MTH207M (2024-25, Odd Semester)

Problem Set 1

Notation: For a matrix A, by A^- we denote a g-inverse of A.

1. If (P, Q) is a rank-factorization of A, show that $(PT, T^{-1}Q)$ is a rank-factorization of A for all non-singular T and that every rank-factorization of A is of this form.

2. If $\mathcal{C}(A) \subseteq \mathcal{C}(B)$ and $\mathcal{R}(A) \subseteq \mathcal{R}(D)$, prove that $A = BCD$ for some matrix C.

3. Show that $Q_1 = Q_2$ if (P, Q_1) and (P, Q_2) are rank-factorizations of A and $P_1 = P_2$ if (P_1, Q) and (P_2, Q) are rank-factorizations of A.

4. Let (P, Q) be a rank-factorization of a non-null square matrix A . Show that $A = A^2$ iff $QP = I$ and that $\rho(A) = \rho(A^2)$ iff QP is non-singular.

5. Does a symmetric matrix A have a symmetric RF, i.e., an RF of the form (P, P^T)?

6. Prove that if A is a symmetrie matrix of rank 1, then $A = \alpha uu^T$ for some non-zero scalar α and some non-null vector u. Can α be dropped?

7. Show that $\rho(A) = \mathrm{tr}(A^-A)$.

8. Let A be an $m \times n$ matrix. Show that an $n \times m$ matrix G is a g-inverse of A iff $\rho(I - GA) = n - \rho(A)$.

9. For two matrices A and C, $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ iff $CA^-A = C$.

10. Show that B^-A^- need not, in general, be a g-inverse of AB. However, if $\rho(AB) = \rho(A)$, show that $B(AB)^-$ is a g-inverse of A and that $B^-B(AB)^-$ is a g-inverse of AB.

11. Let G be a g-inverse of A. Then prove that

$$\{G + (I - GA)U + V(I - AG) \mid U, V \text{ arbitrary }\}$$

is the class of all g-inverses of A. (Hint: If H is a g-inverse of A, take $U = H - G$.)

12. Let B be an $m \times n$ matrix and G an $n \times m$ matrix. Then, for any $r \times m$ full column rank matrix A and $n \times p$ full row rank matrix C, G is a generalized inverse of ABC if and only if $CGA = H$ for some generalized inverse H of B.

13. Show that if G is a g-inverse of $X^T X$, then

    (a) G^T is a g-inverse of $X^T X$, and

    (b) GX^T is a g-inverse of X.

14. Let $\rho(A + B) = \rho(A) + \rho(B)$. Then, show that

    (a) $\mathcal{C}(A) \subseteq \mathcal{C}(A + B)$ and $\mathcal{C}(B) \subseteq \mathcal{C}(A + B)$, and

    (b) for every g-inverse G of $A + B$, show that $AGA = A$ and $AGB = 0$.

15. Let A and B be two matrices such that AB exists. Then, show that

    (a) $B^- A^-$ is a g-inverse of AB iff $A^- ABB^-$ is idempotent.

    (b) If A has full column rank and B has full row rank, then $B^- A^-$ is a g-inverse of AB.

16. Show that for a matrix $A_{m \times n}$, $\mathcal{N}(A) = \mathcal{C}(I_n - A^- A)$. Therefore, the set of all solutions of $Ax = b$ when $b \in \mathcal{C}(A)$ is $\{A^- b + (I_n - A^- A)z \mid z \in \mathbb{R}^n\}$.

17. Let $A_{m \times n}$ and $Q_{p \times n}$ be two matrices. Then Qx has a unique value for all x satisfying $Ax = y$, if and only if $QA^- A = Q$.

18. Let A and B be matrices having the same number of columns, and let $\mathcal{C}(A^T) \cap \mathcal{C}(B^T) = \{0\}$. Then, show that

    (a) $\rho(A^T A + B^T B) = \rho(A) + \rho(B)$,

    (b) $A^T A(A^T A + B^T B)^- A^T A = A^T A$ for any choice of g-inverse of $A^T A + B^T B$.

19. Prove or disprove: For two matrices A and B of the same order, $\mathcal{C}(A + B) = \mathcal{C}(A) + \mathcal{C}(B)$ implies the sum $\mathcal{C}(A) + \mathcal{C}(B)$ is direct.