Particle on oscillating loop

Mechanics Applied to Aerospace Engineering

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Authors:

Sofía Marín Puche

Alejandra Menéndez Lucini

Andrés Velázquez Vela

This report addresses a mechanical problem involving a heavy point particle P of mass m, which slides without friction on a circular wire of center C and radius R. The wire is located on a vertical plane that rotates about the vertical axis with a prescribed frequency. The objective of this assignment is to analyze the system, derive the equations, and implement a numerical solution for the motion of the particle using Matlab.

The study begins by obtaining the position, velocity, and acceleration vectors of the particle, which are expressed in terms of ϕ and its derivatives. Subsequently, the forces acting on the particle P are examined for a generic instant of time t. Next, a second-order differential equation is derived, describing the dynamics of the system without depending on the reaction forces. Additionally, an equation for the forces is formulated in terms of ϕ and $\dot{\phi}$.

The core of this assignment lies in converting the second-order differential equation into two first-order differential equations for the state vector X. These equations are implemented in a Matlab function that will be integrated numerically using the ode45 solver. This step provides a crucial framework for analyzing the particle's motion for all possible combinations of initial conditions and special cases.

1 Introduction

The purpose of this report is to analyze the motion of a heavy particle sliding without friction on a circular wire, which rotates about a vertical axis with a prescribed angular frequency. The system is influenced by gravity and is modeled with one degree of freedom, defined by the angle the particle makes with a reference axis. By deriving the equations of motion and solving them numerically, the position, velocity, and forces acting on the particle can be obtained. The aim of the study is to investigate the particle's dynamics under various initial conditions, exploring how factors such as angular velocity affect the motion and energy of the system.

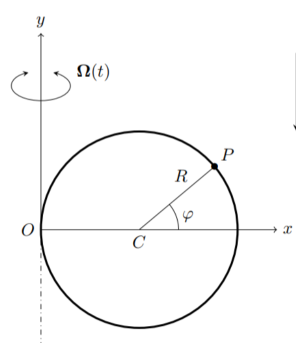

Figure 1: The exercise.

2 Methodology

The dynamics of the particle connected to the spool are modeled using classical mechanics, with the angle ϕ defining the system's configuration. The position, velocity, and acceleration of the particle are derived in terms of ϕ and its derivatives.

Forces acting on the particle include gravitational force $\vec{W} = -mg\vec{j}$ and reaction forces from the wire, along with fictitious forces due to the rotating reference frame. Newton's second law is applied in both radial and tangential directions, resulting in a second-order differential equation for ϕ.

This equation is transformed into a system of first-order differential equations, defining the state vector $\mathbf{X} = [\phi, \dot{\phi}]^T$. The equations are numerically solved using the `ode45` function in MATLAB, exploring various initial conditions to analyze the system's behavior under different scenarios.

3 Results

3.1 Express the vectors: $[\vec{r}_O^P, \vec{v}_O^P, \vec{a}_O^P]$.

The position vector can be constructed by adding together the vector OC plus CP. Where C is the center of the circumference. The velocity and acceleration are obtained by successive differentiation. We define the vectors:

$$\begin{cases} \vec{r}_O^P = R\left[(1+\cos(\theta))\hat{i} + \sin(\theta)\hat{j}\right] \\ \vec{v}_O^P = R\dot{\phi}\left[-\sin(\phi)\hat{i} + \cos(\phi)\hat{j}\right] \\ \vec{a}_O^P = R\ddot{\phi}\left[-\sin(\phi)\hat{i} + \cos(\phi)\hat{j}\right] + R\dot{\phi}^2\left[-\cos(\phi)\hat{i} - \sin(\phi)\hat{j}\right] \end{cases} \tag{1}$$

Also, we define the unit vector: $\vec{e}_r = \cos(\phi)\vec{i} + \sin(\phi)\vec{j}$.

Where:

- $\vec{r}_O^P$ is the position vector of particle P relative to point O in the moving reference frame.
- $\vec{v}_O^P$ is the velocity vector of particle P in terms of its radial and tangential components.
- $\vec{a}_O^P$ is the acceleration vector of particle P, comprising both radial and tangential components.

