DMS625: Introduction to stochastic processes and their applications

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Continuous-time Markov chain

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1 Introduction

So far we have discussed Discrete-time Markov chains (DTMC) with discrete state spaces. We will now consider a generalization where time will become continuous, however the state space will continue to stay discrete.

We say $X(t), t \in [0, \infty)$, is a continuous-time Markov chain (CTMC) if it satisfies the **Markov property** given by,

$$\mathbb{P}(X(t+s) = x_{t+s}|X(s) = x_s, X(u) = x_u, 0 \leq u < s) = \mathbb{P}(X(t+s) = x_{t+s}|X(s) = x_s)$$

Here $x_i \in \mathcal{S}, 0 \leq i \leq t+s$, are the values $X(t)$ attains on the state space $\mathcal{S}$ at each time. This definition of the Markov property is similar to the one for DTMC that we considered earlier. Since in a CTMC time is continuous, this has been accounted for here in the conditioning event on the LHS. The advantages of CTMC are obvious in that now we can model continuously over time. Let us look at an example.

Example 1.1 (Poisson Process). *The Poisson process $N(t)$ with rate λ is a CTMC. We need to show that $N(t)$ satisfies the Markov property,*

$$\begin{aligned}
&\mathbb{P}(N(t+s) = x_{t+s}|N(s) = x_s, N(u) = x_u, 0 \leq u < s)\\
=&\mathbb{P}(N(t+s) = x_{t+s}|N(s) = x_s, N(t_k) = x_{t_k}, N(t_{k-1}) = x_{t_{k-1}}, \ldots, N(t_1) = x_{t_1}, N(0) = 0)\\
&\textit{(For an arbitrary } k\textit{, consider any sequence of times such that } 0 < t_1 < t_2 < \ldots < t_k < s\textit{)}\\
=&\mathbb{P}(N(t+s) - N(s) = x_{t+s} - x_s|N(s) - N(t_k) = x_s - x_{t_k}, N(t_k) - N(t_{k-1}) = x_{t_k} - x_{t_{k-1}}, \ldots, N(t_1) - N(0) = x_{t_1} - 0)\\
=&\mathbb{P}(N(t+s) - N(s) = x_{t+s} - x_s)\\
&\textit{(Follows from the independent increments property)}\\
=&\mathbb{P}(N(t+s) = x_{t+s}|N(s) = x_s)
\end{aligned}$$

Therefore, $N(t)$ is a CTMC.

Example 1.2 (Non-homogeneous Poisson process). *Consider a Non-homogeneous Poisson process N(t) with rate $\lambda(t)$. One can show that it satisfies the Markov property as above. We note that to show the Poisson process satisfied the Markov property above, we only required the independent increments property. Since the non-homogeneous Poisson process possesses the independent increments property, the proof will follow similarly here.*

We say $X(t)$ is a **time-homogeneous** CTMC, if,

$$\mathbb{P}(X(t+s) = x_{t+s}|X(s) = x_s) = \mathbb{P}(X(t) = x_{t+s}|X(0) = x_s)$$

This is similar to the time-homogeneity condition for DTMC, where we say the probability of going from one state to another only depends on the difference of the times of their transition.

Exercises

1. Verify that the Poisson process $N(t)$ with rate λ is a time-homogeneous CTMC.

2. Verify that the non-homogeneous Poisson process with rate $\lambda(t)$ need not be a time-homogeneous CTMC.

Remark 1. *All CTMCs henceforth will be assumed to be time-homogeneous CTMCs unless otherwise stated.*

2 Time spent in a state

Let T_i denote the time spent by a CTMC in state i. T_i is also referred to as the *sojourn time*. Consider a Poisson process $N(t)$ that is modelling the number of Earthquakes. Let's say the first Earthquake occurred on year $t = 1$, and the second earthquake occurred on year $t = 3$. Then the time spent by Poisson process on state $i = 1$ is $T_i = 3 - 1 = 2$ years. We want to understand the distribution of T_i for CTMCs in general.

$$\begin{aligned}
\mathbb{P}(T_i > t+s|T_i > s) &= \mathbb{P}(X(v) = i, v \in (s, s+t]|X(u) = i, i \in [0, s]) \\
&= \mathbb{P}(X(v) = i, v \in (0, t]|X(0) = i) \\
&\text{(Follows from time-homogeneity)} \\
&= \mathbb{P}(T_i > t)
\end{aligned}$$

Therefore, T_i follows the memoryless property and hence T_i is exponentially distributed. Similarly consider,

$$\begin{aligned}
\mathbb{P}(T_i > t_i, T_j > t_j) &= \mathbb{P}(T_j > t_j|T_i > t_i)\mathbb{P}(T_i > t_i) \\
&\text{(Without loss of generality assume that the chain visits } i \text{ before } j\text{)} \\
&= \mathbb{P}(X(v) = j, v \in (s, s+t_j]|X(s) = j, X(u) = i, u \in (r, r+t_i])\mathbb{P}(T_i > t_i) \\
&= \mathbb{P}(X(v) = j, v \in (s, s+t_j]|X(s) = j)\mathbb{P}(T_i > t_i) \\
&\text{(Markov property)} \\
&= \mathbb{P}(T_j > t_j)\mathbb{P}(T_i > t_i)
\end{aligned}$$

We have shown that T_i, T_j are also independent. We can therefore claim the following result.

