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Inspire Mentorship Report

A Topological Approach to Nonlinear Analysis

Submitted in partial fulfillment of the
requirements for the completion of seven weeks summer mentorship program

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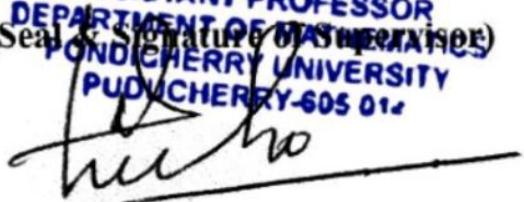
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Chapter 1

Introduction

Topology became a formalized branch of mathematics during the last part of the nineteenth century. Georg Cantor (1845-1918) created point set theory, which was a mix of analysis and geometry, and demonstrated its applications. The beginning of point set topology was filled with controversy and not considered well-developed. Giuseppe Peano (1858-1932) was involved in noticing some of the hazards in working with point set topology [Joh87]. In the early twentieth century, two mathematicians, Maurice Fréchet (1878-1973) and Felix Hausdorff (1868-1942) based topology on the extension and generalization of the idea of continuity. Fréchet in 1906 extended the idea of distance to abstract sets with properties similar to distance on the real number line [Kli72]. Thus, Fréchet introduced the idea of complete metric spaces along with the definition of metric spaces [Wil70]. Hausdorff in 1914 generalized the concept of an open set which is similar to an open interval on the real number line [Bak97].

In 1909, L.E.J. Brouwer introduced his first fixed point theorem using ideas of topology. Brouwer then submitted a paper on vector fields in the same year. Poincaré had made progress on vector fields related to the qualitative theory of differential equations in the 1880's, but Brouwer was unaware of Poincaré's work and approached the subject with a topological viewpoint. Brouwer's investigations were based on Peano's existence theorem

for differential equations. The first order differential equation $\frac{dy}{dx} = f(x, y)$, where f is a continuous function of x and y , possesses at least one integral curve, or solution curve, through each point (x_1, y_1) . Peano's theorem asserts the existence of solution curves that are tangent curves [Joh87].

Since some of Brouwer's work in topology was motivated by working on Peano's theorem, we will work through the ideas of topology needed to prove the Cauchy-Peano existence theorem, a modified version of Peano's theorem. We will start with the idea of complete metric spaces. These abstract spaces retain the idea of a metric, or distance measurement. From the complete metric spaces, we will prove the Ascoli-Arzela theorem which will be used to prove the Cauchy-Peano existence theorem.

Then we work through the ideas that Brouwer explored in fixed point theory. We are interested in when a space is mapped onto itself has a fixed point. Fixed point theory is used in differential equations to determine the existence of a solution [Kli72]. In economics, fixed point theory is used to determine equilibrium of a system. We will explore fixed point theory in the familiar work of Euclidean n-spaces. Then we will extend the ideas to more abstract spaces, namely normed linear spaces. Finally, we will have all the tools needed to approach the Cauchy-Peano existence theorem topologically. For reference, I will include a sketch of the approximation proof for comparison purposes. We will also look at Picard's theorem which has the the additional requirement of uniqueness of the solution and an example of when the solution is not unique.

Chapter 2

Complete Metric Spaces

Since the theorems we will discuss deal mostly with complete metric spaces, we will define a metric space and present some examples of complete metric spaces.

Definition 2.1. Given a set X and a function $d : X \times X \rightarrow \mathbb{R}$, the pair (X, d) is called a *metric space* if d has the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$.
2. $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$.
3. $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If (X, d) is a metric space, then d is called a metric for the space X . Some familiar metric spaces include the following:

1. \mathbb{R}^n with the Euclidean metric where $d(x, y) = \|x - y\| = [(x_1 - y_1)^2 + \dots + (x_n - y_n)^2]^{\frac{1}{2}}$
2. \mathbb{R}^n with the square metric $\rho(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$ is a metric space.

This brings us to a special type of metric space called a complete metric space. Complete metric spaces incorporate aspects of analysis with topological concepts.

Definition 2.2. A metric space is *complete* if every Cauchy sequence in the space converges.

Definition 2.3. A sequence $\{s_n\}_{n=1}^{\infty}$ is *Cauchy* provided that for every $\varepsilon > 0$ there exists a positive integer N such that if $m \geq N$ and $n \geq N$, then $d(s_m, s_n) < \varepsilon$.

To help us get a better picture of complete metric spaces, let us consider a metric space that is not complete. Consider \mathbb{Q} with the Euclidean metric d . The sequence given by $a_n = (1 + \frac{1}{n})^n$ is a Cauchy sequence in \mathbb{Q} , but the sequence converges to e which is not in \mathbb{Q} . Thus (\mathbb{Q}, d) is not complete.

In complete metric spaces, compactness and total boundedness are closely related properties. In order to define the idea of total boundedness, we need to understand a special class of sets called ε -nets.

Definition 2.4. For $\varepsilon > 0$ an ε -net S in a metric space (X, d) is a subset of (X, d) with the property that every point in X is within ε of some point of S .

We can describe this idea with open balls centered at x as $B(x; \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for $\varepsilon > 0$, then S will be an ε -net of X if $X = \bigcup_{s \in S} B(s, \varepsilon)$.

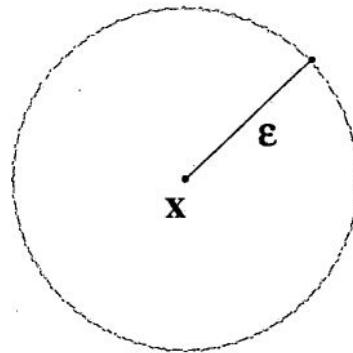


Figure 2.1: Open Ball $B(x; \varepsilon)$

Definition 2.5. A metric space (X, d) is *totally bounded* if given $\varepsilon > 0$, there is a finite ε -net for (X, d) .

In other words, there exists s_1, s_2, \dots, s_n in X such that $X = \bigcup_{i=1}^n B(s_i, \varepsilon)$. Here are a few examples where we examine the total boundedness of various metric spaces.

1. \mathbb{R} with the usual metric $d(x, y) = |x - y|$ is neither bounded nor totally bounded.

Consider the ε -net $(x - \varepsilon, x + \varepsilon)$ for $x \in \mathbb{Q}$. This ε -net has no finite subnet that would cover \mathbb{R} .

2. \mathbb{R} under this metric $d(x, y) = \min\{|x - y|, 1\}$ is bounded, but not totally bounded.

Under the metric, it has an upper bound of one and a lower bound of zero. We can use the same ε -net as above.

3. The subspace $(-1, 1)$ of \mathbb{R} under the usual metric $d(x, y) = |x - y|$ is totally bounded.

Given an ε , divide the interval into segments of length $\frac{\varepsilon}{3}$. Since $(-1, 1)$ is of finite length under the given metric, there will be a finite number of segments. Using the centers of the segments as the centers of the ε -balls, the interval will be covered by a finite number of ε -balls.

4. The subspace $\mathbb{Q} \cap [-1, 1]$ is also totally bounded under the usual metric.

The following lemma states that compactness of any space implies that the space is totally bounded. Thus, if we know that a space is compact, we can assume that it is also totally bounded.

