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- For a matrix A, (P, Q) is an RF of A. Then $\mathcal{C}(A) = \mathcal{C}(P)$, $\mathcal{R}(A) = \mathcal{R}(Q)$, and $\mathcal{N}(A) = \mathcal{N}(Q)$. Further, P has a left inverse and Q has a right inverse.

- In general, RF is not unique, but if (P, Q_1) and (P, Q_2) are both RF of A, then $Q_1 = Q_2$

$$A = PQ_1 \quad \text{also} \quad A = PQ_2$$

$$PQ_1 = PQ_2$$

$$\Rightarrow P_L^{-1} P Q_1 = P_L^{-1} P Q_2$$

$$\Rightarrow Q_1 = Q_2$$

Similarly, if (P_1, Q) and (P_2, Q) are two RF of A. Then $P_1 = P_2$

Theorem: Suppose $A_{m\times n}$ is a matrix with $\rho(A) = r$. Then we can get two n.s. matrices P and Q s.t.

$$A = P_{m\times m} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q_{n\times n}$$

<u>Proof.</u> Let (P_1, Q_1) be an RF of A.

$\mathcal{C}(P_1) \subseteq \mathbb{R}^m$ and the columns are L.I. So, we can extend the columns to a basis of $\mathbb{R}^m$. Lets denote the extension by

$P_{m\times m} = [P_1 \ P_2]$. Note that P is n-s.

Similarly, we can get a matrix $Q_{n\times n} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$

s.t. Q is n-s.

$$A = P_1 Q_1 = [P_1 \ 0] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

$$= [P_1 \ P_2] \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

$$= P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q.$$

Theorem Suppose for a matrix A, $A^2 = A$.

Then $\rho(A) = tr(A)$.

Proof. Take an RF of A, say (P, Q).

$$A^2 = A$$

$$\Rightarrow PQPQ = PQ$$

$$\Rightarrow QP = I_r$$

Further, $tr(A) = tr(PQ) = tr(QP) = r$

This mean $\rho(A) = tr(A)$.

Theorem For two matrices A and B of the

same order, $\rho(A+B) \le \rho(A) + \rho(B)$.

The equality holds if and only if

$$\mathcal{C}(A) \cap \mathcal{C}(B) = \{0\} \quad \text{and} \quad \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$$

Proof: We first show that

$$\mathcal{C}(A+B) \subseteq \mathcal{C}(A) + \mathcal{C}(B).$$

Take $x \in \mathcal{C}(A+B) \Rightarrow x = (A+B)y$ for some y

$$\Rightarrow x = \underset{\in \mathcal{C}(A)}{Ay} + \underset{\in \mathcal{C}(B)}{By}$$

$$\Rightarrow x \in \mathcal{C}(A) + \mathcal{C}(B)$$

$\mathcal{C}(A) = \{Ax \mid x \in \mathbb{R}^{n}\}$

So, $\mathcal{C}(A+B) \subseteq \mathcal{C}(A) + \mathcal{C}(B)$.

$$d(\mathcal{C}(A+B)) \leq d(\underbrace{\mathcal{C}(A)}_{S} + \underbrace{\mathcal{C}(B)}_{T}) \leq d(\mathcal{C}(A)) + d(\mathcal{C}(B))$$

$$\Rightarrow \rho(A+B) \leq \rho(A) + \rho(B) \quad \cdots\cdots (*)$$

Modular law

$d(S+T) = d(S) + d(T) - d(S \cap T)$.

Suppose the equality holds

$$\rho(A+B) = \rho(A) + \rho(B)$$

Then, from $(*)$

$$d(\mathcal{C}(A) + \mathcal{C}(B)) = d(\mathcal{C}(A)) + d(\mathcal{C}(B))$$

This implies, by modular law, $d(\mathcal{C}(A) \cap \mathcal{C}(B)) = 0$

$$\Rightarrow \mathcal{C}(A) \cap \mathcal{C}(B) = \{0\}$$

All the arguments above hold for $\mathcal{R}(A)$ and $\mathcal{R}(B)$ implying $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$.

Now, given $\mathcal{C}(A) \cap \mathcal{C}(B) = \{0\}$ and $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$

we will show that $\rho(A+B) = \rho(A) + \rho(B)$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\mathcal{C}(A) \cap \mathcal{C}(B) = \{0\}$ but $\mathcal{R}(A) = \mathcal{R}(B)$

$$A + B = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$$

$\rho(A+B) = 1$ but $\rho(A) + \rho(B) = 2$

Take an RF of A, say (P_1, Q_1) and an RF of B, say (P_2, Q_2).

$$A = P_1 Q_1 \qquad B = P_2 Q_2$$

$$A + B = P_1 Q_1 + P_2 Q_2$$

$$= [P_1 \quad P_2] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = PQ$$

Note that as $\mathcal{C}(A) = \mathcal{C}(P_1)$, $\mathcal{C}(B) = \mathcal{C}(P_2)$ and $\mathcal{C}(A) \cap \mathcal{C}(B) = \{0\}$,

we have $\mathcal{C}(P_1) \cap \mathcal{C}(P_2) = \{0\}$.

This together with the fact that columns of P_1 and columns of P_2 are L.I implies columns of $P = [P_1 \; P_2]$ are also L.I.

So, P has full column rank.

Using similar arguments and the fact

that $R(A) \cap R(B) = \{0\}$, we have rows of $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ are LI implying Q has full row rank.

So, $A+B = PQ$ where P has full column rank and Q has full row rank $\Rightarrow$ (P, Q) is an RF of $A+B$.

Hence, $\rho(A+B) = \rho(P) =$ numer of columns of $P =$ number of columns of P_1 + number of columns of P_2

$= \rho(P_1) + \rho(P_2) = \rho(A) + \rho(B)$