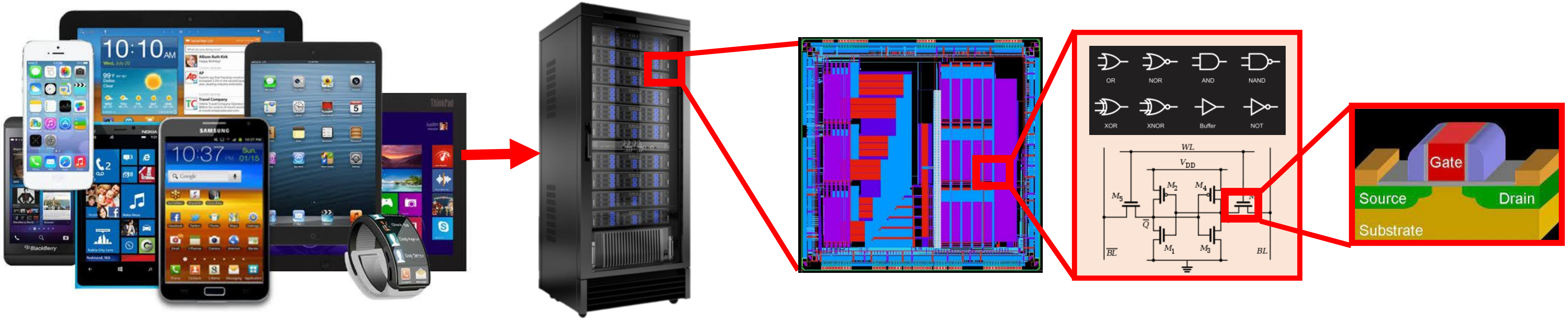

ESC201: Introduction to Electronics

Module 6: Digital Circuits

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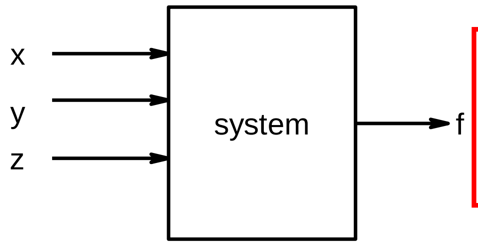
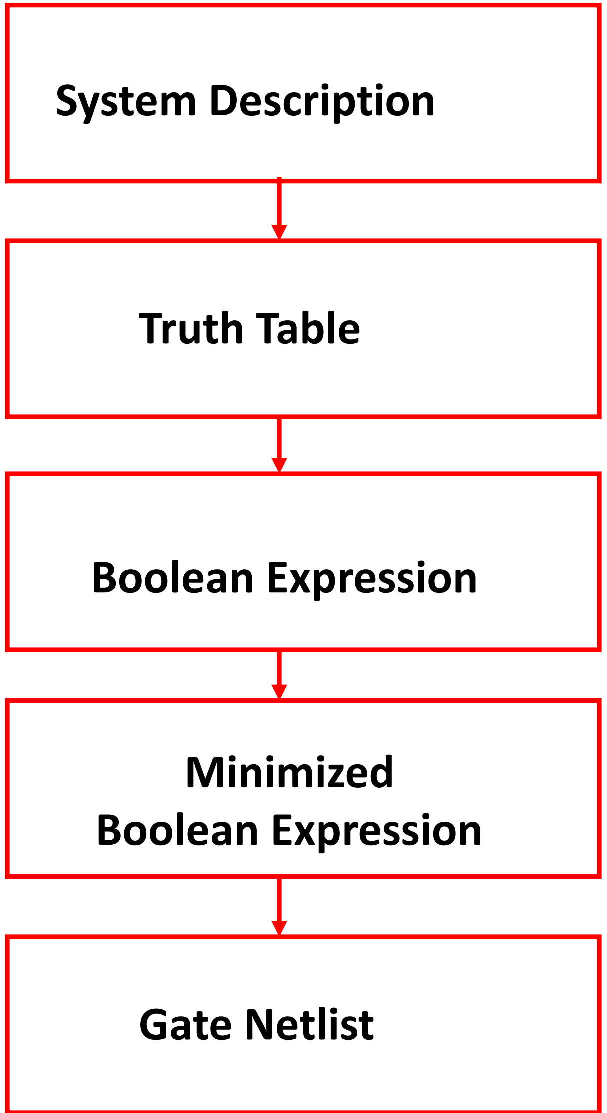
Design Flow

