Acoustical Cloaking using Metamaterials

Underwater Acoustics

Term Project

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Abstract

Over the past few decades, a lot of research has been done on acoustic metamaterials, which have achieved sound properties that aren't possible with regular materials. After showing that locally resonant acoustic metamaterials can work as subwavelength unit cells, researchers have looked into whether it is possible to break the usual limits of the material's bulk stiffness and mass density. When acoustic metamaterials are used in engineering, additive manufacturing, and theoretical research, they show amazing abilities like negative reflection, cloaking, beam formation, and super-resolution imaging. To freely change acoustic propagation in an underwater setting is still hard because of how complicated impedance boundaries and mode transitions are. In this report, we focus on the cloaking characteristics of metamaterials. Particalarly, we reproduce the work of Torrent & Sánchez-Dehesa (2008) and Cummer & Schurig (2007) to achieve acoustical cloaking using COMSOL Multiphysics. Furthermore, we investigate the effectivess of the the acoustical cloaking coating for different frequencies of the background pressure wave.

1 Literature Review

Acoustics is a branch of physics that studies mechanical waves using a variety of evaluation techniques. Acoustics is the study of sound waves' production, propagation, perception, and transformation. Mechanical waves require particles in the medium to transfer their energies, hence they cannot travel into space. It is clear that acoustic technology has many modern-day uses, and the breadth of these uses has grown in tandem with technological development. The fields of music, communications, architecture, ultrasonic imaging, and contemporary engineering are only a few examples of potential application areas (Failla et al. 2024).

Because of their longer wavelength and lack of dampening, acoustic waves are able to penetrate deeper and cause stronger diffraction than other forms of waves (Aydın & San 2024). These characteristics make acoustic wave manipulation more challenging than that of other wave forms. Traditional acoustic materials have shown to have numerous issues with studies examining the transmission, absorption, and reflection of sound waves. One issue is that it doesn't have adequate bandwidth into low frequency for noise reduction because it can't fully absorb and reflect low-frequency sound. Traditional acoustic materials also have an issue with positive sound velocity. Some of the reflected, refracted, and transmitted acoustic waves may even end up in the target acoustic medium. When two acoustic mediums interact, Snell's Law holds true; in this case, the acoustic wave is refracted as it travels down the normal's positive plane at the point of intersection. This explains why both naturally occurring and man-made materials are unable to effectively bend acoustic waves, as they have a positive sound velocity.

A material that is synthetic has been designed to answer issues that do not have a natural counterpart. Metamaterial describes these types of materials. The unique engineering characteristics of these materials are dictated not by their chemical make-up or physical characteristics, but by the micro-scale structures that make them up (Arjunan et al. 2024). Their distinctive and useful technical properties are a result of the careful arrangement of geometry or shapes, which is also called macro-architecture. Their massive synthetic 3D cellular structure enables the production of optimal features that do not present in nature. This novel method of material design paves the way for previously unimaginable levels of customization and the realization of so far unattainable capabilities in materials.

Acoustic metamaterials, a subset of metamaterials created to address issues with traditional acoustic materials, are the subject of this report. Concerning the mechanical substitute In recent years, mechanical metamaterials have gained attention as a promising new way to achieve extraordinary mechanical properties by means of clever structure design. These materials provide specialized mechanical reactions that are out of the reach of homogeneous materials and frequently defy traditional material behavior. Mechanical metamaterials exhibit remarkable qualities including negative stiffness, extreme stiffness amplification, and topological mechanical properties by the careful arrangement of substructures. Their one-of-a-kind characteristics have ignited a surge of research and development in various branches of engineering, with the potential for revolutionary uses in sectors as diverse as aerospace, robotics, civil engineering, and healthcare.

Acoustic metamaterials (AMMS) are artificial structures with periodic topologies that possess exotic features not seen in natural materials and contain extremely small units in relation to the wavelength. So, in comparison to regular acoustic materials, acoustic metamaterials have an extraordinary ability to control sound waves. The acoustic qualities that are not possible with regular materials are created by the periodic arrangement of subcomponents, which is frequently influenced by cellular structures or periodic lattices [14]. Using mass density, stiffness, and geometric configuration as factors, researchers have created acoustic metamaterials with novel methods of sound wave bending, focusing, and directing [15].

The design and production of the metamaterials can be tailored to specific areas of intended use. By doing so, we can tailor the metamaterials to be highly effective at either absorbing or blocking sound (Zhang et al. 2023). If an acoustic metamaterial becomes antiresonant in the low-frequency ranges of the desired wideband, it can block out noise very well; if it becomes resonance in the same regions of the desired broadband, it can absorb sound very well. Negative bulk density (ρ), effective bulk modulus (B), and negative refraction (n) are material qualities that give out-of-the-ordinary effects that are absent in natural materials. It helps advance contemporary acoustics and engineering applications with its remarkable qualities. The Doppler effect in reverse, super lenses, and object cloaking are just a few of the unique topics that have benefited from the research made possible by these negative parameters.

