Assignment 2

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Name: Himanshu Sharma, Roll No.: 230474

1. $|\eta_1\rangle = (1\ 1\ 1)^T \quad |\eta_2\rangle = (1\ 1\ 0)^T \quad |\eta_3\rangle = (1\ 0\ 1)^T$

(a) let $\sum_{i=1}^{3} a_i |\eta_i\rangle = 0$

$$\Rightarrow \quad a_1 + a_2 + a_3 = 0 \qquad -①$$
$$a_1 + a_2 = 0 \qquad -②$$
$$a_1 + a_3 = 0 \qquad -③$$

From ① and ②: $a_3 = 0$
From ① and ③: $a_2 = 0$
$\Rightarrow a_1 = 0$

since $(a_1, a_2, a_3) = (0, 0, 0)$ is the only solution, the set of vectors is linearly independent

(b) $\boxed{|v_1\rangle = \frac{1}{\sqrt{3}}(1\ 1\ 1)^T}$

$$|v_2'\rangle = (1\ 1\ 0)^T - \left[(1\ 1\ 1)\cdot(1\ 1\ 0)^T\right]\left(\frac{1}{\sqrt{3}}\right)^3 (1\ 1\ 1)^T$$

$$= (1\ 1\ 0)^T - \frac{2}{3}(1\ 1\ 1)^T$$

$$= \left(\left(1-\frac{2}{3}\right)\ \left(1-\frac{2}{3}\right)\ \frac{-2}{3}\right)^T$$

$$= \frac{1}{3}(1\ 1\ -2)^T$$

$$\boxed{|v_2\rangle = \frac{1}{\sqrt{6}}(1\ 1\ -2)^T}$$

$$|v_3'\rangle = |\eta_3\rangle - \langle v_1|\eta_3\rangle|v_1\rangle - \langle v_2|\eta_3\rangle|v_2\rangle$$
$$= (1\ 0\ 1)^T - \frac{2}{3}(1\ 1\ 1)^T + \frac{1}{6}(1\ 1\ -2)^T$$

$$\Rightarrow |v_3'\rangle = \frac{1}{6}(3 \quad -3 \quad 0)^T$$

$$\Rightarrow |v_3\rangle = \frac{1}{\sqrt{2}}(1 \quad -1 \quad 0)^T$$

$$\boxed{|v_3\rangle = \frac{1}{\sqrt{2}}(1 \quad -1 \quad 0)^T}$$

(c) $P_1 = |v_1\rangle\langle v_1| = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$P_2 = |v_2\rangle\langle v_2| = \frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}$

$P_3 = |v_3\rangle\langle v_3| = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4) $\bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(a) $\bar{\sigma} \times \bar{\sigma} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sigma_x & \sigma_y & \sigma_z \\ \sigma_x & \sigma_y & \sigma_z \end{vmatrix}$

$= (\sigma_y\sigma_z - \sigma_z\sigma_y, [\sigma_z, \sigma_x], [\sigma_x, \sigma_y])$

$= 2(i\sigma_x, i\sigma_y, i\sigma_z)$

$= \boxed{2i\bar{\sigma}}$ as $[\sigma_i, \sigma_j] = i\,\varepsilon_{ijk}\sigma_k$

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4(b) $(\bar{\sigma}\cdot\bar{A})(\bar{\sigma}\cdot\bar{B}) = A\cdot B + i\bar{\sigma}\cdot(\bar{A}\times\bar{B})$

let $\bar{A} = (A_x, A_y, A_z)$ $\quad \bar{B} = (B_x, B_y, B_z)$

$$LHS = (A_x\sigma_x + A_y\sigma_y + A_z\sigma_z)(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z)$$

now $\sigma_i\sigma_j = i\sigma_k\varepsilon_{ikj} + I\delta_{ij}$

$$\Rightarrow LHS = A_xB_xI + A_xB_y i\sigma_z - A_xB_z i\sigma_y - B_xA_y i\sigma_z + A_yB_yI + A_yB_z i\sigma_z + A_zB_x i\sigma_y - A_zB_y i\sigma_x + A_zB_zI$$

$$= I(A_xB_x + A_yB_y + A_zB_z) + i\sigma_x(A_yB_z - A_zB_y) + i\sigma_y(A_zB_x - A_xB_z) + i\sigma_z(A_xB_y - A_yB_x)$$

$$= \bar{A}\cdot\bar{B}I + i\bar{\sigma}\cdot(\bar{A}\times\bar{B}) = RHS \quad \therefore QED$$

