Computation of a left inverse

$$E_{\alpha} E_{\alpha-1} \ldots E_1 A_{m\times n} = F_{m\times n}$$

$$\Rightarrow E_{m\times m} A_{m\times n} = F_{m\times n}$$

Suppose $\rho(A) = n$

Then

$$\Rightarrow \begin{bmatrix} E^1_{n\times m} \\ E^2_{(m-n)\times m} \end{bmatrix} A = \begin{bmatrix} I_n \\ O_{(m-n)\times n} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} E^1 A \\ E^2 A \end{bmatrix} = \begin{bmatrix} I_n \\ O \end{bmatrix}$$

$$\Rightarrow E^1 A = I_n$$

Hence, E^1 is a left inverse of A.

- Suppose A has a left inverse

$\Rightarrow A^T$ has a right inverse

$A_L^{-1} A = I_n \quad \Rightarrow A^T (A_L^{-1})^T = I_n$

$(A_L^{-1})^T$ is a right inverse of A^T.

Suppose A has a right inverse $\Rightarrow A^T$ has a left inverse. And we can compute a left inverse of A^T to get a right inverse of A.

Rank Factorization of a matrix

Defn: Suppose $A_{m\times n}$ is a matrix with $\rho(A)=r$.
Now, we say a pair of matrices (P, Q) is a RF of A if $A = PQ$ and $P_{m\times r}$ and $Q_{r\times n}$

$$\boxed{\begin{aligned} A_{m\times n} &= X_{m\times k}\, Y_{k\times n}, \quad \rho(A) \le \rho(X) \\ &\Rightarrow r \le \rho(X) \le k \end{aligned}}$$

For, $O_{m\times n}$ RF does not exist.

Theorem: For any matrix $A_{m\times n}$ with $\rho(A) \geq 1$, RF exists.

Proof: Take a column basis of A, say $\{x_1, \ldots, x_r\}$

Consider a matrix $P = [x_1\ x_2\ \ldots\ x_r]_{m\times r}$

$$\mathcal{C}(A) = \mathcal{C}(P)$$
$$\Rightarrow \mathcal{C}(A) \subseteq \mathcal{C}(P)$$
$$\Rightarrow A = PQ \quad \text{for some } Q.$$

where $Q_{r\times n}$.

- RF is not unique.

$\mathcal{C}(A) \subseteq \mathcal{C}(B)$

$A = BC$

$\mathcal{R}(A) \subseteq \mathcal{R}(B)$

$\Leftrightarrow A = CB$

- Suppose (P, Q) is an RF of A and $\rho(A) = r$. $A_{m\times n} = P_{m\times r} Q_{r\times n}$

We claim $\underline{\mathcal{C}(A) = \mathcal{C}(P)}$ and $\mathcal{R}(A) = \mathcal{R}(Q)$

$$A = PQ \Rightarrow \mathcal{C}(A) \subseteq \mathcal{C}(P)$$

$$r = \rho(A) \leq \rho(P) \leq r$$

$$\Rightarrow \underset{= r}{\rho(A) = \rho(P)} \longrightarrow d(\mathcal{C}(A)) = d(\mathcal{C}(P))$$

$$\Rightarrow \mathcal{C}(A) = \mathcal{C}(P)$$

$\boxed{\begin{array}{l} S \subseteq T \\ d(S) = d(T) \\ \text{then } S = T \end{array}}$

__Theorem__ Suppose $A = PQ$ where $A_{m\times n}$ and $P_{m\times k}$ and $Q_{k\times n}$. Then the following statements are equivalent.

(i) (P, Q) is an RF of A, i.e. $k = \rho(A)$

(ii) P has full column rank and Q has full row rank

(iii) Columns of P form a basis of $\mathcal{C}$

(iv) rows of Q form a basis of $\mathcal{R}(A)$.

Example

$A = PQ$ s.t. P has full column rank but (P, Q) is not an RF

$$A_{m\times n} = I_m A$$

(A)

Proof. Suppose $\rho(A) = r$

(i) $\Rightarrow$ (ii) $A_{m\times n} = P_{m\times r} Q_{r\times n}$

As $r = \rho(A) \leq \rho(P) \leq r$,

so, $\rho(P) = r$. Similarly, $\rho(Q) = r$

so, P is of full column rank and Q is of full row rank.

(ii) $\Rightarrow$ (iii) As Q is of full row rank it has a right inverse

$\Rightarrow A Q_R^{-1} = P$

So, $\rho(A) \geq \rho(P)$

Thus, $\rho(P) = \rho(A) = r$

Hence, $\mathcal{C}(A) = \mathcal{C}(P)$

Further, as P has full column rank,

Columns of P form a basis of $\mathcal{C}(A)$.

(iii) $\Rightarrow$ (iv) Columns of P form a basis

of $\mathcal{C}(A)$ $\Rightarrow$ $k = r$

So, $A_{m \times n} = P_{m \times r} \, Q_{r \times n}$

As $\mathcal{R}(A) \subseteq \mathcal{R}(Q)$, and

$r = \rho(A) \leq \rho(Q) \leq r$, it must

be that rows of Q are LI and

they form a basis of $\mathcal{R}(A)$

(iv) $\Rightarrow$ (i) As rows of Q form a

basis of $\mathcal{R}(A)$, we must have $k = r$

implying (P, Q) is an RF of $\mathcal{A}$.