Assignment 3 SIT718 Real World Analytics

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Question 1: Beverage Optimization using Linear Programming

This section requires optimizing the production cost of a beverage made by mixing two products, A and B, while satisfying the given constraints. A linear programming (LP) model is a suitable approach for solving this problem.

(a) Why Linear Programming is Suitable

Linear programming (LP) is an optimization technique used to maximize or minimize an objective function while satisfying constraints. It is suitable for this problem because:

1. **Objective Function is Linear**: The cost of production depends on a linear combination of A and B
2. **Constraints are Linear**: The constraints on lime, orange, and mango contents, as well as total beverage quantity, are linear.
3. **Decision Variables are Continuous**: The amount of A and B can be any real number within the constraints, making it suitable for LP.
4. **Optimization Goal is Clear**: We aim to **minimize the total cost** while ensuring that all ingredient requirements are met.

(b) LP Model Formulation

Step 1: Define Decision Variables

Let:

- x = Amount (in litres) of Product A used per 100 litres of beverage.
- y = Amount (in litres) of Product B used per 100 litres of beverage.

Step 2: Define the Objective Function

The goal is to minimize the total cost of producing the beverage:
Minimize Cost=4x+12y

where:

- Product A costs **$4 per litre**.
- Product B costs **$12 per litre**.

Step 3: Define the Constraints

(i) Orange Concentrate Requirement

The beverage must contain **at least 5 litres of Orange per 100 litres**:

$6x+4y \geq 5(x+y)$

$x-y \geq 0$

(ii) Mango Concentrate Requirement

The beverage must contain **at least 5 litres of Mango per 100 litres**:

$4x+8y \geq 5(x+y)$

$-x+3y \geq 0$

(iii) Lime Concentrate Upper Limit

The beverage must contain **no more than 6 litres of Lime per 100 litres**:

$2x+7y \leq 6(x+y)$

$-4x+y \leq 0$

(iv) Total Beverage Production Requirement

The total production must be **at least 140 litres per week**:

$x+y \geq 140$

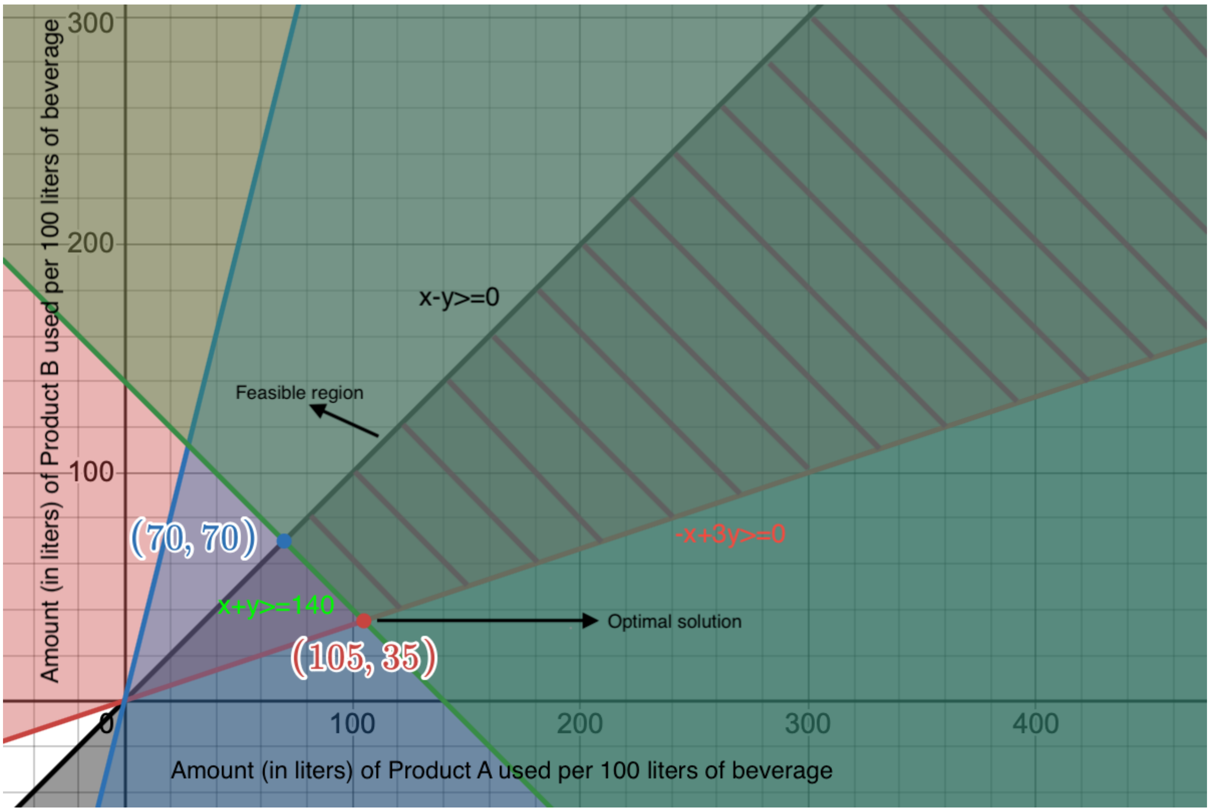
(v) Non-Negativity Constraints

Since the production quantities cannot be negative:

