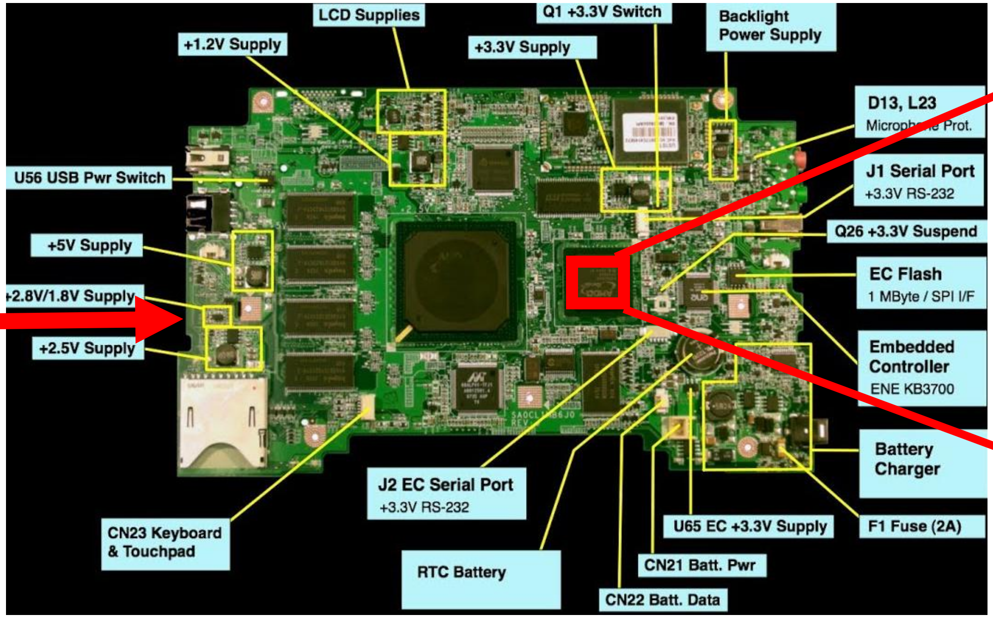

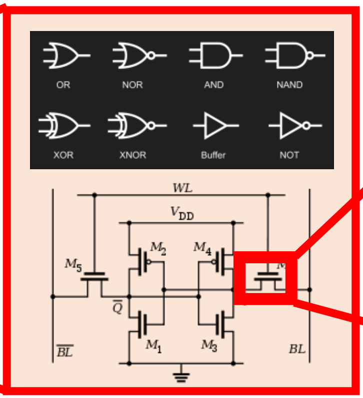

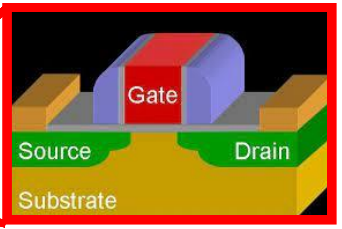

ESC201: Introduction to Electronics

Module 3: Frequency Domain Analysis

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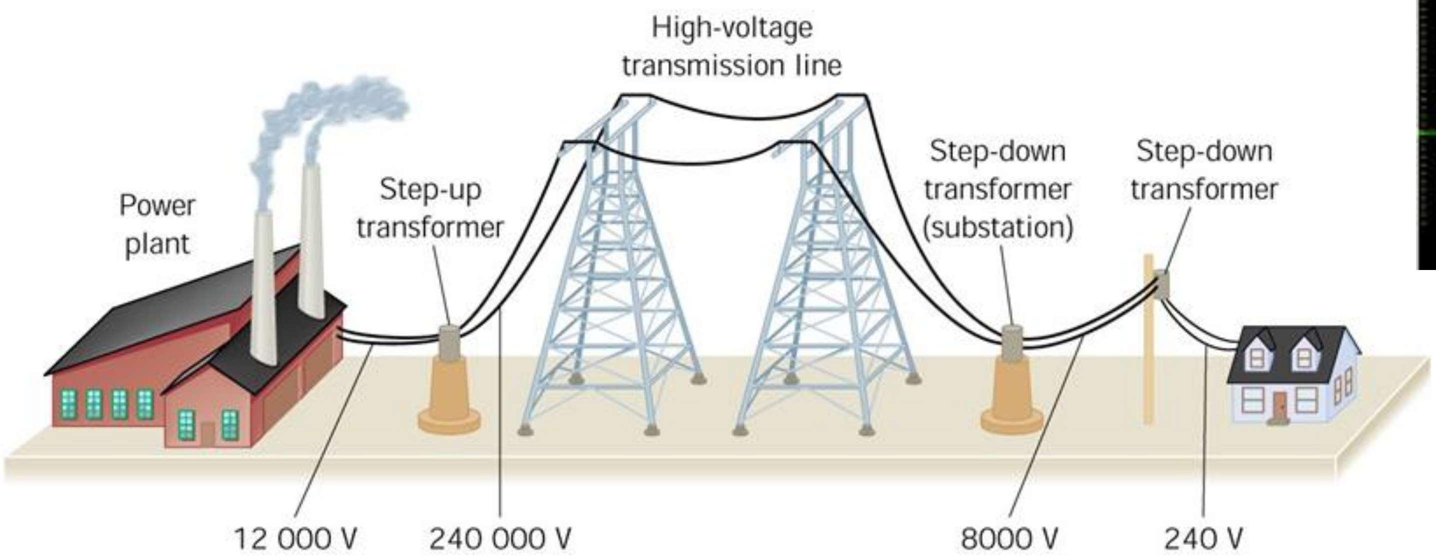
Assistant Professor,
Department of Electrical Engineering,
IIT Kanpur

Recap

- Negative Capacitance: converting MSE problem to electronics domain

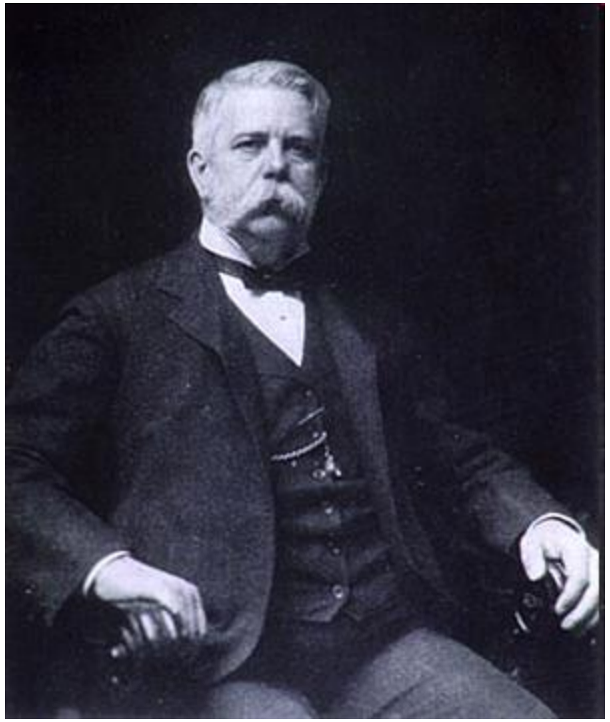
- Series and Parallel RLC circuits: Nightmare to solve them.



War of currents: AC vs DC

Tell Westinghouse to stick to air brakes. He knows all about them.

-Thomas Edison



George Westinghouse formed an alliance with Nikola Tesla

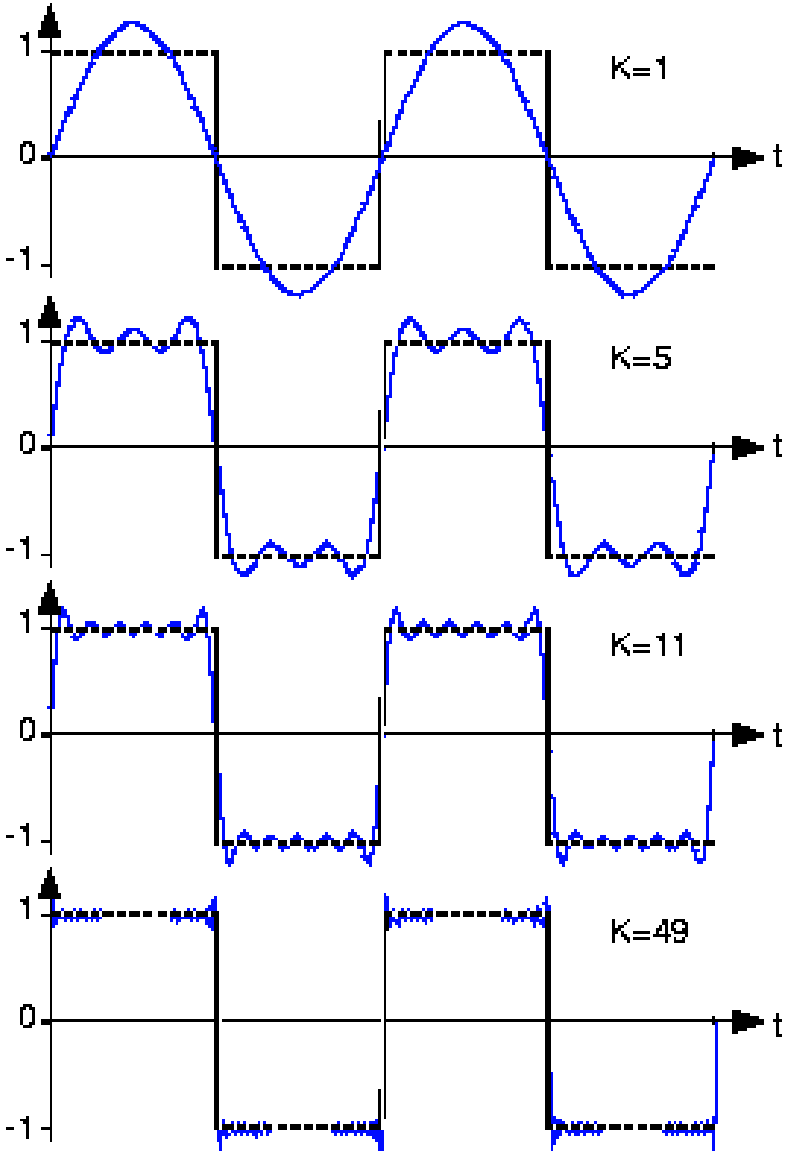
https://en.wikipedia.org/wiki/War_of_Currents

