Restricted Three-body Problem

Modelling in Aerospace Engineering

Assignment III

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This report aims to explain the methods used to solve the systems and to ensure a proper understanding of the results.

Restricted Three-Body Problem Analysis

Transformation to Linear System

The RBTP is converted into a system of ODEs by the creation of new variables that allow the expression of u_1 and u_2 second derivatives as a function of them.

Nevertheless, the system is not linear due to the terms D_1 and D_2, which are related to the gravitational nature of the problem.

The system of differential equations for the state vector X is given by:

$$\frac{dX}{dt} = AX + B(X)$$

where the matrix A and the vector $B(X)$ are defined as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix}$$

$$B(X) = \begin{bmatrix} 0 \\ -\hat{\mu}\frac{x_1+\mu}{D_1} - \mu\frac{x_1-\hat{\mu}}{D_2} \\ 0 \\ -\hat{\mu}\frac{x_3}{D_1} - \mu\frac{x_3}{D_2} \end{bmatrix}$$

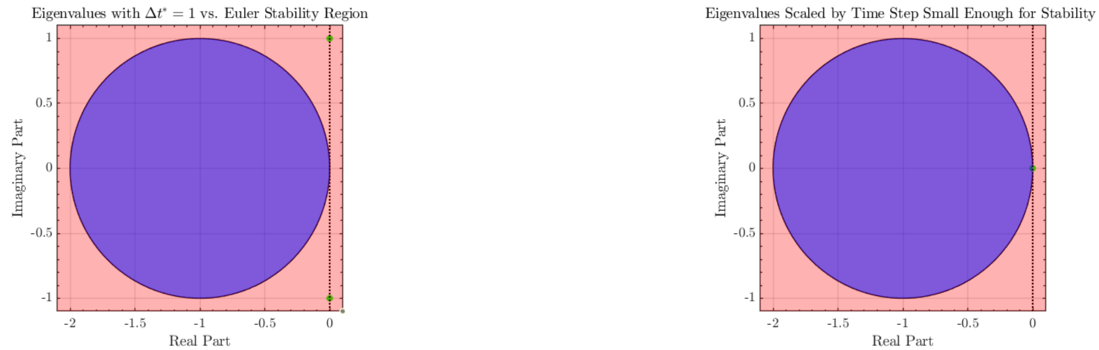
Therefore, to express the system as $\frac{dX}{dt} = AX + B$, A is built as just the linear terms matrix that arises from the expressions of the new variables, while B is a vector containing the non-linear terms of said equations.

Eigenvalue Analysis and Stability

As shown, the initial time step resulted in the eigenvalues being clearly outside of the stability region. In addition, only 2 eigenvalues are visible because the 4 coincide in complex conjugate pairs. Indeed, the eigenvalues going in pairs and being purely imaginary reveals key insights about explicit Euler method validity for the problem.

The process followed to scale the eigenvalues consisted of reducing the time step and therefore the scaled eigenvalues until they fell into the stability region.

Disappointingly, reducing the magnitude of the eigenvalues could only result in their approximation to the origin due to the lack of a real part. For this to happen, the time step needs to be very small, as the only point of the stability region that touches the imaginary axis is the origin.



(a) Eigenvalues scaled with $\Delta t^* = 1$ outside the stability region.

(b) Eigenvalues scaled with reduced Δt^* within the stability region.

Figure 1: Stability region analysis for the explicit Euler method.

As a result, the timestep for which the solution is stable is impractical; its diminutive magnitude would increase computational cost for real problem resolutions.

This enables the conclusion that the Explicit Euler method cannot stabilize systems with purely imaginary eigenvalues unless the timestep tends to 0, highlighting the need for alternative methods.

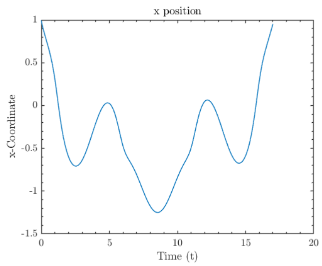
Explicit Euler Method Solution

Regarding the state vector evolution with time, two key aspects should be pointed out.

