Second proof: Our "Lagrangian," $L = f(y)\sqrt{1+y'^2}$, is independent of x. Therefore, in analogy with the conserved energy given in Eq. (6.52), the quantity

$$E \equiv y' \frac{\partial L}{\partial y'} - L = \frac{-f(y)}{\sqrt{1+y'^2}} \tag{6.92}$$

is independent of x. Call it $1/\sqrt{B}$. Then we have easily reproduced Eq. (6.91). For practice, you can also prove this lemma by considering x to be a function of y, as we did in the second solution in the minimal-surface example above. ■

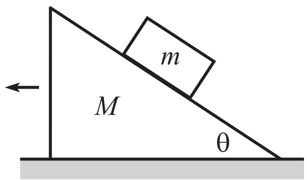

Fig. 6.8

6.9 Problems

Section 6.1: The Euler–Lagrange equations

6.1. **Moving plane** **

A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ (see Fig. 6.8). The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane? (This problem already showed up as Problem 3.8. If you haven't already done so, try solving it using $F = ma$. You will then have a greater appreciation for the Lagrangian method.)

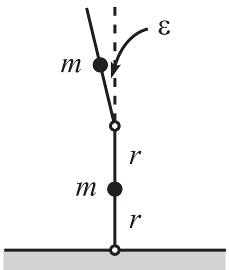

Fig. 6.9

6.2. **Two falling sticks** **

Two massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at an end. One stands on top of the other, as shown in Fig. 6.9. The bottom end of the lower stick is hinged at the ground. They are held such that the lower stick is vertical, and the upper one is tilted at a small angle ϵ with respect to the vertical. They are then released. At this instant, what are the angular accelerations of the two sticks? Work in the approximation where ϵ is very small.

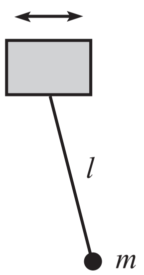

Fig. 6.10

6.3. **Pendulum with an oscillating support** **

A pendulum consists of a mass m and a massless stick of length ℓ. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos(\omega t)$; see Fig. 6.10. What is the general solution for the angle of the pendulum as a function of time?

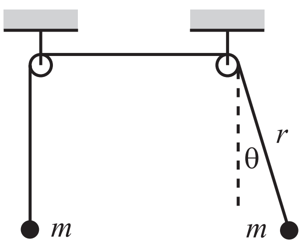

Fig. 6.11

6.4. **Two masses, one swinging** ***

Two equal masses m, connected by a massless string, hang over two pulleys (of negligible size), as shown in Fig. 6.11. The left one moves in a vertical line, but the right one is free to swing back and forth in the plane of the masses and pulleys. Find the equations of motion for r and θ, as shown.

Assume that the left mass starts at rest, and the right mass undergoes small oscillations with angular amplitude ϵ (with $\epsilon \ll 1$). What is the initial average acceleration (averaged over a few periods) of the left mass? In which direction does it move?

6.5. **Inverted pendulum** ****

A pendulum consists of a mass m at the end of a massless stick of length ℓ. The other end of the stick is made to oscillate vertically with a position given by $y(t) = A\cos(\omega t)$, where $A \ll \ell$. See Fig. 6.12. It turns out that if ω is large enough, and if the pendulum is initially nearly upside-down, then surprisingly it will *not* fall over as time goes by. Instead, it will (sort of) oscillate back and forth around the vertical position. If you want to do the experiment yourself, see the 28th demonstration of the entertaining collection in Ehrlich (1994).

Find the equation of motion for the angle of the pendulum (measured relative to its upside-down position). Explain why the pendulum doesn't fall over, and find the frequency of the back and forth motion.

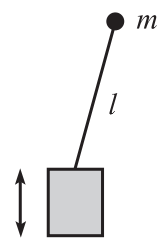

Fig. 6.12

Section 6.2: The principle of stationary action

6.6. **Minimum or saddle** **

(a) In Eq. (6.26), let $t_1 = 0$ and $t_2 = T$, for convenience. And let $\xi(t)$ be an easy-to-deal-with "triangular" function, of the form

$$\xi(t) = \begin{cases} \epsilon t/T, & 0 \le t \le T/2, \\ \epsilon(1 - t/T), & T/2 \le t \le T. \end{cases} \tag{6.93}$$

Under what condition is the harmonic-oscillator ΔS in Eq. (6.26) negative?