Fourier Analysis

$$A\sin\left(\frac{N\pi t}{T}\right) \quad \rightarrow \frac{1}{N}\sin\left(\frac{N\pi t}{T}\right)$$

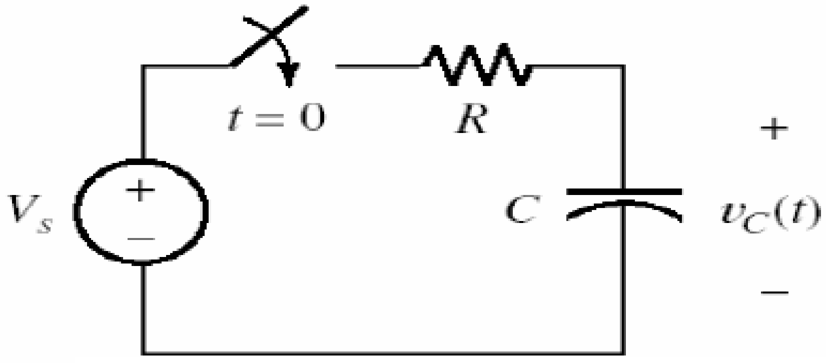
Sinusoids have following interesting property
- Derivative is a sinusoid
- Integral is a sinusoid

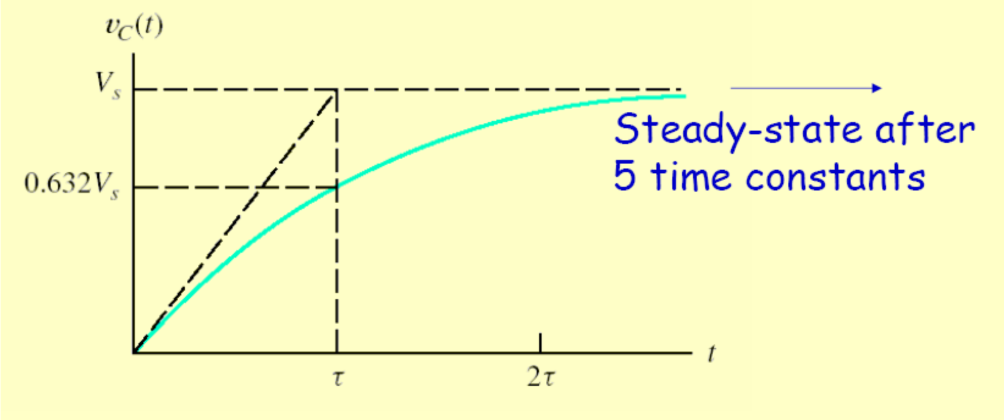
Output is sinusoidal signal with same frequency!



Transient & forced response

- Split solution into two components:
  - Transient (time-dependent component)
  - Forced (steady-state)





$$v_C(t) = \underbrace{V_s}_{\text{Steady-state or forced response}} - \underbrace{V_s e^{-t/\tau}}_{\text{Transient response}}$$

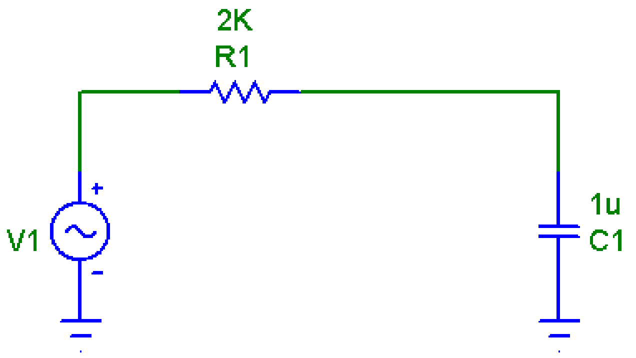
Steady-state or forced response

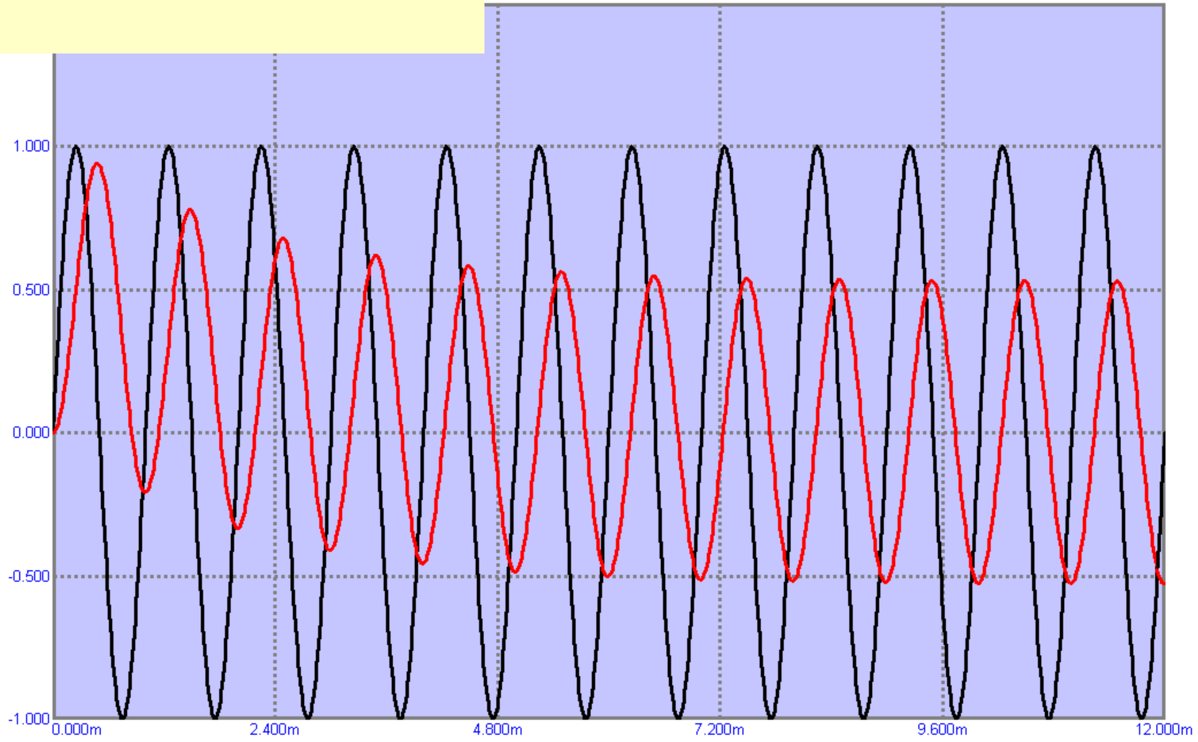
Transient response

Sinusoidal steady state

Sinusoidal Steady-State

- Whenever the forced input to the circuit is sinusoidal the response will be sinusoidal
- If the input persists, the response will persist and we call it steady-state response





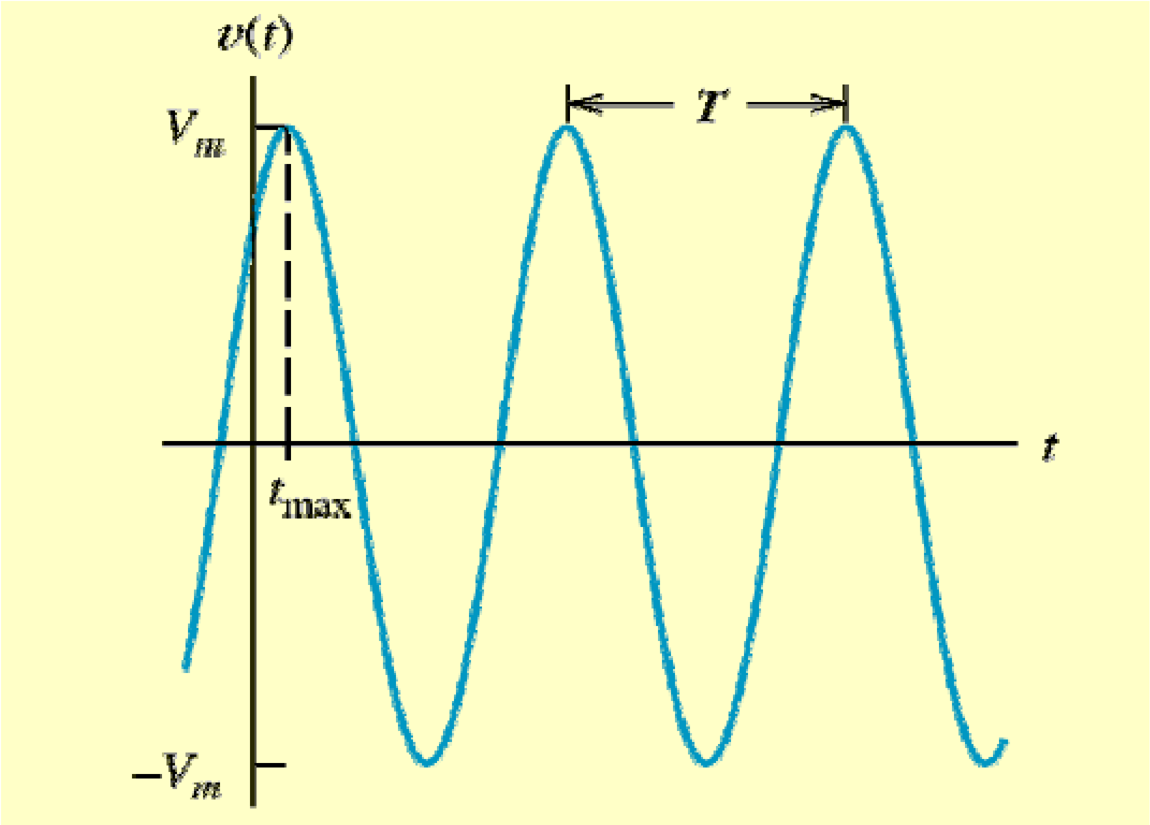
Preliminaries: Representation of Sinusoidal Signals

