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Homework 3

1. $H = \dfrac{p^2(t)}{2m} + \dfrac{1}{2} m\omega^2 x^2(t)$

$z(t) = \sqrt{\dfrac{m\omega}{2\hbar}}\, x(t) + i \dfrac{p(t)}{\sqrt{2m\hbar\omega}}$

$\boxed{\dot{x} = \dfrac{\partial H}{\partial p} = p/m}$

$\dot{p} = \dfrac{-\partial H}{\partial x} = -m\omega^2 x$

now, $\dot{z} = \dfrac{\partial z}{\partial t} = \sqrt{\dfrac{m\omega}{2\hbar}}\, \dot{x} + i \dfrac{\dot{p}}{\sqrt{2m\hbar\omega}}$

$\Rightarrow \dot{z} = \sqrt{\dfrac{\omega}{2\hbar m}}\, p - \dfrac{i m\omega^2 x}{\sqrt{2m\hbar\omega}}$

$= \dfrac{\omega}{i} \dfrac{p(t)}{\sqrt{2m\hbar\omega}} + \dfrac{\omega\,(i)}{i^2} \dfrac{x(t)\sqrt{m\omega}}{\sqrt{2\hbar}}$

$\Rightarrow \boxed{\dot{z} = -i\omega z}$ equation of motion

sol^n of form $z(t) = C \exp(-i\omega t)$

$\Rightarrow \boxed{z(t) = \left(\sqrt{\dfrac{m\omega}{2\hbar}}\, x(0) + i \dfrac{p(0)}{\sqrt{2m\hbar\omega}}\right) e^{-i\omega t}}$

solution to the equation

$x(t) = (z(t) + z^*(t)) \sqrt{\dfrac{\hbar}{2m\omega}}$

$p(t) = (z(t) - z^*(t)) \dfrac{\sqrt{m\hbar\omega}}{i\sqrt{2}}$

let $\sqrt{\dfrac{m\omega}{2\hbar}}\, x(0) + i \dfrac{p(0)}{\sqrt{2m\hbar\omega}} = C \Rightarrow z^* = C^* e^{i\omega t}$

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$$\Rightarrow x(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(Ce^{-i\omega t} + C^* e^{i\omega t}\right)$$

$$p(t) = -i\sqrt{\frac{m\hbar\omega}{2}}\left(Ce^{-i\omega t} - C^* e^{+i\omega t}\right)$$

we can write $C = Ae^{i\phi} \Rightarrow C^* = Ae^{-i\phi}$

substituting and solving, we obtain:

$$x(t) = \frac{\sqrt{\hbar}}{\sqrt{2m\omega}}\left(2A\cos(\omega t - \phi)\right) = \boxed{\sqrt{\frac{2\hbar}{m\omega}}\, A \cos(\omega t - \phi)}$$

$$\boxed{p(t) = -\sqrt{2m\hbar\omega}\; A \sin(\omega t - \phi)}$$

Ans2- $S_p = \sum_{n=0}^{\infty} p^n e^{in\theta} = \frac{1 - pe^{-i\theta}}{1 - 2p\cos\theta + p^2}$, $|p| < 1$

