show that the canonical equations of motion can be written

$$\frac{\partial H}{\partial p_i} + \sum_k \lambda_k \frac{\partial \psi_k}{\partial p_i} = \dot{q}_i, \qquad \frac{\partial H}{\partial q_i} + \sum_k \lambda_k \frac{\partial \psi_k}{\partial q_i} = -\dot{p}_i,$$

where the λ_k are the undetermined Lagrange multipliers. The formulation of the Hamiltonian equations in which t is a canonical variable is a case in point, since a relation exists between p_{n+1} and the other canonical variables:

$$H(q_1, \ldots, q_{n+1};\ p_1, \ldots, p_n) + p_{n+1} = 0.$$

Show that as a result of these circumstances the $2n + 2$ Hamilton's equations of this formulation can be reduced to the $2n$ ordinary Hamilton's equations plus Eq. (8.41) and the relation

$$\lambda = \frac{dt}{d\theta}.$$

Note that while these results are reminiscent of the relativistic covariant Hamiltonian formulation, they have been arrived at entirely within the framework of nonrelativistic mechanics.

10. Assume that the Lagrangian is a polynomial in $\dot{q}$ of no higher order than quadratic. Convert the $2n$ equations (8.2) and (8.14)

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad \dot{p}_i = \frac{\partial L}{\partial q_i},$$

into $2n$ equations for $\dot{q}_i$ and $\dot{p}_i$ in terms of q and p, using the matrix form of the Lagrangian. Show that these are the same equations as would be obtained from Hamilton's equations of motion.

EXERCISES

11. A particle is confined to a one-dimensional box. The ends of the box move slowly towards the middle. By slowly we mean the speed of the ends is small when compared to the speed of the particle. Solve the following using Lagrangian formulation and then using the Hamiltonian.

(a) if the momentum of the particle is p_0 when the walls are a distance x_0 apart, find the momentum of the particle at any later time assuming the collisions with the wall are perfectly elastic. Also assume the motion is nonrelativistic at all times.

(b) When the walls are a distance x apart, what average external force must be applied to each wall in order to move it at a constant speed?

12. Write the problem of central force motion of two mass points in Hamiltonian formulation, eliminating the cyclic variables, and reducing the problem to quadratures.

13. Formulate the double-pendulum problem illustrated by Fig. 1.4, in terms of the Hamiltonian and Hamilton's equations of motion. It is suggested that you find the Hamiltonian both directly from L and by Eq. (8.27).

14. The Lagrangian for a system can be written as

$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2 + y^2},$$

where a, b, c, f, g, and k are constants. What is the Hamiltonian? What quantities are conserved?

15. A dynamical system has the Lagrangian

$$L = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k_1 q_1^2 + k_2 \dot{q}_1 \dot{q}_2,$$

where a, b, k_1, and k_2 are constants. Find the equations of motion in the Hamiltonian formulation.

16. A Hamiltonian of one degree of freedom has the form

$$H = \frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{ba}{2}q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2},$$

where a, b, α, and k are constants.

(a) Find a Lagrangian corresponding to this Hamiltonian.

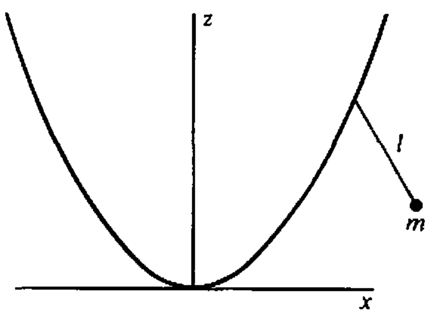
(b) Find an equivalent Lagrangian that is not explicitly dependent on time.

(c) What is the Hamiltonian corresponding to this second Lagrangian, and what is the relationship between the two Hamiltonians?

17. Find the Hamiltonian for the system described in Exercise 19 of Chapter 5 and obtain Hamilton's equations of motion for the system. Use both the direct and the matrix approach in finding the Hamiltonian.

