PHY226M, Special Theory of Relativity, Endsem Exam (April 20th, 2024)

1. Answer the following short questions:
   (a) The force and acceleration vectors are always parallel. True or False? [Marks 1]

   (b) A mirror moves toward you at speed V. You shine a light toward it, and the light beam bounces back at you. What is the speed of the reflected beam? [Marks 1]

   (c) What are the two postulates of special theory of relativity? [Marks 2]

   (d) What is the relation between dual electromagnetic tensor and electromagnetic tensor? [Marks 2]

   (e) P is an object at rest in laboratory frame. Draw its world-line in laboratory frame. [Marks 2]

   (f) In laboratory frame, A and B are moving along X direction with velocity u_A and u_B respectively, where $u_A < c$, $u_B < c$ and $u_A < u_B$. Draw the world-lines of A and B in laboratory frame. Also draw the world-line of a light ray in the same frame. [Marks 2+1]

2. Briefly discuss the results of the Michelson-Morley experiment and point out its importance in the foundation of special theory of relativity. [Marks 4]

3. An observer (O) at rest midway between two sources of light at $x = 0$ meter and $x = 10$ meter observes the two sources to flash simultaneously. According to a second observer (O'), moving at a constant speed (v) parallel to the common $X(X')$-axis, one source of light flashes 13 ns before the other. Find the value of $\beta(= v/c)$. [Marks 6]

4. Use velocity addition theorem (i.e. the transformation of velocity across inertial frames) to prove the relativistic aberration formula:

$$\tan\theta = \frac{\sin\theta' \sqrt{1-\beta^2}}{\cos\theta' + \beta}$$

   You can assume that the primed frame is moving with velocity v w.r.t the unprimed frame along the common $X(X')$ axis. [Marks 5]

5. Start from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and find the (0,2)th component and (1,2)th component

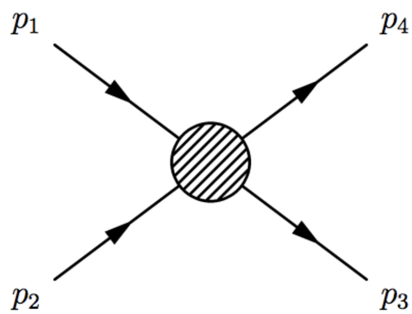
of $F_{\mu\nu}$, i.e F_{02} and F_{12}, in terms of the components of electric field and magnetic field. [Marks 10]

6. Assume that a primed frame (i.e S') is moving w.r.t the unprimed frame (i.e S) with velocity v, along the common $X(X')$ axis. Prove that $B'_y = \gamma(B_y + \frac{v}{c^2}E_z)$. You may start from $F'^{\mu\nu} = \Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}F^{\rho\sigma}$ and the following Lorentz transformation matrix may be used [Marks 5]

$$\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. Show that $\partial_\mu \tilde{F}^{\mu\nu} = 0$ leads to two of the four Maxwell's equations in its usual well-known form. [Marks 8]

8. Consider the case of inelastic scattering of particles: $1 + 2 \rightarrow 3 + 4$ as shown in the figure, where p_1, p_2, p_3 and p_4 are 4-momenta and the corresponding rest masses are m_1, m_2, m_3 and m_4. Assume $c = 1$ and use 4-vectors to solve this problem.



(a) Show that the sum of the Mandelstam variables (s, t, u) is equal to the sum of rest-mass squared of the four particles. [Marks 5]
(b) If particle 2 is at rest, then prove that $s = (m_1^2 + m_2^2 + 2E_1m_2)$, where E_1 is the energy of particle 1. [Marks 5]

9. Consider two arbitrary 4-vectors, A and B. Prove that their inner-product is frame-independent. [Marks 5]

10. Let's define invariant interval as the following: $(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2$.
(a) Using this, discuss what is meant by time-like, light-like and space-like separated events. [Marks 3]
(b) Discuss the physics implications of each of them. [Marks 6]

Expressions that you might need

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{bmatrix}$$

$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$

$A_\mu = \frac{\phi}{c}, -\vec{A}$

$s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$

$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$, $u_y = \frac{u'_y \sqrt{1-\beta^2}}{1 + \frac{u'_x v}{c^2}}$

Endsem exam Solutions

1 (a) False

1 (b) c

1 (c) laws of physics are same in all inertial frames. There is no preferred inertial frame.