(b) Answer the same question, but now with $\xi(t) = \epsilon\sin(\pi t/T)$.

Section 6.3: Forces of constraint

6.7. **Normal force from a plane** **

A mass m slides down a frictionless plane that is inclined at an angle θ. Show, using the method in Section 6.3, that the normal force from the plane is the familiar $mg\cos\theta$.

Section 6.5: Conservation laws

6.8. **Bead on a stick** *

A stick is pivoted at the origin and is arranged to swing around in a horizontal plane at constant angular speed ω. A bead of mass m slides frictionlessly along the stick. Let r be the radial position of the bead. Find the conserved quantity E given in Eq. (6.52). Explain why this quantity is *not* the energy of the bead.

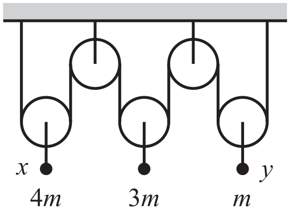

Fig. 6.13

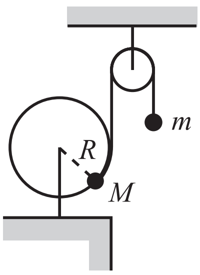

Fig. 6.14

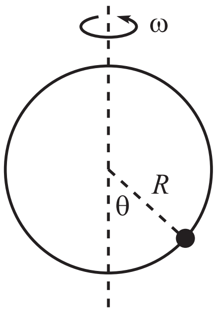

Fig. 6.15

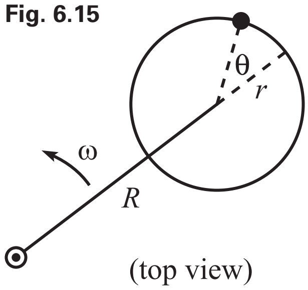

Fig. 6.16

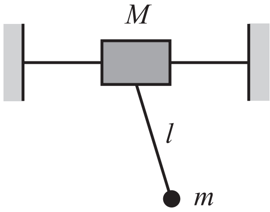

Fig. 6.17

Section 6.6: Noether's theorem

6.9. **Atwood's machine** ∗∗

Consider the Atwood's machine shown in Fig. 6.13. The masses are $4m$, $3m$, and m. Let x and y be the heights of the left and right masses, relative to their initial positions. Find the conserved momentum.

Section 6.7: Small oscillations

6.10. **Hoop and pulley** ∗∗

A mass M is attached to a massless hoop of radius R that lies in a vertical plane. The hoop is free to rotate about its fixed center. M is tied to a string which winds part way around the hoop, then rises vertically up and over a massless pulley. A mass m hangs on the other end of the string (see Fig. 6.14). Find the equation of motion for the angle of rotation of the hoop. What is the frequency of small oscillations? Assume that m moves only vertically, and assume $M > m$.

6.11. **Bead on a rotating hoop** ∗∗

A bead is free to slide along a frictionless hoop of radius R. The hoop rotates with constant angular speed ω around a vertical diameter (see Fig. 6.15). Find the equation of motion for the angle θ shown. What are the equilibrium positions? What is the frequency of small oscillations about the stable equilibrium? There is one value of ω that is rather special; what is it, and why is it special?

6.12. **Another bead on a rotating hoop** ∗∗

A bead is free to slide along a frictionless hoop of radius r. The plane of the hoop is horizontal, and the center of the hoop travels in a horizontal circle of radius R, with constant angular speed ω, about a given point (see Fig. 6.16). Find the equation of motion for the angle θ shown. Also, find the frequency of small oscillations about the equilibrium point.

6.13. **Mass on a wheel** ∗∗

A mass m is fixed to a given point on the rim of a wheel of radius R that rolls without slipping on the ground. The wheel is massless, except for a mass M located at its center. Find the equation of motion for the angle through which the wheel rolls. For the case where the wheel undergoes small oscillations, find the frequency.

