Rather, it is due to the fact that the *other* normal mode, namely $(x_1, x_2) \propto (1, -1)$, gives no contribution to the sum $x_1 + x_2$. There are a few too many 1's floating around in the above example, so it's hard to see which results are meaningful and which results are coincidence. But the following example should clear things up. Let's say we solved a problem using the determinant method, and we found the solution to be

$$\begin{pmatrix} x \\ y \end{pmatrix} = B_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cos(\omega_1 t + \phi_1) + B_2 \begin{pmatrix} 1 \\ -5 \end{pmatrix} \cos(\omega_2 t + \phi_2). \tag{4.58}$$

Then $5x + y$ is the normal coordinate associated with the normal mode $(3, 2)$, which has frequency $\omega_1$. (This is true because there is no $\cos(\omega_2 t + \phi_2)$ dependence in the combination $5x + y$.) And similarly, $2x - 3y$ is the normal coordinate associated with the normal mode $(1, -5)$, which has frequency $\omega_2$ (because there is no $\cos(\omega_1 t + \phi_1)$ dependence in the combination $2x - 3y$).

Note the difference between the types of differential equations we solved in the previous chapter in Section 3.3, and the types we solved throughout this chapter. The former dealt with forces that did not have to be linear in $x$ or $\dot{x}$, but that had to depend on only $x$, or only $\dot{x}$, or only $t$. The latter dealt with forces that could depend on all three of these quantities, but that had to be linear in $x$ and $\dot{x}$.

## 4.6 Problems

*Section 4.1: Linear differential equations*

4.1. **Superposition**

Let $x_1(t)$ and $x_2(t)$ be solutions to $\ddot{x}^2 = bx$. Show that $x_1(t) + x_2(t)$ is *not* a solution to this equation.

4.2. **A limiting case** *

Consider the equation $\ddot{x} = ax$. If $a = 0$, then the solution to $\ddot{x} = 0$ is simply $x(t) = C + Dt$. Show that in the limit $a \to 0$, Eq. (4.2) reduces to this form. *Note*: $a \to 0$ is a sloppy way of saying what we mean. What is the proper way to write this limit?

*Section 4.2: Simple harmonic motion*

4.3. **Increasing the mass** **

A mass $m$ oscillates on a spring with spring constant $k$. The amplitude is $d$. At the moment (let this be $t = 0$) when the mass is at position $x = d/2$ (and moving to the right), it collides and sticks to another mass $m$. The speed of the resulting mass $2m$ right after the collision is half the speed of the moving mass $m$ right before the collision (from momentum conservation, discussed in Chapter 5). What is the resulting $x(t)$? What is the amplitude of the new oscillation?

4.4. **Average tension** $**$

Is the average (over time) tension in the string of a pendulum larger or smaller than $mg$? By how much? As usual, assume that the angular amplitude $A$ is small.

4.5. **Walking east on a turntable** $**$

A person walks at constant speed $v$ eastward with respect to a turntable that rotates counterclockwise at constant frequency $\omega$. Find the general expression for the person's coordinates with respect to the ground (with the $x$ direction taken to be eastward).

*Section 4.3: Damped harmonic motion*

4.6. **Maximum speed** $**$

A mass on the end of a spring (with natural frequency $\omega$) is released from rest at position $x_0$. The experiment is repeated, but now with the system immersed in a fluid that causes the motion to be overdamped (with damping coefficient $\gamma$). Find the ratio of the maximum speed in the former case to that in the latter. What is the ratio in the limit of strong damping ($\gamma \gg \omega$)? In the limit of critical damping?

*Section 4.4: Driven (and damped) harmonic motion*

4.7. **Exponential force** $*$

A particle of mass $m$ is subject to a force $F(t) = ma_0 e^{-bt}$. The initial position and speed are both zero. Find $x(t)$. (This problem was already given as Problem 3.9, but solve it here by guessing an exponential function, in the spirit of Section 4.4.)

4.8. **Driven oscillator** $*$

Derive Eq. (4.31) by guessing a solution of the form $x(t) = A\cos\omega_d t + B\sin\omega_d t$ in Eq. (4.29).

*Section 4.5: Coupled oscillators*

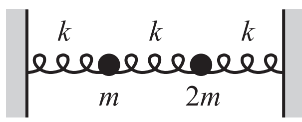


Fig. 4.12

4.9. **Unequal masses** $**$

Three identical springs and two masses, $m$ and $2m$, lie between two walls as shown in Fig. 4.12. Find the normal modes.

