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EE798I : NANOPHOTONICS

Assignment - 1

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Q1) Vector Calculus

a) $(\bar{\nabla} \times \bar{A}) = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$

$$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{A}) = \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_x}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

By continuity of 2^{nd} derivative : $\frac{\partial^2 A_z}{\partial x \partial y} = \frac{\partial^2 A_z}{\partial y \partial x}$ and so on...

$\Rightarrow \bar{\nabla}\cdot(\bar{\nabla}\times\bar{A}) = 0$

example : $\bar{A} =$ ~~$(0, A_y, 0)$~~ $(0, x^2, 0)$

$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{A}) = \bar{\nabla}\cdot(\hat{k}\cdot 2x) = 0$

$\bar{\nabla}\times(\nabla f) = \bar{\nabla}\times\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\right)$

$= \hat{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) + \hat{j}\left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$

$= 0$ by continuity of 2^{nd} derivative

eg: $f = x^2 + y^2$

$\bar{\nabla}\times(\bar{\nabla}f) = \bar{\nabla}\times(2x\hat{i} + 2y\hat{j}) = 0$

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(b) Identity : $\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$

$$\bar{E}_o \times (\bar{k} \times \bar{E}_o) = (\bar{E}_o \cdot \bar{E}_o)\bar{k} - (\bar{E}_o \cdot \bar{k})\bar{E}_o$$
$$= |E_o|^2 \bar{k} - (\bar{E}_o \cdot \bar{k})\bar{E}_o$$

For any 2 arbitrary vectors, $\bar{E}_o$ and $\bar{k}$, the triple cross product lies in the plane defined by $\bar{E}_o$ and $\bar{k}$

in relation to plane wave solutions, the transversality condition dictates that $\vec{E}_o \perp^r \bar{k}$ $\Rightarrow$ $\bar{E}_o \cdot \bar{k} = 0$.

$\Rightarrow$ $\bar{E}_o \times (\bar{k} \times \bar{E}_o) = |E_o|^2 \bar{k}$ which is parallel to $\bar{k}$

we know that $\bar{B} \propto \bar{k} \times \bar{E}_o$, and $\bar{S} \propto (E \times \bar{B})$

$\Rightarrow$ $\bar{S} \propto \bar{E}_o \times (\bar{k} \times \bar{E}_o)$

$\Rightarrow$ the poynting vector is parallel to the direction of propagation of the wave, hence energy travel happens in direction of wave propagation

(c) $$\bar{\nabla} \cdot (\nabla f) = \bar{\nabla} \cdot \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \right) = \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$

$$\bar{\nabla}(\bar{\nabla} \cdot F) = \bar{\nabla}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right)$$
$$= \boxed{\frac{\partial^2 F_x}{\partial x^2}\hat{i} + \frac{\partial^2 F_y}{\partial y^2}\hat{j} + \frac{\partial^2 F_z}{\partial z^2}}$$

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(d) $\bar{E}_{in} = \frac{\rho R}{3\varepsilon_0}\hat{R}$

$\bar{E}_{out} = \frac{\rho r^3}{3\varepsilon_0 R^2}\hat{R}$ where r is radius of sphere

in electrostatics, fields are static hence ther is no time dependence. Moreover, $J=0$. Hence, maxwells eq^{n}s are:

(a) $\bar{\nabla}\cdot\bar{E} = \rho/\varepsilon_0$ (gauss law)

$\bar{\nabla}\cdot\bar{E}_{in} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 E_{in}\right) = \frac{1}{R^2}\cdot\frac{3\rho R^2}{3\varepsilon_0} = \frac{\rho}{\varepsilon_0}$

$\bar{\nabla}\cdot\bar{E}_{out} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\rho r^3}{3\varepsilon_0}\right) = 0$ since $\rho_{out} = 0$

(b) $\bar{\nabla}\times\bar{E} = 0$ (Faradays law)

since ther is no θ or ϕ dependence:

$\bar{\nabla}\times\bar{E} = \frac{-1}{R}\frac{\partial}{\partial R}(RA_\phi)\hat{\theta} + \frac{1}{R}\frac{\partial}{\partial R}(RA_\theta)\cdot\hat{\phi} + \frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)\right) = 0$

$\bar{\nabla}\cdot\bar{B} = 0$ and $\bar{\nabla}\times\bar{B}$ are trivially satisfied as it is an electrostatics problem.