$x \geq 0$, $y \geq 0$

(c) Graphical Solution

To solve this problem graphically:

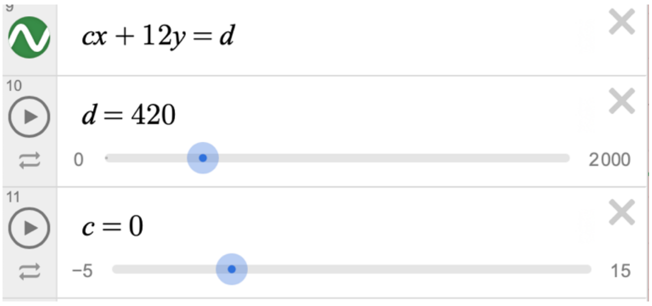


- The shaded region shown above is the feasible region that satisfies all constraints.
- The **optimal solution** is found at the intersection of constraints that minimize cost.
- Substituting the intersection points (70,70) and (105,35) into the cost function, the minimum cost combination is determined which is $840 on intersection point (105,35).

(d) Sensitivity Analysis: Range of Cost for A

We analyse how much the cost of Product A ($4 per litre) can change without affecting the optimal solution.

- **If the cost of A decreases**: The model would favour using more of A.
- **If the cost of A increases**: The model would shift towards using more of B.
- The allowable range for the cost of A is determined by analysing when the optimal point shifts to a different vertex of the feasible region.



z=cx+12y   (c=cost of Product A)

y= -(c/12)x + z

Gradient/slope of objective function should be between gradient of those 2 constraints

$-X+3Y \geq 0$ => c= 1/3
$X+Y \geq 140$ => c= -1

c should lie between -1 and 1/3

$-1 \leq -c/12 \leq 1/3$
$-12 \leq -c \leq 4$
$12 \geq c \geq -4$
$12 \geq c \geq 0$ (As the cost can't be negative)

Question 2: Maximizing Profit for Product Production Using Linear Programming

This section involves formulating and solving a **linear programming (LP) model** to **maximize profit** while ensuring that production constraints related to material proportions and demand are satisfied.

(a) LP Model Formulation

Defining the Decision Variables

Let:

- x_{SC}= Tons of **Spring** product made from **Cotton**
- x_{SW}= Tons of **Spring** product made from **Wool**
- x_{SS}= Tons of **Spring** product made from **Silk**
- x_{AC}= Tons of **Autumn** product made from **Cotton**
- x_{AW}= Tons of **Autumn** product made from **Wool**

- x_{AS} = Tons of **Autumn** product made from **Silk**
- x_{WC} = Tons of **Winter** product made from **Cotton**
- x_{WW}= Tons of **Winter** product made from **Wool**
- x_{WS} = Tons of **Winter** product made from **Silk**

These **9 decision variables** represent how much of each product (Spring, Autumn, Winter) is produced using different materials (Cotton, Wool, Silk).

Defining the Objective Function (Maximize Profit)

The profit for each product is calculated as:

Profit= ∑(Sales Price−Production Cost−Purchase Cost)×Product Quantity

Using given data:

- **Spring:** Sales Price = \$60, Production Cost = \$5
- **Autumn:** Sales Price = \$55, Production Cost = \$3
- **Winter:** Sales Price = \$65, Production Cost = \$8
- **Material Costs:** Cotton = \$30, Wool = \$45, Silk = \$50

Thus, profit per ton for each combination:

$$\begin{aligned}\text{Profit} = {} & (60-5-30)x_{SC} + (60-5-45)x_{SW} + (60-5-50)x_{SS} \\ & + (55-3-30)x_{AC} + (55-3-45)x_{AW} + (55-3-50)x_{AS} \\ & + (65-8-30)x_{WC} + (65-8-45)x_{WW} + (65-8-50)x_{WS}\end{aligned}$$