Lemma 2.6. *If (X, d) is compact, then (X, d) is totally bounded.*

Proof. Let (X, d) be a compact metric space. Given $\varepsilon > 0$, let $B_x = B(x, \varepsilon)$ where $\{B_x\}_{x \in X}$ is a cover for (X, d) . Since (X, d) is compact, there exists a finite subcover $\{B_{x_j}\}$ for $j = 1, \dots, n$. The set $\{x_j : j = 1 \dots n\}$ is a finite ε -net for (X, d) . Therefore, by definition, (X, d) is totally bounded. \square

However, the converse of the lemma does not hold. Total boundedness does not always imply that the space is compact. It turns out that if we add the condition that the space be complete, total boundedness implies compactness. In order to prove the theorem, we will need the following lemma. We state the lemma without proof since it uses information we will not need anywhere else in this paper. The proof can be found in Munkres, page 181 [Mun75].

Lemma 2.7. *Let X be a metrizable space. Then the following are equivalent:*

1. *X is compact.*
2. *X is limit point compact.*
3. *X is sequentially compact.*

Definition 2.8. A space X is *limit point compact* if every infinite subset of X has a limit point.

Definition 2.9. A space X is *sequentially compact* if every sequence in X has a convergent subsequence.

Theorem 2.10. *Let (X, d) be a complete metric space. If (X, d) is totally bounded, then (X, d) is compact.*

Proof. We can show that (X, d) is compact by showing (X, d) is sequentially compact. Let $\{x_n\}$ be a sequence of points in X . We will construct a subsequence of $\{x_n\}$ that is a Cauchy sequence and therefore convergent.

Since (X, d) is totally bounded, we can cover X by finitely many balls of radius 1, i.e., a finite 1-net. At least one of the balls, say B_1 , will contain x_n for infinitely many values of n . Let J_1 be the subset of \mathbb{Z}^+ consisting of those indices n for which $x_n \in B_1$.

Now, cover X by finitely many balls of radius $\frac{1}{2}$, i.e., a finite $\frac{1}{2}$ -net. Because J_1 is infinite, at least one of these balls, say B_2 must contain x_n for infinitely many values of n in J_1 . Choose J_2 to be the set of those indices n for which $n \in J_1$ and $x_n \in B_2$.

In general, given an infinite set J_k of positive integers, choose J_{k+1} to be an infinite subset of J_k such that there is a ball B_{k+1} of radius $\frac{1}{k+1}$ which contains x_n for all $n \in J_{k+1}$.

Choose $n_1 \in J_1$ such that n_1 is the minimum of J_1 . Given n_k , choose $n_{k+1} \in J_{k+1}$ such that n_{k+1} is the minimum of J_{k+1} and $n_{k+1} > n_k$ (we can do this since J_{k+1} is an infinite set). Now for $i, j \geq k$, the indices n_i and n_j both belong to J_k (because $J_1 \supset J_2 \supset \dots$ is a nested sequence of sets). Therefore, for all $i, j \geq k$, the points x_{n_i} and x_{n_j} are contained in a ball B_k of radius $\frac{1}{k}$. It follows that the subsequence $\{x_{n_i}\}$ is a Cauchy sequence and thus converges in X . Hence, X is sequentially compact and (X, d) is compact. \square

An important consequence of this theorem is that total boundedness provides a way to determine if a set is relatively compact.

Definition 2.11. When a subset S in a space (X, d) is contained in a compact subset of X , then S is called *relatively compact*.

Corollary 2.12. A totally bounded subset S of a complete metric space (X, d) is relatively compact.

Proof. Let S be a totally bounded subset of a complete metric space (X, d) . We need to show that the closure of S is also totally bounded. Recall that the closure of S is $\bar{S} = S \cup S'$, where S' is the set of limit points for S . Let $\varepsilon > 0$ be given. Since S is totally bounded, there exists $A_{\frac{\varepsilon}{2}}$ such that $A_{\frac{\varepsilon}{2}}$ is a finite $\frac{\varepsilon}{2}$ -net for S . We will show that $A_{\frac{\varepsilon}{2}}$ is also an ε -net for \bar{S} .

Let $x \in S'$. This means that x is a limit point of some Cauchy sequence in S . Let $\{x_n\}$ be a Cauchy sequence in S that converges to x . Then there exists an $x_j \in S$ such that

$|x - x_j| < \frac{\varepsilon}{4}$. Moreover, there exists $p \in A_{\frac{\varepsilon}{2}}$ such that $x_j \in B(p, \frac{\varepsilon}{2})$. Therefore, $x \in B(p, \varepsilon)$.

Hence, $A_{\frac{\varepsilon}{2}}$ is an ε -net for \bar{S} .

By definition, \bar{S} is totally bounded. Since \bar{S} is closed and a subset of a complete metric space, \bar{S} is compact. Thus, S is relatively compact. \square

Hence, complete metric spaces have enough properties to be able to show compactness of spaces.

Chapter 3

Ascoli-Arzela Theorem

Next, we will explore the Ascoli-Arzela theorem, and its applications, to demonstrate the usefulness of topology in nonlinear analysis, namely in the proof of the Cauchy-Peano existence theorem. Here, we will move from the general metric space to metric spaces of functions. Let (X, d) be a metric space and let $u : X \rightarrow \mathbb{R}$ be a real-valued function. The function u will be bounded if and only if $u(X)$ is a bounded subset of \mathbb{R} . We define the supremum norm (sup norm) $\|u\|$ of u to be the least upper bound of $u(X)$ in \mathbb{R} .

Now consider the set $B(X)$ of all bounded real-valued functions on X . $B(X)$ will be a metric space with the distance between functions u and v given by $d(u, v) = \|u - v\|$. Then, $\|u\|$ is often referred to as the uniform norm [Bro04].

Definition 3.1. Let $f_n : X \rightarrow Y$, $n \in N$, be a sequence of functions from the set X to the metric space (Y, d) . A sequence (f_n) converges uniformly to a function $f : X \rightarrow Y$ if given $\varepsilon > 0$, there exists an integer N such that $d(f_n(x), f(x)) < \varepsilon$ for all $n \geq N$ and for all $x \in X$.

For example, consider the sequence of bounded, continuous functions $f_n : [0, 1] \rightarrow \mathbb{R}$ defined by $f_n(x) = x^n$ [Lay05]. For $x \in [0, 1)$, $\lim_{n \rightarrow \infty} f_n(x) = 0$. However, when $x = 1$,

$\lim_{n \rightarrow \infty} f_n(x) = 1$. Thus, the functions f_n converge pointwise to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

For the functions f_n to converge uniformly, for each $\varepsilon > 0$, we need to find one integer N that will work for all $x \in [0, 1]$. Consider $\varepsilon = \frac{1}{2}$. We can find values for each f_n that are farther than ε away from the function f , in general zero. Given any integer n , we can use any x such that $2^{-\frac{1}{n}} < x < 1$ to get $f_n(x) > \frac{1}{2}$. For example, if $n = 4$, we can choose $x = \frac{9}{10}$. Then $f_4(\frac{9}{10}) = .6561$ which is greater than $\frac{1}{2}$. To visualize the functions, refer to the following figure.

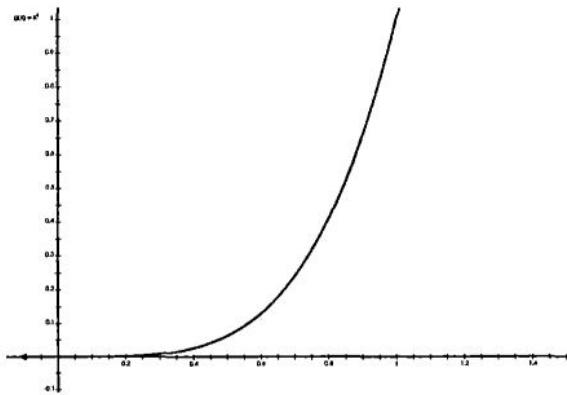


Figure 3.1: Functions f_n

Now, $B(X)$ inherits a complete metric space structure from the reals. Let $\{u_n\}$ be a Cauchy sequence in $B(X)$. For each $x \in X$, $u_n(x)$ is a Cauchy sequence in the reals. Since \mathbb{R} is a complete metric space, $u_n(x)$ converges to a limit $y_x \in \mathbb{R}$. Thus, $\{y | x \in X \text{ and } u_n(x) \rightarrow y\}$ is bounded since every convergent sequence is bounded. Define $u : X \rightarrow \mathbb{R}$ by $u(x) = y_x$. Then $u \in B(X)$ and u_n converges to u . Thus, all Cauchy sequences converge in $B(X)$.