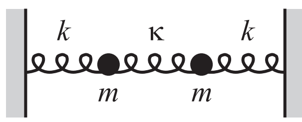


Fig. 4.13

4.10. **Weakly coupled** $**$

Three springs and two equal masses lie between two walls, as shown in Fig. 4.13. The spring constant, $k$, of the two outside springs is much

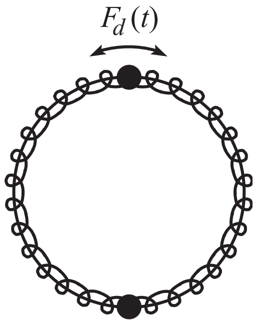


Fig. 4.14

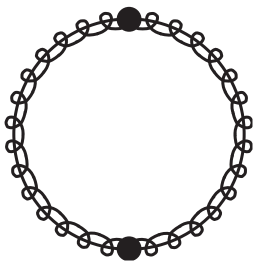

Fig. 4.15

Fig. 4.16

larger than the spring constant, $\kappa$, of the middle spring. Let $x_1$ and $x_2$ be the positions of the left and right masses, respectively, relative to their equilibrium positions. If the initial positions are given by $x_1(0) = a$ and $x_2(0) = 0$, and if both masses are released from rest, show that $x_1$ and $x_2$ can be written as (assuming $\kappa \ll k$)

$$\begin{aligned} x_1(t) &\approx a\cos\big((\omega+\epsilon)t\big)\cos(\epsilon t), \\ x_2(t) &\approx a\sin\big((\omega+\epsilon)t\big)\sin(\epsilon t), \end{aligned} \tag{4.59}$$

where $\omega \equiv \sqrt{k/m}$ and $\epsilon \equiv (\kappa/2k)\omega$. Explain qualitatively what the motion looks like.

4.11. **Driven mass on a circle** $**$

Two identical masses $m$ are constrained to move on a horizontal hoop. Two identical springs with spring constant $k$ connect the masses and wrap around the hoop (see Fig. 4.14). One mass is subject to a driving force $F_\mathrm{d}\cos\omega_\mathrm{d}t$. Find the particular solution for the motion of the masses.

4.12. **Springs on a circle** $****$

(a) Two identical masses $m$ are constrained to move on a horizontal hoop. Two identical springs with spring constant $k$ connect the masses and wrap around the hoop (see Fig. 4.15). Find the normal modes.

(b) Three identical masses are constrained to move on a hoop. Three identical springs connect the masses and wrap around the hoop (see Fig. 4.16). Find the normal modes.

(c) Now do the general case with $N$ identical masses and $N$ identical springs.

## 4.7 Exercises

*Section 4.1: Linear differential equations*

4.13. ***kx* force** $*$

A particle of mass $m$ is subject to a force $F(x) = kx$, with $k > 0$. What is the most general form of $x(t)$? If the particle starts out at $x_0$, what is the one special value of the initial velocity for which the particle doesn't eventually get far away from the origin?

4.14. **Rope on a pulley** $**$

A rope with length $L$ and mass density $\sigma$ kg/m hangs over a massless pulley. Initially, the ends of the rope are a distance $x_0$ above and below their average position. The rope is given an initial speed. If you want the rope to not eventually fall off the pulley, what should this initial speed be? (Don't worry about the issue discussed in Calkin (1989).)

*Section 4.2: Simple harmonic motion*

4.15. **Amplitude** $*$

Find the amplitude of the motion given by $x(t) = C\cos\omega t + D\sin\omega t$.

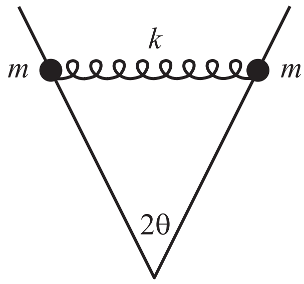

**Fig. 4.17**

4.16. **Angled rails** $*$

Two particles of mass $m$ are constrained to move along two horizontal frictionless rails that make an angle $2\theta$ with respect to each other. They are connected by a spring with spring constant $k$, whose relaxed length is at the position shown in Fig. 4.17. What is the frequency of oscillations for the motion where the spring remains parallel to the position shown?

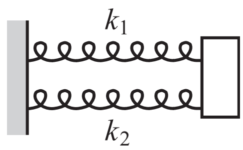

**Fig. 4.18**

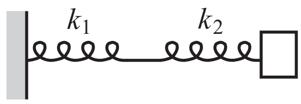

**Fig. 4.19**

4.17. **Effective spring constant** $*$

(a) Two springs with spring constants $k_1$ and $k_2$ are connected in parallel, as shown in Fig. 4.18. What is the effective spring constant, $k_{\text{eff}}$? In other words, if the mass is displaced by $x$, find the $k_{\text{eff}}$ for which the force equals $F = -k_{\text{eff}}x$.

(b) Two springs with spring constants $k_1$ and $k_2$ are connected in series, as shown in Fig. 4.19. What is the effective spring constant, $k_{\text{eff}}$?

