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- For a square matrix A, we say a spectral decomposition exists if $\exists$ a n-s matrix P and a diagonal matrix Δ of order $r = \rho(A)$ s.t.

$$A = P^{-1} \underbrace{\begin{bmatrix} \Delta_{r\times r} & O_{r\times(n-r)} \\ O_{(n-r)\times r} & O_{(n-r)\times(n-r)} \end{bmatrix}}_{D} P$$

So, as $\rho(A) = \rho(D)$, we must have all the diagonal elements of Δ are non-zero.

Also, we have seen the characteristic equation of A is the same as the characteristic equation of D.

- For a diagonal matrix the eigen values are the diagonal elements.

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & d_n \end{bmatrix} x = d_1 x$$

$x = e_1$ satisfies the above equation.

So e_1 is an eigen vector of D corresponding to eigenvalue d_1.

if for a diagonal matrix D, λ appears at k_1th, k_2th, ..., k_nth position at the diagonal of D, then $e_{k_1}, \ldots, e_{k_n}$ are the eigen vectors of D corresponding to λ,

$$GM(D, \lambda) = n$$

the characteristic equation will be

$$(\bar{\lambda} - d_1)(\bar{\lambda} - d_2) \ldots (\bar{\lambda} - d_n) = 0$$

if λ appears at k_1th, k_2th, -- k_nth position at the diagonal of d, the $AM(D, \lambda) = n$

For a diagonal matrix D, $AM(D, \lambda) = GM(D, \lambda)$ for any eigen value λ of D.

$$A = P^{-1} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} P$$

We claim that the columns of P^{-1} will be the eigen vectors of A.

$$A P^{-1} = P^{-1} \underbrace{\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}}_{D}$$

$$\Rightarrow [A(P^{-1})_{\cdot 1} \quad A(P^{-1})_{\cdot 2} \quad \ldots \quad A(P^{-1})_{\cdot n}]$$

$$= [d_1(P^{-1})_{\cdot 1} \quad d_2(P^{-1})_{\cdot 2} \quad \ldots \quad d_n(P^{-1})_{\cdot n}]$$

the columns are eigen vectors of A.

Because (P^{-1}) is a n-s matrix, it implies that A has n LI eigen vectors.

- Further, if a matrix A has n LI eigenvectors the A has a spectral decomposition

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(\lambda - 1)^2 = 0$$

$$(A - I)x = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0$$

Suppose A has n LI eigen vectors, $x_1, \ldots, x_n$.

$$A\underline{[x_1 \ x_2 \ \ldots \ x_n]}$$

$$= [\lambda_1 x_1 \ \lambda_2 x_2 \ \ldots \ \lambda_n x_n]$$

$$= [x_1 \ x_2 \ \ldots \ x_n] \underbrace{\operatorname{diag}(\lambda_1, \ldots, \lambda_n)}_{D}$$

$$P = [x_1, \ldots, x_n]$$

$$AP = PD$$

$$\Rightarrow A = PDP^{-1}$$

$$A = P \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

So, if we arrange $x_1 \ldots x_n$ s.t. the $(n-r)$ vectors are eigenvectors corresponding to 0 then $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0)$

then $A = P \begin{bmatrix} \lambda_1 & 0 & 0 & \\ 0 & \ldots & \lambda_r & 0 \end{bmatrix} P^{-1}$

$$\begin{bmatrix} 0 & \cdots & \\ 0 & & 0 \end{bmatrix}$$

A has a spectral decomposition.

Further, the condition that it has n LI eigen vectors is equivalent to $AM(A,\lambda) = GM(A,\lambda)$ for all eigenvalue λ of A.

$$A = P^{-1} \begin{bmatrix} \Delta_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} P$$

$$= \begin{bmatrix} \underset{n\times r}{R_1} & R_2 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_{1\, r\times n} \\ S_2 \end{bmatrix}$$

$$= R_1 \Delta S_1$$

where $R_{1\, n\times r}$ and $S_{1\, r\times n}$

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \begin{bmatrix} R_1 & R_2 \end{bmatrix} = I_n$$

$$\Rightarrow \begin{bmatrix} S_1 R_1 & S_1 R_2 \\ S_2 R_1 & S_2 R_2 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

$$\Rightarrow S_1 R_1 = I_r$$

this is an equivalent form of spectral decomposition.

$A = R_1 \Delta S_1$ where $S_1 R_1 = I_r$

Take any n-s diagonal matrix $\Delta_{0\, r\times r}$

$$A = R_1 \Delta S_1 = R_1 \Delta_0 \Delta \Delta_0^{-1} S_1$$

$= \bar{R}_1 \Delta \bar{S}_1 \quad$ where $\bar{R}_1 = R_1 \Delta_0$

$\bar{S}_1 = \Delta_0^{-1} S_1$

$$\bar{S}_1 \bar{R}_1 = \Delta_0^{-1} S_1 R_1 \Delta_0$$

$$= \Delta_0^{-1} I_r \Delta_0 = I_r.$$

Spectral decomposition is not unique.

- A is symmetric, then spectral decomposition exists. Further, $A = P^T \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} P$ where $P^{-1} = P^T$

- If A is idempotent, then eigenvalues are 0 and 1 and spectral decomposition exists. Further, $A = P^T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P$ where $P^T = P^{-1}$

Singular value decomposition

Suppose $A_{m \times n}$, then A has a singular value decomposition if $\exists\ U_{m \times m}$, $V_{n \times n}$, and a $\underset{\text{diagonal matrix}}{\Delta_{r \times r}}$ s.t. all the diagonal elements are positive s.t.

$$A = U \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}_{m \times n} V^T$$

where $U^T U = U U^T = I_m$ and $V^T V = V V^T = I_n$