5.6: Isomorphisms

Outcomes

A. Determine if a linear transformation is an isomorphism.
B. Determine if two subspaces of $\mathbb{R}^n$ are isomorphic.

Recall the definition of a linear transformation. Let V and W be two subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A mapping $T: V \to W$ is called a **linear transformation** or **linear map** if it preserves the algebraic operations of addition and scalar multiplication. Specifically, if a, b are scalars and $\vec{x}, \vec{y}$ are vectors,

$$T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$$

Consider the following important definition.

Definition 5.6.1: Isomorphism

A linear map T is called an **isomorphism** if the following two conditions are satisfied.

- T is one to one. That is, if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.
- T is onto. That is, if $\vec{w} \in W$, there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

Two such subspaces which have an isomorphism as described above are said to be **isomorphic.**

Consider the following example of an isomorphism.

Example 5.6.1: Isomorphism

Let $T: \mathbb{R}^2 \mapsto \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$

Show that T is an isomorphism.

Solution

To prove that T is an isomorphism we must show

1. T is a linear transformation;
2. T is one to one;
3. T is onto.

We proceed as follows.

1. T is a linear transformation:

   Let k, p be scalars.

$$\begin{aligned}
T\left(k\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + p\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix} + \begin{bmatrix} px_2 \\ py_2 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} kx_1 + px_2 \\ ky_1 + py_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} (kx_1 + px_2) + (ky_1 + py_2) \\ (kx_1 + px_2) - (ky_1 + py_2) \end{bmatrix} \\
&= \begin{bmatrix} (kx_1 + ky_1) + (px_2 + py_2) \\ (kx_1 - ky_1) + (px_2 - py_2) \end{bmatrix} \\
&= \begin{bmatrix} kx_1 + ky_1 \\ kx_1 - ky_1 \end{bmatrix} + \begin{bmatrix} px_2 + py_2 \\ px_2 - py_2 \end{bmatrix} \\
&= k\begin{bmatrix} x_1 + y_1 \\ x_1 - y_1 \end{bmatrix} + p\begin{bmatrix} x_2 + y_2 \\ x_2 - y_2 \end{bmatrix} \\
&= kT\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + pT\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)
\end{aligned}$$

Therefore T is linear.

2. T is one to one:

   We need to show that if $T(\vec{x}) = \vec{0}$ for a vector $\vec{x} \in \mathbb{R}^2$, then it follows that $\vec{x} = \vec{0}$. Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

   $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   This provides a system of equations given by

   $$\begin{aligned} x+y &= 0 \\ x-y &= 0 \end{aligned}$$

   You can verify that the solution to this system if $x = y = 0$. Therefore

   $$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   and T is one to one.

3. T is onto:

   Let a, b be scalars. We want to check if there is always a solution to

   $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

   This can be represented as the system of equations

   $$\begin{aligned} x+y &= a \\ x-y &= b \end{aligned}$$

   Setting up the augmented matrix and row reducing gives

   $$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & \frac{a+b}{2} \\ 0 & 1 & \frac{a-b}{2} \end{array}\right]$$

   This has a solution for all a, b and therefore T is onto.

Therefore T is an isomorphism.

An important property of isomorphisms is that its inverse is also an isomorphism.

Proposition 5.6.1: Inverse of an Isomorphism

Let $T : V \to W$ be an isomorphism and V, W be subspaces of $\mathbb{R}^n$. Then $T^{-1} : W \to V$ is also an isomorphism.

Proof

Let T be an isomorphism. Since T is onto, a typical vector in W is of the form $T(\vec{v})$ where $\vec{v} \in V$. Consider then for a, b scalars,

$$T^{-1}\left(aT(\vec{v}_1) + bT(\vec{v}_2)\right)$$

where $\vec{v}_1, \vec{v}_2 \in V$. Is this equal to

$$aT^{-1}\left(T(\vec{v}_1)\right) + bT^{-1}\left(T(\vec{v}_2)\right) = a\vec{v}_1 + b\vec{v}_2?$$

Since T is one to one, this will be so if

$$T\left(a\vec{v}_1 + b\vec{v}_2\right) = T\left(T^{-1}\left(aT(\vec{v}_1) + bT(\vec{v}_2)\right)\right) = aT(\vec{v}_1) + bT(\vec{v}_2).$$

However, the above statement is just the condition that T is a linear map. Thus T^{-1} is indeed a linear map. If $\vec{v} \in V$ is given, then $\vec{v} = T^{-1}\left(T(\vec{v})\right)$ and so T^{-1} is onto. If $T^{-1}(\vec{v}) = 0$, then

$$\vec{v} = T\left(T^{-1}(\vec{v})\right) = T(\vec{0}) = \vec{0}$$

and so T^{-1} is one to one.

