Left/Right inverse

- For a matrix A, we say inverse exists iff $\exists$ a matrix B

  $AB = BA = I_n$ where $A_{n\times n}$

(2) $A_{n\times n}$ and $\exists$ a matrix B

s.t. $BA = I_n$

Proof. $\forall A_{n\times n}$ s.t. $\exists B_{n\times n}$

and $BA = I_n$

$$\rho(A) \geq \rho(BA) = \rho(I_n) = n$$

$\Rightarrow$ A is of full rank

$$\mathcal{C}(A) = \mathbb{R}^n$$

$\Rightarrow \mathcal{C}(I_n) \subseteq \mathcal{C}(A) \Leftrightarrow I_n = AC$ for some matrix C

$B = B I_n = BAC = C$

$\mathcal{C}(B) \subseteq \mathcal{C}(A)$

$\Leftrightarrow B = AC$ for some C

$\Rightarrow$ B is the Inverse of A.

Defn For a matrix $A_{n \times n}$, a matrix B is an inverse if $BA = I_n$

- $B_{n \times m} \cdot A_{m \times n} = I_n$ for some cases.

Defn (left inverse) For a matrix $A_{m\times n}$, a matrix $B_{n\times m}$ is called a left inverse of A if

$$BA = I_n$$

Question: What can we say if $AB = I_m$ where B is a left inverse of A?

$$BA = I_n$$

$$\Rightarrow \rho(A) \geq \rho(I_n) = n$$

$$\Rightarrow \rho(A) = n$$

$$AB = I_m$$

$$\Rightarrow \rho(A) \geq \rho(I_m) = m$$

$$\Rightarrow \rho(A) = m$$

$$m = n$$

Result.

$A_{m \times n}$ has a left inverse

$$\Leftrightarrow \rho(A) = n$$

$$R(A) = \mathbb{R}^n = R(I_n)$$

$$\Rightarrow R(I_n) \subseteq R(A)$$

$$\Rightarrow I_n = BA \text{ for some } B$$

$$\boxed{\begin{array}{l} R(B) \subseteq R(A) \\ \Leftrightarrow B = CA \\ \text{for some } A \end{array}}$$

<u>Theorem</u>. Suppose $A_{m \times n}$. Then the following statements are equivalent:

(i) A has a left inverse.

(ii) $\rho(A) = n$

(iii) $R(A) = \mathbb{R}^n$

(iv) $AX = AY \Rightarrow X = Y$

(v) $AX = 0 \Rightarrow X = 0$

Proof. (iv) $\Rightarrow$ (v)

$$AX = 0 = A0$$

Using (iv), $X = 0$

(v) $\Rightarrow$ (i) We will show that

Colums of A are LI.

take a linear combination of the columns of A.

$$\alpha_1 a_{\cdot 1} + \alpha_2 a_{\cdot 2} + \cdots + \alpha_n a_{\cdot n} = O_{m\times 1}$$

$$\Rightarrow A \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = O_{m\times 1}$$

i-th column of $A \equiv a_{\cdot i}$

from (v), we can say that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

Right inverse

Defn For a matrix $A_{m\times n}$, a right inverse exists if $AB = I_m$ for some $B_{n\times m}$.

Theorem

(i) $A_{m\times n}$ has a right inverse

(ii) $\rho(A) = m$

(iii) $\mathcal{C}(A) = \mathbb{R}^m$

(iv) $XA = YA \Rightarrow X = Y$

(v) $XA = 0 \Rightarrow X = 0$

Computing left inverse

$$E_k \cdots \cdots E_2 E_1 A_{m\times n} = F_{m\times n}$$

matrix in

REF

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- For a matrix $A_{m\times n}$, left inverse exists iff $\rho(A) = n$.

- $$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1$$

$\Rightarrow \quad 2x + y = 1$

In general, left inverse is not unique

- For A, B and C are left inverses.

$$[\alpha B + (1-\alpha) C] A = I_n \quad \forall \alpha \in \mathbb{R}$$

- $A_{m \times n}$ has a unique left inverse.

$\Rightarrow \rho(A) = n$

if $b \in \mathcal{C}(A)$, $Ax = b$ has a solution. Furthere as $\rho(A) = n$, the columns are linearly independent, means columns form a basis of the column space.

$$A_L^{-1} A = I_n$$

$$\begin{bmatrix} \sum_{i=1}^{m} \alpha_i^1 a_{i\cdot} \\ \sum_{i=1}^{m} \alpha_i^2 a_{i\cdot} \\ \sum_{i=1}^{m} \alpha_i^{\cdot} \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

Recall.

$\{x_1, \ldots, x_r\}$

$$x = \sum_{i=1}^{r} a_i x_i$$

in a unique way

$\left. \sum_{i=1}^{n} \alpha_i a_{i\cdot} \right]$

$\Rightarrow \{x_1, \ldots, x_r\}$ is L.I.

$\Rightarrow$ as the coefficients are unique, the rows are linearly independent.

$\Rightarrow \rho(A) = m$

So, $m = n$

$A_{m \times n}$

$$f(x) = Ax$$

$$y_{m\times 1} \in \{Ax \mid x \in \mathbb{R}^n\}$$

If y has a unique pre-image then we can get x s.t

$$Ax = y$$

The above holds if the columns of

A are L.I. $\iff \rho(A) = n$

Also, we can trace back x (for a given y) if there is left inverse of A.

existence of left inverse $\iff \rho(A) = n$

- $Ax = b$

$$S_b^A = \{x \mid Ax = b\}$$

$$S_b^A = u + N(A) \quad \text{where } Au = b$$

if A_L^{-1} exists, $A_L^{-1} b$ is a particular solution whenever $b \in \mathcal{E}(A)$.

$$Ax = b$$

$$\Rightarrow A_L^{-1} A x = A_L^{-1} b$$

$$\Rightarrow x = A_L^{-1} b.$$

if A_R^{-1} exists and ~~$Ax = b$ has a solution~~

$$A_R^{-1} b$$

$$A\underbrace{\left(A_R^{-1} b\right)}_{\text{a particular solution}} = b$$

Question: For $Ax = b$ when $b \in C(A)$

is there a matrix, say G, s.t.

Gb will give us a particular

solution to $Ax = b$?