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For a matrix $A_{m\times n}$, G is a LS g-inverse then

$$Gy = \arg\min_{x\in\mathbb{R}^n} \|y - Ax\| \quad \forall y$$

$$AGy = \arg\min_{w\in\mathcal{C}(A)} \|y-w\| \quad \forall y \quad \text{---} (*)$$

$w = Ax$ where $x = Gy$

$w = AGy$

<u>Claim:</u> $\arg\min_{w\in\mathcal{C}(A)} \|y-w\|$ = projection of y into $\mathcal{C}(A)$.

Proof: Suppose projection of y into $\mathcal{C}(A)$ is z

We will show $\|y-w\| \geq \|y-z\| \quad \forall w\in\mathcal{C}(A)$.

Take $w\in\mathcal{C}(A)$

$$\|y-w\|^2 = \|(y-z)-(w-z)\|^2$$

$$= \|y-z\|^2 + \|w-z\|^2 - 2\underbrace{(w-z)^T}_{\in\mathcal{C}(A)}\underbrace{(y-z)}_{\in\mathcal{C}^\perp(A)}$$

$y = z + a$, $z \in \mathcal{C}(A)$, $a \in \mathcal{C}^\perp(A)$

$$= \|y-z\|^2 + \|w-z\|^2$$

$$\geq \|y-z\|^2$$

$\Rightarrow \|y-w\| \geq \|y-z\|$

if P_A is the orthogonal projection matrix into $\mathcal{C}(A)$

then $\arg\min_{w\in\mathcal{C}(A)} \|y-w\| = P_A y \quad \forall y\in\mathbb{R}^m$ ---- $(**)$

$\langle x, y\rangle = y^T x = x^T y$

From $(*)$ and $(**)$

$$AGy = P_A y \quad \forall y\in\mathbb{R}^m$$

2) $AG = P_A$

- Note that G may not be unique but AG is always unique.

- $G = (A^TA)^- A^T$

  $P_A = A(A^TA)^- A^T$

For a matrix D, $\mathcal{C}(A) = \mathcal{C}(D)$

$$P_D = P_A$$

Applying GSOP on the column of A, we will get a set of vectors $\{y_1, \ldots, y_r\}$ s.t.

y_is are orthogonal to each other, $\|y_i\| = 1$ $\forall i = 1(1)r$,

and $Sp(\{y_1, \ldots, y_r\}) = \mathcal{C}(A)$.

$D_{m\times r} = [y_1 \ y_2 \ \ldots \ y_r]$ $\quad \mathcal{C}(D) = \mathcal{C}(A)$

$\Rightarrow P_D = P_A$

$P_D = D\underbrace{(D^TD)^-}_{I_r} D^T$

$= DD^T$.

- For a g-inverse G of $A_{m\times n}$,

  $\rho(A) \le \rho(G) \le \min\{m, n\}$

Theorem: For any s where $\rho(A) \le s \le \min\{m,n\}$, $\exists$ a g-inverse G of A s.t. $\rho(G) = s$.

Proof: See Rao Ishima's exercise.

Eigen value and Eigen Vector

For a square matrix A, we say $\lambda \neq 0$ is an eigen value if $An = \lambda n$ for some $n \neq 0$

$f(n) = An$

$\Rightarrow An = \lambda n$ for some λ

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$An = \lambda n$

$$\Rightarrow \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \lambda n_1 \\ \lambda n_2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 2n_2 &= \lambda n_1 \\ -2n_1 &= \lambda n_2 \end{aligned}$$

$$2n_2 = \lambda \frac{\lambda n_2}{-2} \Rightarrow -4n_2 = \lambda^2 n_2$$

$$\Rightarrow \lambda^2 = -4$$

$\lambda = 2i$ and $-2i$

$\lambda = 2i$, $n = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, $\lambda = -2i$ $n =$

For given λ, $An = \lambda n$

$$(A - \lambda I)n = 0$$

solving for n is basically solving this system of linear equations.

As we are interested in non null x, it

is equivalent to $\det(A - \lambda I) = 0$.