plot $|S_p|^2$ v/s θ for $p = 0.01, 0.5, 0.99$

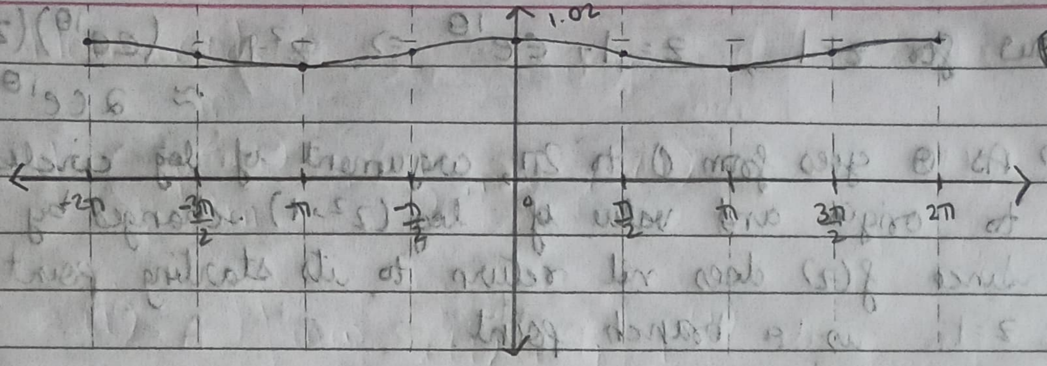
let $N = 1 - pe^{i\theta} \Rightarrow |N|^2 = (1 - pe^{-i\theta})(1 - pe^{i\theta})$

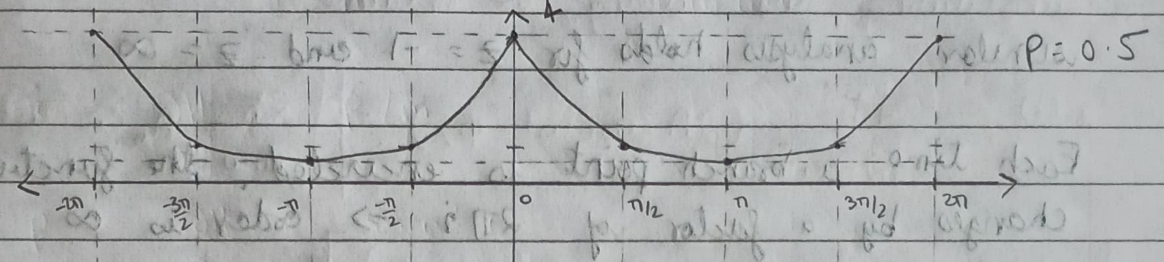
$$\Rightarrow |N|^2 = 1 + p^2 - p(e^{i\theta} + e^{-i\theta}) = 1 + p^2 - 2p\cos\theta$$

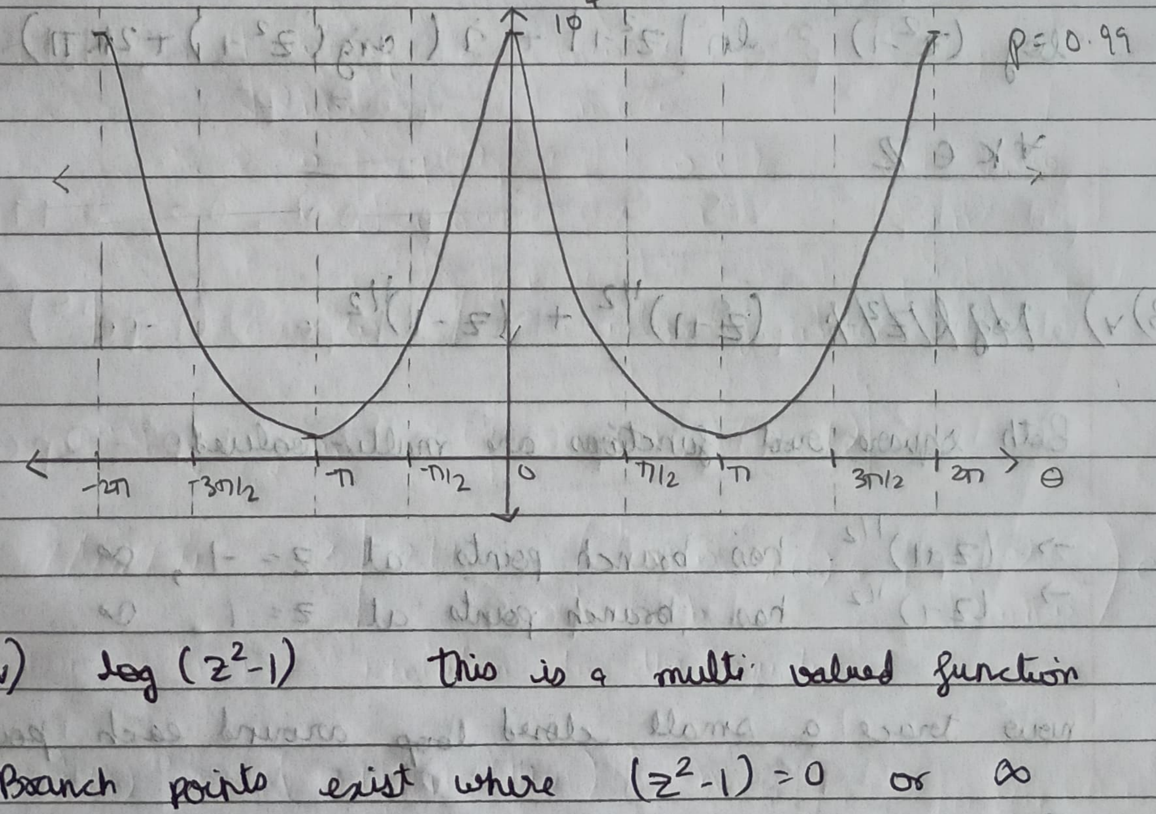
$$\Rightarrow |S_p|^2 = \frac{1}{1 + p^2 - 2p\cos\theta}$$

