So, If $L = \dot{x}\dot{x} + F(t)$

Then, you can write $L = \dots + \frac{\partial}{\partial t}(g(t))$ → Ignoring Remove this this of Lang.

Lef will give the Same Motion Eqn. But don't use for Hamiltonian Formation

So, $E = \frac{1}{2m}\left[F'(x)\right]^2$

$$\boxed{F'(x) = \sqrt{2mE} \qquad F(x) = \underbrace{\sqrt{2mE}}_{\Pi}\, x + C}$$

$S = \Pi x - Et + C$

$S = \Pi x - \frac{\Pi^2}{2m} t + C$

$S = F_2(q, P, t)$

$q \downarrow x$, $\Pi \to P$

Here

$H' = H + \frac{\partial F_2}{\partial t} = H - E = 0$ {$H = E$ is Const}

$\Phi = \left(x - \frac{\Pi t}{m}\right)$ {For this need Relation b/w F_2 and Φ }

$\dot{\Phi} = -\frac{\partial H'}{\partial P}$

$\Rightarrow \dot{\Phi} = 0$

$\boxed{\dot{x} = \Pi/m}$

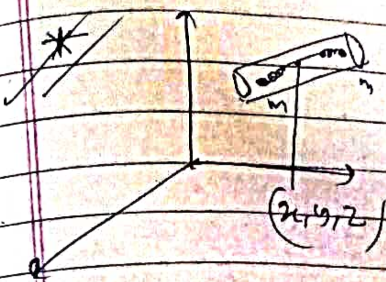
If Force F is given, $F = -\frac{\partial V}{\partial x}$, put this V also in the Langrangian.

To get relation b/w x_1, x_2 go for L. Eqn, for b/w $\dot{x}_1$ and $\dot{x}_2$ go for invariance, Noether's theorem.

* Sol^n of H-Oscillator

* Relaxed length is when spring is not deformed, for convinience can take it to be zero.

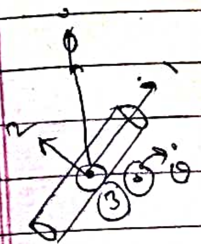
*



One is normal (point mass) and other one is due to Rigid Body Rotation.

$$So,\ L = \overbrace{\frac{1}{2}\times 2m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)}^{normal} + \frac{1}{2}\,2m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

$$+ \underbrace{\frac{1}{2}M\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)}_{normal} + \underbrace{T^{mot}_{mod}}_{\text{Rigid Body Rotation}} - 2\times\frac{1}{2}\times k r^2$$



$$T_{mot} = \frac{1}{2}\,\overbrace{\frac{L^2}{12}}^{I}\,\overbrace{\left(\left(\dot{\phi}\sin\theta\right)^2 + \dot{\theta}^2\right)}^{\omega^2}$$

* We have two angular momenta, normal + RBD (Rotating)

Hamilton Jacobi (Continuation...)

$$* \quad S = \int_{t_1}^{t_2} L\,dt, \quad \frac{\partial S}{\partial q_i} = p_i, \quad \frac{\partial S}{\partial t} = -H$$

$$S = S(q_i, t) \qquad dS = p_i\,dq - H\,dt$$

$$H = H(q_i, p_i, t) = H\left(q_i, \frac{\partial S}{\partial q_i}, t\right)$$

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0$$

$q_i, t \rightarrow N+1$ indpt. variable

A extra const neglected.

$$S = S(t, q_1 \dots q_N, \alpha_1 \dots \alpha_n)$$

$$F_2(q_i, P_i, t) = F_2(q_i, \alpha_i, t)$$

$$H' = H + \frac{\partial F_2}{\partial t} = H + \frac{\partial S}{\partial t} = 0.$$

$$p_i = \frac{\partial F_2}{\partial q_i}, \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$p_i = \frac{\partial S}{\partial q_i}, \qquad Q_i = \frac{\partial S}{\partial P_i} = \frac{\partial S}{\partial \alpha_i}$$

new coordinate when P, Q are momentas.

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} = 0$$

$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i} = 0$$

$$Q_i = \beta_i \qquad \Rightarrow \qquad Q_i = \frac{\partial S}{\partial P_i}$$

$$P_i = \alpha_i$$

$$\beta_i = \frac{\partial S}{\partial \alpha_i}$$

$$\beta_i = \frac{\partial S(t, q_i, \alpha_i)}{\partial \alpha_i}$$

When, H is not dependent on time.

$H(p,q) = E$

$\Rightarrow \frac{\partial s}{\partial t} + E = 0$

$$\boxed{S = S_0(q) - Et}$$

* S_0, $\frac{\partial s}{\partial t} + H = \Phi\left(t, q_i, \frac{\partial s}{\partial q_i}, \frac{\partial s}{\partial t}\right) = 0$

$$\Phi\left\{\underbrace{q_i, t, \partial s/\partial q_i, \partial s'/\partial t}_{i\neq 1}, \underbrace{\phi\left(q_1, \frac{\partial s_1}{\partial q_1}\right)}_{a_1}\right\} = 0$$

$$\longrightarrow \Phi\left(q_i, t, \frac{\partial s'}{\partial q_i}, \frac{\partial s'}{\partial t}, a_1\right) = 0$$

$S = S'(q_i, t) + S_1(q_1)$

$i \neq 1$ Central force.

* Example: $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta} + a(r) + \frac{b(\theta)}{r^2}$

$$\frac{\partial s}{\partial t} + \frac{1}{2m}\left(\frac{\partial s}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial s}{\partial \theta}\right)^2 + \frac{(\partial s/\partial \phi)^2}{2m(r)^2 \sin^2\theta} + a(r) + \frac{b(\theta)}{(r)^2} = 0$$

$$\Phi\left\{r, \theta, t, \frac{\partial s}{\partial r}, \frac{\partial s}{\partial \theta}, \frac{\partial s}{\partial t}, \left(\frac{\partial s}{\partial \phi}\right)^2\right\} = 0$$

Is energy also conserved? Yes.

Harmonic Oscillator ($\partial H/\partial q = 0$)

Ex02: $S = S_0 - Et$ for E = Conserved.

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = E \quad S_0(q)$$

$$S = S_0$$

$$p = \frac{\partial S}{\partial q} \qquad \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + \frac{1}{2} m\omega^2 q^2 = E$$

$$\left(\frac{\partial S_0}{\partial q}\right)^2 = 2mE - m^2\omega^2 q^2$$

new momentum $\boxed{E = P^2/2m}$

$$\left(\frac{dS_0}{dq}\right) = \left[P^2 - m^2\omega^2 q^2\right]^{1/2}$$

Here, P are momentum and they are constant.

$$S_0 = \int dq \left[P^2 - m^2\omega^2 q^2\right]^{1/2}$$

$$= \frac{P^2}{2m\omega} \sin^{-1}\left(\frac{m\omega q}{P}\right) + \frac{q}{2}\sqrt{P^2 - m^2\omega^2 q^2} + C$$

$$S = \frac{P^2}{2m\omega} \sin^{-1}\left(\frac{m\omega q}{P}\right) + \frac{q}{2}\sqrt{P^2 - m^2\omega^2 q^2} - \frac{P^2}{2m}t + C$$

Found S in terms of new momentum β

$$Q = \frac{\partial S}{\partial P} = \frac{P}{m\omega} \sin^{-1}\left(\frac{m\omega q}{P}\right) - \frac{P}{m} t = \beta$$

$$q = A\sin(\omega t + \beta)$$

Φ is const bcoz H' is zero, and bcz $H + \frac{\partial S}{\partial t} = 0$

and, $\frac{\partial H'}{\partial P} = \dot{\Phi} = 0$, $H' = H + \frac{\partial F_2}{\partial t} = 0$

Identified $\alpha \xrightarrow{\text{as}} P$ (new momentum)

Harmonic Oscillator in 2D

Ex02: $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{1}{2}kr^2$

$$\frac{1}{2m}\left[\left(\frac{\partial S_0}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_0}{\partial \theta}\right)^2\right] + \frac{1}{2}kr^2 = E$$

$$r^2\left(\frac{\partial S_0}{\partial r}\right)^2 + \left(\frac{\partial S_0}{\partial \theta}\right)^2 + mkr^2 = 2mEr^2$$

So, as t has been already separated.

