Dynamical Systems in Lean4

Formalizing the Equivalence of Lagrangian and Hamiltonian Mechanics of Harmonic Oscillator

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MA208 Proofs and Programs

April 20, 2025

Setting Up the Background

- The harmonic oscillator is a fundamental physical system.
- Its equation of motion has been derived using Lagrangian and Hamiltonian mechanics, assuming the Newton's laws to hold.
- Proving the relation between Lagrangian and Hamiltonian using Legendre Transformation
- The whole approach is formalized using the Lean4 (with Mathlib4).

System Variables

- $m > 0$: mass of the oscillator.
- $k > 0$: spring constant.
- $x(t)$: position as a function of time.
- $v(t) = \frac{dx}{dt}$: velocity as a function of time.
- $p(t)$: momentum as a function of time.
- p and v are related to each other

Lagrangian Mechanics

The Lagrangian of the harmonic oscillator:

$$L(x, v) = T - V \tag{1}$$
$$= \frac{1}{2}mv^2 - \frac{1}{2}kx^2 \tag{2}$$

where:

- $T = \frac{1}{2}mv^2$ is the kinetic energy.
- $V = \frac{1}{2}kx^2$ is the potential energy.

Lagrange's equation is given as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0$$
$$\implies m\frac{dv}{dt} + kx = 0 \tag{3}$$
$$\implies \boxed{m\frac{d^2x}{dt^2} = -kx = F}$$

Hamitonian Mechanics

The Hamiltonian of the harmonic oscillator:

$$\begin{aligned} H(x,p) &= T + V && (4) \\ &= \frac{p^2}{2m} + \frac{1}{2}kx^2 && (5) \end{aligned}$$

Hamilton's equation is given as

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p} \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial x} \end{aligned} \tag{6}$$

For Harmonic oscillator, it becomes

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \frac{p}{m} \\ \frac{dp}{dt} &= -kx = F \end{aligned}} \tag{7}$$

Legendre Transformation

The **Legendre transformation** is an operator that maps a function defined on a convex domain to another function on the dual domain.

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable convex function. The Legendre transform f^* is a new function defined as:

$$f^* : \mathbb{R} \to \mathbb{R}, \quad f^*(p) = \sup_{x \in \mathbb{R}} (px - f(x))$$

or equivalently (when f is strictly convex and differentiable):

$$f^*(p) = px(p) - f(x(p)), \quad \text{where } p = \frac{df}{dx}$$

Context of Mechanics

- Let $\mathcal{L} : Q \to \mathbb{R}$ be the Lagrangian.
- Then the Legendre transform maps $\mathcal{L}(x, v) \mapsto \mathcal{H}(x, p)$, where $p = \frac{\partial \mathcal{L}}{\partial v}$.
- Formally: $\mathcal{F}(\mathcal{L}) = \mathcal{H}$, the Hamiltonian.
- Legendre transform is the inverse of itself i.e.,

$$\begin{aligned} \mathcal{F}^2(\mathcal{L}) &= \mathcal{L} \\ \implies \mathcal{F}(\mathcal{H}) &= \mathcal{L} \end{aligned}$$

Hamiltonian from Lagrangian

For the harmonic oscillator, recall:

$$L(x, v) = \frac{1}{2}mv^2 - \frac{1}{2}kx^2$$

- Compute momentum: $p = \frac{\partial L}{\partial v} = mv \Rightarrow v = \frac{p}{m}$
- Legendre transform:

$$\begin{aligned} H(x, p) &= pv - L(x, v) \\ &= p\left(\frac{p}{m}\right) - \left(\frac{1}{2}m\left(\frac{p}{m}\right)^2 - \frac{1}{2}kx^2\right) \\ &= \frac{p^2}{m} - \left(\frac{p^2}{2m} - \frac{1}{2}kx^2\right) \\ &= \boxed{\frac{p^2}{2m} + \frac{1}{2}kx^2} \end{aligned} \tag{8}$$

- This is the Hamiltonian!

Lagrangian from Hamiltonian

Given the Hamiltonian of the harmonic oscillator:

$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

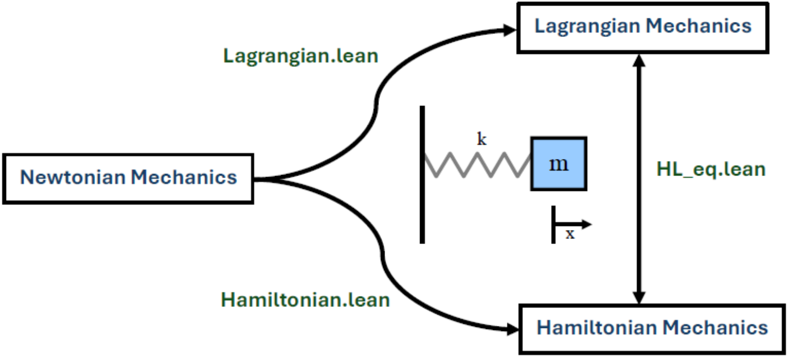
We can recover the Lagrangian using the inverse Legendre transform:

- Compute velocity: $v = \frac{\partial H}{\partial p} = \frac{p}{m} \Rightarrow p = mv$
- Apply inverse Legendre transform:

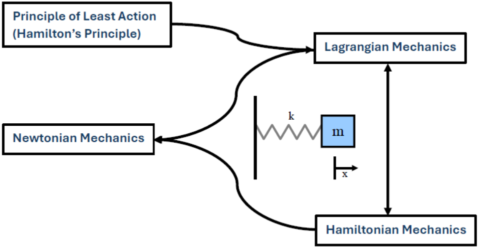
$$\begin{aligned} L(x, v) &= pv - H(x, p) \\ &= mv \cdot v - \left(\frac{(mv)^2}{2m} + \frac{1}{2}kx^2 \right) \\ &= mv^2 - \left(\frac{mv^2}{2} + \frac{1}{2}kx^2 \right) \\ &= \boxed{\frac{1}{2}mv^2 - \frac{1}{2}kx^2} \end{aligned} \tag{9}$$

- This is the original Lagrangian!

Project in Nutshell



Discussions

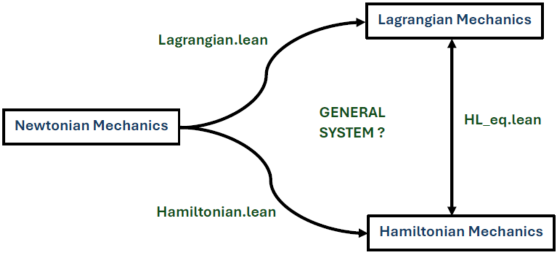


Hamilton's Principle : The motion of system from time t_1 to t_2 is such that the line integral (action)

$$I = \int_{t_1}^{t_2} L dt$$

where $L = T - V$, has a stationary value for the actual path of the motion.

Discussions



Problem :

- Domain has to be convex for applying Legendre Transform (this is not a problem atleast for 1-D domain)
- $p = \frac{\partial L}{\partial v}(v)$ has to be invertible transformation so that we can write $v = v(p)$

Thank You

Suggestions and Questions?