The speed of light in vacuum has the same value (c) in all inertial frame.

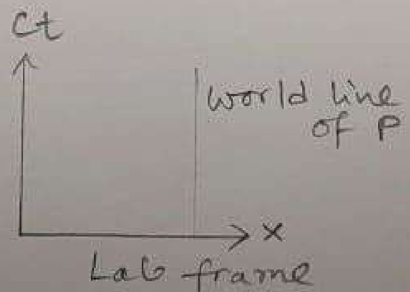
1 (d) $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

OR the following is also fine

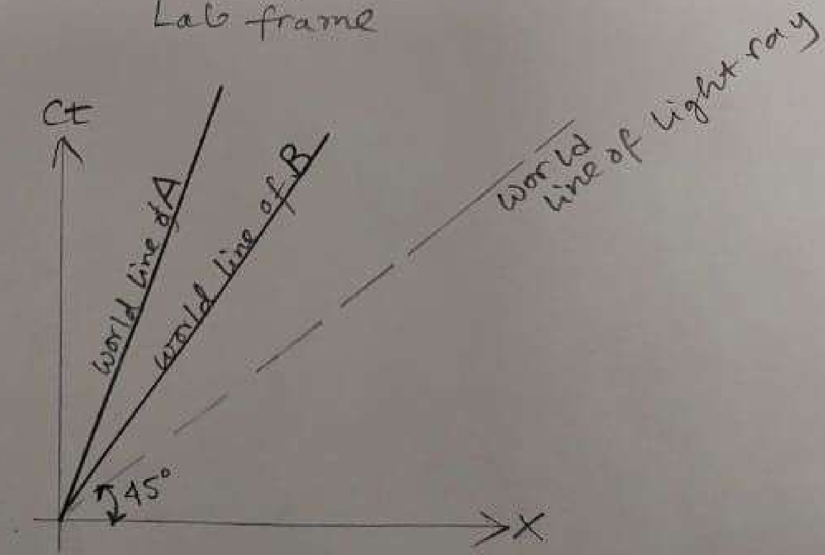
$\Downarrow$

$\frac{1}{4}\tilde{F}^{\mu\nu} F_{\mu\nu} = -\frac{1}{c}\vec{E}\cdot\vec{B}$

1 (e)



1 (f)



(2) $\boxed{\text{Results}}$: Expected to observe fringe shift; but did not find any fringe shift.

Michelson Morley Expt.

$\Rightarrow$ Null result.

$\boxed{\text{Importance}}$: It shed light on ~~on~~ the fact that velocity of light in vaccum is same in all inertial frame and it is c.

(3) In unprimed frame (S) : $\Delta t = 0$; $\Delta x = 10$ meter

In primed frame (S') : $\Delta t' = \pm 13$ ns.

we know time transforms as $t' = \gamma\left(t - \frac{vx}{c^2}\right)$.

So, $t_1' = \gamma\left(t_1 - \frac{vx_1}{c^2}\right)$

$t_2' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right)$

ie, $\Delta t' = t_2' - t_1' = \gamma\left[\underbrace{(t_2 - t_1)}_{0} - \frac{v}{c^2}\underbrace{(x_2 - x_1)}_{10 \text{ meter}}\right]$

$\Rightarrow \Delta t' = \gamma\left[0 - \frac{v}{c^2} \cdot 10 \text{ meter}\right]$

$\Rightarrow \Delta t' = -\frac{\gamma v}{c^2} \cdot 10$

$\Rightarrow -\frac{\gamma v}{c^2} = \frac{\Delta t'}{10} \Rightarrow -\frac{\gamma \beta}{c} = \frac{\Delta t'}{10}$

$\frac{\gamma v}{c^2} = \frac{\gamma \beta}{c}$

$\gamma^2 \beta^2 = \frac{\beta^2}{1-\beta^2}$

Square both sides.

