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Let A be a square matrix of order n, then A has a spectral decomposition iff

(i) $\exists$ a n-s matrix $P_{n\times n}$ and a diagonal matrix $D = \begin{bmatrix} \Delta_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}$ where all the diagonal elements of Δ are non-zero s.t

$$A = P^{-1}\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} P$$

(ii) $\exists$ $R_{n\times r}$ and $S_{r\times n}$ and a diagonal matrix $\Delta_{r\times r}$ with all non-zero diagonal elements s.t $A = R\Delta S$ and $SR = I_r$

(i) $\Rightarrow$ (ii)

$$P^{-1} = [R_{n\times r} \; \bar{R}]$$

$$P = \begin{bmatrix} S_{r\times n} \\ \bar{S} \end{bmatrix}$$

$$P^{-1}\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} P = R\Delta S$$

$$PP^{-1} = I_n$$

$$\Rightarrow \begin{bmatrix} S \\ \bar{S} \end{bmatrix} [R \; \bar{R}] = I_n$$

$$\Rightarrow \begin{bmatrix} SR & S\bar{R} \\ \bar{S}R & \bar{S}\bar{R} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

$$\Rightarrow SR = I_r$$

(ii) $\Rightarrow$ (i) $\quad I - RS = V$

$$V^2 = (I-RS)(I-RS) = I - RS - RS + RSRS$$

$$= I - 2RS + RS = I - RS$$

$$\rho(I - RS) = tr(I - RS) = tr(I) - tr(RS)$$

$$= tr(I) - tr(SR)$$

$$= n - r$$

Suppose (X, Y) is an RF of $(I - RS)$

$$X_{n\times(n-r)} Y_{(n-r)\times n} = I - RS$$

$$P = \begin{bmatrix} R_{n\times r} & X_{n\times(n-r)} \end{bmatrix}_{n\times n}$$

$$Q = \begin{bmatrix} S_{r\times n} \\ Y_{(n-r)\times n} \end{bmatrix}_{n\times n}$$

$$PQ = \begin{bmatrix} R & X \end{bmatrix} \begin{bmatrix} S \\ Y \end{bmatrix} = RS + XY = RS + I - RS$$

$$= I$$

$$Q = P^{-1}$$

$$P \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} R & X \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S \\ Y \end{bmatrix}$$

$$= R \Delta S = A$$

Singular value decomposition

For a matrix $A_{m\times n}$, we say singular value decomposition exists if there are matrices $U_{m\times m}$, $V_{n\times n}$, and $D_{m\times n} = \begin{bmatrix} \Delta_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}$ where all the diagonal elements of Δ are positive s.t

$$A = U\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}V \quad \text{and} \quad U^TU = UU^T = I_m$$

$$V^TV = VV^T = I_n$$

$$U\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}V = [\underset{m\times r}{U_1}\ U_2]\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_{1\,r\times n}\\ V_2\end{bmatrix}$$

$$= U_1 \Delta V_1$$

$$U^TU = I_m$$

$$\Rightarrow \begin{bmatrix}U_1^T\\ U_2^T\end{bmatrix}[U_1\ \ U_2] = I_m$$

$$\Rightarrow \begin{bmatrix}U_1^TU_1 & U_1^TU_2\\ U_2^TU_1 & U_2^TU\end{bmatrix} = \begin{bmatrix}I_r & 0\\ 0 & I_{m-r}\end{bmatrix}$$

$$\Rightarrow U_1^TU_1 = I_r$$

Similarly $V_1V_1^T = I_r$

Second form (singular value decomposition) $\exists$ $R_{m\times r}$, $S_{r\times n}$, and $\Delta_{r\times r}$, a diagonal matrix with all positive elements s.t.

$$A = R\Delta S \quad \text{and} \quad R^TR = I_r \quad \text{and} \quad SS^T = I_r$$

$A = R\Delta_0\Delta\Delta_0^{-1}S$ where Δ_0 is a diagonal matrix with all non-zero diagonal elements

$$A = R\Delta_0\Delta\Delta_0^{-1}S$$

$$= \bar{R}\Delta\bar{S} \qquad\qquad \bar{R}^T\bar{R} = \Delta_0R^TR\Delta_0$$

$$= \Delta_0 I_r \Delta_0$$

$$=$$

$A = R \Delta S = \sum_{i=1}^{r} d_i r_{.i} s_{i.}$ and $\Delta = \text{diag}(d_1, \ldots, d_r)$

$\bar{R} = [r_{.2} \; r_{.1} \; r_{.3} \ldots]$ $\bar{S} = \begin{bmatrix} s_{2.} \\ s_{1.} \\ \vdots \\ s_{r.} \end{bmatrix}$

$\bar{\Delta} = \text{diag}(d_2 \; d_1 \ldots d_r)$

$A = \bar{R} \bar{\Delta} \bar{S}$

In general, singular value decomposition is not unique.

<u>Theorem</u> For any matrix A, singular value decomposition exists.

- $A = U \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} V$ where $U^T U = U U^T = I_m$
  $V^T V = V V^T = I_n$

$A^T A = V^T \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} U^T U \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} V$

$= V^T \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix} V$

$V^T \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix}$ is a spectral decomposition of $A^T A$

Similarly, for $A A^T$, $U \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$ is a spectral decomposition

- Suppose $A = U \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} V$ has a singular value decomposition

$G = V^T \begin{bmatrix} \Delta^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$ is a g-inverse of A.

And this is the MP g-inverse of A.

(i) $AGA = U\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}\underbrace{VV^T}\begin{bmatrix}\Delta^{-1} & 0\\ 0 & 0\end{bmatrix}\underbrace{U^TU}\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}V$

$= U\underbrace{\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta^{-1} & 0\\ 0 & 0\end{bmatrix}}\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}V$

$= U\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}V$

$= U\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}V$

Similarly, show the other properties.

Overview of some multivariable calculus results

$f: \mathbb{R}^n \to \mathbb{R}$

$\frac{\partial f}{\partial x_i}$ is the partial derivative of f w.r.t. x_i

$\nabla f = \begin{bmatrix}\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n}\end{bmatrix}$