2 Physics of acoustic metamaterials

The energy can be transported by the wave, which is known as vibration, as it travels through space and matter. Mechanical waves, which include acoustic waves like sound waves, and electromagnetic waves are the two primary types of waves. As an example of a medium, the collision of air molecules allows the acoustic wave to travel through it. One mathematical formulation that describes the properties and propagation of sound waves in physics is the acoustic wave equation. A second-order partial differential wave equation represents the equation. The second law of Newton, the law of thermodynamics, and the conservation of mass must all be satisfied. Variations in the medium's pressure cause sound waves to be generated. Thus, the acoustic pressure P, particle velocity u, and density variation ρ are the defining characteristics of these entities, and they are interrelated. Acoustic wave equation is given below.

$$\nabla^2 P = \frac{1}{c_0^2}\frac{\partial^2 P}{\partial t^2}$$

where c_0 represents the sound velocity in the free medium.

The electrical permittivity and magnetic permeability of a substance are the usual explanations for its electromagnetic properties. The acoustics community uses density and mass modulus to describe how different substances react to sound waves. If a boundary of the second medium is encountered by the acoustic wave as it travels from the one medium to another, some of its energy is reflected back out into space, while some is refracted as it travels through the second. The explanation for this is that the wave speeds vary according on the media. A high refractive index causes the wave to travel at a slower speed than a low one. A wave's frequency remains constant as it travels across various media. There is a common plane for incident, reflected, and refracted waves. In accordance with the sine (sin) relationship, the wave's refraction angle is defined by the properties of the two media and the angle of incidence. In the optical field of physics, Snell's law states how the speed and direction of a wave alter upon meeting the boundary between two media with differing densities.

The parameters of the bulk modulus B and the bulk density ρ of the medium define the velocity. The bulk elastic modulus B and bulk density ρ are both positive for materials that can be found in nature. Mass per unit volume is the definition of bulk density. Volume modulus and bulk modulus are interchangeable terms. When a material of a given mass is squeezed under the isopressure that surrounds it, its bulk elastic modulus is the amount of resistance it shows to this pressure.

$$c = \sqrt{\frac{B}{\rho}}$$

where the bulk density ρ, bulk elastic modulus B and phase velocity c are given.

The right-hand propagation, also called forward wave propagation, is seen when the bulk density and bulk modulus of the material are positive. When the material's bulk density is negative and its bulk modulus is positive, or when the material's bulk density is positive and its bulk modulus is negative, damped wave propagation is observed. Back wave propagation, often known as the left-hand rule, occurs when the mass density and bulk modulus are negative in conjunction. That is to say, negative refraction is a property of sound waves that allows them to propagate. As the material's bulk density gets closer to zero, its wave group velocity approaches infinity, and it shows great permeability and remarkable acoustic permeability. By doing so, the other side of the metamaterial can be focused more intensely.

Analytical solutions can be found using mathematical formulae for the interplay of transmitted and reflected acoustic waves in a uniform ambient. The examination of acoustic properties, however, becomes quite difficult in non-homogeneous situations. An effective media theory can be developed by using approximations that, under certain conditions, treat the medium as homogenous, thus resolving this issue. By using approach approaches, we can approximate the attributes of a composite material, including its effective sound velocity and mass density, as if it were a homogenized entity, even if it is formed of diverse structures and materials. Assuming this approximation stays the same, it indicates that the material shows very little dispersion in the frequency spectrum. The presence of dispersion characteristics in a material is indicated when, on the other hand, the approximation changes with frequency.

3 Passive acoustic metamaterials

3.1 Locally resonant structure

One typical structural component of locally resonant metamaterials that can achieve band gaps is the use of dipole resonant elements and single-pole resonant elements. A dual-degree-offreedom mass-in-mass arrangement is the standard mathematical depiction of a dipole resonant unit. Band gaps domains of specified frequencies are efficiently blocked by these dipole resonant parts, and they play a crucial role in this process. In most cases, a dipole resonant unit will have two masses that are supported by a framework and can be moved relative to one another. The masses resonate at certain frequencies as a result of their interaction with springs or other flexible components. Building acoustic metamaterials with controlled band gaps relies on the idea of using these dipole resonant elements. Engineered band gap qualities can be achieved by manipulating geometries, spring constants, and masses. Novel uses in waveguiding, vibration isolation, and noise reduction are possible thanks to this method's ability to precisely regulate the propagation of acoustic waves.