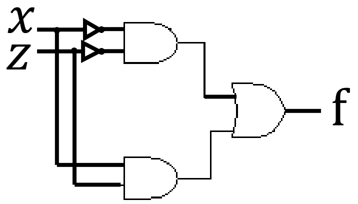




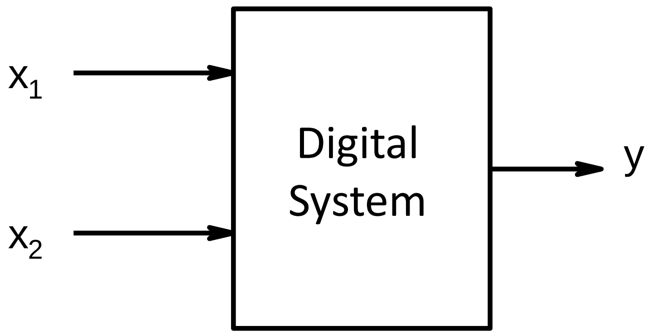
x	y	z	f
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$\mathrm{f} = \overline{x}.\overline{y}.\overline{z} + \overline{x}.y.\overline{z} + x.\overline{y}.z + x.y.z$$

$$\Rightarrow \mathrm{f} = \overline{x}.\overline{z} + x.z$$



Representation of a Digital System



Description in words

$y = 1$ when x_1 is 0 and x_2 is 1

Truth Table

Indicates when response y is 'true'

x_1	x_2	y
0	0	0
0	1	1
1	0	0
1	1	0

Boolean expression

$$y = \overline{x_1} \,.\, x_2$$

SoP Form With Min Terms for Two Inputs

A **min term** is a **product (AND)** that contains all the variables used in a function

The function is the **sum (OR) of min terms for which output function is 'True'**

MSB x	LSB y	min term	
0	0	$\bar{x}.\bar{y}$	m0
0	1	$\bar{x}.y$	m1
1	0	$x.\bar{y}$	m2
1	1	$x.y$	m3

Example

x	y	f_1
0	0	0
0	1	1
1	0	1
1	1	0

$$f_1 = \sum(1,2) = m_1 + m_2 = \bar{x}.y + x.\bar{y}$$

Example

$$f_2 = \sum(0,2,3) = ?$$

$$f_2 = \bar{x}.\bar{y} + x.\bar{y} + x.y$$

We are showing function in **Sum of Products (SoP)** form

Min Terms for Three Inputs

x (MSB)	y	z (LSB)	min terms	
0	0	0	$\overline{x}\,.\,\overline{y}\,.\,\overline{z}$	m0
0	0	1	$\overline{x}\,.\,\overline{y}\,.\,z$	m1
0	1	0	$\overline{x}\,.\,y\,.\,\overline{z}$	m2
0	1	1	$\overline{x}\,.\,y\,.\,z$	m3
1	0	0	$x\,.\,\overline{y}\,.\,\overline{z}$	m4
1	0	1	$x\,.\,\overline{y}\,.\,z$	m5
1	1	0	$x\,.\,y\,.\,\overline{z}$	m6
1	1	1	$x\,.\,y\,.\,z$	m7

$$\mathrm{f}_2 = \sum(1,4,7) = ? \quad \Rightarrow \quad \mathrm{f}_2 = \overline{x}.\,\overline{y}.z + x.\overline{y}.\overline{z} + x.y.z$$

Sum of Products form of function for three input variables

PoS Form With Max Terms for Two Inputs

A **max term** is a **sum (OR)** that contains all the variables used in a function

