9.8 Problems

Section 9.1: Preliminaries concerning rotations

9.1. **Many different ω's** $*$

Consider a particle at the point $(a, 0, 0)$, with velocity $(0, v, 0)$. At this instant, the particle may be considered to be rotating around many different $\boldsymbol{\omega}$ vectors passing through the origin. There isn't just one "correct" $\boldsymbol{\omega}$. Find all the possible $\boldsymbol{\omega}$'s (give their directions and magnitudes).

9.2. **Fixed points on a sphere** $**$

Consider a transformation of a rigid sphere into itself. Show that two points on the sphere end up where they started.

9.3. **Rolling cone** $**$

A cone rolls without slipping on a table. The half-angle at the vertex is α, and the axis has length h (see Fig. 9.40). Let the speed of the center of the base, point P in the figure, be v. What is the angular velocity of the cone with respect to the lab frame at the instant shown? There are many ways to do this problem, so you are encouraged to take a look at the three given solutions, even after solving it.

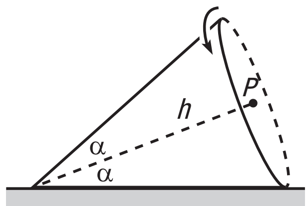

Fig. 9.40

Section 9.2: The inertia tensor

9.4. **Parallel-axis theorem**

Let (X, Y, Z) be the position of an object's CM, and let (x', y', z') be the position relative to the CM. Prove the parallel-axis theorem, Eq. (9.19), by setting $x = X + x'$, $y = Y + y'$, and $z = Z + z'$ in Eq. (9.8).

Section 9.3: Principal axes

9.5. **A nice cylinder** $*$

What must the ratio of height to radius of a cylinder be so that every axis is a principal axis (with the CM as the origin)?

9.6. **Rotating square** $*$

Here's an exercise in geometry. Theorem 9.5 says that if the moments of inertia around two principal axes are equal, then any axis in the plane of these axes is a principal axis. This means that the object will happily rotate around any axis in this plane, that is, no torque is needed. Demonstrate this explicitly for four equal masses in the shape of a square, with the center as the origin, which obviously has two moments equal. Assume that the masses are connected by strings to the axis, as shown in Fig. 9.41, and that they all rotate with the same ω around the axis, so that they remain in the shape of a square. Your task is to show that the tensions in the strings

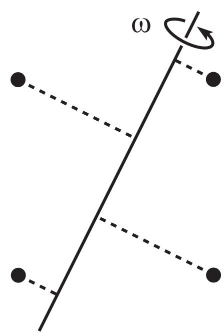

Fig. 9.41

are such that there is no net torque acting on the axis, relative to the center of the square.

9.7. **Existence of principal axes for a pancake** $*$

Given a pancake object in the x-y plane, show that there exist principal axes by considering what happens to the integral $\int xy$ when the coordinate axes are rotated about the origin by an angle of $\pi/2$.

9.8. **Symmetries and principal axes for a pancake** $**$

A rotation of the axes in the x-y plane through an angle θ transforms the coordinates according to (you can accept this)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{9.94}$$

Use this to show that if a pancake object in the x-y plane has a symmetry under a rotation through $\theta \neq \pi$, then $\int xy = 0$ for any choice of axes, which implies that all axes (through the origin) in the plane are principal axes.

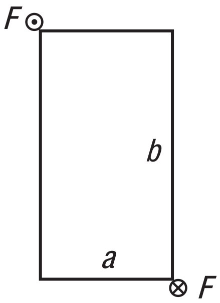

Fig. 9.42

Section 9.4: Two basic types of problems

9.9. **Striking a rectangle** $*$

A flat uniform rectangle with sides of length a and b sits in space, not rotating. You strike the corners at the ends of one diagonal, with equal and opposite forces (see Fig. 9.42). Show that the resulting initial $\boldsymbol{\omega}$ points along the other diagonal.

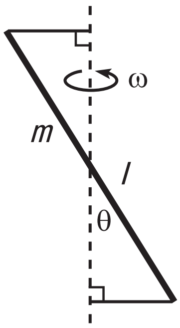

Fig. 9.43

9.10. **Rotating stick** $**$

A stick of mass m and length ℓ spins with frequency ω around an axis, as shown in Fig. 9.43. The stick makes an angle θ with the axis and is kept in its motion by two strings that are perpendicular to the axis. What is the tension in the strings? (Ignore gravity.)