Another important result is that the composition of multiple isomorphisms is also an isomorphism.

Proposition 5.6.2: Composition of Isomorphisms

Let $T : V \to W$ and $S : W \to Z$ be isomorphisms where V, W, Z are subspaces of $\mathbb{R}^n$. Then $S \circ T$ defined by $(S \circ T)(\vec{v}) = S(T(\vec{v}))$ is also an isomorphism.

Proof

Suppose $T : V \to W$ and $S : W \to Z$ are isomorphisms. Why is $S \circ T$ a linear map? For a, b scalars,

$$\begin{aligned} S \circ T(a\vec{v}_1 + b(\vec{v}_2)) &= S(T(a\vec{v}_1 + b\vec{v}_2)) = S(aT\vec{v}_1 + bT\vec{v}_2) \\ &= aS(T\vec{v}_1) + bS(T\vec{v}_2) = a(S \circ T)(\vec{v}_1) + b(S \circ T)(\vec{v}_2) \end{aligned}$$

Hence $S \circ T$ is a linear map. If $(S \circ T)(\vec{v}) = 0$, then $S(T(\vec{v})) = 0$ and it follows that $T(\vec{v}) = \vec{0}$ and hence by this lemma again, $\vec{v} = \vec{0}$. Thus $S \circ T$ is one to one. It remains to verify that it is onto. Let $\vec{z} \in Z$. Then since S is onto, there exists $\vec{w} \in W$ such that $S(\vec{w}) = \vec{z}$. Also, since T is onto, there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. It follows that $S(T(\vec{v})) = \vec{z}$ and so $S \circ T$ is also onto.

Consider two subspaces V and W, and suppose there exists an isomorphism mapping one to the other. In this way the two subspaces are related, which we can write as $V \sim W$. Then the previous two propositions together claim that $\sim$ is an equivalence relation. That is: $\sim$ satisfies the following conditions:

- $V \sim V$
- If $V \sim W$, it follows that $W \sim V$
- If $V \sim W$ and $W \sim Z$, then $V \sim Z$

We leave the verification of these conditions as an exercise.

Consider the following example.

Example 5.6.2: Matrix Isomorphism

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\vec{x}) = A(\vec{x})$ where A is an invertible $n \times n$ matrix. Then T is an isomorphism.

Solution

The reason for this is that, since A is invertible, the only vector it sends to $\vec{0}$ is the zero vector. Hence if $A(\vec{x}) = A(\vec{y})$, then $A(\vec{x} - \vec{y}) = \vec{0}$ and so $\vec{x} = \vec{y}$. It is onto because if

$$\vec{y} \in \mathbb{R}^n, A\left(A^{-1}(\vec{y})\right) = \left(AA^{-1}\right)(\vec{y}) = \vec{y}.$$

In fact, all isomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be expressed as $T(\vec{x}) = A(\vec{x})$ where A is an invertible $n \times n$ matrix. One simply considers the matrix whose i^{th} column is $T\vec{e}_i$.

Recall that a basis of a subspace V is a set of linearly independent vectors which span V. The following fundamental lemma describes the relation between bases and isomorphisms.

Lemma 5.6.1: Mapping Bases

Let $T : V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. If T is one to one, then it has the property that if $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent, so is $\{T(\vec{u}_1), \cdots, T(\vec{u}_k)\}$.

More generally, T is an isomorphism if and only if whenever $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W.

Proof

First suppose that T is a linear transformation and is one to one and $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent. It is required to show that $\{T(\vec{u}_1), \cdots, T(\vec{u}_k)\}$ is also linearly independent. Suppose then that

$$\sum_{i=1}^{k} c_i T(\vec{u}_i) = \vec{0}$$

Then, since T is linear,

$$T\left(\sum_{i=1}^{n} c_i \vec{u}_i\right) = \vec{0}$$

Since T is one to one, it follows that

$$\sum_{i=1}^{n} c_i \vec{u}_i = 0$$

Now the fact that $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent implies that each $c_i = 0$. Hence $\{T(\vec{u}_1), \cdots, T(\vec{u}_n)\}$ is linearly independent.