The function is the **product (AND) of max terms for which output function is 'False'**

x (MSB)	y (LSB)	Max term	
0	0	$x + y$	M0
0	1	$x + \overline{y}$	M1
1	0	$\overline{x} + y$	M2
1	1	$\overline{x} + \overline{y}$	M3

Example

x	y	f_1
0	0	0
0	1	1
1	0	1
1	1	0

$$f_1 = \Pi(0,3)$$

$$f_1 = M_0 . M_3$$

$$f_1 = (x + y).(\overline{x} + \overline{y})$$

Example

$$f_2 = \Pi(1,2) = ? \Rightarrow f_2 = (x + \overline{y}).(\overline{x} + y)$$

Product of Sums (PoS) form of function for two input variables

Max Terms for Three Variables

x (MSB)	y	z (LSB)	Max. terms	
0	0	0	$x + y + z$	M0
0	0	1	$x + y + \overline{z}$	M1
0	1	0	$x + \overline{y} + z$	M2
0	1	1	$x + \overline{y} + \overline{z}$	M3
1	0	0	$\overline{x} + y + z$	M4
1	0	1	$\overline{x} + y + \overline{z}$	M5
1	1	0	$\overline{x} + \overline{y} + z$	M6
1	1	1	$\overline{x} + \overline{y} + \overline{z}$	M7

$f_1 = \Pi(1,5,7) = ?$ $\Rightarrow$ $f_1 = (x + y + \overline{z}).(\overline{x} + y + \overline{z}).(\overline{x} + \overline{y} + \overline{z})$

Product of Sum form of function for three input variables

But Can We Simplify Final Expression?

Term No.	Min term	Max term	x_1	x_2	x_3	y
0.	$\overline{x_1}.\overline{x_2}.\overline{x_3}$	$x_1 + x_2 + x_3$	0	0	0	0
1.	$\overline{x_1}.\overline{x_2}.x_3$	$x_1 + x_2 + \overline{x_3}$	0	0	1	1
2.	$\overline{x_1}.x_2.\overline{x_3}$	$x_1 + \overline{x_3} + x_3$	0	1	0	0
3.	$\overline{x_1}.x_2.x_3$	$x_1 + \overline{x_2} + \overline{x_3}$	0	1	1	1
4.	$x_1.\overline{x_2}.\overline{x_3}$	$\overline{x_1} + x_2 + x_3$	1	0	0	0
5.	$x_1.\overline{x_2}.x_3$	$\overline{x_1} + x_2 + \overline{x_3}$	1	0	1	1
6.	$x_1.x_2.\overline{x_3}$	$\overline{x_1} + \overline{x_2} + x_3$	1	1	0	0
7.	$x_1.x_2.x_3$	$\overline{x_1} + \overline{x_2} + \overline{x_3}$	1	1	1	1

By inspection
$y = x_3$
x_1 and x_2 are "don't cares"

Typical simplifications are not always so obvious!

SOP form: $y = \sum(1,3,5,7)) = \overline{x_1}.\overline{x_2}.x_3 + \overline{x_1}.x_2.x_3 + x_1.\overline{x_2}.x_3 + x_1.x_2.x_3$

POS form: $y = \Pi(0,2,4,6) = (x_1 + x_2 + x_3).(x_1 + \overline{x_2} + x_3).(\overline{x_1} + x_2 + x_3).(\overline{x_1} + \overline{x_2} + x_3)$

On simplification, $y = x_3$ ← How to arrive at simplified form efficiently?

Can use Karnaugh Maps (K-maps) and other approaches to get appropriate expression