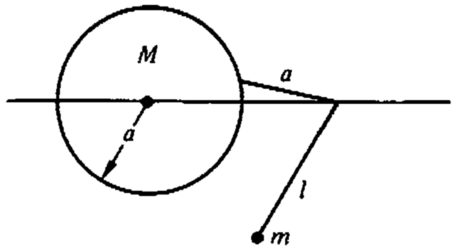
18. Repeat the preceding exercise except this time allow the *pendulum* to move in three dimensions, that is, a spring-loaded spherical pendulum. Either the direct or the matrix approach may be used.

19. The point of suspension of a simple pendulum of length l and mass m is constrained to move on a parabola $z = ax^2$ in the vertical plane. Derive a Hamiltonian governing the motion of the pendulum and its point of suspension. Obtain the Hamilton's equations of motion.



20. Obtain Hamilton's equations of motion for a plane pendulum of length l with mass point m whose radius of suspension rotates uniformly on the circumference of a vertical circle of radius a. Describe physically the nature of the canonical momentum and the Hamiltonian.

21. (a) The point of suspension of a plane simple pendulum of mass m and length l is constrained to move along a horizontal track and is connected to a point on the circumference of a uniform flywheel of mass M and radius a through a massless connecting rod also of length a, as shown in the figure. The flywheel rotates about a center fixed on the track. Find a Hamiltonian for the combined system and determine Hamilton's equations of motion.



(b) Suppose the point of suspension were moved along the track according to some function of time $x = f(t)$, where x reverses at $x = \pm 2a$ (relative to the center of the fly wheel). Again, find a Hamiltonian and Hamilton's equations of motion.

22. For the arrangement described in Exercise 21 of Chapter 2, find the Hamiltonian of the system, first in terms of coordinates in the laboratory system and then in terms of coordinates in the rotating systems. What are the conservation properties of the Hamiltonians, and how are they related to the energy of the system?

23. (a) A particle of mass m and electric charge e moves in a plane under the influence of a central force potential $V(r)$ and a constant uniform magnetic field $\mathbf{B}$, perpendicular to the plane, generated by a static vector potential

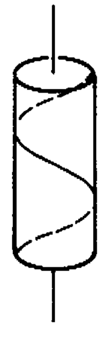
$$\mathbf{A} = \tfrac{1}{2}\mathbf{B} \times \mathbf{r}.$$

Find the Hamiltonian using coordinates in the observer's inertial system.

(b) Repeat part (a) using coordinates rotating relative to the previous coordinate system about an axis perpendicular to the plane with an angular rate of rotation:

$$\omega = -\frac{eB}{m}.$$

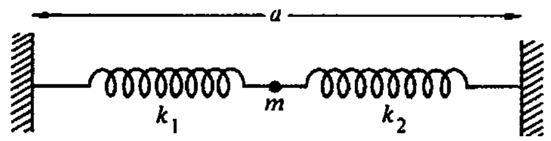
24. A uniform cylinder of radius a and density ρ is mounted so as to rotate freely around a vertical axis. On the outside of the cylinder is a rigidly fixed uniform spiral or helical track along which a mass point m can slide without friction. Suppose a particle starts



at rest at the top of the cylinder and slides down under the influence of gravity. Using any set of coordinates, arrive at a Hamiltonian for the combined system of particle and cylinder, and solve for the motion of the system.

25. Suppose that in the previous exercise the cylinder is constrained to rotate uniformly with angular frequency ω. Set up the Hamiltonian for the particle in an inertial system of coordinates and also in a system fixed in the rotating cylinder. Identify the physical nature of the Hamiltonian in each case and indicate whether or not the Hamiltonians are conserved.

26. A particle of mass m can move in one dimension under the influence of two springs connected to fixed points a distance a apart (see figure). The springs obey Hooke's law and have zero unstretched lengths and force constants k_1 and k_2, respectively.