$\lvert S_p\rvert^2$	$\frac{1}{(1-p)^2}$	$\frac{1}{1+p^2}$	$\frac{1}{(1+p)^2}$	$\frac{1}{1+p^2}$	$\frac{1}{(1-p)^2}$
θ	0	$\pi/2$	π	$3\pi/2$	2π

now plotting for various values of p:







Ans 3) iv) $\log(z^2-1)$ This is a multi valued function

Branch points exist where $(z^2-1) = 0$ or ∞

$\Rightarrow z = 1, -1, \infty$ are branch points

To verify, Trace a small circle around each point.

Now for $z=1$, $z=1+\epsilon e^{i\theta}$ $\Rightarrow$ $z^2-1 = (\epsilon e^{i\theta})(2+\epsilon e^{i\theta})$
$\approx 2\epsilon e^{i\theta}$

$\Rightarrow$ As θ goes from 0 to 2π, argument of log circles to origin and value of $\log(z^2-1)$ changes by $2\pi i$. Since $f(z)$ does not return to its starting point, $z=1$ is a branch point.

similar analysis holds for $z=-1$ and $z=\infty$

Each time a branch point is encircled, the functions changes by a factor of $2\pi i$ $\Rightarrow$ order is ∞

There exist infinite number of brances as:

$$\log(z^2-1) = \ln|z^2-1| + i(\arg(z^2-1) + 2k\pi)$$

$\forall k \in \mathbb{Z}$

Ans3) v) ~~Arg l/z^2-1~~ $(z+1)^{1/2} + (z-1)^{1/2}$

Both square root functions are multi-valued.

$\rightarrow$ $(z+1)^{1/2}$ has branch points at $z=-1, \infty$
$\rightarrow$ $(z-1)^{1/2}$ has branch points at $z=1, \infty$

now trace a small closed loop around each point.

(a) For $z=1$:
As we circle $z=1$, the value of $(z-1)^{1/2}$ changes sign. The function $(z+1)^{1/2}$ is analytic within this loop, so its value does not change. Hence the