Theorem 2.1. *In a CTMC, T_i's are independent and exponentially distributed, $\forall i \in \mathcal{S}$.*

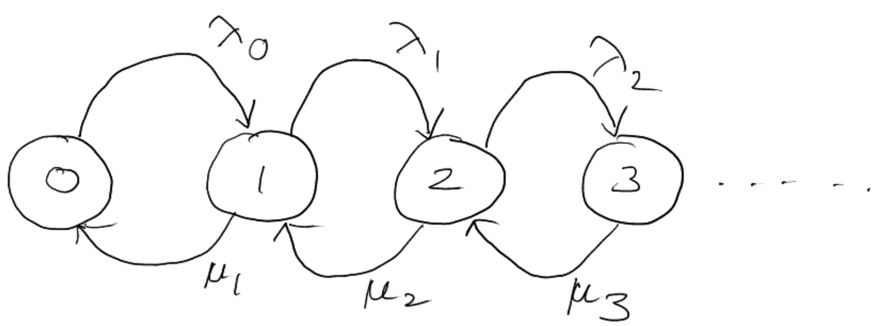
Remark 2. *Note that even though T_i's are exponentially distributed, the rate of the exponential distribution could vary across i. We will see such examples next.*

3 Birth and Death process

We will now look at a class of processes known as the *Birth and Death processes*. These are continuous-time generalisations of the Birth and Death chains we looked at earlier during our study of Markov chains. The state space of the Birth and Death process is $\mathcal{S} = \{0, 1, 2, \ldots\}$.

This means that if the CTMC is at state n, it may either move to state $n-1$ or state $n+1$ with some probability for each, as in the case of Birth and Death chains.

Suppose the CTMC is at state n, then it could move to state $n+1$ after a time $B_n \sim \text{Exp}(\lambda_n)$, i.e, another unit will be *born* after a random time B_n which is exponentially distributed with rate λ_n. Alternatively, it could move to state $n-1$ after a time $D_n \sim \text{Exp}(\mu_n)$, i.e., a unit will *die* after a random time D_n which is exponentially distributed with rate μ_n. We shall assume that the time for a birth and death at state n are independent, i.e., B_n and D_n are independent.



Let's answer a few questions about this system,

- What is the distribution of the time spent by the chain at state n?

  Let us denote this as T_n (time spent in a state, that we introduced above). There could be a birth or a death, therefore the time spent in state n is the time until either a birth or death happens. This is nothing but,

  $$T_n = \min(B_n, D_n)$$

  since B_n, D_n are independent exponential random variables, it follows that $T_n \sim \text{Exp}(\lambda_n + \mu_n)$ (see the section on Exponential distribution, in the Poisson process notes).

- What is the probability that the chain moves to state $n+1$?

  Let us denote the probability of moving to state $n+1$ from state n as $p_{n,n+1}$. If the chain moves to state $n+1$ from n, this implies that a birth happened before death. Therefore,

  $$p_{n,n+1} = \mathbb{P}(B_n < D_n) = \frac{\lambda_n}{\lambda_n + \mu_n}$$

  By similar logic,

  $$p_{n,n-1} = \mathbb{P}(D_n < B_n) = \frac{\mu_n}{\lambda_n + \mu_n}$$

Suppose you record observations from a Birth and Death process such as the one we described above. Each time a transition happens we record the state, say one such realization could be

$0, 1, 2, 1, 2, 3, 2$, where the chain started at 0, followed by a birth to 1, then 2, then a death to 1, followed by births to 2 and 3, and again death to 2. The chain naturally would have spent some time at each of these states but we choose to ignore these times for the moment. Then the transitions from each state to another is called the **embedded Markov chain** with transition matrix for this embedded Markov chain given by $p_{i,j}$ as defined above.

Remark 3. *The idea of the embedded Markov chain can be applied to CTMCs in general. What is implied is that each CTMC can be thought of as a Markov chain (the embedded Markov chain) along with some random time for transition between the states.*

So where do these Birth and Death processes arise?

