03/10/2024

1) $P_{m\times r} T_{r\times r} T^{-1} Q_{r\times n}$

$\rho(A) = r$ $\qquad PTT^{-1}Q = PQ = A$

As T is n-s multiplying by will not change the rank. So, $(PT, T^{-1}Q)$ is an RF.

$P \qquad P_1$

(P_1, Q_1) is an RF $\Rightarrow \rho(P_1) = \rho(Q_1) = r$
$P_1 Q_1 = A$

$\mathcal{C}(P_1) \underset{\subseteq}{=} \mathcal{C}(P)$

$(P_1)_{m\times r} = P_{m\times r} T_{r\times r}$

if $\rho(T) < r$ $\quad \underline{\rho(P_1) \leq \rho(T)} < r$ not possible

$\rho(T) = r$ and $P_1 = PT$

$Q_1 = SQ$.

$P_1 Q_1 = PTSQ = PQ$

$\Rightarrow \qquad PTSQ = PQ$

$TS = I_r$

As T is a square matrix, $T = S^{-1}$

2) If A is a null matrix, take C to be

a null matrix.

If $\rho(A) \geq 1$, $A = PQ$

$\mathcal{C}(A) = \mathcal{C}(P)$, $\mathcal{R}(A) = \mathcal{R}(Q)$

$\mathcal{C}(A) \subseteq \mathcal{C}(B)$	$\mathcal{R}(A) \subseteq \mathcal{R}(D)$
$\Rightarrow \mathcal{C}(P) \subseteq \mathcal{C}(B)$	$\Rightarrow \mathcal{R}(Q) \subseteq \mathcal{R}(D)$
$\Rightarrow P = BX$	$\Rightarrow Q = YD$

$$A = PQ = B\underbrace{XY}_{C}D = BCD$$

4) $A = A^2$

$PQ = PQPQ$

$\Leftrightarrow P_L^{-1} PQ Q_R^{-1} = P_L^{-1} PQPQ Q_R^{-1}$

$\Leftrightarrow I_r = QP$

$QP = I_r$

$PQPQ = PQ$

$\Rightarrow A^2 = A$

$\rho(A) = \rho(A^2) = r$

$\Rightarrow \underline{r} = \rho(A) = \rho(PQPQ) \leq \rho(QPQ)$

$\leq \rho(QP) \leq \underline{r}$

$\rho(QP) = r$

$P_{n\times r}$ $Q_{r\times n}$

$(QP)_{r\times r}$

$\underline{QP \text{ is n-s}} \Rightarrow \rho(A) = \rho(A^2)$

$\rho(QP) = r = \rho(A)$

$\rho(PQPQ) \leq \rho(QP) = r$

$A^2 = PQPQ$

$A^2 Q_R^{-1} = PQP$

$\rho(A^2 Q_R^{-1}) = r$

$r = \rho(A^2 Q_R^{-1}) \leq \rho(A^2)$

$\rho(A^2) \geq r$

$\rho(PQP) = \rho(P) = r$

as QP is n-s

$\rho(A^2) \leq \rho(A) = r$

$\rho(A^2) = r$

$\rho(PQPQ) \geq \rho(P_L^{-1} PQPQ Q_R^{-1}) = \rho(QP) = r$

5) $A_{n\times n}$ Is there an RF (P, P^T)?

$\rho(A) = r$ $\quad P_{n\times r}$

A is symmetric

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$$

$$= \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}$$

$x^2 = 1, \quad xy = 1, \quad y^2 = 1$

$$(x-y)^2 = x^2 - 2xy + y^2 = 1-2+1 = 0$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ w & z \end{bmatrix} \begin{bmatrix} x & w \\ y & z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$$

$$x^2 = 1 \quad xy = 0 \quad y^2 = 0$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$$

$$x^2 = 0, \quad xy = 1, \quad y^2 = 1$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$$

$$x^2 = -1 \qquad y^2 = -1 \quad xy = 1$$

$$x = -i, \quad y = i$$

6) $A = x y^T$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \underline{\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}}$$

$$= \overline{x y^T}$$

$$= \Big[y_1 \begin{bmatrix} x_1 \\ \vdots \end{bmatrix} \quad y_2 \begin{bmatrix} x_1 \\ \vdots \end{bmatrix}$$

$$x_n \qquad x_n$$

$$\cdots \qquad y_m \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \Bigg]$$