$S_0 = R(r) + \Phi(\theta)$ {if θ is cyclic coordinate, we can write it like this}

May all other coordinates

Other coordinates changing

Changing θ, $R \to$ no change, So, they are const

$p_\theta = \frac{d\Phi}{d\theta} = a \Rightarrow \Phi = a\theta + b$ — one momentum, angular momentum (Constant)

another momentum (Constant)

$$\frac{dR}{dr} = \left[2mE - mkr^2 - \frac{a^2}{r^2}\right]^{1/2} \to \text{Find } R.$$

differentiate S w.r.t a and $\sqrt{2mE}$. you get Eqns.

* From Rat, $x_1(0)=0$
$x_2(0)=0$

~~Central coordinates...~~

~~$S_0 = p_\phi \phi + S_1(r) + S_2(\theta)$~~

Central force,

$$* \quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\psi = e^{iS/\hbar}$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - \frac{i\hbar}{2m}\frac{\partial^2 S}{\partial x^2} + V(x) = -\frac{\partial S}{\partial t}$$

For $\hbar \to 0$ it reduces to classical

Classical Eqn,

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r,\theta,\phi) = E$$

$$\Rightarrow \frac{1}{2m}\left(\frac{\partial S_0}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S_0}{\partial \theta}\right)^2 + \frac{1}{2mr^2\sin\theta}\left(\frac{\partial S_0}{\partial \phi}\right)^2$$

$$+ V(r,\theta) = E$$

ϕ cyclic coordinate

$$S_0 = p_\phi \phi + \underbrace{S_1(r) + S_2(\theta)}_{\text{Case Specific}}$$

~~Because, this can't be done always.~~

<u>Small oscillations and Normal-Modes</u>

$\lambda \to$ Damping Coefficient

$$* \quad \ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0, \quad x = Ae^{\alpha t}$$

{ Bcz we know oscillations are sinusoidal }

$$\omega = \sqrt{k/m}$$

exponentially always wins

After Solving,

$$\alpha = -r \pm \sqrt{r^2 - \omega^2}$$

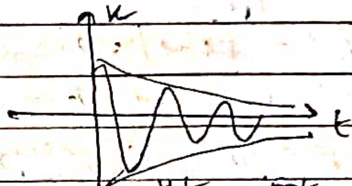
$$x = e^{-rt}\left[A e^{\sqrt{r^2-\omega^2}\,t} + B e^{-\sqrt{r^2-\omega^2}\,t}\right]$$

$$\Omega = \sqrt{r^2 - \omega^2} \quad \{\text{Damped Frequency}\}$$

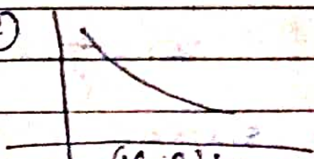
(1) Underdamped $\Omega^2 < 0$

(2) Overdamped $\Omega^2 > 0$

(3) Critically Damped $\Omega^2 = 0$

(1) 

$x(t) = e^{-rt}\, C\cos(\omega t + \phi)$

(2) 

$x = A e^{(-r+\Omega)t} + B e^{-\sqrt{r^2+\Omega^2}\,t}$

(3) $x = e^{-rt}(A + Bt)$

$\Omega^2 > 0 \Rightarrow r > \omega$

eventually it dies out, even though there is competition. because exponentially eventually wins at last.

Forced Oscillation

* $\ddot{x} + 2r\dot{x} + \omega^2 x = F_0 e^{i\omega_0 t}$ Coming From this

General Soln $x = x_h + x_p$ Particular Soln.

x_h Homogeneous Soln — Coming From $\ddot{x} + 2r\dot{x} + \omega^2 x = 0$

Superposition of particular Soln.

$x_p = A e^{i\omega_0 t}$ $\{\omega_0 = $ of driving$\}$

$$x_p = \frac{F_0 e^{i\omega_0 t}}{-\omega_0^2 + 2ir\omega_0 + \omega^2}$$

For, $\ddot{x} + 2r\dot{x} + \omega^2 x = F_1 e^{i\omega_1 t} + F_2 e^{i\omega_2 t} + F_n e^{i\omega_n t}$

$x = x_h + x_{p_1} + x_{p_2} + \dots + x_{p_n}$

With only one F_i term at a time

$x = F_0 e^{i\alpha t}$

? $Dx = Dx_h + Dp_1 + Dp_2$

* $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = F_0 \cos(\omega_0 t) = \frac{F_0}{2} e^{i\omega_0 t} + \frac{F_0}{2} e^{-i\omega_0 t}$

$$x_p = \frac{F_0(\omega^2 - \omega_0^2)\cos(\omega_0 t)}{(\omega^2 - \omega_0^2) + 4\gamma^2\omega_0^2} + \frac{2F_0\gamma\omega_0 \sin(\omega_0 t)}{(\omega^2 - \omega_0^2) + 4\gamma^2\omega_0^2}$$

$x = x_h + x_p$

* So,
$$\left.\begin{aligned} 3\ddot{x} + (5x - 2y)\omega^2 = 0 \\ 3\ddot{y} + (5y - 2x)\omega^2 = 0 \end{aligned}\right\} \text{Coupled}$$

After Solving, for $x = Ae^{i\alpha t}$, $y = Be^{i\alpha t}$

And doing necessary calculations

we get. $(\ddot{x} + \ddot{y}) + (x + y) = 0$

$(\ddot{x} - \ddot{y}) + \frac{7}{3}(x - y) = 0$.

for ~~...~~ $\eta_1 = x + y$, $\eta_2 = x - y$

$\ddot{\eta}_1 + (1)\eta_1 = 0$, $\ddot{\eta}_2 + \left(\frac{7}{3}\right)\eta_2 = 0$

normal coordinate — normal frequency

Decoupled Eqn.

Finding normal modes:

Normal modes are parts in system where parts oscillate at the same frequency.

* $2\ddot{x} + (5x - 3y)\omega^2 = 0$

$2\ddot{y} + (5y - 3x)\omega^2 = 0$

Let, $x = Ae^{i\alpha t}$

$y = Be^{i\alpha t}$

$$\left.\begin{aligned} 2A(-\alpha^2) + (5A - 3B)\omega^2 = 0 \\ 2B(-\alpha^2) + (5B - 3A)\omega^2 = 0 \end{aligned}\right\} \text{(1)}$$

$$\begin{bmatrix} -2\alpha^2 + 5\omega^2 & -3\omega^2 \\ -3\omega^2 & -2\alpha^2 + 5\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, for A, B to be non-trivial

This, Should be non-invertible

* $\omega = \frac{\bar{\omega}+\omega}{2} - \frac{\bar{\omega}-\omega}{2}$

So, $4\alpha^4 - 20\alpha^2\omega^2 + 16\omega^4 = 0$ in (I)

$\Rightarrow \alpha = \pm\omega \Rightarrow$ by substituting $A = B$

$\alpha = \pm 2\omega \Rightarrow$ = " = $A = -B$

$x = A_1 e^{i\omega t} + A_2 e^{i\omega t} + A_3 e^{2i\omega t} + A_4 \cdot e^{-2i\omega t}$

$y = A_1 e^{i\omega t} + A_2 e^{i\omega t} - A_3 e^{2i\omega t} - A_4 e^{2i\omega t}$

normal mode, means, there exist a set of coordinate, when (or parts) systems vibrated with one single frequency.

When done; $(x+y) = A_1 (e^{i\omega t}) + A_2 (e^{-i\omega t})$ · Same frequency in Sino and Cos.