We are interested in the subset $C(X)$ of continuous, bounded, real-valued functions on X . Since convergence in $B(X)$ is uniform convergence, the limit of a sequence of

continuous functions in $B(X)$ is also continuous. So $C(X)$ contains all of its limit functions and hence is a closed subset of $B(X)$. A closed subset of a complete metric space is also complete under the same metric. Thus, $C(X)$ is a complete metric space under the sup norm metric. By corollary 2.12, all totally bounded subsets of $C(X)$ are relatively compact.

However, checking the total boundedness of a set of real-valued functions can be very difficult. So we will show that boundedness and equicontinuity are sufficient conditions for total boundedness.

Definition 3.2. A set A in $C(X)$ is *bounded* if there is a real number β such that $\|u\| < \beta$ for all functions $u \in A$.

Definition 3.3. A set A in $C(X)$ is *equicontinuous* at $x \in X$ if given $\varepsilon > 0$, there exists $\delta_x > 0$ such that if $y \in X$ with $d(x, y) < \delta_x$, then $|u(x) - u(y)| < \varepsilon$ for all $u \in A$.

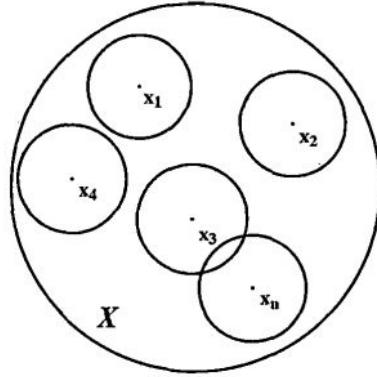
Recall that $|u(x) - u(y)|$ denotes the usual distance metric in \mathbb{R} . We call a set $A \subseteq C(X)$ equicontinuous if the set is equicontinuous at all $x \in X$. Now we are ready to prove the Ascoli-Arzela Theorem [Bro04].

Theorem 3.4. *Let X be a compact metric space. If A is an equicontinuous, bounded subset of $C(X)$, then A is relatively compact.*

Proof. Let X be a compact metric space. Let A be an equicontinuous, bounded subset of $C(X)$. We will show that A is relatively compact by showing A is a totally bounded subset of the complete metric space $C(X)$.

Let $\varepsilon > 0$ be given. Since A is equicontinuous and bounded, for all $x \in X$ there exists a $\delta_x > 0$ such that if $y \in X$ and $d(x, y) < \delta_x$, then $|u(x) - u(y)| < \frac{\varepsilon}{4}$ for all $u \in A$. Thus, $\{B(x, \delta_x) | x \in X\}$ is a cover of the metric space X .

Since X is a compact metric space, there exists a finite subcover of X . That is, there exists x_1, x_2, \dots, x_n in X where $X \subseteq \bigcup_{j=1}^n B(x_j, \delta_{x_j})$. For simplicity of notation, let

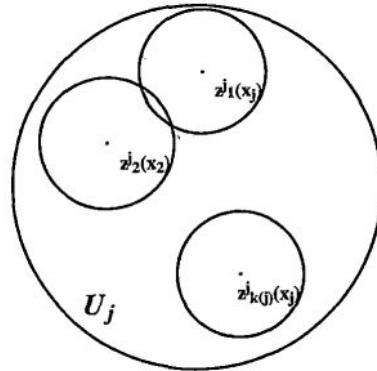
Figure 3.2: The finite cover of X

$\delta_{x_j} = \delta_j$. Then we can conclude that $|u(x) - u(x_j)| < \frac{\varepsilon}{4}$ for all $u \in A$ and $x \in X$.

Now let $U_j = \{u(x_j) | u \in A\}$. U_j will be a bounded subset in \mathbb{R} since A is bounded.

So U_j is relatively compact, or that is, totally bounded. By definition, U_j has a finite $\frac{\varepsilon}{4}$ -net.

So there exists $z_1^j, z_2^j, \dots, z_{k(j)}^j$ in A such that $U_j \subseteq \bigcup_{i=1}^{k(j)} B(z_i^j, \frac{\varepsilon}{4})$.

Figure 3.3: The finite cover of U_j

Let $M = \{\mu | \mu = (\mu_1, \mu_2, \dots, \mu_n), 1 \leq \mu_r \leq k(r) \text{ for } r = 1, \dots, n\}$. In other words μ is an n -tuple of natural numbers and the n comes from our subscripting of the x_j 's. The cardinality of M is given by $|M| = \prod_{r=1}^n k(r) = k(1)k(2)\dots k(n)$. Since each $k(r)$ is a finite number, the product must also be a finite number.

For all $\mu \in M$, let $S_\mu = \{v \in A \mid |v(x_j) - z_{\mu_j}^j(x_j)| < \frac{\varepsilon}{4}$ for all $j \in 1, \dots, n\}$. Now it may happen that some of our S_μ are empty. So let $M^* = \{\mu \in M \mid S_\mu \neq \emptyset\}$. Since each S_μ is not empty, we can choose one $v_\mu \in S_\mu$ for all $\mu \in M^*$. Let $V = \{v_\mu \mid \mu \in M^*\}$.

We claim that V forms a finite ε -net for A . First, we know that V is finite since $|V| \leq |M^*| \leq |M| < \infty$. Let $u \in A$. For all $j \in 1, \dots, n$ choose a μ_j such that $|u(x_j) - z_{\mu_j}^j(x_j)| < \frac{\varepsilon}{4}$. Then $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in M$. So $\mu \in S_\mu$ which implies that $S_\mu \neq \emptyset$. Since S_μ is not empty, $\mu \in M^*$ and so there exists a corresponding $v_\mu \in V$.

Now we need to show that $\|u - v_\mu\| < \varepsilon$. Let $x \in X$, then

$$\begin{aligned} |u(x) - v_\mu(x)| &\leq |u(x) - u(x_j)| + |u(x_j) - z_{u_j}^j(x_j)| \\ &\quad + |z_{u_j}^j(x_j) - v_\mu(x_j)| + \|v_\mu(x_j) - v_\mu(x)\| < \varepsilon. \end{aligned}$$

Thus we have a finite ε -net for A . Hence A is totally bounded. Since A is a subset of the complete metric space $C(X)$, A is relatively compact. \square

Chapter 4

Euclidean n -spaces and Fixed Point Theory

In addition to the Ascoli-Arzela Theorem, fixed point theory is key in proving the Cauchy-Peano Existence theorem topologically. First, we need to know the meaning of the fixed point property.

Definition 4.1. A topological space Y has the *fixed point property* if every continuous function $f : Y \rightarrow Y$ has a fixed point, i.e., $f(y) = y$ for some $y \in Y$.

The fixed point property (fpp) is of interest to mathematicians since it is a property that is preserved by homeomorphisms. That is, if a topological space Y has the fpp and Y is homeomorphic to a topological space Z , then Z also has the fpp [Bro04]. Our goal is to show that a compact, convex subset of a normed linear space has the fpp. We shall first prove the finite dimensional fixed point theorem and then generalize to normed linear spaces.