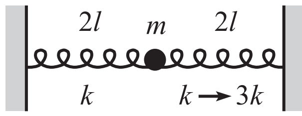

**Fig. 4.20**

4.18. **Changing $k$** $**$

Two springs each have spring constant $k$ and equilibrium length $\ell$. They are both stretched a distance $\ell$ and attached to a mass $m$ and two walls, as shown in Fig. 4.20. At a given instant, the right spring constant is somehow magically changed to $3k$ (the relaxed length remains $\ell$). What is the resulting $x(t)$? Take the initial position to be $x = 0$.

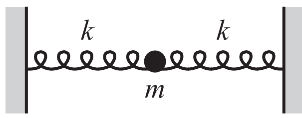

**Fig. 4.21**

4.19. **Removing a spring** $**$

The springs in Fig. 4.21 are at their equilibrium length. The mass oscillates along the line of the springs with amplitude $d$. At the moment (let this be $t = 0$) when the mass is at position $x = d/2$ (and moving to the right), the right spring is removed. What is the resulting $x(t)$? What is the amplitude of the new oscillation?

4.20. **Springs all over** $**$

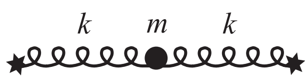

**Fig. 4.22**

(a) A mass $m$ is attached to two springs that have relaxed lengths of zero. The other ends of the springs are fixed at two points (see Fig. 4.22). The two spring constants are equal. The mass sits at its equilibrium position and is then given a kick in an arbitrary

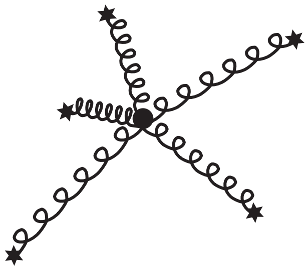

**Fig. 4.23**

direction. Describe the resulting motion. (Ignore gravity, although you actually don't need to.)

(b) A mass $m$ is attached to $n$ springs that have relaxed lengths of zero. The other ends of the springs are fixed at various points in space (see Fig. 4.23). The spring constants are $k_1, k_2, \ldots, k_n$. The mass sits at its equilibrium position and is then given a kick in an arbitrary direction. Describe the resulting motion. (Again, ignore gravity, although you actually don't need to.)

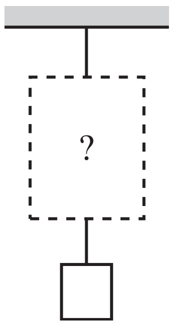


**Fig. 4.24**

4.21. **Rising up** ***

In Fig. 4.24, a mass hangs from a ceiling. A piece of paper is held up to obscure three strings and two springs; all you see is two other strings protruding from behind the paper, as shown. How should the three strings and two springs be attached to each other and to the two visible strings (different items can be attached only at their endpoints) so that if you start with the system at its equilibrium position and then cut a certain one of the hidden strings, the mass will rise up?[5]

4.22. **Projectile on a spring** ***

A projectile of mass $m$ is fired from the origin at speed $v_0$ and angle $\theta$. It is attached to the origin by a spring with spring constant $k$ and relaxed length zero.

(a) Find $x(t)$ and $y(t)$.

(b) Show that for small $\omega \equiv \sqrt{k/m}$, the trajectory reduces to normal projectile motion. And show that for large $\omega$, the trajectory reduces to simple harmonic motion, that is, oscillatory motion along a line (at least before the projectile smashes back into the ground). What are the more meaningful statements that should replace "small $\omega$" and "large $\omega$"?

(c) What value should $\omega$ take so that the projectile hits the ground when it is moving straight downward?

4.23. **Corrections to the pendulum** ***

(a) For small oscillations, the period of a pendulum is approximately $T \approx 2\pi\sqrt{\ell/g}$, independent of the amplitude, $\theta_0$. For finite oscillations, use $dt = dx/v$ to show that the exact expression for $T$ is

$$T = \sqrt{\frac{8\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}. \qquad (4.60)$$

[5] Thanks to Paul Horowitz for this extremely cool problem. For more applications of the idea behind it, see Cohen and Horowitz (1991).

(b) Find an approximation to this $T$, up to second order in $\theta_0^2$, in the following way. Make use of the identity $\cos\phi = 1 - 2\sin^2(\phi/2)$ to write $T$ in terms of sines (because it's more convenient to work with quantities that go to zero as $\theta \to 0$). Then make the change of variables, $\sin x \equiv \sin(\theta/2)/\sin(\theta_0/2)$ (you'll see why). Finally, expand your integrand in powers of $\theta_0$, and perform the integrals to show that[6]

$$T \approx 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{\theta_0^2}{16} + \cdots\right). \tag{4.61}$$

*Section 4.3: Damped harmonic motion*

4.24. **Crossing the origin**

Show that an overdamped or critically damped oscillator can cross the origin at most once.