$\Rightarrow \frac{\gamma^2 \beta^2}{c^2} = \left(\frac{\Delta t'}{10}\right)^2$

$\Rightarrow \frac{\beta^2}{1-\beta^2} = \frac{(13 \text{ ns})^2}{100} \cdot c^2$

$\Rightarrow \frac{\beta^2}{1-\beta^2} = \frac{13 \times 10^{-9} \times 13 \times 10^{-9} \times 3 \times 10^{8} \times 3 \times 10^{8}}{100}$

$\Rightarrow \frac{\beta^2}{1-\beta^2} = 0.15 \Rightarrow \beta^2 = 0.15 - 0.15\beta^2$

$\Rightarrow 1.15\beta^2 = 0.15$

$\Rightarrow \beta^2 = 0.13$

$\Rightarrow \boxed{\beta = 0.36}$ <u>Answer</u>

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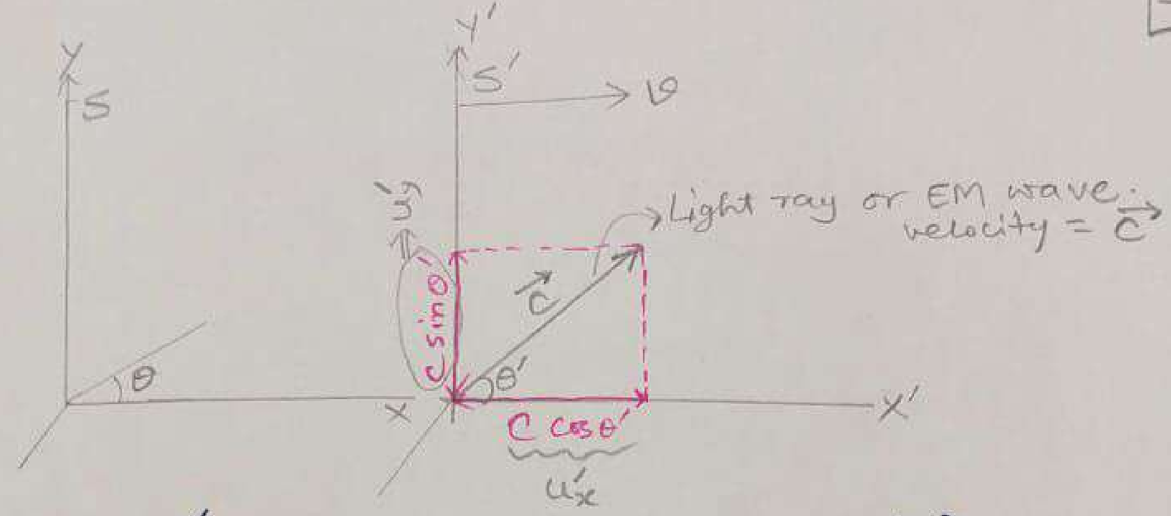


$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}}$$ (velocity addition rule).

$$\Rightarrow u_x = \frac{c\cos\theta' + v}{1 + \frac{c\cos\theta' v}{c^2}} = \frac{c\cos\theta' + v}{1 + \frac{v\cos\theta'}{c}}$$

And $$u_y = \frac{u_y' \sqrt{1-\beta^2}}{1 + \frac{u_x' v}{c^2}}$$ (velocity addition rule).

$$\Rightarrow u_y = \frac{c\sin\theta' \sqrt{1-\beta^2}}{1 + \frac{c\cos\theta' v}{c^2}} = \frac{c\sin\theta' \sqrt{1-\beta^2}}{1 + \frac{v\cos\theta'}{c}}$$

