Problem Sheet 3: 2D Elasticity - II

1. The transformation equations for both stress and strain can be written in the forms:

$$\zeta_{x'x'} = \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} \begin{bmatrix}\zeta_{xx} & \zeta_{xy} \\ \zeta_{xy} & \zeta_{yy}\end{bmatrix} \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix},$$
$$\zeta_{y'y'} = \begin{bmatrix}\cos\left(\theta+\frac{\pi}{2}\right) & \sin\left(\theta+\frac{\pi}{2}\right)\end{bmatrix} \begin{bmatrix}\zeta_{xx} & \zeta_{xy} \\ \zeta_{xy} & \zeta_{yy}\end{bmatrix} \begin{bmatrix}\cos\left(\theta+\frac{\pi}{2}\right) \\ \sin\left(\theta+\frac{\pi}{2}\right)\end{bmatrix},$$
$$\zeta_{x'y'} = \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} \begin{bmatrix}\zeta_{xx} & \zeta_{xy} \\ \zeta_{xy} & \zeta_{yy}\end{bmatrix} \begin{bmatrix}\cos\left(\theta+\frac{\pi}{2}\right) \\ \sin\left(\theta+\frac{\pi}{2}\right)\end{bmatrix},$$

where ζ can be either σ or ε. Verify that the above relations can be written in the following compact form:

$$\begin{bmatrix}\zeta_{x'x'} & \zeta_{x'y'} \\ \zeta_{x'y'} & \zeta_{x'y'}\end{bmatrix} = [Q] \begin{bmatrix}\zeta_{xx} & \zeta_{xy} \\ \zeta_{xy} & \zeta_{yy}\end{bmatrix} [Q]^{\mathsf{T}},$$

where $[Q] = \begin{bmatrix}\cos\theta & \sin\theta \\ \cos\left(\theta+\frac{\pi}{2}\right) & \sin\left(\theta+\frac{\pi}{2}\right)\end{bmatrix}$ is the rotation matrix. Write a computer program utilizing these relations to quickly compute the required values in different problems.

2. In class it was demonstrated how to obtain the strain-displacement relations and the stress equilibrium relations in the polar coordinate system starting from the 2D Cartesian system.

   (a) Obtain the strain-displacement relations and the stress equilibrium relations in the cylindrical coordinate system starting from the 3D Cartesian system.

   (b) Obtain the strain-displacement relations and the stress equilibrium relations in the spherical coordinate system starting from the 3D Cartesian system. This is a bit more challenging than part (a); it is helpful to think of the transformation as being from the Cartesian to the cylindrical first, and then from the cylindrical to the spherical. Note that there are different conventions to denote the spherical coordinate variables. Choose the variables to be ρ, ϕ, and θ, where the θ coincides with the θ of the cylindrical system (measured with reference to the x of the Cartesian).

3. Consider a thick-walled pressure vessel made of a linear elastic isotropic material (E, ν) of inner radius r_i and outer radius r_o subjected to internal pressure p_{in} and external pressure p_{out}. This is a case of plane strain condition.

(a) Starting from the stress equilibrium equation in the radial direction, derive the expressions for the radial and the hoop stress.

(b) Starting from an appropriate Airy stress function, derive the expressions for the radial and the hoop stress.

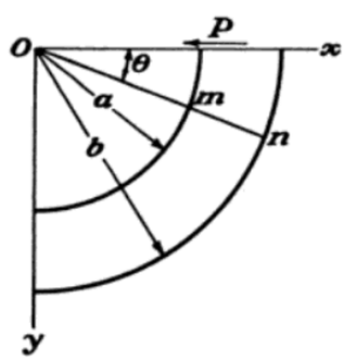
4. Considering the situation in the previous problem when the external pressure is zero and when $t = (b - a) \ll a$, i.e. when the wall is very thin, determine the expression for the hoop stress.

5. In class, when discussing the examples of axisymmetric problems, we had started with the Airy stress function form that is a pure function of the radial coordinate variable, i.e. $\varphi = f(r)$. However, there *are* certain axisymmetric problem solutions which arise from Airy stress functions that *do* have a dependence on θ. For instance, the axisymmetric problem of an annular disk with a fixed inner periphery and loaded with a uniform shear stress τ on the outer periphery is solved starting from the Airy stress function $\varphi = A\theta$, where A is some constant. Determine the stress and the displacement fields in this situation.

6. **Flamant Problem:** A point load is acting vertically downward on an elastic half space. Determine the stresses around the point of application. (Solution available in the GitHub repo.)

7. Consider a curved bar that is subjected to pure bending. Determine the stresses in the bar. (Solution available in the GitHub repo.)

8. Consider a curved bar that is clamped at one end and is subjected to a shear loading on its free horizontal edge. Determine the stresses in the bar. (Solution available in the GitHub repo.)



9. A long composite cylinder of outer radius b is made up of a solid cylinder ($r \in [0,\ a]$) made of material 1 (E_1, ν_1) and an annular cylinder ($r \in [a,\ b]$) made of material 2 (E_2,

ν_2). The two materials are perfectly bonded at $r = a$. Determine the stresses in the two materials. Perfect bonding in this case implies continuity of radial displacement, radial stress, and shear stress at the interface of the two materials.