Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

MTH 207M: Matrix Algebra and Linear Estimation (Module II)

Quiz, Date: October 26, 2024, Saturday

Timing: 10:10 AM to 10:40 AM (Extra 10 minutes for the DAP students)

- Answer all the questions. The exam is for 10 marks.
- **Answer the questions ONLY in the spaces provided after the questions. Answers written anywhere else will not be graded.** You may take additional sheets for rough work.
- Try not to use any result not done in the class. However, if you use any such result, clearly **state and prove** it.
- Write your name, roll No., and program name clearly in the appropriate place.
- **For prove or disprove type questions, clearly state whether it's a prove or a disprove and then provide the arguments.**
- Using a calculator, mobile device, or smartwatch is strictly prohibited.

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Name and Program: ______________________________

Roll number: ______________________________

1. (5 points) Find a reflexive g-inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Answer: We first convert A to its REF by performing necessary row operations on $[A \mid I_3]$.

$$\left(\begin{array}{cccc|ccc} 1 & 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 6 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2(\frac{1}{6})} \left(\begin{array}{cccc|ccc} 1 & 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_{12}(-3)} \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3(\frac{1}{6})} \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{6} \end{array}\right).$$

The matrix we obtain after performing the row operations is $[F \mid E]$, where these notations are introduced in class. From the reduced form, we can say that $\rho(A) = 3$ as F has 3 non-null rows. Further, note that $p_1 = 1$, $p_2 = 2$, and $p_3 = 4$. Thus, a g inverse of A is the following:

$$G = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}.$$

Note that G has only 3 non-null rows and as $\rho(G) \geq \rho(A) = 3$, it must be that $\rho(G) = 3$. Thus, G is a reflexive g inverse of A.

2. (2 points) Prove or disprove: Suppose for a matrix A, A^T is a g inverse of A. Then A^T is the MP g inverse of A.

   Answer: (*Prove*) Note that $\rho(A) = \rho(A^T)$ implying A^T is a reflexive g inverse. Further, as A^TA and AA^T are symmetric, A^T is also a minimum norm and LS g inverse. Thus, A^T us the MP g inverse of A.

3. (3 points) Read the following in bold carefully. It argues that if $Ax = b$ is consistent and u is a solution (i.e., $Au = b$), then a g-inverse, G, of A exists, such that $u = Gb$. If there are flaws in the arguments, find those out or conclude that the result is correct.

Suppose $\boldsymbol{Ax = b}$ is consistent and $\boldsymbol{u}$ is a solution of this system. Let $\boldsymbol{G}$ be a $\boldsymbol{g}$-inverse of $\boldsymbol{A}$. Then, the set of all solutions to $\boldsymbol{Ax = b}$ is $\boldsymbol{\{Gb + (I - GA)z \mid z \in \mathbb{R}^n\}}$. Hence, $\boldsymbol{u = Gb + (I - GA)z}$ for some $\boldsymbol{z \in \mathbb{R}^n}$. Note that we can get a matrix $\boldsymbol{U_{n \times m}}$ such that $\boldsymbol{z = Ub}$. Therefore, $\boldsymbol{u = (G + (I - GA)U)b}$. Now recall that the set of $\boldsymbol{g}$-inverses of $\boldsymbol{A}$ is of the form

$$\{G + (I - GA)X + Y(I - AG) \mid X \text{ and } Y \text{ are arbitrary}\}.$$

Hence, $\boldsymbol{E = G + (I - GA)U}$ is a $\boldsymbol{g}$-inverse of $\boldsymbol{A}$, and $\boldsymbol{u = Eb}$.

Answer: The argument "Note that we can get a matrix $\boldsymbol{U_{n \times m}}$ such that $\boldsymbol{z = Ub}$." is not correct as when $b = 0$, we would not get such a matrix U.

Try to show that if $b \neq 0$, we will always get a matrix $U_{n \times m}$ such that $z = Ub$. Thus, the result holds when $b \neq 0$.