OPTICS (PHY224)
Problem Set 1

Venkata Jayasurya Y.

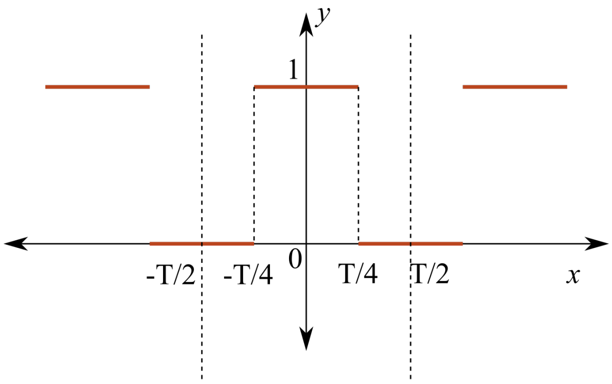
Semester 2024-2025-I

Some problems refer to Peatross and Ware.
You can download it from https://optics.byu.edu/docs/opticsBook.pdf

1. A periodic function $f(t)$ can be expressed as a Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(i\frac{2\pi}{T}kt\right)$$

Determine the Fourier coefficients c_k of the following function (square wave) of periodicity T.



2. Find the Fourier transform of the Gaussian function:

$$f(x) = e^{-ax^2}$$

where a is positive. Call it $F(\kappa)$. Feel free to look up the various way to solve this. This is only for your understanding, and not part of this course.

3. Find the Fourier transform of the “rectangle” function:

$$g(x) = \begin{cases} 1, & \text{if } |x| < 0.5 \\ 0, & \text{otherwise} \end{cases}$$

where a is positive. Call it $G(\kappa)$

4. The Fourier transform of a function $f(x)$ is defined as

$$\mathcal{F}\{f(x)\} = F(k) = \int_{-\infty}^{\infty} dx\ f(x)\ \exp(-i2\pi kx)$$

(a) Determine the Fourier transform of $f(\alpha x)$ where α is a real number

(b) The two-dimensional Fourier transform of $f(x, y)$ is defined as

$$F(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\ f(x, y)\ \exp[-i2\pi(k_x x + k_y y)]$$

What happens to the Fourier transform when x and y are scaled to αx and αy respectively?

(c) Consider the cross section of a Gaussian beam at its waist ($z = 0$).

$$E(x, y) = E_0 \exp(-(x^2 + y^2)/w_0^2)$$

Given that the Fourier transform of $\exp(-ax^2)$ is $\sqrt{\pi/a}\exp(-\pi^2 k_x^2/a^2)$, determine the two-dimensional Fourier transform of $E(x, y)$.

(d) Consider your answer in (c). Each value of (k_x, k_y) corresponds to a plane wave propagating in a specific direction. What is the angle of propagation relative to the z axis corresponding to a specific value of (k_x, k_y)?

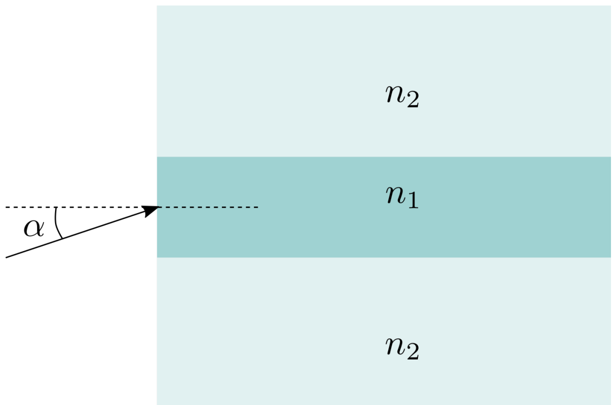
(e) Consider two Gaussian beams, one with waist radius $w_0 = 100\,\mu\text{m}$ and another with $w_0 = 200\,\mu\text{m}$. Deduce using your results in (c) and (d), which one will diverge more (larger angle of divergence).

