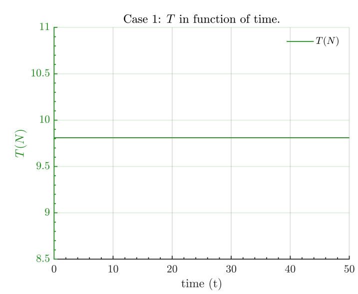
CASE 1



(a) ξ in function of time

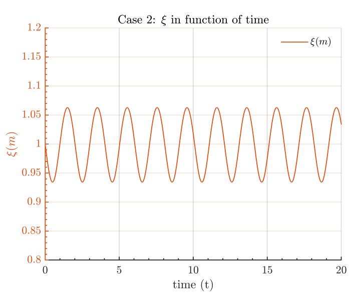


(b) ϕ in function of time

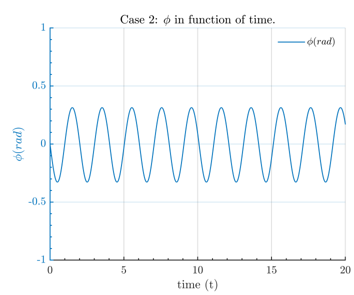


(c) T in function of time

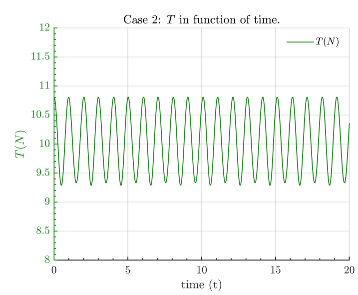
CASE 2



(a) ξ in function of time

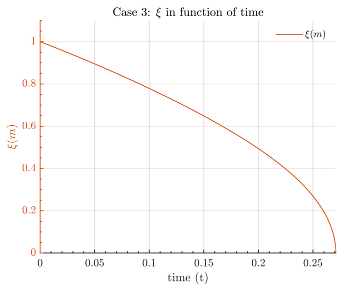


(b) ϕ in function of time

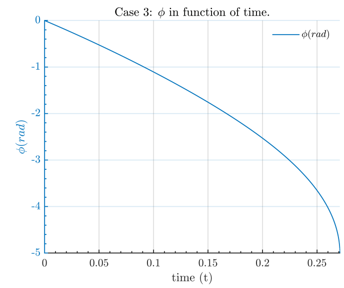


(c) T in function of time

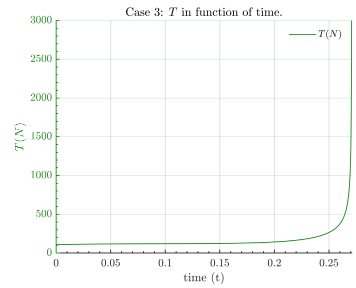
CASE 3



(a) ξ in function of time

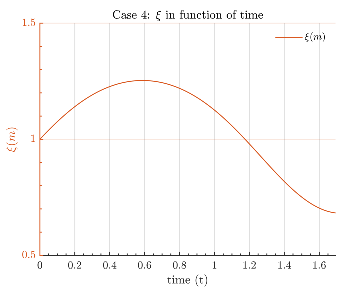


(b) ϕ in function of time

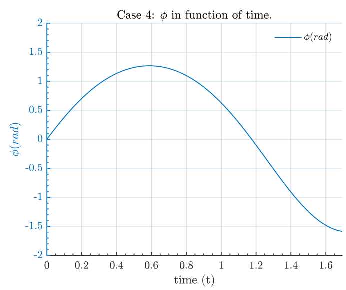


(c) T in function of time

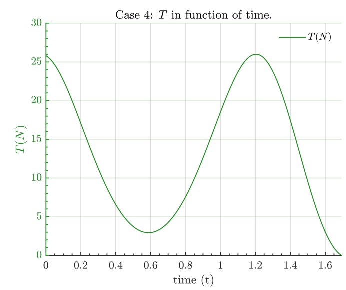
CASE 4



(a) ξ in function of time

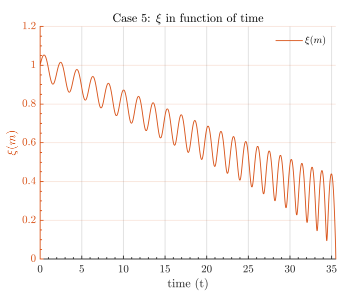


(b) ϕ in function of time

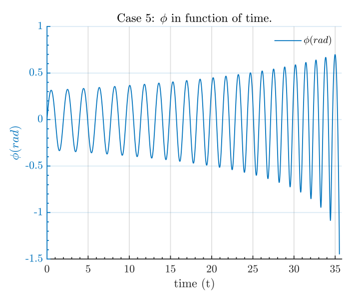


(c) T in function of time

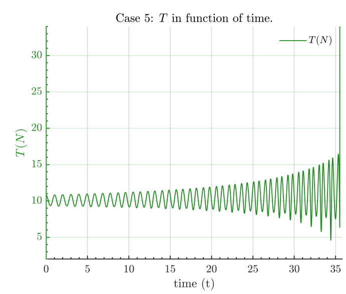
CASE 5



(a) ξ in function of time



(b) ϕ in function of time



(c) T in function of time

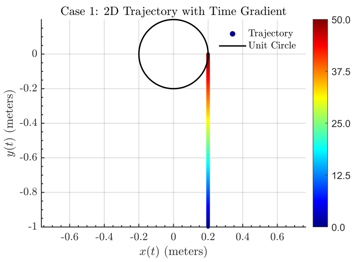
3.8 Trajectory of P in the O_{xy}

Transforming the vectors $\vec{e}_r$ and $\vec{e}_\phi$ into cartesian coordinates, we can plot the trajectory.

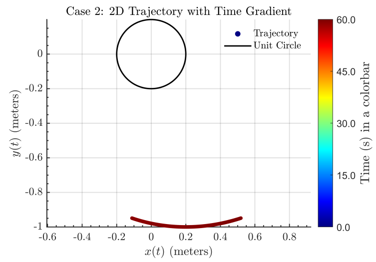
$$\begin{cases} \vec{e}_r = \cos(\phi)\vec{e}_x + \sin(\phi)\vec{e}_y \\ \vec{e}_\phi = -\sin(\phi)\vec{e}_x + \cos(\phi)\vec{e}_y \end{cases} \tag{13}$$

