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$$\begin{bmatrix} E(y_1) \\ E(y_2) \\ E(y_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_1+\beta_2 \end{bmatrix}$$

$$\beta_1 = \gamma_1 \quad , \beta_2 = \gamma_2 \quad , \beta_1+\beta_2 = \gamma_3$$

$$\begin{bmatrix} E(y_1) \\ E(y_2) \\ E(y_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

$$E(y) = X\beta \quad \text{where } \beta \in \mathbb{R}^p$$

$$V(y) = \begin{bmatrix} V(y_1) & Cov(y_1,y_2) & \cdots & Cov(y_1,y_n) \\ Cov(y_2,y_1) & V(y_2) & \cdots & Cov(y_2,y_n) \\ \vdots & & & \\ Cov(y_n,y_1) & Cov(y_n,y_2) & \cdots & V(y_n) \end{bmatrix}$$

$$= \sigma^2 I_n$$

$$Var(Zy) \quad \text{where } Z_{p \times n}$$

$$= Z\, Var(y)\, Z^T = \sigma^2 Z Z^T$$

$$\to Var(Zy) = Var(x)$$

$$= E\left((x - E(x))(x - E(x))^T \right)$$

$$= E\left(\left(Zy - ZE(y)\right)\left(Zy - ZE(y)\right)^T\right)$$

$$= E\left[Z\left(y - E(y)\right)\left(y - E(y)\right)^T Z^T\right]$$

$$= Z\,E\left(\left(y - E(y)\right)\left(y - E(y)\right)^T\right) Z^T$$

$$= Z \operatorname{Var}(y) Z^T$$

$$= \sigma^2 Z Z^T$$

<u>Lemma</u> Consider a linear function $l^T\beta$. Then $l^T\beta$ is estimable iff $l^T \in R(X)$.

<u>Proof</u> First assume that $l^T\beta$ is estimable.

This means $\exists\, C \in \mathbb{R}^n$ s.t. $E(C^T y) = l^T\beta \quad \forall \beta \in \mathbb{R}^p$

$$\Rightarrow C^T E(y) = l^T\beta$$

$$\Rightarrow C^T X\beta = l^T\beta$$

$\boxed{a_{i\cdot}^T \in R(A)}$

$$\Rightarrow (C^T X - l^T)\beta = 0 \quad \forall \beta \in \mathbb{R}^p$$

$$\Rightarrow C^T X = l^T$$

Thus $l^T \in \mathbb{R}(X)$.

Suppose $l^T \in R(X) \Rightarrow l^T = \underline{C}^T X$

Consider the linear function $l^T\beta$ and $C^T y$

$$E(C^T y) = C^T E(y) = C^T X\beta$$

$$= l^T\beta$$

$l^T\beta$ is estimable.

- $$\begin{bmatrix} E(y_1) \\ E(y_2) \\ E(y_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

Is β_1 estimable? ✓

$$= (1, 0, 0) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

- $$\begin{bmatrix} E(y_1) \\ E(y_2) \\ E(y_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

In this model, β_3 is not estimable

- If $R(X) = \mathbb{R}^p$ the all linear functions of β are estimable, i.e., if $\rho(X) = p$.

<u>Theorem</u> Suppose <u>$l^T\beta$ is</u> estimable. Then $l^T G y$, where G is a g. inverse of X, is an unbiased estimator of $l^T\beta$, i.e. $E(l^T G y) = l^T\beta$.

Proof $E(l^T G y) = l^T G\, E(y) = l^T G X \beta$

$$= C^T X G X \beta$$

$$= C^T X \beta$$

$$= l^T \beta$$

$$\left[\begin{array}{l} l^T \in R(G) \\ \Rightarrow l^T \\ = C^T X \end{array}\right]$$

Theorem Suppose $\ell^T\beta$ is estimable. Then $\ell^T G y$, where G is LS g inverse of X, has the minimum variance among all linear unbiased estimators of $\ell^T\beta$.

The estimator $\ell^T G y$ is called the Best linear unbiased estimator (BLUE) of $\ell^T\beta$.

Proof. Take another unbiased estimator of $\ell^T\beta$, say $C^T y$. So, $E(C^T y) = \ell^T\beta$

$$\Rightarrow C^T X = \ell^T$$

$$C^T y = C^T y + \ell^T G y - \ell^T G y = \ell^T G y + \underbrace{(C^T - \ell^T G)}_{W} y$$

$$\begin{aligned} WX &= (C^T - \ell^T G)X \\ &= C^T X - \ell^T G X \\ &= \ell^T - C^T X G X = \ell^T - C^T X = \ell^T - \ell^T = 0 \end{aligned}$$

$$C^T y = \ell^T G y + W y \qquad \text{where } WX = 0$$

$$\begin{aligned} Var(C^T y) &= Var\left((\ell^T G + W) y\right) \\ &= \sigma^2 \left((\ell^T G + W)(G^T \ell + W^T) \right) \\ &= \sigma^2 \left((\ell^T G G^T \ell + \ell^T G W^T + W G^T \ell + W W^T) \right) \\ &= \sigma^2 \left((\ell^T G G^T \ell + C^T X G W^T + W G^T X^T C \right. \end{aligned}$$

$$+ WW^T\big)$$

$$= \sigma^2\Big(\big(l^T G G^T l + C^T G^T \underline{X^T W^T} + \underline{WX} G C + WW^T\big)\Big)$$

$$= \sigma^2\big(l^T G G^T l + WW^T\big) = \sigma^2 l^T G G^T l + \sigma^2 WW^T$$

$$Var\big(l^T G Y\big) = \sigma^2 l^T G G^T l$$

$$W_{1\times n}, \quad WW^T = \sum_{i=1}^{n} w_i^2 \geq 0$$

$$Var\big(l^T G Y + WY\big) \geq Var\big(l^T G Y\big)$$

So, $l^T G y$ has the minimum variance among all the unbiased estimators of $l^T \beta$.