OPTICS (PHY224)
An informal introduction to Fourier Analysis

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August 2024

1 Periodic functions

Consider a periodic function $f(t)$ with a period T. This means that the values of the function repeat themselves after an interval T. In mathematical terms,

$$f(t+T) = f(t) \tag{1}$$

Now, perhaps the simplest function we can think of that is periodic is a sine or cosine function.

$$f(t) = \cos(2\pi t/T) \tag{2}$$

It is straightforward to see that the period of this function is T. Let us now add another term $\sin(4\pi t/T)$ to $f(t)$ to get $g(t)$.

$$g(t) = \cos(2\pi t/T) + \sin(4\pi t/T) \tag{3}$$

This function is plotted in Figure 1. We can observe that this function is also periodic, with the same period T. Upon adding any number of sinusoids whose argument is of the form $2\pi nt/T$ (where n is an integer) to $f(t)$, we still obtain a function whose period is T.

Sines and cosines are ubiquitous in physics. They describe the dynamics of a harmonic oscillator, and are fundamental to how we model vibrations, musical notes and electromagnetic waves. In addition, we have a vast mathematical toolkit we can use with trigonometric functions and complex exponentials. Therefore, it is often of great utility to model the response of oscillatory systems in terms of sines and cosines or their combinations.

We start with an idea that any well-behaved periodic function with period T can be expressed as a summation over suitably weighted sines and cosines of period T and their harmonics (periods $T/2$, $T/3$ and so on). This leads us to the concept of a Fourier series.

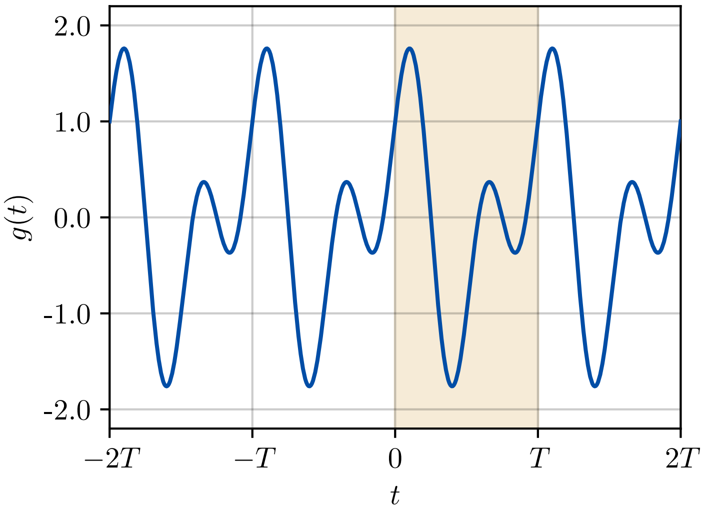

Figure 1: The periodic function $g(t)$ defined in the text. One period is highlighted.

2 Fourier Series

2.1 Definition

A series expansion of a periodic function of period T in terms of sines and cosines, as shown below, is termed a *Fourier series*.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \tag{4}$$

Another way to understand this expansion is that 1, $\cos(2\pi nt/T)$ and $\sin(2\pi nt/T)$ form a basis of the space of all such periodic functions $f(t)$ whose period is T

For cases such as the $g(t)$ defined in Equation 3, it is quite clear that a finite number of terms in the above series is sufficient (two terms, to be precise) to describe the function. For several other commonly encountered periodic functions, an infinite number of terms is required. We note that such a series expansion can be made whenever the function $f(t)$ satisfies the following condition :

$$\int_0^T |f(t)|^2 < \infty \tag{5}$$

This means the integral of $|f(t)|^2$ over one period must be finite. This makes sense practically. Consider an electromagnetic wave $E(t)$ of period T. Asserting that $\int_0^T |E(t)|^2$ is finite implies that the energy density associated with the wave is finite.