We will begin with the normed linear space \mathbb{R}^n . The elements of \mathbb{R}^n are ordered n -tuples $x = (x_1, x_2, \dots, x_n)$ of real numbers. The structure of vector space \mathbb{R}^n has the usual vector addition and scalar multiplication with the inner (dot) product. Recall that the inner

product is defined as $x \cdot y = (x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_n) = x_1y_1 + x_2y_2 + \dots + x_ny_n$. We will define the norm of x as $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{x \cdot x}$.

We begin with a simple convex subset of \mathbb{R}^n , the unit ball. We define the unit ball B^n as the set of all $x \in \mathbb{R}^n$ such that $|x| \leq 1$. Here we are using the usual metric for Euclidean space. So the distance from x to y is $|x - y|$. The following is a well-known result for \mathbb{R}^n , Brouwer's Fixed Point Theorem.

Theorem 4.2. *The unit ball B^n has the fixed point property.*

The proof may be found in Dunford pages 467 through 470, [DS88]. We now proceed to a more general version of the Brouwer Fixed Point Theorem. Rather than considering the unit ball, we want to prove the same result for a compact, convex subset of \mathbb{R}^n . In order to prove this generalized theorem, we will need several lemmas and definitions. The first is the notion of a retract.

Definition 4.3. A subset A of a space X is a *retract* of X if there exists a continuous mapping ρ of X onto A , $\rho : X \rightarrow A$, such that $\rho(x) = x$ for all $x \in A$.

To better understand retracts, let us look at some examples of retracts of various spaces. Points, lines, closed discs, and open or closed squares have the simplest retract which is a point. In the case of the Brouwer Fixed Point Theorem, the unit ball is a retract of the space \mathbb{R}^n . For an annulus, a flat donut shaped object, a retract would be a circle.

Lemma 4.4. *If A is a retract of a space X and X has the fixed point property, then A also has the fixed point property.*

Proof. Since A is a retract of X , by definition, there exists a continuous mapping $\rho : X \rightarrow A$ such that $\rho(x) = x$ for all $x \in A$. Let $i : A \rightarrow X$ be the identity map. Given any continuous map $g : A \rightarrow A$, we will define the map $f : X \rightarrow X$ as

$$f : X \xrightarrow{\rho} A \xrightarrow{g} A \xrightarrow{i} X.$$

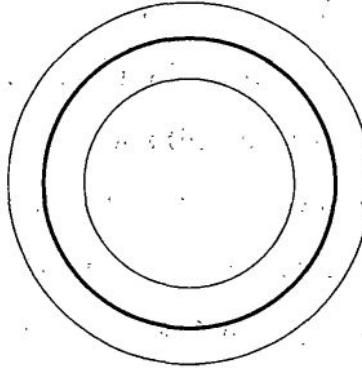


Figure 4.1: The retraction of an annulus

Since X has the fpp, and f is continuous map, then there exists $x \in X$ such that $f(x) = x$. Since $f(X) \in A$, $x \in A$. Therefore, $\rho(x) = x$. Then

$$g(x) = g(\rho(x)) = i(g(\rho(x))) = f(x) = x$$

Thus, the map g also has x as a fixed point. Then all continuous maps of A have the fpp implying that A has the fpp. \square

For example, all retracts of a square have the fpp since a square has the fpp. However, a circle does not have the fpp, thus an annulus which has the circle as a retract does not have the fpp [Sha91]. Now that we know that retracts of spaces retain the fpp, our goal will be to show that a convex, compact subset Q of \mathbb{R}^n is a retract of some n -ball, which in turn is homeomorphic to the unit ball.

Lemma 4.5. *A compact, convex subset Q of the normed linear space \mathbb{R}^n is a retract of some n -ball in \mathbb{R}^n .*

Proof. Since Q is a compact subset of \mathbb{R}^n , then it is closed and bounded. Choose r such that $|x| \leq r$ for all $x \in Q$. Then Q is contained in B_r^n which is defined as all $x \in \mathbb{R}^n$ where $|x| \leq r$. To show that Q is a retract of the B_r^n , we need to show that for each $x \in \mathbb{R}^n$ there

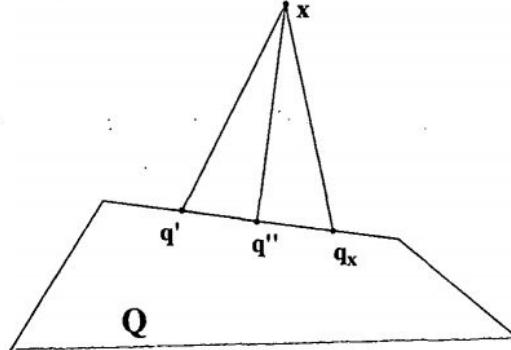


Figure 4.2: The closest point q_x

□

For future reference, q_x will be used to represent the closest point for each $x \in \mathbb{R}^n$, that is, the unique point q_x in Q such that $|x - q_x| < |x - q|$ for all $q \in Q$ and $q \neq q_x$.

Lemma 4.6. *If $x \in \mathbb{R}^n$ and $z \in Q$ where Q is a convex subset of \mathbb{R}^n , then $(z - q_x) \cdot (x - q_x) \leq 0$.*

Proof. Given $x \in \mathbb{R}^n$, $z \in Q$, and q_x as defined in proof of lemma 4.5, define $\phi : [0, 1] \rightarrow \mathbb{R}$ by $\phi(t) = |x - (q_x + t(z - q_x))|^2$. The function $\phi(t)$ is differentiable on $(0, 1)$ and differentiable from the right at $t = 0$. Then

$$\phi'(t) = 2(x - [q_x + t(z - q_x)]) \cdot (-z + q_x).$$

Since Q is a convex subset, $[q_x + t(z - q_x)] \in Q$. So the definition of q_x implies that ϕ is minimized at $t = 0$. Thus, $\phi'(0) \geq 0$. So

$$\begin{aligned}\phi'(0) &= 2(x - [q_x + 0(z - q_x)]) \cdot (-z + q_x) \\ &= 2(x - q_x) \cdot (-z + q_x) \\ &= -2(x - q_x) \cdot (z - q_x) \geq 0\end{aligned}$$

The last statement implies that

$$(x - q_x) \cdot (z - q_x) \leq 0.$$

□

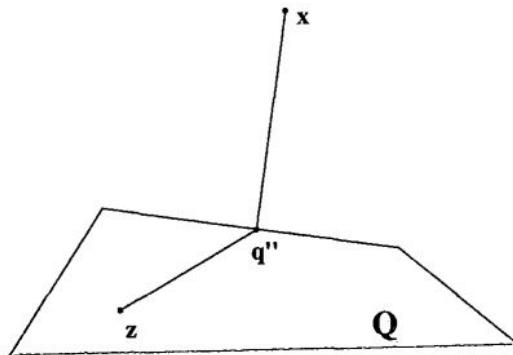


Figure 4.3: The points x , q_x , and z

From lemma 4.6, we now know that the angle formed by x , q_x , and any other point in Q will be obtuse. Using this idea, we can now show that there will be a shrinking of distance when two points in \mathbb{R}^n are replaced with their closest points. To be precise, we have the following lemma.