Simplifying:

$$\textbf{Maximize } Z = 25x_{SC} + 10x_{SW} + 5x_{SS} + 22x_{AC} + 7x_{AW} + 2x_{AS} + 27x_{WC} + 12x_{WW} + 7x_{WS}$$

Defining the Constraints

(i) Demand Constraints

Each product has a **maximum demand** that cannot be exceeded:

$$x_{SC} + x_{SW} + x_{SS} \leq 3200$$
$$x_{AC} + x_{AW} + x_{AS} \leq 3800$$
$$x_{WC} + x_{WW} + x_{WS} \leq 4200$$

(ii) Cotton Proportion Constraints

- **Spring**: At least **55%** Cotton : $x_{SC} \geq 0.55(x_{SC} + x_{SW} + x_{SS})$
- **Autumn**: At least **45%** Cotton : $x_{AC} \geq 0.45(x_{AC} + x_{AW} + x_{AS})$

- **Winter**: At least **30%** Cotton : $x_{WC} \geq 0.30(x_{WC} + x_{WW} + x_{WS})$

Rearranging:

$0.45x_{SC} - 0.55x_{SW} - 0.55x_{SS} \geq 0$
$0.55x_{AC} - 0.45x_{AW} - 0.45x_{AS} \geq 0$
$0.70x_{WC} - 0.30x_{WW} - 0.30x_{WS} \geq 0$

(iii) Wool Proportion Constraints

- **Spring**: At least **30%** Wool : $x_{SW} \geq 0.30(x_{SC} + x_{SW} + x_{SS})$
- **Autumn**: At least **40%** Wool : $x_{AW} \geq 0.40(x_{AC} + x_{AW} + x_{AS})$
- **Winter**: At least **50%** Wool : $x_{WW} \geq 0.50(x_{WC} + x_{WW} + x_{WS})$

Rearranging:

$-0.30x_{SC} + 0.70x_{SW} - 0.30x_{SS} \geq 0$
$-0.40x_{AC} + 0.60x_{AW} - 0.40x_{AS} \geq 0$
$-0.50x_{WC} + 0.50x_{WW} - 0.50x_{WS} \geq 0$

(iv) Silk Proportion Constraints

- **Spring**: At least **1%** Silk : $x_{SS} \geq 0.01(x_{SC} + x_{SW} + x_{SS})$
- **Autumn**: At least **2%** Silk : $x_{AS} \geq 0.02(x_{AC} + x_{AW} + x_{AS})$
- **Winter**: At least **3%** Silk : $x_{WS} \geq 0.03(x_{WC} + x_{WW} + x_{WS})$

Rearranging:

$-0.01x_{SC} - 0.01x_{SW} + 0.99x_{SS} \geq 0$
$-0.02x_{AC} - 0.02x_{AW} + 0.98x_{AS} \geq 0$
$-0.03x_{WC} - 0.03x_{WW} + 0.97x_{WS} \geq 0$

(v) Non-Negativity Constraints

$x_{SC}, x_{SW}, x_{SS}, x_{AC}, x_{AW}, x_{AS}, x_{WC}, x_{WW}, x_{WS} \geq 0$

(b) Optimal Solution for the LP Model

1. Optimal Profit

The **optimal profit** is:

$$\text{Profit} = \$203,620$$

This is the maximum profit achievable given the constraints.

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2. Optimal Production Quantities

Spring Product

- **Cotton** (x_{SC}): 2208 tons
- **Wool** (x_{SW}): 960 tons
- **Silk** (x_{SS}): 32 tons
- **Total**: 2208+960+32=3200 tons (matches demand constraint).

Autumn Product

- **Cotton** (x_{AC}): 2204 tons
- **Wool** (x_{AW}): 1520 tons
- **Silk** (x_{AS}): 76 tons
- **Total**: 2204+1520+76=3800 tons (matches demand constraint).

Winter Product

- **Cotton** (x_{WC}): 1974 tons
- **Wool** (x_{WW}): 2100 tons
- **Silk** (x_{WS}): 126 tons
- **Total**: 1974+2100+126=4200 tons (matches demand constraint).

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3. Interpretation of Results

- **Spring Production**:
  - The majority of Spring product is made from **Cotton (2208 tons)** and **Wool (960 tons)**, with minimal use of **Silk (32 tons)**.
- **Autumn Production**:
  - A balanced use of **Cotton (2204 tons)** and **Wool (1520 tons)**, with some **Silk (76 tons)**.
- **Winter Production**:
  - Relatively high proportions of **Wool (2100 tons)** and **Cotton (1974 tons)**, with slightly more **Silk (126 tons)** compared to Spring and Autumn.