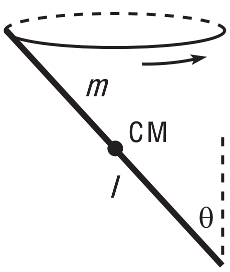

Fig. 9.44

9.11. **Stick under a ring** $**$

A stick of mass m and length ℓ is arranged to have its CM motionless while its top end slides in a circle on a frictionless ring, as shown in Fig. 9.44. The stick makes an angle θ with the vertical. What is the frequency of this motion?

9.12. **Circular pendulum** $**$

Consider a pendulum made of a massless rod of length ℓ with a point mass m on the end. Assume conditions have been set up so that the mass moves in a horizontal circle. Let θ be the constant angle the rod makes with the vertical. Find the frequency, Ω, of this circular motion in three different ways.

(a) Use $\mathbf{F} = m\mathbf{a}$. This method works only if you have a point mass. With an extended object, you have to use one of the following methods involving torque.
(b) Use $\boldsymbol{\tau} = d\mathbf{L}/dt$ with the pendulum pivot as the origin.
(c) Use $\boldsymbol{\tau} = d\mathbf{L}/dt$ with the mass as the origin.

9.13. **Rolling in a cone** **

A fixed cone stands on its tip, with its axis in the vertical direction. The half-angle at the vertex is θ. A small ring of radius r rolls without slipping on the inside surface. Assume that conditions have been set up so that (1) the point of contact between the ring and the cone moves in a circle at height h above the tip, and (2) the plane of the ring is at all times perpendicular to the line joining the point of contact and the tip of the cone (see Fig. 9.45). What is the frequency, Ω, of this circular motion? Work in the approximation where r is much smaller than the radius of the circular motion, $h\tan\theta$.

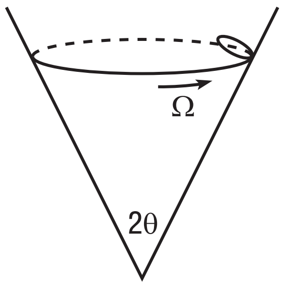

Fig. 9.45

Section 9.5: Euler's equations

9.14. **Tennis racket theorem** ***

If you try to spin a tennis racket (or a book, etc.) around any of its three principal axes, you will find that different things happen with the different axes. Assuming that the principal moments (relative to the CM) are labeled according to $I_1 > I_2 > I_3$ (see Fig. 9.46), you will find that the racket will spin nicely around the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$ axes, but it will wobble in a rather messy manner around the $\hat{\mathbf{x}}_2$ axis. Verify this claim experimentally with a book (preferably lightweight, and wrapped with a rubber band), or a tennis racket, if you happen to study with one on hand.

Now verify this claim mathematically. The main point here is that you can't start the motion off with $\boldsymbol{\omega}$ pointing *exactly* along a principal axis. Therefore, what you want to show is that the motion around the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$ axes is *stable* (that is, small errors in the initial conditions remain small), whereas the motion around the $\hat{\mathbf{x}}_2$ axis is *unstable* (that is, small errors in the initial conditions get larger and larger, until the motion eventually doesn't resemble a rotation around the $\hat{\mathbf{x}}_2$ axis).[24] Your task is to use Euler's equations to prove these statements about stability. (Exercise 9.33 gives another derivation of this result.)

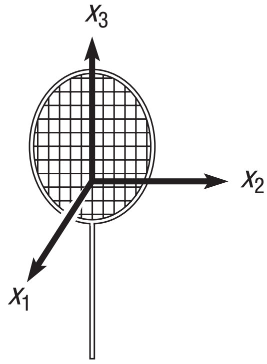

Fig. 9.46

[24] If you try for a long enough time, you'll probably be able to get the initial $\boldsymbol{\omega}$ pointing close enough to $\hat{\mathbf{x}}_2$ so that the book will remain rotating (almost) around $\hat{\mathbf{x}}_2$ for the entire time of its flight. There is, however, undoubtedly a better use for your time, as well as for the book…

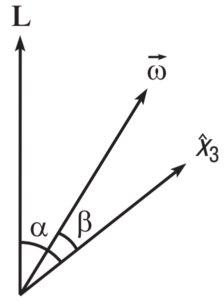

Fig. 9.47

Section 9.6: Free symmetric top

9.15. Free-top angles *

In Section 9.6.2, we showed that for a free symmetric top, the angular momentum $\mathbf{L}$, the angular velocity $\boldsymbol{\omega}$, and the symmetry axis $\hat{\mathbf{x}}_3$ all lie in a plane. Let α be the angle between $\hat{\mathbf{x}}_3$ and $\mathbf{L}$, and let β be the angle between $\hat{\mathbf{x}}_3$ and $\boldsymbol{\omega}$ (see Fig. 9.47). Find the relationship between α and β in terms of the principal moments, I and I_3.