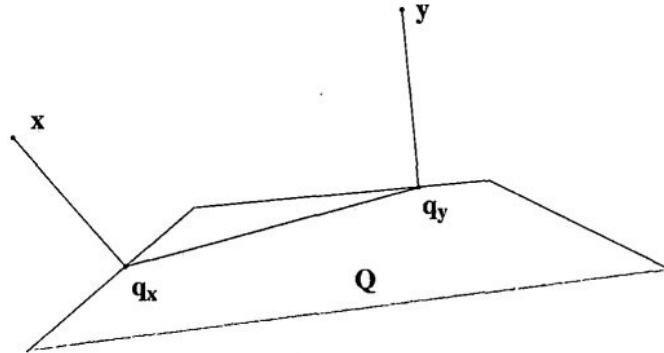
Lemma 4.7. *For all $x, y \in \mathbb{R}^n$, and q_x, q_y , the closest points in the compact, convex set Q , $|q_x - q_y| \leq |x - y|$.*

Proof. It follows from lemma 4.6 for $x \in \mathbb{R}^n$ with $z = q_y$, we get that

$$(q_y - q_x) \cdot (x - q_x) \leq 0$$

It also follows from lemma 4.6 for $y \in \mathbb{R}^n$ with $z = q_x$,

$$(q_x - q_y) \cdot (y - q_y) \leq 0$$

Figure 4.4: Points q_x and q_y

Adding the two inequalities, we get

$$\begin{aligned}
 0 &\geq (q_y - q_x) \cdot (x - q_x) + (q_x - q_y) \cdot (y - q_y) \\
 0 &\geq (q_y - q_x) \cdot [(x - q_x) + (q_y - y)] \\
 0 &\geq (q_y - q_x) \cdot [(q_y - q_x) + (x - y)] \\
 0 &\geq (q_y - q_x) \cdot (q_y - q_x) + (q_y - q_x) \cdot (x - y) \\
 0 &\geq |q_y - q_x|^2 + (q_y - q_x) \cdot (x - y)
 \end{aligned}$$

By the Schwarz inequality,

$$(y - x) \cdot (q_y - q_x) \leq |y - x| |q_y - q_x|$$

Therefore

$$|q_y - q_x|^2 \leq |y - x| |q_y - q_x|$$

If $q_x \neq q_y$, then $x \neq y$. Dividing by $|q_y - q_x|$, we get $|q_y - q_x| \leq |y - x|$. Otherwise, $q_x = q_y$ implies that $|q_y - q_x| = 0$ and since $|y - x|$ is always positive, $|q_y - q_x| \leq |y - x|$. \square

We have now done all of the groundwork for proving the generalized Brouwer fixed point theorem.

Theorem 4.8. *A compact, convex subset Q of \mathbb{R}^n has the fixed point property (fpp).*

Proof. Let Q be a convex, compact subset of \mathbb{R}^n . Define $\rho : B_r^n \rightarrow Q$ by $\rho(x) = q_x$. By lemma 4.7, ρ is continuous. By definition, ρ is a retraction since $\rho(q) = q$ for all $q \in Q$. Thus, Q is retract of B_r^n and has the fixed point property. \square

Chapter 5

Normed Linear Spaces and Fixed Point Theory

In order to prove the Cauchy-Peano theorem, we now need to discuss normed linear spaces having the fixed point property.

Definition 5.1. A *normed (real) linear space* is a linear space X along with a norm $\|\cdot\| : X \rightarrow \mathbb{R}$ which has the following properties:

1. $\|x\| \geq 0$ for all $x \in X$
2. $\|cx\| = |c|\|x\|$ for all real c and $x \in X$
3. $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ for all $x_1, x_2 \in X$

Definition 5.2. Let X be a real normed linear space and $F = \{x_1, x_2, \dots, x_n\}$ a finite subset of X . The *convex hull* of F , $con(F)$, is defined as $con(F) = \{\sum_{j=1}^n t_j x_j : t_j \geq 0, \text{ and } \sum_{j=1}^n t_j = 1\}$.

The convex hull of F lies in a linear space called the span of F which is all of the points $x \in X$ where $x = \sum_{j=1}^n a_j x_j$ for all $a_j \in \mathbb{R}, x_j \in F$. So the $span(F)$ is a finite-dimensional normed linear space of at most dimension n . Then $span(F)$ is linearly

homeomorphic to a Euclidean space. Again, we will assume that this fact is true. The proof of this statement can be found in appendix C of Brown's book [Bro04].

Since $\text{con}(F)$ is a closed and bounded subset of a Euclidean space, $\text{con}(F)$ is compact. The following lemma gives another property of $\text{con}(F)$.

Lemma 5.3. *If $F = \{x_1, x_2, \dots, x_n\}$ is contained in a convex subset C of the normed linear space X , then $\text{con}(F)$ is contained in C . Thus, $\text{con}(F)$ is the intersection of all convex subsets of X containing F .*

Proof. Here, we will use proof by induction. Let C be a convex subset of a normed linear space X and let $F = \{x_1, x_2, \dots, x_n\}$ be contained in C . For the case $F = \{x_1\}$, then $\text{con}(F) = \{x_1\}$. Since x_1 is in C , then $\text{con}(F) \subseteq C$. Assume that when $F = \{x_1, x_2, \dots, x_{n-1}\}$, then $\text{con}(F) \subseteq C$. We need to show that when $F = \{x_1, x_2, \dots, x_n\}$ then $\text{con}(F) \subseteq C$.

Let C be a convex subset of X containing F and let $x = \sum_{j=1}^n t_j x_j \in \text{con}(F)$. We need to show that $x \in C$. There are two possibilities. The first possibility is if $t_n = 1$, then $x = x_n$. Since $x_n \in C$, then $x \in C$. Otherwise, we can rewrite x in the form

$$x = (1 - t_n) \left(\frac{t_1}{1 - t_n} x_1 + \dots + \frac{t_{n-1}}{1 - t_n} x_{n-1} \right) + t_n x_n = (1 - t_n)x^* + t_n x_n$$

Let $F' = \{x_1, x_2, \dots, x_{n-1}\}$. Since $x^* \in \text{con}(F')$, $x^* \in C$ by the induction hypothesis. Therefore, $x \in C$. Hence, all elements in $\text{con}(F)$ are contained in C and $\text{con}(F) \subseteq C$. \square

Now we come to the Schauder projection which shows that compact subsets of normed linear spaces can be mapped to the convex hull of a finite subset.

Lemma 5.4. *Let K be a compact subset of a normed linear space X , with metric d induced by the norm. Given $\varepsilon > 0$, there exists a finite subset F of X and a map $P : K \rightarrow \text{con}(F)$ called the Schauder projection, such that $d(P(x), x) < \varepsilon$ for all $x \in K$.*

Proof. Let $\varepsilon > 0$. Let $F = \{x_1, x_2, \dots, x_m\}$ be a finite ε -net for K . For a particular $i \in \{1, \dots, m\}$, define $\phi_i : K \rightarrow \mathbb{R}$ by

$$\phi_i(x) = \begin{cases} \varepsilon - d(x, x_i) & \text{if } x \in B_\varepsilon(x_i) \\ 0 & \text{otherwise} \end{cases}$$

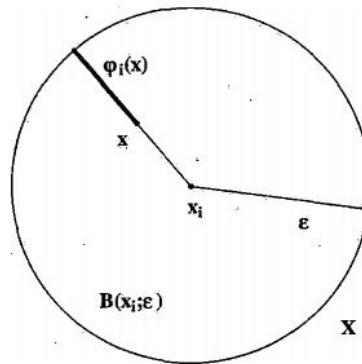


Figure 5.1: The map ϕ_i

Now define $\phi(x) = \sum_{i=1}^m \phi_i(x)$. Since F is an ε -net, then $\phi_i(x) \geq 0$ for all i and at least one $\phi_i(x) > 0$. Thus $\phi(x) > 0$ for all $x \in K$.