Now suppose that T is an isomorphism and $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. It was just shown that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. It remains to verify that $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$. If $\vec{w} \in W$, then since T is onto there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, it follows that there exists scalars $\{c_i\}_{i=1}^n$ such that

$$\sum_{i=1}^{n} c_i \vec{v}_i = \vec{v}.$$

Hence,

$$\vec{w} = T(\vec{v}) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i T(\vec{v}_i)$$

It follows that $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$ showing that this set of vectors is a basis for W.

Next suppose that T is a linear transformation which takes a basis to a basis. This means that if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W. Then if $w \in W$, there exist scalars c_i such that $w = \sum_{i=1}^n c_i T(\vec{v}_i) = T\left(\sum_{i=1}^n c_i \vec{v}_i\right)$ showing that T is onto. If $T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \vec{0}$ then $\sum_{i=1}^n c_i T(\vec{v}_i) = \vec{0}$ and since the vectors $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ are linearly independent, it follows that each $c_i = 0$. Since $\sum_{i=1}^n c_i \vec{v}_i$ is a typical vector in V, this has shown that if $T(\vec{v}) = \vec{0}$ then $\vec{v} = \vec{0}$ and so T is also one to one. Thus T is an isomorphism.

The following theorem illustrates a very useful idea for defining an isomorphism. Basically, if you know what it does to a basis, then you can construct the isomorphism.

Theorem 5.6.1: Isomorphic Subspaces

Suppose V and W are two subspaces of $\mathbb{R}^n$. Then the two subspaces are isomorphic if and only if they have the same dimension. In the case that the two subspaces have the same dimension, then for a linear map $T : V \to W$, the following are equivalent.

1. T is one to one.
2. T is onto.
3. T is an isomorphism.

Proof

Suppose first that these two subspaces have the same dimension. Let a basis for V be $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and let a basis for W be $\{\vec{w}_1, \cdots, \vec{w}_n\}$. Now define T as follows.

$$T(\vec{v}_i) = \vec{w}_i$$

for $\sum_{i=1}^n c_i \vec{v}_i$ an arbitrary vector of V,

$$T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i T\vec{v}_i = \sum_{i=1}^{n} c_i \vec{w}_i.$$

It is necessary to verify that this is well defined. Suppose then that

$$\sum_{i=1}^{n} c_i \vec{v}_i = \sum_{i=1}^{n} \hat{c}_i \vec{v}_i$$

Then

$$\sum_{i=1}^{n}\left(c_i - \hat{c}_i\right)\vec{v}_i = \vec{0}$$

and since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, $c_i = \hat{c}_i$ for each i. Hence

$$\sum_{i=1}^{n} c_i \vec{w}_i = \sum_{i=1}^{n} \hat{c}_i \vec{w}_i$$

and so the mapping is well defined. Also if a, b are scalars,

$$\begin{aligned} T\left(a\sum_{i=1}^{n} c_i\vec{v}_i + b\sum_{i=1}^{n}\hat{c}_i\vec{v}_i\right) = T\left(\sum_{i=1}^{n}\left(ac_i + b\hat{c}_i\right)\vec{v}_i\right) &= \sum_{i=1}^{n}\left(ac_i + b\hat{c}_i\right)\vec{w}_i \\ &= a\sum_{i=1}^{n} c_i\vec{w}_i + b\sum_{i=1}^{n}\hat{c}_i\vec{w}_i \\ &= aT\left(\sum_{i=1}^{n} c_i\vec{v}_i\right) + bT\left(\sum_{i=1}^{n}\hat{c}_i\vec{v}_i\right) \end{aligned}$$

Thus T is a linear transformation.

Now if

$$T\left(\sum_{i=1}^{n} c_i\vec{v}_i\right) = \sum_{i=1}^{n} c_i\vec{w}_i = \vec{0},$$

then since the $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are independent, each $c_i = 0$ and so $\sum_{i=1}^{n} c_i\vec{v}_i = \vec{0}$ also. Hence T is one to one. If $\sum_{i=1}^{n} c_i\vec{w}_i$ is a vector in W, then it equals

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = T\left(\sum_{i=1}^{n} c_i\vec{v}_i\right)$$

showing that T is also onto. Hence T is an isomorphism and so V and W are isomorphic.