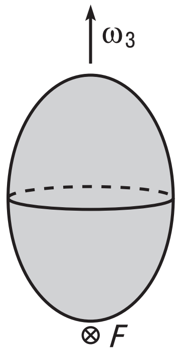

Fig. 9.48

9.16. Staying above **

A top with $I = nI_3$, where n is a numerical factor, is initially spinning around its x_3 axis with angular speed ω_3. You apply a strike at the bottom point, directed into the page as shown in Fig. 9.48 (imagine that you hit a little peg protruding from the bottom). What is the largest value of n for which the total $\boldsymbol{\omega}$ vector never dips below the horizontal axis in the subsequent motion, no matter how hard your strike is?

Section 9.7: Heavy symmetric top

9.17. The top **

This problem deals with the spinning top example in Section 9.7.5. It uses the result for Ω in Eq. (9.82).

(a) What is the minimum value of ω_3 for which circular precession is possible?

(b) Find approximate expressions for $\Omega_\pm$ when ω_3 is very large. The phrase "very large," however, is rather meaningless. What mathematical statement should replace it?

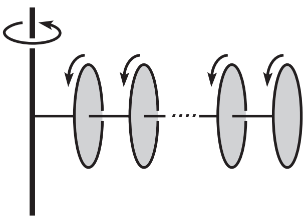

Fig. 9.49

9.18. Many tops **

N identical disks and massless sticks are arranged as shown in Fig. 9.49. Each disk is glued to the stick on its left and attached by a pivot to the stick on its right. The leftmost stick is attached by a pivot to a pole. You wish to set up circular precession with the sticks always forming a straight horizontal line. What should the relative angular speeds of the disks be so that this is possible?

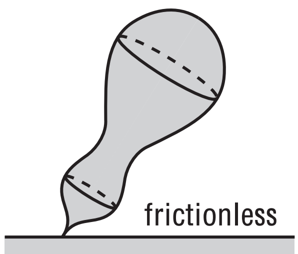

Fig. 9.50

9.19. Heavy top on a slippery table **

Solve the problem of a heavy symmetric top spinning on a frictionless table (see Fig. 9.50). You may do this by simply stating what modifications are needed in the derivation in Section 9.7.3 (or Section 9.7.4).

9.20. **Fixed highest point** $**$

Consider a top made of a uniform disk of radius R, connected to the origin by a massless stick (which is glued perpendicular to the disk) of length ℓ. Paint a dot on the top at its highest point, and label this as point P (see Fig. 9.51). You wish to set up uniform circular precession, with the stick making a constant angle θ with the vertical (θ can be chosen to be any angle between zero and π), and with P always being the highest point on the top. What is the frequency of precession, Ω? What relation between R and ℓ must be satisfied for this motion to be possible?

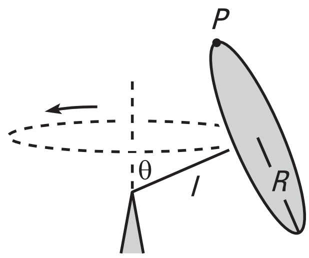

Fig. 9.51

9.21. **Basketball on a rim** $***$

A basketball rolls without slipping around a basketball rim in such a way that the contact points trace out a great circle on the ball, and the CM moves around in a horizontal circle with frequency Ω. The radii of the ball and rim are r and R, respectively, and the ball's radius to the contact point makes an angle θ with the horizontal (see Fig. 9.52). Assume that the ball's moment of inertia around its center is $I = (2/3)mr^2$. Find Ω.

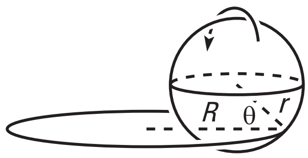

Fig. 9.52

9.22. **Rolling lollipop** $***$

Consider a lollipop made of a solid sphere of mass m and radius r that is radially pierced by a massless stick. The free end of the stick is pivoted on the ground (see Fig. 9.53). The sphere rolls on the ground without slipping, with its center moving in a circle of radius R with frequency Ω. What is the normal force between the ground and the sphere?

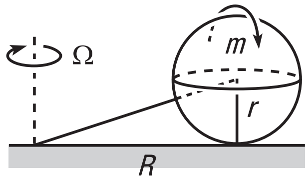

Fig. 9.53

9.23. **Rolling coin** $***$

Initial conditions have been set up so that a coin of radius r rolls around in a circle, as shown in Fig. 9.54. The contact point on the ground traces out a circle of radius R, and the coin makes a constant angle θ with the horizontal. The coin rolls without slipping (assume that the friction with the ground is as large as needed). What is the frequency, Ω, of the circular motion of the contact point on the ground? Show that such motion exists only if $R > (5/6)r\cos\theta$.