Now, $$\tan\theta = \frac{c\sin\theta}{c\cos\theta} = \frac{u_y}{u_x}$$

$$\Rightarrow \tan\theta = \frac{c\sin\theta' \sqrt{1-\beta^2}}{\cancel{\left(1 + \frac{v\cos\theta'}{c}\right)}} \cdot \frac{\cancel{\left(1 + v\cos\theta'/c\right)}}{c\cos\theta' + v}$$

$$\Rightarrow \tan\theta = \frac{c\sin\theta' \sqrt{1-\beta^2}}{c\cos\theta' + v}$$

$$\Rightarrow \boxed{\tan\theta = \frac{\sin\theta' \sqrt{1-\beta^2}}{\cos\theta' + \beta}}$$ Proved

$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \bar{\nabla}\right)$

$A_\mu = \left(\phi/c, -\vec{A}\right)$

$\vec{E} = -\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t}$

so, $E_y = -\frac{\partial}{\partial y}\phi - \frac{\partial A_y}{\partial t}$

$\Rightarrow E_y = -\left(\frac{\partial \phi}{\partial y} + \frac{\partial A_y}{\partial t}\right)$

$\vec{B} = \bar{\nabla}\times A$

$B_z = (\bar{\nabla}\times\bar{A})_z$

$\Rightarrow \boxed{B_z = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x}$

5 $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\Rightarrow F_{02} = \partial_0 A_2 - \partial_2 A_0$

$\Rightarrow F_{02} = -\frac{1}{c}\frac{\partial}{\partial t}A_y - \frac{\partial}{\partial y}\frac{\phi}{c}$

$\Rightarrow F_{02} = -\frac{1}{c}\left(\frac{\partial A_y}{\partial t} + \frac{\partial \phi}{\partial y}\right)$

$\Rightarrow F_{02} = -\frac{1}{c}\cdot(-E_y)$

$\Rightarrow \boxed{F_{02} = E_y/c}$

and $F_{12} = \partial_1 A_2 - \partial_2 A_1$

$\Rightarrow F_{12} = -\frac{\partial}{\partial x}(A_y) + \frac{\partial}{\partial y}(A_x)$

$\Rightarrow F_{12} = \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)$

$\Rightarrow F_{12} = -\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$

$\Rightarrow \boxed{F_{12} = -B_z}$

6. $F'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}$

Take $\mu = 1$, $\nu = 3$

$F'^{13} = B'_y = \Lambda^1_\rho \Lambda^3_\sigma F^{\rho\sigma}$

$= \Lambda^1_1 \Lambda^3_3 F^{13} + \Lambda^1_0 \Lambda^3_3 F^{03}$

$= (\gamma)(1)B_y + (-\gamma\beta)(1)(-E_z/c)$

$\Rightarrow B'_y = \gamma B_y + \gamma\frac{v}{c^2}E_z$

$\Rightarrow \boxed{B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)}$ Proved

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\boxed{7}$ $\partial_\mu \tilde{F}^{\mu\nu} = 0$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$$

First, take $\mu = i$ and $\nu = 0$ $(i = x, y, z$ or $1, 2, 3)$

$\partial_i \tilde{F}^{i0} = 0$

$\Rightarrow \frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y + \frac{\partial}{\partial z} B_z = 0$

$\Rightarrow \boxed{\bar{\nabla} \cdot \bar{B} = 0}$ $\Rightarrow$ This is one of the 4 Maxwell's Equ.

Then take $\mu = \mu$ and $\nu = i$ $(i = 1, 2, 3,$ or $x, y, z)$.

so, $\partial_\mu \tilde{F}^{\mu i} = 0$

$\Rightarrow \partial_0 \tilde{F}^{0i} + \partial_1 \tilde{F}^{1i} + \partial_2 \tilde{F}^{2i} + \partial_3 \tilde{F}^{3i} = 0$