Q2) Plane Waves in a uniform, isotropic, linear, time-inv medium (ε, μ)

(a) general expression for ~~monochromatic~~ plane wave ~~which propagates in z dir~~:

$$E(r,t) = \int_{-\infty}^{\infty} E(\omega)\exp\left[i\left(k(\omega)\cdot x - \omega t\right)\right]d\omega$$

$$H(r,t) = \int_{-\infty}^{\infty} H(\omega) \exp\left[i\left(k(\omega)\cdot r - \omega t\right)\right] d\omega$$

(i) Gauss law for E : $\nabla \cdot E = 0$

$$i\, k(\omega) \cdot E = 0 \quad \text{as} \quad k \cdot E = 0 \quad \text{(transversality)}$$

(ii) Gauss law for H : $\nabla \cdot H = 0$

$$i\, k(\omega) \cdot H = 0 \quad \text{as} \quad k \cdot H = 0 \quad \text{(transversality)}$$

(iii) Faraday's law : $\bar{\nabla} \times \bar{E} = -\mu \dfrac{\partial H}{\partial t}$

$$-\mu \frac{\partial H}{\partial t} = + \int_{-\infty}^{\infty} H(\omega)\,(i\omega\mu) \exp\left[i\left(k(\omega)\cdot r - \omega t\right)\right] d\omega$$

$$= i\,(\bar{k} \times \bar{E})$$

$$= \bar{\nabla} \times \bar{E}$$

$$\Rightarrow \bar{\nabla} \times \bar{E} = -\mu\, \partial H / \partial t$$

(iv) Ampere - Maxwell law : $\nabla \times H = \varepsilon \dfrac{\partial E}{\partial t}$

$$\varepsilon \frac{\partial E}{\partial t} = \int_{-\infty}^{\infty} \underbrace{E(\omega)\,(-\varepsilon i\omega) \exp\left[i\left(k(\omega)\cdot r - \omega t\right)\right] d\omega}_{\nabla \times H(\omega)}$$

$$= \bar{\nabla} \times H$$

hence, verified

(b) (i) For a circularly polarized plane wave

$$\bar{E} = E_0 (\hat{x} + i\hat{y}) e^{i(kz - \omega t)}$$

$$\bar{H} = \frac{1}{Z}(\hat{k} \times \bar{E}) = \frac{E_0}{Z}(\hat{z} \times (\hat{x} + i\hat{y})) e^{i(kz-\omega t)}$$

$$= \frac{E_0}{Z}(\hat{y} - i\hat{x}) e^{i(kz-\omega t)}$$

$$\bar{H}^* = \frac{E_0}{Z}(\hat{y} + i\hat{x}) e^{i(\omega t - kz)}$$

$$\bar{E} \times \bar{H}^* = \frac{E_0^2}{Z}(\hat{x} + i\hat{y}) \times (\hat{y} + i\hat{x}) = \frac{E_0^2}{Z}(2\hat{z})$$

$$\Rightarrow \langle S_{circ} \rangle = \frac{1}{2} \mathrm{Re}\left(\hat{z} \frac{2E_0^2}{Z}\right) = \boxed{\frac{E_0^2}{Z} \hat{z}}$$