* $\ddot{x}_1 = a_1 x_1 + b_1 x_2$ | $x_1 = A e^{i\alpha t}$

$\ddot{x}_2 = a_2 x_2 + b_2 x_2$ | $x_2 = B e^{i\alpha t}$

$x_1 =$

quadratic eqn in $\beta = \alpha^2$, $\alpha = \pm\sqrt{\beta_1}$, $\pm\sqrt{\beta_2}$

always, $B = c_1 A$
$B = c_2 A$

So; $x_1 = A_1 e^{i\sqrt{\beta_1}t} + A_2 e^{i\sqrt{\beta_1}t} + A_3 \sim + A_4 \sim$

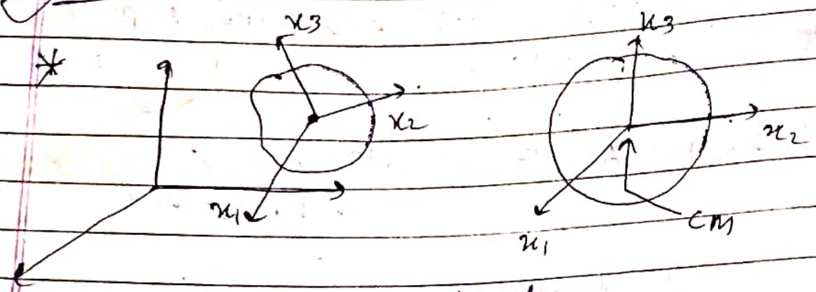
* $x_1 = A_1 \cos(\omega_1 t + \phi_1)$ Similarly, $y_1 = c_1 A_1 \sim + c_1 A \sim + c_2 A_3 e^{i\sqrt{\beta}t} + c_2 \sim$.

Thus, always we get such normal coordinates.

{ normal co-ordinates
normal mode
normal frequency }

left may be not in Quiz. Check.

RBD

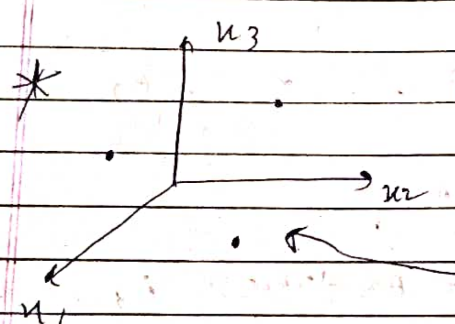


6 degree of freedom
(one body)

For N particle: 3N

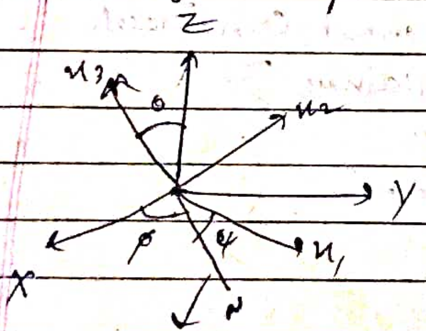
Contraint: $\alpha_{ij} = c_{ij}$

$$\text{dof} = 3N \underbrace{- \frac{1}{2} N(N-1)}_{\text{no of contraints}}$$ (X) goes to -ve when N



need three points to speicifically locate each point, i.e, Contants
So, 6 DOF − 3 contraints
= 3 DoF for Body.

DOF is actually points needed to ~~be~~ define a particle for RBody.

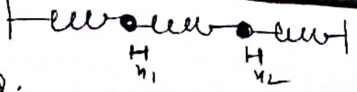


θ: Angle b/w Z and x_3

ϕ, ψ are Euler angle

N is intersection b/w x_2, x_1 plane ~~and~~ XY.

* So,

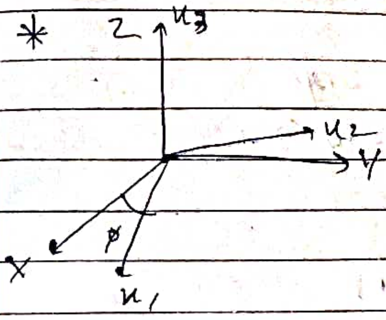


$x_1 > 0$ along
$x_2 > 0$ comp.

$(x_1 - x_2) > 0$ comp, So, m_1 gets $-F$
m_2 gets $+F$ (force).

So, $\theta \to 0$ to π
$\phi, \psi \to 0$ to 2π } Euler Angles.

*



Rotate about z by ϕ
Rotate by θ about x_1
Rotate by ψ about new x_3.

* So, If we choose axis through COM, $\sum m \vec{r} = 0$

$T =$

About Principal Axis: {off diagonal are 0}

$$T_{rot} = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

So, $\Omega_1 = \dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi$

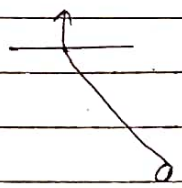
$\Omega_2 = -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi$

$\Omega_3 = \dot\phi\cos\theta + \dot\psi$

For Top, $I_1 = I_2 = I \neq I_3$

$$T_{rot} = \frac{I}{2}\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \frac{I_3}{2}\left(\dot\phi\cos\theta + \dot\psi\right)^2$$

✓ *



$$\boxed{\frac{d\lambda}{dt} \ll \frac{\lambda}{T}}$$

{Means very slowly, adiabatic case}

Time periodic

$\langle A\rangle = \bar{A}$ Now, $H = H(q, p, \lambda)$

$\frac{dE}{dt} = \frac{dH}{dt} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}$ {apply by $\partial H/\partial p_s = \dot q$ no ang cont.}

So, $\overline{\frac{dE}{dt}} = \overline{\left(\frac{\partial H}{\partial \lambda}\right)}\frac{d\lambda}{dt}$ {as $\lambda(t)$} T

$\langle A\rangle = \bar{A}$ So, $\overline{\frac{\partial H}{\partial t}}$ $\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^T \frac{\partial H}{\partial \lambda}\, dt$

Now, $\dot q = \frac{\partial H}{\partial p} \Rightarrow dt = \frac{dq}{(\partial H/\partial p)} \Rightarrow T = \int_0^T dt$

$$T = \oint dq/(\partial H/\partial p)$$

Integral over a period.

$E = H =$ Constant

So, $\frac{dE}{dt} = \frac{d\lambda}{dt} \frac{\oint \partial H/\partial\lambda \, dq/(\partial H/\partial p)}{\oint dq/(\partial H/\partial p)}$ ~~not dependent on t~~

$H(p,q,\lambda) = E$

$\frac{\partial H}{\partial \lambda}\frac{d\lambda}{d\lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} = 0$ {as E is const. in one T}

$\Rightarrow \frac{\partial H/\partial\lambda}{\partial H/\partial p} = -\frac{\partial p}{\partial \lambda}$

$E = E(p,q,\lambda)$

$p = p(E,q,\lambda)$

$\frac{dE}{dt} = -\frac{d\lambda}{dt}\frac{\oint \partial p/\partial\lambda \, dq}{\oint \frac{\partial p}{\partial E} dq}$

How is averaging not out of formula?

$\oint\left[\frac{\partial p}{\partial E}\frac{\overline{dE}}{dt} + \frac{\partial p}{\partial\lambda}\frac{d\lambda}{dt}\right] dq = 0$ From here

→ Adiabatic Invariant

$I = \frac{1}{2\pi}\oint p\,dq = \frac{1}{2\pi}\iint dp\,dq$

Here average value of $\frac{\overline{dI}}{dt}$

$\frac{\overline{dI}}{dt} = \frac{1}{2\pi}\oint\left[\frac{\partial p}{\partial E}\frac{\overline{dE}}{dt} + \frac{\partial p}{\partial\lambda}\frac{d\lambda}{dt}\right] dq = 0$

'0'. So, it's (I) is called adiabatic invariant.

cyclic

$\cancel{\frac{\partial I}{\partial E} = \frac{1}{2\pi}\oint \frac{\partial p}{\partial E}\frac{dE}{dt} + \frac{\partial p}{\partial \lambda}\frac{d\lambda}{dt}}$

→ by differentiating formula of integral.

$2\pi\frac{\partial I}{\partial E} = \frac{1}{2\pi}\oint \frac{\partial p}{\partial E} dq = T$

$\partial E/\partial I = \frac{2\pi}{T} = \omega$, ω is the vibrational frequency of the system.

//

$$I' = \frac{\partial I}{\partial E}$$

$$\Rightarrow \boxed{E = I\omega}$$

$$\tan\theta = \frac{m\omega q}{p} \qquad \frac{\partial \theta}{\partial p} = -\frac{\omega q}{2E} \qquad \frac{\partial \theta}{\partial q} = \frac{\omega p}{2E}$$

$$I'\left(\frac{p}{m}\,\frac{\omega p}{2E} + kq\,\frac{\omega q}{2E}\right) = 1$$

Cyclic: Does not appear in Hamiltonian

$\cancel{\frac{I'\omega}{E}\left(\frac{p^2}{2m}+\frac{1}{2}kq^2\right)}=1$

$I'=\frac{1}{\omega}$

$\frac{dI}{dE}=\frac{1}{\omega}$

$I=\frac{E}{\omega}+\cancel{c}^{0}$

$\boxed{E=I\omega}=H$

Where area,

So, $E=\frac{p^2}{2m}+\frac{1}{2}kq^2 \Rightarrow A=2\pi I \Rightarrow I=\frac{A}{2\pi}$.