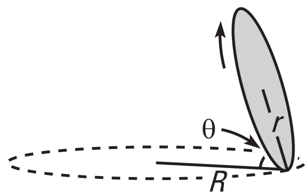

Fig. 9.54

9.24. **Wobbling coin** $***$

If you spin a coin around a vertical diameter on a table, it will slowly lose energy and begin a wobbling motion. The angle between the coin and the table will gradually decrease, and eventually it will come to rest. Assume that this process is slow, and consider the motion when the coin makes an angle θ with the table (see Fig. 9.55). You may assume that the CM is essentially motionless. Let R be the radius of the coin, and let Ω be the frequency at which the contact point on the table traces out its circle. Assume that the coin rolls without slipping.

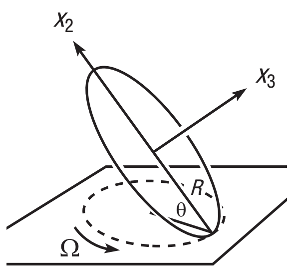

Fig. 9.55

(a) Show that the angular velocity of the coin is $\boldsymbol{\omega} = \Omega \sin\theta\, \hat{\mathbf{x}}_2$, where $\hat{\mathbf{x}}_2$ points upward along the coin, directly away from the contact point.

(b) Show that

$$\Omega = 2\sqrt{\frac{g}{R\sin\theta}}\,. \tag{9.95}$$

(c) Show that Abe (or Tom, Franklin, George, John, Dwight, Susan, or Sacagawea) appears to rotate, when viewed from above, with frequency

$$2(1-\cos\theta)\sqrt{\frac{g}{R\sin\theta}}\,. \tag{9.96}$$

9.25. **Nutation cusps** $**$

(a) Using the notation and initial conditions in the example in Section 9.7.7, prove that kinks occur in nutation if and only if $\Delta\Omega = \pm\Omega_s$. A kink is where the plot of $\theta(t)$ vs. $\phi(t)$ has a discontinuity in its slope.

(b) Prove that these kinks are in fact cusps. A cusp is a kink where the plot reverses direction in the ϕ-θ plane.

9.26. **Nutation circles** $**$

(a) Using the notation and initial conditions in the example in Section 9.7.7, and assuming that $\omega_3 \gg \Delta\Omega \gg \Omega_s$, find (approximately) the direction of the angular momentum right after the sideways kick takes place.

(b) Use Eq. (9.91) to show that the CM then travels (approximately) in a circle around $\mathbf{L}$. And show that this "circular" motion is just what you would expect from the free-top reasoning in Section 9.6.2, in particular, Eq. (9.55).

Additional problems

9.27. **Rolling without slipping** $*$

The standard way that a ball rolls without slipping on a flat surface is for the contact points on the ball to trace out a vertical great circle on the ball. Are there any other ways that a ball can roll without slipping?

9.28. **Rolling straight?** $**$

In some situations, such as the rolling-coin setup in Problem 9.23, the velocity of the CM of a rolling object changes direction as time goes by. Consider a uniform sphere that rolls on the ground without slipping (possibly in the nonstandard way described in the solution to Problem 9.27).

Is it possible for the CM's velocity to change direction? Justify your answer rigorously.

9.29. **Ball on paper** ***

A uniform ball rolls without slipping on a table (possibly in the nonstandard way described in the solution to Problem 9.27). It rolls onto a piece of paper, which you then slide around in an arbitrary (horizontal) manner. You may even give the paper abrupt, jerky motions, so that the ball slips with respect to it. After you allow the ball to come off the paper, it will eventually resume rolling without slipping on the table. Show that the final velocity equals the initial velocity.

9.30. **Ball on a turntable** ****

A uniform ball rolls without slipping on a turntable (possibly in the nonstandard way described in the solution to Problem 9.27). As viewed from the inertial lab frame, show that the ball moves in a circle (not necessarily centered at the center of the turntable) with a frequency equal to $2/7$ times the frequency of the turntable.

9.9 Exercises

Section 9.1: Preliminaries concerning rotations

9.31. **Rolling wheel** **

A wheel with spokes rolls without slipping on the ground. A stationary camera takes a picture of it as it rolls by, from the side. Due to the nonzero exposure time of the camera, the spokes generally appear blurred. At what locations in the picture do the spokes *not* appear blurred? *Hint*: A common incorrect answer is that there is only one point.

