DMS625: Practice Assignment

September 13, 2024

Markov Chains

1. A fair coin is tossed successively. All the tosses are independent of each other. If the outcome is heads, there is a gain of 1 unit and if the outcome is tails, there is a gain of 0 unit. Let Y_i denote the gain for the $i-$th toss. Define $X_i = Y_i + Y_{i-1}, i \geq 2$. Verify if X_i is a Markov chain.

2. *Inventory model.* Consider a bookshop owner who sells copies of *Harry Potter and the seven wizards* (HPSW). The demand for HPSW is random, so the bookshop owner has to stock enough books that he is able to meet the demand but not stock too many books so that he incurs a lot of storage costs. To manage the inventory, he decides to order 3 copies of HPSW, if at the end of the day his inventory for HPSW reaches 1 or 0 copy. He will receive his order by the start of the next day. The probability that there is demand for 3 books on a particular day is 0.2, for 2 books it is 0.3, 1 book it is 0.3, and no sale it is 0.2.

   (a) Identify the transition matrix of the daily stock of HPSW.

   (b) He makes a profit of Rs. 150 on each unit of HPSW sold and spends Rs. 25 a day to store each unsold copy of the book. Identify the long run profit of his inventory management strategy.

3. Every time that the team wins a game, it wins its next game with probability 0.8; every time it loses a game, it wins its next game with probability 0.3. If the team wins a game, then it has dinner together with probability 0.7, whereas if the team loses then it has dinner together with probability 0.2.

   (a) In the long-run, what proportion of the games do the team win?

   (b) In the long-run, what proportions of games result in a team dinner?

   (c) What is the expected number of games the team needs to play for a dinner?

4. The bookshop owner has two computers at the check-out counter. A computer may stop working on any given day with probability 0.1. The repair shop takes 2 days to fix the computer. The repair shop will only accept one computer at a time to repair.

   (a) Construct a Markov chain, with states (a, b) for each day, where a is the number of computers currently functioning and b is the number of days a broken computer has spent at the repair shop. Identify the transition matrix.

   (b) Find the stationary distribution of this Markov chain.

   (c) What is the long-run proportion for a computer to be broken?

5. *Stock price model.* The *tick size* in a market refers to the smallest increment possible between price quotes. For NSE, this is Rs. 0.01, i.e., a stock whose value now is 100, can take a value 100.01 or 99.99. No quotes of 100.005, 100.003, 99.998, etc. are permissible. In other words, the stock price changes in increments of 0.01 only.

   Suppose, the market is open from 10:00 am to 3:00 pm. Assume the tick size of stocks to be 0.01. A stock, *ABC*, starts trading at Rs. 120 at 10:00 am. Every 5 seconds, the probability of ABC moving up a tick is 0.1, staying at the same level is 0.85 and moving down a tick is 0.05.

   (a) Is the stock price recurrent?

   (b) Does the stationary distribution of the stock price exist?

   (c) *American call option.* The American call option is a financial contract on some stock S with a payoff $\max(S_t - K, 0), t \leq T$. Here, S_t denotes the price of the stock at some time t. The call option expires at time T. If the stock price, S_t, is greater than K, the strike price, the owner of the option can exercise it any time, t, before expiry time T to earn $S_t - K$. As an example, suppose, an American option on stock ABC with strike price $K = 110$ expires at the end of the day. If the owner of the option were to exercise it at 10:00 am, because the price of the stock is $S_t = 120$ at 10:00 am, she would earn Rs. $\max(120 - 110, 0) = 10$ at 10:00 am.

   Suppose you own an American option on ABC with strike price, $K = 125$, what is the probability that you will be able to earn a payoff of Rs. 5 before 1:00 pm? You will immediately exercise the option as the price touches Rs. 130.

   Hint: You may want to implement the solution in code to get the answer here.

6. Consider a set of 6 restaurants with imaginative names-A, B, C, D, E, F. The probability of you visiting one of these restaurants each week can be modelled using a Markov chain specified by the transition matrix given below,

$$\begin{array}{c c} & \begin{array}{cccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} \end{array} \\ \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \\ \text{F} \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix} \end{array}$$

   (a) Suppose you visit restaurant B this week, what is the expected number of weeks after which you visit B again?

   (b) What is the probability of visiting C from B in finite time? What is the probability of visiting A from B in finite time?

   (c) Identify the stationary distribution. Does this Markov chain satisfy conditions for long-run convergence to stationary distribution?

Hints/Answers

1. It is not a Markov chain.

2. (a)

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.2 & 0 \\ 0 & 0.2 & 0.3 & 0.3 & 0.2 \\ 0.5 & 0.3 & 0.2 & 0 & 0 \\ 0.2 & 0.3 & 0.3 & 0.2 & 0 \\ 0 & 0.2 & 0.3 & 0.3 & 0.2 \end{bmatrix} \end{array}$$

   (b) $\pi(0) = 0.215, \pi(1) = 0.267, \pi(2) = 0.272, \pi(3) = 0.179, \pi(4) = 0.067.$

$$\text{Expected demand per day} = 0 \times 0.2 + 1 \times 0.3 + 2 \times 0.3 + 3 \times 0.2 = 1.5.$$

   If at the end of the day he has two books, then he will start the next day with two books. He will lose a sale of one book if there is a demand for three books on that day. In no other case he will lose a sale. Therefore,

$$\text{Expected lost sales} = 1 \times \pi(2) \times 0.2$$

$$\text{Net sales} = \text{Expected demand per day} - \text{Expected lost sales}$$

3. (b)

$$\mathbb{P}(\text{Dinner}) = \mathbb{P}(\text{Dinner}|\text{Win})\pi(\text{Win}) + \mathbb{P}(\text{Dinner}|\text{Lose})\pi(\text{Lose})$$

4. (a)

$$\begin{array}{c} \\ (2,0) \\ (1,0) \\ (1,1) \\ (0,1) \\ (0,0) \end{array} \begin{array}{c} \begin{array}{ccccc} (2,0) & (1,0) & (1,1) & (0,1) & (0,0) \end{array} \\ \begin{bmatrix} (0.9)^2 & 2 \times 0.1 \times 0.9 & 0 & 0 & (0.1)^2 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0.9 & 0.1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

5. Formulate it as a Birth and Death chain.

   (c) Compute the hitting time probability to 130. The number of steps in which it has to hit 130 is the number of 5 second intervals between 10:00 am and 1:00 pm.

6. (a) Hint: What kind of a state is B?

   (b) Compute ρ_{BC} and ρ_{BA}. Look at Example 2.11 in the notes.