An alternate, and often much more convenient, way to express the series is in terms of complex exponentials (recollect why we often use complex exponentials in wave optics). We note that

$$\cos\left(\frac{2\pi nt}{T}\right) = \frac{1}{2}\left\{\exp\left(\frac{i2\pi nt}{T}\right) + \exp\left(\frac{-i2\pi nt}{T}\right)\right\}, \text{and}$$

$$\sin\left(\frac{2\pi nt}{T}\right) = \frac{1}{2i}\left\{\exp\left(\frac{i2\pi nt}{T}\right) - \exp\left(\frac{-i2\pi nt}{T}\right)\right\}$$

By suitably rearranging the terms of the series in Equation 4, we arrive at the following:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{i2\pi nt}{T}\right) \tag{6}$$

Hereafter, we shall refer to Equation 6 as a Fourier Series. Say, we have a periodic function $f(t)$. How do we determine the Fourier series coefficients? This is quite straightforward. Consider the following integral.

$$\frac{1}{T}\int_0^T dt f(t)\exp\left(\frac{-i2\pi mt}{T}\right) = \sum_{n=-\infty}^{\infty}\frac{1}{T}c_n\int_0^T dt\exp\left(\frac{i2\pi(n-m)t}{T}\right) \tag{7}$$

The integrand on the right hand side is zero unless $n = m$. Therefore,

$$\frac{1}{T}\int_0^T dt f(t)\exp\left(\frac{-i2\pi mt}{T}\right) = \sum_{n=-\infty}^{\infty}\frac{1}{T}c_n T\delta_{nm} \tag{8}$$

Finally an expression for c_m,

$$c_m = \frac{1}{T}\int_0^T dt f(t)\exp\left(\frac{-i2\pi mt}{T}\right) \tag{9}$$

2.2 An Example: periodic rectangular wave

Let us define a rectangle function $\mathrm{rect}(x/a)$:

$$f(t) = \begin{cases} 1, & |t| < a/2 \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

When we repeat this function with a period T, we obtain the function $g(t)$ (shown in Figure 2 for the case $T = 2a$). It is clear that for $n = 0$, $c_0 = a/T$. It is straightforward to show that the remaining Fourier series coefficients are as follows.

$$\begin{aligned} c_n &= \frac{1}{T}\int_T dt f(t)\exp\left(-\frac{2\pi int}{T}\right) \\ &= \frac{1}{2a}\int_{-a/2}^{a/2} dt\exp\left(-\frac{2\pi int}{T}\right) \\ &= \left[\frac{\exp\left(-\frac{2\pi int}{T}\right)}{2n\pi i}\right]_{-a/2}^{a/2} \\ &= \frac{\sin\left(\frac{n\pi a}{T}\right)}{n\pi} \end{aligned} \tag{11}$$

The series has an infinite number of terms. Including only 9 terms ($n = -4, -3, \cdots, 3, 4$) leads to a crude approximation for $f(t)$, which is plotted in Figure 3. The coefficients for n $= \pm1$,±2, $\pm3$and ±4 are $\mp\frac{2i}{\pi}$, 0, $\mp\frac{2i}{3\pi}$, and 0, respectively. A much better approximation is obtained by including 41 terms ($n = -20, -19, \cdots, 19, 20$). This is also shown in Figure 3 It is interesting to observe patterns in the Fourier series coefficients, c_n. In particular, we note that $c_n^* = c_{-n}$, which is true for all Fourier series of real-valued periodic functions (**try and prove this!**).

