# DMS625: Practice Assignment

October 18, 2024

## Poisson Process

1. Earthquakes hit a country according to a Poisson process with rate 3 per year. Given that two earthquakes hit in the first 6 months, what is the probability that
   (a) both hit in the first two months.
   (b) at least one hit in the first 2 months.

2. A workshop has two machines, $A$ and $B$. Suppose the breakdown of each machine follows an independent Poisson process with rate 2 per year and 3 per year respectively. What is the probability that $A$ breaks down for the second time before $B$ breaks down for the third time?

3. A submarine has three navigational devices but can remain at sea if at least two are working. Suppose that the failure times are exponential with means 1 year, 1.5 years, and 3 years. What is the average length of time the boat can remain at sea?

4. Alice and Bob are at the gym. They will first ride stationary bikes for an exponentially distributed amount of time with means 30 and 20 minutes. Then they will lift weights for an exponentially distributed amount of time with means 10 and 15. What is the probability Alice finishes first?

5. The number of cases of an epidemic follows a Non-homogeneous Poisson with rate $\frac{200}{1+t}$ per day, here $t$ is denoted in days.
   (a) Find the probability of observing 200 cases in the first two days.
   (b) Find the probability density function of the time of arrival of the first case.

6. A warehouse receives high value goods following a Poisson process with rate 2 per hour. The warehouse employs an inspection officer who is supposed to inspect each item that has arrived before sending it to its destination. The officer is known to skimp at his work, sometimes he would send the item ahead without inspecting it. The probability of the officer actually inspecting an item that has arrived at time $t$ is $\sin^2(\frac{\pi t}{12})$, here $t$ is denoted in hours. Find the probability of the officer inspecting 4 goods between $t = 4$ and $t = 6$ hours.

7. An insurer receives claims following a Poisson process with rate 10 per month. The size of each claim is distributed exponentially with the mean amount for the claim being Rs. 1,00,000.
   (a) Find the expected (monetary) amount of claims that the insurer will receive in the first 6 months.
   (b) Find the probability that the insurer atleast received claims totalling Rs. 20,00,000 given that he received 15 number of claims in the first 8 months.

## Hints

1. See Example 3.1.

   (a) $\mathbb{P}(N(2/12) = 2|N(6/12) = 2)$

   (b) $\mathbb{P}(N(2/12) \geq 1|N(6/12) = 2)$

2. Formulate it as a splitting problem. Consider that total breakdowns arrive at a rate of 5 per year and split to A & B. Then $\lambda_A = \lambda p_A = 5p_A$ and $\lambda_A = 2$. Get $p_A$ and $p_B$ accordingly. Now evaluate the following probability,

$$\mathbb{P}(N_A(t) \geq 2|N(t) = 4)$$

   to get the answer. Look at the proof of Theorem 3.3 to figure out how to calculate this.

3. Let $\lambda_i$ denote the rate of failure time of device $i$. Let $T_i$ denote the failure time of device $i$. $F_1$ denotes the time at which the first device fails, $F_1 = \min(T_1, T_2, T_3)$. $F_1$ is exponentially distributed with rate $\lambda_1 + \lambda_2 + \lambda_3$ (Proposition 1.2). Therefore,

$$\mathbb{E}[F_1] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} = 1/2$$

   Look at Example 1.2 to get the probability of each device failing first and then the next device failing. For example, suppose device 1 fails first, then expected time between the first and second failure is $\frac{1}{\frac{2}{3}+\frac{1}{3}} = 1$.

   Similarly, if device 2 fails first, then expected time between first and second failure is $\frac{1}{\frac{1}{1}+\frac{1}{3}} = 3/4$ and if device 3 fails first, then $\frac{1}{\frac{1}{1}+\frac{2}{3}} = 3/5$

   Then the expected time between the first and second failure is,

$$\begin{aligned}\mathbb{E}[F_2 - F_1] &= \sum_i \text{Prob. device } i \text{ failed first} \times \text{Expected time for second failure given device } i \text{ failed first}\\ &= \frac{1}{2} \times 1 + \frac{1}{3} \times \frac{3}{4} + \frac{1}{6} \times \frac{3}{5}\end{aligned}$$

$$\mathbb{E}[F_2] = \mathbb{E}[F_2 - F_1] + \mathbb{E}[F_1]$$

4. Let $T_A = T_{A,B} + T_{A,W}$, where $T_A$ denotes the time spent by Alice on her activities. $T_{A,B}$ denotes the time spent by Alice on Bike and $T_{A,W}$ denotes the time spent by Alice weightlifting. $T_{A,B}$ and $T_{A,W}$ are exponentially distributed with rates say $\lambda_{A,B}$ and $\lambda_{A,W}$. Then,

$$\begin{aligned}\mathbb{P}(T_A < t) &= \mathbb{P}(T_{A,B} + T_{A,W} < t)\\ &= \int_0^t \int_0^{t-x} \lambda_{A,B} e^{-\lambda_{A,B} x} \lambda_{A,W} e^{-\lambda_{A,W} y} dy dx\end{aligned}$$

   You will note that $T_A$ is not exponentially distributed. Get the density $f_{T_A}(t) = \frac{d}{dt}\mathbb{P}(T_A < t)$. Similarly get the density, $f_{T_B}(t)$ for $T_B$. Now we are interested in,

$$\begin{aligned}\mathbb{P}(T_A < T_B) &= \mathbb{P}(T_A < T_B)\\ &= \int_0^\infty \int_x^\infty f_{T_A}(x) f_{T_B}(y) dy dx\end{aligned}$$

5. b) $\mathbb{P}(X_1 > t) = \mathbb{P}(N(t) = 0)$

6. The probability of inspecting a good at time $t$ is $p(t) = \frac{\lambda(t)}{\lambda}$. Therefore, the number of goods being inspected follows a Non-homogeneous Poisson process with rate $\lambda p(t) = 2\sin^2\left(\frac{\pi t}{12}\right)$.

7. (a) Let $Y_i$ denote the size of a claim and $N(t)$ denote the number of claims. Therefore, total amount of claims at time $t$ is,

$$W(t) = \sum_{i=1}^{N(t)} Y_i$$

   This is a compound Poisson process.

   (b) Hint: This has Erlang distribution. Figure out why.