MTH207M (2024-25, ODD SEMESTER)

PROBLEM SET 2

1. Let A and G be two matrices such that G is a g-inverse of A. Further, (P, Q) is an RF of A. Then

   (a) $QGP = I_r$ where $\rho(A) = r$, and

   (b) if G is a reflexive g-inverse then $G = Q_R^{-1} P_L^{-1}$ for some right and left inverse of P and Q, respectively.

2. Let A be an $m \times n$ matrix. Then, G is a minimum norm and reflexive g-inverse of A iff $G = A^T(AA^T)^-$ where $(AA^T)^-$ is a g-inverse of AA^T.

3. Let A be an $m \times n$ matrix. Then, G is a LS and reflexive g-inverse of A iff $G = A^T(A^T A)^-$ where $(A^T A)^-$ is a g-inverse of $A^T A$.

4. (a) For any minimum norm and reflexive g-inverse G of A, prove that $\mathcal{C}(G) = \mathcal{R}(A)$.

   (b) For any reflexive and LS g-inverse G of A, prove that $\mathcal{R}(G) = \mathcal{C}(A)$.

5. Let B^+ be the MP g-inverse of B. Then show the following

   (a) $(A^+)^T = (A^T)^+$

   (b) $(A^T A)^+ = A^T(A^T)^+$,

   (c) $(A^+)^+ = A$,

   (d) $(AA^+)^+ = AA^+$,

   (e) if A is symmetric, then A^+ is symmetric,

   (f) if A is symmetric and idempotent, then $A = A^+$.

6. Let $Ax = b$ be a consistent system. Show that x_j has the same value in all solutions iff $(GA)_{j\cdot} = e_j^T$ for any g-inverse G of A.

7. Show that, for any $m \times n$ matrix A, $A^+ = A^T$ if and only if $A^T A$ is idempotent.

8. Consider a matrix $A = uv^T \neq 0$ where $u, v \in \mathbb{R}^n$. Then show that $B = c^{-1} A^T$ is the MP g-inverse of A where $c = \text{tr}(A^T A)$.

   Let $A = BC$ be a rank factorization of A. Then show that

   (a) $B^+ = (B^T B)^{-1} B^T$ and $C^+ = C^T (CC^T)^{-1}$, and

   (b) $A^+ = C^+ B^+$.

9. Prove or disprove: $A^+ = B^+$ implies $A = B$.

10. Find the MP g-inverse of AA^+ and $A^+ A$.