MTH206M (2024-25, ODD SEMESTER)

PROBLEM SET 3

1. Show that interchanging two rows can be effected by elementary row operations of the other two types, i.e., R_{ij} can be performed using $R_i(\alpha)$ and $R_{ij}(\beta)$ only.

2. Using (1), prove that every square matrix is a product of triangular matrices.

3. Let A be an $m \times n$ matrix with reduced echelon from B. If $m < n$, what is the reduced echelon form of the square matrix $\left[\begin{array}{c} A \\ \hline \mathbf{0} \end{array}\right]$ and if $m > n$ what is the reduced echelon form of the square matrix $\left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right]$?

4. Reduce each of the following matrix to a matrix in reduced echelon form by elementary row operations and find the rank, a row basis, a column basis, and a rank factorization.

   (a) $\begin{bmatrix} 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 3 & 0 & 2 \\ 5 & 7 & -9 & 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & 4 & 3 & 0 \\ 0 & 5 & 10 & 7.5 & 0 \\ 0 & 1 & 2 & 1.5 & 4 \\ 0 & 2 & 4 & 3 & 2 \end{bmatrix}$

5. Prove or disprove: if A is an $m \times n$ matrix such that $m > n$, then $Ax = \mathbf{0}$ has no non-trivial solution.

6. Prove or disprove: every system $Ax = \mathbf{0}$, where A is a 2×3 matrix, has a solution $\boldsymbol{u} = (u_1, u_2, u_3)$ with $u_1 = 0$.

7. If A is a square matrix, show that $Ax = \mathbf{0}$ has a non-trivial solution iff $y^T A = \mathbf{0}$ has a non-trivial solution.

8. Let A and B be matrices with the same number of columns. Show that $Ax = \mathbf{0}$ and $\begin{bmatrix} A \\ \hline B \end{bmatrix} x = \mathbf{0}$ have the same solution space iff $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.

9. Let A be an $m \times n$ matrix of rank $r < m$. Let B be a matrix in echelon form obtained from A by elementary row operations and let E be the transforming matrix (i.e., E is the product of the elementary matrices used in the reduction). Show that the last $m - r$ rows of E form a basis of the solution space of $x^T A = \mathbf{0}$.

10. Let A be an $m \times n$ matrix. Show that $A^T Ax = A^T b$ is consistent for all $b \in \mathbb{R}^m$. Show also that if $Ax = b$ is consistent, then the solution sets of the two systems are the same.

11. Show that $Ax = b$ has a solution belonging to $\mathcal{C}(B)$ iff $ABu = b$ is consistent.

12. Show that for every $b \in \mathcal{C}(A)$, $Ax = b$ has a solution belonging to $\mathcal{C}(A)$ iff (i) A is square and (ii) $\rho(A) = \rho(A^2)$.
(Hint: Try to show that $\mathcal{C}(A) = \mathcal{C}(A^2)$.)