Section 9.2: The inertia tensor

9.32. **Inertia tensor** *

Calculate the $\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$ double cross product in Eq. (9.7) by using the vector identity,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{9.97}$$

Section 9.3: Principal axes

9.33. **Tennis racket theorem** **

Problem 9.14 gives the statement of the "tennis racket theorem," and the solution there involves Euler's equations. Demonstrate the theorem here by writing down the conservation of L^2 and conservation of E statements and then using them in the following way. Produce an equation that says that if ω_2 and ω_3 (or ω_1 and ω_2) start small, then they must remain small. And produce the analogous equation that says that if ω_1 and ω_3 start small, then they need *not* remain small. (It's another matter to show that they actually *won't* remain small. But let's not worry about that here. Anything that *can* happen generally *does* happen in physics.)

9.34. **Moments for a cube** **

In the spirit of Appendix E, calculate the principal moments for a solid cube of mass m and side length ℓ, with the coordinate axes parallel to the edges of the cube, and the origin at a corner.

9.35. **Tilted moments** **

(a) Consider a planar object in the x-y plane. If the x and y axes are principal axes, use the rotation matrix in Eq. (9.94) to show that the moment of inertia around the x' axis, which makes an angle θ with the x axis, is $I_{x'} = I_x \cos^2\theta + I_y \sin^2\theta$.

(b) Consider a general three-dimensional object whose principal axes are the x, y, and z axes. Consider another axis that points along the unit vector (α, β, γ). Show that the moment of inertia around this axis is $\alpha^2 I_x + \beta^2 I_y + \gamma^2 I_z$. *Hint*: The cross product, discussed in Appendix B, provides a nice method of calculating the distance from a point to a line.

9.36. **Quadrupole** **

Consider an arbitrarily shaped body of mass m whose CM is at the origin. Using the law of cosines, the gravitational potential of a mass M at position $\mathbf{R}$ is

$$V(\mathbf{R}) = -\int \frac{GM\,dm}{\sqrt{R^2 + r^2 - 2Rr\cos\beta}}, \tag{9.98}$$

where the integration runs over the volume of the body, and β is the angle that the position vector $\mathbf{r}$ of an arbitrary point in the body makes with the vector $\mathbf{R}$.

(a) Assuming that all points in the body satisfy $r \ll R$, show that an approximate expression for the potential is

$$V(\mathbf{R}) \approx -\frac{GMm}{R} - \frac{GM}{2R^3}\int r^2(3\cos^2\beta - 1)\,dm, \tag{9.99}$$

and then show that this can be written as

$$V(\mathbf{R}) \approx -\frac{GMm}{R} - \frac{GM}{2R^3}(I_1 + I_2 + I_3 - 3I_R), \tag{9.100}$$

where I_1, I_2, and I_3 are the moments around any three orthogonal axes (which we'll take to be principal axes in part (b)), and I_R is the moment around the axis along the $\mathbf{R}$ vector.

(b) Consider now a planet with rotational symmetry around $\hat{\mathbf{x}}_3$, such as the earth which bulges at the equator due to the spinning. Using the result of Exercise 9.35, show that the potential in Eq. (9.100) can be written as

$$V(\mathbf{R}) \approx -\frac{GMm}{R} - \frac{GM}{2R^3}(I_3 - I)(1 - 3\cos^2\theta), \qquad (9.101)$$

where $I \equiv I_1 = I_2$, and θ is the angle that $\mathbf{R}$ makes with $\hat{\mathbf{x}}_3$.

REMARK: The second term here is known as the *quadrupole* term. In electrostatics, a *dipole* consists of equal and opposite charges separated by some distance d. At a given point far away, the forces from these two charges partially cancel. But they don't exactly cancel, because the electrostatic force (which behaves like the gravitational force, with a $1/r^2$ law) depends on the distance and direction to the charges, and the two charges are located at different points. If two dipoles are oriented in opposite directions and then placed side by side, a distance d apart (so that there are charges of q and $-q$ alternating around the corners of a square) then the forces from the dipoles nearly cancel. But again, the cancellation isn't exact, because the dipoles are located in different places. This distribution of charge is called a *quadrupole*, and it is similar to the situation with a spinning (and bulging) planet, because such a planet consists of a spherical ball (which gives rise to the first term in Eq. (9.101)), plus a region of "negative" mass superimposed on the ball at the poles and a region of positive mass superimposed on the ball at the equator. Looking at the above square of charges from far out along the diagonal containing the two negative charges is similar to looking at the earth from far out along its rotation axis. ♣

Section 9.4: Two basic types of problems

9.37. **Sphere and points** *

A uniform sphere of mass m and radius R rotates around the vertical axis with angular speed ω. Two particles of mass $m/2$ are brought close to the sphere at diametrically opposite points, at an angle θ from the vertical, as shown in Fig. 9.56. The masses, which are initially essentially at rest, abruptly stick to the sphere. What angle does the resulting $\boldsymbol{\omega}$ make with the vertical? (If you want, you can check your answer by showing that the θ that makes this angle maximum is $\sin^{-1}\sqrt{7/9} \approx 61.9°$.)

