EXERCISES FOR CHAPTER 2

2.1 A Geometric Way of Thinking

In the next three exercises, interpret $\dot{x} = \sin x$ as a flow on the line.

2.1.1 Find all the fixed points of the flow.

2.1.2 At which points x does the flow have greatest velocity to the right?

2.1.3
a) Find the flow's acceleration $\ddot{x}$ as a function of x.
b) Find the points where the flow has maximum positive acceleration.

2.1.4 (Exact solution of $\dot{x} = \sin x$) As shown in the text, $\dot{x} = \sin x$ has the solution $t = \ln|(\csc x_0 + \cot x_0)/(\csc x + \cot x)|$, where $x_0 = x(0)$ is the initial value of x.
a) Given the specific initial condition $x_0 = \pi/4$, show that the solution above can be inverted to obtain

$$x(t) = 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right).$$

Conclude that $x(t) \to \pi$ as $t \to \infty$, as claimed in Section 2.1. (You need to be good with trigonometric identities to solve this problem.)
b) Try to find the analytical solution for $x(t)$, given an *arbitrary* initial condition x_0.

2.1.5 (A mechanical analog)
a) Find a mechanical system that is approximately governed by $\dot{x} = \sin x$.
b) Using your physical intuition, explain why it now becomes obvious that $x^* = 0$ is an unstable fixed point and $x^* = \pi$ is stable.

2.2 Fixed Points and Stability

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2.2.1 $\dot{x} = 4x^2 - 16$

2.2.2 $\dot{x} = 1 - x^{14}$

2.2.3 $\dot{x} = x - x^3$

2.2.4 $\dot{x} = e^{-x}\sin x$

2.2.5 $\dot{x} = 1 + \frac{1}{2}\cos x$

2.2.6 $\dot{x} = 1 - 2\cos x$

2.2.7 $\dot{x} = e^x - \cos x$ (Hint: Sketch the graphs of e^x and $\cos x$ on the same axes, and look for intersections. You won't be able to find the fixed points explicitly, but you can still find the qualitative behavior.)

2.2.8 (Working backwards, from flows to equations) Given an equation $\dot{x} = f(x)$, we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: For the phase portrait shown in Figure 1, find an equation that is consistent with it. (There are an infinite number of correct answers—and wrong ones too.)

Figure 1

2.2.9 (Backwards again, now from solutions to equations) Find an equation $\dot{x} = f(x)$ whose solutions $x(t)$ are consistent with those shown in Figure 2.

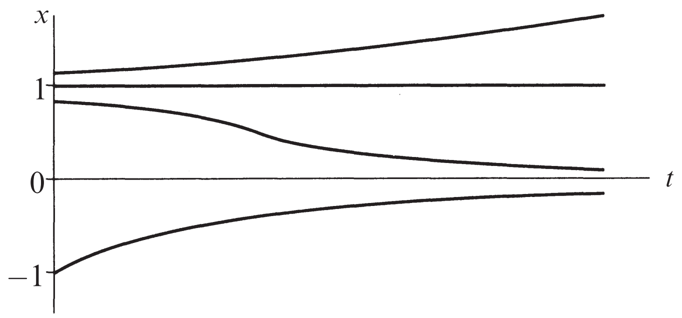

Figure 2

2.2.10 (Fixed points) For each of (a)–(e), find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not. (In all cases, assume that $f(x)$ is a smooth function.)

a) Every real number is a fixed point.
b) Every integer is a fixed point, and there are no others.
c) There are precisely three fixed points, and all of them are stable.
d) There are no fixed points.
e) There are precisely 100 fixed points.

2.2.11 (Analytical solution for charging capacitor) Obtain the analytical solution of the initial value problem $\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$, with $Q(0) = 0$, which arose in Example 2.2.2.