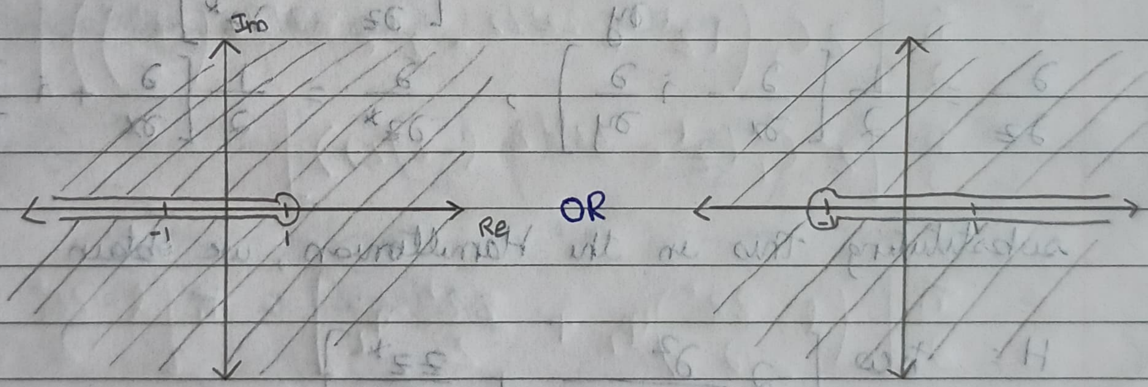
value of the function changes from $(z+1)^{1/2} + (z-1)^{1/2}$
to $(z+1)^{1/2} - (z-1)^{1/2}$.
Hence $z=1$ is a branch point

By symmetric analysis, $z=-1$ is also a branch point

At $z=\infty$, both functions change signs hence the complete function changes sign making $z=\infty$ one of the branch points.

$z = 1, -1, \infty$ are branch points, all of order 1

For each square root function there are two possible distinct values, hence $2 \times 2 = 4$ branches are present in the function



Branch cuts and single-valued domains for 3(iv) & 3(v)

Ans 9 - $H = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + \frac{1}{2}M\omega^2(x^2+y^2)$

$E_{n_r, m} = \hbar\omega(2n_r + |m| + 1) = \hbar\omega(n+1)$

where $n = 2n_r + |m|$, $n_r = 0, 1, 2 \ldots$
$m = 0, \pm 1, \pm 2 \ldots$

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(a) $a_0 = \sqrt{\dfrac{\hbar}{M\omega}}$, $z = \dfrac{x+iy}{a_0}$

oscillator length

we map this problem to z, z^* coordinates

let $\hat{x} = x/a_0$ $\hat{y} = y/a_0$

$$\Rightarrow H = \hbar\omega\left[\frac{-1}{2}\left(\frac{\partial^2}{\partial\hat{x}^2} + \frac{\partial^2}{\partial\hat{y}^2}\right) + \frac{1}{2}(\hat{x}^2 + \hat{y}^2)\right]$$

now $\hat{x} \to x$, $\hat{y} \to y$

also $z = \hat{x} + i\hat{y} \longrightarrow x + iy$

we know that: $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial z^*}$

$$\frac{\partial}{\partial y} = i\left[\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right]$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right], \quad \frac{\partial}{\partial z^*} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$$

substituting this in the Hamiltonian, we obtain:

$$H = \hbar\omega\left[-2\frac{\partial^2}{\partial z\,\partial z^*} + \frac{zz^*}{2}\right]$$

~~(b) For $n = 0$, $|m| = l$, consider:~~

~~$H\,\psi(z, z^*) = E\,\psi(z, z^*)$~~

~~The sol^n is : $\psi_l = z^l \exp(-|z|^2/2)$~~

(b) To show: $H\psi_l = E_l \psi_l$ for $\psi_l(z, z^*) = C_l z^l e^{-|z|^2/2}$

where $C_l = \dfrac{1}{\sqrt{\pi a_0^2 \, l!}}$

$$H\psi_l = \hbar\omega\left[-2\frac{\partial^2}{\partial z \, \partial z^*} + \frac{zz^*}{2}\right]\left(C_l z^l e^{-|z|^2/2}\right)$$

$$\frac{\partial \psi_l}{\partial z^*} = \frac{-C_l}{2} z^{l+1} e^{-|z|^2/2}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \psi_l}{\partial z^*}\right) = \frac{-C_l}{2}\left[(l+1) - \frac{zz^*}{2}\right] z^l e^{-zz^*/2}$$