(ii) For a standing wave:

$$\bar{E} = E_0 \hat{x} (e^{ikz} + e^{-ikz}) e^{-i\omega t}$$

$$= 2E_0 \hat{x} \cos(kz) e^{-i\omega t}$$

$$\bar{H} = \frac{1}{Z}(\hat{k} \times \bar{E}) = \hat{y}\, i \frac{2E_0}{Z} \sin(kz) e^{-i\omega t}$$

$$\bar{E} \times \bar{H}^* = [2E_0 \hat{x} \cos(kz) e^{-i\omega t}] \left[-\hat{y}\, i \frac{2E_0}{Z} \sin(kz) e^{i\omega t}\right]$$

$$= -\hat{z}\, i \frac{4E_0^2}{Z} \cos(kz) \sin(kz)$$

$$= -\hat{z}\, i \frac{2E_0^2}{Z} \sin(2kz)$$

$$\langle S_{stand} \rangle = 0$$

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In a circularly polarized wave, the magnitude of electric field vector is constant as it rotates. This leads to a constant energy density and a constant, steady flow of energy. In a standing wave, energy is stored in oscillating fields between nodes, sloshing back and forth, but there's no net energy propagation.

(c) $\bar{E}(z,t) = A\hat{x}\cos(kz-\omega t) + B\hat{y}\sin(kz-\omega t)$

where $A > B$ as $\hat{x}$ is major axis

(d) $\bar{E}_1(z,t) = E_0\hat{x}\exp(i(kz-\omega t))$

$\bar{E}_2(z,t) = E_0\hat{y}\exp(-ikz-i\omega t)$

$\bar{E}_{tot}(z,t) = E_0 e^{-i\omega t}\left[\hat{x}e^{ikz} + \hat{y}e^{-ikz}\right]$

this is not a standing wave because a standing wave requires fixed spatial nodes where amplitude is always 0, whereas in this case such a condition is only true when $E_0 = 0$, ie when waves don't exist.

(e) $\bar{E}_1(z,t) = E_0\hat{x}\exp(i(kz-\omega t))$
$\bar{E}_2(z,t) = E_0\hat{x}\exp(i(-kz-\omega t+\pi/2))$

$\bar{E}_{tot}(z,t) = E_0\hat{x}(\cos(kz-\omega t) + i\sin(kz-\omega t) +$
$\sin(kz+\omega t) + i\cos(kz+\omega t))$

$= E_0\hat{x}e^{-i\omega t}(e^{ikz} + ie^{-ikz})$

since the expression is linearly separable into ~~functions of its~~ temporal and spatial component, it is a standing waves with nodes at:

$$|e^{ikz} + e^{-ikz}| = 0$$

(f) $\bar{E}_{tot} = \bar{E}_1 + \bar{E}_2$, $\bar{H}_{tot} = \bar{H}_1 + \bar{H}_2$

$\bar{S}_{tot} = \cancel{\bar{H}_{tot} \times \bar{E}_{tot}} = \bar{E}_{tot} \times \bar{H}_{tot} = \bar{E}_1 \times \bar{H}_1 + \bar{E}_1 \times \bar{H}_2 + \bar{E}_2 \times \bar{H}_1 + \bar{E}_2 \times \bar{H}_2$

$\bar{S}_1 = \bar{E}_1 \times \bar{H}_1$, $\bar{S}_2 = \bar{E}_2 \times \bar{H}_2$

$\bar{S}_{tot} - \bar{S}_1 - \bar{S}_2 = \boxed{\bar{E}_1 \times \bar{H}_2 + \bar{E}_2 \times \bar{H}_1}$ which is not always equal to 0

Due to these cross (or interference) terms, poynting vector does not follow superposition

(g) $\bar{E}(x, z, t) = E_0 \hat{y} \exp\left[i\left(\frac{k(x+z)}{\sqrt{2}} - \omega t\right)\right]$

let $\phi = k(x+z)/\sqrt{2} - \omega t$

$\Rightarrow \bar{E}(\phi, t) = E_0 \hat{y} \exp(i\phi)$

$\bar{H} = \frac{1}{Z}(\hat{k} \times E) = \frac{1}{Z}\left(\frac{\hat{x}+\hat{z}}{\sqrt{2}}\right) \times (\hat{y}) E_0 \exp(i\phi)$