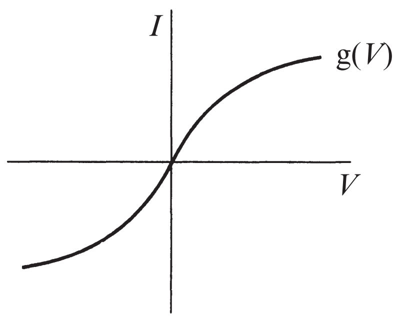

Figure 3

2.2.12 (A nonlinear resistor) Suppose the resistor in Example 2.2.2 is replaced by a nonlinear resistor. In other words, this resistor does not have a linear relation between voltage and current. Such nonlinearity arises in certain solid-state devices. Instead of $I_R = V/R$, suppose we have $I_R = g(V)$, where $g(V)$ has the shape shown in Figure 3.

Redo Example 2.2.2 in this case. Derive the circuit equations, find all the fixed points, and analyze their stability. What qualitative effects does the nonlinearity introduce (if any)?

2.2.13 (Terminal velocity) The velocity $v(t)$ of a skydiver falling to the ground is governed by $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

a) Obtain the analytical solution for $v(t)$, assuming that $v(0) = 0$.
b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the *terminal velocity.* (Beware of bad jokes about the word *terminal* and parachutes that fail to open.)
c) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity.
d) An experimental study (Carlson et al. 1942) confirmed that the equation $m\dot{v} = mg - kv^2$ gives a good quantitative fit to data on human skydivers. Six men were dropped from altitudes varying from 10,600 feet to 31,400 feet to a terminal altitude of 2,100 feet, at which they opened their parachutes. The long free fall from 31,400 to 2,100 feet took 116 seconds. The average weight of the men and their equipment was 261.2 pounds. In these units, $g = 32.2$ ft/sec^2. Compute the average velocity V_{avg}.
e) Using the data given here, estimate the terminal velocity, and the value of the drag constant k. (Hints: First you need to find an exact formula for $s(t)$, the distance fallen, where $s(0) = 0$, $\dot{s} = v$, and $v(t)$ is known from part (a). You should get $s(t) = \frac{V^2}{g} \ln\left(\cosh \frac{gt}{V}\right)$, where V is the terminal velocity. Then solve for V graphically or numerically, using $s = 29{,}300$, $t = 116$, and $g = 32.2$.)

   A slicker way to estimate V is to suppose $V \approx V_{\text{avg}}$, as a rough first approximation. Then show that $gt/V \approx 15$. Since $gt/V >> 1$, we may use the approximation $\ln(\cosh x) \approx x - \ln 2$ for $x >> 1$. Derive this approximation and then use it to obtain an analytical estimate of V. Then k follows from part (b). This analysis is from Davis (1962).

2.3 Population Growth

2.3.1 (Exact solution of logistic equation) There are two ways to solve the logistic equation $\dot{N} = rN(1 - N/K)$ analytically for an arbitrary initial condition N_0.

a) Separate variables and integrate, using partial fractions.
b) Make the change of variables $x = 1/N$. Then derive and solve the resulting differential equation for x.

2.3.2 (Autocatalysis) Consider the model chemical reaction

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$$

in which one molecule of X combines with one molecule of A to form two molecules of X. This means that the chemical X stimulates its own production, a process called *autocatalysis.* This positive feedback process leads to a chain reaction, which eventually is limited by a "back reaction" in which $2X$ returns to $A + X$.

According to the *law of mass action* of chemical kinetics, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants. We denote the concentrations by lowercase letters $x = [X]$ and $a = [A]$. Assume that there's an enormous surplus of chemical A, so that its concentration a can be regarded as constant. Then the equation for the kinetics of x is

$$\dot{x} = k_1 a x - k_{-1} x^2$$

where k_1 and k_{-1} are positive parameters called rate constants.
a) Find all the fixed points of this equation and classify their stability.
b) Sketch the graph of $x(t)$ for various initial values x_0.

2.3.3 (Tumor growth) The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN \ln(bN)$, where $N(t)$ is proportional to the number of cells in the tumor, and $a, b > 0$ are parameters.
a) Interpret a and b biologically.
b) Sketch the vector field and then graph $N(t)$ for various initial values.

The predictions of this simple model agree surprisingly well with data on tumor growth, as long as N is not too small; see Aroesty et al. (1973) and Newton (1980) for examples.