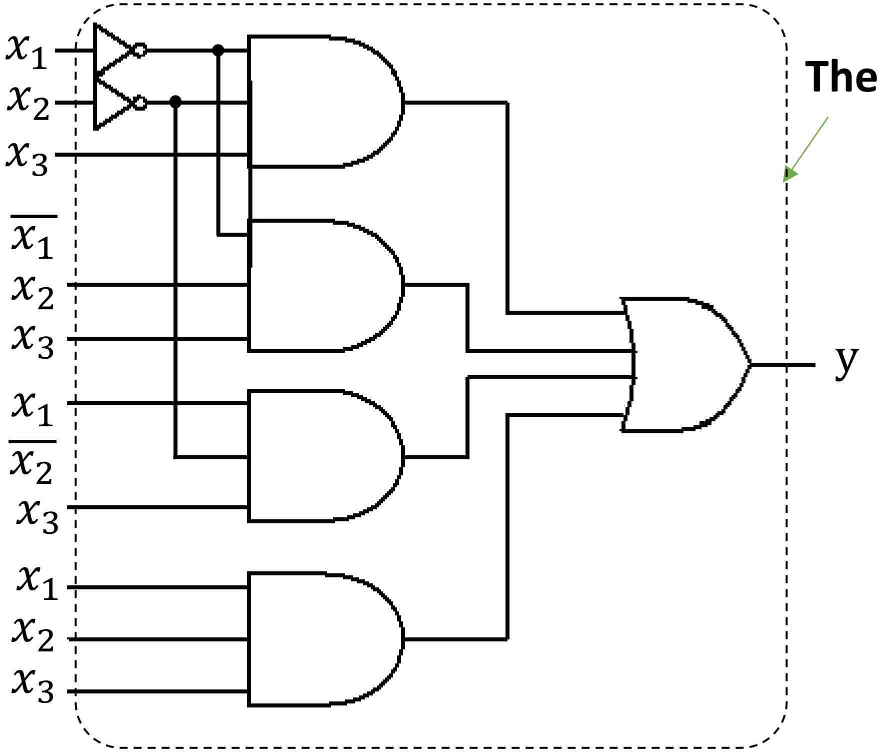
A Case for Simplified Expression

Truth Table

x_1	x_2	x_3	y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Boolean Expression:

$$y = \overline{x_1}.\overline{x_2}.x_3 + \overline{x_1}.x_2.x_3 + x_1.\overline{x_2}.x_3 + x_1.x_2.x_3$$
$$= (x_1 + x_2 + x_3).(x_1 + \overline{x_2} + x_3).(\overline{x_1} + x_2 + x_3).(\overline{x_1} + \overline{x_2} + x_3)$$



The hardware

Simplified Boolean Expression: $y = x_3$

The actual hardware required

This might be an exaggerated example, but the point for simplification of expression is made.

Goal of Simplification

1. Minimize number of product (or sum) terms

2. Minimize number of literals in each term

$$\text{Simplification} \Rightarrow \text{Minimization}$$

If nothing is known about the digital design, the above is a good thumb-rule

There may some exceptions for some design constraints

Minimization

Example

$$y = \overline{x_1}.\overline{x_2}.x_3 + \overline{x_1}.x_2.x_3 + x_1.\overline{x_2}.x_3 + x_1.x_2.x_3$$

$$y = \overline{x_1}.x_3.(\overline{x_2} + x_2) + x_1.x_3.(\overline{x_2} + x_2)$$

$$y = \overline{x_1}.x_3 + x_1.x_3$$

$$y = (\overline{x_1} + x_1).x_3$$

$$\boxed{y = x_3}$$

Principle used: $x + \overline{x} = 1$

Example $f=\bar{x}.\bar{y}+\bar{x}.y+x.\bar{y}$

Apply the Principle: $x+\bar{x}=1$ to simplify

$$f=\bar{x}.(\bar{y}+y)+x.\bar{y}$$

$$f=\bar{x}+x.\bar{y}$$

How do we simplify further?

Principle used : $x + x = x$

$$f=\bar{x}.\bar{y}+\bar{x}.y+x.\bar{y}=\bar{x}.\bar{y}+\bar{x}.\bar{y}+\bar{x}.y+x.\bar{y}$$

$$f=\bar{x}.\bar{y}+\bar{x}.y+\bar{x}.\bar{y}+x.\bar{y}$$

$$=\bar{x}.(\bar{y}+y)+(\bar{x}+x).\bar{y}=\boxed{\bar{x}+\bar{y}}$$

Example

Simplify

$$f=\bar{x}_1.x_2.\bar{x}_3.x_4+\bar{x}_1.x_2.x_3.x_4+x_1.x_2.\bar{x}_3.x_4+x_1.x_2.x_3.x_4+$$
$$x_1.\bar{x}_2.\bar{x}_3.x_4+x_1.\bar{x}_2.x_3.x_4$$

Work this out!