↳ Ellipse

$I=\frac{1}{2\pi}\oint p\,dq$

Canonical Transformation:

$Q=\theta,\ P=I \qquad F_2(q,P,t),\ p=\partial F_2/\partial q$

$Q=\partial F_2/\partial P$

$\theta=\frac{\partial F_2}{\partial I} \qquad p=\frac{\partial F_2}{\partial q}$

So, $H(q,\partial F_2/\partial q)=E \Rightarrow \frac{1}{2m}\left(\frac{\partial F_2}{\partial q}\right)^2+V(q)=E$.

$\frac{dF_2}{dt}=\frac{\partial F_2}{\partial q}\frac{dq}{dt}=p\dot{q}$

$F_2=\cancel{F_2}\Big|_0^{0}+\int p\,dq=\int_0^q\sqrt{2m(E-V)}\,dq$

This F_2 gives $\theta \equiv$ cyclic, $I =$ Const,

$$F_2 = \int_0^q \sqrt{2m\left(I\omega - V(q)\right)}\, dq$$

$$\frac{\partial F_2}{\partial q} = \sqrt{2m(E-V)} = p$$

$$\theta = \frac{\partial F_2}{\partial P} = \frac{\partial F_2}{\partial I} = \frac{\partial}{\partial I}\int_0^q \sqrt{2m(I\omega - V)}\, dq$$

$$= \sin^{-1}\left\{ q\sqrt{\frac{k}{2I\omega}} \right\}$$

This gives, $F_2(q, I) = \frac{q}{2}\sqrt{2m\left(I\omega - \frac{kq^2}{2}\right)} + I\sin^{-1}\left(q\sqrt{\frac{k}{2I\omega}}\right)$

* For action angle variable, we need to have n-cyclic coordinate for n-degree of freedom, i.e, we need n-conserved quantities.

$I = \oint p\, dq$ $\qquad I = -i\hbar\, \partial/\partial\theta$

$H(I) = E$

$H(I)\psi = E\psi$ $\qquad \psi = e^{iI\theta/\hbar}$

$e^{2\pi i I/\hbar} = 1$ {for periodic}

$I = n\hbar$

$$\boxed{\oint p\, dq = n\hbar}$$

$H = \omega I = n\omega\hbar$

Small Oscillations

* $m\ddot{x} + kx = 0$ Let, $m = m(t)$, $k = k(t)$

$\frac{d}{dt}(m\dot{x}) + kx = 0$

Let,

$d\tau = \frac{dt}{m(t)}$

$\frac{1}{m(t)} \frac{d}{d\tau}\left(m(t) \frac{dx}{dt}\right) + kx = 0$

$\Rightarrow \frac{d}{d\tau}\left(\frac{dx}{d\tau}\right) + m(t) k(t) x = 0$

$\frac{d^2x}{d\tau^2} + (km) x = 0$

analogy of Let

$\frac{d^2x}{d\tau^2} + \omega^2(\tau) x = 0$

Redefinition of variable

$\frac{d^2x}{dt^2} + \omega^2(t) x = 0$

$\frac{d^2x}{dt^2} + \omega^2(t) x = 0$ $\omega(t+T) = \omega(t)$

i.e, periodic.

take $t \to t+T$

$x(t)$ is a soln.

$x(T+t)$ is a soln

$\frac{d^2}{dt^2} x(t+T) + \omega^2(t+T) x(t+T) = 0$

$\frac{d^2 x(t+T)}{dt^2} + \omega^2(t) x(t+T) = 0$

assumed to be So, Let $x_1(t), x_2(t)$ be the Soln.

x_1, x_2 are Real.

Then, $x_1(t+T) = a_1 x_1(t) + a_2 x_2(t)$

$x_2(t+T) = a_3 x_1(t) + a_4 x_2(t)$

Because also Soln, why $x_1(t)$ / also basis

For above to be valid,

So,

$$x_1(T+t) = \mu_1 x_1(t)$$

$$x_2(T+t) = \mu_2 x_2(t)$$

The most general form with prop. are

$$x_1(t) = \mu_1^{t/T} \Pi_1(t) \qquad \Pi_{1,2}(t+T) = \Pi_{12}(t)$$

$$x_2(t) = \mu_2^{t/T} \Pi_2(t) \qquad \dot{\Pi}_{1,2}(t+T) = \dot{\Pi}_{12}(t)$$

$$\ddot{x}_1 + \omega^2 x_1 = 0 \quad - (1) \times x_2$$

$$\ddot{x}_2 + \omega^2 x_2 = - (1) \times x_1$$

$$\ddot{x}_1 x_2 - \ddot{x}_2 x_1 = 0 \Rightarrow \frac{d}{dt}(\dot{x}_1 x_2 - \dot{x}_2 x_1) = 0$$

$$\boxed{\dot{x}_1 x_2 - \dot{x}_2 x_1 = C}$$

$$\dot{x}_1 = \frac{\mu_1^{t/T} (\ln \mu_1)}{T} \Pi_1 + \mu_1^{t/T} \dot{\Pi}_1$$

$$\dot{x}_1 x_2 = \frac{(\mu_1 \mu_2)^{t/T}}{T} \Pi_1 \Pi_2 + (\mu_1 \mu_2)^{t/T} \dot{\Pi}_1 \Pi_2$$

Similarly, $\dot{x}_2 x_1 =$

Then, $\dot{x}_1 x_2 \big|_{t+T} = (\mu_1 \mu_2) \times \dot{x}_1 x_2$

$\dot{x}_2 x_1 \big|_{t+T} = (\mu_1 \mu_2) \times \dot{x}_2 x_1$

Can see $\mu_1 \mu_2$ a extra appearing when evaluated at $t+T$

So, $\boxed{\mu_1 \mu_2 = 1}$

$$(z^n)^* = (z^*)^n$$

$$x_1^* = (\mu_1^*)^{t/T} \Pi_1(t) = x_1(t) \quad \{\text{Because } x_1 \text{ is Real}\}$$

$$x_2^* = (\mu_2^*)^{t/T} \Pi_2(t) = x_2(t)$$

$$\begin{pmatrix} x_1(t+T) \\ x_2(t+T) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \doteq \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Eigen Values

Eigen Value

So, $\mu_1^* = \mu_1$, i.e, Real.

$\mu_2^* = \mu_2$

Cases,

(1) $\mu_1 = \mu_2 = 1$,

Simply, periodic Solns.

(2) $\mu_1 = -1$, $\mu_2 = -1$

$\Rightarrow \quad x_{12} = (-1)^{t/T} R_{1,2}$

$x_{12}(t+2T) = (-1)^t R_{1,2} = x_{1,2}(t)$

↳ called periodic doubled Soln.

(3) $\mu_1 = \frac{1}{\mu_2}$

$x_1 = (\mu_1)^{t/T} R_1(t)$

$x_2 = (\mu_1)^{-t/T} R_2(t)$ } One Rises, and other one dies out.

So, deviation takes one way to ∞.

Parametric Resonance. $\to w(t), k(t)$

{To bring parameter has been changed.}

Now, let's define parametric resonance soln.