Define the Schauder projection by

$$P(x) = \sum_{i=1}^m \frac{\phi_i(x)}{\phi(x)} x_i.$$

Since all of the ϕ_i are continuous, the Schauder projection is also continuous. To show that the subset is almost finite dimensional, we will calculate the distance from the original x and the image of x under the Schauder projection.

$$\begin{aligned} &= \left| \sum_{i=1}^m \frac{\phi_i(x)}{\phi(x)} x_i - \sum_{i=1}^m \frac{\phi_i(x)}{\phi(x)} x \right| \\ &= \left| \sum_{i=1}^m \frac{\phi_i(x)}{\phi(x)} (x_i - x) \right| \\ &\leq \sum_{i=1}^m \frac{\phi_i(x)}{\phi(x)} |x_i - x| \end{aligned}$$

By definition of the ϕ_i 's, if then $\phi_i(x) = 0$. And so

$$d(P(x), x) < \sum_{i=1}^m \frac{\phi_i(x)}{\phi(x)} \varepsilon$$

Since $|x_i - x| < \varepsilon$, then $\phi_i(x) < \varepsilon$. Hence,

$$\sum_{i=1}^m \frac{\phi_i(x)}{\phi(x)} \varepsilon = \varepsilon.$$

□

In the generalized Brouwer Fixed Point Theorem, a compact, convex subset of a Euclidean space has the fixed point property. Also, remember that closed and bounded subsets of a Euclidean space are compact. However, in a more general normed linear space, closed and bounded subsets are not necessarily compact. Closed, bounded, and convex subsets of a normed linear space do not necessarily inherit the fixed point property. Consider Kakutani's example.

Kakutani's Example: *There is a closed, bounded convex subset C of a normed linear space X and a map $f : C \rightarrow C$ without fixed points.*

Let the normed linear space X be Hilbert space l^2 which is all infinite sequences of reals $x = \{x_1, x_2, \dots\}$ for which the series $\sum_{j=1}^{\infty} x_j^2$ converges. The space is a normed vector space with term by term addition and the natural scalar product. The norm is defined as

$$|x| = \sqrt{\sum_{j=0}^{\infty} x_j^2}$$

Consider the unit ball C in our space X . C consists of the set of the points x such that $|x| \leq 1$. The unit ball is a closed, bounded, and convex subset.

Define $f(x) = f(\{x_1, x_2, \dots\}) = \{\sqrt{1 - |x|^2}, x_1, x_2, \dots\}$

Calculating the norm of $f(x)$,

$$|f(x)| = \sqrt{(\sqrt{1 - |x|^2})^2 + x_1^2 + x_2^2 + \dots} = \sqrt{(1 - |x|^2) + |x|^2} = 1$$

Thus, f is injective. Actually, f maps C to the unit sphere in X , which is all sequences whose norm is exactly one. Since f can be written as a composition of functions that are continuous, then $f : C \rightarrow S \subset C$ is continuous. We now ask the question, does f have a fixed point?

Suppose that f has a fixed point. Then there exists $x' = \{x'_1, x'_2, \dots\}$ such that $f(x') = x'$. By definition of f , $|x'| = |f(x')| = 1$ and

$$\begin{aligned} f(x') &= f(\{x'_1, x'_2, \dots\}) = \{\sqrt{1 - |x'|^2}, x'_1, x'_2, \dots\} \\ &= \{0, x'_1, x'_2, \dots\} = x' = \{x'_1, x'_2, \dots\} \end{aligned}$$

This statement implies that $x'_1 = 0$. If this is true, then $x'_2 = 0$, and so on. Thus x' must be the zero sequence. Thus, $|x'| = 0 \neq |f(x')|$. So f does not have a fixed point. \square

Since f in Kakutani's example maps the unit ball to the unit sphere, the image of f is closed and bounded. Yet, the unit sphere is not compact. Consider the sequence $\{e_1, e_2, \dots\}$ where e_j consists of zeros except for a 1 in the j th place. Recall that every convergent sequence is Cauchy. So if a subsequence was convergent, then the distance between consecutive terms must be getting closer together. However, this sequence does not have a convergent subsequence since any two points are $\sqrt{2}$ distance apart by definition of the norm. Therefore, there is no convergent subsequence. Since there is no convergent subsequence, the unit sphere is not compact [Bro04]. We want the image of the map to be compact. Schauder built compactness into his maps so that the image would be compact. Here is Schauder's Fixed Point Theorem.

Definition 5.5. A map $f : X \rightarrow Y$ is a *compact map* if the image $f(x)$ is a relatively compact subset of Y .

Theorem 5.6. Let C be a closed convex subset of a normed linear space and let $f : C \rightarrow C$ be a compact map, then f has a fixed point.

Proof. Let K denote the closure of $f(C)$. Since K is a compact set, then $f(C)$ is relatively compact. For each natural number n , let F_n be a finite $\frac{1}{n}$ -net for K and let $P_n : K \rightarrow \text{con}(F_n)$ be the Schauder projection. F_n is a subset of K which is in C since C is closed. By the lemma 5.4 and the fact that C is convex, $\text{con}(F) \subseteq C$. Define $f_n : \text{con}(F_n) \rightarrow \text{con}(F_n)$ by restricting f to $\text{con}(F_n)$ and composing with P_n . That is $f_n(x) = P_n(f(x))$ for $x \in \text{con}(F)$.

Recall that $\text{con}(F_n)$ is contained in the span of F_n which is homeomorphic to Euclidean space \mathbb{R}^n . Thus, we can use the generalized Brouwer fixed point theorem to state that each f_n has a fixed point. Choose one of the fixed points for each f_n and call it y_n . Since K is compact, the sequence $\{f(y_n)\}$ has a convergent subsequence which we will also refer to as $\{f(y_n)\}$ for ease of notation. Since the subsequence is convergent, call the limit of the subsequence y . Note that $y \in C$ since C is a closed set.

Now we claim that y is a fixed point of the map f . Using the approximation property of the Schauder projection, $d(P_n(x), x) < \frac{1}{n}$. When $x = f(y_n)$, then $d(f_n(y_n), f(y_n)) < \frac{1}{n}$. So the sequence $\{f_n(y_n)\} = \{y_n\}$ must converge to the same point y . Since $\{y_n\}$ converges to y and f is continuous, $\{f(y_n)\}$ converges to $f(y)$. Since convergent sequences have only one limit, $f(y) = y$. \square

As a result of this theorem, we finally reach the very generalized Brouwer Fixed Point Theorem. This theorem will be key in proving the Cauchy-Peano Existence Theorem topologically.

Theorem 5.7. *A compact, convex subset of a normed linear space has the fixed point property.*

Proof. Let C be a compact, convex subset of a normed linear space. Any map on a compact domain is compact, so by the Schauder fixed point theorem, C has the fixed point property. \square

Chapter 6

Cauchy-Peano Existence Theorem

We are now ready to prove the Cauchy-Peano Existence theorem in two ways, using topology and then analysis.

Theorem 6.1. *Given a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is continuous in a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$, there exists $\alpha > 0$ and a continuous function $\phi : [x_0 - \alpha, x_0 + \alpha] \rightarrow \mathbb{R}$ such that $\phi(x_0) = y_0$ and $\phi'(x) = f(x, \phi(x))$ for all x in the interval.*

Proof. Let the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous in the neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$. We need to show that there exists $\alpha > 0$ and a solution to the initial value problem $\phi' = f(x, \phi(x))$, $\phi(x_0) = y_0$ on the interval $[x_0 - \alpha, x_0 + \alpha]$.