$$= \frac{-1}{2}\left(l+1 - \frac{zz^*}{2}\right)\psi_l$$

substituting this back, we obtain:

$$H\psi_l = \hbar\omega\left[-2\left(\frac{-1}{2}\left(l+1-\frac{zz^*}{2}\right)\psi_l\right) + \frac{zz^*}{2}\psi_l\right]$$

$$= \hbar\omega\left[(l+1 - zz^*/2) + zz^*/2\right]\psi_l$$

$$= \hbar\omega(l+1)\psi_l$$

$\Rightarrow \psi_l$ is an eigenfunction with $E_l = \hbar\omega(l+1)$

Now we know that $H = H^*$ since the system is invariant under time-reversal operation

we have $H\psi_l = E_l \psi_l$

$\Rightarrow H^* \psi_l^* = E_l^* \psi_l^*$, $E_l = \hbar\omega(l+1) \in \mathbb{R}$

$\Rightarrow \boxed{H\psi_l^* = E_l \psi_l^*}$ hence ψ_l^* is also a eigen f^n with same eigenvalue E_l

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(c) $L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

$= -i\hbar\left[\left(\frac{z+z^*}{2}\right)\left[i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)\right] - \left(\frac{z-z^*}{2i}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right)\right]$

$= \frac{-i\hbar}{2}\left[zi\frac{\partial}{\partial z} - zi\frac{\partial}{\partial z^*} + iz^*\frac{\partial}{\partial z} - iz^*\frac{\partial}{\partial z^*}\right.$

$\left. - \frac{z}{i}\frac{\partial}{\partial z} - \frac{z}{i}\frac{\partial}{\partial z^*} + \frac{z}{i}\frac{\partial}{\partial z} + \frac{z}{i}\frac{\partial}{\partial z^*}\right]$

$= \boxed{\frac{\hbar}{2}\left[z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right]}$ QED

(d)(i) $L_z \psi_l (z, z^*)$

$= \frac{\hbar}{2}\left[z\frac{\partial\psi_l}{\partial z} - z^*\frac{\partial\psi_l}{\partial z^*}\right]$

$= \frac{\hbar}{2}\left[zC_l\left(lz^{l-1}e^{-zz^*/2} + z^l e^{-zz^*/2}\left(\frac{-z^*}{2}\right)\right)\right]$

$- \frac{\hbar}{2}\left[z^* C_l \left(z^l e^{-zz^*/2}\left(-z/2\right)\right)\right]$

$= \frac{\hbar}{2}\left[\left(l - \frac{zz^*}{2}\right)\psi_l + \frac{zz^*}{2}\psi_l\right] = \hbar\left[l - \frac{zz^*}{2} + \frac{zz^*}{2}\right]\psi_l$

$\Rightarrow \boxed{L_z\psi_l = l\hbar\psi_l}$

(ii) we know $L_z\psi_l(z, z^*) = l\hbar\psi_l(z, z^*)$

$L_z^* = \frac{\hbar}{2}\left[z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right] = -L_z$

now $L_z^*\psi_l^* = l\hbar\psi_l^*$

$\Rightarrow \boxed{L_z\psi_l^* = -l\hbar\psi_l^*}$ QED

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(e) $\Psi_l(z, z^*) = \frac{1}{(\pi a_0^2 l!)^{1/2}} z^l e^{-|z|^2/2}$

$\Psi_l(z) = \frac{1}{\sqrt{\pi a_0^2}} \frac{1}{(l!)^{1/2}} z^l e^{-|z|^2/2}$

$|\Psi_l(z)|^2 = \frac{1}{\pi a_0^2 l!} (|z|^2)^l e^{-|z|^2}$

$\Rightarrow S = \frac{e^{-|z|^2}}{\pi a_0^2} \sum_{l=0}^{\infty} \frac{(|z|^2)^l}{l!} = \frac{e^{-|z|^2} e^{|z|^2}}{\pi a_0^2}$