* Let $\omega^2(t) = \omega_0^2(1 + h\cos \Omega t)$, $\{ h \ll 1, h > 0$

small purturbation

Checks periodicity

$\ddot{x} + \omega_0^2 [1 + h\cos \Omega t] x = 0$

$h = 0$

Let, $x = x_0 = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

~~with $B = i$ it's imaginary Soln.~~

Let approximal x_0 be Sin, bcz x_0 is already occurring when $h=0$.

$$\ddot{x}_0 + \omega_0^2 \left[A\cos(\omega_0 t) + B\sin(\omega_0 t) \right] + \omega_0^2 h \cos \Omega t \left[A\cos(\omega_0 t) + B\sin(\omega_0 t) \right] \overset{!}{=} 0$$

Now,

to See ^ the system responding by h-factor, put, $\Omega = 2\omega_0$

$$\ddot{x}_0 + \omega_0^2 A\cos\omega_0 t \left(1+\frac{h}{2}\right) + \omega_0^2 B \sin\omega_0 t \left(1-\frac{h}{2}\right) + \left[(\cos 3\omega_0 t + \sin 3\omega_0 t)\right] \overset{!}{=} 0$$

now, Let,

$\Omega = 2\omega_0 + \varepsilon$, $\varepsilon \ll \omega_0$

$$\frac{d^2x}{dt^2} + \omega_0^2\left(1 + h\cos\left(2\omega_0+\varepsilon\right)t\right)x = 0$$

$$x = a(t)\cos\left(\omega_0+\frac{\varepsilon}{2}\right)t + b(t)\sin\left(\omega_0+\frac{\varepsilon}{2}\right)t$$

$a(t), b(t)$ are not oscillating {Changing Slowly}

Let $K_1 = (\omega_0 + \varepsilon/2)$, $K = K_1 t$

Then, $x = a\cos K + b\sin K$

$\left.\begin{matrix}\ddot{a} \ll \dot{a} \\ \ddot{b} \ll \dot{b}\end{matrix}\right\}$ If a, b are changing slowly, $\ddot{a}$ is more less.

$$\dot{x} = \dot{a}\cos K - aK_1\sin K + \dot{b}\sin K + bK_1\cos K$$

$$\ddot{x} = -2\dot{a}K_1\sin K - aK_1^2\cos K + 2\dot{b}K_1\cos K - bK_1^2\sin K \quad -(1)$$

$$\omega_0^2 x = \omega_0^2 a\cos K + \omega_0^2 b\sin K \quad -(2)$$

$$\omega_0^2 x\, h\cos(\Omega t) = \omega_0^2 h\cos(2\omega_0+\varepsilon)t\left[a\cos K + b\sin K\right]$$
$$= \omega_0^2 h\cos(2K)\left[a\cos K + b\sin K\right]$$

how?

$\omega_0^2 h x \cos(\nu t) = \omega_0^2 h \left[a/2 \cos k - \frac{b}{2} \sin k \right]$ — (3)

~~$\sin k \left[-2\dot a k_1 - b k_1^2 + b\omega_0^2 - \frac{b}{2}\omega_0^2 h \right] + \cos k \left[2\dot b k_1 - a k_1^2 + a\omega_0^2 + \frac{a}{2}\omega_0^2 h \right] = 0$~~

$$\sin k \left[-2a k_1 - b k_1^2 + b\omega_0^2 - \frac{b}{2}\omega_0^2 h \right]$$

$$+ \cos k \left[2\dot b k_1 - a k_1^2 + a\omega_0^2 + \frac{a}{2}\omega_0^2 h \right] = 0$$

~~$2\dot a\left(\omega_0 + \frac{\varepsilon}{2}\right) + b\left(\omega_0 + \frac{\varepsilon}{2}\right)^2 + \frac{b}{2}\omega_0^2 h = 0$~~

~~$2\dot b\left(\omega_0 + \frac{\varepsilon}{2}\right) - a\left(\omega_0 + \frac{\varepsilon}{2}\right)^2 +$~~

has to be '0' for holding.

Ignore ε^2

$$2\dot a\left(\omega_0 + \frac{\varepsilon}{2}\right) + b\left(\omega_0 + \frac{\varepsilon}{2}\right)^2 - b\omega_0^2 + \frac{b}{2}\omega_0^2 h = 0$$

$$2\dot b\left(\omega_0 + \frac{\varepsilon}{2}\right) - a\left(\omega_0 + \frac{\varepsilon}{2}\right)^2 + a\omega_0^2 + \frac{a}{2}\omega_0^2 h = 0$$

$$\left(\omega_0 + \frac{\varepsilon}{2}\right)^{-1} = \frac{1}{\omega_0}\left(1 - \frac{\varepsilon}{2\omega_0}\right)$$

$$\dot a + \frac{b}{2}\omega_0 \varepsilon \frac{1}{\omega_0}\left(1 - \frac{\varepsilon}{2\omega_0}\right) + \frac{1}{4}\omega_0 h = 0$$

$$\dot a + b\left[\frac{\varepsilon}{2} + \frac{\omega_0 h}{4}\right] = 0 \quad -(4)$$

$$\dot b + a\left(-\frac{\varepsilon}{2} + \frac{\omega_0 h}{4}\right) = 0 \quad -(5)$$

Sung Ansartz,
~~Absatz~~,
for linear differential eqn.
$a(t) = a_1 e^{st}$
$b(t) = b_1 e^{st}$

$$a_1 s + \frac{b_1}{2}\left(\varepsilon + \frac{\omega_0 h}{2}\right) = 0$$

$$b_1 s + \frac{a_1}{2}\left(-\varepsilon + \frac{\omega_0 h}{2}\right) = 0$$

$$S = -\frac{b_1}{2a_1}\left(\varepsilon + \frac{\omega_0 h}{2}\right)$$

$$S = \frac{-a_1}{2b_1}\left(-\varepsilon + \frac{\omega_0 h}{2}\right)$$

$$S^2 = \frac{1}{4}\left[\left(\frac{\omega_0 h}{2}\right)^2 - \varepsilon^2\right], \quad S = \pm\frac{1}{2}\sqrt{\left(\frac{\omega_0 h}{2}\right)^2 - \varepsilon^2} = \pm S_1$$

where, $-\frac{h\omega_0}{2} < \varepsilon < \frac{h\omega_0}{2}$

So, ~~let~~, an'dhot oscillatiy, comming from Simple Oscillation. $\frac{h\omega_0}{2} < \varepsilon < \frac{h\omega_0}{2}$ So, we get the Range of parametric Resonance.

So, all four could have been soln but same 'S' does not give anything interesting, Rest gives the same complex soln,

$$x = a_1 e^{S_1 t}\cos\left(\omega_0 + \frac{\varepsilon}{2}\right)t + b_1 e^{-S_1 t}\sin\left(\omega_0 + \frac{\varepsilon}{2}\right)t$$

If

$x_1 = \mu^{t/T} R_1$

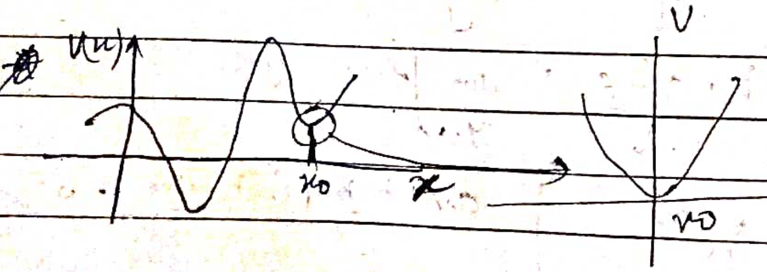
$x_2 = \mu^{-t/T} R_2$

then, $\mu = \left(e^{2S\pi/\omega_0}\right)^{t/T}$

$$\mu^{t/T} = e^{St}\, e^{\frac{t\,\omega_0}{2\pi}\times\frac{2S\pi}{\omega_0}} = e^{St}$$

Anharmonic Oscillations $\left\{ \text{So, } F = -kx - \alpha x^3 - \beta x^3 \right\}$

* $\ddot{x} + \omega_0^2 x + \beta x^3 = 0 \rightarrow$ Duffing oscillator



$\partial V/\partial x = 0$ at max or min i.e. x_0,

$$V(x) = V(x_0) + \cancel{(x-x_0)\frac{\partial V}{\partial x}} + \frac{1}{2}(x-x_0)^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{6}(x-x_0)^3\frac{\partial^3 V}{\partial x^3} + \frac{1}{24}(x-x_0)^4\frac{\partial^4 V}{\partial x^4} + \cdots$$

* So, x^3 term neglected because it gives unbounded potential.