Canonical Form

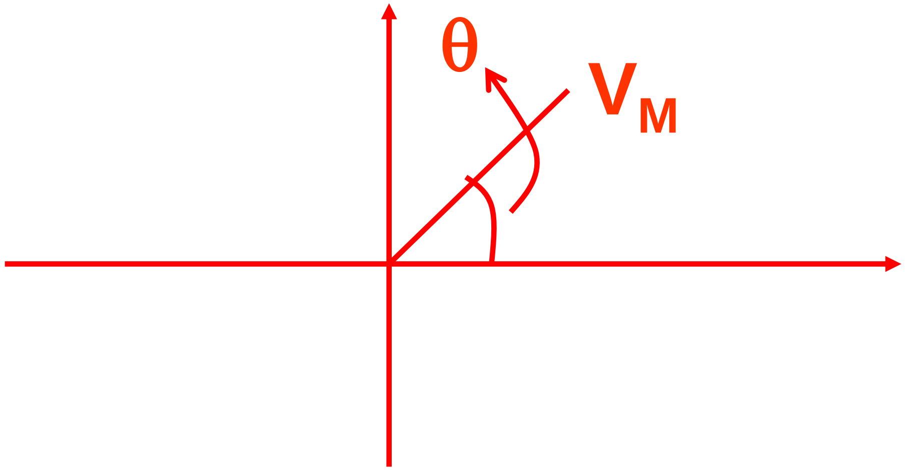
$$v(t) = V_m \cos(\omega t + \theta)$$

peak value

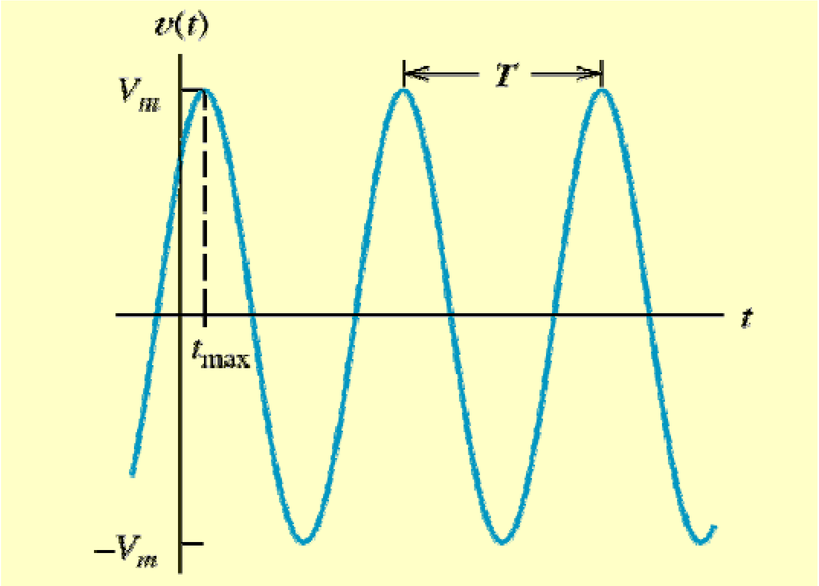
phase



Representation of Sinusoidal Signals



$$v(t) = V_m \cos(\omega t + \theta)$$



ω is the angular frequency in radians per second

T is the period , where $f = \dfrac{1}{T}$ is the frequency

$$\omega = \frac{2\pi}{T} \qquad \omega = 2\pi f$$

θ is the phase angle

Representation of Sinusoidal Signals…

$$v(t) = V_m \cos(\omega t + \theta)$$

$$\sin(z) = \cos(z - 90^\circ)$$

$$5\sin(4\pi t - 60^o)$$

$$= 5\cos(4\pi t - 60^o - 90^o)$$

Amplitude = 5 ;
Phase = -150°

Phase in radians: $360^o = 2\pi$

$$\theta = \frac{-150}{360} \times 2\pi = -2.618 \quad \text{radians}$$

$\omega = 4\pi\ rad/s$	$\omega = \frac{2\pi}{T} = 4\pi \Rightarrow T = 0.5s$	$f = \frac{1}{T} = 2Hz$

Preliminaries:

Find the phase difference between the two currents

$$i_1 = 4\sin(377t + 25^o)$$

$$i_2 = -5\cos(377t - 40^o)$$

Canonical Form $x(t) = x_m \cos(\omega t + \theta)$

$$i_1 = 4\cos(377t + 25^o - 90^o)$$

$$\theta_1 = -65^o$$

$$i_2 = 5\cos(377t - 40^o + 180^o)$$

$$\theta_2 = 140^o$$

$$\theta_1 - \theta_2 = -205^o$$

$$\sin(\omega t \pm 180^o) = -\sin \omega t$$

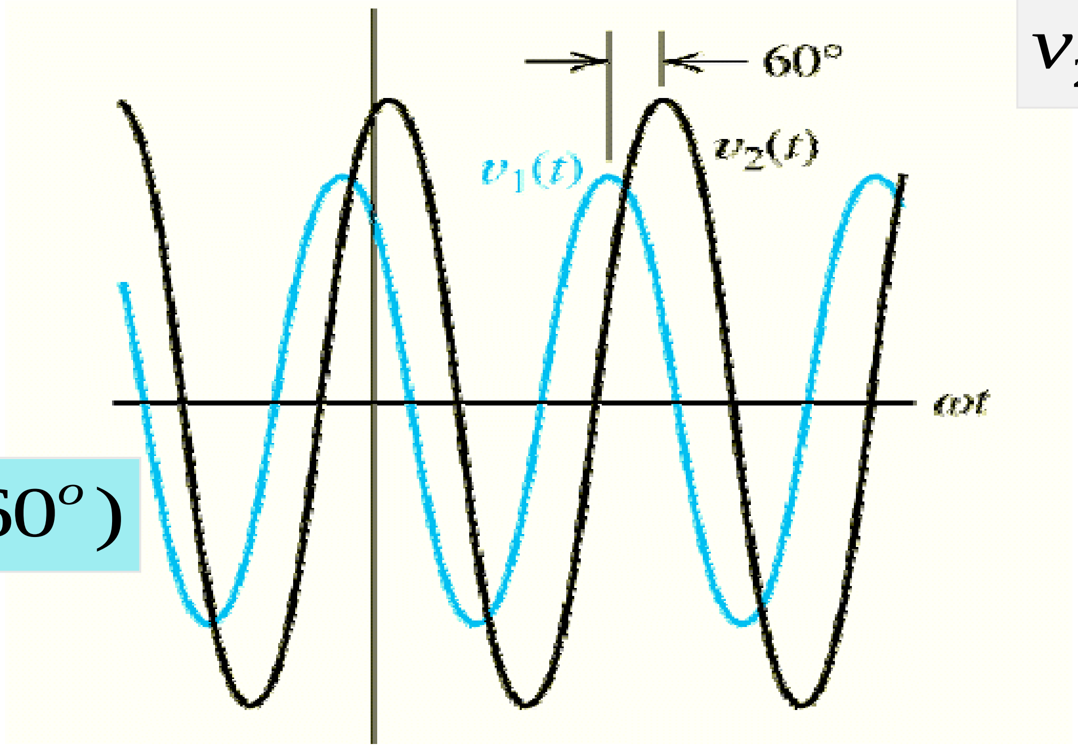
$$\cos(\omega t \pm 180^o) = -\cos \omega t$$