Liu et al. (2000) were able to reduce noise by more than 60% at 400 Hz using a 2 cm thickness. Achieving negative mass density and establishing an environment with locally resonant structural elements were necessary for this success. An 8x8x8 meta-atom cube, containing tiny lead balls coated with silicone, meant to mimic mass-spring resonators,

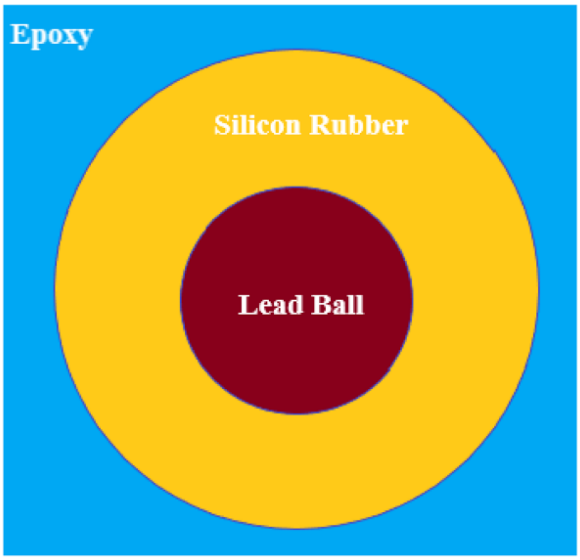

Figure 1: Schematic of the meta-atom structure utilized by Liu et al. (2000).

was measured as part of the experiment. Subwavelength scattering were provided by these microscopic meta-atoms, shown in fig 1. In order to generate destructive interference, oscillations with a phase difference that were outside of the resonance frequency were also employed. An important step towards understanding and making use of acoustic metamaterials' characteristics has been taken here in this study (Aydın & San 2024).

3.2 Helmholtz resonator

Applications pertaining to acoustic absorption and noise reduction rely heavily on the Helmholtz resonator, an essential notion in acoustics. Its usual configuration consists of a spherical, volume-controlled housing for an air chamber and a neck. There is a smaller opening on one end and a narrower one on the other. When it comes to dampening certain frequencies of sound, the Helmholtz resonator shines (see fig. 2). As a mass-spring system, the Helmholtz resonator absorbs resonances. Mass elements absorb and dampen sound by oscillating in response to applied pressure, which in turn dissipates energy via the spring.

A negative effective mass phenomena was demonstrated by Fang et al. (2006) in a metamaterial with a waveguide-attached array of Helmholtz resonators. The bandgap and transmission coefficient were subsequently determined using equivalent media theories. Although this research is limited to a one-dimensional model of wave propagation, it

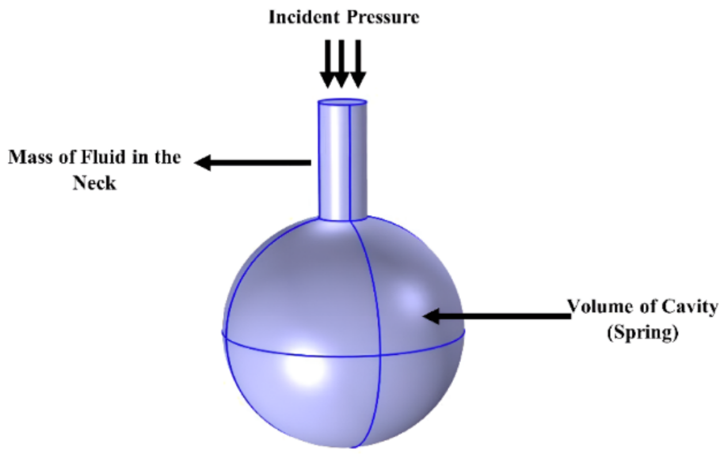

Figure 2: Schematic of a Helmholtz resonator (Zhang et al. 2023).

opened the door to investigations into multi-dimensional Helmholtz resonator arrays. A particular sort of anisotropic metamaterial (AM) with a 1D Helmholtz resonance cavity was suggested for use in the subwavelength regime. At the resonant frequency, the equivalent modulus turns negative, which they also studied. The fascinating characteristics of Helmholtz resonators were illuminated and new ways to understand their complex arrayed use were made possible by this research. Potentially useful in acoustic and material design contexts are the ideas of negative effective mass and other unusual wave propagation scenarios.

3.3 Membrane type

A membrane structure permits the insertion of a center mass block by means of a fixed membrane attached to a hard boundary. A flexible membrane is symbolized by the vibrating mass, and the enclosed air volume sustaining the membrane is represented by the spring. Because the air spring within the enclosed volume is both stiff and massive, sound waves contact the membrane's surface, causing it to vibrate. A substantial amount of absorption is caused by the superior corpuscle velocities that are generated by the membrane vibration. Having said that, the frequency bandwidth that membrane absorbers can use to block out sound is somewhat narrow. To increase the bandwidth, damping is used.