$\Rightarrow \boxed{\bar{H}(\phi, t) = \frac{E_0}{\sqrt{2} Z} \exp(i\phi) (\hat{z} - \hat{x})}$

~~The H E H~~ time domain expressions are real part of the complex fields

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$$\bar{S} = \bar{E} \times \bar{H} = \frac{E_0^2}{Z\sqrt{2}} (\hat{x} + \hat{z}) \cos^2\phi = \boxed{\hat{k} \frac{E_0^2}{Z} \cos^2\phi}$$

Electric polarization, $P = (\varepsilon - \varepsilon_0)\, \hat{y}\, E_0 \cos\phi$
Magnetization density, $M = 0$.

(h) polarization direction $= \hat{n} \times \hat{y} = (\hat{z} - \hat{x}) / \sqrt{2}$

$$\bar{E}(x,z,t) = E_0 \frac{(\hat{z} - \hat{x})}{\sqrt{2}} \cos\left[\frac{k_0(x+z)}{\sqrt{2}} - \omega t\right]$$

$$\bar{H}(x,z,t) = \frac{-\hat{y}\, E_0}{Z} \cos\left[\frac{k(x+z)}{\sqrt{2}} - \omega t\right]$$

energy conservation relation : $\boxed{\nabla \cdot S = -\frac{\partial u}{\partial t}}$

$$S = \bar{E} \times \bar{H} = \frac{E_0^2}{Z} \cos^2\phi \left(\frac{\hat{x} + \hat{z}}{\sqrt{2}}\right)$$

$$\nabla \cdot S = \nabla \cdot \left(\frac{E_0^2}{Z} \cos^2\left(\frac{kx + kz}{\sqrt{2}} - \omega t\right)\left(\frac{\hat{x} + \hat{z}}{\sqrt{2}}\right)\right)$$

$$= -\frac{E_0^2}{Z} \cdot \frac{1}{\sqrt{2}} \left(2\cos\phi \sin\phi \cdot \left(\frac{k_0}{\sqrt{2}} + \frac{k_0}{\sqrt{2}}\right)\right)$$

$$= \boxed{-\frac{kE_0^2}{Z} \sin 2\phi}$$

now, energy density, $u = \frac{1}{2}\varepsilon |E|^2 + \frac{1}{2}\mu |H|^2$

$$\Rightarrow u = \frac{1}{2}\left(\varepsilon E_0^2 \cos^2\phi + \mu \frac{E_0^2}{Z^2}\cos^2\phi\right), \quad Z = \sqrt{\mu/\varepsilon}$$

$$= \frac{E_0^2 \cos^2\phi}{2}\left(\varepsilon + \frac{\mu\varepsilon}{\mu}\right) = \varepsilon E_0^2 \cos^2\phi$$

$$\Rightarrow -\frac{\partial u}{\partial t} = -2 E_0^2 \varepsilon \cos\phi \sin\phi\, \omega = -E_0^2 \varepsilon \sin 2\phi\, \omega$$

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here, $\omega = kv = \frac{k}{\sqrt{\varepsilon\mu}} = \frac{k}{\varepsilon Z}$

substituting, we get : $\frac{-\partial u}{\partial t} = \frac{-E_0^2 k}{Z} \sin 2\phi = \nabla . S$

hence, verified.

Q3) <u>Linearity of Maxwells equations</u>

in source free medium, maxwells eq^ns :

$\bar{\nabla} \cdot \bar{E} = 0$

$\bar{\nabla} \cdot \bar{B} = 0$ or $\nabla \cdot H = 0$

$\bar{\nabla} \times \bar{E} = -\partial B / \partial t$ or $\bar{\nabla} \times \bar{E} = -\mu \, \partial H / \partial t$

$\bar{\nabla} \times H = \partial D / \partial t$ or $\bar{\nabla} \times B = (\partial E / \partial t) \mu\varepsilon$

in order to prove linearity, we just need to show that all operations on E and H are linear.