6.14. **Pendulum with a free support** ∗∗

A mass M is free to slide along a frictionless rail. A pendulum of length ℓ and mass m hangs from M (see Fig. 6.17). Find the equations of motion. For small oscillations, find the normal modes and their frequencies.

6.15. **Pendulum support on an inclined plane** $**$

A mass M is free to slide down a frictionless plane inclined at an angle β. A pendulum of length ℓ and mass m hangs from M; see Fig. 6.18 (assume that M extends a short distance beyond the side of the plane, so the pendulum can hang down). Find the equations of motion. For small oscillations, find the normal modes and their frequencies.

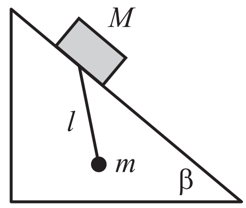

Fig. 6.18

6.16. **Tilting plane** $***$

A mass M is fixed at the right-angled vertex where a massless rod of length ℓ is attached to a very long massless rod (see Fig. 6.19). A mass m is free to move frictionlessly along the long rod (assume that it can pass through M). The rod of length ℓ is hinged at a support, and the whole system is free to rotate, in the plane of the rods, about the hinge. Let θ be the angle of rotation of the system, and let x be the distance between m and M. Find the equations of motion. Find the normal modes when θ and x are both very small.

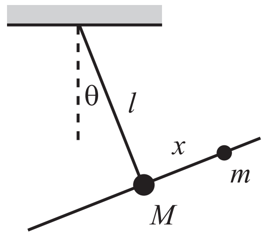

Fig. 6.19

6.17. **Rotating curve** $***$

The curve $y(x) = b(x/a)^{\lambda}$ is rotated around the y axis with constant frequency ω (see Fig. 6.20). A bead moves frictionlessly along the curve. Find the frequency of small oscillations about the equilibrium point. Under what conditions do oscillations exist? (This problem gets a little messy.)

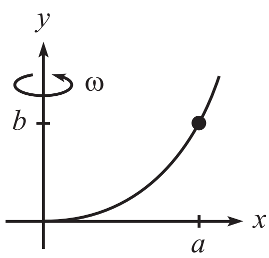

Fig. 6.20

6.18. **Motion in a cone** $***$

A particle slides on the inside surface of a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The half-angle at the tip is α (see Fig. 6.21). Let r be the distance from the particle to the axis, and let θ be the angle around the cone. Find the equations of motion.

If the particle moves in a circle of radius r_0, what is the frequency, ω, of this motion? If the particle is then perturbed slightly from this circular motion, what is the frequency, Ω, of the oscillations about the radius r_0? Under what conditions does $\Omega = \omega$?

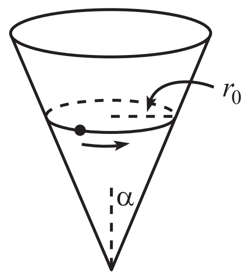

Fig. 6.21

6.19. **Double pendulum** $****$

Consider a double pendulum made of two masses, m_1 and m_2, and two rods of lengths ℓ_1 and ℓ_2 (see Fig. 6.22). Find the equations of motion.

For small oscillations, find the normal modes and their frequencies for the special case $\ell_1 = \ell_2$ (and consider the cases $m_1 = m_2$, $m_1 \gg m_2$, and $m_1 \ll m_2$). Do the same for the special case $m_1 = m_2$ (and consider the cases $\ell_1 = \ell_2$, $\ell_1 \gg \ell_2$, and $\ell_1 \ll \ell_2$).

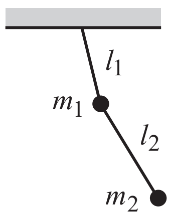

Fig. 6.22

Section 6.8: Other applications

6.20. **Shortest distance in a plane** *

In the spirit of Section 6.8, show that the shortest path between two points in a plane is a straight line.

