Problem Sheet 1: Elasticity Recap

1. The constitutive law for a linear, elastic, isotropic solid is given by $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}$. From this relation find the inverse relation, i.e. ε_{ij} in terms of σ_{ij}. To do this, first set $i = j = p$, utilize the fact that $\varepsilon_{pp} \equiv \varepsilon_{kk}$, and then obtain the expression for ε_{kk}; next substitute this expression back in the above equation to find ε_{ij}.

2. By choosing specific values of the indices in the expression of C_{ijkl} for a linear, elastic, isotropic material, express λ and G in terms of the various components of C_{ijkl}. Are these expressions unique?
4444/5555/6666 = G, 1122/1133 = lambda

3. Compare the relation found in Q1 with the following:

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

to show that $E = \dfrac{G\left(3\lambda + 2G\right)}{\lambda + G}$ and $\nu = \dfrac{\lambda}{2\left(\lambda + G\right)}$. Subsequently show that $\lambda = \dfrac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)}$ and $G = \dfrac{E}{2\left(1+\nu\right)}$.

4. For a hydrostatic state of compression $\sigma_{ij} = -p\delta_{ij}$ (p being a positive constant), show that the dilatation is given by $-p/K$, where K is referred to as the bulk modulus. Also show

$$K = \frac{E}{3\left(1-2\nu\right)} \equiv \frac{3\lambda + 2G}{3}.$$

If $\nu \to 1/2$, what happens to K? Also, what happens to the volumetric deformation? What can you say about the compressibility of the material? How does the first invariant of strain feature in all this?

5. Consider an incompressible elastic material (refer to the previous question).

   (a) Show that the constraint of $\varepsilon_{kk} = 0$ implies that the Poisson's ratio, $\nu = \dfrac{1}{2}$.

   (b) Referring to the expressions of the Lamé parameter, λ and the bulk modulus, K found earlier, show that both λ and K tend to infinity when $\nu \to \dfrac{1}{2}$.

   (c) Show that the constitutive relation, $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}$ for such an incompressible material should contain an indeterminate term.

   (d) The constitutive relation with the indeterminate term is usually written as $\sigma_{ij} = -p\delta_{ij} + 2G\varepsilon_{ij}$; show that $p = -\dfrac{1}{3}\sigma_{kk}$.

6. If the Young's modulus E, the bulk modulus K, and the shear modulus μ are required to be positive, show that the Poisson's ratio ν must satisfy the inequality

$$-1 < \nu < \frac{1}{2}.$$

For most real materials, however, $0 < \nu < 1/2$. Show that this more restrictive inequality implies $\lambda > 0$. Materials that have negative Poisson's ratio are referred to as *auxetic* materials.

7. Using the constitutive relation for a linear, elastic, isotropic solid in the mechanical equilibrium equations ($\nabla \cdot \boldsymbol{\sigma} = 0$ OR $\dfrac{\partial \sigma_{ij}}{\partial X_j} = 0$) where body forces are absent, show that the following equation is obtained:

$$(\lambda + G)\nabla(\nabla \cdot \boldsymbol{u}) + G\nabla^2 \boldsymbol{u} = 0 \quad \text{OR} \quad (\lambda + G)\frac{\partial}{\partial X_i}\left(\frac{\partial u_k}{\partial X_k}\right) + G\frac{\partial^2 u_i}{\partial X_j^2} = 0.$$

8. Consider a strain field such that

$$\varepsilon_{11} = Ax_2^2, \quad \varepsilon_{22} = Ax_1^2, \quad \varepsilon_{12} = Bx_1x_2, \quad \varepsilon_{33} = \varepsilon_{32} = \varepsilon_{31} = 0.$$

Find the relationship between A and B such that it is possible to obtain a single-valued continuous displacement field which corresponds to the given strain field. [$B = 2A$]

9. Consider the strain-displacement relations in a rectangular Cartesian coordinate system and verify that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{1}$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \tag{2}$$

Extend the ideas of these two equations to obtain

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \tag{3}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \tag{4}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right) \tag{5}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right) \tag{6}$$

These six equations are referred to as the compatibility equations.

10. The six compatibility equations in the previous question are not actually independent. To see this, first obtain from Eqs. (2), (5), and (6) the following:

$$\frac{\partial^4 \varepsilon_{xx}}{\partial y^2 \partial z^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \tag{7}$$

$$\frac{\partial^4 \varepsilon_{yy}}{\partial z^2 \partial x^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right) \tag{8}$$

$$\frac{\partial^4 \varepsilon_{zz}}{\partial x^2 \partial y^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right). \tag{9}$$

Next, add Eqs. (7) and (8) and compare with what you obtain after differentiating Eq. (1) w.r.t z twice. *This comparison shows that Eqs. (7), (8), and (9) are really the three independent equations.*