OPTICS (PHY224)
Problem Set 3

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1. The Fourier transform of a function $f(x)$ is defined as

$$\mathcal{F}\{f(x)\} = F(\kappa) = \int_{-\infty}^{\infty} dx\ f(x)\ \exp(-i2\pi\kappa x).$$

Consider the unit-width rectangle function:

$$f(x) = \begin{cases} 1, & \text{for } |x| \leq 0.5 \\ 0, & \text{otherwise.} \end{cases}$$

It's Fourier transform is $F(\kappa) = \sin(\pi\kappa)/(\pi\kappa)$

(a) Calculate the Fourier transform of a rectangular step function whose width is a.

$$g(x) = \begin{cases} 1, & \text{for } |x| \leq a/2 \\ 0, & \text{otherwise.} \end{cases}$$

(b) In the Fraunhofer approximation, the diffracted field from an aperture at $z = 0$ with a field distribution $E(x', y')$ is is given by the following Fourier transform.

$$E(x, y, z) \propto \iint E(x', y') e^{-i\frac{k(x'x+y'y)}{z}}\, dx'\, dy'$$

where the integral is carried out in the plane of the aperture. Here, z is the position of the detection plane and $k = 2\pi\kappa = 2\pi/\lambda_0$ is the wavenumber, where λ_0 is the wavelength. Using your result from (a), determine the shape of the Fraunhofer diffraction pattern of a long thin slit of width h. Determine the position of its first minimum.

2. Consider the standard Fraunhofer diffraction experiment shown in Figure 1 with the screen located far away (large z_0). The width of the slit (centered at $x = 0$) is a. The diffracted field at the screen in the Fraunhofer regime is given by the following expression, which is essentially a Fourier transform:

$$E(x, z = z_0) \propto \int dx'\, E(x', z = 0) \exp\left(-i\frac{k}{z_0}xx'\right)$$

With this knowledge

(a) Given illumination with a plane wave from the left as shown in Figure 1(a), obtain an expression for the intensity distribution at the screen.

(b) The illumination direction is slightly tilted as shown in as shown in Figure 1(b). How will the diffraction pattern change?

3. Consider a ("scalar") Gaussian beam whose electric field has the following functional form at the beam waist located at $z = 0$ (where the beam diameter is minimum and equal to w_0^2).

$$E(r) = E_0 \exp\left(\frac{-r^2}{w_0^2}\right)$$

Using your knowledge of Fourier transforms, determine the angle of divergence of the beam as it propagates over long distance. The far-field (Fraunhofer) diffraction pattern is simply a scaled Fourier transform

4. Consider two materials with refractive indices $n_1(\omega)$ and $n_2(\omega)$ that vary with frequency ω as shown in Figure 2 below. A short pulse of light, of carrier frequency ω_0 and frequency band width $\delta\omega$ (shaded region in the figure) travels through the media. In which medium does the pulse envelope travel slower? Explain why.

5. Consider two y-polarized plane waves (each containing two frequencies) given by:

$$E(x,z,t) = E_0\left(e^{i(k_{1x}x + k_{1z}z - \omega_1 t)} + e^{i(k_{2x}x + k_{2z}z - \omega_2 t)}\right)$$
$$E'(x,z,t) = E_0\left(e^{i(k'_{1x}x + k'_{1z}z - \omega_1 t)} + e^{i(k'_{2x}x + k'_{2z}z - \omega_2 t)}\right)$$

   (a) Obtain an expression for the total electric field $E + E'$ in the plane $z = 0$.

   (b) Obtain and simplify the expression for the total electric field $|E + E'|^2$ in the plane $z = 0$

   (c) Obtain an expression for the time-invariant interference pattern $I \propto \langle |E + E'|^2 \rangle_T$ produced by the superposition of these two waves at $z = 0$, correct to a proportionality factor. Average over all temporal variations.