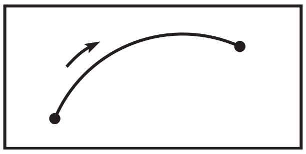

Fig. 6.23

6.21. **Index of refraction** **

Assume that the speed of light in a given slab of material is proportional to the height above the base of the slab.[12] Show that light moves in circular arcs in this material; see Fig. 6.23. You may assume that light takes the path of least time between two points (Fermat's principle of least time).

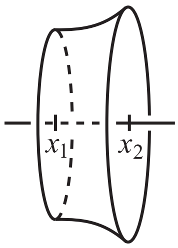

Fig. 6.24

6.22. **Minimal surface** **

Derive the shape of the minimal surface discussed in Section 6.8, by demanding that a cross-sectional "ring" (that is, the region between the planes $x = x_1$ and $x = x_2$) is in equilibrium; see Fig. 6.24. *Hint*: The tension must be constant throughout the surface (assuming that we're ignoring gravity, which we are).

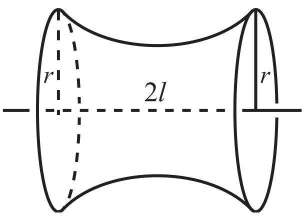

Fig. 6.25

6.23. **Existence of a minimal surface** **

Consider the minimal surface from Section 6.8, and look at the special case where the two rings have the same radius r (see Fig. 6.25). Let 2ℓ be the distance between the rings. What is the largest value of ℓ/r for which a minimal surface exists? You will need to solve something numerically here.

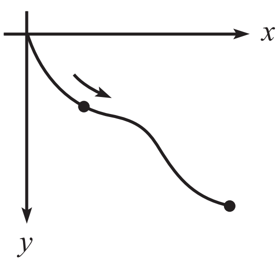

Fig. 6.26

6.24. **The brachistochrone** ***

A bead is released from rest at the origin and slides down a frictionless wire that connects the origin to a given point, as shown in Fig. 6.26. You wish to shape the wire so that the bead reaches the endpoint in the shortest possible time. Let the desired curve be described by the function $y(x)$, with downward taken to be positive. Show that $y(x)$ satisfies

$$1 + y'^2 = \frac{B}{y}, \tag{6.94}$$

where B is a constant. Then show that x and y may be written as

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta). \tag{6.95}$$

[12] If you want to make the equivalent statement in terms of the material's "index of refraction," commonly denoted by n, then you can say: As a function of the height y, the index n is given by $n(y) = y_0/y$, where y_0 is some length that is larger than the height of the slab. This is equivalent to the original statement because the speed of light in a material equals c/n.

This is the parametrization of a *cycloid*, which is the path taken by a point on the rim of a rolling wheel.

6.10 Exercises

Section 6.1: The Euler–Lagrange equations

6.25. **Spring on a T** **

A rigid T consists of a long rod glued perpendicular to another rod of length ℓ that is pivoted at the origin. The T rotates around in a horizontal plane with constant frequency ω. A mass m is free to slide along the long rod and is connected to the intersection of the rods by a spring with spring constant k and relaxed length zero (see Fig. 6.27). Find $r(t)$, where r is the position of the mass along the long rod. There is a special value of ω; what is it, and why is it special?

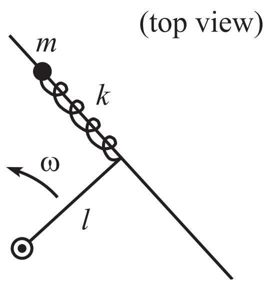

Fig. 6.27

6.26. **Spring on a T, with gravity** ***

Consider the setup in the previous exercise, but now let the T swing around in a vertical plane with constant frequency ω. Find $r(t)$. There is a special value of ω; what is it, and why is it special? (You may assume $\omega < \sqrt{k/m}$.)

6.27. **Coffee cup and mass** **

A coffee cup of mass M is connected to a mass m by a string. The coffee cup hangs over a frictionless pulley of negligible size, and the mass m is initially held with the string horizontal, as shown in Fig. 6.28. The mass m is then released. Find the equations of motion for r (the length of string between m and the pulley) and θ (the angle that the string to m makes with the horizontal). Assume that m somehow doesn't run into the string holding the cup up.

