Computation of a g-inverse.

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & -2 & -4 \\ 2 & -1 & 1 & -1 & -2 \end{bmatrix}$

we want to compute a g inverse of A. we augment $[A|I_4]$ and perform row operations on both the matrices to reduce A to a matrix in REF.

$$\left[\begin{array}{ccccc|cccc} 1 & 2 & 3 & 1 & 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & -4 & 0 & 0 & 1 & 0 \\ 2 & -1 & 1 & -1 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_{21}(1)$$

$$\left[\begin{array}{ccccc|cccc} 1 & 2 & 3 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & -4 & 0 & 0 & 1 & 0 \\ 2 & -1 & 1 & -1 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_{41}(-2)$$

$$\left[\begin{array}{ccccc|cccc} 1 & 2 & 3 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & -4 & 0 & 0 & 1 & 0 \\ 0 & -5 & -5 & -3 & -6 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|cccc} 0 & 3 & 3 & -2 & -4 & 0 & 0 & 1 & 0 \\ 0 & -5 & -5 & -3 & -6 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_2\left(\tfrac{1}{3}\right)$$

$$\left[\begin{array}{ccccc|cccc} 1 & 2 & 3 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1/3 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 3 & 3 & -2 & -4 & 0 & 0 & 1 & 0 \\ 0 & -5 & -5 & -3 & -6 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_{12}(-2)$$

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 1/3 & 2/3 & 1/3 & -2/3 & 0 & 0 \\ 0 & 1 & 1 & 1/3 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 3 & 3 & -2 & -4 & 0 & 0 & 1 & 0 \\ 0 & -5 & -5 & -3 & -6 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_{32}(-3)$$

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 1/3 & 2/3 & 1/3 & -2/3 & 0 & 0 \\ 0 & 1 & 1 & 1/3 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 & -1 & -1 & 1 & 0 \\ 0 & -5 & -5 & -3 & -6 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow R_{42}(-5)$$

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 1/3 & 2/3 & 1/3 & -2/3 & 0 & 0 \\ 0 & 1 & 1 & 1/3 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -4/3 & -8/3 & -11/3 & -5/3 & 0 & 1 \end{array}\right]$$

$R_3(-1/3)$

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 1/3 & 2/3 & 1/3 & -2/3 & 0 & 0 \\ 0 & 1 & 1 & 1/3 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1/3 & 1/3 & -1/3 & 0 \\ 0 & 0 & 0 & -4/3 & -8/3 & -11/3 & -5/3 & 0 & 1 \end{array}\right]$$

$R_{13}(-1/3)$

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 0 & 0 & 2/9 & -7/9 & 1/9 & 0 \\ 0 & 1 & 1 & 1/3 & 2/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1/3 & 1/3 & -1/3 & 0 \\ 0 & 0 & 0 & -4/3 & -8/3 & -11/3 & -5/3 & 0 & 1 \end{array}\right]$$

$R_{23}(-1/3)$

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 0 & 0 & 2/9 & -7/9 & 1/9 & 0 \end{array}\right.$$

$$\left[\begin{array}{ccccc|cccc} 0 & 1 & 1 & 0 & 0 & 2/9 & 2/9 & 1/9 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1/3 & 1/3 & -1/3 & 0 \\ 0 & 0 & 0 & -4/3 & -8/3 & -11/3 & -5/3 & 0 & 1 \end{array}\right]$$

$\downarrow R_{43}(4/3)$

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 1 & 0 & 0 & 2/9 & -7/9 & 1/9 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2/9 & 2/9 & 1/9 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1/3 & 1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -29/9 & -11/9 & -4/3 & 1 \end{array}\right]$$

So, $E = \begin{bmatrix} 2/9 & -7/9 & 1/9 & 0 \\ 2/9 & 2/9 & 1/9 & 0 \\ 1/3 & 1/3 & -1/3 & 0 \\ -29/9 & -11/9 & -4/3 & 1 \end{bmatrix}$

and $\rho(A) = 3$, $p_1 = 1$, $p_2 = 2$, and $p_3 = 4$.

Thus $G = \begin{bmatrix} 1\text{st row of } E \\ 2\text{nd row of } E \\ \text{null vector} \\ 3\text{rd row of } E \\ \text{null vector} \end{bmatrix}_{5\times 4}$

$$= \begin{bmatrix} 2/9 & -7/9 & 1/9 & 0 \\ 2/9 & 2/9 & 1/9 & 0 \\ 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} & & & \\ 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$