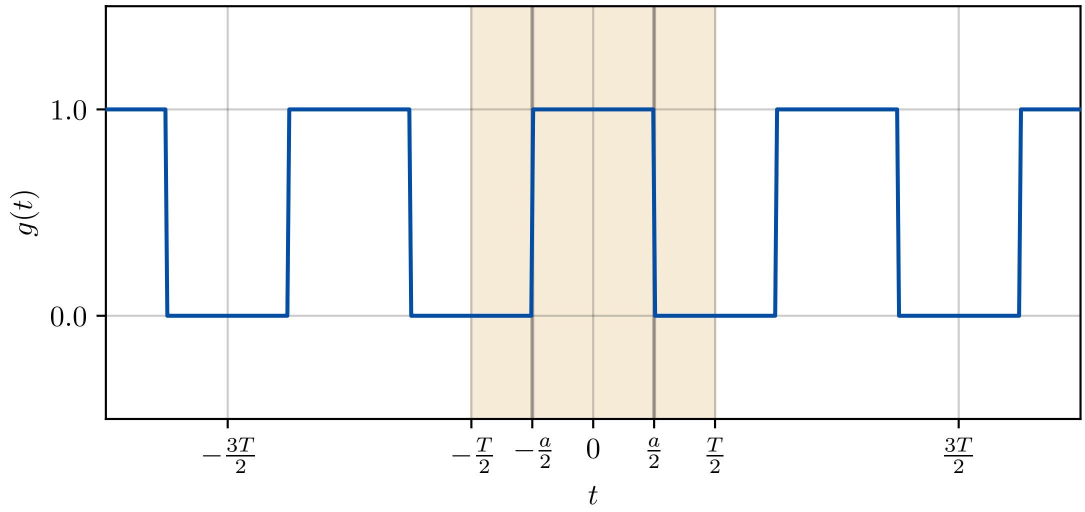

Figure 2: A periodic function constructed by repeating the $\text{rect}(t/a)$ function with a period $2a$

3 Fourier Transforms

3.1 Motivation

While motivating our discussion of Fourier analysis, we mentioned vibrations of solid objects, musical notes and electromagnetic waves. However, we know that these physical signals are not infinite in spatial extent or duration, and therefore not strictly periodic.

The context of music and sound is perhaps the most accessible. Many musical instruments produce notes of long duration (compared to their time period), such as a piano, veena or sitar, and we can readily discern the frequency of the sound they produce (the *pitch*). In these cases, perhaps we can justify our use of Fourier series to analyze the signals. Other instruments such as a xylophone or a bell produce sound signals of much shorter duration. Yet, we can clearly associate a pitch with such a sound. This leads us to a question - can we analyze non-periodic signals in a manner similar to the preceding section?

A second example, from optics. Mathematically, the interference pattern produced by the superimposition of two plane waves is of infinite spatial extent. But in practice, most intensity distributions we encounter are finite. We have observed (and probably even measured) the intensity profile of light emitted by a laser. There is clearly no periodicity associated with the intensity distribution. It is not a single plane wave, since its intensity is concentrated to a small region and Can such an intensity pattern be generated by superimposition of (possibly an infinitely number of) plane waves?

3.2 The Fourier Transform

Recall our definition of a Fourier Series from Equations 6 and 9:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{i2\pi nt}{T}\right) \tag{6 revisited}$$

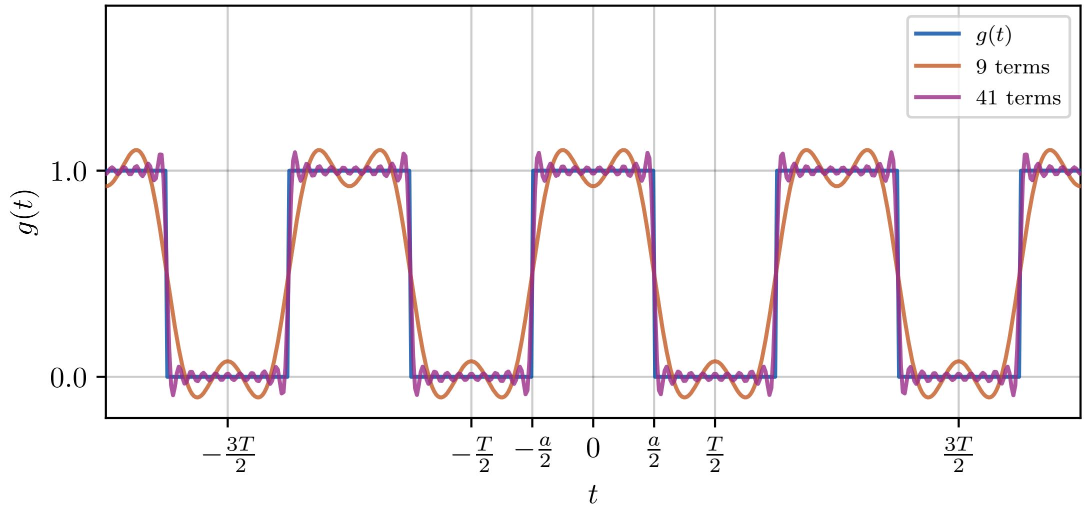

Figure 3: Truncated Fourier series approximations to the periodic function shown in Figure 2. A better approximation is obtained using more terms in the series. The red curve uses $n = -4, -3, \cdots, 3, 4$ and the magenta curve uses $n = -20, -19, \cdots, 19, 20$.