Example 3.1 (M/M/1 Queue)**.** *Consider a shop with a clerk and customers line up at the shop to get their requirements serviced. Customers arrive at a rate of λ_n and customers are serviced by the clerk at a rate of μ_n. Here n denotes the number of customers in the system, i.e., the customer getting serviced and those waiting in the queue to get serviced.*

Note here that the arrival rate, λ_n, is state dependent (depends on n). This in application means that if a customer were to see a queue with 100 people, they would be less likely to join it, than if there were 5 people in the queue (given that customers generally prefer to wait less). A similar interpretation can be made for the service rate being state dependent.

This is the simplest kind of queue, referred to in the queuing notation (also referred to as the Kendall notation) as M/M/1 queue. The first "M" denotes that the arrival rate has the Memoryless or Markovian property. The second "M" denotes that the service rate has the Memoryless or Markovian property. The third "1" denotes the number of servers in the system. Several other choices and parameters can be used to describe more general queues, but we shall not cover them in this course.

Example 3.2 (M/M/s Queue)**.** *A simple extension of the M/M/1 queue is one with s servers instead of just 1. Customers arrive at rate λ_n. Suppose each of the s servers can service with rate μ. Then convince yourself that,*

$$\mu_n = \begin{cases} n\mu & 1 \le n \le s \\ s\mu & n > s \end{cases}$$

Calculate $p_{n,n+1}$ and $p_{n,n-1}$.

Example 3.3 (Pure Birth process)**.** *Suppose $\mu_n = 0, \forall n$. Then this is called a pure birth process.*

The Birth and Death processes are a general framework for modelling in continuous-time. They can be applied to model queues as considered above, stock prices, limit order books, epidemics and many more phenomena.

Let T_i^{i+1} denote the time taken by the Birth and Death process to go from state i to $i + 1$. We want to understand $\mathbb{E}[T_i^{i+1}]$. The idea behind the subsequent calculations is the following. There are two possible cases,

- There is a direct transition from state i to $i + 1$.
- There is a transition first from i to $i - 1$, in which case the CTMC would first have to return to i and then go to $i + 1$.

Define a random variable I_i,

$$I_i = \begin{cases} 1 & i \to i+1 \\ 0 & i \to i-1 \end{cases}$$

I_i takes the value 1, if the chain first moves from i to $i+1$, and 0 if it moves to $i-1$. Note also that $\mathbb{P}(I_i = 1) = p_{i,i+1}$ and $\mathbb{P}(I_i = 0) = p_{i,i-1}$. Therefore,

- $\mathbb{E}[T_i^{i+1}|I_i = 1] = \mathbb{E}[T_i] = \frac{1}{\lambda_i+\mu_i}$
- $\mathbb{E}[T_i^{i+1}|I_i = 0] = \mathbb{E}[T_i] + \mathbb{E}[T_{i-1}^i] + \mathbb{E}[T_i^{i+1}] = \frac{1}{\lambda_i+\mu_i} + \mathbb{E}[T_{i-1}^i] + \mathbb{E}[T_i^{i+1}]$

And,

$$\mathbb{E}[T_i^{i+1}] = \mathbb{P}(I_i = 1)\mathbb{E}[T_i^{i+1}|I_i = 1] + \mathbb{P}(I_i = 0)\mathbb{E}[T_i^{i+1}|I_i = 0]$$

Also, note that $\mathbb{E}[T_0^1] = \frac{1}{\lambda_0}$.
Therefore, you can show that (Verify!) for $\lambda_n = \lambda$ and $\mu_n = \mu\ \forall n$,

$$\mathbb{E}[T_i^{i+1}] = \begin{cases} \frac{1-(\mu/\lambda)^{i+1}}{\lambda-\mu} & \lambda \neq \mu \\ \frac{(i+1)}{\lambda} & \lambda = \mu \end{cases}$$

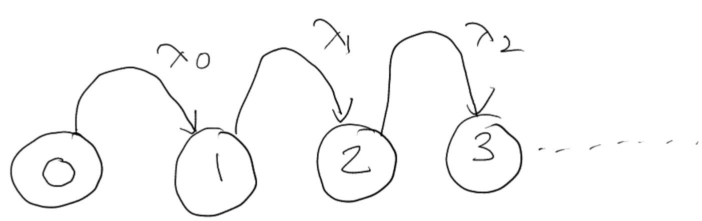
Transition Probability Function

The **transition probability function** of a CTMC is defined as,

$$P_{i,j}(t) = \mathbb{P}(X(t+s) = j|X(s) = i)$$

Note that $\sum_j P_{ij}(t) = 1$.

The transition probability function can be thought of as the analogous quantity for a CTMC to the $n-$step transition probability function, $\mathbf{P}^{(n)}(i, j)$ that we came across while studying Markov chains.



Consider a pure birth process defined as above. If $\lambda_1 = \lambda_2 = \ldots = \lambda$, then the pure birth process is a Poisson process (Verify!).