(a) Using the position of the particle from one fixed point as the generalized coordinate, find the Lagrangian and the corresponding Hamiltonian. Is the energy conserved? Is the Hamiltonian conserved?

(b) Introduce a new coordinate Q defined by

$$Q = q - b\sin\omega t, \qquad b = \frac{k_2 a}{k_1 + k_2}.$$

What is the Lagrangian in terms of Q? What is the corresponding Hamiltonian? Is the energy conserved? Is the Hamiltonian conserved?

27. **(a)** The Lagrangian for a system of one degree of freedom can be written as

$$L = \frac{m}{2}(\dot{q}^2 \sin^2 \omega t + \dot{q}q\omega \sin 2\omega t + q^2\omega^2).$$

What is the corresponding Hamiltonian? Is it conserved?

(b) Introduce a new coordinate defined by

$$Q = q \sin \omega t.$$

Find the Lagrangian in terms of the new coordinate and the corresponding Hamiltonian. Is H conserved?

28. Consider a system of particles interacting with each other through potentials depending only on the scalar distances between them and acted upon by conservative central forces from a fixed point. Obtain the Hamiltonian of the particle with respect to a set of axes, with origin at the center of force, which is rotating around some axis in an inertial system with angular velocity ω. What is the physical significance of the Hamiltonian in this case? Is it a constant of the motion?

29. Obtain the Hamiltonian of a heavy symmetrical top with one point fixed, and from it the Hamilton's equations of motion. Relate these to the equations of motion discussed in Section 5.7 and, in particular, show how the solution may be reduced to quadratures. Also use the Routhian procedure to eliminate the cyclic coordinates.

30. In Exercise 16 of Chapter 1, there is given the velocity-dependent potential assumed in Weber's electrodynamics. What is the Hamiltonian for a single particle moving under the influence of such a potential?

31. Treat the nutation of a "fast" top as an example of small oscillations about steady motion, here precession at constant θ. Find the frequency of nutation.

32. A symmetrical top is mounted so that it pivots about its center of mass. The pivot in turn is fixed a distance r from the center of a horizontal disk free to rotate about a vertical axis. The top is started with an initial rotation about its figure axis, which is initially at an angle θ_0 to the vertical. Analyze the possible nutation of the top as a case of small oscillations about steady motion.

33. Two mass points, m_1 and m_2, are connected by a string that acts as a Hooke's-law spring with force constant k. One particle is free to move without friction on a smooth horizontal plane surface, the other hangs vertically down from the string through a hole in the surface. Find the condition for steady motion in which the mass point on the plane rotates uniformly at constant distance from the hole. Investigate the small oscillations in the radial distance from the hole, and in the vertical height of the second particle.

34. A possible covariant Lagrangian for a system of one particle interacting with a field is

$$\Lambda = \tfrac{1}{2} m u_\lambda u_\lambda + D_{\lambda_\nu}(x_\mu) m_{\lambda\nu},$$

where $D_{x\nu}(x_\mu)$ is an antisymmetric field tensor and $m_{\lambda\nu}$ is the antisymmetric angular momentum tensor,

$$m_{\lambda\nu} = m(x_\lambda u_\nu - x_\nu u_\lambda).$$

What are the canonical momenta? What is the corresponding covariant Hamiltonian?

35. Consider a Lagrangian of the form

$$L = \tfrac{1}{2} m(\dot{x}^2 - \omega^2 x^2) e^{\gamma t},$$

where the particle of mass m moves in one direction. Assume all constants are positive.

(a) Find the equations of motion.

(b) Interpret the equations by giving a physical interpretation of the forces acting on the particle.

(c) Find the canonical momentum and construct the Hamiltonian. Is this Hamiltonian a constant of the motion?

(d) If initially $x(0) = 0$ and $dx/dt = 0$, what is $x(t)$ as t approaches large values?