5. Find the period, wavelength, velocity and direction of a wave represented as $A(x, t) = \cos[2(3t + 15x - 5y + 8z)]$

6. Show that the reflected ($r_\perp$) and transmitted($t_\perp$) amplitudes for light polarized normal to the plane of incidence obey $(-r_\perp) + t_\perp = 1$

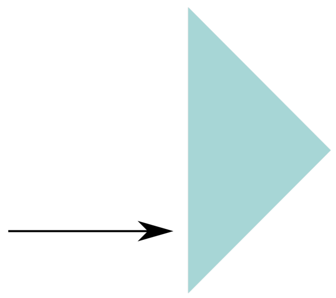
7. The reflectance (R) is defined as the ratio of the power reflected at an interface to the incident power. Similarly, the transmittance is the ratio of the transmitted power to the incident power. Derive expressions relating T and R to the reflected and transmitted amplitudes (r and t) and the angles of incidence and transmission (θ_i and θ_t). Show that for an interface involving non-absorbing (transparent) dielectric media, $R+T = 1$ for both polarizations of light.

8. Only the light incident within the cone of half-angle α can be guided along a glass fiber, by total internal reflection (see the cross section of the fiber below). This angle is known as the acceptance angle and $\sin\alpha$ is known as the numerical aperture of the fiber. If n_1 and n_2 are the refractive indices of the core and cladding materials respectively with $n_1 > n_2$, and light is incident from air, show that $\sin\alpha = \sqrt{n_1^2 - n_2^2}$



9. Show that when $\theta_i > \theta_c$ at a dielectric interface, $r_\parallel$ and $r_\perp$ are complex and that $r_\parallel r_\parallel^* = r_\perp r_\perp^* = 1$

10. A beam of light enters a right-angle glass prism ($n = 1.5$) held in air normal to the hypotenuse (see the figure below). Explain what happens. What happens if the prism is in water ($n = 1.33$) instead of air?



11. Express the following waves as Jones vectors and determine the state of polarization.

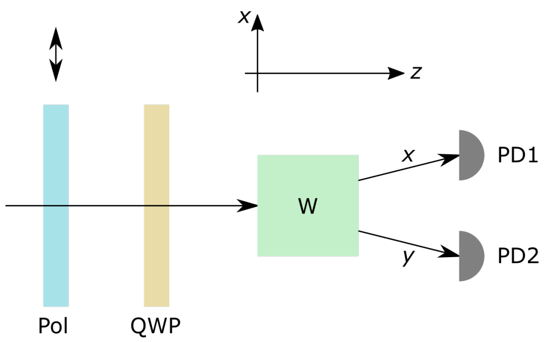
    (a) $\mathbf{E} = \mathbf{x}E_0 \cos(kz - \omega t) + \mathbf{y}E_0 \sin(kz - \omega t)$

    (b) $\mathbf{E} = \mathbf{x}E_0 \cos(kz - \omega t) - \mathbf{y}E_0 \cos(kz - \omega t + 3\pi/4)$

    (c) $\mathbf{E} = \mathbf{x}E_0 \sin(kz - \omega t) - \mathbf{y}E_0 \cos(kz - \omega t)$

12. Consider a plane polarized light wave with equal s ($\perp$) and p ($\parallel$) components propagating in glass ($n = 1.5$). Using the Fresnel equations, determine the polarization state of the reflected light when this wave meets a glass-air interface at (a) an angle of incidence of 45° and (b) an angle of incidence of 30°. Obtain numerical values of the reflected amplitudes first, and then proceed to find the polarization state as a Jones vector.

13. A Wollaston prism is a device that splits the light passing through it into two parts, one horizontally polarized and one vertically polarized, with a small angle between them. Consider the arrangement shown in the following figure. A linear polarizer (Pol) polarizes



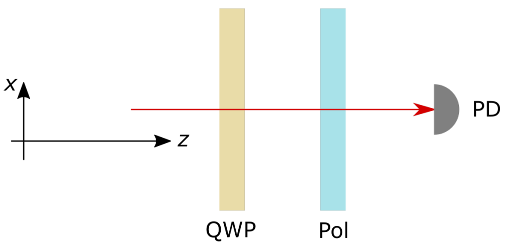
the incoming light along the x direction. The light then passes through a quarter-wave plate (QWP) whose fast axis makes an angle θ with the x axis. A Wollaston prism then splits the x and y components into two beams. These two parts are detected by two photodiodes PD1 and PD2, respectively. Determine:

    (a) the dependence of the signal detected at PD1 on θ

    (b) the dependence of the difference between PD1 and PD2 signals on θ

Plot (sketch by hand) the functional forms of the two results (a) and (b) together.

