MTH206M (2024-25, ODD SEMESTER)

PROBLEM SET 1

1. Check whether the subsets given form subspaces of the vector space V?

   (a) $V = \mathbb{R}^3$, $S = \{(x_1, x_2, x_3) : 2x_1 + x_2 + x_3 = 1\}$.

   (b) $V = \mathbb{R}^2$, $S = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0\}$.

   (c) $V = \mathbb{R}^2$, $S = \{(x_1, x_2) \mid x_1 x_2 \geq 0\}$.

   (d) $V = \mathbb{R}^2$, $S = \{c(2,3) + d(-3,1) \mid c, d \in \mathbb{R} \text{ and } c = d\}$.

2. Gilbert Strang Book: Problem Set 1.1 (Page 8)

   Problems 15-19 and Problems 20, 22, 23, 24, and 27.

3. Find a subspace $V \subseteq \mathbb{R}^3$ that contains the vectors $(1,0,1)$ and $(2,3,0)$ but does not contain the vector $(1,0,0)$.

4. For any two subsets A and B of a vector space V, show that

   (i) $Sp(A) \cup Sp(B) \subseteq Sp(A \cup B)$,

   (ii) $Sp(A \cap B) \subseteq Sp(A) \cap Sp(B)$,

   and the proper inclusion is possible in each.

5. Prove or disprove: $Sp(A) \cap Sp(B) \neq \emptyset \implies A \cap B \neq \emptyset$.

6. Suppose V is a vector space and S is a subspace of V. If x and y are vectors in V such that $x + y \in S$ then show that either both x and y are in S or none of x and y are in S. What if $x + y + z \in S$? Further if α is a non-zero scalar and $\alpha x \in S$ then show that $x \in S$.

7. Show that the intersection of any family of subspaces is a subspace (the intersection of the empty family of subsets of V is defined to be V).

8. Show that for any set $A \subseteq V$, $Sp(A)$ is the intersection of all subspaces of V containing A.

9. Show that $x_1, x_2, \ldots, x_k$ are linearly independent iff

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$$

implies

$$\alpha_1 = \beta_1, \ldots, \alpha_k = \beta_k.$$

10. Prove or Disprove: Suppose a vector u can be written in a unique way in terms of $x_1, \ldots, x_k$. Then $x_1, \ldots, x_k$ are independent.

11. If x and y are linearly independent show that $x + \alpha y$ and $x + \beta y$ are linearly independent whenever $\alpha \neq \beta$.

12. Let A be a linearly independent subset of a subspace S. If $x \notin S$, show that $A \cup \{x\}$ is linearly independent. If $B \subseteq S \setminus A$ and B is linearly independent, does it follow that $A \cup B$ is linearly independent?

13. Let $Sp(A) = S$. Then show that no proper subset of A generates S iff A is linearly independent.

14. For what values of α, are the veetors $(0, 1, \alpha)$, $(\alpha, 1, 0)$ and $(1, \alpha, 1)$ in $\mathbb{R}^3$ linearly independent?

15. For two subspaces S and T, show that $S \cup T$ is a subspace if and only if either $S \subseteq T$ or $T \subseteq S$.

16. Let the set of vectors $A = \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ be linearly independent. Show that

$$Sp(\{x_1, \ldots, x_r, y_1, \ldots, y_s\}) = Sp(\{x_1, \ldots, x_r\}) + Sp(\{y_1, \ldots, y_s\})$$

holds and this sum is direct.

17. From Rao and Bhimasankaram's book:

- Section 1.4: $5(a)$.
- Section 1.5: $2,\ 3,\ 7,\ 8,\ 9,\ 12,\ 15,\ 16,\ 17$.
- Section 1.6: $2,\ 3,\ 5,\ 6,\ 7,\ 8$.
- Section 1.7: $1,\ 2,\ 5,\ 6,\ 7$.