$$\sin(\omega t \pm 90^o) = \pm\cos \omega t$$

$$\cos(\omega t \pm 90^o) = \mp\sin \omega t$$

Phase relationship

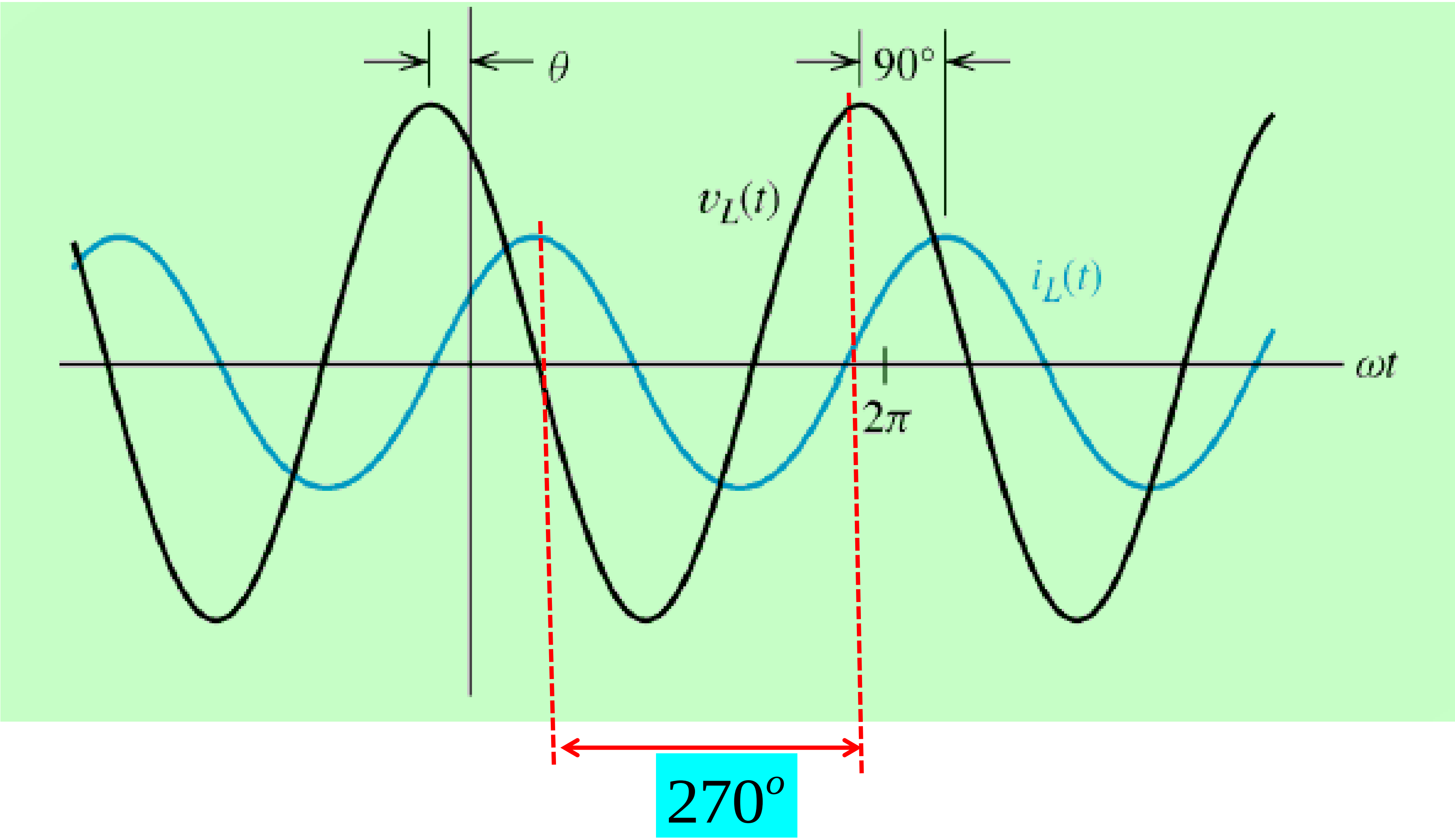
$$v_2(t) = v_{2m} \cos(\omega t)$$



$$v_1(t) = v_{1m} \cos(\omega t + 60^o)$$

The peaks of $v_1(t)$ occur 60^o before the peaks of $v_2(t)$.

In other words, $v_1(t)$ leads $v_2(t)$ by 60^o .



Voltage leads current by 90° or lags current by 270° ?

Phase difference is usually considered between -180 to 180°

Add or subtract 360° to bring the phase between -180 to 180°

$$i_1 = 4\cos(377t - 65^o) \qquad i_2 = 5\cos(377t + 140^o)$$

Does i_2 lead i_1 ?

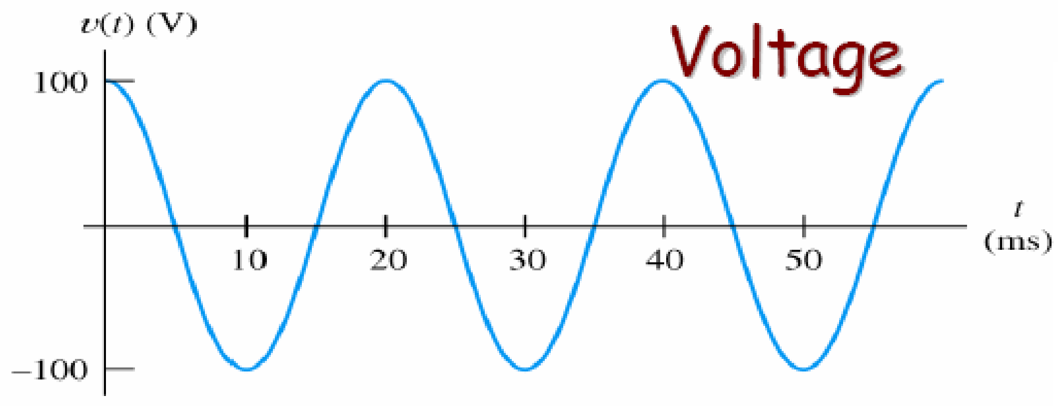
$$\theta_1 - \theta_2 = -205^o$$
$$\theta_1 - \theta_2 = -205^o + 360^o = 155^o$$

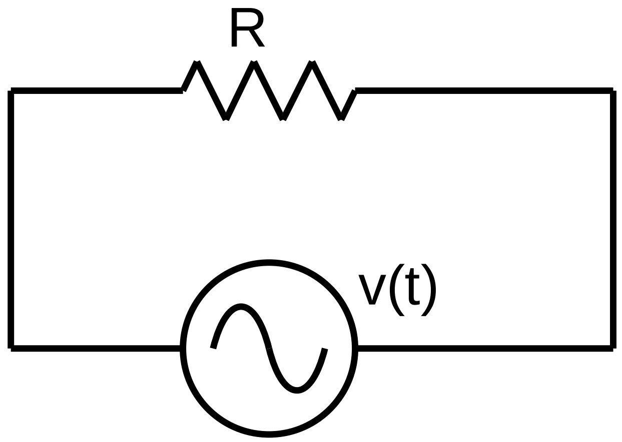
i_1 leads i_2 by 155^o

Circuit Analysis under Sinusoidal Signals

Given

$$v(t) = V_0 \cos(200t + 45^o)$$

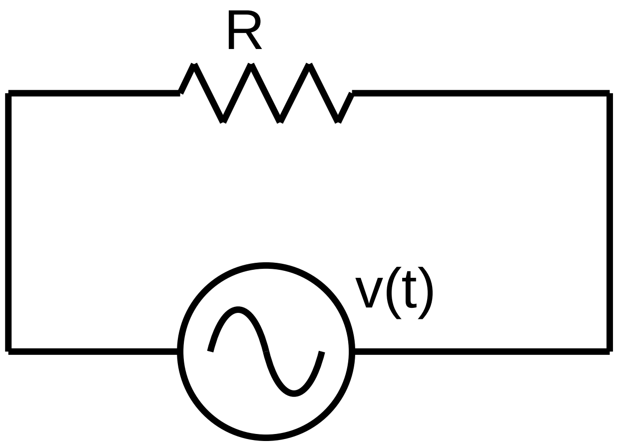




Compute current in R

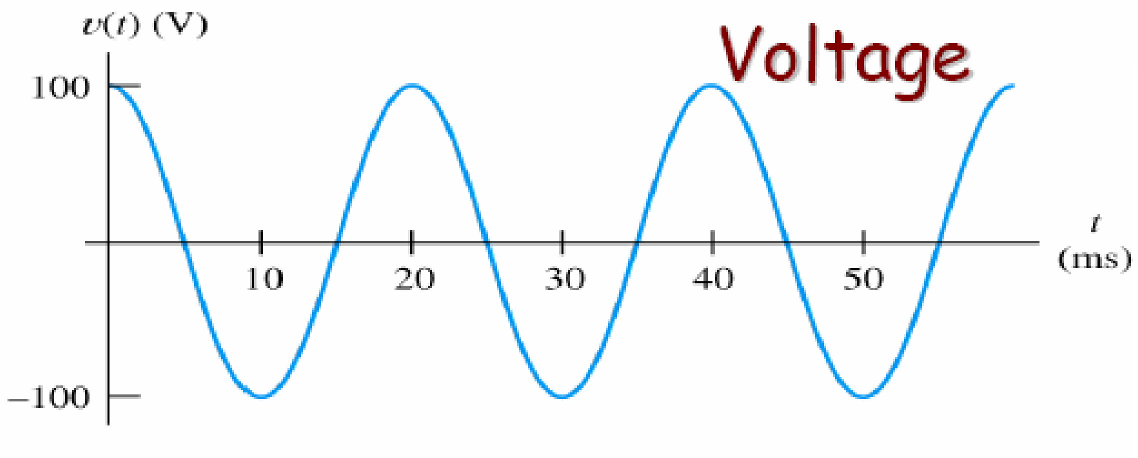
$$i(t) = \frac{v(t)}{R} = \frac{V_o}{R}\cos(200t + 45^o)$$

Power dissipation with sinusoidal Voltage

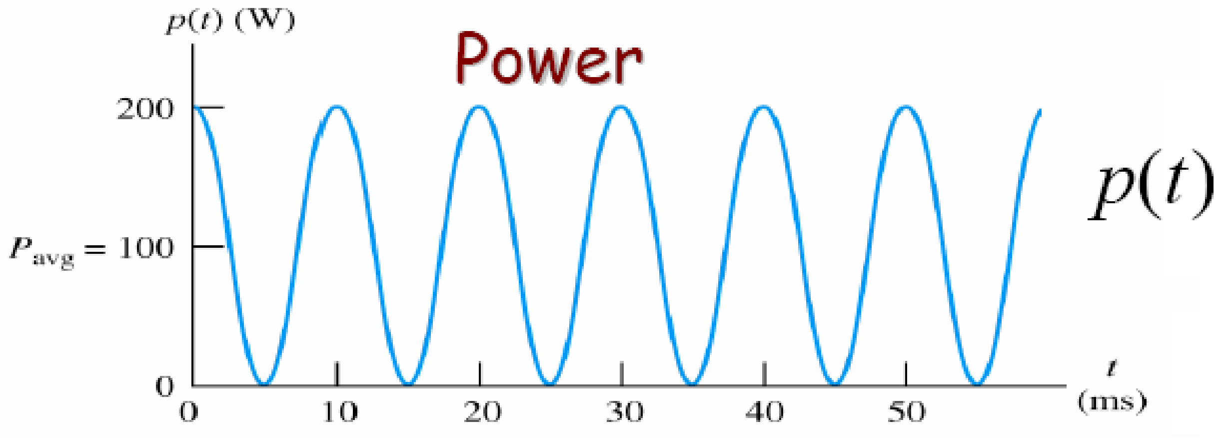


$$p = v(t)i(t) = \frac{v(t)^2}{R}$$

$$p(t) = 200\cos^2 100\pi t \;\; \mathrm{W}$$



(a)