14. Consider a light wave (described by its Jones vector $\mathbf{j}_{\text{in}}$) propagating along the z direction and through a quarter-wave plate with its fast axis oriented at an angle θ w.r.t. the x-axis, followed by a polarizer oriented at an angle ψ w.r.t. the x-axis, as shown below.



Calculate the transmitted intensity for the following cases.

(a) $\mathbf{j}_{\text{in}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$; $\theta = \pi/4$; $\psi = 0$

(b) $\mathbf{j}_{\text{in}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$; $\theta = \pi/4$; $\psi = \frac{\pi}{2}$

(c) $\mathbf{j}_{\text{in}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$; $\theta = \pi/4$; $\psi = 0$

(d) $\mathbf{j}_{\text{in}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$; $\theta = \pi/4$; $\psi = \frac{\pi}{2}$

(e) $\mathbf{j}_{\text{in}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\(1+i)/\sqrt{2}\end{bmatrix}$; $\theta = \pi/4$; $\psi = 0$

(f) $\mathbf{j}_{\text{in}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\(1+i)/\sqrt{2}\end{bmatrix}$; $\theta = \pi/4$; $\psi = \frac{\pi}{2}$

(g) $\mathbf{j}_{\text{in}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\(1-i)/\sqrt{2}\end{bmatrix}$; $\theta = \pi/4$; $\psi = 0$

(h) $\mathbf{j}_{\text{in}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\(1-i)/\sqrt{2}\end{bmatrix}$; $\theta = \pi/4$; $\psi = \frac{\pi}{2}$

Based on these results, can you determine the handedness of polarization using this configuration?

15. Write down the Jones vectors corresponding to right and left circularly polarized light. Call them $\mathbf{j}_{\text{R}}$ and $\mathbf{j}_{\text{L}}$, respectively. Consider a linear combination $\alpha\mathbf{j}_{\text{R}}+\beta\mathbf{j}_{\text{L}}$ where $|\alpha|^2+|\beta|^2 = 1$. For what values of α and β does the above linear combination represent linear polarization? What is the orientation of the resulting linear polarization?

16. You are given a source of light of vacuum wavelength λ_0, propagating along the positive z axis.

(a) A polarizer, whose axis is oriented along the x-axis, is placed in the path of light. Write down the Jones vector of light after it has passed through the polarizer.

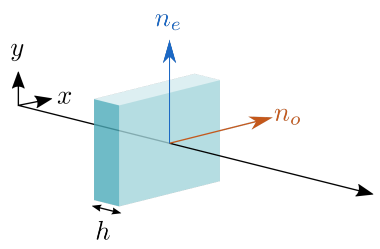
(b) A quarter-wave plate (QWP), is placed in the path after the polarizer such that its axis (it is not known if this axis is fast or slow) makes 45° with the positive x-axis. What are the possible Jones vectors of the resulting light?

(c) Derive the Jones matrix of a QWP whose fast axis makes an angle of 45° with the positive x-axis.

(d) Given a second QWP with a known fast and slow axis orientation, and a second polarizer, how would you determine the orientation of the fast and slow axis of the first QWP? Show it mathematically using the Jones formalism.

17. Derive for yourself, Example 5.3 in Peatross and Ware.

18. Consider a sheet of uniaxial birefringent material of thickness h, normal to the z axis, with light propagating along an axis such that x-polarized light experiences n_o and y-polarized light experiences n_e, as shown in the figure below.



Light of vacuum wavelength λ_0 propagates through it. Assume $n_e > n_o$.