The coffee cup will initially fall, but it turns out that it will reach a lowest point and then rise back up. Write a program (see Section 1.4) that numerically determines the ratio of the r at this lowest point to the r at the start, for a given value of m/M. (To check your program, a value of $m/M = 1/10$ yields a ratio of about 0.208.)

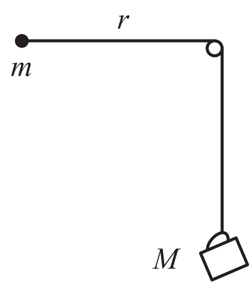

Fig. 6.28

6.28. **Three falling sticks** ***

Three massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at their ends, as shown in Fig. 6.29. The bottom end of the lower stick is hinged at the ground. They are held such that the lower two sticks are vertical, and the upper one is tilted at a small angle ϵ with respect to the vertical. They are then released. At this instant, what are the angular accelerations of the three sticks? Work

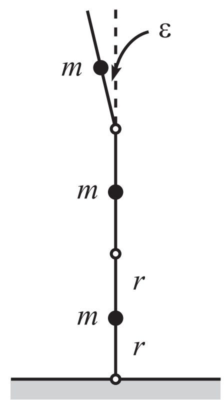

Fig. 6.29

in the approximation where ϵ is very small. (You may want to look at Problem 6.2 first.)

6.29. **Cycloidal pendulum** $****$

The standard pendulum frequency of $\sqrt{g/\ell}$ holds only for small oscillations. The frequency becomes smaller as the amplitude grows. It turns out that if you want to build a pendulum whose frequency is independent of the amplitude, you should hang it from the cusp of a cycloid of a certain size, as shown in Fig. 6.30. As the string wraps partially around the cycloid, the effect is to decrease the length of string in the air, which in turn increases the frequency back up to a constant value. In more detail:

A cycloid is the path taken by a point on the rim of a rolling wheel. The upside-down cycloid in Fig. 6.30 can be parametrized by $(x, y) = R(\theta - \sin\theta,\ -1 + \cos\theta)$, where $\theta = 0$ corresponds to the cusp. Consider a pendulum of length $4R$ hanging from the cusp, and let α be the angle the string makes with the vertical, as shown.

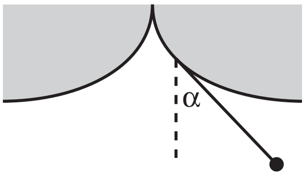

Fig. 6.30

(a) In terms of α, find the value of the parameter θ associated with the point where the string leaves the cycloid.
(b) In terms of α, find the length of string touching the cycloid.
(c) In terms of α, find the Lagrangian.
(d) Show that the quantity $\sin\alpha$ undergoes simple harmonic motion with frequency $\sqrt{g/4R}$, independent of the amplitude.
(e) In place of parts (c) and (d), solve the problem again by using $F = ma$. This actually gives a much quicker solution.

Section 6.2: The principle of stationary action

6.30. **Dropped ball** $*$

Consider the action, from $t = 0$ to $t = 1$, of a ball dropped from rest. From the E–L equation (or from $F = ma$), we know that $y(t) = -gt^2/2$ yields a stationary value of the action. Show explicitly that the particular function $y(t) = -gt^2/2 + \epsilon t(t-1)$ yields an action that has no first-order dependence on ϵ.

6.31. **Explicit minimization** $*$

For a ball thrown upward, guess a solution for y of the form $y(t) = a_2t^2 + a_1t + a_0$. Assuming that $y(0) = y(T) = 0$, this quickly becomes $y(t) = a_2(t^2 - Tt)$. Calculate the action between $t = 0$ and $t = T$, and show that it is minimized when $a_2 = -g/2$.

6.32. **Always a minimum** $*$

For a ball thrown up in the air, show that the stationary value of the action is always a global minimum.