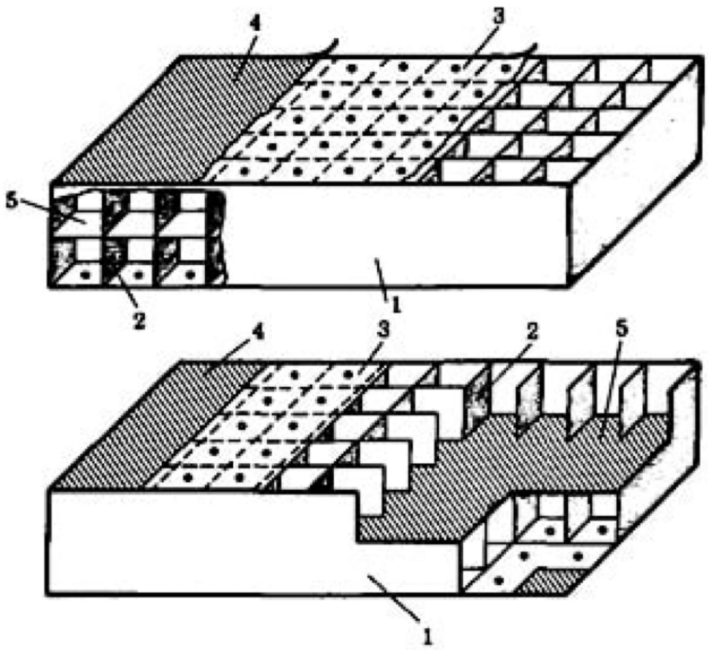

Figure 3: Example of a membrane type metamaterial (Frommhold et al. 1994).

Use of membrane absorbers allows for the absorption of low-frequency sound energy. Due of the confined air's resistance to compression and the membrane's resistance to fast flexure, they oscillate at low frequencies. By transforming the energy of sound into heat, it is able to absorb it. The usage of these for low-frequency absorption is described in this brief description by Frommhold et al. (1994). It has been observed that the resistance of the membrane to quick flexure and the contained air to compression are the sources of low-frequency vibration. Because of this technique, the energy of sound can be more easily transformed into heat. When it comes to controlling and absorbing low-frequency sound energy, such absorbers are crucial.

3.4 Space-coiled structures

Metamaterials constructed into a spiral or coiled shape are known as space-coiled structures. These structures are designed to reduce the equivalent sound velocities of acoustic waves while simultaneously increasing their distance covered. Because of this, larger refractive indices can be achieved, and the limitations of acoustic materials with membrane-type or local resonant properties, such as short lifetimes and narrow frequency ranges, can be

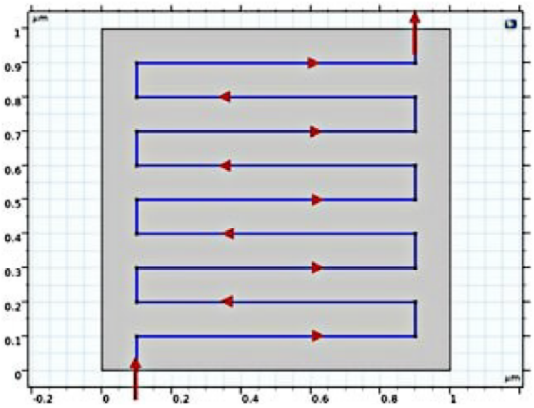

Figure 4: Example of a space-coiled resonator (Long et al. 2020).

overcome (refer fig. 4).

4 Active acoustic metamaterials

Noisy places in real life are often complicated and change over time. It is hard to change the features of a passive acoustic metamaterial after it has been made. When working with complex, changing acoustic environments, this is a big problem. When you use active acoustic metamaterials, you can change the acoustic properties by applying outside forces, like piezoelectricity, mechanical pressure, fluid filling, and temperature changes. Active acoustic metamaterials have a lot of room for growth when it comes to controlling noise.

The concept of active acoustic metamaterials was presented for the first time by Akl & Baz (2011) and active control methods for acoustic metamaterials were presented to control sound waves. They designed a group of structures that consisted of liquid cavities separated by multilayer piezoelectric membranes (fig. 5(a)). The vibration states of the piezoelectric membranes were controlled using an applied voltage. An equivalent mass density was produced without changing its structure. In addition, the transmission of the sound wave was controlled, and its operating frequency band was expanded. Subsequently, piezoelectric sensors were combined with Helmholtz resonators to produce an adjustable acoustic-stealth metamaterial (fig 5(b))(Popa et al. 2013, Popa & Cummer 2014). A piezoelectric sensor was used to measure the pressure of the incident wave, and the electrical signal generated by an internal circuit produced membrane vibrations. Thus, it is possible to realize active control of the acoustic properties of the material.