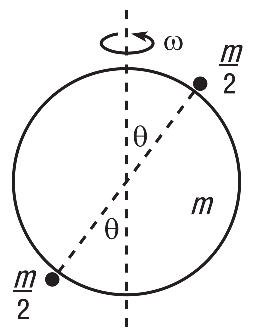

Fig. 9.56

9.38. **Striking a triangle** **

Consider the rigid object in Fig. 9.57. Four masses lie at the points shown on a rigid isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $3m$, and the other three masses are m. Label them A, B, C, D, as shown. Assume that the object is floating freely in outer space. Mass C is struck with a quick blow, directed into the page. Let the impulse have magnitude $\int F\,dt = P$. What are the velocities of all the masses immediately after the blow?

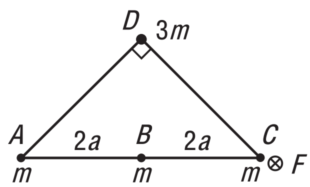

Fig. 9.57

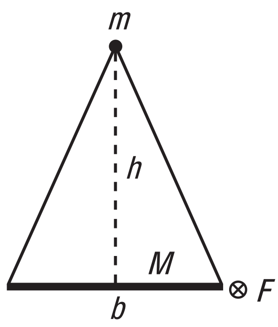

Fig. 9.58

9.39. **Striking another triangle** **

Consider the rigid object in Fig. 9.58. A uniform stick of mass M lies along the base of an isosceles triangle, and a mass m lies at the opposite vertex. The base has length b, and the height is h. Assume that the object is floating freely in outer space. The right end of the stick is struck with a quick blow, directed into the page. Let the impulse have magnitude $\int F\,dt = P$. What is the velocity of mass m immediately after the blow?

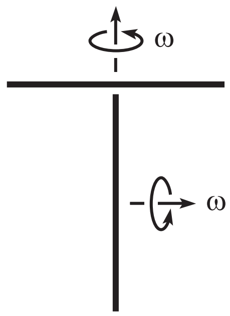

Fig. 9.59

9.40. **Sticking sticks** **

Two identical uniform sticks spin around their stationary centers with equal angular speeds, as shown in Fig. 9.59. The bottom stick is slowly raised until its top end collides with the center of the top stick. The sticks stick together to form a rigid "T." Assume that the collision takes place when the top stick lies in the plane of the paper. Immediately after the collision, one point (in addition to the CM) on the T will instantaneously be at rest. Where is this point?

9.41. **Circling stick again** **

Solve the problem in Section 9.4.2 again, but now use the CM as the origin.

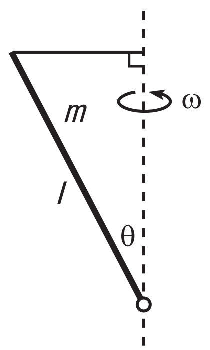

Fig. 9.60

9.42. **Pivot and string** **

A stick of mass m and length ℓ spins with frequency ω around an axis, as shown in Fig. 9.60. The stick makes an angle θ with the axis. One end is pivoted on the axis, and the other end is connected to the axis by a string that is perpendicular to the axis. What is the tension in the string, and what is the force that the pivot applies to the stick? (Ignore gravity.)

9.43. **Rotating sheet** **

A uniform flat rectangular sheet of mass m and side lengths a and b rotates with angular speed ω around a diagonal. What torque is required? Given a fixed area A, what should the rectangle look like if you want the required torque to be as large as possible? What is the upper bound on the torque?

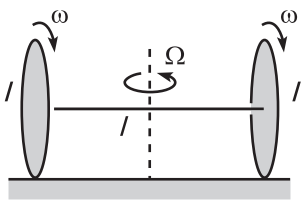

Fig. 9.61

9.44. **Rotating axle** **

Two wheels of mass m and moment of inertia I are connected by a massless axle of length ℓ, as shown in Fig. 9.61. The system rests on a frictionless surface, and the wheels rotate with frequency ω around the axle. Additionally, the whole system rotates with frequency Ω around the vertical axis through the center of the axle. What is the largest value of Ω for which both wheels stay on the ground?