6.33. **Second-order change** *

Let $x_a(t) \equiv x_0(t) + a\beta(t)$. Equation (6.19) gives the first derivative of the action with respect to a. Show that the second derivative is

$$\frac{d^2}{da^2}S[x_a(t)] = \int_{t_1}^{t_2}\left(\frac{\partial^2 L}{\partial x^2}\beta^2 + 2\frac{\partial^2 L}{\partial x \partial \dot{x}}\beta\dot{\beta} + \frac{\partial^2 L}{\partial \dot{x}^2}\dot{\beta}^2\right) dt. \tag{6.96}$$

6.34. **$\ddot{x}$ dependence** *

Assume that there is $\ddot{x}$ dependence (in addition to $x, \dot{x}, t$ dependence) in the Lagrangian in Theorem 6.1. There will then be the additional term $(\partial L/\partial \ddot{x}_a)\ddot{\beta}$ in Eq. (6.19). It is tempting to integrate this term by parts twice, and then arrive at a modified form of Eq. (6.22):

$$\frac{\partial L}{\partial x_0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}_0}\right) = 0. \tag{6.97}$$

Is this a valid result? If not, where is the error in the reasoning?

Section 6.3: Forces of constraint

6.35. **Constraint on a circle** *

A bead of mass m slides with speed v around a horizontal hoop of radius R. What force does the hoop apply to the bead? (Ignore gravity.)

6.36. **Atwood's machine** *

Consider the standard Atwood's machine in Fig. 6.31, with masses m_1 and m_2. Find the tension in the string.

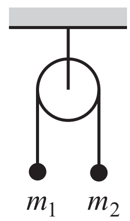

Fig. 6.31

6.37. **Cartesian coordinates** **

In Eq. (6.35), take two time derivatives of the $\sqrt{x^2+y^2} - R = 0$ equation to obtain

$$R^2(x\ddot{x} + y\ddot{y}) + (x\dot{y} - y\dot{x})^2 = 0, \tag{6.98}$$

and then combine this with the other two equations to solve for F in terms of $x,\ y,\ \dot{x},\ \dot{y}$. Convert the result to polar coordinates (with θ measured from the vertical) and show that it agrees with Eq. (6.32).

6.38. **Constraint on a curve** ***

Let the horizontal plane be the x-y plane. A bead of mass m slides with speed v along a curve described by the function $y = f(x)$. What force does the curve apply to the bead? (Ignore gravity.)

Section 6.5: Conservation laws

6.39. **Bead on stick, using $F = ma$** $*$

After doing Problem 6.8, show again that the quantity E is conserved, but now use $F = ma$. Do this in two ways:

(a) Use the first of Eqs. (3.51). *Hint*: multiply through by $\dot{r}$.

(b) Use the second of Eqs. (3.51) to calculate the work done on the bead, and use the work–energy theorem.

Section 6.6: Noether's theorem

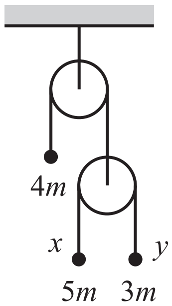

Fig. 6.32

6.40. **Atwood's machine** $**$

Consider the Atwood's machine shown in Fig. 6.32. The masses are $4m$, $5m$, and $3m$. Let x and y be the heights of the right two masses, relative to their initial positions. Use Noether's theorem to find the conserved momentum. (The solution to Problem 6.9 gives some other methods, too.)

Section 6.7: Small oscillations

6.41. **Spring and a wheel** $*$

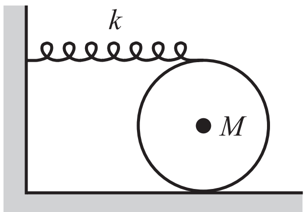

Fig. 6.33

The top of a wheel of mass M and radius R is connected to a spring (at its equilibrium length) with spring constant k, as shown in Fig. 6.33. Assume that all the mass of the wheel is at its center. If the wheel rolls without slipping, what is the frequency of (small) oscillations?

6.42. **Spring on a spoke** $**$

A spring with spring constant k and relaxed length zero lies along a spoke of a massless wheel of radius R. One end of the spring is attached to the center, and the other end is attached to a mass m that is free to slide along the spoke. When the system is in its equilibrium position with the spring hanging vertically, how far (in terms of R) should the mass hang down (you are free to adjust k) so that for small oscillations, the frequency of the spring oscillations equals the frequency of the rocking motion of the wheel? Assume that the wheel rolls without slipping.