2.3.4 (The Allee effect) For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N. This is called the Allee effect (Edelstein–Keshet 1988). For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.
a) Show that $\dot{N}/N = r - a(N-b)^2$ provides an example of the Allee effect, if r, a, and b satisfy certain constraints, to be determined.
b) Find all the fixed points of the system and classify their stability.
c) Sketch the solutions $N(t)$ for different initial conditions.
d) Compare the solutions $N(t)$ to those found for the logistic equation. What are the qualitative differences, if any?

2.3.5 (Dominance of the fittest) Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$ and $\dot{Y} = bY$, respectively, with initial conditions $X_0, Y_0 > 0$ and growth rates $a > b > 0$. Here X is "fitter" than Y in the sense that it reproduces faster, as reflected by the inequality $a > b$. So we'd expect X to keep increasing its share of the total population $X + Y$ as $t \to \infty$. The goal of this exercise is to demonstrate this intuitive result, first analytically and then geometrically.

a) Let $x(t) = X(t)/[X(t) + Y(t)]$ denote X's share of the total population. By solving for $X(t)$ and $Y(t)$, show that $x(t)$ increases monotonically and approaches 1 as $t \to \infty$.
b) Alternatively, we can arrive at the same conclusions by deriving a differential equation for $x(t)$. To do so, take the time derivative of $x(t) = X(t)/[X(t) + Y(t)]$ using the quotient and chain rules. Then substitute for $\dot{X}$ and $\dot{Y}$ and thereby show that $x(t)$ obeys the logistic equation $\dot{x} = (a - b)x(1 - x)$. Explain why this implies that $x(t)$ increases monotonically and approaches 1 as $t \to \infty$.

2.3.6 (Language death) Thousands of the world's languages are vanishing at an alarming rate, with 90 percent of them being expected to disappear by the end of this century. Abrams and Strogatz (2003) proposed the following model of language competition, and compared it to historical data on the decline of Welsh, Scottish Gaelic, Quechua (the most common surviving indigenous language in the Americas), and other endangered languages.

Let X and Y denote two languages competing for speakers in a given society. The proportion of the population speaking X evolves according to

$$\dot{x} = (1 - x)P_{YX} - xP_{XY}$$

where $0 \le x \le 1$ is the current fraction of the population speaking X, $1 - x$ is the complementary fraction speaking Y, and P_{YX} is the rate at which individuals switch from Y to X. This deliberately idealized model assumes that the population is well mixed (meaning that it lacks all spatial and social structure) and that all speakers are monolingual.

Next, the model posits that the attractiveness of a language increases with both its number of speakers and its perceived status, as quantified by a parameter $0 \le s \le 1$ that reflects the social or economic opportunities afforded to its speakers. Specifically, assume that $P_{YX} = sx^a$ and, by symmetry, $P_{XY} = (1 - s)(1 - x)^a$, where the exponent $a > 1$ is an adjustable parameter. Then the model becomes

$$\dot{x} = s(1 - x)x^a - (1 - s)x(1 - x)^a .$$

a) Show that this equation for $\dot{x}$ has three fixed points.
b) Show that for all $a > 1$, the fixed points at $x = 0$ and $x = 1$ are both stable.
c) Show that the third fixed point, $0 < x^* < 1$, is unstable.

This model therefore predicts that two languages cannot coexist stably—one will eventually drive the other to extinction. For a review of generalizations of the model that allow for bilingualism, social structure, etc., see Castellano et al. (2009).

2.4 Linear Stability Analysis

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

2.4.1 $\dot{x} = x(1-x)$ **2.4.2** $\dot{x} = x(1-x)(2-x)$

2.4.3 $\dot{x} = \tan x$ **2.4.4** $\dot{x} = x^2(6-x)$

2.4.5 $\dot{x} = 1-e^{-x^2}$ **2.4.6** $\dot{x} = \ln x$

2.4.7 $\dot{x} = ax - x^3$, where a can be positive, negative, or zero. Discuss all three cases.

2.4.8 Using linear stability analysis, classify the fixed points of the Gompertz model of tumor growth $\dot{N} = -aN\ln(bN)$. (As in Exercise 2.3.3, $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters.)

