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- Consider the standard linear model $y = X\beta + \epsilon$ and $L\beta = z$

$$\tilde{\beta} = \beta^{LS} - (X^T X)^- L^T \left(L (X^T X)^- L^T\right)^- (L\beta^{LS} - z)$$

Lemma: $\underline{\rho(L) = \rho(L (X^T X)^- L^T)}$

Assumptions: we can take $L_{m \times p}$ s.t. $\rho(L) = m$.

Also, suppose $\rho(X) = p$.

$$\rho\left(L (X^T X)^{-1} L^T\right) = m$$

$$\tilde{\beta} = \beta^{LS} - (X^T X)^{-1} L^T \left(L (X^T X)^{-1} L^T\right)^{-1} (L\beta^{LS} - z)$$

if $m = 1$

$$\tilde{\beta} = \beta^{LS} - \frac{(X^T X)^{-1} L^T (L\beta^{LS} - z)}{(L (X^T X)^{-1} L^T)}$$

- $y_i = \theta_i + \epsilon_i \quad \forall i = 1(1)4$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$$

$$L = (1, 1, 1, 1)$$

$$L\theta = 0 \Rightarrow (1, 1, 1, 1) \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} = 0$$

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\theta^{LS} = (X^T X)^{-1} X^T y = y$$

$$\tilde{\theta} = y - \begin{pmatrix} \bar{y} \\ \bar{y} \\ \bar{y} \\ \bar{y} \end{pmatrix}$$

- $y_{ij} = \mu + \theta_i + \epsilon_{ij} \quad \forall i = 1(1)k \quad j = 1, \ldots, n_i$

the restriction is $\theta_k - \theta_l = 0 \quad \forall k \neq l$

$$\min_{\beta} (y - X\beta)^T (y - X\beta) \quad \text{subject to } L\beta = z$$

$$(y - X\beta)^T (y - X\beta)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \mu - \theta_i)^2$$

$$\mu + \theta_i \overset{LS}{=} \bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$$

$\sum_{i=1}^{n} (y_i - \theta)^2$

is minimized when $\theta = \bar{y}$

$$\theta_i = \theta_j \quad \forall i \neq j$$

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \mu - \theta)^2$$

$$\mu + \theta \overset{LS}{=} \bar{y} = \frac{1}{k} \sum_{i=1}^{k} \left(\frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} \right)$$

- $\rho(X) = n$, $p = n$ and $\rho(X) = n$

$$\beta^{LS} = (X^T X)^{-1} X^T y = X^{-1} y$$

$$RSS = (y - X\beta^{LS})^T (y - X\beta^{LS}) = 0.$$

- Fisher - Cochran Theorem (Matrix Theoretic Version)

Theorem: Let $A_1, \dots, A_k$ be $n \times n$ s.t $\sum_{i=1}^{k} A_i = I_n$. Then the following conditions are equivalent.

(i) $\sum_{i=1}^{k} \rho(A_i) = n$

(ii) $A_i^2 = A_i \quad \forall i = 1 (1) k$

(iii) $A_i A_j = 0 \quad \forall i \neq j$

Proof: See any Linear Algebra Book.

$S + T = \mathbb{R}^n$

projection into S along T

P is a projection matrix

Multivariate Normal distribution

- Let $u_1, \dots, u_n$ be standard Normal variables which are independent

$u_i \sim N(0,1)$ and u_is are independent.

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \sim N(0, I_n)$$

Consider $X_{m \times n}$ and a vector $\mu_{m \times 1}$

$$y = Xu + \mu$$

Then y follows a multivariate Normal distribution

$$E(y) = \mu$$

$$Var(y) = XX^T = \Sigma$$

$\mathrm{Var}(Y) = XX^T = \Sigma$

$$f_Y(w) = \frac{1}{(\sqrt{2\pi})^n \sqrt{|\Sigma|}} \, e^{-\frac{1}{2}\left[(w-\mu)^T \Sigma^{-1} (w-\mu)\right]}$$

Well exist iff X is of full row rank.

- For a random vector y, characteristic function

$$E\left(e^{it^T y}\right) \qquad t \in \mathbb{R}^n.$$