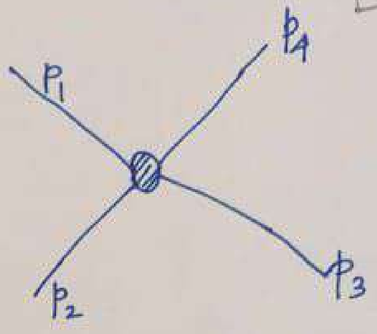
$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}(\vec{B}) + \frac{\partial}{\partial x}\left(\frac{E_z}{c} - \frac{E_y}{c}\right) + \frac{\partial}{\partial y}\left(-\frac{E_z}{c} + \frac{E_x}{c}\right) + \frac{\partial}{\partial z}\left(\frac{E_y}{c} - \frac{E_x}{c}\right) = 0$$

$$\Rightarrow \frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \frac{1}{c}\left[\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right) + \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)\right] = 0$$

$$\Rightarrow \frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \frac{1}{c}\left[(\bar{\nabla} \times \bar{E})_y + (\nabla \times E)_x + (\bar{\nabla} \times \bar{E})_z\right] = 0$$

$$\Rightarrow \frac{\partial \bar{B}}{\partial t} + \bar{\nabla} \times \bar{E} = 0$$

$\Rightarrow \boxed{\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}}$ $\Rightarrow$ This is another Maxwell's Equ.



8. (a) $1+2 \rightarrow 3+4$

We assume $c=1$.

$$s+t+u = (p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2$$
$$= p_1^2 + p_2^2 + 2p_1p_2$$
$$+ p_1^2 + p_3^2 - 2p_1p_3$$
$$+ p_1^2 + p_4^2 - 2p_1p_4$$

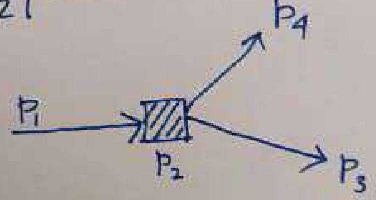
From 4-momentum conservation we know

$$p_1 + p_2 = p_3 + p_4$$
$$\Rightarrow p_2 - p_3 - p_4 = -p_1$$

$p_1^2 = p_1^\mu p_{1\mu} = m_1^2$ same for p_2, p_3, p_4

$$= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1(p_2 - p_3 - p_4)$$
$$= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1(-p_1)$$
$$= p_1^2 + p_2^2 + p_3^2 + p_4^2$$
$$= m_1^2 + m_2^2 + m_3^2 + m_4^2 \quad \text{(Proved)}.$$

8(b) If particle (2) at rest then $|\vec{p_2}| = 0$



In this scenario

$$s = (p_1+p_2)^2$$
$$= p_1^2 + p_2^2 + 2p_1p_2$$
$$= m_1^2 + m_2^2 + 2(E_1E_2 - \underbrace{\vec{p_1}\cdot\vec{p_2}}_{0})$$

$p_1 = (E_1, \vec{p_1})$

$p_2 = (E_2, \vec{p_2})$

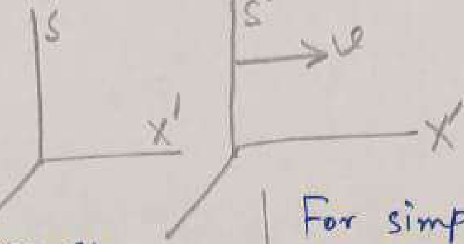
$\vec{p_2}$ is 0

$$\Rightarrow s = m_1^2 + m_2^2 + 2E_1E_2$$

Since (2) is at rest its energy (E_2) is due to its rest mass; so. $E_2 = m_2$

$$\Rightarrow \boxed{s = m_1^2 + m_2^2 + 2E_1 m_2} \text{ Proved.}$$

9. If A and B are 4-vectors then their components transform according to Lorentz transformation.



For simplicity, assume that S' moves along common $x-x'$ axis with velocity v.
In this case,

$$ct = x_0 = \gamma(ct' + \beta x')$$
$$x = x_1 = \gamma(x' + \beta ct')$$
$$y = x_2 = y'$$
$$z = x_3 = z'$$

So, A, B transform as,

$$A_0 = \gamma(A_0' + \beta A_1')$$
$$A_1 = \gamma(A_1' + \beta A_0')$$
$$A_2 = A_2'$$
$$A_3 = A_3'$$

