Problem Sheet 2: 2D Elasticity - I

1. During lectures, the *plane stress* formulation was shown to give rise to the equation:

$$\nabla^4 \varphi = -(1-\nu)\, \nabla^2 V,$$

where φ is the Airy stress function and V is the potential associated with the conservative body force. Proceeding in a similar way, show that the *plane strain* formulation gives rise to the equation:

$$\nabla^4 \varphi = -\frac{1-2\nu}{1-\nu} \nabla^2 V.$$

2. For the plane stress case, show that the expressions for the stresses are:

$$\sigma_{xx} = \frac{E}{1-\nu^2} \left(\varepsilon_{xx} + \nu \varepsilon_{yy}\right),$$
$$\sigma_{yy} = \frac{E}{1-\nu^2} \left(\varepsilon_{yy} + \nu \varepsilon_{xx}\right).$$

3. For the plane strain case, show that the expressions for the strains are:

$$\varepsilon_{xx} = \frac{1+\nu}{E} \left[(1-\nu)\sigma_{xx} - \nu \sigma_{yy}\right],$$
$$\varepsilon_{yy} = \frac{1+\nu}{E} \left[(1-\nu)\sigma_{yy} - \nu \sigma_{xx}\right].$$

4. For the example problem solved using SymPy in class (Simply-supported beam under uniformly-distributed loading), determine the expressions of the stress components, σ_{xx}, σ_{xy}, and σ_{yy}. Plot the stress fields considering the values: $a = 10$, $h = 1$, and $q_0 = 1$. You will have to use NumPy and Matplotlib for plotting.

5. Again referring to the same example problem, determine the expression of σ_{xx} using the Mechanics of Materials approach, i.e. based on the flexure formula, $\sigma_{xx} = -\dfrac{My}{I}$. Compare with the expression found above. On the beam cross-section represented by $x = 0$, plot both the expressions of σ_{xx} for $a = 10$, $h = 1$, and $q_0 = 1$. Extract the maximum magnitude of the difference between the stresses and plot that magnitude coresponding to different values of $a \in [5,\ 50]$, keeping $h = 1$. What do you conclude from this trend?

Important: For each of the following problems, an Airy stress function is mentioned to help you attempt the problem "by hand". However, **this problem sheet is actually meant to give you practice in using SymPy within Jupyter Notebook**. When using SymPy, consider polynomials up to 4th, 5th or 6th degree (including lower degree terms) and use the framework demonstrated in class for the example problem.

6. The Airy stress function required to find the stresses in the cantilever beam (of unit width) as shown in Figure 1 is $\varphi = C_1x^2 + C_2x^2y + C_3y^3 + C_4y^5 + C_5x^2y^3$. Find the values of the constants using the requirements that φ should satisfy the biharmonic equation and the boundary conditions (pointwise boundary conditions on the top and bottom of the beam and integrated boundary conditions on the ends $x = 0$ and $x = L$).

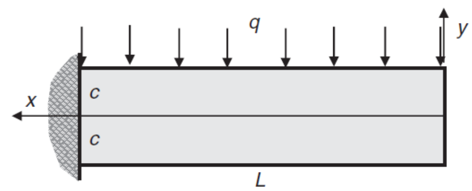

Figure 1

[Answer: $C_1 = -q/4$, $C_2 = -3q/8c$, $C_3 = q/20c$, $C_4 = -q/40c^3$, $C_5 = q/8c^3$]

7. Consider the cantilever beam (of unit width) loaded by uniform shear along its bottom edge as shown in Figure 2. Use appropriate conditions (pointwise boundary conditions on the horizontal surfaces and resultant conditions on the vertical surfaces) to find the stress field given that the required Airy stress function is of the form $\varphi = Axy + By^2 + Cy^3 + Dxy^2 + Exy^3$.

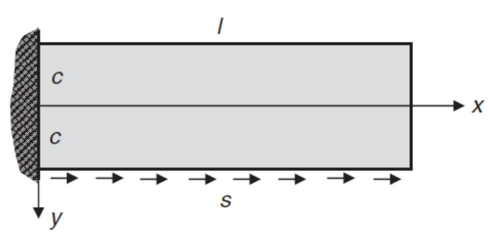

Figure 2

[Answer: $\sigma_{xx} = \frac{s}{2c}\left(1 + \frac{3y}{c}\right)(l - x)$, $\sigma_{yy} = 0$, $\sigma_{xy} = -\frac{s}{4}\left(1 + \frac{y}{c}\right)\left(1 - \frac{3y}{c}\right)$]

8. To solve the problem of a cantilever beam (of unit width) carrying a uniformly varying loading as shown in Figure 3, the following Airy stress function form is proposed: $\varphi = C_1 xy + C_2\frac{x^3}{6} + C_3\frac{x^3 y}{6} + C_4\frac{xy^3}{6} + C_5\frac{x^3y^3}{9} + C_6\frac{xy^5}{20}$. Determine the values of the various constants such that all conditions on the problem are satisfied. Use resultant force boundary conditions at the beam-ends.

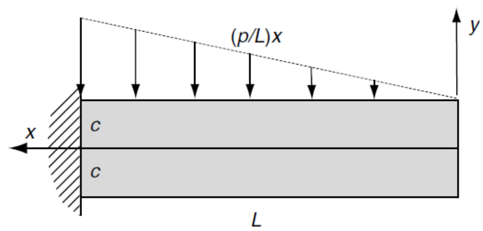

Figure 3

[Answer: $C_1 = -\frac{pc}{40L}$, $C_2 = -\frac{p}{2L}$, $C_3 = -\frac{3p}{4Lc}$, $C_4 = \frac{3p}{10Lc}$, $C_5 = \frac{3p}{8Lc^3}$, $C_6 = -\frac{p}{2Lc^3}$]

9. The cantilever beam (of unit width) shown in Figure 4 is subjected to a distributed shear stress $\tau_o x/l$ on the upper face. The following Airy stress function is given

$$\varphi = c_1y^2 + c_2y^3 + c_3y^4 + c_4y^5 + c_5x^2 + c_6x^2y + c_7x^2y^2 + c_8x^2y^3.$$

Determine the constants and find the stress distribution in the beam. Use resultant force boundary conditions at the ends.

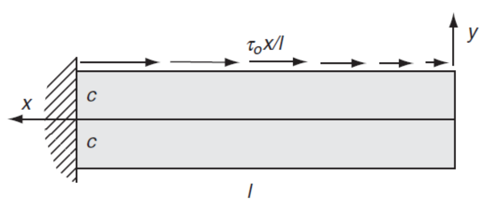

Figure 4

[Answer: $c_1 = \frac{\tau_o c}{12l}$, $c_2 = \frac{\tau_o}{20l}$, $c_3 = -\frac{\tau_o}{24cl}$, $c_4 = -\frac{\tau_o}{40c^2l}$, $c_5 = -\frac{\tau_o c}{8l}$, $c_6 = -\frac{\tau_o}{8l}$, $c_7 = \frac{\tau_o}{8cl}$, $c_8 = \frac{\tau_o}{8c^2l}$]