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where

$$c_m = \frac{1}{T}\int_0^T dt\, f(t) \exp\left(\frac{-i2\pi m t}{T}\right) \tag{9 revisited}$$

How can we extend this treatment to non-periodic functions. Perhaps we can consider non-periodic functions as periodic functions with a period equal to ∞. However, in that case, the coefficients become ill-defined in the limit $T \to \infty$ due to the T in the denominator in the expression for c_n.

Let us define

$$F\left(\frac{m}{T}\right) = \int_0^T dt\, f(t) \exp\left(-i2\pi t \frac{m}{T}\right) \tag{12}$$

Here we can identify m/T as the frequency of the m^{th} harmonic of the fundamental frequency $1/T$. Now, 9 becomes:

$$c_m = \frac{1}{T} F\left(\frac{m}{T}\right)$$

We identify $1/T$ as the separation in frequency between successive harmonics in the Fourier series. As we increase the period of the function T, the frequency separation between successive harmonics becomes smaller. For $T \to \infty$, this separation becomes infinitesimally small and approaches zero, and the discrete set of frequencies that described the periodic function approach a continuous range of frequencies from $-\infty$ to ∞. The limits of integration too, which covered one period, now span the entire real line from $-\infty$ to ∞. If we denote the continuous frequency variable ν and the infinitesimally small frequency difference between successive frequencies by $d\nu$, we can express $f(t)$ as

$$f(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \left(\frac{1}{T}\right) F\left(\frac{n}{T}\right) \exp\left(i2\pi t \frac{n}{T}\right) \tag{13}$$

Identifying the right-hand-side of above expression as a Riemann sum leads to

$$f(t) = \int_{-\infty}^{\infty} d\nu F(\nu) \exp(i2\pi\nu t) dt \tag{14}$$

Similarly we can express $F(\nu)$ as

$$F(\nu) = \int_{-\infty}^{\infty} f(t) \exp(-i2\pi\nu t) dt \tag{15}$$

Equations 15 denotes the *Fourier transform*, and Equation 14 denotes the *inverse Fourier transform*. Similar to the case of Fourier series, in order to define a Fourier transform this way, the function $f(t)$ needs to satisfy the condition

$$\int_{-\infty}^{\infty} |f(t)|^2 < \infty \tag{16}$$

Such functions are said to be square integrable on $(-\infty, \infty)$. In addition, Fourier transforms can also be defined for periodic functions such as sines and cosines.

Note: It is important to realise that the sign in the complex exponent of the Fourier series and transform has been chosen arbitrarily. It would have been equally convenient to use $\exp(+i2\pi\nu t)$ for the Fourier transform with equal convenience. This is largely a matter of convenience. But in all cases, the sign in the exponent is opposite for the Fourier transform and its inverse. One particular situation where this sign ambiguity might cause confusion is the case of propagating waves and wave packets, where often different sign conventions are used to transform the spatial ($\exp(i\mathbf{k}\cdot\mathbf{r})$) and the temporal parts ($\exp(i\omega t)$) of the propagating field. This is primarily because in our convention, for a forward-propagating wave ($\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t]$), the sign on the temporal frequency is negative.

3.3 Some Examples

3.3.1 The rectangle function

Consider the simple rectangle function:

$$f(t) = \text{rect}\left(\frac{t}{a}\right) = \begin{cases} 1, & |t| < \frac{a}{2} \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

Using an intergation technique similar to Section 2.2, it is straightforward to show that the Fourier transform of $f(t)$ is

$$\begin{aligned} F(\nu) &= \int_{-\infty}^{\infty} f(t) \exp(-i2\pi\nu t) dt \\ &= \left[\frac{\exp(i2\pi\nu t)}{-2\pi i\nu}\right]_{-a/2}^{a/2} \\ &= \frac{\exp(-i\pi\nu a) - \exp(i\pi\nu a)}{-2i(\pi\nu)} \\ &= \frac{\sin(\pi\nu a)}{\pi\nu} \end{aligned} \tag{18}$$