2.4.9 (Critical slowing down) In statistical mechanics, the phenomenon of "critical slowing down" is a signature of a second-order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. Here's a mathematical version of the effect:

a) Obtain the analytical solution to $\dot{x} = -x^3$ for an arbitrary initial condition. Show that $x(t) \to 0$ as $t \to \infty$, but that the decay is not exponential. (You should find that the decay is a much slower algebraic function of t.)
b) To get some intuition about the slowness of the decay, make a numerically accurate plot of the solution for the initial condition $x_0 = 10$, for $0 \le t \le 10$. Then, on the same graph, plot the solution to $\dot{x} = -x$ for the same initial condition.

2.5 Existence and Uniqueness

2.5.1 (Reaching a fixed point in a finite time) A particle travels on the half-line $x \ge 0$ with a velocity given by $\dot{x} = -x^c$, where c is real and constant.

a) Find all values of c such that the origin $x = 0$ is a stable fixed point.
b) Now assume that c is chosen such that $x = 0$ is stable. Can the particle ever reach the origin in *a finite* time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c?

2.5.2 ("Blow-up": Reaching infinity in a finite time) Show that the solution to $\dot{x} = 1 + x^{10}$ escapes to $+\infty$ in a finite time, starting from any initial condition. (Hint: Don't try to find an exact solution; instead, compare the solutions to those of $\dot{x} = 1 + x^2$.)

2.5.3 Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \neq 0$.

2.5.4 (Infinitely many solutions with the same initial condition) Show that the initial value problem $\dot{x} = x^{1/3}$, $x(0) = 0$, has an infinite number of solutions. (Hint: Construct a solution that stays at $x = 0$ until some arbitrary time t_0, after which it takes off.)

2.5.5 (A general example of non-uniqueness) Consider the initial value problem $\dot{x} = |x|^{p/q}$, $x(0) = 0$, where p and q are positive integers with no common factors.

a) Show that there are an infinite number of solutions for $x(t)$ if $p < q$.
b) Show that there is a unique solution if $p > q$.

2.5.6 (The leaky bucket) The following example (Hubbard and West 1991, p. 159) shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see an empty bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Here's a crude model of the situation. Let $h(t)$ = height of the water remaining in the bucket at time t; a = area of the hole; A = cross-sectional area of the bucket (assumed constant); $v(t)$ = velocity of the water passing through the hole.

a) Show that $av(t) = A\dot{h}(t)$. What physical law are you invoking?
b) To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount Δh and that the water has density ρ. Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation $v^2 = 2gh$.
c) Combining (b) and (c), show $\dot{h} = -C\sqrt{h}$, where $C = \sqrt{2g}\left(\frac{a}{A}\right)$.
d) Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is non-unique *in backwards time*, i.e., for $t < 0$.

2.6 Impossibility of Oscillations

2.6.1 Explain this paradox: a simple harmonic oscillator $m\ddot{x} = -kx$ is a system that oscillates in one dimension (along the x-axis). But the text says one-dimensional systems can't oscillate.

2.6.2 (No periodic solutions to $\dot{x} = f(x)$) Here's an analytic proof that periodic solutions are impossible for a vector field on a line. Suppose on the contrary that $x(t)$ is a nontrivial periodic solution, i.e., $x(t) = x(t + T)$ for some $T > 0$, and $x(t) \neq x(t+s)$ for all $0 < s < T$. Derive a contradiction by considering $\int_t^{t+T} f(x)\frac{dx}{dt}\,dt$.

2.7 Potentials

For each of the following vector fields, plot the potential function $V(x)$ and identify all the equilibrium points and their stability.

2.7.1 $\dot{x} = x(1-x)$ **2.7.2** $\dot{x} = 3$

2.7.3 $\dot{x} = \sin x$ **2.7.4** $\dot{x} = 2 + \sin x$

2.7.5 $\dot{x} = -\sinh x$ **2.7.6** $\dot{x} = r + x - x^3$, for various values of r.