Next suppose $T : V \mapsto W$ is an isomorphism, so these two subspaces are isomorphic. Then for $\{\vec{v}_1, \cdots, \vec{v}_n\}$ a basis for V, it follows that a basis for W is $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ showing that the two subspaces have the same dimension.

Now suppose the two subspaces have the same dimension. Consider the three claimed equivalences.

First consider the claim that $1.) \Rightarrow 2.)$. If T is one to one and if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, then $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. If it is not a basis, then it must fail to span W. But then there would exist $\vec{w} \notin \text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ and it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n), \vec{w}\}$ would be linearly independent which is impossible because there exists a basis for W of n vectors.

Hence $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$ and so $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis. If $\vec{w} \in W$, there exist scalars c_i such that

$$\vec{w} = \sum_{i=1}^{n} c_i T(\vec{v}_i) = T\left(\sum_{i=1}^{n} c_i\vec{v}_i\right)$$

showing that T is onto. This shows that $1.) \Rightarrow 2.)$.

Next consider the claim that $2.) \Rightarrow 3.)$. Since $2.)$ holds, it follows that T is onto. It remains to verify that T is one to one. Since T is onto, there exists a basis of the form $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$. Then it follows that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent. Suppose

$$\sum_{i=1}^{n} c_i\vec{v}_i = \vec{0}$$

Then

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

Hence each $c_i = 0$ and so, $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. Now it follows that a typical vector in V is of the form $\sum_{i=1}^{n} c_i\vec{v}_i$. If $T\left(\sum_{i=1}^{n} c_i\vec{v}_i\right) = \vec{0}$, it follows that

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

and so, since $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$ is independent, it follows each $c_i = 0$ and hence $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$. Thus T is one to one as well as onto and so it is an isomorphism.

If T is an isomorphism, it is both one to one and onto by definition so 3.) implies both 1.) and 2.).

Note the interesting way of defining a linear transformation in the first part of the argument by describing what it does to a basis and then "extending it linearly" to the entire subspace.

Example 5.6.4: Isomorphic Subspaces

Let $V = \mathbb{R}^3$ and let W denote

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Show that V and W are isomorphic.

Solution

First observe that these subspaces are both of dimension 3 and so they are isomorphic by Theorem 5.6.1. The three vectors which span W are easily seen to be linearly independent by making them the columns of a matrix and row reducing to the reduced row-echelon form.

You can exhibit an isomorphism of these two spaces as follows.

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

and extend linearly. Recall that the matrix of this linear transformation is just the matrix having these vectors as columns. Thus the matrix of this isomorphism is

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

You should check that multiplication on the left by this matrix does reproduce the claimed effect resulting from an application by T.

Consider the following example.

Example 5.6.5: Finding the Matrix of an Isomorphism

Let $V = \mathbb{R}^3$ and let W denote

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Let $T : V \mapsto W$ be defined as follows.

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

Find the matrix of this isomorphism T.

Solution

First note that the vectors

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$$

are indeed a basis for $\mathbb{R}^3$ as can be seen by making them the columns of a matrix and using the reduced row-echelon form.

Now recall the matrix of T is a 4×3 matrix A which gives the same effect as T. Thus, from the way we multiply matrices,

$$A\begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\2 & 1 & 1\\1 & 0 & 2\\1 & 1 & 0\end{bmatrix}$$

Hence,

$$A = \begin{bmatrix}1 & 0 & 1\\2 & 1 & 1\\1 & 0 & 2\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 2 & -1\\2 & -1 & 1\\-1 & 2 & -1\end{bmatrix}$$

Note how the span of the columns of this new matrix must be the same as the span of the vectors defining W.

This idea of defining a linear transformation by what it does on a basis works for linear maps which are not necessarily isomorphisms.

✔ Example 5.6.6: Finding the Matrix of an Isomorphism

Let $V = \mathbb{R}^3$ and let W denote

$$\text{span}\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\2\end{bmatrix}\right\}$$

Let $T : V \mapsto W$ be defined as follows.

$$T\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, T\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, T\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\\2\end{bmatrix}$$

Find the matrix of this linear transformation.

Solution

Note that in this case, the three vectors which span W are not linearly independent. Nevertheless the above procedure will still work. The reasoning is the same as before. If A is this matrix, then

$$A\begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 2\end{bmatrix}$$

and so

$$A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\1 & 0 & 0\\1 & 0 & 1\end{bmatrix}$$

The columns of this last matrix are obviously not linearly independent.