3.2 Mathematical expression of the forces acting on P.

The real forces acting on P are the weight and the (unknown) reaction of the wire:

$$\begin{cases} \vec{W} = -mg\hat{j} \\ \vec{N} = N_1\hat{e}_r + N_2\hat{k} \end{cases} \tag{2}$$

The $Oxyz$ reference frame is rotating, so it is not inertial. As we will write down the equations of motion in $Oxyz$, we will also have to consider the following fictitious forces:

$$\begin{cases} \vec{F}_{I1} = mA^2R\omega^2(1+\cos(\phi))\sin^2(\omega t)\hat{i}, \\ \vec{F}_{I2} = -2mAR\omega\dot{\phi}\sin(\phi)\sin(\omega t)\hat{k}, \\ \vec{F}_{I3} = mAR\omega^2(1+\cos(\phi))\cos(\omega t)\hat{k}. \end{cases} \tag{3}$$

We have used $\alpha = A\omega^2 cos(\omega t)\vec{j}$

The forces that exert work on the particle are only W and $\mathbf{F}_{I1}$. While W derives from the potential $mgR\sin(\phi)$, $\mathbf{F}_{I1}$ does not have a potential form that is independent from time. Thus, $\mathbf{F}_{I1}$ is not conservative. Then, the mechanical energy of P is not conserved, because there are forces that exert work that are not conservative.

3.3 Analysis of the Dynamics in the Rotating Reference Frame

The required second-order differential equation for ϕ can be found by applying Newton's second law, ($\sum \vec{F} = m\vec{a}$), in the $\vec{i}$ and $\vec{j}$ directions. By combining these equations to eliminate the unknown reaction N_1, we effectively project this law along $\mathbf{e}_\phi$:

$$\begin{cases} mR\ddot{\phi}(-\sin(\phi)) + mR\dot{\phi}^2(-\cos(\phi)) = mA^2R\omega^2(1+\cos(\phi))\sin^2(\omega t) + N_1\cos(\phi) \\ mR\ddot{\phi}\cos(\phi) + mR\dot{\phi}^2(-\sin(\phi)) = N_1\sin(\phi) - mg \\ 0 = -2mA\omega R\dot{\phi}\sin^2(\phi)\sin(\omega t) + mAR\omega^2(1+\cos(\phi))\cos(\omega t) + N_2 \end{cases} \tag{4}$$

To obtain the second-order differential equation for ϕ, we proceed by solving the system as follows:

$$\ddot{\phi} + \frac{g\cos(\phi)}{R} + A^2\omega^2(1+\cos(\phi))\sin^2(\phi t)\sin(\phi) = 0 \tag{5}$$

The two equations needed are the projections of Newton's second law along $\mathbf{e}_r$ and $\mathbf{k}$. Similarly to the previous system, we solve as follows:

$$N_1 = mg\sin(\phi) - mA^2R\omega^2(1+\cos(\phi))\sin^2(\omega t)\cos(\phi) - mR\dot{\phi}^2 \tag{6}$$

For N_2, we isolate equation 3 from (**4**):

$$N_2 = 2mAR\omega\dot{\phi}\sin(\phi)\sin(\omega t) - mAR\omega^2(1+\cos(\phi))\cos(\omega t) \tag{7}$$

3.4 Two first-order differential equations for the state vector: $\mathbf{X} = [\phi, \dot{\phi}]^T$

To express the second-order equation as a system of first-order equations, we define the state vector $\mathbf{X}$:

$$\mathbf{X} = \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} \quad \frac{d}{dt}\mathbf{X} = f(t, \mathbf{X}) \tag{8}$$

Then, we can write the derivative of the state vector:

$$\frac{d}{dt}\mathbf{X} = \begin{bmatrix} \dot{\phi} \\ -\frac{g\cos(\phi)}{R} - A^2\omega^2(1+\cos(\phi))\sin^2(\omega t)\sin(\phi) \end{bmatrix} \tag{9}$$

This leads to the system of first-order differential equations:

$$\begin{cases} \frac{d}{dt}X_1 = X_2 \\ \frac{d}{dt}X_2 = -\frac{g\cos(X_1)}{R} - A^2\omega^2(1+\cos(X_1))\sin^2(\omega t)\sin(X_1) \end{cases} \tag{10}$$