The transition probability function if $X(t)$ is a Poisson process can be calculated,

$$\begin{aligned}\mathbb{P}(X(t+s) = j|X(s) = i) &= \mathbb{P}(X(t+s) - X(s) = j - i|X(s) = i) \\ &= \mathbb{P}(X(t+s) - X(s) = j - i) \\ &\text{(From the independent increments property)} \\ &= \frac{e^{-\lambda t}(\lambda t)^{(j-i)}}{(j-i)!}\end{aligned}$$

Now let us consider the general case of a pure birth process, where $\lambda_i's$ need not all be identical. It is easy to see that for a pure birth process,

$$T_i \sim \text{Exp}(\lambda_i)$$

We now want to understand the transition probability function of a pure birth process.

$$\begin{aligned}
&P_{i,j}(t) \\
=&\mathbb{P}(X(t+s) = j | X(s) = i) \\
=&\mathbb{P}(X(t+s) < j+1 | X(s) = i) - \mathbb{P}(X(t+s) < j | X(s) = i) \\
=&\mathbb{P}(T_i + T_{i+1} + \ldots + T_j > t) - \mathbb{P}(T_i + T_i i + 1 + \ldots + T_{j-1} > t)
\end{aligned}$$

We know T_i's are independent, but are not identically distributed since λ_i's could be different. The sum of exponential distribution with non-identical parameters follows what is called as the Hypo-exponential distribution. We shall consider a more general approach towards studying transition probability function and return to this question for the pure birth process.

Remark 4. *Note that $p_{i,j}$ that we discussed earlier is the probability of the CTMC going to j from i before any other state. However $P_{ij}(t)$ is the probability of going from i to j in some time t. These both are different probabilities, don't confuse them.*

Chapman-Kolmogorov Equation

Just like we had the Chapman-Kolmogorov equation for the Markov chain, we have an analogous result for CTMCs.

Proposition 3.1. $P_{i,j}(t+s) = \sum_{k=0}^{\infty} P_{i,k}(t) P_{k,j}(s)$

Proof. Left as an exercise. The proof follows similarly to that of Chapman-Kolmogorov equation for the Markov chain. □

Recall that if we wanted to calculate the probability of going from state i to j in n steps, i.e., $\mathbf{P}^{(n)}(i,j)$, we would multiply the transition matrix $\mathbf{P}(i,j)$ to itself n times and consider its $(i,j)-th$ entry. This result was a consequence of the Chapman-Kolmogorov equation for Markov chain.

For CTMCs, unfortunately there is no such simple way for evaluating $P_{i,j}(t)$. However, like the Markov chain we will use the Chapman-Kolmogorov equation to study the transition probability function for CTMCs.

Instantaneous transition rates

We define the **instantaneous transition rate** of a CTMC as,

$$q_{i,j} = v_i p_{i,j}$$

Here v_i is the rate parameter of the distribution of T_i, i.e.,

$$T_i \sim \text{Exp}(v_i)$$

It will be shortly clear why $q_{i,j}$ is referred as the *instantaneous* transition rate.

Example 3.4. *1. Poisson process*

$T_i \sim Exp(\lambda)$, therefore, $v_i = \lambda_i$ and $p_{i,i+1} = 1$ (Why?). Therefore,

$$q_{i,i+1} = \lambda$$

Find $q_{i,i-1}$

2. *Pure birth process*

   Find $q_{i,i+1}$ and $q_{i,j}$, where $j > i + 1$.

3. *Birth and Death process*

   $T_i \sim Exp(\lambda_i + \mu_i)$, therefore, $v_i = \lambda_i + \mu_i$ and $p_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}$. Therefore,

   $$q_{i,i+1} = \lambda_i$$

   Find $q_{i,i-1}$.

Lemma 3.1.

$$\lim_{h \to 0} \frac{1 - P_{i,i}(h)}{h} = v_i$$

Proof. Assume that h is small, such that after time h, the CTMC which starts at state i continues to be in state i.

Then,

$$1 - P_{i,i}(h) = \mathbb{P}(T_i < h) = \int_0^h v_i e^{-v_i t} dt = 1 - e^{-v_i h}$$

It follows that $1 - P_{i,i}(h) = v_i h + o(h)$ (Verify!).

Therefore,

$$\lim_{h \to 0} \frac{1 - P_{i,i}(h)}{h} = \lim_{h \to 0} \frac{v_i h + o(h)}{h} = v_i$$

□

and the following Lemma is why we refer to $q_{i,j}$ as the instantaneous transition rate.

