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$$f(X) = tr(AX)$$

$$\frac{\partial f(x)}{\partial x_{ij}} = e_j^T A e_i$$
$$= a_{ji}$$

$$\frac{\partial f(x)}{\partial X} = A^T$$

$$\frac{\partial^2 f(x)}{\partial X \partial X^T} = 0_{n \times n}$$

$$\frac{\partial f(x)}{\partial x_{11}} = a_{11} \qquad \frac{\partial^2 f(x)}{\partial x_{11} \partial x_{11}} = 0$$

if $X$ is symmetric,

$$\frac{\partial f(x)}{\partial x_{11}} = a_{11}$$

$$\frac{\partial f(x)}{\partial x_{12}} =$$

$$\frac{\partial f(x)}{\partial x_{ij}} = \frac{\partial}{\partial x_{ij}} tr(Ax) = tr\left(A \frac{\partial X}{\partial x_{ij}}\right)$$

$i \neq j$

$$= tr\left(A\left(e_i e_j^T + e_j e_i^T\right)\right)$$
$$= tr\left(A e_i e_j^T + A e_j e_i^T\right)$$
$$= tr\left(e_j^T A e_i + e_i^T A e_j\right)$$
$$= tr\left(e_j^T A e_i\right) + tr\left(e_i^T A e_j\right)$$
$$= a_{ji} + a_{ij}$$

$i = j$

$$\frac{\partial f(x)}{\partial x_{ii}} = tr\left(A \frac{\partial X}{\partial x_{ii}}\right) = tr\left(A e_i e_i^T\right)$$

$= tr(e_i^T A e_i)$

$= a_{ii}$

if $X$ is symmetric

$$\frac{\partial f(x)}{\partial X} = A + A^T - dia(a_{11}, \ldots a_{nn})$$

$$X = \begin{bmatrix} & & x_{ij} \\ & x_{ij} & \end{bmatrix}$$

· $X$ is a square matrix and $X^{-1}$ exists

$$\frac{\partial X^{-1}}{\partial x_{ij}}$$

$$X X^{-1} = I_n$$

$$\Rightarrow \frac{\partial}{\partial x_{ij}} (X X^{-1}) = \frac{\partial}{\partial x_{ij}} (I_n) = 0$$

$$\Rightarrow \frac{\partial X}{\partial x_{ij}} X^{-1} + X \frac{\partial X^{-1}}{\partial x_{ij}} = 0$$

$$\Rightarrow X \frac{\partial X^{-1}}{\partial x_{ij}} = - \frac{\partial X}{\partial x_{ij}} X^{-1}$$

$$\Rightarrow \frac{\partial X^{-1}}{\partial x_{ij}} = - X^{-1} \frac{\partial X}{\partial x_{ij}} X^{-1}$$

$$\Rightarrow \frac{\partial X^{-1}}{\partial x_{ij}} = - X^{-1} (e_i e_j^T) X^{-1}$$

if $X$ is symmetric,

$$\frac{\partial X^{-1}}{\quad} = - X^{-1} (e_i e_j^T + e_j e_i^T) X^{-1} \quad \text{if } i \neq j$$

$\partial x_{ij}$

$$= -X^{-1}\left(e_i e_i^T\right)X^{-1} \quad \text{if } i=j$$

Linear Estimation.

- $Y_1, \ldots, Y_n \sim N(\theta, 1)$ and $Y_1, \ldots Y_n$ are independent.

$\bar{Y} = \frac{1}{n}\sum Y_i$, $Y_i$ for some $i$, $\frac{Y_1+Y_2}{2}$

$$\boxed{E(Y_i) = \theta \quad \forall i = 1(1)n}$$

and $Var(Y_i) = 1 \; \forall i$

and $cov(Y_i, Y_j) = 0$ $\forall\, i \neq j$

$$\begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1} \theta_{1\times 1}$$

$$\begin{bmatrix} E(Y_1) \\ \vdots \\ E(Y_{n_1}) \\ E(Y_{n_1+1}) \\ \vdots \\ E(Y_{n_2}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix}_{n_2\times 2} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}_{2\times 1}$$

$$\begin{bmatrix} E(Y_1) \\ \vdots \\ E(Y_{n_1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ E(Y_{n_1+1}) \\ \vdots \\ E(Y_{n_2}) \end{bmatrix} \quad \begin{bmatrix} \vdots & \vdots \\ 1 & -1 \\ 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix}$$

whether we can estimate $2\theta_1 + \theta_2$?

whethere we can estimate $\theta_1$ and $\theta_2$?

$$\begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix} = \underbrace{\begin{bmatrix} & & \\ & X & \\ & & \end{bmatrix}_{n\times p}}_{\text{Known matrix}} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}$$

- The question is whether we can estimate $\beta_1, \dots, \beta_p$ or any function of $\beta_1, \dots \beta_p$.

- Let $y_1, \dots, y_n$ be such that

$E(\underset{\sim}{y}) = X\beta$ where $X_{n\times p}$ is a known matrix and $\beta$ is a set of unknown parameters. Furthere $Var(y_i) = \sigma^2$ $\forall i = 1(1)n$ (unknown) and $cov(y_i, y_j) = 0$ $\forall i \neq j$

Our objectives are

(i) Finding estimators of $\beta_1, \dots, \beta_p$ and there <u>linear</u> function

(ii) finding an estimator of $\sigma^2$.

This model is similar to the following model

$Y = X\beta + \underset{\sim}{\epsilon}$ where $\underset{\sim}{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$ are

random variables with $E(\epsilon_i) = 0 \; \forall i$,

$Var(\epsilon_i) = \sigma^2 \; \forall i$, and $cov(\epsilon_i, \epsilon_j) = 0$.
$\forall i \neq j$

And $X$ is a know matrix and $\beta$ is vector of

unknowns.

- A linear function of $\beta$, $l^T\beta$, is estimable if $\exists$ a linear function of $y$, $C^T y$, s.t

$$E(C^T y) = l^T \beta \quad \forall \beta \in \mathbb{R}^p$$

- Example

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\beta_1 + \beta_2 = 1^T \beta$$

$$y_1 + y_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$E(y_1 + y_2) = \beta_1 + \beta_2 \quad \forall \beta \in \mathbb{R}^2$$

$$E(y_3) = \beta_1 + \beta_2 \quad \forall \beta \in \mathbb{R}^2$$

$$E\left(2(y_1 + y_2) - y_3\right) = \beta_1 + \beta_2$$