5(c) To prove: $e^{\pm i\bar{\sigma}\cdot\bar{v}} = \cos v \pm i(\bar{\sigma}\cdot\hat{v})\sin v$

$\hat{v} = \bar{v}/v$, $\quad v = |v|$

$$\exp(\pm i\bar{\sigma}\cdot\bar{v}) = 1 \pm \frac{i\bar{\sigma}\cdot\bar{v}}{1!} + \frac{(i\bar{\sigma}\cdot\bar{v})^2}{2!} \pm \frac{(i\bar{\sigma}\cdot\bar{v})^3}{3!} + \ldots$$

$$\bar{\sigma}\cdot\bar{v} = \sigma_xv_x + \sigma_yv_y + \sigma_zv_z$$
$$(\bar{\sigma}\cdot\bar{v})^2 = (v_x^2 + v_y^2 + v_z^2)I$$
$$(\bar{\sigma}\cdot\bar{v})^3 = (v_x^2 + v_y^2 + v_z^2)(\sigma_xv_x + \sigma_yv_y + \sigma_zv_z)$$
$$(\bar{\sigma}\cdot\bar{v})^4 = (v_x^2 + v_y^2 + v_z^2)^2 \quad \text{and so on}$$

① $-\cos v = 1 + \frac{(iv)^2}{2!} + \frac{(iv)^4}{4!} + \ldots = 1 + \frac{(i\bar{\sigma}\cdot\bar{v})^2}{2!} + \frac{(i\bar{\sigma}\cdot\bar{v})^4}{4!} + \ldots$

$$\sin v = \frac{v^1}{1!} + \frac{v^3}{3!} + \frac{v^5}{5!} + \cdots$$

$$\pm\left(\frac{i\bar{\sigma}\cdot\bar{v}}{1!} \pm \frac{(i\bar{\sigma}\cdot\bar{v})^3}{3} + \frac{(i\bar{\sigma}\cdot\bar{v})^5}{5!} + \cdots\right)$$

$$= \pm(i\bar{\sigma}\cdot\hat{v})\left(\frac{v}{1!} + \frac{v^2 v}{3!} + \frac{v^4 v}{5!} + \cdots\right)$$

$$= \pm(i\bar{\sigma}\cdot\hat{v}) \sin v \quad — ②$$

combining ① and ②:

$$\exp(\pm i\,\bar{\sigma}\cdot\bar{v}) = \cos v \pm i(\bar{\sigma}\cdot\hat{v}) \sin v \qquad : \text{QED}$$

(8) $\ln \sigma_j = \frac{i\pi}{2}[\mathbb{I} - \sigma_j]$

exponentiating both sides

$$\sigma_j = \exp\left(\frac{i\pi}{2}[\mathbb{I} - \sigma_j]\right)$$

solving the RHS

$$\left(\begin{array}{l}\exp(i\theta M) = \mathbb{I}(\cos\theta) + (i\sin\theta)M \\ \text{for any matrix that satisfies } M^2 = \mathbb{I}\end{array}\right)$$

~~Here, $(\mathbb{I} - \sigma_j)^2 = \mathbb{I}^2 - 2\sigma_j + \sigma_j^2$~~

$$\text{RHS} = \exp\left(\frac{i\pi}{2}\mathbb{I}\right)\exp\left(\frac{-i\pi}{2}\sigma_j\right)$$

$$= \left(\mathbb{I}\cos\pi/2 + i\,\mathbb{I}\sin\pi/2\right)\left(\mathbb{I}\cos\pi/2 + i\sigma_j\sin(-\pi/2)\right)$$

$$= (i^2\,\mathbb{I})(-1)\sigma_j = \boxed{\sigma_j} = \text{LHS}$$

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(4) $H = \frac{-\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2 y^2$

$x = y/a \quad a = \sqrt{\hbar/(m\omega)}$

$H = \frac{\hbar\omega}{2}\left[-\frac{d^2}{dx^2} + x^2\right]$

$a = \frac{1}{\sqrt{2}}\left[x + \frac{d}{dx}\right], \quad a^\dagger = \frac{1}{\sqrt{2}}\left[x - \frac{d}{dx}\right]$

where $\left(\frac{d}{dx}\right)^\dagger = \frac{-d}{dx}, \quad H = \hbar\omega(a^\dagger a + 1/2)$

