Problem Sheet 5: Plate Theory-II

1. Determine the deflection of a circular plate that is simply-supported around the periphery and subjected to uniform loading in the vertically downward direction.
   (The solution to this problem is presented in this link.)

2. Determine the deflection of a circular plate that is simply-supported around the periphery and subjected to a point load acting vertically downward at the centre.

3. Consider a rectangular plate of dimensions: a parallel to the x-axis and b parallel to the y-axis. All edges are simply-supported. The plate is subjected to a sinusoidal loading in the transverse (z) direction of the form $q = q_0 \sin \dfrac{\pi x}{a} \sin \dfrac{\pi y}{b}$.

   (a) Determine the deflection, w.

   (b) Compare the total reaction force along the four edges with the total load applied on the plate and determine the difference.

   (c) Account for the difference in part (b) in terms of the contribution from the corner points.

   (The solution to this problem is presented in this link.)

4. Consider a rectangular plate of dimensions: a parallel to the x-axis and b parallel to the y-axis. All edges are simply-supported. The edges at $x = 0$ and $x = a$ are uniformly compressed by a force (per unit length) $\bar{N}_x$.

   (a) Show that the critical loadings are given by: $\bar{N}_x = D\left(\dfrac{a}{m\pi}\right)^2 \left[\left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2\right]^2$.

   (b) Divide the equation in part (a) throughout by h (the plate thickness) to obtain an expression for the critical stress (σ_{cr}).

   (c) In the expression of part (a), set $n = 1$, and show that $\sigma_{\mathrm{cr}} = \dfrac{E k_{\mathrm{cr}} \pi^2}{12(1-\nu^2)} \left(\dfrac{h}{b}\right)^2$, where $k_{\mathrm{cr}} = \left(m\dfrac{b}{a} + \dfrac{1}{m}\dfrac{a}{b}\right)^2$.

   (d) Plot k_{cr} vs the aspect ratio a/b for different values of m and highlight the minimum value of k_{cr} as a/b increases and m varies, showing that k_{cr} approaches the value 4.

   (e) Plot the buckled shape of the plate for $a/b = 2$ and $m = 2$. Explore what happens for other values of a/b and m.

   (The solution to this problem is available in Section 9.13 of the book by Dym and Shames. The plots are presented in this link)

5. Problems 6, 7, and 8 are concerned with the buckling of rectangular plates that are simply-supported along $x = 0$ and $x = a$, compressed along those edges by a force per unit length of magnitude P, and having various edge conditions along $y = 0$ and $y = b$. In the following, we shall create the common part of the formulation required for the three problems.

   (a) Show that the governing differential equation reduces to $D\nabla^4 w = -P\dfrac{\partial^2 w}{\partial x^2}$.

   (b) Assuming that the plate buckles in m sinusoidal half-waves, take the solution of the governing equation in part (a) in the form $w = f(y)\sin\dfrac{m\pi x}{a}$, where $f(y)$ is a pure function of y and m is a positive integer. Note that this form automatically satisfies the boundary conditions along the simply-supported edges $x = 0$ and $x = a$:

$$w = 0 \quad \text{and} \quad M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0.$$

   Substituting this solution form in the governing equation obtain the following differential equation to determine $f(y)$:

$$\frac{\mathrm{d}^4 f}{\mathrm{d}y^4} - 2\left(\frac{m\pi}{a}\right)^2\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} + \left\{\left(\frac{m\pi}{a}\right)^4 - \frac{P}{D}\left(\frac{m\pi}{a}\right)^2\right\} f = 0.$$

   (c) Assuming $\dfrac{P}{D} > \left(\dfrac{m\pi}{a}\right)^2$, show that the general solution of the differential equation for $f(y)$ in part (b) may be written in the form $f(y) = C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3\cos(\beta y) + C_4\sin(\beta y)$, where C_1, C_2, C_3, C_4 are constants, and

$$\alpha = \left\{\left(\frac{m\pi}{a}\right)^2 + \sqrt{\frac{P}{D}\left(\frac{m\pi}{a}\right)^2}\right\}^{1/2}, \qquad \beta = \left\{-\left(\frac{m\pi}{a}\right)^2 + \sqrt{\frac{P}{D}\left(\frac{m\pi}{a}\right)^2}\right\}^{1/2}.$$

6. Consider the situation of Problem 5 together with the conditions: the edge at $y = 0$ is simply-supported and the edge at $y = b$ is free.

