$$D = \frac{dN}{dV}$$

must also be constant in time, that is,

$$\frac{dD}{dt} = 0,$$

which proves Liouville's theorem. An alternative statement of the theorem follows from Eq. (9.149) as

$$\frac{\partial D}{\partial t} = -[D, H]. \tag{9.150}$$

When the ensemble of systems is in statistical equilibrium, the number of systems in a given state must be constant in time, which is to say that the density of system points at a given spot in phase space does not change with time. The variation of D with time at a fixed point corresponds to the partial derivative with respect to t, which therefore must vanish in statistical equilibrium. By Eq. (9.150), it follows that the equilibrium condition can be expressed as

$$[D, H] = 0.$$

We can ensure equilibrium therefore by choosing the density D to be a function of those constants of the motion of the system not involving time explicitly, for then the Poisson bracket with H must vanish. Thus, for conservative systems D can be any function of the energy, and the equilibrium condition is automatically satisfied. The characteristics of the ensemble will be determined by the choice of function for D. As an example, one well-known ensemble, the *microcanonical* ensemble, occurs if D is constant for systems having a given narrow energy range and zero outside the range.

The considerations have been presented here to illustrate the usefulness of the Poisson bracket formulation in classical statistical mechanics. Further discussion of these points would carry us far outside our field.

DERIVATIONS

1. One of the attempts at combining the two sets of Hamilton's equations into one tries to take q and p as forming a complex quantity. Show directly from Hamilton's equations of motion that for a system of one degree of freedom the transformation

$$Q = q + ip, \qquad P = Q^*$$

is not canonical if the Hamiltonian is left unaltered. Can you find another set of coordinates Q', P' that are related to Q, P by a change of scale only, and that are canonical?

2. Show that the transformation for a system of one degree of freedom,

$$Q = q\cos\alpha - p\sin\alpha,$$
$$P = q\sin\alpha + p\cos\alpha,$$

satisfies the symplectic condition for any value of the parameter α. Find a generating function for the transformation. What is the physical significance of the transformation for $\alpha = 0$? For $\alpha = \pi/2$? Does your generating function work for both of these cases.

3. In Section 8.4 some of the problems of treating time as one of the canonical variables are discussed. If we are able to sidestep these difficulties, show that the equations of transformation in which t is considered a canonical variable reduce to Eqs. (9.14) if in fact the transformation does not affect the time scale.

4. Show directly that the transformation

$$Q = \log\left(\frac{1}{q}\sin p\right), \quad P = q\cot p$$

is canonical.

5. Show directly that for a system of one degree of freedom the transformation

$$Q = \arctan\frac{\alpha q}{p}, \quad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

is canonical, where α is an arbitrary constant of suitable dimensions.

6. The transformation equations between two sets of coordinates are

$$Q = \log(1 + q^{1/2}\cos p),$$
$$P = 2(1 + q^{1/2}\cos p)q^{1/2}\sin p.$$

(a) Show directly from these transformation equations that Q, P are canonical variables if q and p are.

(b) Show that the function that generates this transformation is

$$F_3 = -(e^Q - 1)^2\tan p.$$

7. (a) If each of the four types of generating functions exist for a given canonical transformation, use the Legendre transformation to derive relations between them.

(b) Find a generating function of the F_4 type for the identify transformation and of the F_3 type for the exchange transformation.

(c) For an orthogonal point transformation of q in a system of n degrees of freedom, show that the new momenta are likewise given by the orthogonal transformation of an n-dimensional vector whose components are the old momenta plus a gradient in configuration space.

8. Prove directly that the transformation

$$Q_1 = q_1, \quad P_1 = p_1 - 2p_2,$$
$$Q_2 = p_2, \quad P_2 = -2q_i - q_2$$

is canonical and find a generating function.

9. (**a**) For a single particle show directly (that is, by direct evaluation of the Poisson brackets), that if u is a scalar function only of r^2, p^2, and $\mathbf{r} \cdot \mathbf{p}$, then

$$[u, \mathbf{L}] = 0.$$

(**b**) Similarly show directly that if $\mathbf{F}$ is a vector function,

$$\mathbf{F} = u\mathbf{r} + v\mathbf{p} + w(\mathbf{r} \times \mathbf{p}),$$

where u, v, and w are scalar functions of the same type as in part (a), then

$$[F_i, L_j] = \epsilon_{ijk} F_k.$$

10. Find under what conditions

$$Q = \frac{\alpha p}{x}, \quad P = \beta x^2,$$

where α and β are constants, represents a canonical transformation for a system of one degree of freedom, and obtain a suitable generating function. Apply the transformation to the solution of the linear harmonic oscillator.

