11/11/2024.

- For a random vector Y, the characteristic function (CF), denoted by $\phi_Y(t)$, is defined as $E(e^{it^T Y})$ $\forall\ t \in \mathbb{R}^n$ where Y is a n-dimensional random vector.

It is known that CF of a random vector always exists and uniquely characterizes the distribution.

- Suppose $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is a n-dimensional random vector. Then Y_1 and Y_2 are independent iff
$$\phi_Y(t) = \phi_{Y_1}(t_1)\,\phi_{Y_2}(t_2) \quad \text{for all}$$
$t = (t_1, t_2) \in \mathbb{R}^n$.

- For a random vector Y with $Y \sim MVN(\mu, \Sigma)$. Then the
$$\phi_Y(t) = \underline{e^{\left(it^T\mu - \frac{1}{2} t^T \Sigma t\right)}}.$$

(R-1) Suppose $Y \sim MVN(\mu, \Sigma)$ then for any matrix B,
$$BY \sim MVN(B\mu, B\Sigma B^T)$$

Proof $\phi_{BY}(t) = E\left(e^{it^T BY}\right)$

X and Y are rvs.

$P(X \leq x, Y \leq y)$

$= P(X \leq x)\, P(Y \leq y)$

- For n rvs

$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$

$= P(X_1 \leq x_1) \cdots P(X_n \leq x_n)$

$$= E\left(e^{i(B^T t)^T Y}\right)$$

$$= E\left(e^{is^T Y}\right) \left(= \phi_Y(s)\right) \quad s = B^T t$$

$$= e^{\left(is^T \mu - \frac{1}{2} s^T \Sigma s\right)}$$

$$= e^{\left(it^T B\mu - \frac{1}{2} t^T B \Sigma B^T t\right)}$$

$$= e^{\left(it^T \mu^* - \frac{1}{2} t^T \Sigma^* t\right)}$$

where $\mu^* = B\mu$

and $\Sigma^* = B\Sigma B^T$

$$BY \sim MVN(B\mu, B\Sigma B^T).$$

R-2 $\quad Y \sim MVN(\mu, \Sigma)$

$$\underset{n\times 1}{Y} = \begin{pmatrix} Y_{1\, q\times 1} \\ Y_2 \end{pmatrix} \qquad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Then $\quad Y_1 \sim MVN(\mu_1, \Sigma_{11})$ ✓

$\qquad Y_2 \sim MVN(\mu_2, \Sigma_{22})$

$$B = \left[I_q \mid 0 \right]$$

$$\underline{BY} = \left[I_q \mid 0 \right] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = Y_1$$

$B\mu = \mu_1 \qquad B\Sigma B^T = \Sigma_{11}$

So, from R-1, we have $Y_1 \sim MVN(\mu_1, \Sigma_{11})$

$B = \left[0 \mid I_{n-q} \right] \qquad BY = Y_2$, $B\mu = \mu_2$, and

$B\Sigma B^T = \Sigma$

$B \Sigma B^T = \Sigma_{22}$

$$Y_2 \sim MVN(\mu_2, \Sigma_{22})$$

(R-3) In (R-2) setup, Y_1 and Y_2 are independent iff $\Sigma_{12} = 0$.

Proof: Suppose Y_1 and Y_2 are independent then

$$\operatorname{cov}(Y_1, Y_2) = 0 \Rightarrow \Sigma_{12} = 0$$

Now if $\Sigma_{12} = 0$ then $\Sigma_{21} = 0$

$$t^T \Sigma t = [t_1^T \ t_2^T] \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$= t_1^T \Sigma_{11} t_1 + t_2^T \Sigma_{22} t_2$$

$$\phi_Y(t) = e^{\left(i t^T \mu - \frac{1}{2} t^T \Sigma t\right)}$$

$$= e^{\left(i t_1^T \mu_1 + i t_2^T \mu_2 - \frac{1}{2}(t_1^T \Sigma_{11} t_1) - \frac{1}{2}(t_2^T \Sigma_{22} t_2)\right)}$$

$$= e^{\left(i t_1^T \mu_1 - \frac{1}{2}(t_1^T \Sigma_{11} t_1)\right)} e^{\left(i t_2^T \mu_2 - \frac{1}{2} t_2^T \Sigma_{22} t_2\right)}$$

$$= \phi_{Y_1}(t_1) \ \phi_{Y_2}(t_2)$$

So, Y_1 and Y_2 are independent.

(R-4) $Y \sim MVN(\mu, \Sigma)$, AY and BY are independent iff $A \Sigma B^T = 0$

Proof Suppose AY and BY are independent

then $\operatorname{cov}(AY, BY) = 0$

$$\Rightarrow A \Sigma B^T = 0$$

$\Rightarrow A\Sigma B^T = 0$

$$\begin{bmatrix} A \\ B \end{bmatrix} Y \sim MVN\left[\begin{pmatrix} A\mu \\ B\mu \end{pmatrix} \begin{pmatrix} A\Sigma A^T & A\Sigma B^T \\ B\Sigma A^T & B\Sigma B^T \end{pmatrix}\right]$$

$$\begin{bmatrix} AY \\ BY \end{bmatrix}$$

Now from R-3, we can say that

AY and BY are indebondent if $A\Sigma B^T = 0$.

<u>Fisher Cochran Theorem</u>

$Y_1, \ldots, Y_n \sim N(0,1)$

$Y^T Y = \sum Y_i^2 \sim \chi^2_n$

A is symetric and idempotent

$Y^T A Y \sim \chi^2_r$ when $r = \rho(A)$

$A = P^T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P$

$$A = P^T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P$$

$$Y^T A Y = Y^T P^T \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P Y$$

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$Y^T A Y = P_1 Y$$

<u>Theorem</u> Suppose $Y \sim MVN(0, I_n)$. Further, suppose $\{A_i\}_{i=1}^k$ are symmetric and idempotent matrices with $\rho(A_i) = r_i$. Then if $\sum_{i=1}^k r_i = n$, the following statements are equivalent.

- $A_i A_j = 0 \quad \forall i \neq j$
- $Y^T A_i Y \sim \chi^2_{r_i}$ and $Y^T A_i Y$ and $Y^T A_j Y$ are independent.

$Y^T Y =$