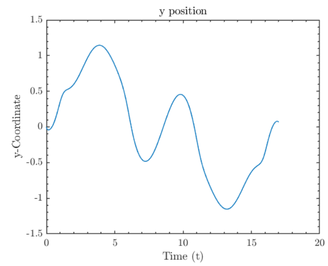
First of all, the behaviour of the variables is bounded and does not blow up, showing how the small time step obtained in part 2 ensures stability. Although it is highly dependent on the tolerance selected in the iterative method for the timestep.

Secondly, the wording mentions the critical points $(-\mu, 0)$ and $(\mu_n, 0)$ at which D_1 and D_2 become 0. At the start, the moon is close to one of the critical points, therefore the non-linear term related to gravitational force dominates and provokes a sudden and strong change in the velocities. Although physically correct, Euler's first order approximation exaggerates it, and a higher order method could handle it better.

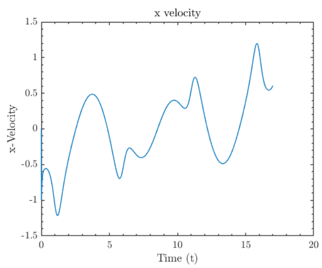
Finally, each time the x position is close to the critical points, maximum and minimum are seen in the velocity plots, which aligns with inverse proportionality of gravitational force and distance to the attractor.



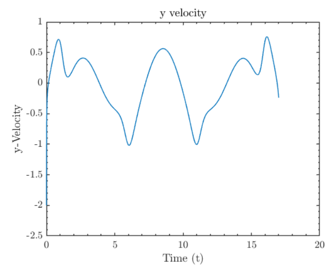
(a) x-Position vs Time



(b) x-Velocity vs Time



(c) y-Position vs Time



(d) y-Velocity vs Time

Figure 2: Temporal evolution of the third body state variables.

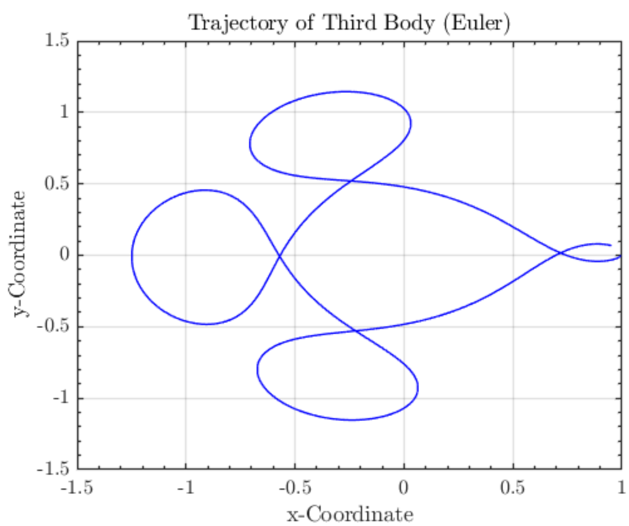


Figure 3: Trajectory of the third body in the xy-plane using explicit Euler method.

With respect to the trajectory plot, it is similar to the one in the wording, confirming a stable solution which resembles the analytical solution. There exist minor errors resulting from the method itself which are small thanks to the time step selection, but yet existent.

RK4 Method and Comparison

The plot shows how both methods are able to compute a trajectory similar to the analytical one. Despite this, RK4 generates a trajectory with tighter loops which effectively reduces the differences with the analytical one. This is especially noticeable near the critical point $(\hat{\mu}, 0)$. Therefore, although both methods are able to reproduce orbital dynamics, RK4 shows better accuracy for the same time step as expected based on the theory classes.