Lemma 3.2.

$$\lim_{h \to 0} \frac{P_{i,j}(h)}{h} = q_{i,j}$$

Proof. Again assume that h is small, such that you reach j from i in one jump in some time h and stay at j.

$$P_{i,j}(h) = \mathbb{P}(X(h) = j | X(0) = i) = p_{i,j} \mathbb{P}(T_i < h)$$

This therefore becomes the probability of making the jump ($p_{i,j}$) multiplied by the probability that the jump happens in some time less than h ($\mathbb{P}(T_i < h)$). Hence,

$$\begin{aligned} P_{i,j}(h) &= p_{i,j} \mathbb{P}(T_i < h) \\ &= p_{i,j}(v_i h + o(h)) \\ \text{(See the previous proof)} \qquad &= q_{i,j} h + o(h) \end{aligned}$$

Therefore,

$$\lim_{h \to 0} \frac{P_{i,j}(h)}{h} = q_{i,j}$$

□

Kolmogorov Backward Equation

We will now establish an important tool for computing the transition probability function.

Theorem 3.3. *The Kolmogorov Backward equation for a CTMC is given by,*

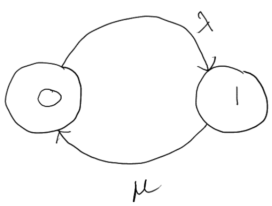
$$P'_{i,j}(t) = \sum_{k \neq i} q_{i,k} P_{k,j}(t) - v_i P_{i,j}(t)$$

Proof.

$$\begin{aligned} P_{i,j}(t+h) - P_{i,j}(t) &= \sum_{k=0}^{\infty} P_{i,k}(h) P_{k,j}(t) - P_{i,j}(t) \\ &\text{(Applying the Chapman-Kolmogorov equation to the first term)} \\ &= P_{i,i}(h) P_{i,j}(t) + \sum_{k \neq i} P_{i,k}(h) P_{k,j}(t) - P_{i,j}(t) \\ &= \sum_{k \neq i} P_{i,k}(h) P_{k,j}(t) - P_{i,j}(t)(1 - P_{i,i}(h)) \end{aligned}$$

Now divide by h on both sides and take the limit $h \to 0$ to obtain the result of the theorem. □

Example 3.5. *Consider the following two-state CTMC with states $\{0\}$ and $\{1\}$. The transitions and their rates are shown in the diagram below. Find $P_{0,0}(t)$.*



Note that $q_{0,1} = v_0 \times p_{0,1} = \lambda \times 1$ and $q_{1,0} = v_1 \times p_{1,0} = \mu \times 1$.
Now the Kolmogorov Backward equations are given by,

$$\begin{aligned} P'_{0,0}(t) &= q_{0,1} P_{1,0}(t) - v_0 P_{0,0}(t) \\ &= \lambda P_{1,0}(t) - \lambda P_{0,0}(t) \qquad (1) \\ P'_{1,0}(t) &= q_{1,0} P_{0,0}(t) - v_1 P_{1,0}(t) \\ &= \mu P_{0,0}(t) - \mu P_{1,0}(t) \qquad (2) \end{aligned}$$

Multiplying (1) by μ and multiplying (2) by λ, and adding them,

$$\mu P'_{0,0}(t) + \lambda P'_{1,0}(t) = 0$$

Integrating the above we get,

$$\mu P_{0,0}(t) + \lambda P_{1,0}(t) = c$$

$P_{0,0}(0) = 1$, i.e., the probability of starting at the state $\{0\}$ and staying at state $\{0\}$ when no time has passed is 1. $P_{1,0}(0) = 0$, i.e., the probability of starting at state $\{0\}$ and moving to state $\{1\}$ when no time has passed is 0.

Therefore, $c = \mu$. And rearranging the above we obtain,

$$\lambda P_{1,0}(t) = \mu[1 - P_{0,0}(t)]$$

Substituting in (1),

$$
\begin{aligned}
P'_{0,0}(t) &= \mu[1 - P_{0,0}(t)] - \lambda P_{0,0}(t) \\
P'_{0,0}(t) + (\mu+\lambda)P_{0,0}(t) &= \mu \\
P'_{0,0}(t) + (\mu+\lambda)\left(P_{0,0}(t) - \frac{\mu}{\mu+\lambda}\right) &= 0 \\
(\textit{Let } h(t) = P_{0,0}(t) - \frac{\mu}{\mu+\lambda}) \\
h'(t) + (\mu+\lambda)h(t) &= 0 \\
h(t) &= Ke^{-(\mu+\lambda)t}
\end{aligned}
$$

Therefore,

$$P_{0,0}(t) = Ke^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda}$$

Since $P_{0,0}(0) = 1$*. Hence,* $K = \frac{\lambda}{\mu+\lambda}$ *and therefore,*

$$P_{0,0}(t) = \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda}$$

Exercises

1. Verify that the Kolmogorov Backward equation for the Pure Birth Process is,

$$P'_{i,j} = \lambda_i P_{i+1,j}(t) - \lambda_i P_{i,j}(t)$$

2. Verify that the Kolmogorov Backward equations for the Birth and Death process are,

$$
\begin{aligned}
P'_{0,j}(t) &= \lambda_0 P_{i,j}(t) - \lambda_0 P_{0,j}(t) \\
P'_{i,j}(t) &= \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i)P_{i,j}(t), \forall i > 0
\end{aligned}
$$

