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For a square matrix A of order n

λ is an eigenvalue if $\exists\, x \neq 0$ s.t

$$Ax = \lambda x$$

And x is an eigenvector corresponding to λ.

- if x is an eigenvector then cx is also an eigenvector. $A(cx) = cAx = c\lambda x = \lambda(cx)$

- if x and y are two eigenvectors corresponding to λ, $A(x+y) = Ax + Ay = \lambda x + \lambda y = \lambda(x+y)$
  then $x+y$ is also an eigenvector corresponding to λ.

So, the set of all eigenvectors corresponding to λ, and the null vector form a subspace $\mathbb{C}^n$, denoted by $ES(A,\lambda)$.

$$\boxed{\begin{aligned} &A(\alpha x+\beta y) \\ &= \alpha Ax + \beta Ay \\ &= \alpha\lambda x + \beta\lambda y \\ &= \lambda(\alpha x+\beta y)\end{aligned}}$$

- λ is an eigenvalue of $A \iff (A-\lambda I)x = 0$ for some $x \neq 0$

$$\iff \rho(A-\lambda I) < n$$

$$\iff |A-\lambda I| = 0$$

$|A-\lambda I|$ is a polynomial in λ with degree n.

This is called the characteristic polynomial of A.

and $|A-\lambda I| = 0$ is called the characteristic equation.

Also, the set of all eigen values of A is the set of all roots of the equation $|A-\lambda I| = 0$.

- For a matrix of order n, there can be at most n eigenvalues.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$|A - \lambda I_2| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(1-\lambda) = 0$$

$$\Rightarrow \lambda = 1,\ \lambda = 0$$ are the two solutions

For $\lambda = 1$

$$(A - \lambda I) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$(A - \lambda I)x = 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_2 = 0$$

For $\lambda = 0$,

$$Ax = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 = 0$$

this matrix A has two linearly independent eigen vectors.

- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 = 0$$

$$\Rightarrow \lambda = 1$$

this matrix has only one eigen value.

$$(A - \lambda I_2)x = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_2 = 0$$

The number of linearly independent eigen vectors is 1.

- Suppose $\bar{\lambda}$ is an eigenvalue of A. Then $ES(A, \bar{\lambda})$ is subspace. The dimension of $ES(A, \bar{\lambda})$ is called the Geometric multiplicity of $\bar{\lambda}$, denoted by $GM(A, \bar{\lambda})$

- For an eigenvalue $\bar{\lambda}$ of A, the algebraic multiplicity of $\bar{\lambda}$ is the number of $\bar{\lambda}$ appears as the root of characteristic equation, denoted by $AM(A, \bar{\lambda})$

$I_n x = x$, the eigen value is 1

$$|I_n - \lambda I_n| = 0$$

$$\Rightarrow (1-\lambda)^n = 0$$

$$GM(I_n, 1) = n = AM(I_n, 1)$$

- Suppose a matrix has $\lambda_1, \ldots, \lambda_k$ as the distinct eigenvalues $(k \leq n)$, then

$$\sum_{i=1}^{k} AM(A, \lambda_i) = n$$

- $AM(A, \bar{\lambda}) \geq GM(A, \bar{\lambda})$ $\forall A$ and $\forall \bar{\lambda}$, an eigen value of A.

- $\sum_{i=1}^{k} GM(A, \lambda_i) \leq n$, the equality holds iff
$$AM(A, \lambda_i) = GM(A, \lambda_i) \quad \forall i = 1(1)k.$$

Spectral decomposition of a square matrix

- For a square matrix A, we say that a spectral decomposition exists if $\exists$ a n-s P and a diagonal matrix Δ of order r, where $r = \rho(A)$, s.t.

$$A = P^{-1} \underbrace{\begin{bmatrix} \Delta_{r \times r} & O_{r \times (n-r)} \\ O_{(n-r) \times r} & O_{(n-r) \times (n-r)} \end{bmatrix}}_{D} P$$

- Suppose a matrix has a spectral decomposition

$$|A - \lambda I_n| = 0$$

$$\Leftrightarrow |P^{-1}DP - \lambda I_n| = 0$$

$$\Leftrightarrow |P^{-1}DP - \lambda P^{-1}P| = 0$$

$$\Leftrightarrow |P^{-1}(D - \lambda I_n)P| = 0$$

$$\Leftrightarrow |P^{-1}|\,|D - \lambda I_n|\,|P| = 0$$

$$\Leftrightarrow |D - \lambda I_n| = 0$$

So, the of set of eigen values of A is the same as the set of eigenvalues of D