$E = \frac{1}{2}kx^2 + \frac{1}{4}\beta x^4$ {at turning point all E is potential}

$$x_{tp} = \sqrt{\frac{2E}{k}} - \frac{\beta E}{2k^2}\sqrt{\frac{2E}{k}} + \frac{7\beta^2 E^2}{8\,k^4}\sqrt{\frac{2E}{k}} + \ldots$$

$\ddot{x} + \omega_0^2 x + \beta x^3 = 0$, let $\beta x^3 << \omega_0^2 x$

because it's like perturbation.

$x(t) = x_0 + \beta x_1$ why

x_0 can't be sol because bcz, $\ddot{x}_0 + \omega_0^2 x_0 = 0$ what about βx_0^3.

Absurd: $x(t) = x_0 + \beta x_1$

$$\frac{d^2}{dt^2}(x_0 + \beta x_1) + \omega_0^2(x_0 + \beta x_1)$$

$$+\beta(x_0 + \beta x_1)^3 = 0$$

as $\beta <<$ x_0 is sum of harmonic.

Upto $O(\beta)$, {no β terms equal to zero, soln to sum}

$$\ddot{x}_1 + \omega_0^2 x_1 + x_0^3 = 0.$$

$$\ddot{x}_1 + \omega_0^2 x_1 = (A_0 e^{i\omega_0 t} + B_0 e^{-i\omega_0 t})^3$$

the reason for shooting the amplitude.

$$\boxed{\ddot{x}_1 + \omega_0^2 x_1 = C_1 \cos\omega_0 t + C_2 \cos(3\omega_0)t}$$

(natural frequency equals Driving)

How?

Can't hold because $\omega = \omega_0$ shoots the amplitude, But how small change can do that, ~~as~~ ~~even~~ even energy is not being added.

Let's try commenting,

$x = x_0 + \beta x_1$

and $\omega = \omega_0 + \beta\omega_1$, now, {S.t $x_0 = A\cos(\omega t)$}

$$\ddot{x}_0 + \omega^2 x_0 = 0$$

If $\beta = 0$,

$$\ddot{x}_0 + \omega_0^2 x_0 = 0 \quad \text{\{Satisfies the Eqn\}}$$

$$\frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = -\beta x^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\ddot{x}$$

$$\frac{\omega_0^2}{\omega^2}\left(\ddot{x}_0 + \beta\ddot{x}_1\right) + \omega_0^2\left(x_0 + \beta x_1\right) = -\beta\left(x_0 + \beta x_1\right)^3 - \frac{\beta\omega_1\left(2\omega_0 + \beta\omega_1\right)\left(\ddot{x}_0 + \beta\ddot{x}_1\right)}{\omega^2}$$

$$\frac{\omega_0^2}{\omega^2}\ddot{x}_1 + \omega_0^2 x_1 = -x_0^3 + 2\omega_0\omega_1 x_0$$

$$x_0 = A\cos(\omega t) \quad \text{\{Already Real part\}}$$

$$R.H.S = -A^3\cos^3(\omega t) + 2\omega_0\omega_1 A\cos\omega t$$
$$= -A^3\left[\frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t\right] + 2\omega_0\omega_1 A\cos\omega t$$

put $\omega_1 = \frac{3}{8}\frac{A^2}{\omega_0}$

$R.H.S = -\frac{A^3}{4}\cos 3\omega t$, no amplitude is ok.

Damped - Harmonic - Forced - Oscillation

* $\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = C_0\cos(\omega t) - \beta x^3$

$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x = C_0\cos(\omega t) = C_0 e^{i\omega t}$ ↳ take Real part.

Because

$x_p(t) = Ae^{i\omega t} \Rightarrow A = \dfrac{C_0}{\omega_0^2 - \omega^2 + 2i\mu\omega}$

$x_p(t) = \dfrac{C_0}{(\omega_0^2 - \omega^2 + 2i\mu\omega)} e^{i\omega t}$

Let $A = ae^{i\delta}$, if $\delta = 0$, A = Real

s.t. $ae^{i\delta} = \dfrac{C_0}{(\omega_0^2 - \omega^2) + 2i\mu\omega}$

Find $\cos(\delta)$, $\sin(\delta)$ by Comparing.

$\tan\delta = \dfrac{2\mu\omega}{(\omega^2 - \omega_0^2)}$ $\quad a = \dfrac{C_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\mu^2\omega^2}}$

$x_p = Ae^{i\omega t} = ae^{i(\delta + \omega t)}$

$x = \cancel{x_h}^{\,0} + x_p \doteq a\cos(\omega t + \delta)$ $\Big\}$ This such Eqn that after long time.

Now, see what happens after Resonance.

$\omega^2 = (\omega_0 + \varepsilon)^2 \quad \varepsilon \ll \omega_0$

$\omega_0^2 + 2\omega_0\varepsilon + \varepsilon^2 = \omega^2$

$\omega_0^2 - \omega^2 = -2\omega_0\varepsilon - \varepsilon^2$

$a = \dfrac{C_0}{\sqrt{4\varepsilon^2\omega_0^2 + 4\mu^2\omega_0^2}}$ $\Big\{$ Ignore $\varepsilon^3, \varepsilon^4 \Big\}$

$\tan\delta = \frac{\gamma}{\varepsilon}$ { put $\omega = \omega_0 + \varepsilon$ in previous Eqn }

$$a^2(\varepsilon^2+\gamma^2) = C_0^2/4\omega_0^2$$

~~non-linear~~ part changes the fundamental frequency, ~~Not distorted by forcing or~~ dissipation.

ω_0 (before) was replaced by

$$\omega = \omega_0 + \frac{3}{8}\frac{A^2}{\omega_0}\beta = \omega_0 + \kappa A^2$$ { non-linear solved before }

In (1) $\Rightarrow$ $\omega = \omega_0$ (before) $\to \omega_0 + \kappa a^2$ Now ↳ Why?

shifted

$$\varepsilon = \omega - \omega_0 \to \varepsilon - \kappa a^2$$

$$a^2(\varepsilon^2+\gamma^2) = \frac{C_0^2}{4\omega_0^2}$$

→ Same because approx when bring it here.

$$a^2\left((\varepsilon-\kappa a^2)^2+\gamma^2\right) = \frac{C_0^2}{4\omega_0^2}$$

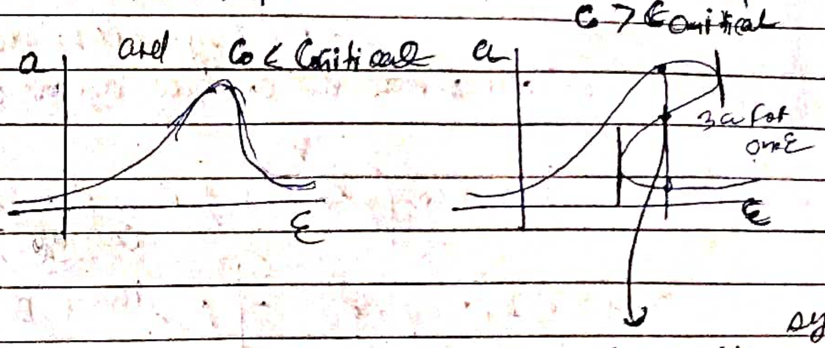
$$(\varepsilon-\kappa a^2)^2 = C_0^2/4\omega_0^2 a^2 - \gamma^2$$

After ignoring a^3, a^4 -- terms
we get, $c_0 \ll, \kappa a \ll$.

Same Eqn, $a^2(\varepsilon^2+\gamma^2) = \frac{C_0^2}{4\omega_0^2}$

Resonance = Max Amplitude

① If a is increased.



here, the sys is very unstable
Why?