Kolmogorov Forward equation

The Kolmogorov Forward equation is given by,

$$P'_{i,j}(t) = \sum_{k\neq j} q_{k,j} P_{i,k}(t) - v_j P_{i,j}(t)$$

The Kolmogorov Forward equation is only valid for certain CTMCs like the Birth-Death processes and finite state CTMCs, this would cover all CTMCs considered in this course. The Kolmogorov Backward equation on the other hand is well defined for all CTMCs.

Exercises

1. Verify that the Kolmogorov Forward equation for the Pure birth process is,

$$P'_{i,j}(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{i,j}(t)$$

2. Using the forward equation, show that for a Pure Birth process,

$$P_{i,i}(t) = e^{-\lambda_i t}$$

   Is there an alternate way to show the above?

3. Verify that the Kolmogorov Forward equations for the Birth and Death process are,

$$
\begin{aligned}
P'_{i,0}(t) &= \mu_1 P_{i,1}(t) - \lambda_0 P_{i,0}(t) \\
P'_{i,j}(t) &= \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t) - (\lambda_j + \mu_j)P_{i,j}(t), \forall j > 0
\end{aligned}
$$

4 Limiting Probability

The notion of **limiting probability** is the analogue of the stationary distribution for a CTMC. It is defined as follows,

$$P_j \equiv \lim_{t\to\infty} P_{i,j}(t)$$

P_j can be interpreted as the probability of a CTMC to be in state $\{j\}$ after a long time has passed. In other words, it is the long-run proportion of the CTMC to be in state $\{j\}$.

One can consider a more explicit example. Consider a M/M/1 queue. Suppose this queue has a limiting probability. Then P_5 denotes the probability of this queue to have 5 people after a long time has passed.

Exercises

1. Show that if the $\lim_{t\to\infty} P_{i,j}(t)$ exists, then $\lim_{t\to\infty} P'_{i,j}(t) = 0$.

We will use the result from the above exercise to obtain conditions for the existence of limiting probability. Consider the Kolmogorov Forward equation,

$$\begin{aligned} P'_{i,j}(t) &= \sum_{k\neq j} q_{k,j} P_{i,k}(t) - v_j P_{i,j}(t) \\ &\text{(Taking limit)} \\ \lim_{t\to\infty} P'_{i,j}(t) &= \lim_{t\to\infty} \sum_{k\neq j} q_{k,j} P_{i,k}(t) - v_j P_{i,j}(t) \\ 0 &= \sum_{k\neq j} q_{k,j} P_k - v_j P_j \end{aligned}$$

Therefore, the set of equations below can be used to determine the limiting probabilities of a CTMC,

$$\sum_{k\neq j} q_{k,j} P_k = v_j P_j, \forall j \tag{3}$$

and $\sum_j P_j = 1$. Let us apply this to evaluate the limiting probability of a CTMC.

Limiting probability of Birth and Death process

Applying (3) to the Birth and Death process leads to the following set of equations (Verify!),

$$\begin{aligned} \mu_1 P_1 &= \lambda_0 P_0 \\ \mu_2 P_2 + \lambda_0 P_0 &= (\lambda_1 + \mu_1) P_1 \\ \mu_3 P_3 + \lambda_1 P_1 &= (\lambda_2 + \mu_2) P_2 \\ &\vdots \\ \mu_{n+1} P_{n+1} + \lambda_{n-1} P_{n-1} &= (\lambda_n + \mu_n) P_n, n \geq 1 \end{aligned}$$

It follows from the above that,

$$\begin{aligned} \lambda_0 P_0 &= \mu_1 P_1 \\ \lambda_1 P_1 &= \mu_2 P_2 \\ \lambda_2 P_2 &= \mu_3 P_3 \\ &\vdots \\ \lambda_n P_n &= \mu_{n+1} P_{n+1} \end{aligned}$$

Therefore, verify that the following also holds,

$$P_n = \frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\dots\mu_2\mu_1}P_0$$

and since, $\sum_{n=0}^{\infty} P_n = 1$. It follows that,

$$P_0 = \frac{1}{1+\sum_{n=1}^{\infty}\frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\dots\mu_2\mu_1}} \tag{4}$$

Hence, it follows that the limiting probabilities in a Birth and Death chain exists if and only if,

$$\sum_{n=1}^{\infty}\frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\dots\mu_2\mu_1} < \infty \tag{5}$$

Example 4.1 (Limiting probability of a M/M/1 queue)**.** *Recall that a M/M/1 queue is a Birth and Death process where $\lambda_0 = \lambda_1 = \ldots = \lambda_{n-1} = \lambda$ and $\mu_1 = \mu_2 = \ldots = \mu_n = \mu$. From above it follows that,*

$$P_0 = \frac{1}{1+\sum_{n=1}^{\infty}\frac{\lambda^n}{\mu^n}} = 1-\frac{\lambda}{\mu}$$
$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

Note that the above probabilities are well defined if and only if,

$$\lambda < \mu$$

Therefore, the limiting probabilities in a M/M/1 queue exists if and only if, $\lambda < \mu$. That is the clerk services the customers at a faster rate than the rate at which the customers join the queue.