Parameters:

$X_1 = \phi$: angle
$X_2 = \dot{\phi}$: angular velocity
R: radius of the circular path
m: mass of point P
g: acceleration due to gravity
A, ω: constants of the angular frequency

These equations are converted to first-order differential equations in MATLAB and solved using the `ode45` function.

```
function Xdot = diffeq(t, X, g, omega, A, R)
phi = X(1);
phidot = X(2);
Xdot = [phidot; ((-g*cos(phi))/R) - (A^2) * (omega^2) * (1 + cos(phi))
* (sin(omega*t)^2) * sin(phi)];
end
```

4 Numerical Solution and Analysis of Motion

To solve for the motion of the particle and analyze its energy and forces, we implement the integration of the system's equations of motion using MATLAB's `ode45` solver. We consider the parameters provided:

$$A = \pi, \quad g = 9.81\,\mathrm{m/s}^2, \quad m = 1\,\mathrm{kg}, \quad R = 1\,\mathrm{m}, \quad \omega = 0.1\,\mathrm{rad/s}$$

with initial conditions $\phi(0) = 0\,\mathrm{rad}$ and $\dot{\phi}(0) = 0\,\mathrm{rad/s}$. The system is analyzed over the interval $t = [0, 3T]$, where T is the period of the rotating reference frame given by $T = \frac{2\pi}{\omega}$.

4.1 Energy Analysis

The potential energy V and kinetic energy T of the system are given by:

$$\begin{cases} V = mg(y - y_0) + \frac{1}{2}m\omega^2\left(x^2 + z^2\right) \\ T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \end{cases} \tag{11}$$

The mechanical energy E is then:

$$E = V + T \tag{12}$$

4.2 Results

We plot the evolution of the angle $\phi(t)$, reaction forces N_1 and N_2, and the mechanical energy E, showing the potential and kinetic components. We have reduced the time coordinate to $\frac{1}{3}$ of the total $3T$ for better visualizing

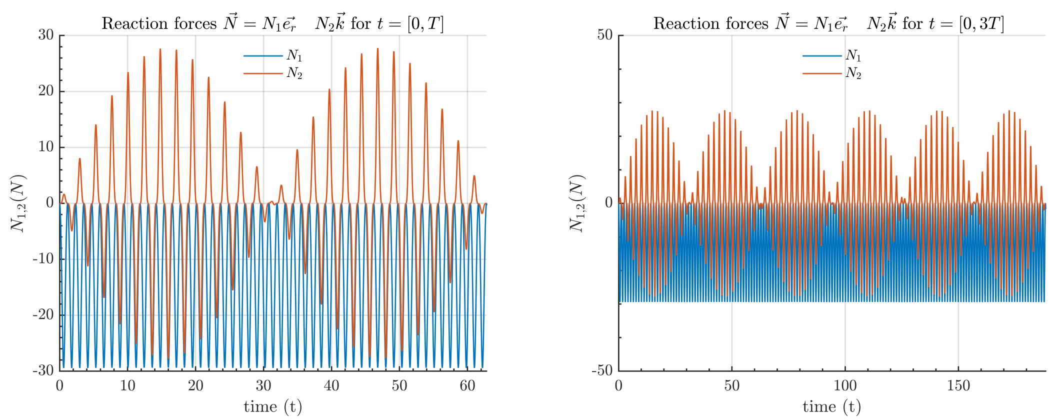

Figure 2: The normal forces.

