MTH207M (2024-25, ODD SEMESTER)

PROBLEM SET 3

1. Consider the model $E(y_1) = \beta_1 + \beta_2$, $E(y_2) = \beta_1 - \beta_2$, $E(y_3) = \beta_1 + 2\beta_2$ with the usual assumptions. Obtain the BLUE of $2\beta_1 + \beta_2$ and find its variance. Also, find $\hat{\beta}$.

2. Consider the model $E(y_1) = 2\beta_1 + \beta_2$, $E(y_2) = \beta_1 - \beta_2$, $E(y_3) = \beta_1 + \alpha\beta_2$ with the usual assumptions. Determine α such that the BLUEs of β_1, β_2 are uncorrelated.

3. Consider the model $E(y_1) = \beta_1 + \beta_2$, $E(y_2) = 2\beta_1$, $E(y_3) = \beta_1 - \beta_2$ with the usual assumptions. Find the RSS.

4. Consider the one-way Anova model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij},\ i = 1, \ldots, k,\ j = 1, \ldots, n_i,$$

   where ϵ_{ij} are independent with the mean 0 and variance σ^2. What are the estimable functions? Is the grand mean $\bar{y}$ an unbiased estimator of μ?

5. Consider the model $E(y_1) = \beta_1 + 2\beta_2$, $E(y_2) = 2\beta_1$, $E(y_3) = \beta_1 + \beta_2$ with the usual assumptions. Find the RSS subject to the restriction $\beta_1 = \beta_2$.

6. Let $x_1, \ldots, x_n$ be real numbers with mean $\bar{x}$. Consider the linear model

$$y_i = \alpha + \beta(x_i - \bar{x})$$

   with the usual assumptions. Show that the BLUEs of α and β are uncorrelated.

7. Suppose for a linear model, there is no linear function $c^T y$ such that $E(c^T y) = 0$. Further, suppose for $l^T\beta$, $E(d^T y) = l^T\beta$. What can we say about the BLUE of $l^T\beta$?

8. Consider the standard linear model with usual assumptions. Let $\mathcal{S} = \{c \in \mathbb{R}^n \mid E(c^T y) = 0\}$. Show that

   (a) S is a subspace, [all functions $c^T y$ with $c \in S$ are called the *error* functions]

   (b) if $d(S) = 1$ and $\{d\}$ is a basis of S, then for any $p^T \beta$ the BLUE is

   $$u^T y - \frac{\operatorname{cov}(u^T y, d^T y)}{\operatorname{var}(d^T y)} d^T y$$

   where $E(u^T y) = p^T \beta$.

9. Prove or disprove: Consider a standard linear model with usual assumptions. The for any estimable linear function $l^T \beta$, every unbiased estimator is of the form $l^T G y$ for some g-inverse G of X.