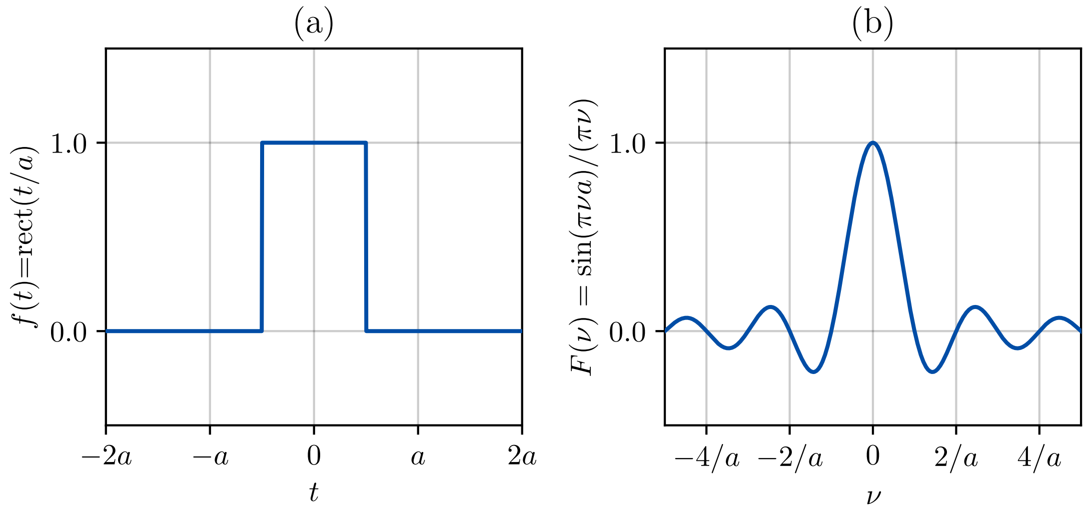

Figure 4: (a) The rectangle function defined in Equation 17; (b) its Fourier Transform.

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The function, together with its Fourier transform are shown in Figure 4.

3.3.2 Fourier series with an increasing period

Consider the periodic rectangular wave we discussed in Section 2.2. It is simply a periodic version of the function whose Fourier transform we calculated in Section 3.3.1 Its Fourier coefficients are given by

$$c_n = \begin{cases} 0.5, & n = 1 \\ \dfrac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}, & \text{otherwise} \end{cases} \tag{19}$$

The function itself can be reconstructed by evaluating the Fourier Series.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{i2\pi nt}{T}\right)$$

where $T = 2a$. c_n are independent of the period T and remain unchanged unless we change the width of the rectangle function (equal to a in our example). They will only correspond to a different frequency It is instructive to observe what happens to the Fourier coefficients when we increase the period. .0.0Consider the cases of $T = 2a$, $T = 4a$ and $T = 8a$. The Fourier

3.3.3 The Gaussian

Another commonly encountered function in optics, particularly in the context of lasers, is the Gaussian:

$$f(t) = \exp(-at^2) \tag{20}$$

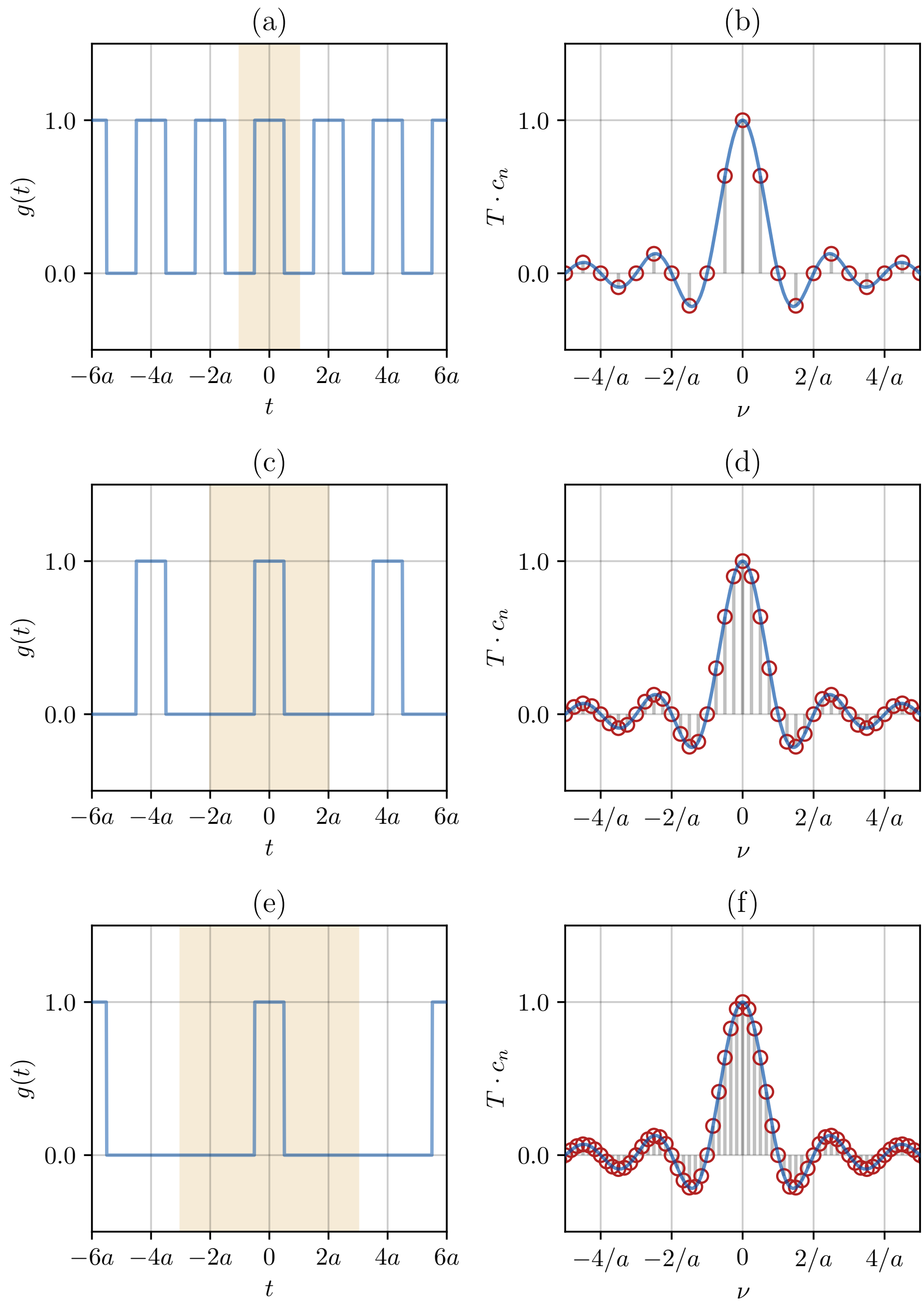

Figure 5: The effect of increasing the period in case of a rectangle function of width a repeated with periods $T = 2a$, $T = 4a$ and $T = 6a$, shown in (a),(c) and (e), respectively. One period is highlighted. The corresponding Fourier series coefficients scaled by T are shown as red markers in (b),(d) and (f). The Fourier transform of the rectangle function (without periodic repetition) is shown as a blue line for comparison.

where a is a positive real number. In order to compute its Fourier transform it is convenient to consider the real and imaginary parts of the exponential.

$$F(\nu) = \int_{-\infty}^{\infty} e^{-at^2} e^{-i2\pi\nu t} dt \tag{21}$$

$$= \int_{-\infty}^{\infty} e^{-at^2} [\cos(2\pi\nu t) - i\sin(2\pi\nu t)] dt \tag{22}$$

$$= \int_{-\infty}^{\infty} e^{-at^2} \cos(2\pi\nu t) dt - i \int_{-\infty}^{\infty} e^{-at^2} \sin(2\pi\nu t) dt \tag{23}$$

The second integrand is odd, so integration over a symmetrical range gives 0. Let us define $\omega = 2\pi\nu$. Then, upon differentiating the first integral w.r.t. ω , we get,

$$\frac{d}{d\omega} \int_{-\infty}^{\infty} e^{-at^2} \cos(\omega t)\, dt = -\int_{-\infty}^{\infty} e^{-at^2} t \sin(\omega t)\, dt \tag{24}$$