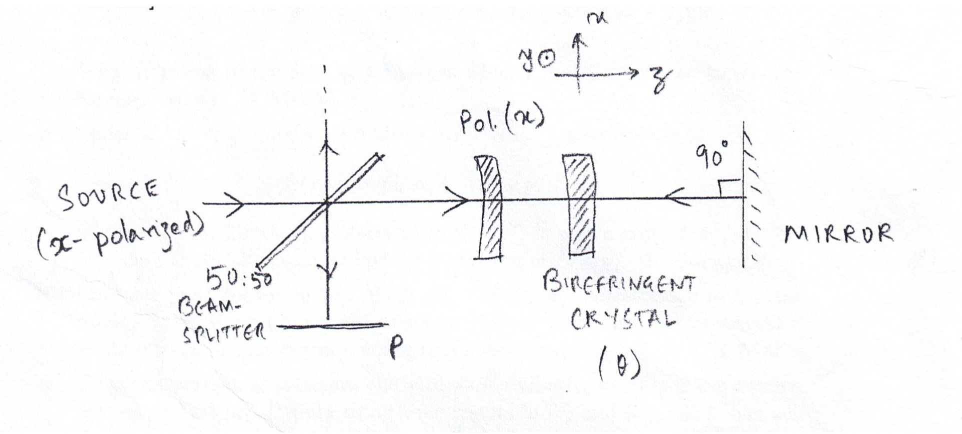
(a) Obtain an expression for the phase difference $\Delta\phi_{fs}$ between light polarized along the fast and the slow axes as a function of λ_0. Derive the Jones Matrix for the sheet in this configuration.

(b) Calculate the thickness for which the birefringent material can be used as a quarter-wave plate (QWP) for λ_0. How many such values for the thickness exist? Calculate some numerical values for $\lambda_0 = 500$ nm, $n_o = 1.5$ and $n_e = 1.51$.

(c) You just realised that your laser emits $\lambda_0 \pm \Delta\lambda_0$, which is close but not exactly equal to λ_0. You would like to choose a thickness from the results in (b) such that $\Delta\phi_{fs}$ deviates the least from QWP behaviour. What is the best choice of thickness from (b), which gives a value of $\Delta\phi_{fs}$ closest to $\frac{\pi}{4}$?

19. Problem 5.3 from Peatross and Ware.
Fresnel's equation here refers to Equation 5.19 in Peatross and Ware, which is similar to what we derived in class, but using the normalized direction vector $\mathbf{u}$.

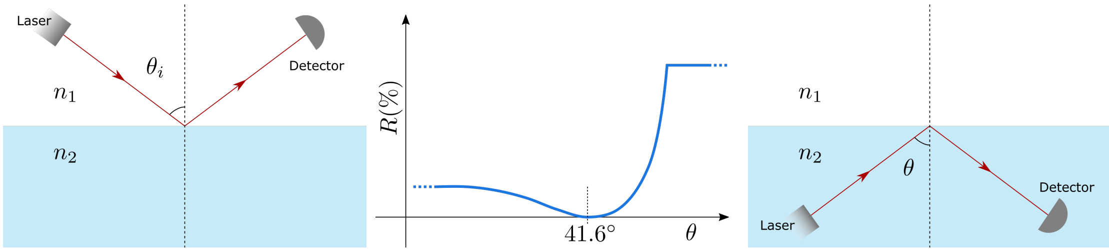
20. Consider the geometry in the figure below. The birefringent crystal (refractive indices: n (fast) and $n + \Delta n$ (slow), thickness h) is oriented with its fast axis making an angle θ with the positive x axis. The polarizer axis is oriented along the x axis.



(a) Calculate the intensity/irradiance recorded at the plane P in case of the following geometry as a function of h and θ.

(b) For what value of h and θ will there be no irradiance detected at P?

21. You are given a laser pointer of wavelength $\lambda_0 = 532$ nm whose output is linearly polarized. Consider the interface between two liquids and the configuration shown in the Figure below (a). Upon measuring the variation of reflected power vs angle of incidence (θ), you have found that the reflectance (R) depends on θ as shown in Figure below (b).