$$\vec{r}_O^P = a(\cos(\phi)\vec{e}_x + \sin(\phi)\vec{e}_y) + [-\xi_0 + a(\omega t - (\phi - \phi_0))](-\sin(\phi)\vec{e}_x + \cos(\phi)\vec{e}_y) \tag{14}$$

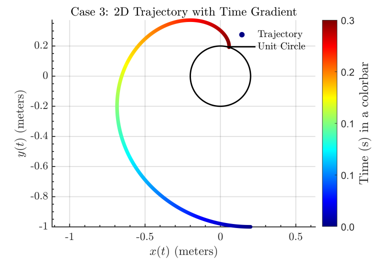
2D Trajectory with the time represented as a gradient



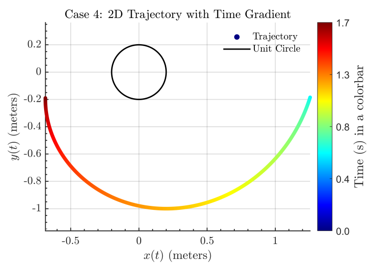
(a) CASE 1



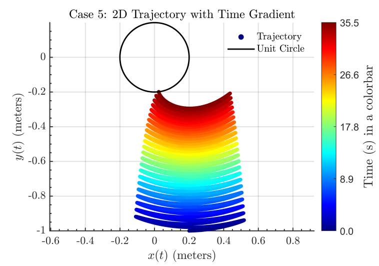
(b) CASE 2



(c) CASE 3



(a) CASE 4



(b) CASE 5

We have also attached the visdeos of the trayectories in the *.zip* file.

3.9 Mechanical energy E of P

We can express the mechanical energy as the sum of the kinetic and potential energies. In this case the mechanical energy is expressed respect to the initial point $P(-1, 0)$.

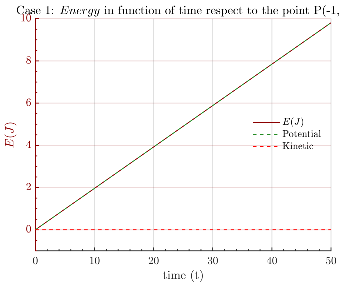
$$E = \frac{1}{2}m|\vec{v}_O^P|^2 + mg(x - x_0) \tag{15}$$

The conservation of mechanical energy E depends, essentially, on the rotation of the spool. If the spool rotates, it modifies the energy of the system, therefore it is not conserved.

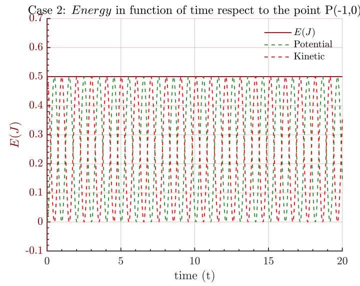
Key constraints of the system:

- **Case 1:** E is not conserved because the rotating spool adds energy continuously.
- **Case 2:** E is conserved because the spool is not rotating and the particle moves downward.
- **Case 3:** The particle hit the spool, causing loss of energy, therefore it is not conserved.
- **Case 4:** The particle moves upward and no rotation in the spool so the energy is conserved.
- **Case 5:** The rotating spool adds energy to the system, so it is not conserved.