The right hand side can be integrated by parts using the formula:

$$\int_a^b u(t) v'(t) dt = [u(t)v(t)]_a^b - \int_a^b u'(t) v(t) dt \tag{25}$$

Identifying $u = \sin(\omega t)$ and $v = e^{-at^2}$, we have:

$$\frac{d}{d\omega} \int_{-\infty}^{\infty} e^{-at^2} \cos(\omega t)\, dt = \frac{1}{2a} \int_{-\infty}^{\infty} \sin(\omega t)\, d(e^{-at^2}) \tag{26}$$

$$= -\frac{1}{2a} \int_{-\infty}^{\infty} e^{-at^2}\, d(\sin(\omega t)) \tag{27}$$

$$= -\frac{\omega}{2a} \int_{-\infty}^{\infty} e^{-at^2} \cos(\omega t)\, dt \tag{28}$$

The above result is of the form $g'(\omega) = -\frac{\omega}{2a} g(\omega)$ which has the solution $g(\omega) = c\exp(-\omega^2/4a)$. Using the result that $g(0) = \int_{-\infty}^{\infty} e^{-at^2}\, dt = \sqrt{\pi/a}$, we have

$$F(\nu) = \int_{-\infty}^{\infty} e^{-at^2} e^{-i2\pi\nu t} dt \tag{29}$$

$$= \sqrt{\frac{\pi}{a}} \exp\left(-\frac{\pi^2 \nu^2}{a}\right) \tag{30}$$

The result; the Fourier transform of a Gaussian is a Gaussian!

4 Some properties of Fourier Transforms

It is important to understand the properties of the Fourier transform, particularly those relevant to the study of electromagnetic waves and their superpositions. Let us denote the Fourier transform of a function by $\mathcal{F}\{\}$ and the inverse Fourier transform by $\mathcal{F}^{-1}\{\}$. So the Fourier transform of a function $f(t)$ would be denoted by $\mathcal{F}\{f(t)\}$, i.e.

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) \exp(-i2\pi\nu t) dt \tag{31}$$

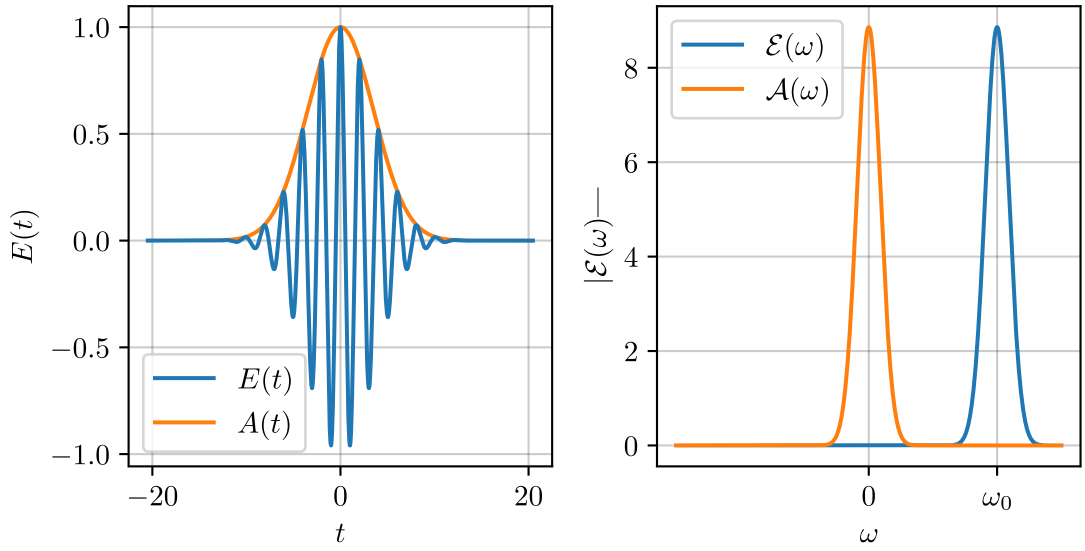

Figure 6: (a) The wave packet in time; (b) and its Fourier Transform. For an explanation of $A(t)$, $E(t)$, $\mathcal{A}(\omega)$ and $\mathcal{E}(\omega)$, follow the main text.

4.1 Linearity

$$\mathcal{F}\{f(t)+g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\} \tag{32}$$

This should not be too difficult to prove. Just substitute the expressions into the integral and it is quite straightforward.