Since f is continuous in a neighborhood of $(x_0, y_0) \in \mathbb{R}^2$, then there exists $a > 0$ such that if $(x, y) \in \mathbb{R}^2$ with $|x - x_0| \leq a$ and $|y - y_0| \leq a$, then f is continuous at (x, y) . Let Q be the square in the plane defined by

$$Q = \{(x, y) \in \mathbb{R}^2 : |x - x_0| \leq a \text{ and } |y - y_0| \leq a\}.$$

Now let us choose $M > 1$ such that $M \geq |f(x, y)|$ for all $x, y \in Q$. We know that the set of values $|f(x, y)|$ is closed and bounded since Q is a compact set and f is a continuous map. Thus M is an upper bound for $f(Q)$. Let $\alpha = \frac{a}{M}$.

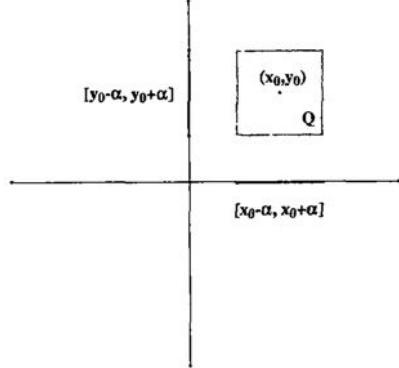


Figure 6.1: The region Q

Let C be the set of all real valued functions that are continuous on the closed interval $[x_0 - \alpha, x_0 + \alpha]$. The set C inherits a linear space structure from the reals by defining for $u, v \in C$ and $r \in \mathbb{R}$, the sum $(u + v)(x) = u(x) + v(x)$ and the scalar product $ru(x) = u(rx)$. Consider the subset A of C defined as:

1. $|u(x) - y_0| \leq a$ for all $x \in [x_0 - \alpha, x_0 + \alpha]$
2. $|u(x_1) - u(x_2)| \leq M|x_1 - x_2|$ for all $x_1, x_2 \in [x_0 - \alpha, x_0 + \alpha]$

A is a convex subset of C . That is, if $u, v \in A$ and $0 \leq t \leq 1$, then $tu + (1-t)v \in A$. We define the norm $\|u\|$ of a function u in A by $\|u\| = \max\{|u(x)| : x_0 - \alpha \leq x \leq x_0 + \alpha\}$. This norm induces the topology on C with metric d defined as $d(u, v) = \|u - v\|$. Thus, C is a normed linear space.

From condition (1) of the definition of set A , we have a bound $\beta = \max(|y_0 + a|, |y_0 - a|)$ for all $u \in A$. From condition (2), we obtain the condition of equicontinuity when we choose $\delta_x = \frac{\varepsilon}{M}$ for all $x \in [x_0 - \alpha, x_0 + \alpha]$ and a given $\varepsilon > 0$. Since A is a bounded, equicontinuous subset of C , we can use the Ascoli-Arzela theorem to conclude that A is relatively compact.

We know that A is a closed subset of C . Let $\{u_n\}$ be a sequence in A that converges

uniformly to $g \in C$. That is, given $\varepsilon > 0$, there exists $N > 0$ such that $|u_n - g(x)| < \varepsilon$ for $n > N$ and all $x \in [x_0 - \alpha, x_0 + \alpha]$. Using the triangle inequality and condition (1),

$$|g(x) - y_0| \leq |g(x) - u_n(x)| + |u_n(x) - y_0| \leq \varepsilon + a.$$

Since ε is arbitrary, it follows that $|g(x) - y_0| \leq a$. Using the triangle inequality again and condition (2) of set A , for all $x_1, x_2 \in [x_0 - \alpha, x_0 + \alpha]$,

$$|g(x_1) - g(x_2)| \leq |g(x_1) - u_n(x_1)| + |u_n(x_1) - u_n(x_2)| + |g(x_2) - u_n(x_2)| \leq \varepsilon + M|x_1 - x_2| + \varepsilon$$

Again, since ε is infinitely small, $|g(x_1) - g(x_2)| \leq M|x_1 - x_2|$. Thus, the limit function g of the sequence satisfies the conditions of A and $g \in A$. Hence, A contains its limit functions and is therefore closed. Since A is closed and relatively compact, A is compact.

Now, we define a function $T : C \rightarrow C$ by $Tu(x) = y_0 + \int_{x_0}^x f(t, u(t))dt$ for $u \in C$.

We claim that the map T has a fixed point ϕ in C . In theorem 5.7, we proved that a convex, compact subset of a normed linear space has the fixed point property. Since the function f is continuous on Q , T is continuous on C and therefore on A . By definition of T and the conditions on the compact set A , the image of T is contained in A . Hence, by theorem 5.6, the map T has a fixed point, that is $T\phi = \phi$.

Since $T\phi = \phi$, we have $\phi(x) = y_0 + \int_{x_0}^x f(t, \phi(t))dt$. By definition of f , f is continuous on the closed square Q and $\phi(x)$ is an antiderivative of f . We can use the fundamental theorem of calculus to rewrite the equation to get $\phi'(x) = f(x, \phi(x))$. Also, ϕ satisfies the initial condition $\phi(x_0) = y_0$. Thus, we have our desired function ϕ and we are done. \square

Here, we will examine a brief sketch of the approximation proof of the same theorem.

Proof. Again, we will choose the values a , M , and α as we did in the topological proof. For each integer $n \geq 1$, choose $\delta_n > 0$ small enough so that $|x - \bar{x}| < \delta_n$ and $|y - \bar{y}| < \delta_n$ implies

$|f(x, y) - f(\bar{x}, \bar{y})| < \frac{1}{n}$. Then we will choose the points $x_0 - \alpha = x_{-k_n}^{(n)} < x_{-k_n+1}^{(n)} < \dots < x_{-1}^{(n)} < x_0 < x_1^{(n)} < \dots < x_{k_n-1}^{(n)} < x_{k_n}^{(n)} = x_0 + \alpha$ to create intervals of equal length such that $|x_{j+1}^{(n)} - x_j^{(n)}| \leq \frac{\delta_n}{M}$. Now we can define a piecewise, linear function $\phi_n : [x_0 - \alpha, x_0 + \alpha] \rightarrow \mathbb{R}$ as follows.

On the interval of $[x_0, x_1^{(n)}]$, set $\phi_n(x_0) = y_0$ and set the slope of the line equal to $f(x_0, y_0)$. Then on the interval of $[x_1^{(n)}, x_2^{(n)}]$, the slope of the line will be $f(x_1^{(n)}, y_1^{(n)})$, where $y_1^{(n)} = \phi_n(x_1^{(n)})$. We can continue to define ϕ_n in this manner moving to the right until $x_{k'_n}^{(n)} = x_0 + \alpha$.

Then on the interval $[x_{-1}^{(n)}, x_0]$, set $\phi_n(x_{-1}^{(n)}) = y_{-1}^{(n)}$ and set the slope of the line equal to $f(x_{-1}^{(n)}, y_{-1}^{(n)})$. On the interval $[x_{-2}^{(n)}, x_{-1}^{(n)}]$, the slope of the line will be $f(x_{-2}^{(n)}, y_{-2}^{(n)})$, where $y_{-2}^{(n)} = \phi_n(x_{-1}^{(n)})$. We can continue to define ϕ_n in this manner moving to the left until $x_{-k'_n}^{(n)} = x_0 - \alpha$. The function $\phi_n(t)$ is differentiable for all $t \neq x_j^{(n)}$.