2.7.7 (Another proof that solutions to $\dot{x} = f(x)$ can't oscillate) Let $\dot{x} = f(x)$ be a vector field on the line. Use the existence of a potential function $V(x)$ to show that solutions $x(t)$ cannot oscillate.

2.8 Solving Equations on the Computer

2.8.1 (Slope field) The slope is constant along horizontal lines in Figure 2.8.2. Why should we have expected this?

2.8.2 Sketch the slope field for the following differential equations. Then "integrate" the equation manually by drawing trajectories that are everywhere parallel to the local slope.
a) $\dot{x} = x$ b) $\dot{x} = 1 - x^2$ c) $\dot{x} = 1 - 4x(1-x)$ d) $\dot{x} = \sin x$

2.8.3 (Calibrating the Euler method) The goal of this problem is to test the Euler method on the initial value problem $\dot{x} = -x$, $x(0) = 1$.
a) Solve the problem analytically. What is the exact value of $x(1)$?
b) Using the Euler method with step size $\Delta t = 1$, estimate $x(1)$ numerically—call the result $\hat{x}(1)$. Then repeat, using $\Delta t = 10^{-n}$, for $n = 1, 2, 3, 4$.
c) Plot the error $E = |\hat{x}(1) - x(1)|$ as a function of Δt. Then plot $\ln E$ vs. $\ln t$. Explain the results.

2.8.4 Redo Exercise 2.8.3, using the improved Euler method.

2.8.5 Redo Exercise 2.8.3, using the Runge–Kutta method.

2.8.6 (Analytically intractable problem) Consider the initial value problem $\dot{x} = x + e^{-x}$, $x(0) = 0$. In contrast to Exercise 2.8.3, this problem can't be solved analytically.
a) Sketch the solution $x(t)$ for $t \geq 0$.
b) Using some analytical arguments, obtain rigorous bounds on the value of x at $t = 1$. In other words, prove that $a < x(1) < b$, for a, b to be determined. By being clever, try to make a and b as close together as possible. (Hint: Bound the given vector field by approximate vector fields that can be integrated analytically.)

c) Now for the numerical part: Using the Euler method, compute x at $t = 1$, correct to three decimal places. How small does the stepsize need to be to obtain the desired accuracy? (Give the order of magnitude, not the exact number.)
d) Repeat part (b), now using the Runge–Kutta method. Compare the results for stepsizes $\Delta t = 1$, $\Delta t = 0.1$, and $\Delta t = 0.01$.

2.8.7 (Error estimate for Euler method) In this question you'll use Taylor series expansions to estimate the error in taking one step by the Euler method. The exact solution and the Euler approximation both start at $x = x_0$ when $t = t_0$. We want to compare the exact value $x(t_1) \equiv x(t_0 + \Delta t)$ with the Euler approximation $x_1 = x_0 + f(x_0)\Delta t$.
a) Expand $x(t_1) = x(t_0 + \Delta t)$ as a Taylor series in Δt, through terms of $O(\Delta t^2)$. Express your answer solely in terms of x_0, Δt, and f and its derivatives at x_0.
b) Show that the local error $|x(t_1) - x_1| \sim C(\Delta t)^2$ and give an explicit expression for the constant C. (Generally one is more interested in the global error incurred after integrating over a time interval of fixed length $T = n\Delta t$. Since each step produces an $O(\Delta t)^2$ error, and we take $n = T/\Delta t = O(\Delta t^{-1})$ steps, the global error $|x(t_n) - x_n|$ is $O(\Delta t)$, as claimed in the text.)

2.8.8 (Error estimate for the improved Euler method) Use the Taylor series arguments of Exercise 2.8.7 to show that the local error for the improved Euler method is $O(\Delta t^3)$.

2.8.9 (Error estimate for Runge–Kutta) Show that the Runge–Kutta method produces a local error of size $O(\Delta t^5)$. (Warning: This calculation involves massive amounts of algebra, but if you do it correctly, you'll be rewarded by seeing many wonderful cancellations. Teach yourself *Mathematica*, *Maple*, or some other symbolic manipulation language, and do the problem on the computer.)