For such points,

$$\frac{da}{d\varepsilon} = \frac{-\varepsilon a + Ka^3}{\varepsilon^2 + m^2 - 4K\varepsilon a^2 + 3K^2a^4}$$

So, Denominator → 0
$da/d\varepsilon \to \infty$

$$\varepsilon^2 + m^2 - 4K\varepsilon a^2 + 3K^2a^2 = 0$$

$$2\varepsilon = 4Ka^2 \pm \sqrt{(4Ka^2)^2 - 4(3K^2a^2 + m^2)}$$

$$\frac{da}{d\varepsilon} = 0 \Rightarrow \varepsilon = Ka^2$$

$$a^2 = \frac{c^2}{4\omega_0^2 m^2} \Rightarrow a_{max} = \frac{c_0}{2\omega_0 m}$$

For, $\varepsilon^2 = 3m^2$ (Roots Coincide) {Given, c_0 also → critical and same point also}

So, $c_{0\,critical} = \frac{32}{3\sqrt{3}} \frac{m^2\omega_0^3}{K}$

Now, Let's See for x^2, did for x^3 before,

$$* \ \ddot{x} + \omega_0^2 x + \alpha x^2 = 0 \quad \Big| \quad \ddot{x}_0 + \omega_0^2 x_0 = 0$$

1st try with $x_0 = x_0 + \alpha x_1$ } Done afterwards

$x = x_0 + \alpha x_1$; $\omega = \omega_0$

$$(\ddot{x}_0 + \alpha \ddot{x}_1) + \omega_0^2 (\cancel{x_0} + \alpha x_1) + \alpha (x_0 + \alpha x_1)^2 = 0$$

$\cos(\omega_0 t)$

$$\ddot{x}_1 + \omega_0^2 x_1 = -x_0^2 = -\left[A_0 e^{i\omega t} + B_0 e^{i\omega t}\right]^2$$

↳ Here we get $\cos(2\omega t)$ terms does not shoot. So Good But.

Let, $x = x_0 + \alpha x_1 + \alpha^2 x_2$

Then,

$$(\ddot{x}_0 + \alpha \ddot{x}_1 + \alpha^2 \ddot{x}_2) + \omega_0^2 (x_0 + \alpha x_1 + \alpha^2 x_2) + \alpha (x_0 + \alpha x_1 + \alpha^2 x_2)^2 = 0$$

$$\alpha^2 (\ddot{x}_2 + \omega_0^2 x_2) + \alpha \left[x_0^2 + \alpha^2 x_1^2 + \alpha^4 x_2^2 + 2\alpha x_0 x_1 + 2\alpha^2 x_0 x_2 + 2\alpha^3 x_1 x_2 \right] = 0$$

So, Retaining α term:

$$\ddot{x}_2 + \omega_0^2 x_2 + 2 x_0 x_1 = 0$$

$\cos(\omega_0 t)$ $\cos(2\omega_0 t)$ $\cos(\omega_0 t)$

Here, again amplitude shoots up.

$$* \text{ Let, } \omega = \frac{\omega_0}{2} + \varepsilon \Rightarrow \ddot{x} + \omega_0^2 x = C_0 \cos(\omega_0/2 + \varepsilon) t$$

in

So, $x_p = a \cos(\omega t + \delta)$

$$x_p = \frac{C_0}{(\omega_0^2 - \omega^2)} \cos(\omega t + \delta)$$

$\to 0$

$$\omega_0^2 - \omega^2 = \omega_0^2 - \frac{\omega_0^2}{4} - \omega_0 \varepsilon$$
$$= \frac{3}{4}\omega_0^2 - \omega\varepsilon = \frac{3}{4}\omega_0^2$$

? $x_0 = \frac{4C_0}{3\omega_0^2} \cos\left(\frac{\omega_0}{2} + \varepsilon\right)t$

is it p.

Let, Forced-Damped-Oscillator with non-linear term,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = C_0\cos(\omega t) - \alpha x^2$$

How to choose?

Abt, $x = x_0 + x_1$ ← because.

$$\text{s.t.}, \quad \ddot{x}_0 + 2\gamma\dot{x}_0 + \omega_0^2 x_0 = C_0\cos(\omega t)$$
$$\alpha\ddot{x}_1 + 2\gamma\alpha\dot{x}_1 + \omega_0^2\alpha x_1 + \alpha(x_1)^2 = -\alpha x_0^2 - \alpha^2 x_0 x_1$$

Ignore. Exp later.

$$\alpha x_1 = x_1$$

$$\therefore \ddot{x}_1 + 2\gamma\dot{x}_1 + \omega_0^2 x_1 + \alpha x_1^2 = -\alpha x_0^2$$
$$= -\alpha\frac{8C_0^2}{9\omega_0^4}\left[1 + \cos(\omega_0 + 2\varepsilon)\right]$$

This is the non-linear effect we get.

Where did x_1 come from?

$*$ For, $\ddot{x} + \omega_0^2 x + \alpha x^2 = 0$

$$\ddot{x}_1 + \omega_0^2 x_1 + x_0^2 = 0$$

homogeneous; (superposition)

Let, $x = x_0 + \alpha x_1$ | $x_0 = a\cos(\omega_0 t) = a\cos(\omega_0 t + \phi)$

$$\text{s.t.}, \quad \ddot{x}_0 + \omega_0^2 x_0 = 0$$

$$\Rightarrow \ddot{x}_1 + \omega_0^2 x_1 = -a^2\cos^2(\omega_0 t) = -\frac{a^2}{2}(1 + \cos 2\omega_0 t)$$

Let $x_1 = A\cos(2\omega_0 t)$

Then, add the constant.

$$(-4\omega_0^2 + \omega_0^2)A = -\frac{a^2}{2} \Rightarrow A = \frac{a^2}{6\omega_0^2}$$

$$x_1 = \frac{a^2}{6\omega_0^2}\left[\cos(2\omega_0 t) - 3\right]$$

Now, Let's See for x^2, did for x^3 before,

* $\ddot{x} + \omega_0^2 x + \alpha x^2 = 0$ | $\ddot{x}_0 + \omega_0^2 x_0 = 0$

→ Ist try with $x_0 = x_0 + \alpha x_1$ {Done after Manch}

$x = x_0 + \alpha x_1$; $\omega = \omega_0$

$(\ddot{x}_0 + \alpha \ddot{x}_1) + \omega_0^2 (x_0 + \alpha x_1) + \alpha (x_0 + \alpha x_1)^2 = 0$

$\cos(\omega_0 t)$

$\ddot{x}_1 + \omega_0^2 x_1 = -x_0^2 = -\left[A_0 e^{i\omega t} + B_0 e^{i\omega t}\right]^2$

→ Here we get $\cos(2\omega t)$ terms does not shoot. So Good But.

Let, $x = x_0 + \alpha x_1 + \alpha^2 x_2$

Then,

$(\ddot{x}_0 + \alpha \ddot{x}_1 + \alpha^2 \ddot{x}_2) + \omega_0^2 (x_0 + \alpha x_1 + \alpha^2 x_2)$

$+ \alpha (x_0 + \alpha x_1 + \alpha^2 x_1)^2 = 0$

$\alpha^2 (\ddot{x}_2 + \omega_0^2 x_2) + \alpha [x_0^2 + \alpha^2 x_1^2 + \alpha^4 x_2^2 + 2\alpha x_0 x_1 + 2\alpha^2 x_0 x_2 + 2\alpha^3 x_1 x_2] = 0$

So, Retaining α term:

$\ddot{x}_2 + \omega_0^2 x_2 + 2 x_0 x_1 = 0$

$\cos(\omega_0 t)$ $\cos(2\omega_0 t)$ → $\cos(\omega_0 t)$

Here, again amplitude shoots up.

* Let, $\omega = \frac{\omega_0}{2} + \varepsilon \Rightarrow \ddot{x} + \omega_0^2 x = C_0 \cos(\omega_0/2 + \varepsilon) t$

in

So, $x_p = a \cos(\omega t + \delta)$

$x_p = \frac{C_0}{(\omega_0^2 - \omega^2)} \cos(\omega t + \delta)$ → 0

$$\omega_0^2 - \omega^2 = \omega_0^2 - \frac{\omega_0^2}{4} - \omega_0 \varepsilon$$
$$= \tfrac{3}{4}\omega_0^2 - \omega_0\varepsilon = \tfrac{3}{4}\omega_0^2$$

?

$$x_0 = \frac{4C_0}{3\omega_0^2} \cos\left(\frac{\omega_0}{2} + \varepsilon\right)t$$

is it p.

Now, Forced-Damped-Oscillator with non-linear term,

$$\ddot{x} + 2\gamma \dot{x}_0 + \omega_0^2 x = C_0\cos(\omega t) - \alpha x^2$$

Abut, $x = x_0 + \alpha x_1$ ← because.

