17/10/24

$$A = \begin{bmatrix} x_1 \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} & x_2 \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} & \cdots & x_n \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

$A = \alpha u u^T$

$$= \begin{bmatrix} u_1 x_1 & u_1 x_2 & \cdots & u_1 x_n \\ u_2 x_1 & u_2 x_2 & \cdots & u_2 x_n \\ \vdots & & & \\ u_n x_1 & u_n x_2 & \cdots & u_n x_n \end{bmatrix}$$

$u_1 x_2 = u_2 x_1$

$\vdots$

$u_1 x_n = u_n x_1$

we need to show

$u = \alpha x$

$\Leftrightarrow u_i = \alpha x_i$ for some α and $\forall i \in 1(1)n$

$$\frac{u_1}{x_1} = \frac{u_2}{x_2}$$

$u_i = \alpha x_i$

$$\frac{u_1}{x_1} = \frac{u_n}{x_n}$$

suppose $x_1 = 0$, $u_1 = 0$

if d is dropped $\quad A = uu'$

$$a_{11} = u_1^2$$

we can get a symmetric matrix with rank 1 s.t, the first element is negative, a contradiction.

$$A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}$$

7) we know GA is idempotent $\Rightarrow \rho(GA) = tr(GA)$

Also, $\rho(GA) = \rho(A) \Rightarrow \rho(A) = tr(GA)$.

8) Let G be a g-inverse of A.

$$(I-GA)(I-GA) = I - GA - GA + GAGA$$
$$= I - GA - GA + GA$$
$$= I - GA$$

$$\underline{\rho(I-GA)} = tr(I-GA) = tr(I) - tr(GA)$$
$$= n - \rho(GA)$$
$$= \underline{n - \rho(A)}$$

For a matrix G, $\rho(I-GA) = n - \rho(A)$

we need to show G is a g-inverse of A.

$\rho(A) \cdot \rho(I - GA) \ldots$

$\rho(A) + \rho(I - GA) = n$

$$\Rightarrow \mathcal{C}(I-GA) = \mathcal{N}(A)$$

$$\rho(A(I-GA)) = \rho(I-GA) - d(\mathcal{C}(I-GA) \cap \mathcal{N}(A))$$

$$= \rho(I-GA) - d(\mathcal{C}(I-GA))$$

$$= \rho(I-GA) - \rho(I-GA)$$

$$= 0$$

$$A(I-GA) = 0$$

$$\Rightarrow A = AGA$$

$$\rho(I-GA) + \rho(A) = n$$

$$d(\mathcal{C}(I-GA)) = n - \rho(A) = d(\mathcal{N}(A))$$

$$x \in \mathcal{N}(A) \Rightarrow Ax = 0$$

$$\Rightarrow GAx = 0$$

$$\Rightarrow x = (I-GA)x \Rightarrow x \in \mathcal{C}(I-GA)$$

$$\mathcal{N}(A) \subseteq \mathcal{C}(I-GA)$$

Also $d(\mathcal{N}(A)) = d(\mathcal{C}(I-GA))$

So, $\mathcal{N}(A) = \mathcal{C}(I-GA)$

9) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$

$C = XA$ for some X

$$CA^{-}A = XAA^{-}A = XA = C$$

given $\underbrace{CA^{-}A = C}$

$R(C) \subseteq R(A)$

10) $\rho(AB) = \rho(A) \Rightarrow C(A) = C(AB)$

$\Rightarrow A = ABX$

$A \underbrace{B(AB)^{-}}_{} A = AB(AB)^{-} ABX = ABX = A$

11) $A\left(G + (I - GA)U + V(I - AG)\right)A$

$= AGA + A(I - GA)UA + AV(I - AG)A$

$= A + (A - AGA)UA + AV(A - AGA)$

$= A$

Let $\underline{H}$ be a g-inverse of A

$U = H - G$

$G + (I - GA)(H - G) + V(I - AG)$

$= \cancel{G} + (H - \cancel{G} - GAH + GAG) + V - VAG$

$= \underline{H - GAH + GAG + V - VAG}$

$= H - GAH + GAG + GAH - GAHAG \quad \boxed{V = GAH}$

$= H - G\cancel{AH} + G\cancel{A}G + \cancel{GA}H - \cancel{GA}G$

$= H$

13 (b) GX^T where $G = (X^TX)^-$

$$X(GX^T)X = X(X^TX)^- X^TX$$

$$= YX^TX(X^TX)^- X^TX$$

$$= YX^TX = X$$

$\rho(X) = \rho(X^TX)$

$\Rightarrow R(X) = R(X^TX)$

$X = YX^TX$

$\rho(X^TX)$

$= \rho(X) - d(\mathcal{C}(X) \cap N(X^T))$

$= \rho(X) - d(\mathcal{C}(X) \cap R^{\perp}(X^T))$

$= \rho(X) - d(\mathcal{C}(X) \cap \mathcal{C}^{\perp}(X))$

$= \rho(X)$

$\rho(X) = \rho(XX^T)$