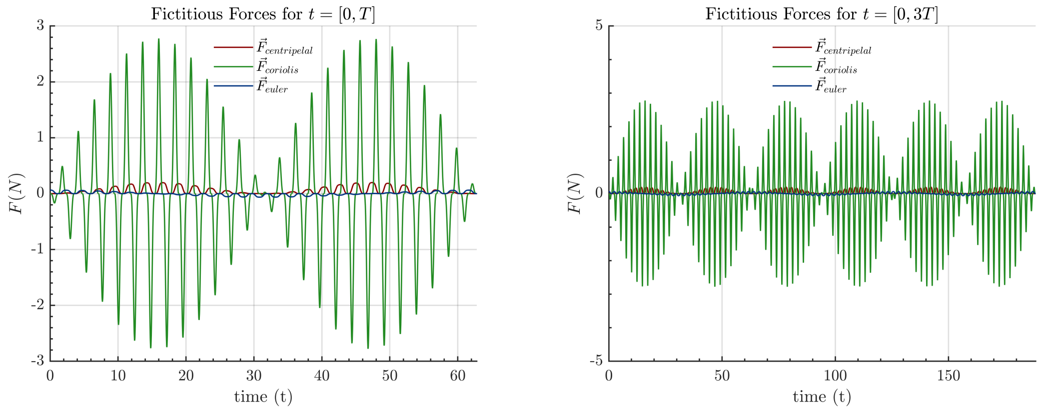

Figure 3: The fictitious forces.

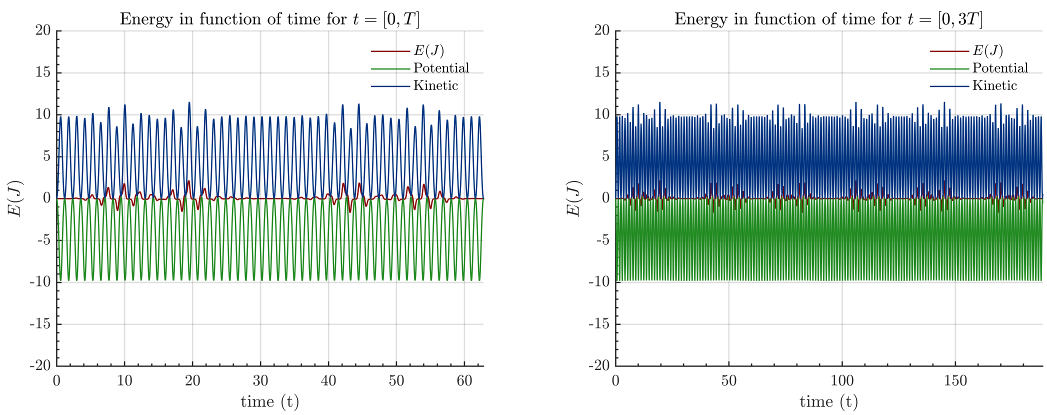

Figure 4: E , T , K as a function of time. **Respect to the static reference frame.**

4.3 Trajectory with Time Gradient

The trajectory of the particle is displayed in a 3D plot with a color gradient representing velocity. This provides an intuitive visualization of how the particle's speed varies along its path.