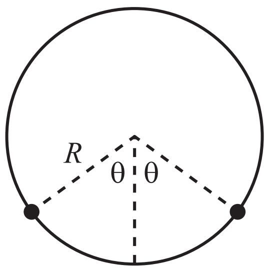

Fig. 6.34

6.43. **Oscillating hoop** $**$

Two equal masses are glued to a massless hoop of radius R that is free to rotate about its center in a vertical plane. The angle between the masses is 2θ, as shown in Fig. 6.34. Find the frequency of small oscillations.

6.44. **Oscillating hoop with a pendulum** $***$

A massless hoop of radius R is free to rotate about its center in a vertical plane. A mass m is attached at one point, and a pendulum of length $\sqrt{2}R$

(and also of mass m) is attached at another point 90° away, as shown in Fig. 6.35. Let θ be the angle of the hoop relative to the position shown, and let α be the angle of the pendulum with respect to the vertical. Find the normal modes for small oscillations.

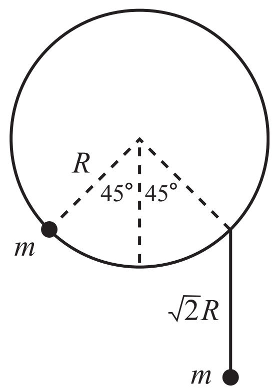

Fig. 6.35

6.45. **Mass sliding on a rim** **

A mass m is free to slide frictionlessly along the rim of a wheel of radius R that rolls without slipping on the ground. The wheel is massless, except for a mass M located at its center. Find the normal modes for small oscillations.

6.46. **Mass sliding on a rim, with a spring** ***

Consider the setup in the previous exercise, but now let the mass m be attached to a spring with spring constant k and relaxed length zero, the other end of which is attached to a point on the rim. Assume that the spring is constrained to run along the rim, and assume that the mass can pass freely over the point where the spring is attached to the rim. To keep things from getting too messy here, you can set $M = m$.

(a) Find the frequencies of the normal modes for small oscillations. Check the $g = 0$ limit, and (if you've done the previous exercise) the $k = 0$ limit.
(b) For the special case where $g/R = k/m$, show that the frequencies can be written as $\sqrt{k/m}(\sqrt{5} \pm 1)/2$. This numerical factor is the golden ratio (and its inverse). Describe what the normal modes look like.

6.47. **Vertically rotating hoop** ***

A bead is free to slide along a frictionless hoop of radius r. The plane of the hoop is vertical, and the center of the hoop travels in a vertical circle of radius R with constant angular speed ω about a given point (see Fig. 6.36). Find the equation of motion for the angle θ shown. For large ω (which implies small θ), find the amplitude of the "particular" solution with frequency ω. What happens if $r = R$?

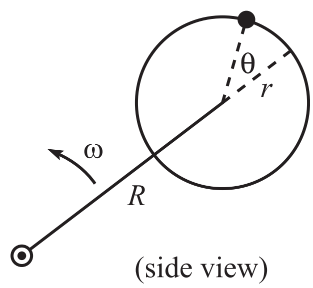

Fig. 6.36

6.11 Solutions

6.1. **Moving plane**

Let x_1 be the horizontal coordinate of the plane (with positive x_1 to the left), and let x_2 be the horizontal coordinate of the block (with positive x_2 to the right); see Fig. 6.37. The relative horizontal distance between the plane and the block is $x_1 + x_2$, so the height fallen by the block is $(x_1 + x_2)\tan\theta$. The Lagrangian is therefore

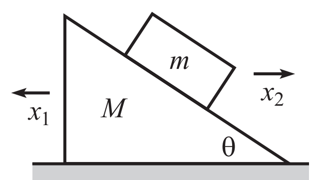

Fig. 6.37

$$L = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\Big(\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2\tan^2\theta\Big) + mg(x_1 + x_2)\tan\theta. \tag{6.99}$$