24|10|2024

1) $AGA = A$

$\Rightarrow PQGPQ = PQ$

$\Rightarrow P_L^{-1} P Q G P Q Q_R^{-1} = P_L^{-1} P Q Q_R^{-1}$

$\Rightarrow \quad QGP = I_r$

G is a reflexive g-inverse. we have to show that $G = Q_R^{-1} P_L^{-1}$ for some Q_R^{-1} and P_L^{-1}

Suppose we want to show $Q_R^{-1} P_L^{-1}$ is a reflexive g-inverse

$$A Q_R^{-1} P_L^{-1} A$$
$$\Rightarrow P \underbrace{Q Q_R^{-1}} \underbrace{P_L^{-1} P} Q$$
$$\Rightarrow PQ$$
$$\Rightarrow A$$

$\rho(Q_R^{-1} P_L^{-1}) \leq \rho(Q_R^{-1}) = r$

because $Q_R^{-1} P_L^{-1}$ is a g-inverse $\rho(Q_R^{-1} P_L^{-1}) \geq r$

so $\rho(Q_R^{-1} P_L^{-1}) = r$.

$\Rightarrow Q_R^{-1} P_L^{-1}$ is a reflexive g-inverse.

$QGP = I_r$ we have QG is a left inverse of P and GP is a right inverse of Q

$$G = GAG = \underbrace{(GP)}_{\text{right inverse of } Q} \underbrace{(QG)}_{\text{left inverse of } P}$$

2) Suppose G is a minimum norm g-inverse

$\Rightarrow G(AA^T) = A^T$

$\Rightarrow G = A^T(AA^T)^-$

$G = A^T(AA^T)^-$

AGA

$\Rightarrow AA^T(AA^T)^-A \Rightarrow \underline{AA^T}\,\underline{(AA^T)^-}\,\underline{AA^T}X$ for some X

$\Rightarrow AA^TX$

$\Rightarrow A$

$A = AA^TX$

$GAG = A^T(AA^T)^-AA^T(AA^T)^-$ $\qquad A^T = YAA^T$

$= Y\underline{AA^T}\,\underline{(AA^T)^-}\,\underline{AA^T}(AA^T)^-$

$= YAA^T(AA^T)^-$

$= A^T(AA^T)^- = G$

$GAA^T = A^T(AA^T)^-AA^T$ $\qquad GAA^T = A^T$

$= YAA^T(AA^T)^-AA^T$

$= YAA^T = A^T \Rightarrow G$ is a minimum norm g-inverse

$G = GAG = A^TG^TG$

$\underline{G^TG}$ is a g-inverse of AA^T

If this holds then $G = A^T(AA^T)^-$

$A(A^TG^T)GAA^T = AGAGAA^T$

$= HGHH' = HH'$

3) G is a LS g-inverse

$(A^T A) G = A^T$

$G = (A^T A)^- A^T$

4) (a) $GAG = G$, $\mathcal{C}(G) \subseteq \mathcal{C}(GA) \subseteq \mathcal{C}(A^T G^T)$

$\subseteq \mathcal{C}(A^T)$

$d(\mathcal{C}(G)) = d(R(A))$ $= R(A)$

$\mathcal{C}(G) = R(A)$

| if $S \subseteq T$ and $d(S) = d(T)$ then $S = T$ |

5 (a) $(A^+)^T = (A^T)^+$

$A^T = B$

$B^+ = (A^+)^T$

(i) $BB^+B = B$

(ii) $B^+BB^+ = B^+$

(iii) B^+B is symmetric

(iv) BB^+ is symmetric

(i) $BB^+B = A^T (A^+)^T A^T$

$= (AA^+A)^T = A^T = B$

(c) we know $(A^+)^T = (A^T)^+$

if A is symmetric

$(A^+)^T = (A^T)^+ = A^+ \quad \ldots \quad A^+ \ldots$

[illegible], so A is also symmetric

(f) from (b)

$$(A^T A)^+ = A^T (A^T)^+$$

if A is symmetric and idempotent

$$\Rightarrow A^+ = A A^+$$

$$\Rightarrow (A^+)^+ = (A A^+)^+$$

$$\Rightarrow A = A A^+$$