Problem Sheet 4: Plate Theory-I

1. The fundamental difference between the classical plate theory (CPT) and the Föppl-von Kármán (FvK) plate theory is that in CPT, the linear strain-displacement relations are used whereas in the FvK plate theory, the quadratic terms are included in the strain-displacement relations. Rework the steps of the FvK plate theory derivation but using the linear strain-displacement relations to obtain the final governing differential equation for CPT as: $D\nabla^4 w = q$. Note: You are required to carry out this derivation only with reference to the Cartesian coordinate system.

2. For a circular plate under axisymmetric conditions, obtain from $D\nabla^4 w = q$, the CPT governing equation referred to a cylindrical coordinate system with origin at the plate centre.

3. Using the generalized Hooke's law corresponding to the plane stress case, show that the expressions for $M_x := \int_{-h/2}^{h/2} \sigma_{xx} z \, \mathrm{d}z$, $M_y := \int_{-h/2}^{h/2} \sigma_{yy} z \, \mathrm{d}z$, $M_{xy} := \int_{-h/2}^{h/2} \sigma_{xy} z \, \mathrm{d}z$ are $M_x = -D\left(\dfrac{\partial^2 w}{\partial x^2} + \nu \dfrac{\partial^2 w}{\partial y^2}\right)$, $M_y = -D\left(\dfrac{\partial^2 w}{\partial y^2} + \nu \dfrac{\partial^2 w}{\partial x^2}\right)$, and $M_{xy} = -D(1-\nu)\dfrac{\partial^2 w}{\partial x \partial y}$.

4. For a circular plate referred to a cylindrical coordinate system with origin at its centre, show using appropriate transformation that $M_r = M_x \cos^2\theta + M_y \sin^2\theta + 2M_{xy}\sin\theta\cos\theta$. Furthermore, if the circular plate is under axisymmetric conditions, show that $M_r = -D\left(\dfrac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \dfrac{\nu}{r}\dfrac{\mathrm{d}w}{\mathrm{d}r}\right)$.

5. In this question, we shall derive the expressions for the transverse plane shear forces defined as $Q_x := \int_{-h/2}^{h/2} \cancel{z}\sigma_{xz} \, \mathrm{d}z$ and $Q_y := \int_{-h/2}^{h/2} \cancel{z}\sigma_{yz} \, \mathrm{d}z$. For Q_x, do the following steps:

   - Consider the stress equilibrium equation in the x-direction and multiply throughout by z
   - Then integrate w.r.t. z between the limits $z = -h/2$ and $z = h/2$
   - Then use integration by parts over the integral containing $z\sigma_{xz}$ and use the fact that $\sigma_{zx} = \sigma_{xz} = 0$ at $z = \pm h/2$.
   - Show that $Q_x = \dfrac{\partial M_x}{\partial x} + \dfrac{\partial M_{xy}}{\partial y}$.
   - Finally, substituting the expressions of M_x and M_{xy} from Question ~~2~~ 3 in the above

expression, show that $Q_x = -\dfrac{\partial}{\partial x}(\nabla^2 w)$.

Following similar steps, show $Q_y = \dfrac{\partial M_{xy}}{\partial x} + \dfrac{\partial M_y}{\partial y} = -\dfrac{\partial}{\partial y}(\nabla^2 w)$.
Think whether the above relations are true only for the CPT or only for the FvK plate theory or for both.

6. For a circular plate show using appropriate transformations that $Q_r = Q_x \cos\theta + Q_y \sin\theta$. Furthermore, if the circular plate is under axisymmetric conditions, show that $Q_r = -D\dfrac{\mathrm{d}}{\mathrm{d}r}\left\{\dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r\dfrac{\mathrm{d}w}{\mathrm{d}r}\right)\right\}$.

7. In Question 1, while working out the derivation for CPT, the boundary terms will turn out to be:

$$\oint \left[\left\{-M_x\frac{\partial \delta w}{\partial x} - M_{xy}\frac{\partial \delta w}{\partial y} + Q_x\delta w\right\} n_x + \left\{-M_y\frac{\partial \delta w}{\partial y} - M_{xy}\frac{\partial \delta w}{\partial x} + Q_y\delta w\right\} n_y\right] \mathrm{d}s,$$

where use has been made of the expressions for Q_x and Q_y in terms of M_x, M_y, and M_{xy} obtained in Question 5. Now, consider an edge Γ of a rectangular plate with $x =$ constant, such that $\dfrac{\partial()}{\partial x} = 0$, $n_x = 1$, $n_y = 0$, and $\mathrm{d}s \equiv \mathrm{d}y$. Show that for this edge, the aforementioned integral involving the boundary terms reduce to:

$$-\left[M_{xy}\delta w\right]_{y_1}^{y_2} + \int_\Gamma \left(\frac{\partial M_{xy}}{\partial y} + Q_x\right)\delta w \;\mathrm{d}y,$$

where y_1 and y_2 are the y-coordinates of the vertices of the edge Γ. The special combination $\left(\dfrac{\partial M_{xy}}{\partial y} + Q_x\right)$ is referred to as the "effective shear".

8. **Question for thinking:** In both the CPT and the FvK plate theory, we start with the assumption that the transverse shear strains are zero, ie. $\varepsilon_{xz} = E_{xz} = 0$ and $\varepsilon_{yz} = E_{yz} = 0$. This may give us the impression that the corresponding shear stresses are zero. But, in reality, they cannot be! In fact we found expressions for Q_x and Q_y earlier by integrating those transverse shear stresses. And, yet, we claim that we use *"plane stress"* conditions while writing the stress-strain relations. Think, if there is any better way to describe the assumptions and/or to resolve these internal inconsistencies.

   Hints for thinking: Instead of saying that the normal stress $\sigma_{zz} = 0$, it is better to say that it is much smaller than σ_{xx} and σ_{yy}. So, instead of *setting equal to zero*, we *neglect it*, relatively speaking. This neglecting leads to expressions for σ_{xx} and σ_{yy} that are akin to the plane stress expressions, but physically it is not really so. Furthermore, the responsibility of "balancing" the transverse load is taken by the transverse shear stresses (which is why they *cannot* be zero). But, then what about the inconsistency regarding the transverse shear strain being zero? There is no way to fix that within CPT or FvK plate theory! We have to go to the Mindlin plate theory (also referred to as the First-order Shear Deformation Theory).

9. The Föppl-von Kármán equations are given by:

$$
\begin{aligned}
D\nabla^4 w &= q + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \\
\nabla^4 F &= Eh\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]
\end{aligned}
$$

It would be useful to convert these equations into a form that does not depend on the choice of the coordinate system. Examples of such forms include terms written using the Laplacian operator ($\nabla^2(\cdot)$) and the biharmonic operator ($\nabla^4(\cdot)$).

In a similar vein, define the "diamond" operator as

$$
\diamondsuit(f,g) = \frac{1}{2}\left\{(\nabla^2 f)(\nabla^2 g) + \nabla^2\left(f\nabla^2 g + g\nabla^2 f\right)\right\} - \frac{1}{4}\left\{\nabla^4(fg) + f\nabla^4 g + g\nabla^4 f\right\}
$$

Now, verify that the Föppl-von Kármán equations can be written as:

$$
\begin{aligned}
D\nabla^4 w &= q + \diamondsuit(F,w) \\
\nabla^4 F &= -\frac{1}{2}Eh\,\diamondsuit(w,w)
\end{aligned}
$$

You will find it easier to do this verification using SymPy.