Lecture 3

Gaussian Elimination - An Example

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3.1 Introduction

In Gaussian Elimination we try to solve certain equations, say n, by eliminating variables from these equations. The equations are of the form:

$$A\overrightarrow{X} = \overrightarrow{b} \tag{3.1}$$

Where, A is a matrix of size n×n that consist of coefficients of the variables that are contained in $\overrightarrow{X}$ and $\overrightarrow{b}$ is a column vector that consists of equations R.H.S values. In this lecture we will pick up an example to understand the algorithm for same.

3.2 Understanding Through Example

Set of Equations

We will be solving following set of equations.

$$2x_1 - 3x_2 + 2x_3 + 5x_4 = 3 \tag{3.2}$$

$$x_1 - x_2 + x_3 + 2x_4 = 1 \tag{3.3}$$

$$3x_1 + 2x_2 + 2x_3 - x_4 = 0 \tag{3.4}$$

$$x_1 + x_2 - 3x_3 - x_4 = 0 \tag{3.5}$$

We need to solve for:

$$\overrightarrow{X}^T = (x_1, x_2, x_3, x_4)^T \tag{3.6}$$

Elimination Steps

Step 1:

From equation 3.2, we get:

$$x_1 = \frac{3x_2 - 2x_3 - 5x_4 + 3}{2} \tag{3.7}$$

Substituting x_1 from 3.7 in equations 3.2, 3.3, 3.4 and 3.5.New set of equations is:

$$2x_1 - 3x_2 + 2x_3 + 5x_4 = 3 \tag{3.8}$$

$$0x_1 + 0.5x_2 + 0x_3 - 0.5x_4 = -0.5 \tag{3.9}$$

$$0x_1 + 6.5x_2 - 1x_3 - 8.5x_4 = -4.5 \tag{3.10}$$

$$0x_1 + 2.5x_2 - 4x_3 - 3.5x_4 = -1.5 \tag{3.11}$$

Observation : x_1 is eliminated from equations 3.3, 3.4 and 3.5 using equation 3.1 and coefficient of x_1 from equation 3.2, i.e, 2 is pivot.

Step 2:

From equation 3.9 we get:

$$x_2 = -1 + x_4 \tag{3.12}$$

Now, we substitute this in equations 3.10 and 3.11 and our set of equations change to following:

$$2x_1 - 3x_2 + 2x_3 + 5x_4 = 3 \tag{3.13}$$

$$0x_1 + 0.5x_2 + 0x_3 - 0.5x_4 = -0.5 \tag{3.14}$$

$$0x_1 + 0x_2 - 1x_3 - 2x_4 = 2 \tag{3.15}$$

$$0x_1 + 0x_2 - 4x_3 - 1x_4 = 1 \tag{3.16}$$

Observation : x_2 is eliminated from equations 3.10 and 3.11 and pivot is the coefficient of x_2 from equation 3.9, i.e, 0.5.

Step 3:

From equation 3.15 we get:

$$x_3 = -2 - 2x_4 \tag{3.17}$$

Now, substituting this x_3 in equation 3.16, our final set of equation becomes:

$$2x_1 - 3x_2 + 2x_3 + 5x_4 = 3 \tag{3.18}$$

$$0x_1 + 0.5x_2 + 0x_3 - 0.5x_4 = -0.5 \tag{3.19}$$

$$0x_1 + 0x_2 - 1x_3 - 2x_4 = 2 \tag{3.20}$$

$$0x_1 + 0x_2 + 0x_3 + 7x_4 = -7 \tag{3.21}$$

Observation : x_3 is eliminated from equation 3.16 and pivot is coefficient of x_3 from equation 3.20, i.e., -1.

Originally, matrix A was:

$$A = \begin{pmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & -1 \\ 1 & 1 & -3 & -1 \end{pmatrix} \tag{3.22}$$

Final matrix after applying the above four elimination steps is:

$$U = \begin{pmatrix} 2 & -3 & 2 & 5 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 7 \end{pmatrix} \tag{3.23}$$

This final A matrix is U, i.e, upper triangular matrix. Similarly, originally $\overrightarrow{b}$ was:

$$\overrightarrow{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.24}$$

After applying the three steps, new formed vector is:

$$\overrightarrow{c} = \begin{pmatrix} 3 \\ -0.5 \\ 2 \\ 7 \end{pmatrix} \tag{3.25}$$

Final Step:

Now, we need to solve the upper triangular system, U$\overrightarrow{X}$=$\overrightarrow{c}$ by backward substitution. Substituting x_4=-1, from equation 3.21 in equations 3.18, 3.19 and 3.20, we get:

$$x_3 = 0 \tag{3.26}$$

$$x_2 = -2 \tag{3.27}$$

$$x_1 = -4 \tag{3.28}$$

Hence, final solution is:

$$\overrightarrow{X}^T = (x_1, x_2, x_3, x_4)^T = (-4, -2, 0, -1) \tag{3.29}$$

3.3 Idea of Elimination Algorithm

Operations performed during elimination algorithm are multiplication and subtraction. For multiplication, a multiplier is required. Let the multiplier be m_{ij}, where i and j represent row number and column number of matrix A. In $\boldsymbol{step1}$, following elementary row operations are performed to eliminate x_1 from 3.3, 3.4 and 3.5:

