14/10/2024

$\{ G + (I - GA)U + V(I - AG) \mid U$ and V are arbitrary matrices$\}$

In general $\rho(G) \geq \rho(AG) = \rho(A)$

We want a g-inverse which has the same rank as the original matrix. Such a g-inverse is called a reflexive g-inverse.

$$\rho(G) = \rho(A)$$

- For a matrix A,

G is a reflexive g-inverse $\Leftrightarrow$ $GAG = G$

$\boxed{\text{Any g-inverse } Q_R^{-1} P_L^{-1} \text{ has rank same as } A}$

Proof: We are given $\rho(A) = \rho(G)$

Recall that $\mathcal{C}(GA) \subseteq \mathcal{C}(G)$

Because $\rho(A) = \rho(GA)$, we have $\rho(G) = \rho(GA)$

implying $\mathcal{C}(GA) = \mathcal{C}(G)$

$\Rightarrow G = GAX$ for some X

$$GAG = GAGAX = GAX = G.$$

Now, we are given $GAG = G$

$$\rho(G) \leq \rho(GA) = \rho(A).$$

Also, $\rho(A) \leq \rho(G)$

So, $\rho(A) = \rho(G)$

- $\rho(Q_R^{-1} P_L^{-1}) \leq \rho(Q_R^{-1}) = \rho(A)$

$Q_R^{-1} P_L^{-1}$ is a g-inverse, $\rho(Q_R^{-1} P_L^{-1}) \geq \rho(A)$

So, we have $\rho(Q_R^{-1} P_L^{-1}) = \rho(A)$.

- For every $b \in \mathcal{C}(A)$, Gb is a solution to $Ax = b$

We want a g-inverse such that Gb has smallest norm among all solutions to $Ax = b$

$$\|Gb\| \leq \|Gb + (I - GA)z\| \quad \forall z \in \mathbb{R}^n$$
$$\text{and } \forall b \in \mathcal{C}(A)$$

Such a g-inverse is called a minimum norm g-inverse.

- For a matrix A, the following statements are equivalent

(i) G is a minimum norm g-inverse

(ii) $(GA)^T = GA$

(iii) $A A^T G^T = A$

Proof: (ii) $\Rightarrow$ (i) We are given $(GA)^T = GA$ and we have to show

$$\|Gb\| \leq \|Gb + (I - GA)z\|$$

$\forall z \in \mathbb{R}^n$ and $\forall b \in \mathcal{C}(A)$.

$$\|Gb + (I-GA)z\|^2$$
$$= \|Gb\|^2 + \|(I-GA)z\|^2 + 2b^TG^T(I-GA)z$$

$\|x\|^2 = \langle x, x\rangle$

$\|x+y\|^2$

$= \langle x+y, x+y\rangle$

$= \langle x, x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \|y\|^2 + 2x^Ty$

$\langle w, z\rangle = w^Tz = z^Tw$

$$b^TG^T(I-GA)z$$
$$= u^TA^TG^T(I-GA)z$$
$$= u^TGA(I-GA)z$$
$$= u^T(GA - GA.GA)z$$
$$= u^T(GA-GA)z$$
$$= 0$$

$b \in \mathcal{C}(A)$

$b = Au$

$$\|Gb + (I-GA)z\|^2 = \|Gb\|^2 + \|(I-GA)z\|^2$$
$$\geq \|Gb\|^2$$

$$\|Gb + (I-GA)z\| \geq \|Gb\|$$

(i) $\Rightarrow$ (ii)

$$\|Gb + (I-GA)z\| \geq \|Gb\| \quad \forall b \in \mathcal{C}(A)$$
$$\forall z \in \mathbb{R}^n$$

$$\Rightarrow \|Gb\|^2 + \|(I-GA)z\|^2 + 2b^TG^T(I-GA)z$$
$$\geq \|Gb\|^2 \quad \forall b \in \mathcal{C}(A)$$

$$\Rightarrow \|(I-GA)z\|^2 + 2b^TG^T \quad \forall z \in \mathbb{R}^n$$

$(I-GA)z \leq 0$ $\quad \forall b \in \mathcal{C}(A)$ and $z \in \mathbb{R}^n$.

$$\Rightarrow \|(I-GA)z\|^2 + 2u^T A^T G^T (I-GA)z \geq 0$$

$\forall u \in \mathbb{R}^n$ and $\forall z \in \mathbb{R}^n$

Suppose for some u and z,

$$2u^T A^T G^T (I-GA)z > 0$$

we can take $\alpha < 0$ large enough s.t.

$$2\alpha u^T A^T G^T (I-GA)z + \|(I-GA)z\|^2 < 0$$

but this contradicts that G is a minimum norm g-inverse. So, $2u^T A^T G^T (I-GA)z \leq 0$

$\forall u \in \mathbb{R}^n$ $\forall z \in \mathbb{R}^n$

If $2u^T A^T G^T (I-GA)z < 0$,

we can take $\alpha > 0$ large enough s.t.

$$\|(I-GA)z\|^2 + 2\alpha u^T A^T G^T (I-GA)z < 0,$$

a contradiction.

So, we have $u^T A^T G^T (I-GA)z = 0$ $\forall z \in \mathbb{R}^n$ $\forall u \in \mathbb{R}^n$

$$\Rightarrow u^T A^T G^T (I-GA) = 0 \quad \forall u \in \mathbb{R}^n$$

$$\Rightarrow A^T G^T (I-GA) = 0$$

$$\Rightarrow (GA)^T = (GA)^T GA$$

$GA = (GA)^T GA$ $\quad$ So, $(GA)^T = GA$.

(ii) (=) (iii) Try to show it