(a) $[a, a^\dagger]\psi = \frac{1}{2}\left[x^2\psi - x\frac{d\psi}{dx} + \frac{d}{dx}(x\psi) - \frac{d^2}{dx^2}(\psi)\right] - \frac{1}{2}\left[x^2\psi + x\frac{d\psi}{dx} - \frac{d}{dx}(x\psi) - \frac{d^2}{dx^2}\psi\right]$

$= \frac{1}{2}\left[\underline{-x\frac{d\psi}{dx}} + \psi + \underline{x\frac{d\psi}{dx}} - \underline{x\frac{d\psi}{dx}} + \psi + \underline{x\frac{d\psi}{dx}}\right]$

$= \psi \Rightarrow \boxed{[a, a^\dagger] = 1}$

(b) $e^a a^\dagger e^{-a} = a^\dagger + 1, \quad e^{a^\dagger} a e^{-a^\dagger} = a - 1$

By Hadamard lemma:

$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots$

$\Rightarrow e^a a^\dagger e^{-a} = a^\dagger + [a, a^\dagger] + \frac{1}{2!}[a,[a,a^\dagger]] + \dots$

$= a^\dagger + 1 + 0 + 0 \dots = \boxed{a^\dagger + 1}$

now :

$$e^{a^\dagger} a e^{-a^\dagger} = \cancel{a-1}$$

$$= a + [a^\dagger, a] + \frac{1}{2!}[a^\dagger, [a^\dagger, a]] + \dots$$

$$= a - 1 + 0 + 0$$

$$= \boxed{a-1} \qquad \text{QED}$$

(c) $U \equiv U(\lambda, \lambda^*) = e^{\lambda a^\dagger - \lambda^* a}$

To ~~prove~~ prove: $U^\dagger U = UU^\dagger = 1$

$U = e^{\lambda a^\dagger} e^{-\lambda^* a}$ let $X = \lambda a^\dagger - \lambda^* a$

now $X^\dagger = \lambda^* a - \lambda a^\dagger = -X$

$U^\dagger U = e^{X^\dagger} e^{X} = e^{-X} e^{X} = 1$ —

$U U^\dagger = e^{X} e^{X^\dagger} = e^{X} e^{-X} = 1$

$\Rightarrow U^\dagger U = UU^\dagger = 1$

(d) To show : $e^{\lambda^* a - \lambda a^\dagger} a e^{\lambda a^\dagger - \lambda^* a} = a + \lambda$

again, we use the Hadamard lemma :

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots$$

let $X = \lambda^* a - \lambda a^\dagger$ now $\lambda a^\dagger - \lambda^* a = -X = X^\dagger$

$$\text{LHS} = a + [\lambda^* a - \lambda a^\dagger, a] + \frac{1}{2!}[X, [X, a]] + \dots$$

$$= a + \lambda^*[a, a] - \lambda[a^\dagger, a] + \dots$$

$$= a - \lambda(-1) + \dots$$

$$= a + \lambda + \underbrace{\frac{1}{2!}[X, \lambda] + \dots}_{\to 0}$$

$$\Rightarrow e^{\lambda^* a - \lambda a^\dagger} a e^{\lambda a^\dagger - \lambda^* a} = a + \lambda$$

similarly:

$$e^{\lambda^* a - \lambda a^\dagger} a^\dagger e^{\lambda a^\dagger - \lambda^* a} = a^\dagger + \left(\lambda^* [a, a^\dagger] - \lambda [a^\dagger, a^\dagger]\right) + 0 + 0 + \dots$$

$$= \boxed{a^\dagger + \lambda^*}$$

obtaining second exp from first, take a conj. on both sides:

$$\left(e^{\lambda^* a - \lambda a^\dagger} a e^{\lambda a^\dagger - \lambda^* a}\right)^\dagger = (a + \lambda)^\dagger$$

$$\Rightarrow e^{\lambda^* a - \lambda a^\dagger} a^\dagger e^{\lambda a^\dagger - \lambda^* a} = a^\dagger + \lambda^*$$