Average Power

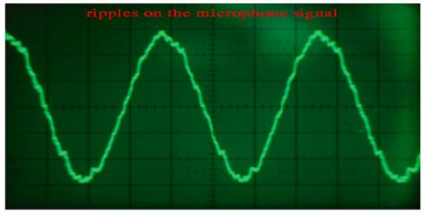
$\boldsymbol{X}$: $x_1, x_2, x_3, \dots \dots \dots .. x_N$

$$x_{avg} = \frac{1}{N} \sum x_i$$

If X is continuous, its average over a time t_1

$$x_{avg} = \frac{1}{t_1} \int_0^{t_1} x(t)dt$$

For periodic signals



$$x_{avg} = \frac{1}{T} \int_0^{T} x(t)dt$$

Average Power

$$x_{avg} = \frac{1}{T}\int_0^T x(t)dt$$

$$p(t) = \frac{v(t)^2}{R}$$

$$p_{avg} = \frac{1}{T}\int_0^T \frac{v(t)^2}{R}dt$$

$$p_{avg} = \frac{\frac{1}{T}\int_0^T v(t)^2 dt}{R}$$

$$p_{avg} = \frac{\left[\sqrt{\frac{1}{T}\int_0^T v(t)^2 dt}\right]^2}{R}$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2 dt}$$

$$p_{avg} = \frac{V_{rms}^2}{R}$$

This is true for any periodic waveform

RMS voltage and current

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2 dt}$$

$$v(t) = V_m \cos(\omega t + \theta)$$

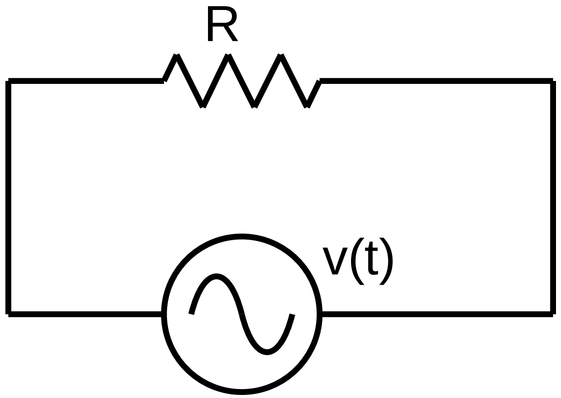
$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

$$\int_0^T \cos^2(\omega t + \theta)dt = \int_0^T \frac{1 + \cos(2\omega t + 2\theta)}{2} dt$$

$$= \frac{T}{2} + \frac{1}{4\omega}\sin(2\omega t + 2\theta)\Big|_0^T = \frac{T}{2}$$

The RMS value for a sinusoid is the peak value divided by the square root of 2

Power dissipation simplified



$$v(t) = V_m \cos(\omega t + \theta)$$

$$i(t) = I_m \cos(\omega t + \theta)$$

$$p_{avg} = \frac{V_{rms}^2}{R}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

$$p_{avg} = \frac{V_m^2}{2R}$$

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i(t)^2\, dt}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

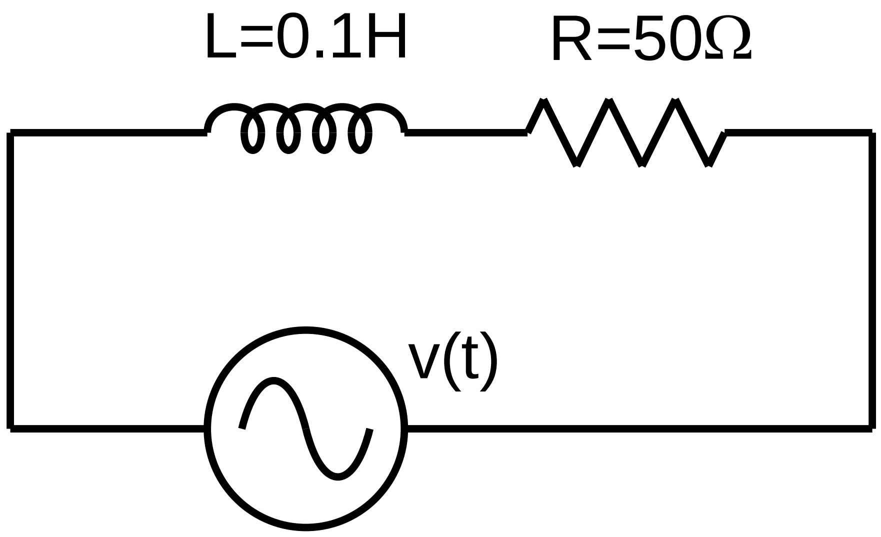
$$p_{avg} = \frac{1}{2} I_m^2 R$$

Circuit Analysis under Sinusoidal Signals

Given

$$v(t) = 2\cos(200t + 45)$$

$$v_R(t) = 1.85\cos(200t + 23.2)$$

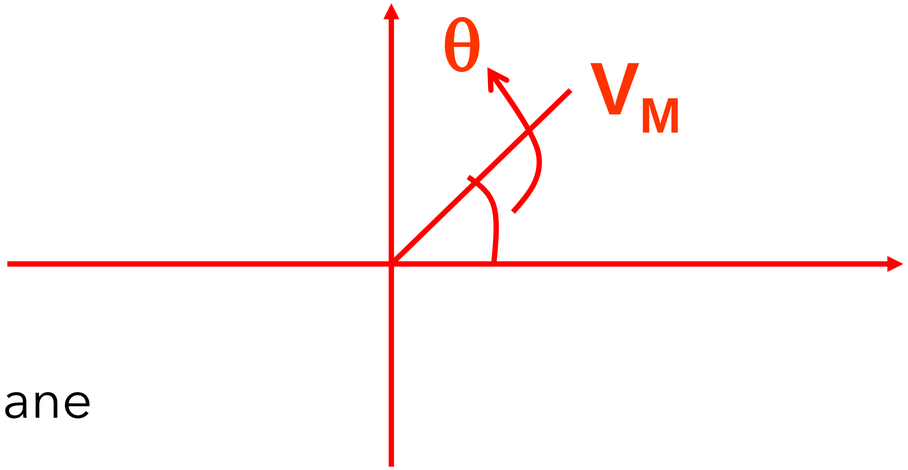


Compute $v_L(t)$

$$v_L(t) = v(t) - v_R(t)$$
$$= 2\cos(200t + 45) - 1.85\cos(200t + 23.2)$$

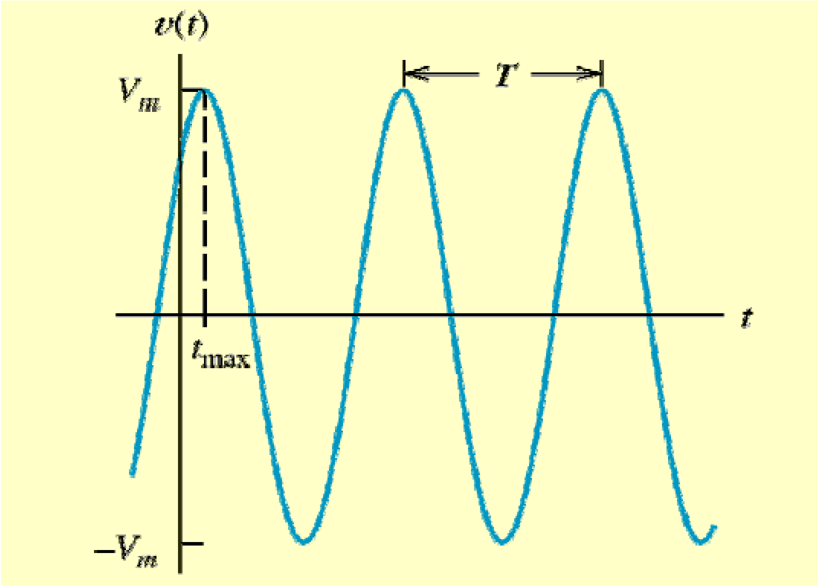
Solving such circuits requires us to add/subtract sinusoids !

Representation of Sinusoidal Signals



Complex plane

$$v(t) = V_m \cos(\omega t + \theta)$$



Performing algebra on sinusoids by representing them as complex numbers

Sinusoidal variables → Complex variables → Perform Algebra on Complex variables → Transform complex variable back to sinusoid

Complex Numbers

$$z = x + jy$$

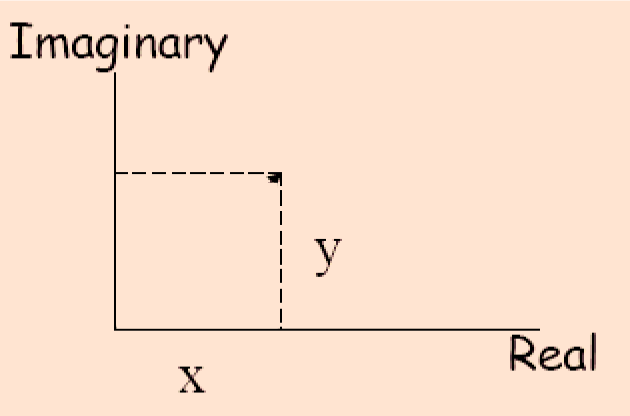
Real part

$\sqrt{-1}$

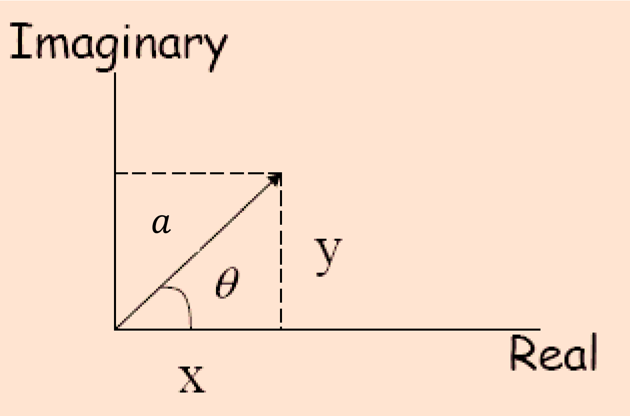
Imaginary part

$$z = a\angle\theta$$

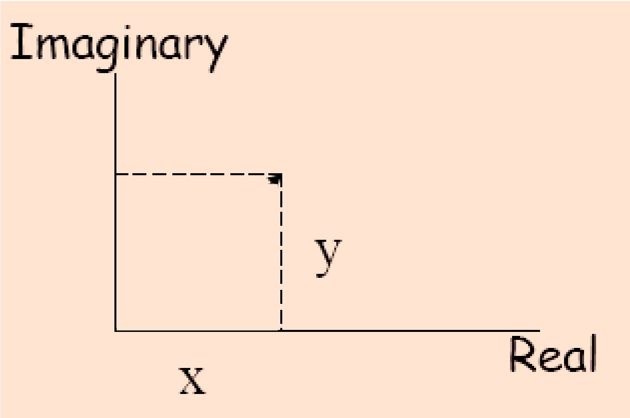
- Complex number can be represented as a point in the complex plane (2D)