9.45. **Stick on a ring** **

(a) A stick of mass m and length $2r$ is arranged to make a constant angle θ with the horizontal, with its bottom end sliding in a circle on a frictionless ring of radius r, as shown in Fig. 9.62. What is the frequency of this motion? It turns out that there is a minimum θ for which this motion is possible; what is it?

(b) If the radius of the ring is now R, what is the largest value of r/R for which this motion is possible for $\theta \to 0$?[25]

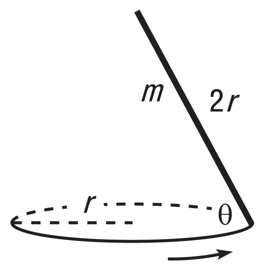

Fig. 9.62

Section 9.6: Free symmetric top

9.46. **Slightly wobbling** *

A coin of mass m and radius R is initially spinning around the axis perpendicular to its plane, with angular speed ω_3. It is supported by a pivot at its center. You apply an infinitesimal downward strike at a point on the rim, as shown in Fig. 9.63, giving the coin an infinitesimal angular velocity component $\omega_\perp$ in the plane of the coin. When the plane of the coin returns to its original plane for the first time, what (approximately) is the orientation of the coin?

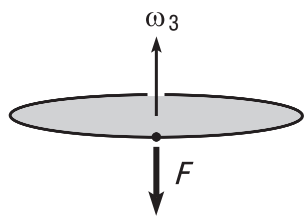

Fig. 9.63

9.47. **Original orientation** **

A coin of mass m and radius R is initially spinning around the axis perpendicular to its plane, with angular speed ω_3. It is supported by a pivot at its center. You apply a (nonzero) downward strike at a point on the rim, as shown in Fig. 9.63, giving the coin an angular velocity component $\omega_\perp$ in the plane of the coin. Consider the nth time the plane of the coin returns to its original plane. What is the minimum value of n for which it is possible for the coin to have *exactly* the same orientation as when it started? What should $\omega_\perp$ be in terms of ω_3 to achieve this?

9.48. **Seeing tails** **

A coin (floating in outer space) of mass m and radius R is initially spinning around the axis perpendicular to its plane, with angular speed ω_3. You view the coin directly from above, and you apply a downward strike at a point on the rim, as shown in Fig. 9.63. What is the minimum impulse, $\int F\,dt$, you must apply in order to be able to barely see the underside of the coin at some later time in its wobbling motion? Assuming that you apply this minimum impulse, how far will the center of the coin have moved by the time you are able to see the underside?

[25] You can also play around with both parts of this problem for the setup where the stick swings around below the ring, with its top end running along the ring.

9.49. Flipping a coin **

Imagine flipping an initially horizontal heads-up coin. If the initial angular velocity is horizontal, then the coin will rotate around this horizontal diameter the entire time in the air. The fraction of the flight time that the coin spends heads-up is therefore 1/2. In practice, however, it is impossible to make the initial $\boldsymbol{\omega}$ be *exactly* horizontal, so assume that the initial components are $\omega_\perp$ and ω_3, with $\omega_3 \ll \omega_\perp$. Show that in this limit, the fraction of the flight time that the coin spends heads-up is $1/2 + (4\omega_3^2)/(\pi\omega_\perp^2)$.[26]

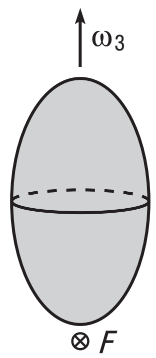

Fig. 9.64

9.50. Dipping low **

A top with $I = 3I_3$ floats in outer space and initially spins around its x_3 axis with angular speed ω_3. You apply a strike at the bottom point, directed into the page, as shown in Fig. 9.64, producing an angular velocity component, $\omega_\perp$, directed to the right. What should $\omega_\perp$ be in terms of ω_3 in order to have the total $\boldsymbol{\omega}$ vector dip as far below the horizontal as possible in the subsequent motion?

Section 9.7: Heavy symmetric top

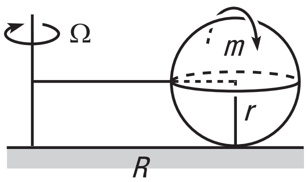

Fig. 9.65

9.51. Rolling lollipop *

Consider a lollipop made of a solid sphere of mass m and radius r that is radially pierced by a massless horizontal stick. The free end of the stick is pivoted on a pole (see Fig. 9.65), and the sphere rolls on the ground without slipping, with its center moving in a circle of radius R with frequency Ω. What is the normal force between the ground and the sphere?