Example 4.2 (Limiting probability of a M/M/s queue)**.** *Recall that in a M/M/s queue there are s servers*

$$\mu_n = \begin{cases} n\mu & 1 \le n \le s \\ s\mu & n > s \end{cases}$$

and $\lambda_n = \lambda, \forall n$. From (5) it follows that the limiting probabilities exist if and only if,

$$\sum_{n=1}^{\infty}\frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\dots\mu_2\mu_1} = \sum_{n=1}^{s}\frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\dots\mu_2\mu_1} + \sum_{n=s+1}^{\infty}\frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\dots\mu_2\mu_1} < \infty$$

Therefore,

$$\sum_{n=s+1}^{\infty}\frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\dots\mu_2\mu_1} < \infty$$

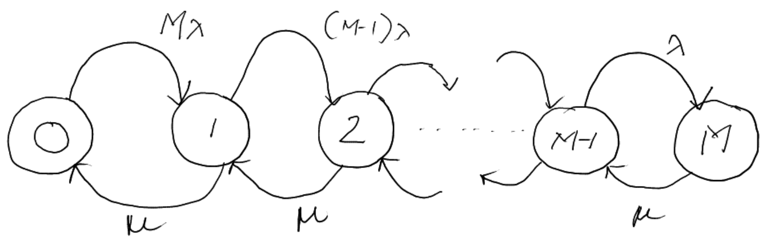
Now,

$$\sum_{n=s+1}^{\infty}\frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\dots\mu_2\mu_1} = \sum_{n=s+1}^{\infty}\frac{\lambda^n}{(s\mu)^{n-s}(s\mu)((s-1)\mu)((s-2)\mu)\dots\mu} = \frac{\lambda^s}{s!\mu^s}\sum_{n=s+1}^{\infty}\frac{\lambda^{n-s}}{(s\mu)^{n-s}} < \infty$$

This sum exists when, $\lambda < s\mu$. That is all the s clerks in the queue service the customers faster than the rate at which the customers join the queue.

Example 4.3. *There are M machines and 1 serviceman. Each machine runs for an exponentially distributed amount of time with mean $\frac{1}{\lambda}$ before breaking down. The serviceman takes an exponentially distributed amount of time to repair each broken machine with mean $\frac{1}{\mu}$.*

First define the states of the system as $\mathcal{S} = \{0, 1, 2, \ldots, M\}$, which denotes the number of broken machines. It follows that this system is a CTMC. See the figure below to understand the transitions.



In state $\{0\}$ there are M functioning machines, the time the first machine breaks down is $\min(T_1, T_2, \ldots, T_M)$, where T_i denotes the time the $i-th$ machine broke down. Note that,

$$\min(T_1, T_2, \ldots, T_M) \sim Exp(M\lambda)$$

Therefore, the rate from $\{0\}$ to $\{1\}$ is $M\lambda$. Verify the other rates in the figure given above. Now from (4) it follows that for this system,

$$P_0 = \frac{1}{1 + \sum_{n=1}^{M} \left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!}}$$

Find P_n for this system.

Uniformization

Knowing $P_{i,j}(t)$ for a CTMC leads to an entire understanding of the CTMC. The tools to obtain CTMC so far that we have seen are the Kolmogorov Backward and Kolmogorov Forward equations. Solving them analytically is not always possible. Uniformization is a technique to computationally evaluate $P_{i,j}(t)$ for a CTMC.

Consider a CTMC, $X(t)$, in which mean time spent in a state is identically distributed across all states, i.e.,

$$T_i \sim \text{Exp}(v), \forall i \in \mathcal{S}$$

Let $N(t)$ denote the number of transitions in this CTMC by time t. Can you guess what kind of a process $N(t)$ is?