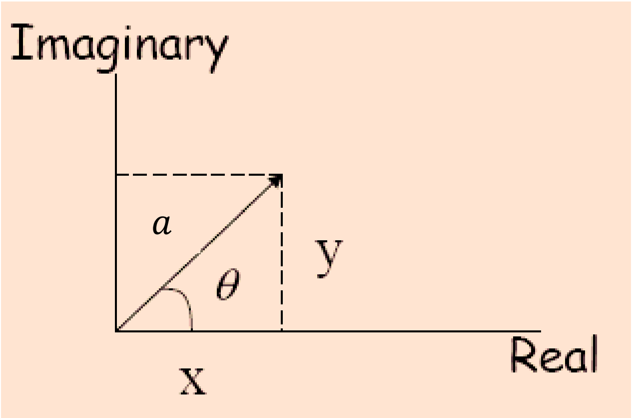


- Polar representation



Complex Numbers





$$z = x + jy$$

$$z = a\angle\theta$$

absolute value

$$|z| = a$$

$$|z| = \sqrt{x^2 + y^2}$$

phase/angle

$$\mathrm{Angle}(z) = \theta$$

$$\tan\theta = \frac{y}{x}$$

$$x = a\cos\theta$$

$$y = a\sin\theta$$

$$z = x + jy$$

$$z = a\cos\theta + ja\sin\theta$$

$$z = a(\cos\theta + j\sin\theta) \quad = ae^{j\theta}$$

Euler Identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{j\theta} = 1\angle\theta$$

$$z = a\angle\theta = ae^{j\theta} = a(\cos\theta + j\sin\theta)$$

Examples

$$z_1 = 5\angle 30^\circ$$

$$z_1 = 5\cos(30^\circ) + j5\sin(30^\circ)$$

$$= 4.33 + j2.5 = x + jy$$

$$1\angle 90^\circ = \cos 90 + j\sin 90 = j$$

$$z_2 = 10 + j5$$

$$z_2 = \sqrt{(10)^2 + (5)^2}\angle \tan^{-1}\left(\frac{5}{10}\right)$$

$$= 11.18\angle 26.57^\circ$$

$$z_3 = -10 + j5$$

$$z_3 = \sqrt{(10)^2 + (5)^2}\angle 153.43^\circ$$

Complex Numbers

$$z_1 = 5 + j5 \qquad z_2 = 3 - j4$$

$$z_1 + z_2 = (5 + j5) + (3 - j4) = 8 + j1$$

$$z_1 - z_2 = (5 + j5) - (3 - j4) = 2 + j9$$

To **add** or **subtract** two complex numbers, convert them first into rectangular form and then perform the operations

Complex Numbers Multiplication/Division

$z_1 = 5 + j5 \quad z_2 = 3 - j4$

$$
\begin{aligned}
z_1 z_2 &= (5 + j5)(3 - j4) \\
&= 15 - j20 + j15 - j^2 20 \\
&= 15 - j20 + j15 + 20 \\
&= 35 - j5
\end{aligned}
$$

Complex conjugate of z is:

$z^* = x - jy$

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{5 + j5}{3 - j4} \times \frac{z_2^*}{z_2^*} \\
&= \frac{5 + j5}{3 - j4} \times \frac{3 + j4}{3 + j4} \\
&= \frac{15 + j20 + j15 + j^2 20}{9 + j12 - j12 - j^2 16} \\
&= \frac{15 + j20 + j15 - 20}{9 + j12 - j12 + 16} \\
&= \frac{-5 + j35}{25} \\
&= -\frac{5}{25} + j\frac{35}{25} \\
&= 0.2 + j1.4
\end{aligned}
$$

Complex Numbers Multiplication/Division

$$z_1 z_2 = |z_1| e^{j\theta_1} \times |z_2| e^{j\theta_2}$$
$$= |z_1||z_2| e^{j(\theta_1+\theta_2)}$$

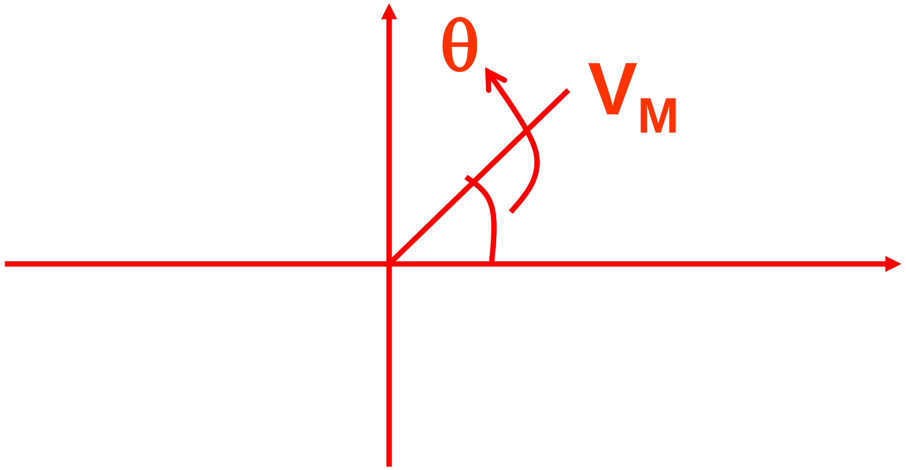
$$\frac{z_1}{z_2} = \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}}$$
$$= \frac{|z_1|}{|z_2|} e^{j(\theta_1-\theta_2)}$$

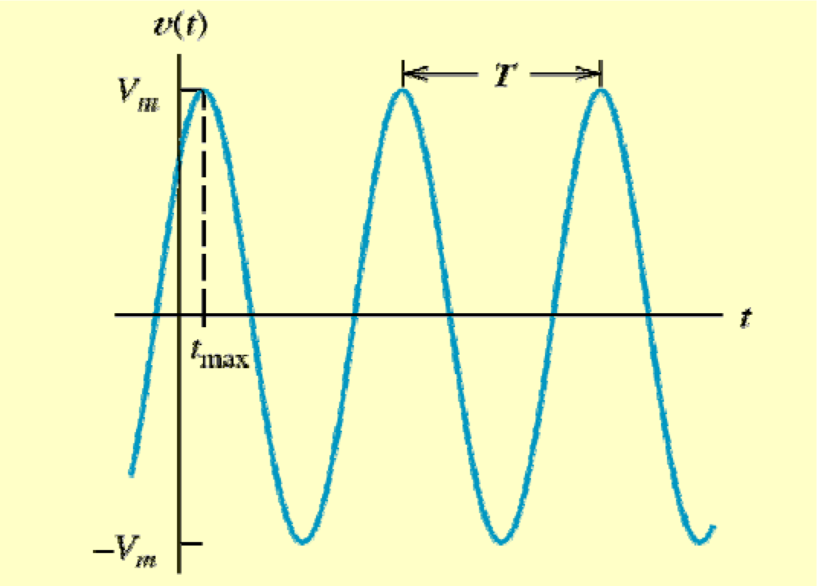
$$z_1 z_2 = |z_1| \angle\theta_1 \times |z_2| \angle\theta_2$$
$$= |z_1||z_2| \angle(\theta_1+\theta_2)$$

$$\frac{z_1}{z_2} = \frac{|z_1| \angle\theta_1}{|z_2| \angle\theta_2}$$
$$= \frac{|z_1|}{|z_2|} \angle(\theta_1-\theta_2)$$

To multiply and divide complex numbers, it is easier to use polar coordinates

Representation of Sinusoidal Signals: Phasors





Complex plane

$v(t) = V_m \cos(\omega t + \theta)$

$\rightarrow V_m \angle \theta = V_m e^{j\theta}$

$V_m \angle \omega t + \theta$ → $\mathrm{Re}(V_m \angle \omega t + \theta)$ → $V_m \cos(\omega t + \theta)$

Example

$v_1(t) = 20\cos(200t - 45^o)$

$v_2(t) = 10\cos(200t + 60^o)$

$$20\cos(\omega t - 45^\circ) \longrightarrow \mathbf{V}_1 = 20\angle -45^\circ \qquad 14.14 - j14.14$$

$$10\sin(\omega t + 60^\circ) \longrightarrow \mathbf{V}_2 = 10\angle -30^\circ \qquad 8.660 - j5$$

$$\begin{aligned}
\mathbf{V}_s &= \mathbf{V}_1 + \mathbf{V}_2 \\
&= 20\angle -45^\circ + 10\angle -30^\circ \\
&= 14.14 - j14.14 + 8.660 - j5 \\
&= 23.06 - j19.14 \\
&= 29.97\angle -39.7^\circ
\end{aligned}$$

$$v_s(t) = 29.97\cos(\omega t - 39.7^\circ)$$

Impedance Model

- Let us try to understand the “complex world” in time domain only

$$v(t) = V_m \cos(\omega t + \theta)$$

$$\mathrm{Re}(V_m \angle \omega t + \theta)$$

- Actual signal is the real part of the phasor.

