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For a minimum norm g-inverse, we have

$$G A A^T = A^T$$

$$\boxed{\begin{array}{l} AGA = A \\ AA^TG^T = A \\ GAA^T = A^T \end{array}}$$

$A^T(AA^T)^-$ will satisfy this equation

$$\boxed{\begin{array}{l} A^T = XAA^T \\ \text{for some } X \\ \rho(A^T) = \rho(AA^T) \\ R(A^T) \supseteq R(AA^T) \end{array}}$$

$$A^T(AA^T)^- AA^T$$

$$= X(AA^T)(AA^T)^- AA^T$$

$$= X AA^T = A^T$$

Also, $A^T(AA^T)^-$ is a g-inverse of A.

$$\boxed{\begin{array}{l} R(A^T) \\ \quad = R(AA^T) \\ R(A^T) \\ \quad \subseteq R(AA^T) \\ A^T = XAA^T \end{array}}$$

So, a minimum norm g-inverse exists.

$$A\left(A^T(AA^T)^-\right)A$$

$$= AA^T(AA^T)^- A$$

$$A = (AA^T)Y$$

$$= AA^T(AA^T)^- AA^T Y$$

$$= (AA^T)Y = A$$

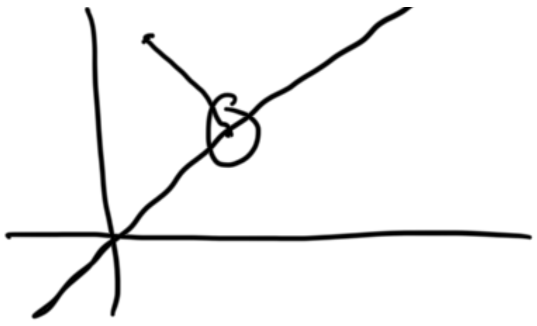
- $y \notin \mathcal{C}(A)$ $\qquad \|y - Ax\| > 0 \quad \forall x \in \mathbb{R}^n$

$A_{m\times n}$

$$\arg\left(\min_{x\in\mathbb{R}^n} \|y - Ax\|\right)$$

- A g-inverse is a least square g-inverse (LS) if for all y and x

$$\|y - AGy\| \le \|y - Ax\|$$



Theorem: Suppose A is a matrix and G is a g-inverse of A. Then, the following statements are equivalent.

(i) G is a LS g-inverse

(ii) AG is a symmetric matrix.

(iii) $(A^TA)G = A^T \longrightarrow$ we can not write $G = (A^TA)^- A^T$ but $(A^TA)^- A^T$ will satisfy $(A^TA)G = A^T$

Proof: (i) $\Rightarrow$ (ii) G is a LS g-inverse

$\Rightarrow \forall x, y$

$$\|y - AGy\| \leq \|y - Ax\|$$

$$\boxed{\begin{array}{c} Ax = b \\ x = Gb \end{array}}$$

$$x - Gy = w$$

$$\|y - AGy\| \leq \|y - Aw - AGy\| \quad \forall w, \forall y$$

$$\Rightarrow \|y - AGy\|^2 \leq \|y - Aw - AGy\|^2 \quad \forall w, \forall y$$

$$= \|y - AGy\|^2 + \|Aw\|^2 - 2w^TA^T(y - AGy)$$

$$\Rightarrow \|Aw\|^2 - 2w^TA^T(y - AGy) \geq 0 \quad \forall w \text{ and } \forall y$$

$$\Rightarrow \|Aw\|^2 - 2w^TA^T(I - AG)y \geq 0 \quad \forall w, \forall y$$

$$\Rightarrow w^TA^T(I - AG)y = 0 \quad \forall w, \forall y$$

$$\Rightarrow \quad w^T A^T (I - AG) = 0 \quad \forall w$$

$$\Rightarrow \quad (I - AG)^T A w = 0 \quad \forall w$$

$$\Rightarrow \quad (I - AG)^T A = 0$$

$$\Rightarrow \quad A - (AG)^T A = 0$$

$$\Rightarrow \quad A = (AG)^T A$$

$$\Rightarrow \quad \underline{AG = (AG)^T AG}$$

$\Rightarrow$ AG is a symmetric matrix

(ii) $\Rightarrow$ (i) $\quad$ AG is symmetric

We have to show

$$\|y - AGy\| \le \underline{\|y - Ax\|} \quad \forall x, \forall y$$

$$\|y - Ax\|^2 = \|y - Aw - AGy\|^2 \quad \Big| \; x - Gy = w$$

$$\Rightarrow \|y - Ax\|^2 = \|y - AGy\|^2 + \|Aw\|^2 - \underline{2 w^T A^T (I - AG) y} \; \ldots (*)$$

