Class Notes| Week 1

MSO: Introduction to Probability Theory

Fall 2024

Contents

1 Basics of Probability

1.1 Axioms and Definitions

Definition 1 (Random Experiment)**.** An experiment in which:

- all possible outcomes of the experiment are known in advance
- outcome of a particular trial/performance of the experiment cannot be specified in advanced
- the experiment can be repreated under identical conditions

Denoted via ε

Definition 2 (Sample Space)**.** The collection of all possible outcomes of the random experiment ε is called its sample space. Example, for the random experiment of tossing three coins at the same time, the sample space shall be:

$$\Omega = \{(x, y, z) : x, y, z \in H, T\}$$

Definition 3 (Events)**.** If the outcome of a random experiment ε is an element of a subset $\mathcal{E}$ of Ω, then we say that the event $\mathcal{E}$ has occured. An event is an set object.

Collection of all events is denoted as $\mathcal{F}$ and is called the **Event Space** This means $\mathcal{F}$ is a set of sets. Empty Set $\emptyset$ and the sample space Ω will always be an element in $\mathcal{F}.\mathcal{F}$ is the power set of Ω.

Definition 4 (Event Space)**.** We say that $\mathcal{F}$ is an event space if:

1 $\Omega \in \mathcal{F}$

2 If $A \in \mathcal{F}$, then $A' \in \mathcal{F}$

3 If $A_1, A_2, ..., A_n \in \mathcal{F}$, then $A_1 \cup A_2 \cup ... \cup A_n \in \mathcal{F}$

Definition 5 (Probability, Classical (A priori) Definition)**.** Suppose that a random experiment results in n (a finite number) outcomes. Given an event $A \in \mathcal{F}$, if it apears in m ($0 \leq m \leq n$) outcomes, then the probability of A is $\frac{m}{n}$.

The classical definition only works when there are finitely many outcomes. Due to the limitations of this definition, we look for other ways to understand the notion of probability.

Definition 6 (Probability, Relative Frequency(A posteriori) Definition)**.** If a random experiment $\mathcal{E}$ is repeared a large number, say n, of times and an event A occurs m many times, then the relative frequency $\frac{m}{n}$ may be taken as an approximate value of a probability of A.

The a posteriori definition of probability works only after performing the random experiment.

Definition 7 (Set Function)**.** A set function is a function whose domain is a collection/class of sets

Definition 8 (Probability functiion/measure)**.** Suppose that Ω and $\mathcal{F}$ are the sample space and the event event space of an event $\mathcal{E}$ repspectively. A real valued set function $\mathbb{P}$, defined on event space $\mathcal{F}$, is said to be the Probability function/measure if it satisfies the following properties:

- $\mathbf{P}(\emptyset) = 0$
- (non-negativity) $\mathbf{P}(E) \geq 0$ for any event E in $\mathcal{F}$
- (Countable additivity) If $\{E_n\}_n$ is a sequence of events in $\mathcal{F}$ such that $E_i \bigcap E_j = \emptyset, \forall i \neq j$, then $\mathbb{P}(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$.

Definition 9 (Probability Space)**.** If $\mathbb{P}$ is a probability function defined on the event space $\mathcal{F}$ of a random experiment $\mathcal{E}$, then the triple (Ω , $\mathcal{F}$, $\mathbb{P}$) is said to be a probability space. Here, Ω denotes the sample space of $\mathcal{E}$.

Definition 10 (Mutually Exclusive/Pairwise Disjoint Events)**.** Let $\mathcal{I}$ be an Indexing set. A collection of events $\{E_i : i \in \mathcal{I}\}$ is said to be mutually exclusive or pairwise disjoint if $E_i \bigcap E_j = \emptyset, \forall i \neq j$.

1.2 Primary Propositions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space associated with a random experiment $\mathcal{E}$.

Proposition 1.1. $\mathbb{P}(\emptyset) = 0$.