We know that for a Poisson process the interarrival times are independent and identically distributed as exponential distribution. It turns out the converse is also true, that if the interarrival times are iid exponential, then the arrivals follow a Poisson process. For a proof of this fact refer Durrett (1999) chapter on Poisson processes. Therefore, $N(t)$ is a Poisson process with rate v.

$$
\begin{aligned}
P_{i,j}(t) &= \mathbb{P}(X(t) = j | X(0) = i) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(X(t) = j | X(0) = i, N(t) = n)\mathbb{P}(N(t) = n | X(0) = i) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(X(t) = j | X(0) = i, N(t) = n) e^{-vt} \frac{(vt)^n}{n!} \\
&\text{(Note that the term in blue is the just the n-step transition probability of the embedded Markov chain)} \\
&= \sum_{n=0}^{\infty} p_{i,j}^{(n)} e^{-vt} \frac{(vt)^n}{n!} \qquad (6)
\end{aligned}
$$

Note that (6) holds only for CTMCs where $v_i = v, \forall i \in \mathcal{S}$.

However, we can adapt this for CTMCs where v_i's are not all identical. What we do is we will *uniformize* all states to have the same rate for the time spent in each state. We will uniformize it by rate v such that,

$$v_i \leq v, \forall i$$

This will require us to adjust the transition matrix of the embedded Markov chain of the CTMC. The new transition matrix $p^*_{i,j}$ is given by,

$$p^*_{i,j} = \begin{cases} 1 - \frac{v_i}{v} & j = i \\ \frac{v_i}{v} p_{i,j} & j \neq i \end{cases}$$

Now (6) can be rewritten as,

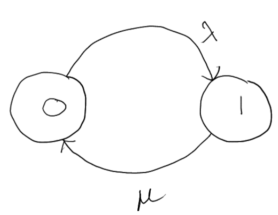
$$P_{i,j}(t) = \sum_{n=0}^{\infty} p_{i,j}^{*(n)} e^{-vt} \frac{(vt)^n}{n!} \qquad (7)$$

Note that (7) is computationally feasible always, i.e., it can be readily evaluated on a computer to get a numerical estimate of $P_{i,j}(t)$. This is the advantage of uniformization, that it gives us a numerical procedure to understand the transition function of the CTMC.

Remark 5. *The readjustment of $p_{i,j}$ to $p^*_{i,j}$ is based on the idea that in the embedded Markov chain of the original CTMC $p_{i,i} = 0$, i.e., the probability of going from the state to itself is 0. This is a consequence of the way we define the embedded Markov chain.*

In the uniformized CTMC since we allot "extra"($v \geq v_i$) $v - v_i$ to every state for the time spent, we account for this by introducing a fictitious possibly non-zero probability of going from state $\{i\}$ to $\{i\}$. This is given by $1 - \frac{v_i}{v}$. The other states are adjusted accordingly.

Example 4.4 (2-state CTMC)**.** *Consider the following two-state CTMC with states $\{0\}$ and $\{1\}$ which we considered earlier. To recall, the transitions and their rates are shown in the diagram below.*



Verify the following, $v_0 = \lambda, v_1 = \mu, p_{0,1} = 1, p_{1,0} = 1.$

Verify also, $p^*_{0,0} = \frac{\mu}{\lambda+\mu}, p^*_{0,1} = \frac{\lambda}{\lambda+\mu}, p^*_{1,1} = \frac{\lambda}{\lambda+\mu}, p^*_{1,0} = \frac{\mu}{\lambda+\mu}$.

$$p^* = \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}$$

Verify that,

$$p^{*(2)} = \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}^2 = \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}$$

Therefore,

$$p^{*(n)} = \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}$$

From (7),

$$\begin{aligned}
P_{0,0}(t) &= \sum_{n=0}^{\infty} P^{*(n)}_{0,0} e^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} \\
&= e^{-(\lambda+\mu)t} + \sum_{n=1}^{\infty} \left(\frac{\mu}{\lambda+\mu}\right) e^{-(\lambda+\mu)t} \frac{[(\lambda+\mu)t]^n}{n!} \\
&= e^{-(\lambda+\mu)t} + \left(\frac{\mu}{\lambda+\mu}\right) e^{-(\lambda+\mu)t} \sum_{n=1}^{\infty} \frac{[(\lambda+\mu)t]^n}{n!} \\
&= e^{-(\lambda+\mu)t} + \left(\frac{\mu}{\lambda+\mu}\right) e^{-(\lambda+\mu)t} \left(e^{(\lambda+\mu)t} - 1\right) \\
&\textit{(Follows from the Taylor expansion of } e^{(\lambda+\mu)t}\textit{)} \\
&= \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda} \textit{(Rearranging)}
\end{aligned}$$

We obtained the same solution for $P_{0,0}(t)$ *by solving the Backward equation, we obtain the same answer through uniformization here. Note that here we were lucky to be able to obtain an analytical solution through uniformization itself, however that need not always be the case. But through uniformization we can always evaluate (7) on a computer to numerically obtain* $P_{i,j}(t)$.

References

1. Sheldon Ross, Introduction to Probability Models, Academic Press, 2024.
2. Rick Durrett, Essentials of Stochastic Processes, Springer, 1999.