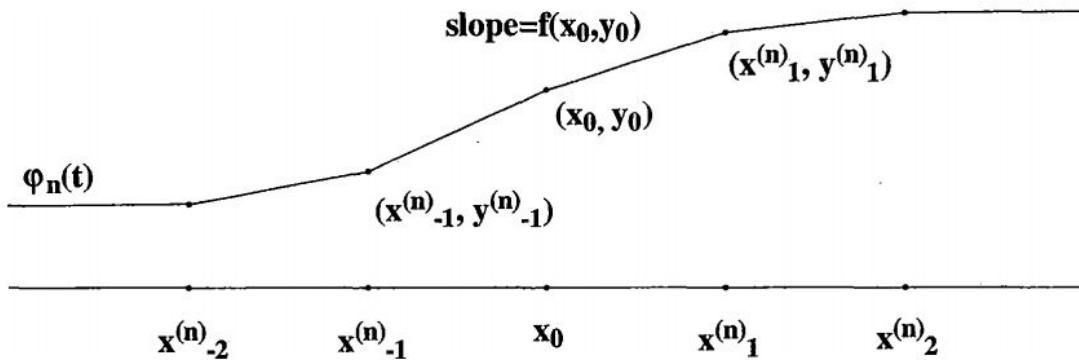


Figure 6.2: The piecewise linear function ϕ_n

The sequence of functions ϕ_n can be shown to be equicontinuous and bounded. Then, by the Ascoli-Arzela theorem, the sequence ϕ_n contains a subsequence that converges uniformly on the interval $[x_0 - \alpha, x_0 + \alpha]$. The limit of the subsequence is a continuous

function and we denote it ϕ . Rewriting the ϕ_n function, we get

$$\phi_n(x) = y_0 + \int_{x_0}^x f(t, \phi_n(t)) + \Delta_n(t) dt$$

with

$$\Delta_n = \begin{cases} \phi'_n(t) - f(t, \phi_n(t)), & \text{if } t \neq x_j^{(n)} \\ 0, & \text{if } t = x_j^{(n)} \end{cases}$$

Since each interval length is less than $\frac{1}{n}$, then $|\Delta_n(t)| < \frac{1}{n}$. Again, we can use the fundamental theorem of calculus to conclude that ϕ is our desired solution to the initial value problem. \square

Chapter 7

Applications

7.1 Picard's Theorem

The Cauchy-Peano Existence Theorem is helpful in determining if a solution exists, however it fails to show uniqueness of the solution. For those using differential equations to make predictions about a system, uniqueness is key to making correct predictions [BDH02]. A similar theorem that includes the condition of uniqueness of the solution is Picard's Theorem by Charles Emile Picard (1856-1941). Picard's Theorem can be proven for a first order differential equation and a higher order differential equation [TP63]. We will work through the theorem for the case of a first order differential equation. Before we begin, we need the following definition [Cro80].

Definition 7.1. A function $f(x, y)$ satisfies the *Lipschitz condition* with respect to y if there is a constant k so that

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$$

for all (x, y_1) and (x, y_2) in the domain of f .

Also, in the proof we will use the following definition of a contraction mapping [DD02].

Definition 7.2. Let X be a subset of a normed vector space. A map $T : X \rightarrow X$ is called a *contraction* on X if there is a positive constant $c < 1$ so that

$$|Tx - Ty| \leq c|x - y|$$

for all $x, y \in X$. That is, T is Lipschitz with constant $c < 1$.

Theorem 7.3. Given a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is defined and continuous in a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ and satisfies the Lipschitz condition, there exists $\alpha > 0$ and a unique continuous function $\phi : [x_0 - \alpha, x_0 + \alpha] \rightarrow \mathbb{R}$ such that $\phi(x_0) = y_0$ and $\phi' = f(x, \phi(x))$ for all $x \in [x_0 - \alpha, x_0 + \alpha]$.

Proof. The proof for the existence of α and ϕ follows the same argument in the Cauchy-Peano theorem. However, we will be more particular about choosing a and M . Let $k > 0$ be the Lipschitz constant for f on Q and choose a small enough so that $ka < 1$. Rather than setting M to be an upper bound for the values of $|f(x, y)|$, we will set $M = \sup\{|f(x, y)| : (x, y) \in Q\}$. Thus, M is the least upper bound of the values of $|f(x, y)|$.

Recall from our proof of the Cauchy-Peano theorem that we defined $T : C \rightarrow C$ by $Tu(x) = y_0 + \int_{x_0}^x f(t, u(t))dt$ for $u \in C$. Since f satisfies the Lipschitz condition with a Lipschitz constant $k > 0$ on Q , then for $u_1, u_2 \in A$

$$\begin{aligned} |Tu_1(x) - Tu_2(x)| &\leq \int_{x_0}^x |f(t, u_1(t)) - f(t, u_2(t))|dt \\ &\leq k \int_{x_0}^x |u_1(t) - u_2(t)|dt \\ &\leq k \left| \int_{x_0}^x |u_1 - u_2| dt \right| \\ &= k|u_1 - u_2| \left| \int_{x_0}^x dt \right| \\ &= k|u_1 - u_2||x - x_0| \\ &\leq ka|u_1 - u_2| \end{aligned}$$

Since $ka < 1$, by definition of a , T is a contraction mapping on A .

In the proof of the Cauchy-Peano theorem, we found ϕ to be a fixed point of T . That is, $T\phi = \phi$. Suppose that ω is also a fixed point of T and $T\omega = \omega$. Then

$$|\phi - \omega| = |T\phi - T\omega| \leq ka|\phi - \omega|$$

Since $ka < 1$, then $|\phi - \omega| = 0$. Thus, $\phi = \omega$ and ϕ is the unique fixed point of T . \square

7.2 An Example of non-Uniqueness

To demonstrate the difference of the Cauchy-Peano theorem and Picard's theorem, consider the following differential equation [BDH02].

$$y' = y^{\frac{2}{3}}, \quad (0) = 0, \quad 0 \leq x \leq 2$$

The function $f(x, y) = y^{\frac{2}{3}}$ does not satisfy the Lipschitz condition. Thus, we cannot use Picard's theorem to guarantee the uniqueness of the solution. However, we can use the Cauchy-Peano theorem to guarantee the existence of a solution or solutions. We continue working on the problem by separating the variables

$$y^{-\frac{2}{3}}y' = 1$$

and integrating

$$3y^{\frac{1}{3}} = x + C.$$

Since $y(0) = 0$, then $C = 0$. Thus, we have

$$y = \frac{x^3}{27}.$$

However, the solution $y = 0$ also works for our problem. Thus, we have two solutions for our differential equation.

Chapter 8

Conclusion

Thus, we have completed our journey through nonlinear analysis by the means of topology. We began in the familiar world of complete metric spaces and Euclidean space. We proved the Ascoli-Arzela theorem which was key to both proofs of the Cauchy-Peano Existence Theorem. We were able to extend the notion of the fixed point property of a subset of a normed linear space with the conditions that the subset be convex and compact. We tied the ideas of complete metric spaces and fixed point theory together in the proof of the Cauchy-Peano Existence Theorem.

In Picard's theorem, we guaranteed the uniqueness of the solution by adding the Lipschitz condition. However, Picard's theorem and the Cauchy-Peano Existence Theorem are not specific about the size of the interval for solutions to exist. The next question to study is what is the largest interval on which a unique solution exists for the differential equation. The name for this concept of expanding an interval to its largest size while guaranteeing a unique solution is continuation. Could the size of the interval be determined and proven? Could the size be proven topologically or only by approximation? What other ideas in nonlinear analysis could be proven using topology?

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