11. Determine whether the transformation

$$Q_1 = q_1 q_2, \qquad P_1 = \frac{p_1 - p_2}{q_2 - q_1} + 1,$$
$$Q_2 = q_1 + q_2, \qquad P_2 = \frac{q_2 p_2 - q_1 p_1}{q_2 - q_1} - (q_2 + q_1)$$

is canonical.

12. Show that the direct conditions for a canonical condition are given immediately by the symplectic condition expressed in the form

$$\mathsf{J}\mathsf{M} = \mathsf{M}^{-1}\mathsf{J}.$$

13. The set of restricted canonical transformations has a group-property. Verify this statement once using the invariance of Hamilton's principle under canonical transformation (cf. Eq. (9.11)), and again using the symplectic condition.

14. Prove that the transformation

$$Q_1 = q_1^2, \qquad Q_2 = q_2 \sec p_2,$$
$$P_1 = \frac{p_1 \cos p_2 - 2q_2}{2q_1 \cos p_2}, \qquad P_2 = \sin p_2 - 2q_1$$

is canonical, by any method you choose. Find a suitable generating function that will lead to this transformation.

15. **(a)** Using the fundamental Poisson brackets find the values of α and β for which the equations

$$Q = q^\alpha \cos \beta p, \quad P = q^\alpha \sin \beta p$$

represent a canonical transformation.

(b) For what values of α and β do these equations represent an *extended* canonical transformation? Find a generating function of the F_3 form for the transformation.

(c) On the basis of part (b), can the transformation equations be modified so that they describe a canonical transformation for all values of β?

16. For a symmetric rigid body, obtain formulas for evaluating the Poisson brackets

$$[\dot{\phi}, f(\theta, \phi, \psi)], \quad [\dot{\psi}, f(\theta, \phi, \psi)]$$

where θ, ϕ, and ψ are the Euler angles, and f is any arbitrary function of the Euler angles.

17. Show that the Jacobi identity is satisfied if the Poisson bracket sign stands for the commutator of two square matrices:

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}.$$

Show also that for the same representation of the Poisson bracket that

$$[\mathbf{A}, \mathbf{BC}] = [\mathbf{A}, \mathbf{B}]\mathbf{C} + \mathbf{B}[\mathbf{A}, \mathbf{C}].$$

18. Prove Eq. (9.83) using the symplectic matrix notation for the Lagrange and Poisson brackets.

19. Verify the analog of the Jacobi identity for Lagrange brackets,

$$\frac{\partial\{u, v\}}{\partial w} + \frac{\partial\{v, w\}}{\partial u} + \frac{\partial\{w, u\}}{\partial v} = 0,$$

where u, v, and w are three functions in terms of which the (q, p) set can be specified.

20. **(a)** Verify that the components of the two-dimensional matrix **A**, defined by Eq. (9.141), are constants of the motion for the two-dimensional isotropic harmonic oscillator problem.

(b) Verify that the quantities S_i, $i = 1, 2, 3$, defined by Eqs. (9.144), (9.145), (9.146), have the properties stated in Eqs. (9.147) and (9.148).

EXERCISES

21. **(a)** For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2},$$

show that there is a constant of the motion

$$D = \frac{pq}{2} - Ht.$$

(b) As a generalization of part (a), for motion in a plane with the Hamiltonian

$$H = |\mathbf{p}|^n - ar^{-n},$$

where **p** is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of the motion

$$D = \frac{\mathbf{p} \cdot \mathbf{r}}{n} - Ht.$$

(c) The transformation $Q = \lambda q$, $p = \lambda P$ is obviously canonical. However, the same transformation with t time dilatation, $Q = \lambda q$, $p = \lambda P$, $t' = \lambda^2 t$, is not. Show that, however, the equations of motion for q and p for the Hamiltonian in part (a) are invariant under this transformation. The constant of the motion D is said to be associated with this invariance.

22. For the point transformation in a system of two degrees of freedom,

$$Q_1 = q_1^2, \quad Q_2 = q_1 + q_2,$$

find the most general transformation equations for P_1 and P_2 consistent with the overall transformation being canonical. Show that with a particular choice for P_1 and P_2 the Hamiltonian

$$H = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$$

can be transformed to one in which both Q_1 and Q_2 are ignorable. By this means solve the problem and obtain expressions for q_1, q_2, p_1, and p_2 as functions of time and their initial values.

23. By any method you choose, show that the following transformation is canonical:

$$x = \frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right), \qquad p_x = \frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right),$$
$$y = \frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right), \qquad p_y = -\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right),$$

where α is some fixed parameter.