- Complex signals will represent actual signal via their real part

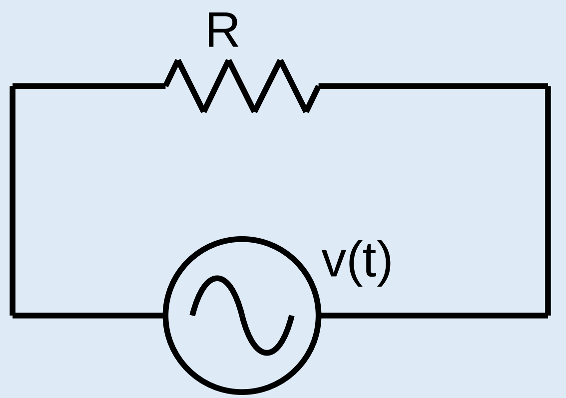
- If ωt is not written then

$$v(t) = V_m \cos(\omega t + \theta)$$

$$V_m \angle \theta$$

- shows the frequency coefficient

Impedance Model (R)



$v(t) = V_M \cos(\omega t + \theta)$

- Fix the input sinusoid frequency & consider steady state
- All currents & voltages are sinusoids at the same frequency

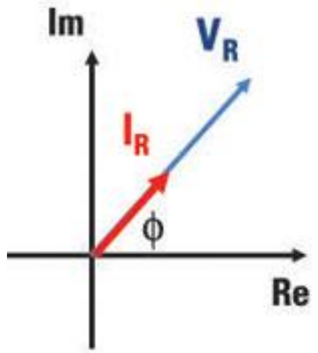
$v_R(t) = V_M \cos(\omega t + \theta)$

$i_R(t) = \dfrac{V_M}{R} \cos(\omega t + \theta)$

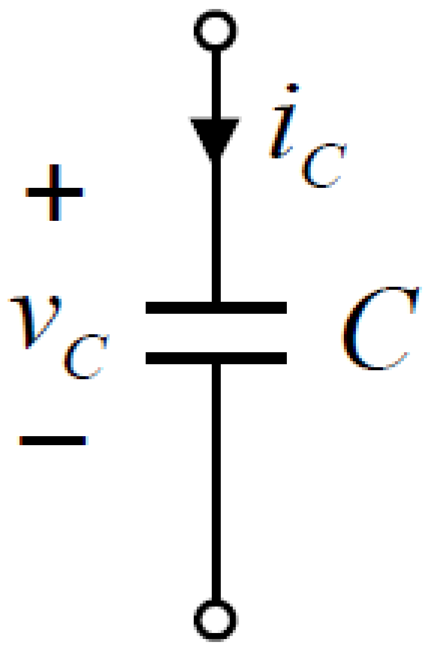
$V_R = V_M \angle \theta$

$I_R = \dfrac{V_M}{R} \angle \theta$

$$I_R = \frac{V_R}{R}$$



Impedance Model (C)



$$v_C(t) = V_M \cos(\omega t + \theta)$$

$$i_C(t) = C\frac{dv_C}{dt}$$

$$i_C(t) = \omega C V_M \cos(\omega t + \theta + 90^o)$$

$$V_C = V_M \angle \theta$$

$$I_C = \omega C V_M \angle \theta + 90$$

$$I_C = \omega C \angle 90 \times V_M \angle \theta$$

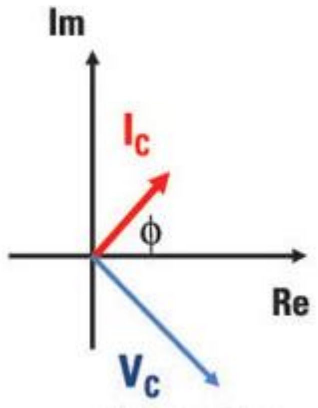
$$I_C = j\omega C V_C$$

In a capacitor, current leads voltage by 90^0

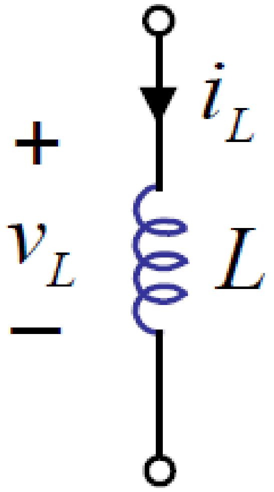
Generalized Ohm's Law for Capacitors in terms of Phasors

$$V_C = I_C Z_C$$

$$Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$$



Impedance Model (L)



$$i_L(t) = I_M \cos(\omega t + \theta)$$

$$v_L(t) = L\frac{di_L}{dt}$$

$$v_L(t) = \omega L I_M \cos(\omega t + \theta + 90^o)$$

$$I_L = I_M\angle\theta$$

$$V_L = \omega L I_M\angle\theta + 90$$

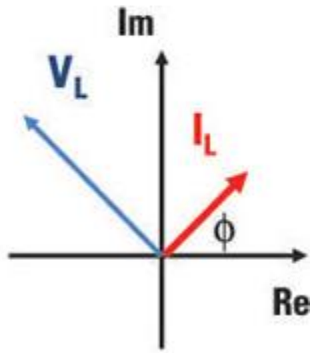
$$V_L = \omega L\angle 90 \times I_M\angle\theta$$

$$V_L = j\omega L I_L$$

In a capacitor, current lags voltage by 90^0

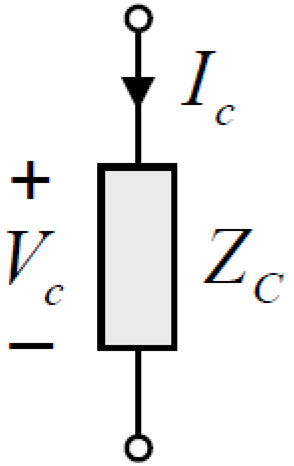
$$V_L = I_L Z_L$$

$$Z_L = j\omega L$$



Impedance Model

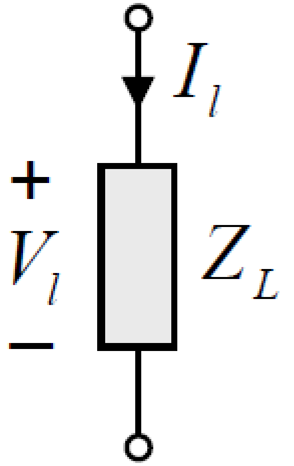
Generalized Ohm's Law for in terms of Phasors