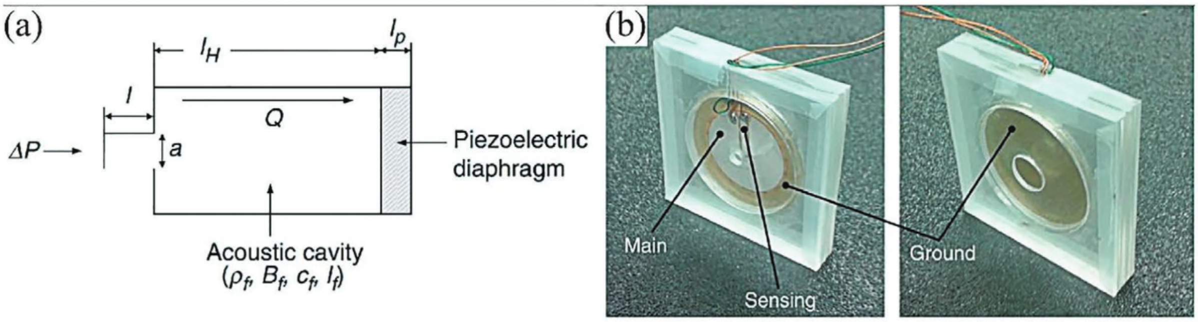

Figure 5: Active acoustic metamaterials (Akl & Baz 2011, Popa et al. 2013).

Studies on active sound-insulation metamaterials have been interdisciplinary, integrating acoustics, mechanics, materials science, and control science. At present, most of the research is based on piezoelectric control, probably because piezoelectric control is a well-established and mature method. Active soundinsulation metamaterials are still in early development, but can more flexibly control the sound insulation performance at the target frequency than passive structures. Owing to the high cost of electroacoustic components and the fast aging of piezoelectric materials in extreme environments, the application of active sound insulation metamaterials is challenging.

5 Acoustic cloaking using metamaterial

5.1 Model setup

Consider a cylindrical tube of inner radius R_2 and outer radius R_3 filled with air. The inner periphery of the tube is coated with an acoustic metamaterial, such that the inner radius of coating is R_1. In this study, we investigate four distinct cases: i) Homogenized cloak with effective properties given by eq. 1, ii) without any cloak, iii) cloak with 50 layers of materials 1 and 2, and iv) cloak with 20 layers of materials 1 and 2. The geometry and mesh for the problem is shown in fig. 6. The grey color shows the background material which is considered to be air. And the metamaterial coating is shown in the black color. The maximum mesh size of taken to be $\lambda/6$, where λ is the wavelength of the background

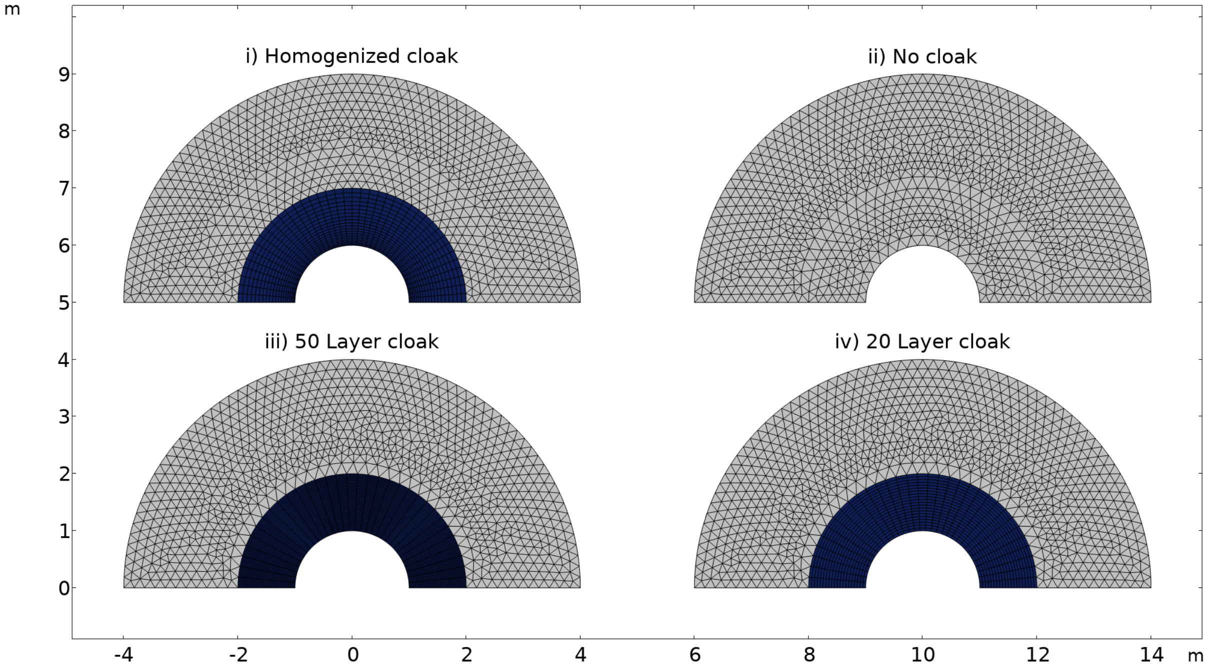

Figure 6: Geometry and mesh of the problem: i) homogenized cloak (upper left), ii) no cloak (upper right), iii) 50 layer cloak (lower left), and iv) 20 layer cloak (lower right).

pressure wave (defined below). It is important to mention that, for each case the radius r is defined from its respective geometrical center.