Apply this transformation to the problem of a particle of charge q moving in a plane that is perpendicular to a constant magnetic field **B**. Express the Hamiltonian for this problem in the (Q_i, P_i) coordinates letting the parameter α take the form

$$\alpha^2 = \frac{qB}{c}.$$

From this Hamiltonian, obtain the motion of the particle as a function of time.

24. **(a)** Show that the transformation

$$Q = p + iaq, \quad P = \frac{p - iaq}{2ia}$$

is canonical and find a generating function.

(b) Use the transformation to solve the linear harmonic oscillator problem.

25. **(a)** The Hamiltonian for a system has the form

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right).$$

Find the equation of motion for q.

(b) Find a canonical transformation that reduces H to the form of a harmonic oscillator. Show that the solution for the transformed variables is such that the equation of motion found in part (a) is satisfied.

26. A system of n particles moves in a plane under the influence of interaction forces derived from potential terms depending only upon the scalar distances between particles.

(a) Using plane polar coordinates for each particle (relative to a common origin), identify the form of the Hamiltonian for the system.

(b) Find a generating function for the canonical transformation that corresponds to a transformation to coordinates rotating in the plane counterclockwise with a uniform angular rate ω (the same for all particles). What are the transformation equations for the momenta?

(c) What is the new Hamiltonian? What physical significance can you give to the difference between the old and the new Hamiltonians?

27. **(a)** In the problem of small oscillations about steady motion, show that at the point of steady motion all the Hamiltonian variables $\boldsymbol{\eta}$ are constant. If the values for steady motion are $\boldsymbol{\eta}_0$ so that $\boldsymbol{\eta} = \boldsymbol{\eta}_0 + \boldsymbol{\xi}$, show that to the lowest nonvanishing approximation the effective Hamiltonian for small oscillation can be expressed as

$$H(\boldsymbol{\eta}_0, \boldsymbol{\zeta}) = \tfrac{1}{2}\boldsymbol{\xi}\mathbf{S}\boldsymbol{\xi},$$

where $\mathbf{S}$ is a square matrix with components that are functions of $\boldsymbol{\eta}_0$ only.

(b) Assuming all frequencies of small oscillation are distinct, let $\mathbf{M}$ be a square $2n \times 2n$ matrix formed by the components of a possible set of eigenvectors (for both positive and negative frequencies). Only the directions of the eigenvectors are fixed, not their magnitudes. Show that it is possible to apply conditions to the eigenvectors (in effect fixing their magnitudes) that make $\mathbf{M}$ the Jacobian matrix of a canonical transformation.

(c) Show that the canonical transformation so found transforms the effective Hamiltonian to the form

$$H = i\omega_j q_j p_j,$$

where ω_j is the magnitude of the normal frequencies. What are the equations of motion in this set of canonical coordinates?

(d) Finally, show that

$$F_2 = q_j P_j + \frac{i}{2}\frac{P_j^2}{\omega_j} - \frac{i}{4}\omega_j q_j^2$$

leads to a canonical transformation that decomposes H into the Hamiltonians for a set of uncoupled linear harmonic oscillators that oscillate in the normal modes.

28. A charged particle moves in space with a constant magnetic field $\mathbf{B}$ such that the vector potential, $\mathbf{A}$, is

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{B} \times \mathbf{r})$$

(a) If v_j are the Cartesian components of the velocity of the particle, evaluate the Poisson brackets

$$[v_i, v_j], \quad i \neq j = 1, 2, 3.$$

(b) If p_i is the canonical momentum conjugate to x_i, also evaluate the Poisson brackets

$$[x_i, v_j], \quad [p_i, v_j],$$
$$[x_i, \dot{p}_j], \quad [p_i, \dot{p}_j].$$

29. The semimajor axis a of the elliptical Kepler orbit and the eccentricity e are functions of first integrals of the motion, and therefore of the canonical variables. Similarly, the mean anomaly

$$\phi \equiv \omega(t - T) = \psi - e \sin\psi$$

is a function of r, θ, and the conjugate momenta. Here T is the time of periapsis passage and is a constant of the motion. Evaluate the Poisson brackets that can be formed of a, e, ϕ, ω, and T. There are in fact only nine nonvanishing distinct Poisson brackets out of these quantities.

30. **(a)** Prove that the Poisson bracket of two constants of the motion is itself a constant of the motion even when the constants depend upon time explicitly.

(b) Show that if the Hamiltonian and a quantity F are constants of the motion, then the nth partial derivative of F with respect to t must also be a constant of the motion.