4.25. **Strong damping** $*$

In the strong damping ($\gamma \gg \omega$) case discussed in the remark in the overdamping subsection, we saw that $x(t) \propto e^{-\omega^2 t/2\gamma}$ for large $t$. Using the definitions of $\omega$ and $\gamma$, this can be written as $x(t) \propto e^{-kt/b}$, where $b$ is the coefficient of the damping force. By looking at the forces on the mass, explain why this makes sense.

4.26. **Maximum speed** $*$

A critically damped oscillator with natural frequency $\omega$ starts out at position $x_0 > 0$. What is the maximum initial speed (directed toward the origin) it can have and not cross the origin?

4.27. **Another maximum speed** $**$

An overdamped oscillator with natural frequency $\omega$ and damping coefficient $\gamma$ starts out at position $x_0 > 0$. What is the maximum initial speed (directed toward the origin) it can have and not cross the origin?

4.28. **Ratio of maxima** $**$

A mass on the end of a spring is released from rest at position $x_0$. The experiment is repeated, but now with the system immersed in a fluid that causes the motion to be critically damped. Show that the maximum speed of the mass in the first case is $e$ times the maximum speed in the second case.[7]

[6] If you like this sort of thing, you can show that the next term in the parentheses is $(11/3072)\theta_0^4$. But be careful, this fourth-order correction comes from two terms.

[7] The fact that the maximum speeds differ by a fixed numerical factor follows from dimensional analysis, which tells us that the maximum speed in the first case must be proportional to $\omega x_0$. And

*Section 4.4: Driven (and damped) harmonic motion*

4.29. **Resonance**

Given $\omega$ and $\gamma$, show that the $R$ in Eq. (4.33) is minimum when $\omega_d = \sqrt{\omega^2 - 2\gamma^2}$ (unless this is imaginary, in which case the minimum occurs at $\omega_d = 0$).

4.30. **No damping force** *

A particle of mass $m$ is subject to a spring force, $-kx$, and also a driving force, $F_d \cos \omega_d t$. But there is no damping force. Find the particular solution for $x(t)$ by guessing $x(t) = A \cos \omega_d t + B \sin \omega_d t$. If you write this in the form $C \cos(\omega_d t - \phi)$, where $C > 0$, what are $C$ and $\phi$? Be careful about the phase (there are two cases to consider).

*Section 4.5: Coupled oscillators*

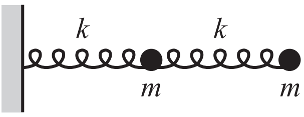


**Fig. 4.25**

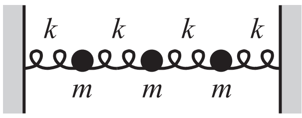


**Fig. 4.26**

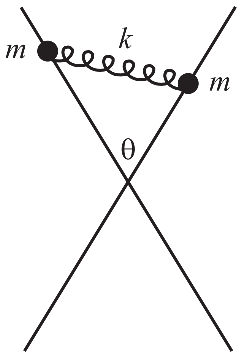


**Fig. 4.27**

4.31. **Springs and one wall** **

Two identical springs and two identical masses are attached to a wall as shown in Fig. 4.25. Find the normal modes, and show that the frequencies can be written as $\sqrt{k/m}(\sqrt{5} \pm 1)/2$. This numerical factor is the golden ratio (and its inverse).

4.32. **Springs between walls** **

Four identical springs and three identical masses lie between two walls (see Fig. 4.26). Find the normal modes.

4.33. **Beads on angled rails** **

Two horizontal frictionless rails make an angle $\theta$ with each other, as shown in Fig. 4.27. Each rail has a bead of mass $m$ on it, and the beads are connected by a spring with spring constant $k$ and relaxed length zero. Assume that one of the rails is positioned a tiny distance above the other, so that the beads can pass freely through the crossing. Find the normal modes.

4.34. **Coupled and damped** **

The system in the example in Section 4.5 is modified by immersing it in a fluid so that both masses feel a damping force, $F_f = -bv$. Solve for $x_1(t)$ and $x_2(t)$. Assume underdamping.

4.35. **Coupled and driven** **

The system in the example in Section 4.5 is modified by subjecting the left mass to a driving force $F_d \cos(2\omega t)$, and the right mass to a driving

since $\gamma = \omega$ in the critical-damping case, the damping doesn't introduce a new parameter, so the maximum speed has no choice but to again be proportional to $\omega x_0$. But showing that the maximum speeds differ by the nice factor of $e$ requires a calculation.