Similarly,
$$B_0 = \gamma(B_0' + \beta B_1')$$
$$B_1 = \gamma(B_1' + \beta B_0')$$
$$B_2 = B_2'$$
$$B_3 = B_3'$$

So, $A \cdot B = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3$

$$\Rightarrow A \cdot B = \gamma(A_0' + \beta A_1')\gamma(B_0' + \beta B_1') - \gamma(A_1' + \beta A_0')\gamma(B_1' + \beta B_0') - A_2' B_2' - A_3' B_3'$$

$$\Rightarrow A \cdot B = \gamma^2\left(A_0' B_0' + \cancel{\beta A_0' B_1'} + \cancel{\beta A_1' B_0'} + \beta^2 A_1' B_1'\right) - \gamma^2\left(A_1' B_1' + \cancel{\beta A_1' B_0'} + \cancel{\beta A_0' B_1'} + \beta^2 A_0' B_0'\right) - A_2' B_2' - A_3' B_3'$$

$$\Rightarrow A \cdot B = \gamma^2 A_0' B_0' + \gamma^2 \beta^2 A_1' B_1' - \gamma^2 A_1' B_1' - \gamma^2 \beta^2 A_0' B_0' - A_2' B_2' - A_3' B_3'$$

$$\Rightarrow A \cdot B = A_0' B_0' \gamma^2 (1 - \beta^2) - A_1' B_1' \gamma^2 (1 - \beta^2) - A_2' B_2' - A_3' B_3'$$

$$\Rightarrow A \cdot B = A_0' B_0' - A_1' B_1' - A_2' B_2' - A_3' B_3'$$

$$\Rightarrow A \cdot B = A' \cdot B' \quad \text{(Proved)}$$

$$\gamma^2 = \frac{1}{1-\beta^2}$$
$$\Rightarrow \gamma^2(1-\beta^2) = 1$$

8

10 (a) $(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2$

Time like : $(\Delta s)^2 > 0$

$\Rightarrow c^2(\Delta t)^2 > (\Delta x)^2$

$\Rightarrow \frac{\Delta x}{\Delta t} < c$

Space-like : $(\Delta s)^2 < 0$

$\Rightarrow c^2(\Delta t)^2 < (\Delta x)^2$

$\Rightarrow \frac{\Delta x}{\Delta t} > c$

Light like : $(\Delta s)^2 = 0$

$\Rightarrow c^2(\Delta t)^2 = (\Delta x)^2$

$\Rightarrow \frac{\Delta x}{\Delta t} = c$

10 (b) we know, $\Delta x' = \gamma(\Delta x - v\Delta t)$ — (1)

For time like, $\frac{\Delta x}{\Delta t} < c$ so lets assume $v = \frac{\Delta x}{\Delta t}$.

From (1) $\Delta x' = \gamma\left(\Delta x - \frac{\Delta x}{\Delta t}\cdot\Delta t\right) = 0$

i.e. if 2 events are time-like separated it is possible to find a frame S' in which the 2 events happen at the same place.

we know, $\Delta t' = \gamma\left(\Delta t - \frac{\boxed{v}}{c^2}\Delta x\right)$

$\Rightarrow \Delta t' = \gamma\left(\Delta t - \frac{1}{c^2}\cdot \boxed{c^2\frac{\Delta t}{\Delta x}}\Delta x\right)$

$\Rightarrow \Delta t' = \gamma \times 0 = 0$

If 2 events are space-like separated it is possible to find a frame S' where the 2 events happen at the same time.

Light like is a limiting case.

For space like

$\frac{\Delta x}{\Delta t} > c$

$\Rightarrow \frac{\Delta t}{\Delta x} < \frac{1}{c}$

$\Rightarrow c^2\frac{\Delta t}{\Delta x} < c$

Lets assume

$c^2\frac{\Delta t}{\Delta x} = v$

10(b) can also be answered using Space time diagrams.