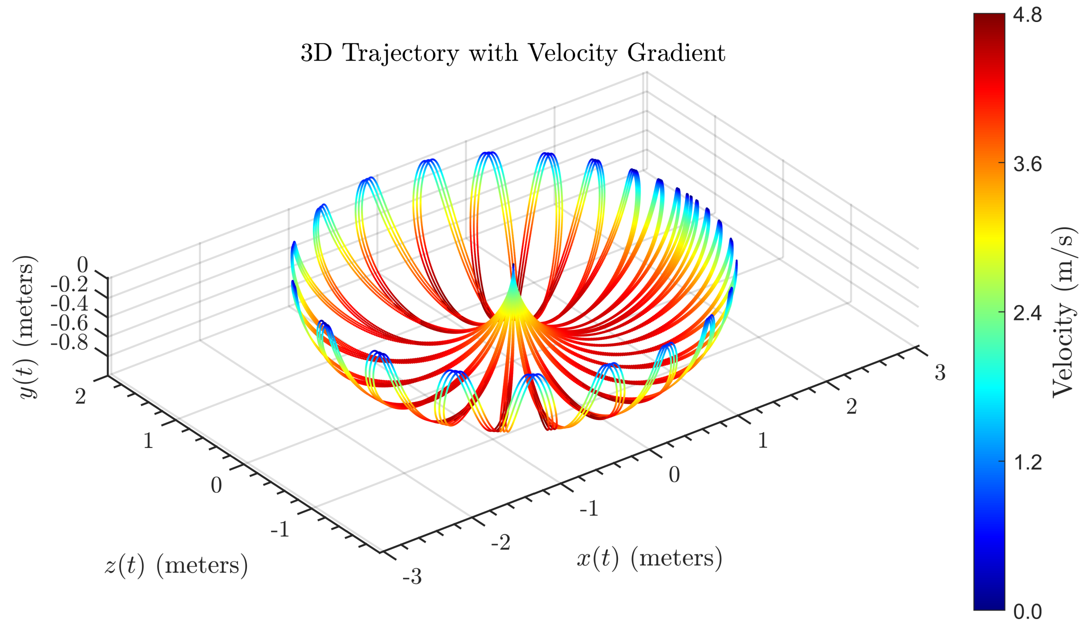

Figure 5: Trajectory with **velocity** gradient.

The analysis provides insights into the dynamics of the trajectory of P as seen from a stationary reference frame $Ox_1y_1z_1$, on a rotating circular wire under gravitational and fictitious forces. By integrating the equations of motion, we obtain the time evolution of ϕ, energy distribution, and reaction forces, illustrating the effects of rotational frequency on the particle's trajectory and energy. The color gradient in the image represents the particle's velocity, with increasing shades of red indicating higher speeds.

4.4 Combinations of ϕ_0 and $\dot{\phi}_0$ over time

In this part, examining $\dot{\phi}$ as a function of ϕ with varying initial conditions allows us to distinguish between an oscillatory and a non-oscillatory movement of the particle. This indicates that with a large initial kinetic energy, the particle moves continuously along the loop: orange, black and blue lines.

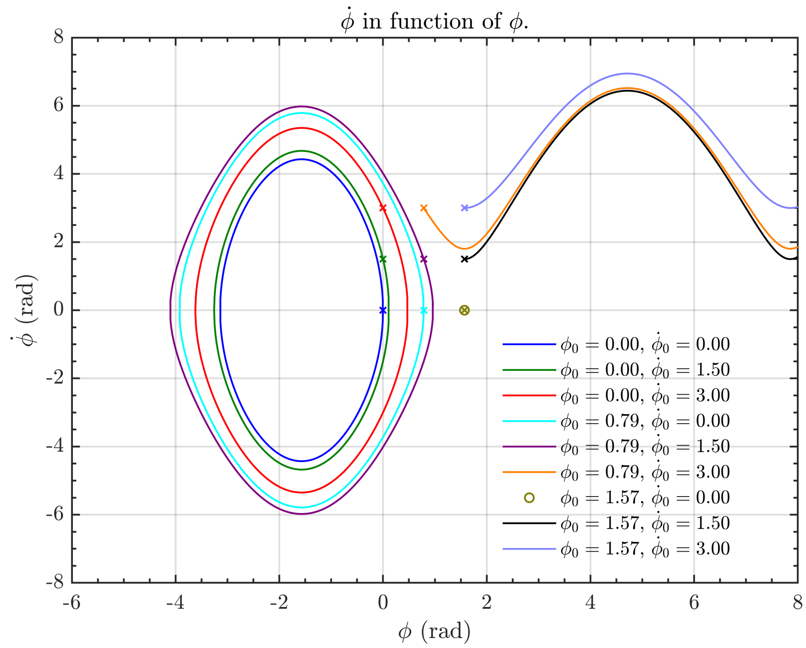

Figure 6: $\dot{\phi}$ as a function of ϕ .

Conversely, with smaller values of ϕ_0 and $\dot{\phi}_0$, the particle demonstrates oscillations, remaining bounded within an angular range. By analyzing the potential and total mechanical energy, we can derive a relationship between the initial values of ϕ_0 and $\dot{\phi}_0$ that dictates whether the motion will be oscillatory or non-oscillatory.

$$\begin{cases} \dot{\phi}_0 = \cos\phi_0 = 0 & , \phi_0 = k\left(-\frac{\pi}{2}\right) \quad \text{or} \quad k\left(\frac{3\pi}{2}\right), \quad k \in \mathbb{N} \\ \dot{\phi}_0 = \sqrt{\frac{2g(1-\sin\phi_0)}{R}} & , otherwise \end{cases} \tag{13}$$

If $\dot{\phi}_0 > \sqrt{\dots}$ it will be a non oscilatory movement,'the 3 branches koving to the right'.
If $\dot{\phi}_0 = \sqrt{\dots}$ it is a stationary point, stable(only when $\dot{\phi}_0 = \cos\phi_0 = 0$) or unstable.
If $\dot{\phi}_0 < \sqrt{\dots}$ it will be an oscilatory movement,'the 6 'circle-shaped' cases.

