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$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & -2 & -4 \\ 2 & -1 & 1 & -1 & -2 \end{bmatrix}_{4\times 5}$$

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}_{4\times 4}$$

$$F = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 5}$$

$\mathcal{R}(A)$ basis $= \{(1,0,1,0,0), (0,1,1,0,0),$
$(0,0,0,1,2)\}$

$\mathcal{C}(A)$ basis $= \{(1,-1,0,2), (2,1,3,-1),$
$(1,0,-2,-1)\}$

$A = PQ$

$$P = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 0 & 3 & -2 \\ 2 & -1 & -1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$Q_R^{-1} P_L^{-1}$ is a g-inverse of A.

$$G = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ 0 & 0 & 0 & 0 \\ e_{31} & e_{32} & e_{33} & e_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 4} \qquad \begin{array}{l} p_1 = 1 \\ p_2 = 2 \\ \\ p_3 = 4 \end{array}$$

$$EA = F$$

$$\Leftrightarrow \; e_{i\cdot} A = f_{i\cdot} \qquad \forall \, i = 1(1)m$$

$$(GA)_{p_1\cdot} = g_{p_1\cdot} A$$

$$= e_{1\cdot} A = f_{1\cdot}$$

$$(GA)_{p_k\cdot} = g_{p_k\cdot} A = e_{k\cdot} A = f_{k\cdot}$$

$$\forall \, k = 1(1)r$$

$$(GA)_{i\cdot} = 0 \qquad \forall \, i \notin \{p_1, \ldots, p_r\}$$

$$GA = \begin{array}{c} \begin{array}{ccccc} p_1 & p_2 & \downarrow & p_3 & \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 5} \end{array}$$

$$\rho(GA) = \rho(F) = \rho(A)$$

For GA

1st to (p_1-1)th column — null vector.

p_1 th column — $e^n_{p_1}$

(p_1+1) th to (p_2-1) th columns — $\alpha_i e^n_{p_1}$ $\quad i = (p_1+1)(1)(p_2-1)$

p_2 th Column $\qquad e^n_{p_2}$

(p_2+1) th to (p_3-1) columns $\qquad \beta_i e^n_{p_1} + \gamma_i e^n_{p_2} \qquad i = (p_2+1)$ th (p_3-1)

$\vdots$

p_r th Column $\qquad e^n_{p_r}$

(p_r+1) th to n th columns $\qquad \delta_i^{p_1} e^n_{p_1} + \delta_i^{p_2} e^n_{p_2} + \cdots + \delta_i^{p_r} e^n_{p_r}$

$i = p_r+1\,(1)\,n$

$$(GA)(GA)_{\cdot p_1} = (GA)_{\cdot p_1}$$

$$(GA)(GA)_{\cdot p_k} = (GA)_{\cdot p_k} \qquad k = 1\,(1)\,r$$

(p_1+1) th column

$$\begin{aligned}(GA)(GA)_{\cdot(p_1+1)} &= (GA)\, \alpha_{p_1+1} e^n_{p_1} \\ &= \alpha_{p_1+1} (GA) e^n_{p_1} \\ &= \alpha_{p_1+1} (GA)_{\cdot p_1} \\ &= \alpha_{p_1+1} e^n_{p_1} = (GA)_{\cdot(p_1+1)}\end{aligned}$$

(p_2+1) column

$$\begin{aligned}(GA)(GA)_{\cdot(p_2+1)} &= (GA)\, \beta_{p_2+1} e^n_{p_1} + (GA)\, \gamma_{p_2+1} e^n_{p_2} \\ &= \beta_{p_2+1} (GA) e^n_{p_1} + \gamma_{p_2+1} (GA) e^n_{p_2} \\ &= \beta_{p_2+1} (GA)_{\cdot p_1} + \gamma_{p_2+1} (GA)_{\cdot p_2} \\ &= \beta_{p_2+1} e^n_{p_1} + \gamma_{p_2+1} e^n_{p_2} \\ &= (GA)_{\cdot(p_2+1)}\end{aligned}$$

So, finally we have

$$(GA)(GA)_{\cdot j} = (GA)_{\cdot j} \quad \forall j = 1(1)n$$

$$GA\,GA = \left[GA(GA)_{\cdot 1} \quad GA(GA)_{\cdot 2} \; \cdots \; GA(GA)_{\cdot n} \right]$$

$$= \left[(GA)_{\cdot 1} \quad (GA)_{\cdot 2} \; \cdots \; (GA)_{\cdot n} \right]$$

$$= GA$$

So, G is a g-inverse of A.

$$\left[\begin{array}{c|c} A_{m\times n} & I_m \end{array} \right]$$

$$= \left[\begin{array}{c|c} F & E \end{array} \right]$$