4.2 Time and frequency shifts

Consider a function $f(t)$ whose Fourier transform is $F(\nu)$.

$$\mathcal{F}\{f(t)\exp(i2\pi\nu_0 t)\} = \int_{-\infty}^{\infty} f(t)\exp(i2\pi\nu_0 t)\exp(-i2\pi\nu t)dt \tag{33}$$

$$= \int_{-\infty}^{\infty} f(t)\exp(-i2\pi(\nu-\nu_0)t)dt \tag{34}$$

$$= F(\nu-\nu_0) \tag{35}$$

A similar property exists for the inverse Fourier tranform as well.

$$\mathcal{F}^{-1}\{F(\nu)\exp(i2\pi\nu t_0)\} = \int_{-\infty}^{\infty} F(\nu)\exp(i2\pi\nu t_0)\exp(i2\pi\nu t)dt \tag{36}$$

$$= \int_{-\infty}^{\infty} F(\nu)\exp(i2\pi\nu(t+t_0)dt) \tag{37}$$

$$= f(t+t_0) \tag{38}$$

4.3 Scaling the variables

Consider a function $f(t)$ whose Fourier transform is $\mathcal{F}\{f(t)\} = F(\nu)$, and a real number a. The Fourier transform of $f(at)$ is given by

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} dt\, f(at) \exp(-i2\pi\nu t) \tag{39}$$

If $a > 0$, This is simply:

$$\begin{aligned}\int_{-\infty}^{\infty} dt\, f(at) \exp(-i2\pi\nu t) &= \frac{1}{a}\int_{-\infty}^{\infty} du\, f(u) \exp(-i2\pi\nu u/a) \\ &= \frac{1}{a} F\left(\frac{\nu}{a}\right) \end{aligned} \tag{40}$$

If $a > 0$, the integral evaluates to:

$$\begin{aligned}\int_{-\infty}^{\infty} dt\, f(at) \exp(-i2\pi\nu t) &= \frac{1}{a}\int_{\infty}^{-\infty} du\, f(u) \exp(-i2\pi\nu u/a) \\ &= -\frac{1}{a}\int_{-\infty}^{\infty} du\, f(u) \exp(-i2\pi\nu u/a) \\ &= \frac{-1}{a} F\left(\frac{\nu}{a}\right) \end{aligned} \tag{41}$$

By combining Equations 40 and 41, we can write:

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\nu}{a}\right) \tag{42}$$

This scaling property has several interesting consequences in optics, especially in the context of diffraction.

4.4 The Dirac Delta function

We will often encounter a rather peculiar entity called the Dirac delta. We may loosely think of it as the following:

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \tag{43}$$

which also obeys

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \tag{44}$$

Let us briefly consider the integral

$$I(t) = \int_{-\infty}^{\infty} e^{i2\pi\nu t} d\nu \tag{45}$$

which you may understand as the inverse Fourier transform of 1. The result would be a signal which consists of all frequencies with a unit amplitude. We can see that this integral yields the Dirac delta. For any $\nu \neq 0$, the integrand is an oscillating function which integrates to zero, and for $\nu = 0$, the integrand equals 1 and the result is infinite. This corresponds to extremely short impulse (of zero width). You can see how this seems to agree with the scaling rule discussed in the preceding section.

4.5 Convolution theorem

A convolution of two functions $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau \tag{46}$$

Here we shall state (without proof, for now) that

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\} \tag{47}$$

That seems like a seemingly arbitrary looking property. But we will encounter such integrals in numerous scenarios in optics, particularly in the context of diffraction and optical imaging.

4.6 Parseval's theorem

Consider a function $f(t)$ and its Fourier transform $F(\nu)$. Parseval's theorem states that

$$\int_{-\infty}^{\infty} |f(t)|^2\ dt = \int_{-\infty}^{\infty} |F(\nu)|^2\ d\nu \tag{48}$$

In the context of optics, $f(t)$ typically corresponds to a time-dependent (and often periodic) electric field. In that case $|f(t)|^2$ is the intensity of the field, and is a measure of the energy carried by the field. In this context, $|F(\nu)|^2$ is the spectral power density - the energy carried by each frequency component (or *colour*) of the field. Parseval's identity therefore relates the energy density to its distribution among the various frequencies that constitute the signal, and this quantity remains the same whether analysed in the time domain or in the frequency domain.