MTH206M (2024-25, Odd Semester)

Problem Set 2

1. Show that if A is a non-null real matrix then so is AA^T.

2. Let A be a real $m \times n$ matrix. Show that each diagonal element of AA^T is non-negative. If the i-th diagonal element of AA^T is 0, show that the i-th rows of A and AA^T are null and the i-th column of AA^T is null.

3. Let A be an $n \times n$ matrix such that CAC^T is symmetric for all matrices C of order $m \times n$ where m is a fixed positive integer ≥ 2. Show that A is symmetric.

4. Do there exist square matrices A and B of the same order such that $AB - BA = I$? Why?

5. Prove or disprove the following statements:

   (a) Given any non-null (column) vector x, there exists a (column) vector y such that $y^T x = 1$.

   (b) If x and y are (column) vectors, then each column of xy^T is a scalar multiple of x.

   (c) Let x and y are non-null (column) vectors such that $y^T x = 0$. Then $z^T x = 0$ implies z is a scalar multiple of y.

6. Show that for an upper triangular matrix, the rank is not less than the number of non-zero diagonal elements.

7. Let A be a square matrix and $B = \frac{1}{2}(A + A^T)$. Prove the following.

   (a) B is symmetric.

   (b) $x^T Bx = x^T Ax$ for all $n \times l$ vectors x.

(c) If C is a symmetric matrix such that $x^T Cx = x^T Ax$ for all x, then $C = B$.

8. (a) If A is an $m \times n$ matrix and if $Ax_1 = 0, Ax_2 = 0, \ldots, Ax_n = 0$, for some basis $\{x_1, \ldots, x_n\}$ of $\mathbb{R}^n$, show that $A = 0$.

   (b) If A is an $n \times n$ matrix and if $Ax_1 = x_1, Ax_2 = x_2, \ldots, Ax_n = x_n$, for some basis $\{x_1, \ldots, x_n\}$ of $\mathbb{R}^n$, show that $A = I_n$.

9. (a) If $y^T Ax = 0$ for all A and if $x \neq 0$, prove that $y = 0$. (b) If $y^T Ax = 0$ for all A and if $y \neq 0$, prove that $x = 0$.

10. Let A be an $m \times n$ matrix of rank r. Determine the possible values for the rank of the matrix obtained by (i) changing exactly one element and (ii) changing two elements.

11. Show that an $m \times n$ matrix A has rank at most 1 iff $A = xy^T$ for some column vectors x and y. Show further that $\rho(A) = 1$ iff both x and y are non-null.

12. Prove or disprove: If $A_{m\times n}$ has rank m and $B_{n\times m}$ has rank m, then AB has an inverse.

13. If A is a square matrix such that $3A^4 - 4A^3 + 2A + 5I = 0$, prove that A has an inverse.

14. Let A and B be $n \times n$ matrices such that AB is diagonal with non-zero diagonal entries. Show that in general, A and B may not commute but if the diagonal entries of AB are all equal then A and B commute.

15. A square matrix is said to be idempotent if $A^2 = A$. Show that if a matrix A is non-singular and idempotent then $A = I$.

16. If A and B commute and B is non-singular then show that A and B^- also commute.

17. Prove or disprove: $\rho(ABC) \leq \rho(AC)$.

18. If A and B have the same number of rows then show that $\mathcal{C}(A : B) = \mathcal{C}(A) + \mathcal{C}(B)$.

19. Prove that $\mathcal{N}(A) \subseteq \mathcal{N}(B)$ iff $\mathcal{R}(A) \supseteq \mathcal{R}(A)$. Also, show that null space of a matrix A is not altered if you premultiply A with a non-singular matrix.

20. For a square matrix A of order n, prove that

    (a) $\mathcal{N}(A) \subseteq \mathcal{C}(I_n - A)$ and

(b) $\mathcal{C}(A) + \mathcal{C}(I_n - A) = \mathbb{R}^n$.

21. (a) If A and B are two matrices of the same number of rows then show that $\mathcal{C}([A : B]) = \mathcal{C}(A) + \mathcal{C}(B)$.

    (b) If A and B are two matrices of the same number of columns then show that $\mathcal{R}(C) = \mathcal{R}(A) + \mathcal{R}(B)$ and $\mathcal{N}(C) = \mathcal{N}(A) \cap \mathcal{N}(B)$ where

$$C = \left[\begin{array}{c} A \\ \hline B \end{array}\right]$$

22. (a) If A and B are two matrices of the same number of rows then show that $\rho[A : B] = \rho(A)$ iff $B = AC$ for some matrix C.

    (b) If A and B are two matrices of the same number of columns then show that $\rho(C) = \rho(A)$ iff $B = DA$ for some matrix D where

$$C = \left[\begin{array}{c} A \\ \hline B \end{array}\right]$$

23. For $n \times n$ matrices A and B, show that the rank of $\left[\begin{array}{c|c} A & I_n \\ \hline I_n & B \end{array}\right]$ is n iff $B = A^{-1}$.

24. Prove that $\mathcal{N}(A) \subseteq \mathcal{N}(B)$ iff $\mathcal{R}(B) \subseteq \mathcal{R}(B)$. Deduce that null space does not change if we pre-multiply by a non-singular (inverse exists) matrix.

25. Show that $\rho(AB) = \rho(A)$ when B is non-singular.

26. Prove that $\rho(PAQ) = \rho(A)$ iff $\rho(A) = \rho(PA) = \rho(AQ)$.

27. Show that $\mathcal{C}(A) \subseteq \mathcal{C}(B)$ iff $\mathcal{C}(RAX) \subseteq \mathcal{C}(RBY)$ where R, X, and Y are non-singular matrices.

28. If (P, Q) is a rank-factorization of A, show that $(PT, T^{-1}Q)$ is a rank-factorization of A for all non-singular T and that every rank-factorization of A is of this form.

29. Show that $Q_1 = Q_2$ if (P, Q_1) and (P, Q_2) are rank-factorizations of A and $P_1 = P_2$ if (P_1, Q) and (P_2, Q) are rank-factorizations of A.

30. Show that a matrix A is of rank 1 iff $A = xy^T$ for some non-null column vectors x and y.

31. Let (P, Q) be a rank-factorization of a non-null square matrix A . Show that $A = A^2$ iff $QP = I$ and that $\rho(A) = \rho(A^2)$ iff QP is non-singular.