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Computation of a Rank Factorization of a matrix.

$E_a \ldots\ldots E_2 E_1 A_{m\times n} = F_{m\times n}$ (F is in REF)

$\Longleftrightarrow \underline{E_{m\times m} A_{m\times n} = F_{m\times n}}$

Defn A matrix $F_{m\times n}$ with $\rho(F) = r$ is in Reduced Echelon Form (REF) if

(i) the first r rows of F are non-null and the remaining rows are null.

(ii) the first non-null element in i-th row is at p_i th position where

$$p_1 < p_2 < \ldots < p_r$$

(iii) $f_{\cdot p_i} = e_i^m \quad \forall i = 1(1)r$

For a matrix A, $a_{i\cdot}$ denotes the i-th row and $a_{\cdot j}$ denotes the j-th column of A

Example

$$F = \begin{bmatrix} 0 & 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$EA = F$$

- A row basis of A is the first r rows of F

$$F = \begin{bmatrix} D \\ 0 \end{bmatrix}$$

- A column basis of A is the collection of p_1th, p_2th, ... p_rth columns of A.

$$EA = F$$

$$\Leftrightarrow \quad E a_{\cdot j} = f_{\cdot j} \qquad \forall j = 1(1)n$$

$$\Rightarrow \quad E a_{\cdot p_i} = e_i^m \qquad \forall i = 1(1)r$$

$$C_{m \times r} = [a_{\cdot p_1} \ a_{\cdot p_2} \ \cdots \ a_{\cdot p_r}]$$

$$EC = [e_1^m \ e_2^m \ \cdots \ e_r^m]$$

$$E^{-1} = [C \ T]$$ for some other matrix T.

$$EA = F$$

$$\Leftrightarrow \quad A = E^{-1} F$$

$$\Leftrightarrow \quad A = [C \ T] \begin{bmatrix} D \\ O \end{bmatrix}$$

$$= CD$$

E is n-s.

$Ex = e_1$

the first colum of E^{-1} is $y \neq x$

$Ex = e_1$

$x = E^{-1} e_1$

as the rows of D give us a row basis A, (C, D) is an RF of A.

$$\boxed{EA = F}$$

$EA = F$ where F is in REF

then F is unique.

$$\bar{E} A = F_1$$

$$A = CD = CD_1$$
$$D = D_1$$

- If a matrix has a left (or right) inverse then there exist a matrix G s.t Gb is a particular solution to $Ax = b$ whenever $b \in \mathcal{C}(A)$.

$$A(Gb) = b \quad \forall b \in \mathcal{C}(A)$$
$$\Leftrightarrow \quad A(GAz) = Az \quad \forall z \in \mathbb{R}^n$$
$$\Leftrightarrow \quad AGAz = Az \quad \forall z \in \mathbb{R}^n$$
$$\Leftrightarrow \quad (AGA - A)z = 0 \quad \forall z \in \mathbb{R}^n$$
$$\mathcal{N}(AGA - A) = \mathbb{R}^n \Leftrightarrow \mathcal{R}(AGA - A) = \{0\}$$
$$\Leftrightarrow AGA = A$$

For a matrix A, $\mathcal{N}(A) = \mathcal{R}^{\perp}(A)$

$d(\mathcal{N}(AGA - A)) = n$

Suppose (P, Q) is an RF of A.

Then $PQGPQ = PQ$

$$\Leftrightarrow \quad QGP = I_r \quad \text{when } r = \rho(A)$$

$G = Q^{-1} P^{-1}$

$\geq n$

$n - \rho(AGA - A) = n$

$\rho(AGA - A) = 0$

$AGA = A$

If we set $G = Q_R P_L$

$$Q Q_R^{-1} P_L^{-1} P = I_r$$

Defn For a matrix A, a matrix G is called a g-inverse of A if Gb is a particular solution to $Ax = b$ whenever $b \in \mathcal{C}(A)$.

Take $G = Q_R^{-1} P_L^{-1}$. Consider $b \in \mathcal{C}(A)$

$$A(Gb)$$
$$\Rightarrow A Q_R^{-1} P_L^{-1} b$$
$$\Rightarrow P P_L^{-1} b$$
$$\Rightarrow P P_L^{-1} A z$$
$$\Rightarrow P P_L^{-1} P Q z$$
$$\Rightarrow P Q z \Rightarrow A z = b$$

$b \in \mathcal{C}(A)$

$\Rightarrow b = Az$ for some z

Theorem Suppose for a matrix $A_{m \times n}$, G is another matrix of order $n \times m$. Then the following statements are equivalent.

(i) Gb is a particular solution to $Ax = b$ whenever $b \in \mathcal{C}(A)$

(ii) $AGA = A$

(iii) AG is idempotent and $\rho(AG) = \rho(A)$

(iv) GA is idempotent and $\rho(GA) = \rho(A)$.

Proof:

(i) $\Rightarrow$ (ii) we have already proved it.

(ii) $\Rightarrow$ (iii) $AGA = A \Rightarrow AGAG = AG$

So, AG is idempotent

Also $AGA = A \Rightarrow \rho(A) \leq \rho(AG)$.

and we know $\rho(AG) \leq \rho(A)$

Hence $\rho(AG) = \rho(A)$.

(iii) $\Rightarrow$ (iv) $AGAG = AG$

$GAGAG = GAG$ ... *

Now $\rho(AG) = \rho(A)$

$\Leftrightarrow \mathcal{C}(A) = \mathcal{C}(AG)$

$\Leftrightarrow AGB = A$ for som

in (*), if we post multiply by B, we have

$GAGAGB = GAGB$

$GAGA = GA$ [as $AGB = A$]

[Recall that the

Can be directly
Claimed using
Rank cancellation law]

So, GA is idempotent.

Further $AGAG = AG$

$\Rightarrow AGAGB = AGB$ [as $AGB = A$]

$\Rightarrow AGA = A$

So, $\rho(A) \leq \rho(GA)$. Also, we know $\rho(A) \geq \rho(GA)$ implying $\rho(A) = \rho(GA)$.

(iv) $\Rightarrow$ (i)

As $\rho(GA) = \rho(A)$

$\Rightarrow R(GA) = R(A)$

$\Rightarrow A = CGA$

so, $GAGA = GA$

$\Rightarrow CGAGA = CGA$

$\Rightarrow AGA = A$

Thus, whenever $b \in \mathcal{C}(A)$

$A(Gb) = AGAz$ for some z

$= Az = b$ [as $AGA = A$]