$$\begin{aligned}(2) &\longleftarrow (2) - m_{21}\,(1)\,; \\ m_{21} &= \tfrac{a_{21}}{a_{11}} = \tfrac{1}{2} = 0.5\end{aligned} \tag{3.30}$$

$$\begin{aligned}(3) &\longleftarrow (3) - m_{31}\,(1)\,; \\ m_{31} &= \tfrac{a_{31}}{a_{11}} = \tfrac{3}{2} = 1.5\end{aligned} \tag{3.31}$$

$$\begin{aligned}(4) &\longleftarrow (4) - m_{41}\,(1)\,; \\ m_{41} &= \tfrac{a_{41}}{a_{11}} = \tfrac{1}{2} = 0.5\end{aligned} \tag{3.32}$$

In above equations, a_{ij} are values from matrix A from row i and column j respectively and (1), (2), (3), (4) are rows of matrix A. Elementary matrix thus obtained will be:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 \\ -1.5 & 0 & 1 & 0 \\ -0.5 & 0 & 0 & 1 \end{pmatrix} \tag{3.33}$$

Multiplying E_1 and A we get:

$$E_1 A = \begin{pmatrix} 2 & -3 & 2 & 5 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 6.5 & -1 & -8.5 \\ 0 & 2.5 & -4 & -3.5 \end{pmatrix} \tag{3.34}$$

And, on multiplying E_1 and $\overrightarrow{b}$, we get:

$$E_1 \overrightarrow{b} = \begin{pmatrix} 3 \\ -0.5 \\ -4.5 \\ -1.5 \end{pmatrix} \tag{3.35}$$

Elementary row operations performed while eliminating x_2 from last two equations are:

$$(1) \longleftarrow (1) \tag{3.36}$$

$$(2) \longleftarrow (2) \tag{3.37}$$

$$\begin{aligned}(3) &\longleftarrow (3) - m_{32}\,(2)\,; \\ m_{32} &= \tfrac{a_{32}}{a_{22}} = \tfrac{6.5}{0.5} = 13\end{aligned} \tag{3.38}$$

$$\begin{aligned}(4) &\longleftarrow (4) - m_{42}\,(2)\,; \\ m_{42} &= \tfrac{a_{42}}{a_{22}} = \tfrac{2.5}{0.5} = 5\end{aligned} \tag{3.39}$$

Elementary matrix obtained by these elementary row operations is as follows:

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -13 & 1 & 0 \\ 0 & -5 & 0 & 1 \end{pmatrix} \tag{3.40}$$

Now, we multiply this with $E_1 A$ and $E_1 \overrightarrow{b}$.

$$E_2 \left(E_1 A \right) = \begin{pmatrix} 2 & -3 & 2 & 5 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -4 & -1 \end{pmatrix} \tag{3.41}$$

and,

$$E_2 \left(E_1 \overrightarrow{b} \right) = \begin{pmatrix} 3 \\ -0.5 \\ 2 \\ 1 \end{pmatrix} \tag{3.42}$$

Elementary operations required for eliminating x_3 from last equation are:

$$(1) \longleftarrow (1) \tag{3.43}$$

$$(2) \longleftarrow (2) \tag{3.44}$$

$$(3) \longleftarrow (3) \tag{3.45}$$

$$\begin{aligned} (4) \longleftarrow (4) - m_{43} \left(3 \right); \\ m_{43} = \tfrac{a_{43}}{a_{33}} = \tfrac{-4}{-1} = 4 \end{aligned} \tag{3.46}$$

Elementary matrix thus obtained is:

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{pmatrix} \tag{3.47}$$

Multiplying this with $E_2 E_1 A$ and $E_2 E_1 \overrightarrow{b}$, we get following:

$$E_3 \left(E_2 E_1 A \right) = \begin{pmatrix} 2 & -3 & 2 & 5 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 7 \end{pmatrix} = U \tag{3.48}$$

And,

$$E_3 \left(E_2 E_1 \overrightarrow{b} \right) = \begin{pmatrix} 3 \\ -0.5 \\ 2 \\ -7 \end{pmatrix} = \overrightarrow{c} \tag{3.49}$$

Since,

$$
\begin{aligned}
&(E_3E_2E_1)\,A = U \\
&\Rightarrow A = (E_3E_2E_1)^{-1}\,U = LU
\end{aligned}
\tag{3.50}
$$

Observation : 'n-1' steps are required for an $n \times n$ matrix.

3.4 Computational Analysis

Let A$\overrightarrow{X} = \overrightarrow{b}$; A is $n \times n$ matrix. Suppose we are in step number 'k', then following two computations will take place:

1. Multiplier Computation (involving division):

$$
m_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}
\tag{3.51}
$$

2. Sub-matrix Computation for A and $\overrightarrow{b}$:

$$
\begin{aligned}
a_{ij}^{(k+1)} &= a_{ij}^{(k)} - m_{ik}a_{kj}^{(k)} \\
b_i^{(k+1)} &= b_i^{(k)}b_k^{(k)}
\end{aligned}
\tag{3.52}
$$

Multiplications and divisions are significant computations as compared to additions and subtractions.