$\bar{\nabla} \cdot \bar{E}_{net} = \bar{\nabla} \cdot (a_1 \bar{E}_1 + a_2 \bar{E}_2) = a_1 \bar{\nabla} \cdot \bar{E}_1 + a_2 \bar{\nabla} \cdot \bar{E}_2 = 0$

$\bar{\nabla} \cdot \bar{H}_{net} = \bar{\nabla} \cdot (a_1 \bar{H}_1 + a_2 \bar{H}_2) = a_2 \bar{\nabla} \cdot \bar{H}_2 + a_1 \bar{\nabla} \cdot \bar{H}_1 = 0$

$\bar{\nabla} \times \bar{E} = a_1 \bar{\nabla} \times \bar{E}_1 + a_2 \bar{\nabla} \times \bar{E}_2 = -a_1 \partial B_1 / \partial t - a_2 \partial B_2 / \partial t$

$= -\mu \frac{\partial}{\partial t} (a_1 H_1 + a_2 H_2) = -\mu \frac{\partial \bar{H}_{net}}{\partial t}$

$\bar{\nabla} \times \bar{H}_{net} = \mu\varepsilon \left[\frac{a_1 \partial \bar{E}_1}{\partial t} + \frac{a_2 \partial E_2}{\partial t} \right]$

$= \mu\varepsilon \left[\frac{\partial}{\partial t} (a_1 \bar{E}_1 + a_2 \bar{E}_2) \right] = \mu\varepsilon \frac{\partial \bar{E}_{net}}{\partial t}$

Hence proved that superposition principle holds for Maxwells equations in source free medium.

When sources are present:

(i) $\bar{\nabla} \cdot \bar{E} = \rho / \varepsilon$

(ii) $\bar{\nabla} \cdot \bar{H} = 0$

(iii) $\bar{\nabla} \times \bar{E} = -\mu \, \partial H / \partial t$

} remain the same upon source introduction

(iv) $\bar{\nabla} \times \bar{H} = J_{free} + \varepsilon \dfrac{\partial \bar{E}}{\partial t}$

let $\rho_{net} = a_1 \rho_1 + a_2 \rho_2$, $J_{net} = a_1 J_1 + a_2 J_2$

$$\bar{\nabla} \cdot \bar{E}_{net} = \bar{\nabla} \cdot (a_1 \bar{E}_1 + a_2 \bar{E}_2)$$

$$= a_1 \bar{\nabla} \cdot \bar{E}_1 + a_2 \bar{\nabla} \cdot \bar{E}_2$$

$$= a_1 \rho_1 / \varepsilon + a_2 \rho_2 / \varepsilon = \rho_{net} / \varepsilon$$

where its given that $\bar{\nabla} \cdot \bar{E}_1 = \rho_1 / \varepsilon$, $\bar{\nabla} \cdot E_2 = \rho_2 / \varepsilon$

$$\bar{\nabla} \times \bar{H}_{net} = \bar{\nabla} \times (a_1 \bar{H}_1 + a_2 \bar{H}_2)$$

$$= a_1 \bar{\nabla} \times \bar{H}_1 + a_2 \bar{\nabla} \times \bar{H}_2$$

$$= a_1 \left(J_1 + \varepsilon \frac{\partial \bar{E}_1}{\partial t} \right) + a_2 \left(J_2 + \varepsilon \frac{\partial \bar{E}_2}{\partial t} \right)$$

$$= (a_1 J_1 + a_2 J_2) + \varepsilon \left(a_1 \frac{\partial \bar{E}_1}{\partial t} + a_2 \frac{\partial \bar{E}_2}{\partial t} \right)$$

$$= \boxed{J_{net} + \varepsilon \frac{\partial \bar{E}_{net}}{\partial t}}$$