5.2 Metamaterial

Cummer & Schurig (2007) showed that acoustical cloaking of a cylindrical body can be attained by an anisotropic coating with inner radius R_1 and outer radius R_2 whose density, ρ_r and ρ_θ, and bulk modulus B vary as follows:

$$\rho_\mathrm{r} = \rho_\mathrm{b}\frac{r}{r - R_1}, \tag{1a}$$

$$\rho_\theta = \rho_\mathrm{b}\frac{r - R_1}{r}, \tag{1b}$$

$$B = B_\mathrm{b}\left(\frac{R_2 - R_1}{R_2}\right)^2 \frac{r}{r - R_1}, \tag{1c}$$

where ρ_r and ρ_θ are densities along the r and θ directions, respectively; r is the coordinate from the center of the cylinder; ρ_b B_b denote the density and bulk modulus of the background surrounding fluid, provided in table 1.

It was proposed by Torrent & Sánchez-Dehesa (2008) that such spatially varying anistropic properties can be achieved by coating the targeted body with two alternating materials having the properties

$$\rho_1 = \frac{r + \sqrt{(2rR_1 - R_1^2)}}{r - R_1}\rho_{\mathrm{b}}, \tag{2}$$

$$c_1 = \frac{R_2 - R_1}{R_2}\frac{r}{r - R_1}c_{\mathrm{b}}, \tag{3}$$

$$\rho_2 = \rho_{\mathrm{b}}^2/\rho_1, \tag{4}$$

$$c_2 = c_1 = c, \tag{5}$$

$$B_1 = \rho_1 c_1^2, \tag{6}$$

$$B_2 = \rho_2 c_2^2. \tag{7}$$

As the number of layers is increased, homogenization analysis shows that the effective properties of the coating approaches the properties required for acoustical cloaking (given in eq. 1). This derivation is provided in the appendix A.

5.3 Conservation Laws

The conservation of mass and momentum for an anisotropic media leads to,

$$\nabla \cdot (-\boldsymbol{\rho}^{-1}\nabla p_{\mathrm{t}}) - \frac{\omega^2}{B_{\mathrm{b}}}p_{\mathrm{t}} = 0, \tag{8}$$

where $\boldsymbol{\rho}$ is the density tensor of the anisotropic material, p_{t} is the total pressure, ω is the angular frequency and B_{b} is the bulk modulus of the background medium.

5.4 Boundary and initial conditions

The initial pressure flactuations are assumed to be zero. Moreover, at $r = R_1$, the velocity is taken to be zero. At $r = R_2$, there exists a pressure field $p = p_0\mathrm{e}^{-ik_{\mathrm{b}}x}$. The edges $\theta = 0$ and $\theta = \pi$ are give symmetric boundary condition.

Table 1: Values of material properties and operating parameters

Material property or parameter	Value
ρ_b, Density of surrounding background media (air)	1.25 kg/m^3
c_b, Speed of sound in surrounding background media (air)	343 m/s
f_0, Frequency of background pressure wave	300 Hz
R_1, Inner cloak radius	1 m
R_2, Outer cloak radius	2 m
R_3, Outer radius of the cylindrical tube	4 m
p_0, Parameter in the pressure boundary condition	1 Pa

6 Results and discussion

The total acoustic pressure for the four different cases can be seen in fig.7. The top right figure shows the pressure field without the cloak, when the cylinder is surrounded only by air. The incident pressure wave is scattered in all directions and is significantly influenced by the cylinder. In the upper-left figure, we see the homogenized cloak in use. The incident wave is undisturbed outside the cloak and it is not possible to determine that there is a cylinder present at all. The two bottom figures show how the cloak gets better when the number of layers is increased and the model is more similar to the homogenized cloak.

