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- $l^T\beta$ is estimable, then $l^T G y$ where G is a LS g-inverse of X is the BLUE of $l^T\beta$.

$$\mathrm{Var}(l^T G y) = \sigma^2 l^T G G^T l$$
$$= \sigma^2 l^T (X^T X)^- l \quad \text{for all g-inverse of } (X^T X)$$

$$X^T X G G^T X^T X = \underbrace{X^T G^T X^T}_{X^T} \underbrace{X G X}_{X} = X^T X = X^T X (X^T X)^- X^T X$$

for all g-inverse of $X^T X$

$$X^T X G G^T X^T X = X^T X (X^T X)^- X^T X$$
$$\Rightarrow X G G^T X^T X = X (X^T X)^- X^T X$$
$$\Rightarrow \underline{X G G^T X^T} = X (X^T X)^- X^T$$

$W X^T X = X$ for some W

$X^T X Y = X^T$ for some Y

$$l^T G G^T l = u^T \underline{X G G^T X^T} u$$
$$= u^T X (X^T X)^- X^T u$$
$$= l^T (X^T X)^- l$$

as $l^T \in R(X)$ $\Rightarrow l^T = u^T X$

- $l^T G y$ is the BLUE of $l^T\beta$ given $l^T\beta$ is estimable.

Suppose $c^T y$ and $d^T y$ are two BLUEs of $l^T\beta$

$\alpha c^T y + (1-\alpha) d^T y$ is an unbiased estimator of $\ell^T\beta$ $\quad$ $E(a^T y) = \ell^T\beta$

$$\text{Var}\left(\left(\alpha c^T + (1-\alpha) d^T\right) y\right)$$

$$V(a^T y) = \sigma^2 a^T a$$

is a quadratic in α and is minimum when $\alpha = 0$ and $\alpha = 1$. This means in the expression, the coefficient of α^2 is 0. Conclude that $c = d$.

$$V(c^T y) = \boxed{\sigma^2 \sum c_i^2}$$

$$E(c^T y) = \ell^T\beta \Rightarrow c^T X\beta = \ell^T\beta$$

$$\Rightarrow \boxed{c^T X = \ell^T}$$

- If $\rho(X) = p$ i.e, $R(X) = \mathbb{R}^p$ the all linear functions are estimable.

<u>Theorem</u> Let $\rho(X) = p$ and let $\hat\beta$ be the BLUE of the column vector $\beta = \begin{pmatrix}\beta_1\\ \vdots \\ \beta_p\end{pmatrix}$.

Gauss-Markov Theorem

Then $\hat\beta = (X^TX)^{-1}X^Ty$ and $\text{Var}(\hat\beta) = \sigma^2 (X^TX)^{-1}$.

Further, BLUE of $\ell^T\beta$ is $\ell^T\hat\beta$ and $\text{var}(\ell^T\hat\beta)$ is $\sigma^2 \ell^T (X^TX)^{-1}\ell$

<u>Proof.</u>

$$\beta = \begin{pmatrix}\beta_1\\ \beta_2 \\ \vdots \\ \beta_p\end{pmatrix} = \begin{pmatrix} e_1^T\beta \\ e_2^T\beta \\ \vdots \\ e_p^T\beta\end{pmatrix}$$

BLUE of β is $= \begin{pmatrix} e_1^T G y \\ e_2^T G y \\ \\ e_p'^T G y \end{pmatrix} = I_p G y = \underline{G y}$

$X G X = X \Rightarrow \underline{X^T G^T X^T} = X^T$

$\Rightarrow X^T X G = X^T$

$\Rightarrow G = (X^T X)^{-1} X^T$

BLUE of β is $\hat{\beta} = G y = (X^T X)^{-1} X^T y$

$\mathrm{Var}\left((X^T X)^{-1} X^T y\right) = \sigma^2 (X^T X)^{-1} X^T X (X^T X)^{-1}$

$= \sigma^2 (X^T X)^{-1}$

For $l^T \beta$, the BLUE is $l^T G y = l^T \underbrace{(X^T X)^{-1} X^T y}$

$= l^T \hat{\beta}$

$\mathrm{Var}(l^T \hat{\beta}) = \mathrm{Var}\left(l^T (X^T X)^{-1} X^T y\right)$

$= \sigma^2 l^T (X^T X)^{-1} X^T X (X^T X)^{-1} l$

$= \sigma^2 l^T (X^T X)^{-1} l$

$l^T G y$

This theorem is known as Gauss-Markov theorem

<u>Method of leas sqaure</u>

$y = X\beta + \epsilon$

$\epsilon^T \epsilon = (y - X\beta)^T (y - X\beta) = \|y - X\beta\|^2$

$\|y - P_x y\|^2 \le \|y - X\beta\|^2 \quad \forall \beta \in \mathbb{R}^p$

$\|y - X\beta\|^2$ is minimized when $X\beta$ is the orthogonal projection of y into $C(X)$, i.e $X\beta = P_x y$

$\Rightarrow X\beta = X(X^T X)^- X^T y$

$\Rightarrow X\beta = X(X^TX)^- X^T y$

we take $(X^TX)^- X^T y$ as the least square estimator of β, denoted by β_{LS}

$$\beta_{LS} = (X^TX)^- X^T y$$

<u>Alternative way</u>

$$\epsilon^T\epsilon = (y - X\beta)^T(y - X\beta)$$

$$= y^Ty - \underline{y^TX\beta} - \overline{\beta^TX^Ty} + \beta^TX^TX\beta$$

$$= y^Ty - 2y^TX\beta + \beta^TX^TX\beta$$

$$\frac{\partial \epsilon^T\epsilon}{\partial \beta} = 0 - 2(y^TX)^T + 2(X^TX)\beta$$

$$\frac{\partial}{\partial y} a^Ty = a$$

if we equate it to 0.

$$(X^TX)\beta = X^Ty$$

$X^Ty \in \mathcal{C}(X^T) = \mathcal{C}(X^TX)$, so $(X^TX)\beta = X^Ty$ has a solution,

$$\hat{\beta} = (X^TX)^- X^Ty$$