(a) (b) (c)

(a) What is the polarization of the laser?

(b) What is the ratio of refractive indices n_1/n_2?

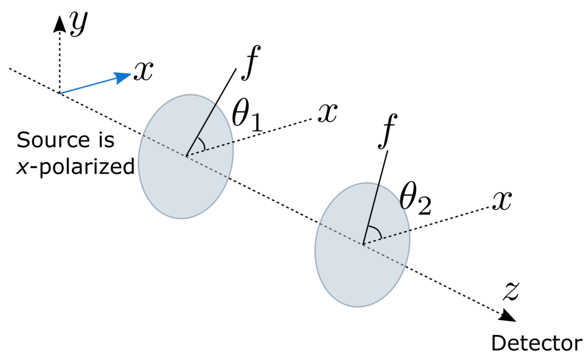
(c) Calculate the angle above which the reflectance is 100%.

(d) If the liquid at the bottom is water (i.e. $n_2 = 1.33$), Calculate the index n_1.

(e) Sketch the reflectance vs angle of incidence in a scenario where the light is incident from the bottom liquid as shown in Figure above (c).

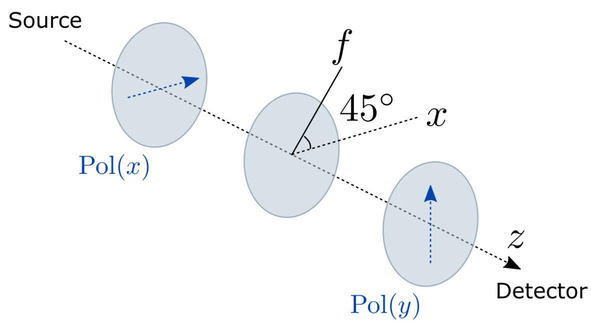
22. (a) You are given a birefringent material, which when transmitting light along the specified direction, exhibits two refractive indices, $n_1 = 1.5$ and $n_2 = 1.6$ for orthogonal polarizations. Explain how you would use this material to make a quarter-wave plate for a free-space wavelength $\lambda_0 = 633$ nm. Identify the fast and slow axes, and determine the thickness you would use.

(b) You are given a source of light polarized along the x axis and two quarter wave plates as shown in the figure below. You are not allowed to remove or rearrange any optical element, but you can rotate the quarter-wave plates. Explain how you would use this

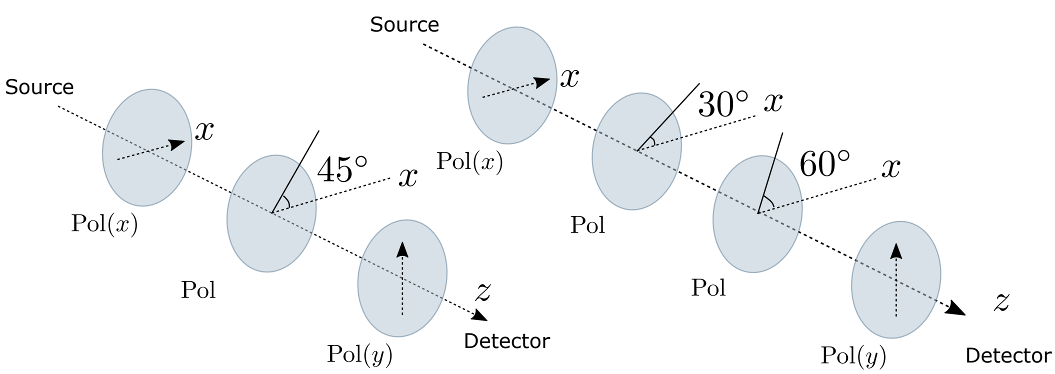


configuration to obtain (i) y-polarized light, (ii) light polarized at 45° w.r.t. the x axis, and (iii) Right Circularly Polarized (RCP) light.

