Left/Right inverse

- For a matrix A, we say inverse exists iff $\exists$ a matrix B

$$AB = BA = I_n \quad \text{where } A_{n\times n}$$

(Z) $A_{n\times n}$ and $\exists$ a matrix B

s.t. $BA = I_n$

Proof: $\forall_{n\times n}$ s.t. $\exists B_{n\times n}$

and $BA = I_n$

$$\rho(A) \geq \rho(BA) = \rho(I_n) = n$$

$\Rightarrow$ A is of full rank

$$\mathcal{C}(A) = \mathbb{R}^n$$

$\Rightarrow \mathcal{C}(I_n) \subseteq \mathcal{C}(A) \Leftrightarrow I_n = AC$ for some matrix C

$B = B I_n = BAC = C$

$\Rightarrow$ B is the Inverse of A.

$\mathcal{C}(B) \subseteq \mathcal{C}(A)$

$\Leftrightarrow B = AC$ for some C

Defn For a matrix $A_{n \times n}$, a matrix B is an inverse if

$BA = I_n$

$B_{n \times m} A_{m \times n} = I_n$ for some cases.

Defn (left inverse) For a matrix $A_{m\times n}$, a matrix $B_{n\times m}$ is called a left inverse of A if

$$BA = I_n$$

Question: What can we say if $AB = I_m$ where B is a left inverse of A?

$$BA = I_n$$

$$\Rightarrow \rho(A) \geq \rho(I_n) = n$$

$$\Rightarrow \rho(A) = n$$

$$AB = I_m$$

$$\Rightarrow \rho(A) \geq \rho(I_m) = m$$

$$\Rightarrow \rho(A) = m$$

$$m = n$$

Result.

$A_{m\times n}$ has a left inverse

$$\Leftrightarrow \rho(A) = n$$

$$R(A) = \mathbb{R}^n = R(I_n)$$

$$\Rightarrow R(I_n) \subseteq R(A)$$

$$\Rightarrow I_n = BA \text{ for some } B$$

$$\boxed{\begin{aligned} & R(B) \subseteq R(A) \\ \Leftrightarrow\ & B = CA \\ & \text{for some } A \end{aligned}}$$

<u>Theorem</u>. Suppose $A_{m \times n}$. Then the following statements are equivalent:

(i) A has a left inverse.

(i) A has a [illegible]

(ii) $\rho(A) = n$

(iii) $R(A) = \mathbb{R}^n$

(iv) $AX = AY \Rightarrow X = Y$

(v) $AX = 0 \Rightarrow X = 0$

<u>Proof.</u> (iv) $\Rightarrow$ (v)

$$AX = 0 = A0$$

Using (iv), $X = 0$

(v) $\Rightarrow$ (i) We will show that

Colums of A are LI.

take a linear combination of the columns of A.

$$\alpha_1 a_{\cdot 1} + \alpha_2 a_{\cdot 2} + \cdots + \alpha_n a_{\cdot n} = 0_{m \times 1}$$

$$\Rightarrow A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = 0_{m \times 1}$$

$\boxed{\text{i-th column of } A \equiv a_{\cdot i}}$

from (v), we can say that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

Right inverse

Defn For a matrix $A_{m\times n}$, a right inverse exists if $AB = I_m$ for some $B_{n\times m}$.

Theorem

(i) $A_{m\times n}$ has a right inverse

(ii) $\rho(A) = m$

(iii) $\mathcal{C}(A) = \mathbb{R}^m$

(iv) $XA = YA \Rightarrow X = Y$

(v) $XA = 0 \Rightarrow X = 0$

Computing left inverse

$$E_k \cdots\cdots E_2 E_1 A_{m\times n} = F_{m\times n}$$

$\uparrow$

matrix in

KEY