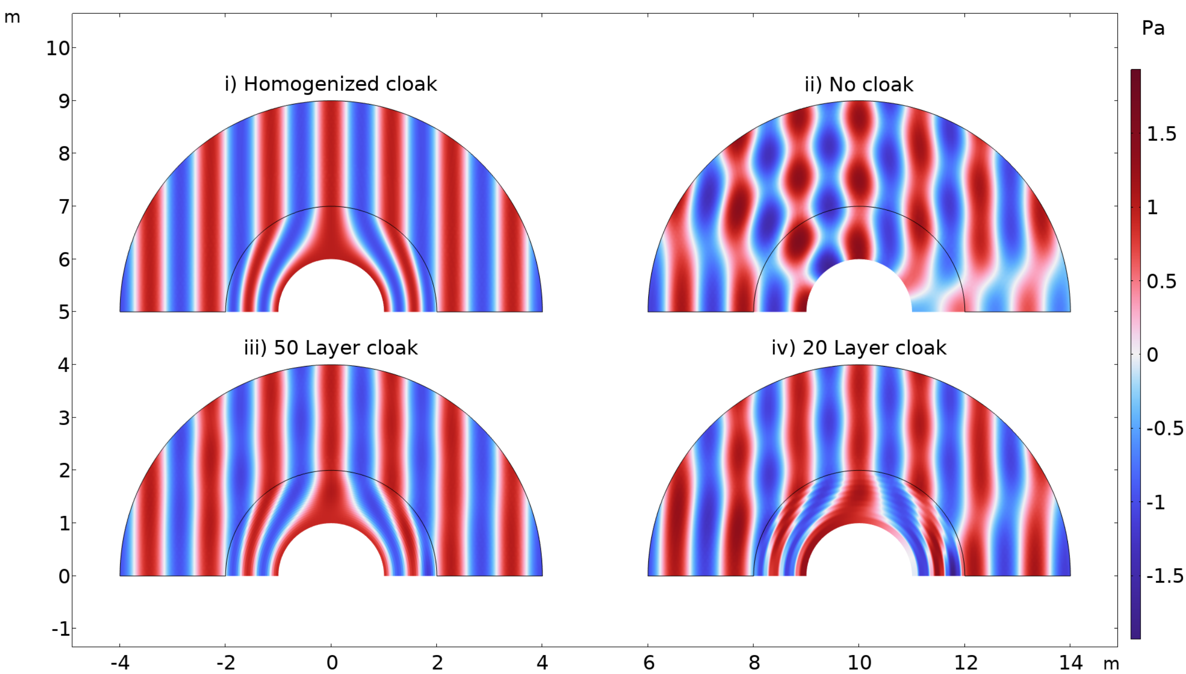

Figure 7: Surface plot of the total acoustic pressure, p_t (Pa): i) homogenized cloak (upper left), ii) no cloak (upper right), iii) 50 layer cloak (lower left), and iv) 20 layer cloak (lower right).

Another way to illustrate the effect of the cloak is to look at the total acoustic pressure along the cloak boundary. This is shown in fig.8. where we can see that the background pressure field curve coincides with the curve for the homogenized cloak as expected for an effective cloak.

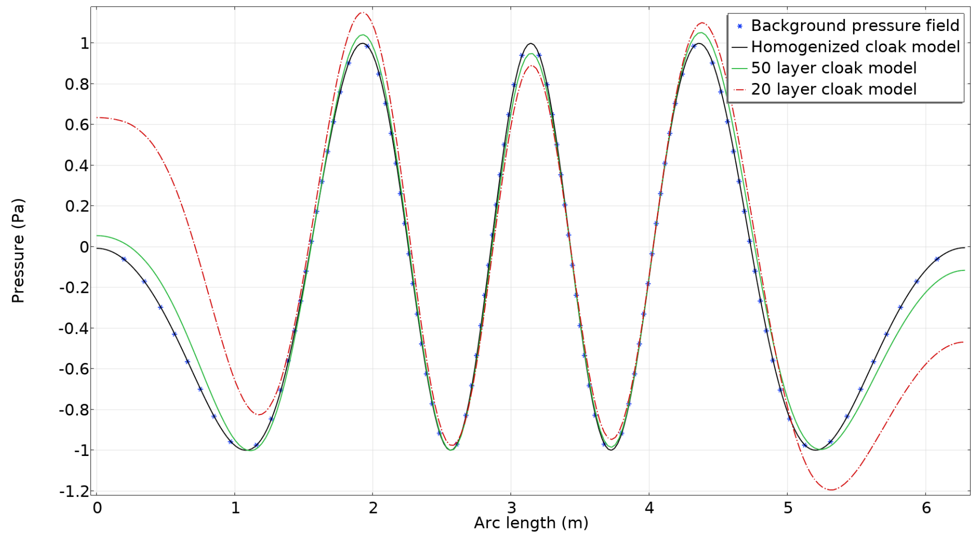

Figure 8: Total Acoustic Pressure Along Cloak Boundary (Pa)

However, this cloaking effect is not observed for all frequencies of the background pressure wave. Fig. 9 shows the variation of $|p_{\mathrm{b}} - p_{\mathrm{t}}|$ at (R_1, π). As the frequency of the p_{b} is increased the coating gets ineffective towards cloaking.