(II) $M = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$, $a \in [0,1]$, $b \in [0,1]$

(a) $M = \eta_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \eta_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \eta_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \eta_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \quad a = \eta_3 + \eta_0$$
$$b = \eta_1 - i\eta_2$$
$$1-a = \eta_1 + i\eta_2$$
$$1-b = \cancel{\eta_1 + i\eta_2} \; \eta_0 - \eta_3$$

~~$\Rightarrow a = 1 - \eta_1 - i\eta_2$~~ ~~$\Rightarrow b = 1 - \eta_0 + \eta_3$~~

$$\Rightarrow \boxed{\frac{1+b-a}{2} = \eta_1}, \quad \boxed{\frac{1-a-b}{2i} = \eta_2}$$

$$\boxed{\frac{1+a-b}{2} = \eta_0}, \quad \boxed{\frac{a-1+b}{2} = \eta_3}$$

(b) $M = a|0\rangle\langle 0| + b|0\rangle\langle 1| + (1-a)|1\rangle\langle 0| + (1-b)|1\rangle\langle 1|$

(c) let $M|v\rangle = \lambda|v\rangle$ where $|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

~~now, from (b):~~

$$\cancel{M|v\rangle = a|0\rangle\langle 0|v\rangle + b|0\rangle\langle 1|v\rangle + (1-a)|1\rangle\langle 0|v\rangle + (1-b)|1\rangle\langle 1|v\rangle}$$

$$\cancel{\Rightarrow \begin{pmatrix} av_1 + bv_2 \\ (1-a)v_1 + (1-b)v_2 \end{pmatrix} =}$$

eigenvalue eqn: $|M - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ 1-a & 1-b-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)(1-b-\lambda) - (b)(1-a) = 0$$

$$\Rightarrow a - \cancel{ab} - \lambda a - \lambda + b\lambda + \lambda^2 - b + \cancel{ab} = 0$$

$$\Rightarrow (a-b) + \lambda^2(1) + \lambda(b-a-1) = 0$$

$$\Rightarrow \lambda = \frac{(a+1-b) \pm \sqrt{a^2+b^2+1+2a-2b-2ab-4a+4b}}{2}$$

$$\Rightarrow \lambda = \frac{(a+1-b) \pm \sqrt{(b-a+1)^2}}{2}$$

$$\Rightarrow \lambda_\pm = 1, (a-b)$$

$$\begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix} \begin{pmatrix} v_+^{(1)} \\ v_+^{(2)} \end{pmatrix} = \begin{pmatrix} (a-b)\, v_+^{(1)} \\ (a-b)\, v_+^{(2)} \end{pmatrix}$$

$$\Rightarrow a v_+^{(1)} + b v_+^{(2)} = a v_+^{(1)} - b v_+^{(1)}$$

$$\Rightarrow v_+^{(1)} = - v_+^{(2)}$$

also, $(1-a)\,v_+^{(1)} + (1-b)\,v_+^{(2)} = a v_+^{(2)} - b v_+^{(2)}$

$\Rightarrow (v_+^{(1)} + v_+^{(2)}) - a v_+^{(1)} - b v_+^{(2)} = a v_+^{(2)} - b v_+^{(2)}$

$\Rightarrow a v_+^{(2)} + b v_+^{(2)} - b v_+^{(2)} - a v_+^{(2)} = 0$

which is true

$\Rightarrow |v_+\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

now for $\lambda = 1$

$\begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow ax + by = x \quad \Rightarrow y = \dfrac{1-a}{b} x$

$\Rightarrow \boxed{|v_-\rangle = \begin{pmatrix} b/(1-a) \\ 1 \end{pmatrix}}$

(d) let $x \begin{pmatrix} -1 \\ 1 \end{pmatrix} + y \begin{pmatrix} b/(1-a) \\ 1 \end{pmatrix} = 0$

$\Rightarrow \dfrac{by}{1-a} - x = 0, \quad y + x = 0$

$\Rightarrow \dfrac{by}{1-a} + y = 0 \quad \Rightarrow y = -x$

$\Rightarrow \dfrac{(1-a+b)\,y}{1-a} = 0$

$\Rightarrow y = 0 \quad \Rightarrow x = 0 \quad \forall a, b$

Since $(x, y) = (0, 0)$ is the only solution, $|v_\pm\rangle$ are linearly independent