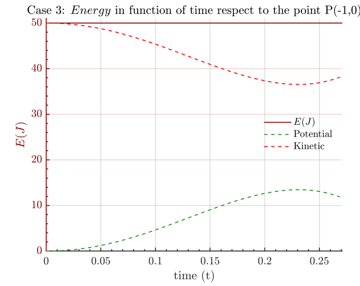
Energy in function of time respect to the point P(-1,0)



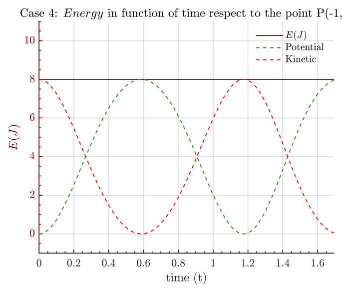
(a) CASE 1



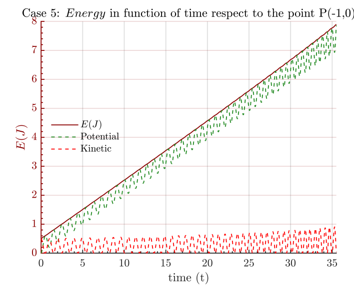
(b) CASE 2



(c) CASE 3



(a) CASE 4



(b) CASE 5

4 Discussion

4.1 When and where does the particle hit the spool in case 3? Discuss what happens to $\dot{\phi}$ near the end in case 3. When does the string lose tension in case 4? What is the stop condition in case 5?

For **Case 3**, the particle starts with a backward initial velocity and the spool is not rotating ($\omega = 0\,rad/s$). The particle will hit the spool when the unspooled length of the string becomes zero, that is exactly at $t = 0.2709$ seconds and at an angle $\phi = -5.0000$ rad.

As the particle approaches the spool, $\dot{\phi}$ increases rapidly due to the tension in the string, causing it to start rotating. The final magnitude of $\dot{\phi}$ that the program computated is $-5,387.10^7$ rad.

In **Case 4**, the particle begins with an rightward initial velocity and a stationary spool. At the first moment when the velocity is equal to 0 the tension T isn´t able to drop to 0. It is not until it reach the left side of the oscilation that by the time we reach the top of it, when the particle falls the tension is lost at $t = 1.6935$ seconds. To acomplish this result we were forced to reduce the tolerances with `RelTol', 1e-8,'AbsTol', 1e-10` to have a more precise `ode45` computation.

For **Case 5**, the spool is rotating and the particle has an initial rightward velocity. The stop condition occurs when the pasticle hits the spool, meaning $\xi = 0$ at time $t = 35.5094$ seconds.

4.2 Open discussion: discuss what changes would need to be made to the code to integrate the motion of the particle after tension is lost in case 4.

In **Case 4**, the particle begins with an initial rightward velocity while the spool remains stationary. As the particle ascends, gravity decelerates it, leading to a reduction in the tension of the string. Eventually, the string loses all tension, when it reach the left side of the oscilation that by the time we reach the top of it, when the particle falls the tension is lost at $t = 1.6935$ seconds. Currently, the code stops the simulation when this condition is met, as it is considered a stop criterion. However, to model the motion of the particle beyond this point, it is necessary to make some modifications to continue the simulation after the string becomes loose.

The only change that we will have to make for the simulation to don´t stop when the tension T reach 0 will be the following. In the stop function we have to change the value of terminate from 1 to 0: `terminate = [1, 0]`.

5 Conclusions

In this exercise, we analyzed the motion of a particle connected to a rotating spool and examined the result of the motion in different cases. Through mathematical analysis of the equations of motion, we studied the different trajectories of the particle under various initial conditions, including both rotating and stationary spools, as well as particles with upward and downward velocities.

Our results show that mechanical energy is conserved when no external forces influence the system, in other words, when $\omega = 0$ initially. In Case **2**, **3** and **4**, energy is conserved as the spools are stationary, excluding the time when they touch the spool, with a continuous exchange between kinetic and potential energy. In these cases, the forces acting on the particle are internal, and no energy is lost or introduced from external sources.

On the contrary, energy is not conserved in Cases **1** and **5** for the rotating spool adds energy to the system. This demonstrates how external inputs, such as the spool's rotation, disrupt the conservation of mechanical energy.

In conclusion, after doing this exercise, we have achieved a better understanding of the motion of particles and the different outcomes that can occur when the conditions are modified. Furthermore, by using `ode45`, we have learned to solve differential equations both of first and second order in Matlab.