30/09/24

- Every matrix has a g-inverse.

<u>Proof:</u> Suppose A is a null matrix of order $m \times n$. Then any matrix of order $n \times m$ is a g-inverse. [will satisfy $AGA = A$]

if $\rho(A) \geq 1$. Then $G = Q_R^{-1} P_L^{-1}$ where (P, Q) is an RF of A.

$$AGA = P \underbrace{Q Q_R^{-1}}\, \underbrace{P_L^{-1} P} Q$$
$$= PQ = A.$$

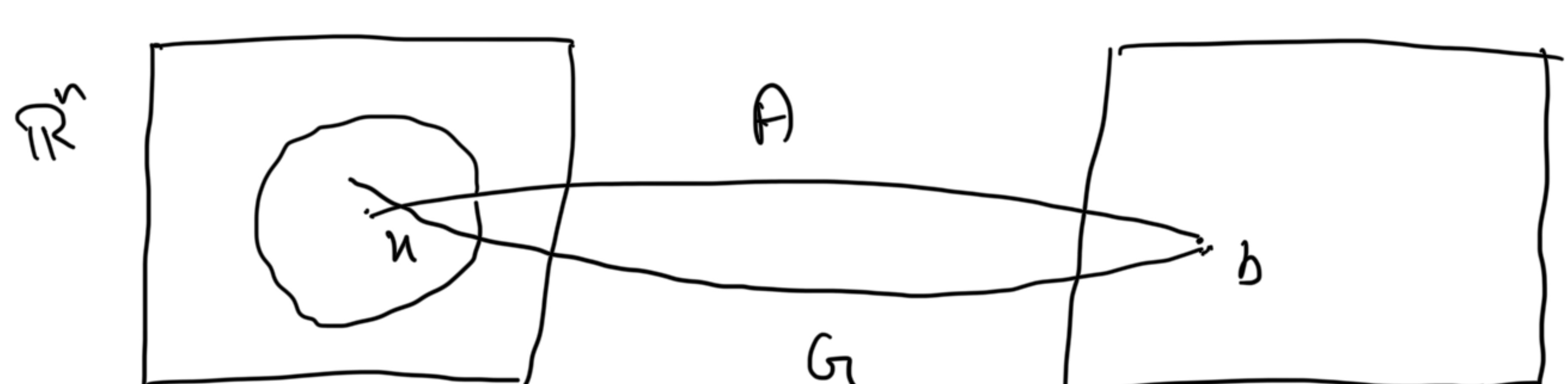
- g-inverse is not unique.

Suppose A has a left inverse the $A_L^{-1} A = I_n$

$$A A_L^{-1} A = A$$

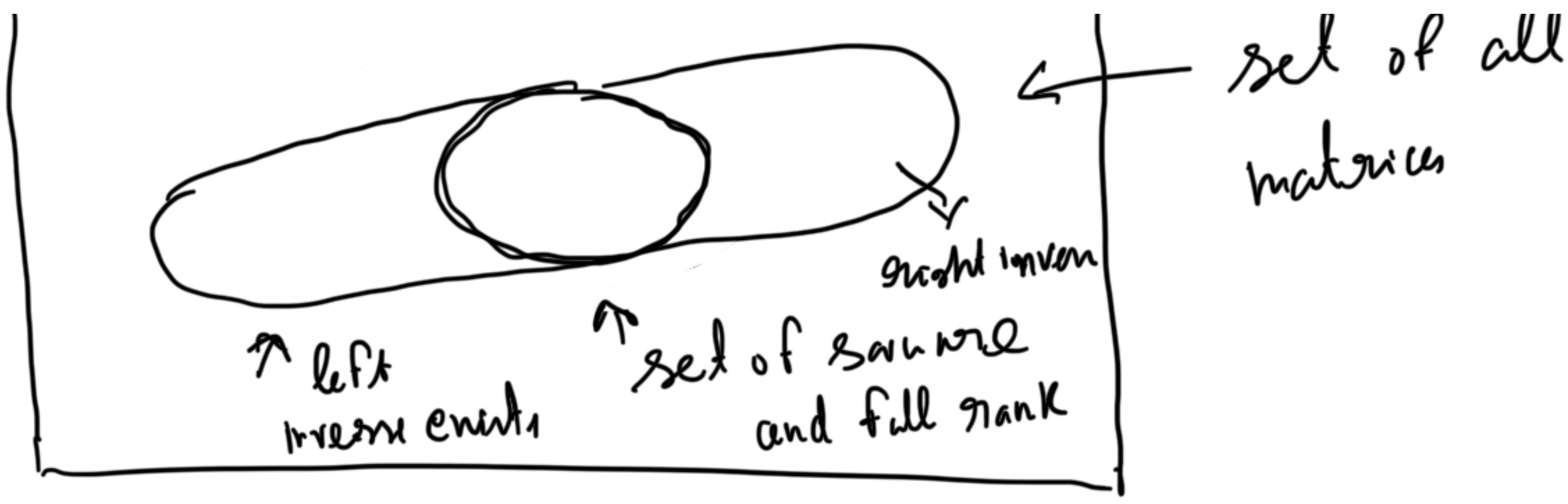
$$\Rightarrow A_L^{-1} \text{ is a g-inverse}$$