(c) As an illustration of this result, consider the uniform motion of a free particle of mass m. The Hamiltonian is certainly conserved, and there exists a constant of the motion

$$F = x - \frac{pt}{m}.$$

Show by direct computation that the partial derivative of F with t, which is a constant of the motion, agrees with $[H, F]$.

31. Show by the use of Poisson brackets that for a one-dimensional harmonic oscillator there is a constant of the motion u defined as

$$u(q, p, t) = \ln(p + im\omega q) - i\omega t, \quad \omega = \sqrt{\frac{k}{m}}.$$

What is the physical significance of this constant of the motion?

32. A system of two degrees of freedom is described by the Hamiltonian

$$H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2.$$

Show that

$$F_1 = \frac{p_1 - a q_1}{q_2} \quad \text{and} \quad F_2 = q_1 q_2$$

are constants of the motion. Are there any other independent algebraic constants of the motion? Can any be constructed from Jacobi's identity?

33. Set up the magnetic monopole described in Exercise 28 (Chapter 3) in Hamiltonian formulation (you may want to use spherical polar coordinates). By means of the Poisson bracket formulation, show that the quantity D defined in that exercise is conserved.

34. Obtain the motion in time of a linear harmonic oscillator by means of the formal solution for the Poisson bracket version of the equation of motion as derived from Eq. (9.116). Assume that at time $t = 0$ the initial values are x_0 and p_0.

35. A particle moves in one dimension under a potential

$$V = \frac{mk}{x^2}.$$

Find x as a function of time, by using the symbolic solution of the Poisson bracket form for the equation of motion for the quantity $y = x^2$. Initial conditions are that at $t = 0$, $x = x_0$, and $v = 0$.

36. **(a)** Using the theorem concerning Poisson brackets of vector functions and components of the angular momentum, show that if $\mathbf{F}$ and $\mathbf{G}$ are two vector functions of the coordinates and momenta only, then

$$[\mathbf{F}\cdot\mathbf{L}, \mathbf{G}\cdot\mathbf{L}] = \mathbf{L}\cdot(\mathbf{G}\times\mathbf{F}) + L_i L_j [F_i, G_j].$$

(b) Let $\mathbf{L}$ be the total angular momentum of a rigid body with one point fixed and let L_μ be its component along a set of Cartesian axes fixed in the rigid body. By means of part (a) find a general expansion for

$$[L_\mu, L_\nu], \qquad \mu, \nu = 1, 2, 3.$$

(*Hint:* Choose for $\mathbf{F}$ and $\mathbf{G}$ unit vectors along the μ and ν axes.)

(c) From the Poisson bracket equations of motion for L_μ derive Euler's equations of motion for a rigid body.

37. Set up the problem of the spherical pendulum in the Hamiltonian formulation, using spherical polar coordinates for the q_i. Evaluate directly in terms of these canonical variables the following Poisson brackets:

$$[L_x, L_y], \qquad [L_y, L_z], \qquad [L_z, L_x],$$

showing that they have the values predicted by Eq. (9.128). Why is it that p_θ and p_ψ can be used as canonical momenta, although they are perpendicular components of the angular momentum?

38. In Section 9.7, it is shown that if any two components of the angular momentum are conserved, then the total angular momentum is conserved. If two of the components are identically zero, the third must be conserved. From this it would appear to follow that in any motion confined to a plane, so that the components of the angular momentum in the plane are zero, the total angular momentum is constant. There appear to be a number of obvious contradictions to this prediction; for example, the angular momentum of an oscillating spring in a watch, or the angular momentum of a plane disk rolling down an inclined plane all in the same vertical plane. Discuss the force of these objections and whether the statement of the theorem requires any restrictions.

39. **(a)** Show from the Poisson bracket condition for conserved quantities that the Laplace–Runge–Lenz vector **A**,

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r},$$

is a constant of the motion for the Kepler problem.

(b) Verify the Poisson bracket relations for the components of **A** as given by Eq. (9.131).

40. Consider a system that consists of a rigid body in three-space with one point fixed. Using cylindrical coordinates find the canonical transformation corresponding to new axes rotating about the z-axis with an arbitrary time-dependent angular velocity. Verify that your proposed solution is canonical.

41. We start with a time independent Hamiltonian $H_o(q, p)$ and impose an external oscillating field making the Hamiltonian

$$H = H_o(q, p) - \varepsilon \sin \omega t$$

where ε and ω are given constants.

(a) How are the canonical equations modified?

(b) Find a canonical transformation that restores the canonical form of the equations of motion and determine the "new" Hamiltonian.

(c) Give a possible physical interpretation of the imposed field.