$$V_c = Z_C\, I_c$$

$$Z_C = \frac{1}{j\omega C}$$

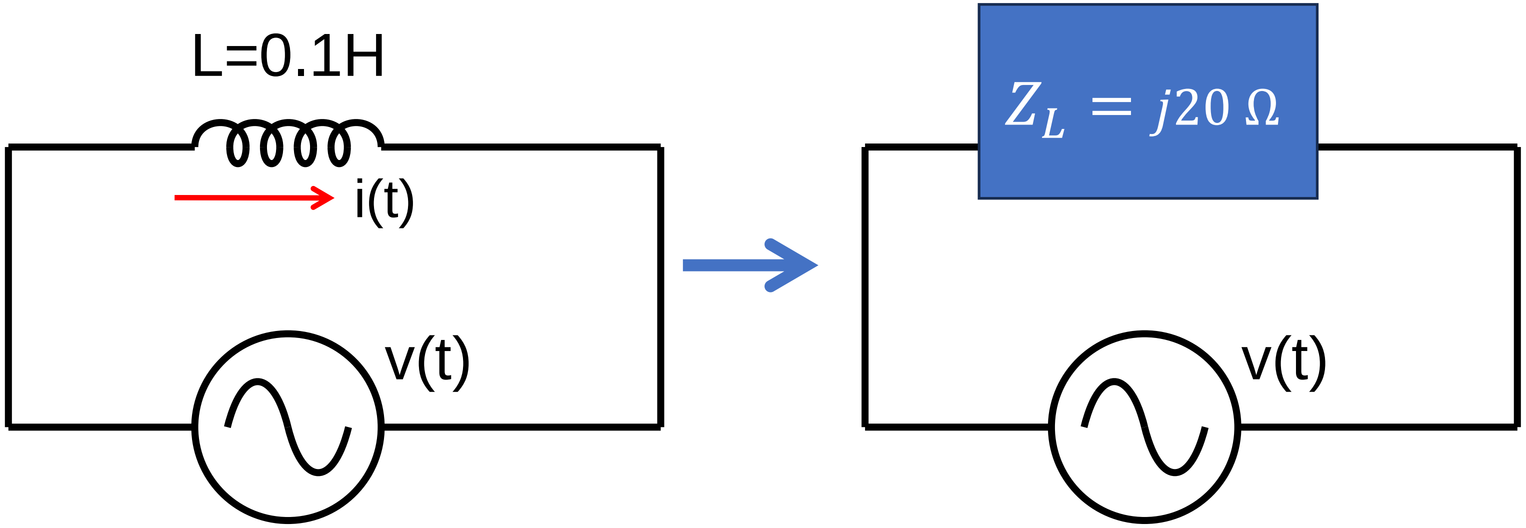
impedance



$$V_l = Z_l I_l$$

$$Z_l = j\omega L$$

Example



$$v(t) = 2\cos(200t + 45)\ \mathrm{V}$$

$$\omega = 200\ rad/s$$

$$V_L = 2\angle 45\ \mathrm{V}$$

$$Z_L = j\omega L = j20\ \Omega$$

$$i(t) = ?$$

$$i(t) = 0.1\cos(200t - 45)\ \mathrm{A}$$

$$V_L = I_L Z_L$$

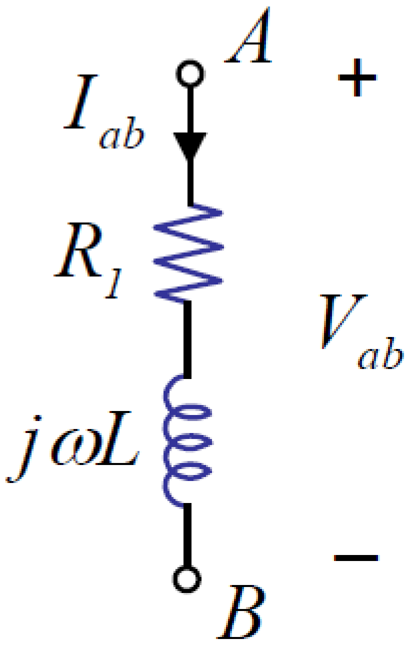
$$\Rightarrow I_L = \frac{V_L}{Z_L}$$

$$I_L = \frac{2\angle 45\ \mathrm{V}}{j20\ \Omega}$$

$$= \frac{2\angle 45}{20\angle 90}\mathrm{A}$$

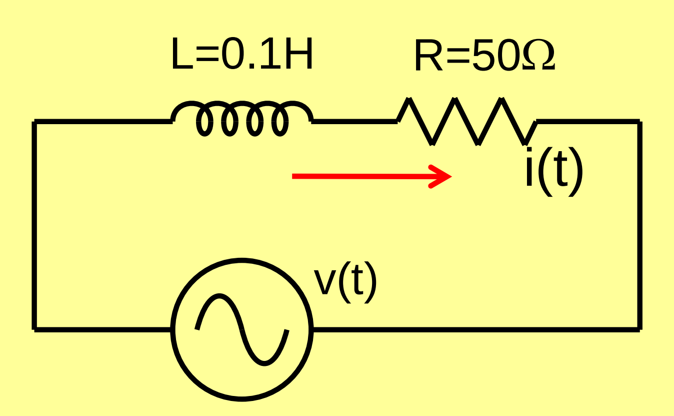
$$= 0.1\angle -45\ \mathrm{A}$$

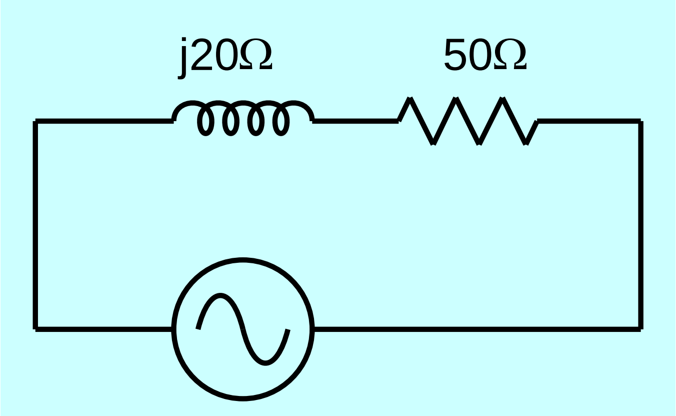
Series-parallel operations

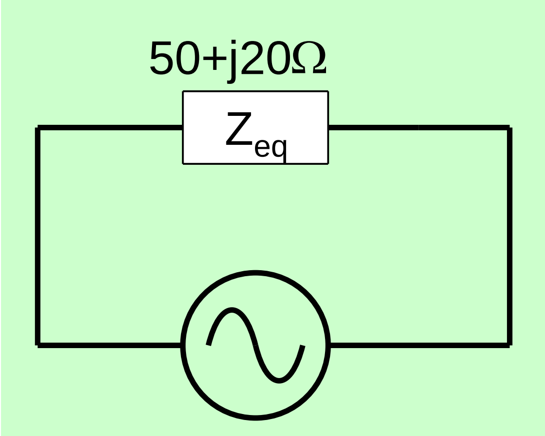


$$Z_{AB} = \frac{V_{ab}}{I_{ab}} = R_1 + j\omega L$$

Example: RL Circuit







$$v(t) = 2\cos(200t + 45)\ \mathrm{V}$$

$$i(t) = ?$$

$$i(t) = 0.037\cos(200t + 23.2)\ \mathrm{A}$$

$$\omega = 200\ \mathrm{rad/s}$$

$$V = 2\angle 45\ \mathrm{V}$$

$$Z_L = j\omega L = j20\ \Omega$$

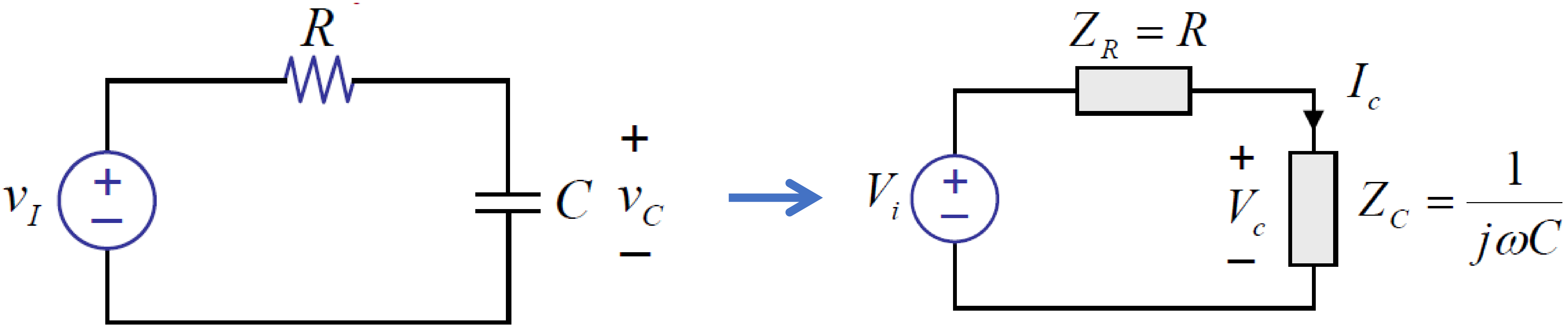
$$I = \frac{2\angle 45}{50 + j20}$$

$$= \frac{2\angle 45}{53.85\angle 21.8}$$

$$= 0.037\angle 23.2\ \mathrm{A}$$

$$V_R = 2\angle 45 \times \frac{50}{50 + j20}\ \mathrm{V}$$

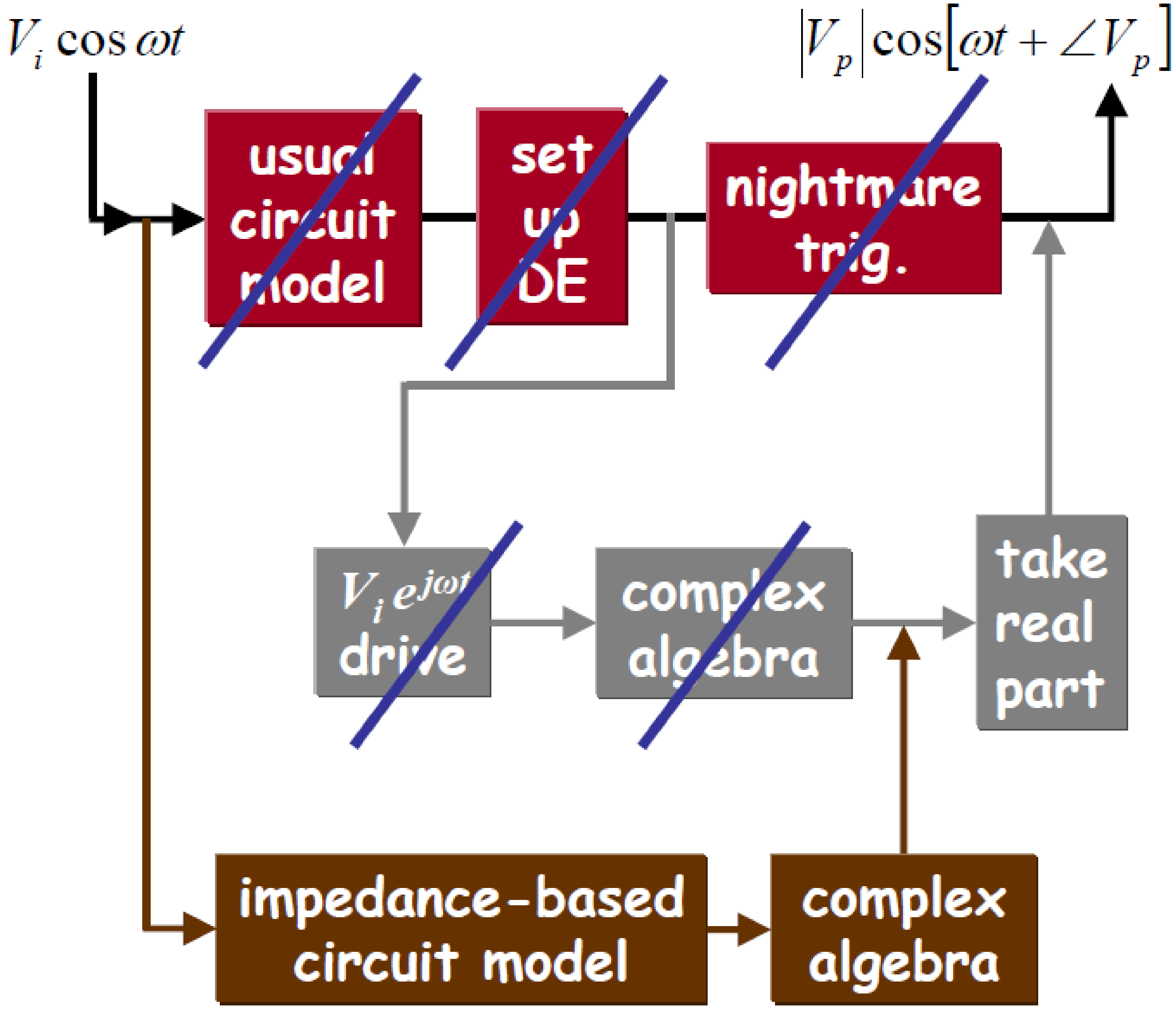
Example: RC



$$V_c = \frac{1}{1 + j\omega RC} V_i$$

Can apply KCL, KVL, Thevenin, Norton equivalents, superposition series/parallel, node method

Big Picture



$V_i \cos \omega t$
$|V_p| \cos[\omega t + \angle V_p]$
usual circuit model
set up DE
nightmare trig.
$V_i e^{j\omega t}$ drive
complex algebra
take real part
impedance-based circuit model
complex algebra