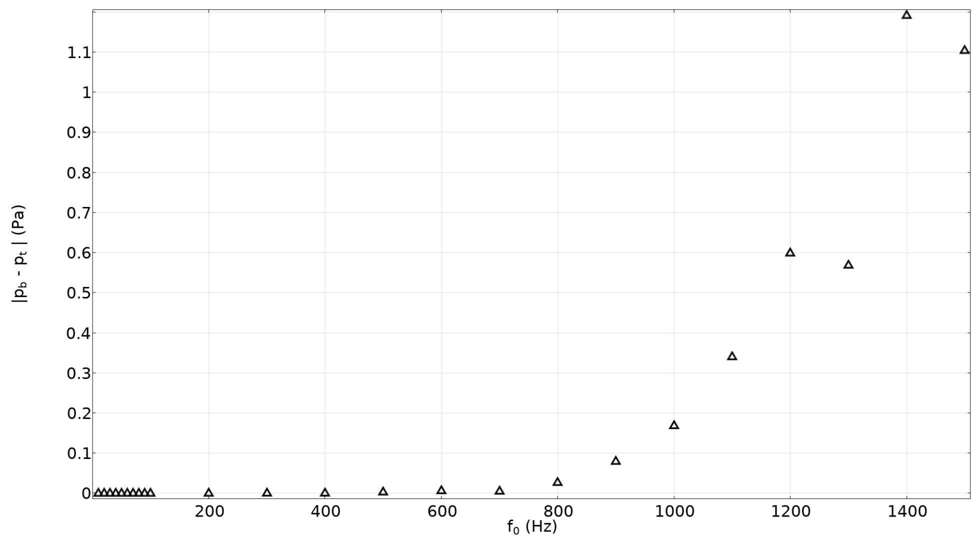

Figure 9: $|p_{\mathrm{b}} - p_{\mathrm{t}}|$ at (R_1, π).

To mitigate this issue, we propose that the material 1 or material 2 should be replaced by a fluid such as electrorheological or magnetorhelogial fluid. This would allow the properties of material 1 and 2 to be changed on the fly through some external stimuli (such as, electric and magnetic field).

7 Summary

Compared to regular acoustic materials, research into acoustic metamaterials has led to big steps forward in controlling low-frequency noise. It has also led to the creation of a number of theory models and new ideas for better sound wave control. But there are still some problems with the frequency band that can be used, checking samples, building strength, and active/passive composites that need to be looked into more thoroughly.

1. Current noise-reduction acoustic metamaterials are very good at blocking or absorbing noise in the moderate and low frequency ranges, but their working frequency band is often very narrow, so they can't fully meet the needs of processing broadband noise.

2. It hasn't been looked into in detail how strong these suggested acoustic metamaterials are, and most of them are hard to use in real-world engineering situations.

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A Appendix

$$
\begin{aligned}
\rho_\mathrm{r} &= \frac{\rho_1+\rho_2}{2} && \text{(A.1)}\\
&= \frac{1}{2}\left[\rho_1 + \rho_\mathrm{b}^2/\rho_1\right]\\
&= \frac{1}{2\rho_1}\left[\rho_1^2 + \rho_\mathrm{b}^2\right]\\
&= \frac{\rho_\mathrm{b}^2}{2\rho_1}\left[\left(\frac{r+\sqrt{(2rR_1-R_1^2)}}{r-R_1}\right)^2 + 1\right]\\
&= \frac{\rho_\mathrm{b}^2}{2\rho_1}\left[\frac{r^2+(2rR_1-R_1^2)+2r\sqrt{(2rR_1-R_1^2)}+r^2+R_1^2-2rR_1}{(r-R_1)^2}\right]\\
&= \frac{\rho_\mathrm{b}^2}{\rho_1}\left[\frac{r^2+r\sqrt{(2rR_1-R_1^2)}}{(r-R_1)^2}\right]\\
\implies \rho_\mathrm{r} &= \rho_\mathrm{b}\frac{r}{r-R_1} && \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
\rho_\theta &= \frac{2\rho_1\rho_2}{\rho_1+\rho_2} && \text{(A.3)}\\
&= \frac{\rho_1\rho_2}{\rho_\mathrm{r}}\\
&= \frac{\rho_\mathrm{b}^2}{\rho_\mathrm{b}\frac{r}{r-R_1}}\\
\implies \rho_\theta &= \rho_\mathrm{b}\frac{r-R_1}{r} && \text{(A.4)}\\
B &= \frac{2B_1B_2}{B_1+B_2} && \text{(A.5)}\\
&= \frac{2\rho_1\rho_2c^4}{\rho_1c^2+\rho_2c^2}\\
&= \frac{2\rho_1\rho_2c^2}{\rho_1+\rho_2}\\
&= \rho_\theta c^2\\
&= \rho_\mathrm{b}\frac{r-R_1}{r}\left(\frac{R_2-R_1}{R_2}\frac{r}{r-R_1}c_\mathrm{b}\right)^2\\
&= B_\mathrm{b}\left(\frac{R_2-R_1}{R_2}\right)^2\frac{r}{r-R_1} && \text{(A.6)}
\end{aligned}
$$