$$w^T A^T (I - AG) y$$

$A = AGA$

$\Rightarrow A^T = A^T G^T A^T$

$$\Rightarrow w^T (A^T - A^T AG) y$$

$$\Rightarrow w^T (A^T - \overline{A^T G^T A^T}) y$$

$$\Rightarrow w^T (A^T - A^T) y$$

$$\Rightarrow 0$$

From $(*)$, we have $\|y - Ax\|^2 \ge \|y - AGy\|^2$

$$\Rightarrow \|y - Ax\| \ge \|y - AGy\|$$

(ii) $\Rightarrow$ (iii) $\quad$ AG is symmetric and $AGA = A$

$$G^T A^T A = A$$
$$\Rightarrow (A^T A) G = A^T$$

- We want a g-inverse that satisfies all the three properties, i.e, a g-inverse which is reflexive, minimum norm, LS. Equivalently, we want a matrix G s.t.
  - $AGA = A$
  - $GAG = G$ (reflexive)
  - $(GA)^T = GA$ (minimum norm)
  - $(AG)^T = AG$ (LS g-inverse)

  Such a g-inverse is called a Moore-Penrose (MP) g-inverse.

<u>Theorem</u> For a matrix A, MP g-inverse always exists and is unique. It is denoted by A^+.

Proof: (Existence) First assume that $A = O_{m \times n}$. Then $O_{n \times m}$ satisfies all the properties of a MP g-inverse.

Now, suppose A is a non-null matrix. Consider $G = Q^T (P^T A Q^T)^{-1} P^T$ where

(P, Q) is an RF of A.

Note that $P^T A Q^T = (P^T P)(Q Q^T)$ is an invertible matrix. We show that G satisfies all the properties of a MP g-inverse.

- $$\begin{aligned} AGA &= A Q^T (P^T A Q^T)^{-1} P^T A \\ &= P(QQ^T)(QQ^T)^{-1}(P^T P)^{-1}(P^T P)Q \\ &= PQ = A \end{aligned}$$

- $$\begin{aligned} GAG &= Q^T (QQ^T)^{-1}(P^T P)^{-1} P^T PQ \\ &\quad Q^T (QQ^T)^{-1}(P^T P)^{-1} P^T \\ &= Q^T (QQ^T)^{-1}(P^T P)^{-1} P^T \\ &= G \end{aligned}$$

- $$\begin{aligned} GA &= Q^T (QQ^T)^{-1}(P^T P)^{-1} P^T P Q \\ &= Q^T (QQ^T)^{-1} Q \end{aligned}$$

Note that $Q^T (QQ^T)^{-1} Q$ is a symmetric matrix. So GA is a symmetric matrix.

- $$\begin{aligned} AG &= PQ Q^T (QQ^T)^{-1}(P^T P)^{-1} P^T \\ &= P(P^T P)^{-1} P^T \end{aligned}$$

Again $P(P^T P)^{-1} P^T$ is a symmetric matrix.

So, AG is a symmetric matrix.

(Uniqueness) Let G_1 and G_2 be two MP g-inverses. Then

$$\begin{aligned}
G_1 &= G_1 A G_1 \\
&= G_1 G_1^T A^T && (\text{as } AG_1 = G_1^T A^T) \\
&= G_1 G_1^T A^T G_2^T A^T && (\text{as } A^T = A^T G_2^T A^T) \\
&= G_1 A G_1 G_2^T A^T && (\text{as } AG_1 = G_1^T A^T) \\
&= G_1 A G_2 && (\text{as } G_1 A G_1 = G_1)
\end{aligned}$$

Now
$$\begin{aligned}
G_2 &= G_2 A G_2 \\
&= A^T G_2^T G_2 && (\text{as } G_2 A = A^T G_2^T) \\
&= A^T G_1^T A^T G_2^T G_2 && (\text{as } A^T = A^T G_1^T A^T) \\
&= G_1 A A^T G_2^T G_2 && (\text{as } A^T G_1^T = G_1 A) \\
&= G_1 A G_2 A G_2 && (\text{as } A^T G_2^T = G_2 A) \\
&= G_1 A G_2 && (\text{as } G_2 A G_2 = G_2)
\end{aligned}$$

Thus $G_1 = G_2$.