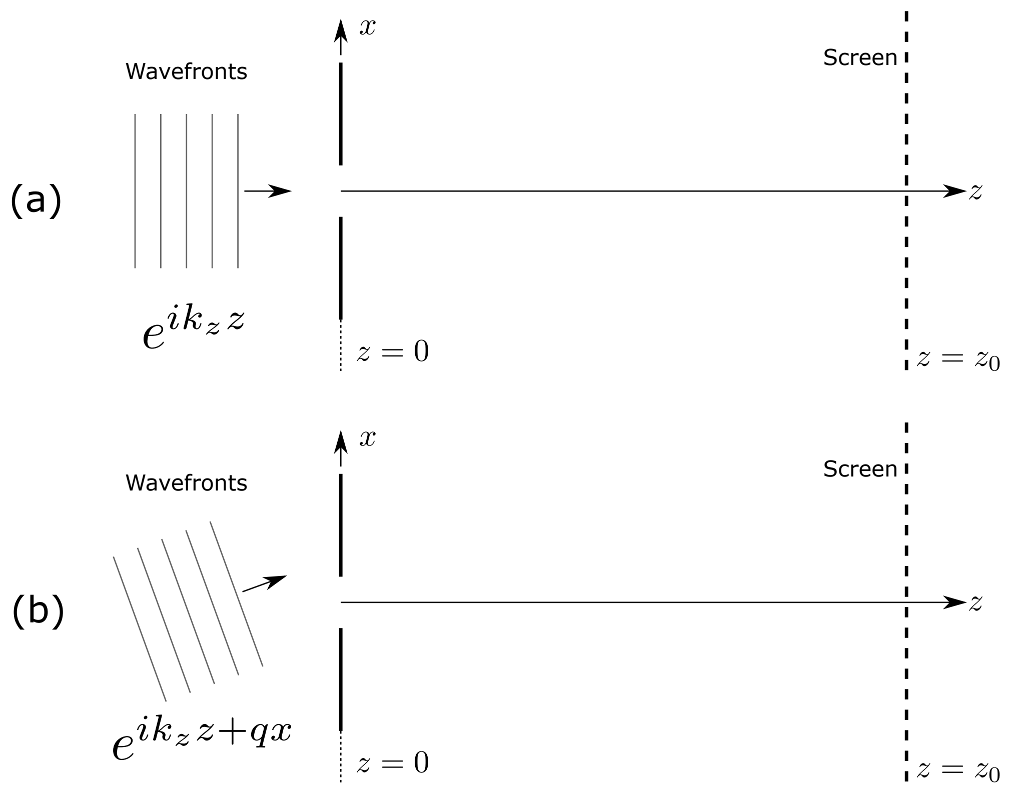

Figure 1: Fraunhofer diffraction geometry

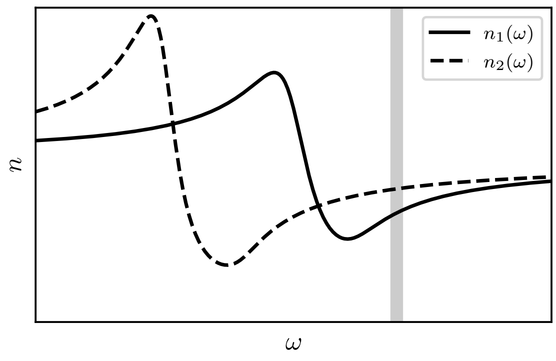

Figure 2: Refractive index vs wavelength for two dielectrics

6. (a) The inverse Fourier transform of a function $f(x)$ is defined as

$$f(t) = \int_{\infty}^{\infty} d\nu \, F(\nu) \, e^{i2\pi\nu t}.$$

Determine the inverse Fourier transform of $F(\nu)e^{-i2\pi\nu T_0}$ in terms of $f(t)$

(b) Show that the group velocity can be expressed as

$$v_g = \frac{c}{n(\lambda_0) - \lambda_0 \dfrac{dn(\lambda_0)}{d\lambda_0}}$$

where c is the speed of light in vacuum, $n(\lambda_0)$ is the wavelength-dependent refractive index of the medium, and λ_0 is the vacuum wavelength.

(c) Derive an expression for the group velocity v_g in a medium whose refractive index at a wavelength λ_0 is given by $n(\lambda_0) = A + B/\lambda_0^2$

7. Fraunhofer double slit diffraction pattern is observed in the focal plane of a lens of focal length 0.5m. The wavelength of incident light is 500 nm. The distance between two maxima adjacent to the maximum of zero order is 5 mm and the fourth order maximum is missing. Find the width of each slit and the distance between their centers.

8. The objective of a telescope has a diameter of 2.54 m. Assuming the mean wavelength of incident light to be 550 nm. Calculate the least angular separation of two stars which can be resolved by it.

9. Calculate the inner and outer radii of the 10th half-period zone for a plane wavefront with respect to a point at a distance 0.5m from it. Assume the wavelength of light $\lambda = 500$nm.

10. Calculate the ratio of spontaneous emission rate to stimulated emission rate at $T = 103\ K$ for visible light of frequency 5×10^{14} Hz and microwave of frequency 109 Hz. Comment on the result.