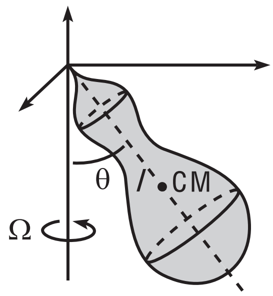

Fig. 9.66

9.52. Horizontal $\boldsymbol{\omega}$ **

A top (with mass m, moments I and I_3, and distance ℓ from the pivot to CM) undergoes uniform precession, with its axis making a constant angle θ with the negative z axis, as shown in Fig. 9.66. If conditions are set up so that the top's $\boldsymbol{\omega}$ is always horizontal, what is the frequency of precession? Can such motion exist if the top is up above the horizontal, making an angle θ with the positive z axis?

[26] If the coin starts truly horizontal (or is at least not biased to tilt in any particular direction on average, which is still a condition that seems difficult to guarantee), and if you catch the coin in your hand to reduce random table-bouncing effects, then the result of this exercise implies that the coin is biased to come up heads. This effect is analyzed in detail in a paper by Persi Diaconis, Susan Holmes, and Richard Montgomery (to be published), who estimate that the probability of obtaining heads for a normally flipped coin is about 0.51.

9.53. Sliding lollipop ***

Consider a lollipop made of a solid sphere of mass m and radius r that is radially pierced by a massless stick. The free end of the stick is pivoted on the ground, which is frictionless (see Fig. 9.67). The sphere *slides* along the ground, with the same point on the sphere always touching the ground. The center moves in a circle of radius R with frequency Ω. Show that the normal force between the ground and the sphere is $N = mg + mr\Omega^2$, which is independent of R. Solve this by:

(a) Using an $\mathbf{F} = m\mathbf{a}$ argument.[27]
(b) Using the more complicated $\boldsymbol{\tau} = d\mathbf{L}/dt$ argument.

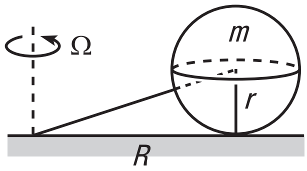

Fig. 9.67

9.54. Rolling wheel and axle ***

A massless axle has one end attached to a wheel (a uniform disk of mass m and radius r), with the other end pivoted on the ground (see Fig. 9.68). The wheel rolls on the ground without slipping, with the axle inclined at an angle θ. The point of contact on the ground traces out a circle with frequency Ω.

(a) Show that $\boldsymbol{\omega}$ points horizontally to the right (at the instant shown), with magnitude $\omega = \Omega/\tan\theta$.
(b) Show that the normal force between the ground and the wheel is

$$N = mg\cos^2\theta + mr\Omega^2\left(\frac{1}{4}\cos\theta\sin^2\theta + \frac{3}{2}\cos^3\theta\right). \quad (9.102)$$

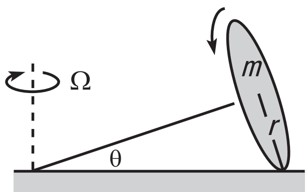

Fig. 9.68

9.55. Ball under a cone ***

A hollow ball (with $I = (2/3)mR^2$) rolls without slipping on the inside surface of a fixed cone, whose tip points upward, as shown in Fig. 9.69. The angle at the vertex of the cone is 60°. Initial conditions have been set up so that the contact point on the cone traces out a horizontal circle of radius ℓ at frequency Ω, while the contact point on the ball traces out a circle of radius $R/2$. Assume that the coefficient of friction between the ball and the cone is sufficiently large to prevent slipping. What is the frequency of precession, Ω? What does it reduce to in the limits $\ell \gg R$ and $\ell \to (\sqrt{3}/2)R$ (the ball has to fit inside the cone, of course). What relation between ℓ and R must be satisfied if the setup is to work with a solid ball with $I = (2/5)mR^2$?

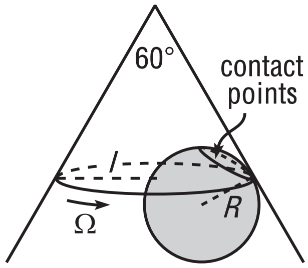

Fig. 9.69

9.56. Ball in a cone ****

A ball (with $I = (2/5)MR^2$) rolls without slipping on the inside surface of a fixed cone, whose tip points downward. The half-angle at the vertex

[27] This method happens to work here, due to the unusually nice nature of the sphere's motion. For more general motion (for example, in Problem 9.22, where the sphere is spinning), you must use $\boldsymbol{\tau} = d\mathbf{L}/dt$.