$\Rightarrow \boxed{S = 1/\pi a_0^2}$ QED

Ans 11) $\phi_m(z) = \frac{1}{\sqrt{2\pi 2^m m!}} z^m \quad m = 0, 1, 2 \ldots$

$\mu[z, z^*] = \exp(-|z|^2/2)$

$\langle \phi_n | \phi_m \rangle = \int \phi_n^*(z) \phi_m(z) \mu[z, z^*] dx dy$

(a) $\langle \phi_n | \phi_m \rangle = \int \frac{(z^*)^n z^m}{2\pi \sqrt{2^{n+m} n! m!}} \exp(-|z|^2/2) dx dy$

now convert this problem to polar coordinates

$dx dy \rightarrow r dr d\theta$, $|z|^2 = r^2$

$z = re^{i\theta} \Rightarrow \langle \phi_n | \phi_m \rangle = C_{m,n} \int r^{n+m} e^{(m-n)i\theta} e^{-r^2/2} r dr d\theta$

$\rightarrow \langle \phi_n | \phi_m \rangle = C_{m,n} \int e^{i(n+m)\theta} d\theta \int r^{n+m+1} e^{-r^2/2} dr$

Now if $n \neq m$:

$$\int_0^{2\pi} \exp[i(m-n)\theta]\, d\theta = \left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi} = \frac{1-1}{i(m-n)} = 0$$

$\Rightarrow \langle \phi_n | \phi_m \rangle = 0$ if $n \neq m$

In $m = n$:

$$\int_0^{2\pi} e^0\, d\theta = 2\pi$$

now $\int_0^{\infty} r^{n+m+1} \exp(-r^2/2)\, dr$ let $r^2/2 = u$ $\Rightarrow r\,dr = du$ $\Rightarrow (2u)^{1/2} = r$

$$= \int_0^{\infty} r^{m+n} \exp(-u)\, du$$

$$= \int_0^{\infty} 2^{\frac{m+n}{2}} u^{\frac{m+n}{2}} \exp(-u)\, du$$

$$= \sqrt{2^{m+n}} \underbrace{\int_0^{\infty} u^m \exp(-u)\, du}_{\text{gamma function}} = 2^m m!$$

substituting: $\langle \phi_m | \phi_m \rangle = \frac{1}{2\pi\, m!\, 2^m} (2\pi)(m!\, 2^m) = 1$

$$\Rightarrow \boxed{\langle \phi_m | \phi_n \rangle = \delta_{mn}} \quad \text{QED}$$

(b) $b^\dagger = z/\sqrt{2}$ $\quad b = \sqrt{2}\, d/dz$

To show $b^\dagger \phi_m \propto \phi_{m+1}$

$b\, \phi_m \propto \phi_{m-1}$

$$b^\dagger \phi_m = \frac{1}{\sqrt{2\pi\, 2^{m+1}\, m!}} z^{m+1} = \phi_{m+1} \sqrt{(m+1)}$$

Hence $b^{\dagger}$ raises the state from ϕ_m to ϕ_{m+1}

$$b\phi_m = \sqrt{2}\frac{d}{dz}\phi_m = \frac{m z^{m-1}}{\sqrt{2\pi 2^{m-1}(m-1)!\, m}} = \sqrt{m}\,\phi_{m-1}$$

Hence b lowers the state from ϕ_m to ϕ_{m-1}

(c) $[b, b^{\dagger}] = \left[\sqrt{2}\frac{d}{dz}, \frac{z}{\sqrt{2}}\right]$

$$[b, b^{\dagger}]\psi = \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}\frac{d}{dz}(z\psi) - \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} z\frac{d\psi}{dz}$$

$$= z\frac{d\psi}{dz} + \psi - z\frac{d\psi}{dz} = \psi$$

$\Rightarrow \boxed{[b, b^{\dagger}] = 1}$ QED