Proof. Consider an infinite sequence of events $\{E_i\}_\infty$ wherein $E_1 = \Omega$, and $E_i \forall i \geq 2 = \emptyset$. We know that $E_i \bigcap E_j = \emptyset, \forall i \neq j$. Hence, by definition, we have:

$$\begin{aligned}
\mathbb{P}(\bigcup_{n=1}^{\infty} E_n) &= \sum_{n=1}^{\infty} \mathbb{P}(E_n) \\
\implies 1 &= 1 + \sum_{n=2}^{\infty} \mathbb{P}(E_n) \\
\implies 0 &= \lim_{k\to\infty} \sum_{n=2}^{k} \mathbb{P}(E_n) \\
\implies 0 &= \lim_{k\to\infty} (k-1)\mathbb{P}(\emptyset) \\
\implies \mathbb{P}(\emptyset) &= 0
\end{aligned}$$

QED

Proposition 1.2 (Finite Additivity)**.** Let $E_1, E_2..., E_n \in \mathcal{F}$ for some integer $n \geq 2$ be **mutually exclusive** events. Then $\mathbb{P}(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mathbb{P}(E_i)$.

Proof. Hint: Take all elements after the n^{th} elements to be $\emptyset$ QED

Proposition 1.3. $\mathbb{P}(E) + \mathbb{P}(E^c) = 1$ for all events $E \in \mathcal{F}$.

Proof. Hint: The two events are mutually exclusive and their union is Ω QED

Proposition 1.4. $0 \leq \mathbb{P}(E) \leq 1 \forall$ events $E \in \mathcal{F}$.

Proof. Hint: Lower limit via definition, upper limit via Proposition 1.3 QED

Proposition 1.5 (Monotonicity)**.** Suppose $A, B \in \mathcal{F}$, and $A \subseteq B$, then $\mathcal{P}(A) \leq \mathcal{P}(B)$

Proof. Hint: $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(A^c \bigcap B)$, also mention finite additivity and the fact that A and $A^c \bigcap B$ are mutually exclusive. QED

Proposition 1.6 (Inclusion-Exclusion principle for two events)**.** For $A, B \in \mathcal{F}$, we have:

$$\mathbb{P}(A \bigcup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \bigcap B)$$

Proof. Hint: Divide the sets into parts, show mutual exclusivity, hence use Finite Additivity to prove. QED

Proposition 1.7 (Boole's inequality for two events)**.** For $A, B \in \mathcal{F}$, we have: $\mathbb{P}(A) + \mathbb{P}(B) \geq \mathbb{P}(A \bigcup B)$

Proof. Hint: Follows from Inclusion-Exclusion. QED

Proposition 1.8 (Bonferroni's inequality for two events)**.** For $A, B \in \mathcal{F}$, we have: $\mathbb{P}(A \bigcup B) \geq max\{0, \mathbb{P}(A) + \mathbb{P}(B) - 1\}$

Proof. Hint: Follows from Inclusion-Exclusion and boundary condition of $\mathbb{P}(E)$ QED

Definition 11 (Rigorous definition of Probability Function)**.** Let Ω be any finite or countably infinite set. Consider $\mathcal{F} = 2^{\omega}$ the power set. let $p : \Omega \to [0, 1]$ be a function such that:

$$\sum_{\omega \in \Omega} p_{\omega} = 1$$

Now consider a real valued set function $\mathbb{P}$ on $\mathcal{F}$ defined by:

$$\mathbb{P}(\mathbb{A}) = \sum_{\omega \in \mathbb{A}} p_{\omega}$$

Proposition 1.9. The function $\mathbb{P}$ defined above is the probability function on $\mathcal{F}$.

Proof. Hint: Verify all the 3 definitions for the function defined. QED

Definition 12 (Discrete Probability spaces)**.** Let Ω be a finite or countable set. We refer to a probability space of the form $(\Omega, 2^{\Omega}, \mathbb{P})$ as a discrete probability space.

Definition 13 (Elementary Events)**.** We may refer to the singleton events in a discrete probability space as elementary events.

Proposition 1.10 (Generalized Inclusion-Exclusion Principlle)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ define a probability space and let $A_1, ..., A_n$ be the events. Then:

$$\mathbb{P}(\bigcap_{i=1}^{n} A_i) = S_{1,n} - S_{2,n} + S_{3,n} - ... + -1^{n-1} S_{n,n},$$

where

$$S_{n,k} := \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \mathbb{P}(A_{i_1} \cup A_{i_2} \cup ... \cup A_{i_n})$$

Proof. Hint: Already proven for $n = 2$. Use induction now. QED