23. You are given a thin film of birefringent material, such that it generates exactly half a wave of path difference between its fast and slow axes when used with a free-space wavelength of L. Determine the behaviour of this birefringent film for an arbitrary free-space wavelength λ. Plot the transmission *spectrum* (transmission vs λ) when this film is placed between two crossed polarizers as shown in the Figure below (f represents the fast axis). Assume the polarizers are ideal at all wavelengths, and that none of the materials involved is dispersive.



24. Consider the arrangement of three polarizers shown in Figure (a) below.The source emits light polarized along the x-axis, of intensity I_0.



(a) (b)

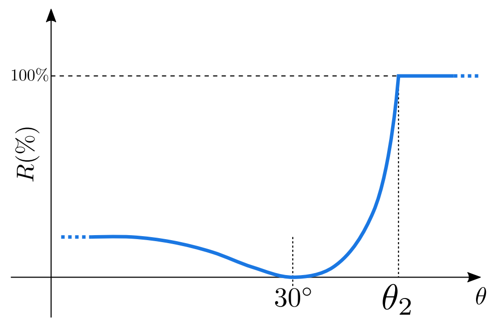
(a) Calculate the intensity transmitted after the third polarizer (oriented along the y-axis) in terms of I_0 in the arrange

(b) Calculate the intensity transmitted after the fourth polarizer (oriented along the y-axis) in terms of I_0 in the arrangement of Figure (b).

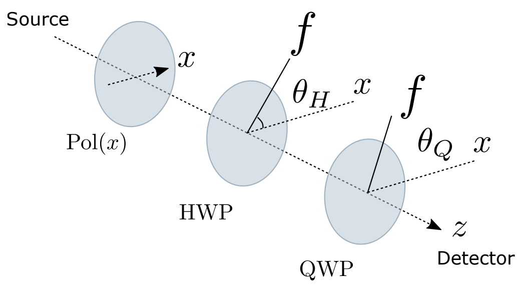
(c) Consider the case where the light propagates through $N+1$ polarizers ($n = 0, 1, 2, \cdots, N$), where the first polarizer ($n = 0$) is oriented along the x-axis, and the axis of n^{th} polarizer makes an angle $n\pi/(2N)$ with the x axis. This way, the last ($n = N$) polarizer is oriented perpendicular to the x axis. Find the transmitted intensity in this case in terms of I_0.

(d) What would happen if the number of polarizers N in (c) is very large? Determine the transmitted intensity in the limit $N \to \infty$.

25. (a) In an experiment, light of a certain polarization propagates from a liquid of refractive index n_1 to a liquid of index n_2. The reflectance at the interface between the two liquids has been found to vary with the angle of incidence (θ) as shown below: Determine the (i) polarization, (ii) the ratio of refractive indices n_1/n_2, and (iii) the angle θ_2.



(b) You are given a source emitting x-polarized (linear) light. The light passes through a half-wave plate (fast axis oriented at θ_H w.r.t. the x axis) and a quarter-wave plate (fast axis oriented at θ_Q w.r.t. the x axis) as shown in the. What should be the orientations of the wave plates θ_H and θ_Q to obtain: (i) light that is linearly polarized along the y direction and (ii) circularly polarized light. List as many configurations as you can.



26. Consider a light wave (wavelength λ) propagating along the positive z-axis, whose electric field components are as follows.

$$E_x = E_0 \cos(kz - \omega t) E_y \qquad = E_0 \cos(kz - \omega t + \frac{\pi}{4})$$

(a) Using a graph paper, plot the polarization ellipse corresponding to the above field. (Determine E_x, E_y at various phases between 0 and 2π, and plot these points on the graph paper)

(b) What is the state of polarization of this wave? Specify handedness if relevant.

(c) Suppose you were provided a birefringent crystal with refractive indices $n_f = 1.6$ and $n_s = 1.7$ for the orthogonal fast and slow axes, respectively. You are asked to design a wave plate that generates the polarization state described in this question.You are provided light of wavelength $\lambda_0 = 633$ nm which is linearly polarized at $45°$ relative to the x axis.Calculate the thickness (along direction of propagation) of the birefringent crystal that you would use.