   (a) Show that the expressions for the boundary conditions at $y = 0$ and $y = b$ are:

$$y = 0: \quad w = 0 \quad \text{and} \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = 0,$$

$$y = b: \quad M_y = 0 \quad \text{and} \quad Q_{\text{eff}} = Q_y + \frac{\partial M_{xy}}{\partial x} = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right] = 0.$$

   (b) Using the boundary conditions along $y = 0$, show that the general solution form for $f(y)$ may be reduced to

$$f(y) = A\sinh(\alpha y) + B\sin(\beta y),$$

   where A and B are constants.

(c) Using the boundary conditions along $y = b$ and noting that A and B should be non-zero to obtain non-trivial solutions, obtain the following equation:

$$\beta \left\{ \alpha^2 - \nu \left(\frac{m\pi}{a} \right)^2 \right\}^2 \tanh(\alpha b) = \alpha \left\{ \beta^2 + \nu \left(\frac{m\pi}{a} \right)^2 \right\}^2 \tanh(\beta b)$$

Since the expressions of α and β contain P, the above equation provides a recipe for determining the critical values of P.

7. Consider the situation of Problem 5 together with the conditions: the edge at $y = 0$ is clamped (or, built-in) and the edge at $y = b$ is free.

   (a) Show that the expressions for the boundary conditions at $y = 0$ and $y = b$ are:

$$y = 0: \quad w = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 0,$$

$$y = b: \quad M_y = 0 \quad \text{and} \quad Q_{\text{eff}} = Q_y + \frac{\partial M_{xy}}{\partial x} = -D \left[\frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right] = 0.$$

   (b) Using the boundary conditions along $y = 0$, show that the general solution form for $f(y)$ may be reduced to $f(y) = A(\cos\beta y - \cosh\alpha y) + B\left(\sin\beta y - \frac{\beta}{\alpha}\sinh\alpha y\right)$.

   (c) Using the boundary conditions along $y = b$, and noting that A and B should be non-zero to obtain non-trivial solutions, obtain the following equation:

$$2ts + (s^2 + t^2)\cos\beta b \cosh\alpha b = \frac{1}{\alpha\beta}\left(\alpha^2 t^2 - \beta^2 s^2\right)\sin\beta b \sinh\alpha b,$$

   where $t = \beta^2 + \nu\frac{m^2\pi^2}{a^2}$ and $s = \alpha^2 - \nu\frac{m^2\pi^2}{a^2}$. Again, as in the previous problem, the above equation provides a recipe for determining the critical values of P.

8. Consider the situation of Problem 5 together with the conditions: both the edges $y = 0$ and $y = b$ are clamped (or, built-in). Following the procedure presented in Problems 6 and 7, show that the equation that provides a recipe for determining the critical values of P is:

$$2(1 - \cos\beta b \cosh\alpha b) = \left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right)\sin\beta b \sinh\alpha b$$

(Problems 6, 7, and 8 together with the preliminary framework of Problm 5 are discussed in the book "Theory of Elastic Stability" by Timoshenko and Gere in Section 9.4.)

9. A thin circular plate clamped around its periphery is loaded by a radial compressive force per unit length, P. Determine the critical value of this load for buckling in the transverse direction.
(Study the detailed solution of this problem in this link.)