Simplification of Boolean Expressions

Principle: $x + \overline{x} = 1$ and $x + x = x$

But factorising appropriately is a challenge!

Need a systematic and simpler method for applying these two principles

1881		1952		1954
Marquand diagram	→	**Veitch chart**	→	**Karnaugh maps**
Allen Marquand		Edward W. Veitch		Maurice Karnaugh
				
history-computer.com		wikipedia		alchetron.com

Karnaugh Map (K map) is a popular technique for carrying out simplification

It represents the information in problem in such a way that the two principles become easy to apply

K-map Representation of Truth Table

x	y	min term	
0	0	$\bar{x} . \bar{y}$	m0
0	1	$\bar{x} . y$	m1
1	0	$x . \bar{y}$	m2
1	1	$x . y$	m3

x \ y	0	1
0	m_0	m_1
1	m_2	m_3

Example

x	y	f_1
0	0	0
0	1	1
1	0	1
1	1	0

$\Rightarrow$

x \ y	0	1
0	0	1
1	1	0

Example

$$f_2 = \sum(1,2,3) \quad \Longrightarrow$$

x \ y	0	1
0	0	1
1	1	1

Example

x \ y	0	1
0	1	0
1	0	1

$$\Longrightarrow \quad f = \bar{x}.\bar{y} + x.y$$

3-variable K-map representation

x	y	z	min terms	
0	0	0	$\bar{x} \cdot \bar{y} \cdot \bar{z}$	m0
0	0	1	$\bar{x} \cdot \bar{y} \cdot z$	m1
0	1	0	$\bar{x} \cdot y \cdot \bar{z}$	m2
0	1	1	$\bar{x} \cdot y \cdot z$	m3
1	0	0	$x \cdot \bar{y} \cdot \bar{z}$	m4
1	0	1	$x \cdot \bar{y} \cdot z$	m5
1	1	0	$x \cdot y \cdot \bar{z}$	m6
1	1	1	$x \cdot y \cdot z$	m7

x \ yz	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Example

x	y	z	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

x \ yz	00	01	11	10
0	0	1	1	0
1	0	1	1	0

Example

x \ yz	00	01	11	10
0	1	0	1	0
1	0	1	1	0

$$f = \bar{x}.\bar{y}.\bar{z} + \bar{x}.y.z + x.\bar{y}.z + x.y.z$$

Please give it a try to find the simplest expression from the K-map.

4-variable K-map representation

w	x	y	z	min terms
0	0	0	0	m_0
0	0	0	1	m_1
0	0	1	0	m_2
0	0	1	1	m_3
⋮	⋮	⋮	⋮	⋮
1	1	1	0	m_{14}
1	1	1	1	m_{15}

wx \ yz	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

Example

wx \ yz	00	01	11	10
00	1	0	1	0
01	0	1	1	0
11	1	0	0	1
10	1	0	0	0

$$f = \overline{w}.\overline{x}.\overline{y}.\overline{z} + \overline{w}.\overline{x}.y.z + \overline{w}.x.\overline{y}.z + \overline{w}.x.y.z$$
$$+ w.x.\overline{y}.\overline{z} + w.x.y.\overline{z} + w.\overline{x}.\overline{y}.\overline{z}$$

Minimisation using K-map

Example

$$f_2 = \sum(2,3)$$

x \ y	0	1
0	0	0
1	1	1

$$f = x.\overline{y} + x.y$$

$$f = x.(\overline{y} + y)$$

$$f = x$$

Combine terms which differ in only one bit position. As a result, whatever is common remains.