How it done?

$$\text{s.t., } \ddot{x}_0 + 2\gamma\dot{x}_0 + \omega_0^2 x_0 = C_0\cos(\omega t)$$

$$\alpha\ddot{x}_1 + 2\gamma\cdot\alpha\dot{x}_1 + \omega_0^2 \alpha x_1 + \alpha(\alpha x_1)^2 = -\alpha x_0^2 \; - \alpha^2 x_0 x_1$$

Ignore. Exp later.

$$\alpha x_1 = x_1$$

$$\therefore \ddot{x}_1 + 2\gamma\dot{x}_1 + \omega_0^2 x_1 + \alpha x_1^2 = -\alpha x_0^2$$

$$= -\alpha \frac{8C_0^2}{9\omega_0^4}\left[1 + \cos(\omega_0 + 2\varepsilon)\right]$$

This is the non-linear effect we get.

Where did x_1 come from?

* FOR, $\ddot{x} + \omega_0^2 x + \alpha x^2 = 0$

$$\ddot{x}_1 + \omega_0^2 x_1 + x_0^2 = 0$$

homogeneous; (Superposition)

Let, $x = x_0 + \alpha x_1$ | $x_0 = a\cos(\omega_0 t) = a\cos(\omega_0 t + \phi)$

$$\text{s.t., } \ddot{x}_0 + \omega_0^2 x_0 = 0$$

$$\Rightarrow \ddot{x}_1 + \omega_0^2 x_1 = -a^2\cos^2(\omega_0 t) = -\frac{a^2}{2}(1 + \cos 2\omega_0 t)$$

Let $x_1 = A\cos(2\omega_0 t)$

Then, add the constant.

$$(-4\omega_0^2 + \omega_0^2)A = -\frac{a^2}{2} \Rightarrow A = \frac{a^2}{6\omega_0^2}$$

$$x_1 = \frac{a^2}{6\omega_0^2}\left[\cos(2\omega_0 t) - 3\right]$$

Now, Let $x = x_0 + \alpha x_1 + \alpha^2 x_2$

put in Eq 7,

$$(\ddot{x}_0 + \alpha \ddot{x}_1 + \alpha^2 \ddot{x}_2) + \omega_0^2 (x_0 + \alpha x_1 + \alpha^2 x_2)$$

$$+ \alpha (x_0 + \alpha x_1 + \alpha^2 x_2)^2 = 0$$

$$\Rightarrow \alpha^2 (\ddot{x}_2 + \omega_0^2 x_2 + 2x_0 x_1) = 0$$

$$\ddot{x}_2 + \omega_0^2 x_2 + 2x_0 x_1 = 0$$

$$2x_0 x_1 = \underbrace{2a\cos(\omega_0 t) \frac{a^2}{6\omega_0^2} [\cos 2\omega_0 t - 3]}$$

So, by combining we get $\cos(\omega_0 t)$

Thus, Let $x = x_0 + \alpha x_1 + \alpha^2 x_2$ ~~$x_0 = a\cos\omega t$~~ Set $\ddot{x}_0 + \omega^2 x_0 = 0$

$\omega = \omega_0 + \alpha \omega_1 + \alpha^2 \omega_2$ $x_0 = a\cos(\omega t)$

This makes calculation Easier,

$$\frac{\omega_0^2}{\omega^2} \ddot{x} + \omega_0^2 x = -\alpha x^2 - \left(1 - \frac{\omega_0^2}{\omega^2}\right) \ddot{x}$$

$$\frac{\omega_0^2}{\omega^2} (\ddot{x}_0 + \alpha \ddot{x}_1 + \alpha^2 \ddot{x}_2) + \omega_0^2 (x_0 + \alpha x_1 + \alpha x_2)$$

$$= -\alpha (x_0 + \alpha x_1 + \alpha^2 x_2)^2 - \frac{(\alpha\omega_1 + \alpha^2\omega_2)(2\omega_0 + \alpha\omega_1 + \alpha^2\omega_2)}{\omega^2} \times (\ddot{x}_0 + \alpha \ddot{x}_1 + \alpha^2 \ddot{x}_2)$$

Retaining upto order α,

$$\frac{\omega_0^2}{\omega^2} \ddot{x}_1 + \omega_0^2 x_1 = -x_0^2 - \frac{2\omega_0 \omega_1 \ddot{x}_0}{\omega^2}$$

Gives Resonance

So, $\omega_1 = 0$ bc3 at $O(\epsilon)$ we don't need.

upto $O(\alpha^2)$

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$$\frac{\omega_0^2}{\omega^2}\ddot{x}_2 + \omega_0^2 x_2 = -2\left[x_0 x_1 - \frac{\omega_0 \omega_2}{\omega^2}\omega^2 x_0\right]$$

RHS of $O(\epsilon^2)$: $-2\left[x_0 x_1 - \omega_0 \omega_2 x_0\right]$

$= -2a\cos\omega t\left[x_1 - \omega_0\omega_2\right]$

as $\omega_1 = 0$

$$\frac{\omega_0^2}{\omega^2}\ddot{x}_1 + \omega_0^2 x_1 = -x_0^2 = -a^2\cos^2(\omega t)$$

$$\ddot{x}_1 + \omega^2 x_1 = \frac{-\omega^2 a^2}{2\omega_0^2}(1+\cos 2\omega t)$$

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{\omega^2 a^2}{2\omega_0^2}(1+\cos 2\omega t)$$

as we have a const.

Find particular soln with

Then add the constant in

R.H.S of Soln.

$$\cancel{\text{So, } x_1 = \frac{a^2}{6\omega_0^2}(\cos 2\omega t - 3) - 2a\cos(\omega t)\left[\frac{a^2}{6\omega_0^2}(\cos 2\omega t - 3) - \omega_0\omega_2\right] = -2a\left[\frac{a^2}{6}\right.}$$

$$x_1 = \frac{a^2}{6\omega_0^2}(\cos 2\omega t - 3) \text{ put in } O(\epsilon^2) \text{ Eqn}$$

$$\text{R.H.S} = -2a\cos\omega t\ \frac{a^2}{6\omega_0^2}\left\{(\cos 2\omega t - 3) - \omega_0\omega_2\right\}$$

$$= -2a\left(\frac{a^2}{6\omega_0^2}\times\frac{1}{2}\left(\cos 3\omega t + \cos\omega t\right) \times -\cos\omega t\right.$$

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*Take cebsants of cos type.

$$\boxed{-\frac{a^2}{2m\omega_0^2} x - \cos\omega t \; m\omega_0\omega_2}$$

$$\frac{a^2}{12m\omega_0^2} - \frac{a^2}{2m\omega_0^2} - m\omega_0\omega_2 = 0$$

$$\Rightarrow m\omega_0\omega_2 = \frac{a^2}{2m\omega_0^2}\left(\frac{-1+1}{6}\right) = \frac{-5a^2}{12m\omega_0^2}$$

$$\boxed{\omega_2 = \frac{-5a^2}{12m\omega_0^2}}$$

~~Here, For Finding non-Resonance, we needed x_1 to be solved.~~

Q Solve for x_1

* $\ddot{x}_1 + \omega_0^2 x_1 = -a^3 \cos^3(\omega_0 t)$

$\ddot{x}_1 + \omega_0^2 x_1 = -\frac{a^3}{4}\left[\cos 3\omega_0 t + 3\cos\omega_0 t\right]$

particular :

For $\ddot{x}_1 + \omega_0^2 x_1 = -\frac{a^3}{4} \cos\omega_0 t$

Same

$x_1 = A\cos(\omega_0 t)$ {Gets Cancelled}

So then,

Let $x = a\cos(\omega t) + \omega^2 b$

$\omega^2 x = \cancel{a\cos(\omega t)\omega^2} + \cancel{\omega^2 b} t \sin(\omega t)$

$\dot{x} = -a\omega \sin(\omega t) + b \sin(\omega t) + bt\omega \cos(\omega t)$

$\ddot{x} = -a\omega^2 \cos\omega t + b\omega\cos(\omega t) - \cancel{bt\omega^2 \sin(\omega t)}$

$+ b\omega \cos(\omega t)$

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$$x_1 = \frac{-a^3}{8\omega_0^2}\left[9\cos(\omega_0 t) + 12\omega_0 t \sin(\omega_0 t)\right]$$

$$\ddot{x}_1 + \omega_0^2 x_1 = \frac{-a^3}{4}\cos(\omega_0 t)$$