10. A thin circular annular plate clamped around its periphery is loaded by a radial compressive force per unit length, P. The inner radius is b and the outer radius is a. Determine the critical value of the load for buckling in the transverse direction.
(Study the detailed solution of this problem in this link.)

11. A circular annular plate of inner radius b and outer radius a is simply supported at the outer periphery and is loaded by a moment of magnitude M_0 along the inner periphery. The plate is governed by the classical plate equation $\nabla^4 w = 0$, where w is the transverse direction displacement. Assume axisymmetric conditions. Solve the equation and use the two pairs of boundary conditions at the inner and outer radii to obtain the expression for the deflection, w.

12. This problem is concerned with the transformation rules from the (x, y) coordinate system to the (s, n) coordinate system where s is the coordinate along the periphery of the plate and n is the coordinate perpendicular to it. For any point on the periphery $x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, the unit outward normal is $\hat{\mathbf{e}}_n = n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}} = \frac{\mathrm{d}y}{\mathrm{d}s}\hat{\mathbf{i}} - \frac{\mathrm{d}x}{\mathrm{d}s}\hat{\mathbf{j}}$. Then show that $\hat{\mathbf{e}}_n = -n_y\hat{\mathbf{i}} + n_x\hat{\mathbf{j}}$. Next, comparing the expressions of a general scalar gradient $\nabla\phi$ in the (x, y) and the (s, n) systems, show that:

$$\frac{\partial \phi}{\partial x} = -n_y\frac{\partial \phi}{\partial s} + n_x\frac{\partial \phi}{\partial n},$$
$$\frac{\partial \phi}{\partial y} = n_x\frac{\partial \phi}{\partial s} + n_y\frac{\partial \phi}{\partial n}.$$

Subsequently, show that $\frac{\partial \phi}{\partial n} = n_x\frac{\partial \phi}{\partial x} + n_y\frac{\partial \phi}{\partial y}$.

13. Starting from the intermediate step in the classical plate theory variational formulation:

$$\int_A \left(\nabla^4 w - q\right)\delta w \mathrm{d}A - \oint \left[\left(-M_x\frac{\partial \delta w}{\partial x} - M_{xy}\frac{\partial \delta w}{\partial y} + Q_x\delta w\right)n_x \right.$$
$$\left. + \left(-M_{xy}\frac{\partial \delta w}{\partial x} - M_y\frac{\partial \delta w}{\partial y} + Q_y\delta w\right)n_y\right]\mathrm{d}s = 0,$$

transform to the (s, n) system mentioned in the previous problem. Use the following transformations:

$$\begin{aligned}
M_n &= n_x^2 M_x + 2n_x n_y M_{xy} + n_y^2 M_y,\\
M_{ns} &= n_x n_y(M_y - M_x) + (n_x^2 - n_y^2)M_{xy},\\
Q_n &= Q_x n_x + Q_y n_y.
\end{aligned}$$

Finally, obtain the boundary conditions in the form:

$$\text{Either} \quad \frac{\partial M_{ns}}{\partial s} + Q_n = 0, \quad \text{or,} \quad w \text{ is specified,}$$
$$\text{Either} \quad M_n = 0, \quad \text{or,} \quad \frac{\partial w}{\partial n} \text{ is specified.}$$

14. Consider an elliptical plate with a clamped periphery and loaded by uniformly ditributed load q [N/m^2]. The equation of the boundary is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. The governing differential equation for plate deflection, w is $D\nabla^4 w = q$ and the boundary conditions for the clamped periphery are $w = 0$ and $\frac{\partial w}{\partial n} = 0$ (see previous problem). Taking inspiration from the solution of the corresponding problem for the clamped circular plate ($w = \beta(r^2 - R^2)$; refer class notes), take the form for the deflection of the elliptical plate as $w = w_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2$, with w_0 to be determined. Verify that this form for the deflection satisfies the boundary conditions on the periphery. Determine w_0 by substituting the solution form in the governing differential equation.