4.5 Analysis of Oscillation Errors

Assuming $A^2 \ll \frac{g}{R\omega^2}$, $\phi_0 = -\frac{\pi}{2}$ and $\delta_0 = 0$, we compare solutions to the full differential equation against the linearized version, which is valid for small oscillations.

$$\%_{\text{error}} = \frac{\phi - \phi_{\text{linearized}}}{\phi} \times 100 \tag{14}$$

$$\phi_{\text{linearized}} = \delta - \frac{\pi}{2} \tag{15}$$

The percentage error between these solutions is plotted over time for different angular frequencies.

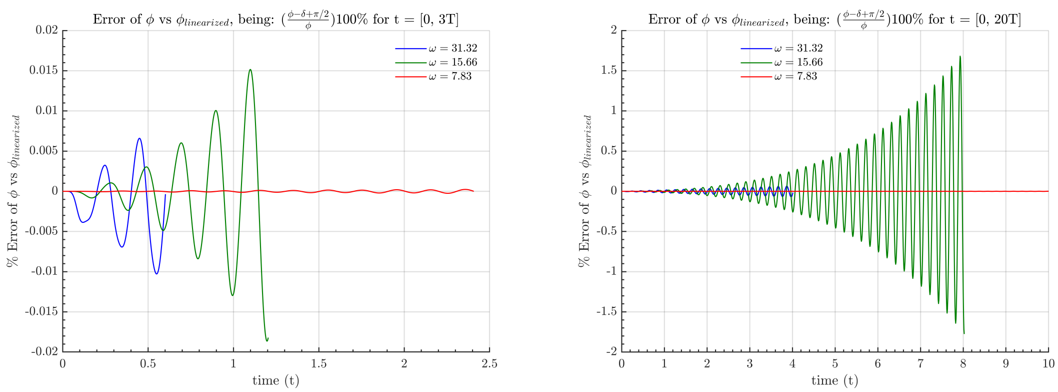

Figure 7: The error of ϕ versus $\phi_{\text{linearized}}$.

Results show that the error is smaller when the system is further to the critical frequency. However, as the frequency converges from this critical point, $2\omega = \omega_n = \sqrt{\frac{g}{R}}$, in this case $\omega \approx 15.66$, errors accumulate. This indicates that nonlinear effects become significant and the linear approximation fails to capture the particle's full behavior. This highlights the importance of choosing the appropriate approximation model: the linear approximation is not effective near critical conditions, while the full nonlinear model is necessary for more general cases.

5 Discuss of the results

The oscillating loop setup illustrates key aspects of dynamics. For small oscillations, the linearized equations provide accurate predictions. However, as parameters move away from these condition, nonlinearity becomes dominant, requiring the complete differential model for a better representation. This analysis highlights the value of the full model in capturing complex oscillatory behaviors and the limitations of linear approximations in dynamic systems involving rotational and gravitational effects.

6 Conclusion

In this report, we studied the dynamics of a particle sliding on a rotating circular wire, looking at how different starting conditions and rotation speeds affect its motion and energy. We derived the equations governing the system, turned them into first-order differential equations, and solved them numerically using MATLAB's ode45 solver.

Our results indicate that the particle's motion can be either oscillatory or non-oscillatory, depending on its initial speed and position. With moderate initial conditions, the particle oscillates within a limited range, exchanging energy between kinetic and potential forms. However, at higher initial speeds, the particle moves continuously around the loop, disrupting energy conservation due to the rotational effects of the system when $A \neq 0$.

Additionally, we tested a linear approximation model for small oscillations, finding it accurate only for small oscilations.

In conclusion, after doing this exercise, we have achieved a better understanding of the motion of particles and the different outcomes that can occur when the conditions are modified. Furthermore, by using ode45, we have learned to solve differential equations both of first and second order in Matlab.