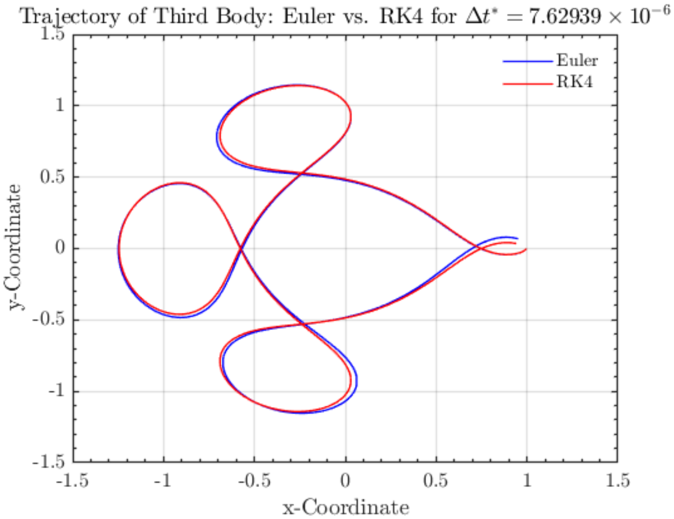


Figure 4: Comparison between Euler and RK4. For $\Delta t^* = 7.629394415521064 \times 10^{-6}$

Optimal Time-Step for RK4

After computing the reduction by a factor of 2 eleven times, we have reached the conclusion that at the eighth iteration we can obtain a very faithful value.

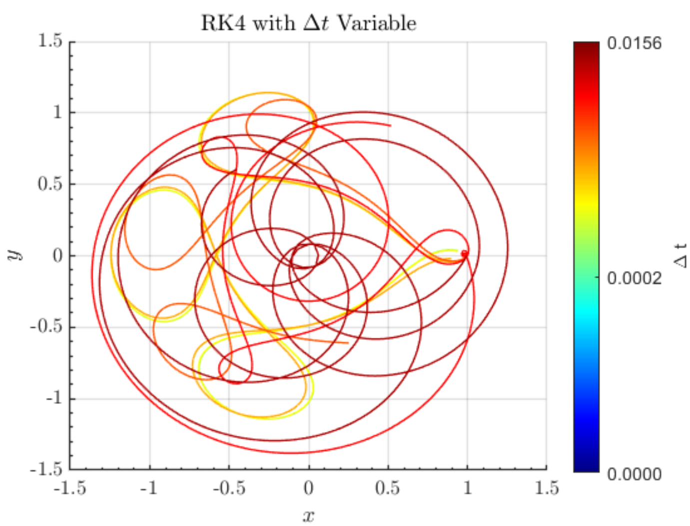


Figure 5: Comparison between Euler and RK4. For Δt^* being multiplied by 2 11 times.

Euler with Large Time-Step

As expected, the plot shows how the Euler method is unstable and grows unbounded for time steps greater than the one used in part 3.

This follows the explanation of part 2 for the Euler method. With purely imaginary eigenvalues, the only way to scale them in such a way that they fall into the stable region is to make them as close to 0 as possible. Therefore, if the timestep is not practically 0, the eigenvalues fall out of the stability region.

Furthermore, it is proven how the stability region of RK4 contains part of the imaginary axis, resulting in higher stability, which summed to its higher accuracy enables the selection of RK4 as the correct method choice out of both for the RTBP.

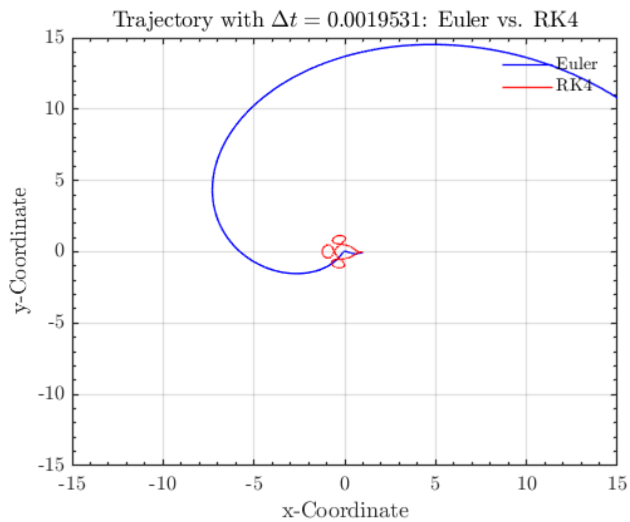


Figure 6: Comparison between Euler and RK4. For $\Delta t = 0.001953$.