Example

x \ y	0	1
0	0	1
1	0	1

$$f = \bar{x}.y + x.y$$

$$f = (\bar{x} + x).y \qquad \Rightarrow f = y$$

Example

x \ y	0	1
0	1	0
1	1	0

$$\Rightarrow f = \bar{y}$$

Example

x \ y	0	1
0	1	1
1	0	0

$$\Rightarrow f = \bar{x}$$

Example

$$f_2 = \sum(1,2,3)$$

x \ y	0	1
0	0	1
1	1	1

$$f = x.\bar{y} + x.y + \bar{x}.y$$

$$f = x.(\bar{y} + y) + \bar{x}.y$$

$$= x + \bar{x}.y$$

$$f = x + \bar{x}.y + x.y$$

$$= x + (\bar{x} + x).y$$

$$= x + y$$

The idea is to cover <u>all the 1's</u> with <u>as few terms</u> as possible

3-variable Minimisation

Example

x \ yz	00	01	11	10
0	1	0	1	0
1	0	1	1	0

$$y.z$$

$$x.z$$

$$f = \bar{x}.\bar{y}.\bar{z} + \bar{x}.y.z + x.y.z + x.\bar{y}.z$$

$$f = \bar{x}.\bar{y}.\bar{z} + y.z + x.z$$

Example

x \ yz	00	01	11	10
0	1	0	0	1
1	0	1	1	0

$$\bar{x}.\bar{z}$$

$$x.z$$

$$f = \bar{x}.\bar{y}.\bar{z} + \bar{x}.y.\bar{z} + x.y.z + x.\bar{y}.z$$

$$f = \bar{x}.\bar{z} + x.z$$

Remember the K-map folds around!

Example

x \ yz	00	01	11	10
0	0	0	0	0
1	1	1	1	1

$x.\bar{y}$

$x.y$

$$f = x.\bar{y}.\bar{z} + x.\bar{y}.z + x.y.z + x.y.\bar{z}$$

$$f = x.\bar{y} + x.y$$

x \ yz	00	01	11	10
0	0	0	0	0
1	1	1	1	1

x

$$f = x.(\bar{y} + y) = x$$

Example

x \ yz	00	01	11	10
0	1	1	1	1
1	0	0	0	0

$\bar{x}$

Example

x \ yz	00	01	11	10
0	1	0	0	1
1	1	0	0	1

$\bar{z}$

Example

x \ yz	00	01	11	10
0	0	1	1	0
1	0	1	1	0

z

Example

x \ yz	00	01	11	10
0	1	0	0	1
1	1	1	1	1

$\bar{z}$

x

$$f = x + \bar{z}$$

Can We Group 3 Min Terms?

Can we do this ?

x \ yz	00	01	11	10
0	0	0	0	0
1	1	1	1	0

x \ yz	00	01	11	10
0	0	0	0	0
1	1	1	1	0

Note that each encirclement should represent a single product term.

In this case it does not represent a single product term**.**

$$
\begin{aligned}
f &= x.\overline{y}.\overline{z} + x.\overline{y}.z + x.y.z \\
&= x.\overline{y} + x.z
\end{aligned}
$$

We do not get a single product term.

Grouping in 3 terms will not help in minimization of terms in function.

Reordering of Numbering not Beneficial for Simplification

Can we use K-map with the following ordering of variables?

NOT A GOOD IDEA!

x \ yz	00	01	10	11
0	0	0	0	0
1	0	1	1	0

Can we combine these two terms into a single term ?

$$f = x.\overline{y}.z + x.y.\overline{z}$$

$$= x.(\overline{y}.z + y.\overline{z})$$

Note that no simplification is possible here.

K-map requires variable to change one bit between adjacent cells

- Continued -

x \ yz	00	01	10	11
0	0	1	0	1
1	0	0	0	0

NOT A GOOD IDEA!

These two terms can be combined into a single term but it is not easy to show that on the diagram.

$$f = \bar{x}.\bar{y}.z + \bar{x}.y.z$$
$$= \bar{x}.(\bar{y} + y).z = \bar{x}.z$$

Kmap requires information to be represented in such a way that it is easy to apply the principle $x + \bar{x} = 1$