And as left inverse is not unique, g-inverse is not unique.



$f(x) = Ax$

$$\boxed{AGA = A}$$



$$\mathcal{G} = \{ G + (I - GA)U + V(I - AG) \mid U \text{ and } V \text{ are arbitrary matrices} \}$$

G: a particular g-inverse.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1$$

$\Rightarrow 2x + y = 1$ $\Rightarrow$ left inverse is not unique.

<u>Theorem</u> Suppose A is a matrix and G is a g-inverse of A. Then

(i) a system $Ax = b$ is consistent iff $AGb = b$

(ii) $\mathcal{N}(A) = \{ (I - GA)x \mid x \in \mathbb{R}^n \}$

(iii) The set of all solution of $Ax = b$ is $Gb + (I - GA)x$ where $x \in \mathbb{R}^n$.

<u>Proof</u>: (i) Suppose $Ax = b$ is consistent, then Gb is a solution to $Ax = b \Rightarrow AGb = b$

Further, if $AGb = b$, $A(Gb) = b$, $Az = b$ where $z = Gb$

$b \in \mathcal{C}(A) \Rightarrow Ax = b$ is consistent.

(ii) $x \in \{(I - GA)y \mid y \in \mathbb{R}^n\}$

$$x = (I - GA)z \quad \text{for some } z \in \mathbb{R}^n$$

$$Ax = A(I - GA)z = (A - AGA)z = 0$$

$$x \in N(A)$$

Take $x \in N(A)$. $\quad Ax = 0 \Rightarrow GAx = 0$

$$\Rightarrow \quad \underline{(I_n - GA)}x = \underline{x}$$

$$x \in \{(I_n - GA)y \mid y \in \mathbb{R}^n\}$$

$$N(A) = \{(I_n - GA)y \mid y \in \mathbb{R}^n\}$$

(iii) The set of all solutions to $Ax = b$ is $u + N(A)$ where u is a particular solution. As Gb is a particular solution and $N(A) = \{(I_n - GA)x \mid x \in \mathbb{R}^n\}$, we have $Gb + \{(I_n - GA)x \mid x \in \mathbb{R}^n\}$ is the set of all solutions.

<u>Theorem</u> (i) If a matrix is non-singular then a g-inverse to that matrix is the inverse of that matrix.

(ii) If a matrix has a left inverse

then every g-inverse is a left inverse and every left inverse is a g-inverse.

(ii) Similar results for right inverse.

<u>Proof</u> (i) $A^{-1} A = I_n$

$\Rightarrow A A^{-1} A = A \Rightarrow A^{-1}$ is a g-inverse

For any g-inverse G,

$$AGA = A$$

$$\Rightarrow GA = I_n$$

$\Rightarrow G = A^{-1}$, every g inverse is the inverse of the matrix.

(ii) $A_L^{-1} A = I_n$

$\Rightarrow A A_L^{-1} A = A \Rightarrow A_L^{-1}$ is a g-inverse

$$AGA = A \Rightarrow A_L^{-1} AGA = A_L^{-1} A$$

$\Rightarrow GA = I_n \Rightarrow G$ is a left inverse of A.

Computation of a g-inverse.

$$E_{m\times m} A_{m\times n} = F_{m\times n}$$

Construct a matrix $G_{n\times m}$ such that

$a_{h_i i} = e_{i\cdot}$ $\forall i = 1(1)r$