8.6 Coupled Oscillators and Quasiperiodicity

8.6.1 ("Oscillator death" and bifurcations on a torus) In a paper on systems of neural oscillators, Ermentrout and Kopell (1990) illustrated the notion of "oscillator death" with the following model:

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1 \cos\theta_2, \qquad \dot{\theta}_2 = \omega_2 + \sin\theta_2 \cos\theta_1,$$

where $\omega_1, \omega_2 \geq 0$.

a) Sketch all the qualitatively different phase portraits that arise as ω_1, ω_2 vary.
b) Find the curves in ω_1, ω_2 parameter space along which bifurcations occur, and classify the various bifurcations.
c) Plot the stability diagram in ω_1, ω_2 parameter space.

8.6.2 Reconsider the system (8.6.1):

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1), \qquad \dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2).$$

a) Show that the system has no fixed points, given that $\omega_1, \omega_2 > 0$ and $K_1, K_2 > 0$.
b) Find a conserved quantity for the system. (Hint: Solve for $\sin(\theta_2 - \theta_1)$ in two ways. The existence of a conserved quantity shows that this system is a non-generic flow on the torus; normally there would not be any conserved quantities.)
c) Suppose that $K_1 = K_2$. Show that the system can be nondimensionalized to

$$d\theta_1/d\tau = 1 + a\sin(\theta_2 - \theta_1), \qquad d\theta_2/d\tau = \omega + a\sin(\theta_1 - \theta_2).$$

d) Find the *winding number* $\lim_{\tau\to\infty} \theta_1(\tau)/\theta_2(\tau)$ analytically. (Hint: Evaluate the long-time averages $\langle d(\theta_1 + \theta_2)/d\tau \rangle$ and $\langle d(\theta_1 - \theta_2)/d\tau \rangle$, where the brackets are defined by $\langle f \rangle \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\tau)d\tau$. For another approach, see Guckenheimer and Holmes (1983, p. 299).)

8.6.3 (Irrational flow yields dense orbits) Consider the flow on the torus given by $\dot{\theta}_1 = \omega_1, \dot{\theta}_2 = \omega_2$, where ω_1/ω_2 is irrational. Show each trajectory is ***dense***; i.e., given any point p on the torus, any initial condition q, and any $\varepsilon > 0$, there is some $t < \infty$ such that the trajectory starting at q passes within a distance ε of p.

8.6.4 Consider the system

$$\dot{\theta}_1 = E - \sin\theta_1 + K\sin(\theta_2 - \theta_1), \qquad \dot{\theta}_2 = E + \sin\theta_2 + K\sin(\theta_1 - \theta_2)$$

where $E, K \geq 0$.

a) Find and classify all the fixed points.
b) Show that if E is large enough, the system has periodic solutions on the torus. What type of bifurcation creates the periodic solutions?

c) Find the bifurcation curve in (E, K) space at which these periodic solutions are created.

A generalization of this system to $N >> 1$ phases has been proposed as a model of switching in charge-density waves (Strogatz et al. 1988, 1989).

8.6.5 (Plotting Lissajous figures) Using a computer, plot the curve whose parametric equations are $x(t) = \sin t$, $y(t) = \sin \omega t$, for the following rational and irrational values of the parameter ω:

(a) $\omega = 3$ (b) $\omega = \frac{2}{3}$ (c) $\omega = \frac{5}{3}$

(d) $\omega = \sqrt{2}$ (e) $\omega = \pi$ (f) $\omega = \frac{1}{2}(1+\sqrt{5})$.

The resulting curves are called *Lissajous figures.* In the old days they were displayed on oscilloscopes by using two ac signals of different frequencies as inputs.

8.6.6 (Explaining Lissajous figures) Lissajous figures are one way to visualize the knots and quasiperiodicity discussed in the text. To see this, consider a pair of uncoupled harmonic oscillators described by the four-dimensional system $\ddot{x} + x = 0$, $\ddot{y} + \omega^2 y = 0$.

a) Show that if $x = A(t) \sin \theta(t)$, $y = B(t) \sin \phi(t)$, then $\dot{A} = \dot{B} = 0$ (so A, B are constants) and $\dot{\theta} = 1$, $\dot{\theta} = \omega$.
b) Explain why (a) implies that trajectories are typically confined to two-dimensional tori in a four-dimensional phase space.
c) How are the Lissajous figures related to the trajectories of this system?

8.6.7 (Mechanical example of quasiperiodicity) The equations

$$m\ddot{r} = \frac{h^2}{mr^3} - k, \quad \dot{\theta} = \frac{h}{mr^2}$$

govern the motion of a mass m subject to a central force of constant strength $k > 0$. Here r, θ are polar coordinates and $h > 0$ is a constant (the angular momentum of the particle).

a) Show that the system has a solution $r = r_0$, $\dot{\theta} = \omega_\theta$, corresponding to uniform circular motion at a radius r_0 and frequency ω_θ. Find formulas for r_0 and ω_θ.
b) Find the frequency ω_r of small radial oscillations about the circular orbit.
c) Show that these small radial oscillations correspond to quasiperiodic motion by calculating the winding number ω_r / ω_θ.
d) Show by a geometric argument that the motion is either periodic or quasiperiodic for *any* amplitude of radial oscillation. (To say it in a more interesting way, the motion is never chaotic.)
e) Can you think of a mechanical realization of this system?

8.6.8 Solve the equations of Exercise 8.6.7 on a computer, and plot the particle's path in the plane with polar coordinates r, θ.