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5. Summary

- **Optimal Profit**: $203,620
- **Optimal Production Quantities**:
  - Spring: 3200 tons
  - Autumn: 3800 tons
  - Winter: 4200 tons
- The solution satisfies all demand and material proportion constraints, ensuring maximum profit.

Question 3: Game Theory Solution

(a) Payoff Matrix

The given payoff matrix is symmetric and zero-sum, meaning the payoffs for Player 1 are the negatives of the payoffs for Player 2. The rows represent Player 1's strategies (x1 to x6), and the columns represent Player 2's strategies (y1 to y6).
The table specifies the payoffs Player 1 gains when specific strategies are played by both players.

Player1/Player2		y1	y2	y3	y4	y5	y6
		WRB	WBR	RWB	RBW	BRW	BWR
x1	WRB	0	0	0	-75	0	75
x2	WRB	0	0	-75	0	75	0
x3	WRB	0	75	0	0	-75	0
x4	WRB	75	0	0	0	0	-75
x5	WRB	0	-75	75	0	0	0
x6	WRB	-75	0	0	75	0	0

There are no saddle points in the payoff matrix since the lower value of the game is -75 and the upper value is 75 which is not equal.

(b) Linear Programming Models

Player 1's Linear Programming Model
Player 1's goal is to **maximize their minimum guaranteed payoff** (v).

Objective Function:
Maximize Z=v

Constraints: From the payoff matrix, the constraints are derived to ensure vvv is less than or equal to the expected payoff for any strategy combination:

$v \le 75x_4 - 75x_6$ or equivalently $v - 75x_4 + 75x_6 \le 0$
$v \le 75x_3 - 75x_5$ or equivalently $v - 75x_3 + 75x_5 \le 0$
$v \le -75x_2 + 75x_5$ or equivalently $v + 75x_2 - 75x_5 \le 0$
$v \le -75x_1 + 75x_6$ or equivalently $v + 75x_1 - 75x_6 \le 0$
$v \le 75x_2 - 75x_3$ or equivalently $v - 75x_2 + 75x_3 \le 0$
$v \le 75x_1 - 75x_4$ or equivalently $v - 75x_1 + 75x_4 \le 0$

Probability Constraint: The probabilities must sum to 1:
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$
And all probabilities must be non-negative:
$x_i \ge 0$ for all i

Player 2's Linear Programming Model
Player 2's goal is to **minimize Player 1's maximum guaranteed payoff** (v).

Objective Function:

Minimize Z=v

Constraints: From the payoff matrix, the constraints are derived to ensure vvv is greater than or equal to the expected payoff for any strategy combination:

$v \geq -75y_4 + 75y_6$ or equivalently $v + 75y_4 - 75y_6 \geq 0$
$v \geq -75y_3 + 75y_5$ or equivalently $v + 75y_3 - 75y_5 \geq 0$
$v \geq 75y_2 - 75y_5$ or equivalently $v - 75y_2 + 75y_5 \geq 0$
$v \geq 75y_1 - 75y_6$ or equivalently $v - 75y_1 + 75y_6 \geq 0$
$v \geq -75y_2 + 75y_3$ or equivalently $v + 75y_2 - 75y_3 \geq 0$
$v \geq -75y_1 + 75y_4$ or equivalently $v + 75y_1 - 75y_4 \geq 0$

Probability Constraint: The probabilities must sum to 1:

$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 1$

And all probabilities must be non-negative:

$y_i \geq 0$ for all i

(c) Done in R

(d) Solving for Optimal Strategies

Using the linear programming models for Player 1 and Player 2, the following probabilities were computed:

Player 1's Optimal Mixed Strategy:

- WRB (x_1) = **0.33**
- WBR (x_2) = **0**
- RWB (x_3) = **0**
- RBW (x_4) = **0.33**
- BWR (x_5) = **0**
- BRW (x_6) = **0.33**

Player 2's Optimal Mixed Strategy:

- WRB (y_1) = **0.33**
- WBR (y_2) = **0**
- RWB (y_3) = **0**
- RBW (y_4) = **0.33**
- BWR (y_5) = **0**
- BRW (y_6) = **0.33**

Interpretation of Results

1. **Both players adopt the same mixed strategy**:
   - They randomly play each of the strategies WRB, RBW, BRW with equal probability (33% each).
   - They avoid playing WBR, RWB, BWR, as those strategies provide less advantage.
2. **The value of the game is 0**:
   - The game is fair because both players employ optimal strategies that neutralize each other's advantage.
   - Neither player can guarantee a win if the other plays optimally.

The optimal solution is **not unique** because:

- Multiple combinations of probabilities for WRB, RBW, and BRW yield the same expected payoff.
- The linear programming model allows for multiple valid solutions that satisfy the constraints.