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$\langle v_+ | v_- \rangle = \frac{b}{a-1} + 1 \neq 0$, hence they are not orthogonal

$$\boxed{|v_+'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}}$$

$$|v_-''\rangle = \begin{pmatrix} b/(1-a) \\ 1 \end{pmatrix} - \left(\frac{b+a-1}{a-1}\right)\cdot\frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$= \begin{pmatrix} \frac{2b}{2(1-a)} + \frac{b+a-1}{2(a-1)} \\ 1 + \frac{b+a-1}{2(1-a)} \end{pmatrix} = \begin{matrix} \frac{b-a+1}{2(1-a)} \\ \frac{b-a+1}{2(1-a)} \end{matrix}$$

$$= \cancel{3b+a-1}$$

$$\Rightarrow \boxed{|v_-'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}$$

the orthogonal vectors are $\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

(f) $M = U D U^\dagger$ $\qquad \lambda_+ = 1$, $\lambda_- = (a-b)$

$$D \equiv \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

$$U = (|v_+'\rangle \; |v_-'\rangle) = \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}$$

For M to be normalizable, $M^\dagger M = M M^\dagger$

$$\Rightarrow \begin{pmatrix} a^2+b^2 & a-a^2+b-b^2 \\ a-a^2+b-b^2 & 2+a^2+b^2-2a-2b \end{pmatrix} = \begin{pmatrix} 2a^2+1-2a & 2ab+1-2a-2b \\ 2ab+1-2a-2b & 2b^2+1-2b \end{pmatrix}$$

$$\Rightarrow \boxed{a+b=1}$$

15) $R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$

$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$

$R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(a) Rotation about x axis:

$$x' = x$$
$$y' = y\cos\phi + z\sin\phi \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$z' = -y\sin\phi + z\cos\phi$$

Rotation about y axis:

$$x' = x\cos\phi - z\sin\phi$$
$$y' = y \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$z' = x\sin\phi + z\cos\phi$$

Rotation about z axis:

$$x' = x\cos\phi + y\sin\phi \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$y' = -x\sin\phi + y\cos\phi$$
$$z' = z$$

(b) we use $\cos\delta\phi \to 1$, $\sin\delta\phi \to \delta\phi$

now $R_x(\delta\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\phi \\ 0 & -\delta\phi & 1 \end{pmatrix} = I + i\delta\phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$= I + i\delta\phi J_x$

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we get similar results for $R_y(\delta\phi)$ and $R_z(\delta\phi)$ as $\cos\delta\phi$ terms lie on the diagonal. simplifying, we get:

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) let $\delta\phi = \phi/n$ as $n \to \infty$

now $R_k(\phi)$ is a result of application of 'n' $\delta\phi$

$\Rightarrow R_k(\phi) = (R_k(\delta\phi))^n$ as $n \to \infty$

$\Rightarrow R_x(\phi) = \lim_{n\to\infty} [R_k(\delta\phi)]^n = \lim_{n\to\infty} \left[I + \frac{i\phi J_k}{n}\right]^n$

This is the identity for the exponential

$\Rightarrow \boxed{R_x(\phi) = \exp(i\phi J_k)}$

(d) $$[J_x, J_y] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iJ_z$$

similarly $$[J_y, J_z] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = iJ_x$$

$$[J_z, J_x] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = iJ_y$$

now since $[A, B] = -[B, A]$ $\Rightarrow$ $\boxed{[J_k, J_l] = i\epsilon_{klm} J_m}$

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Homework 3

1. $H = \frac{p^2(t)}{2m} + \frac{1}{2} m\omega^2 x^2(t)$

$z(t) = \sqrt{\frac{m\omega}{2\hbar}}\, x(t) + i \frac{p(t)}{\sqrt{2m\hbar\omega}}$

$\boxed{\dot{x} = \frac{\partial H}{\partial p} = p/m}$

$\dot{p} = \frac{-\partial H}{\partial x} = -m\omega^2 x$

now, $\dot{z} = \frac{\partial z}{\partial t} = \sqrt{\frac{m\omega}{2\hbar}}\, \dot{x} + i \frac{\dot{p}}{\sqrt{2m\hbar\omega}}$

$\Rightarrow \dot{z} = \sqrt{\frac{\omega}{2\hbar m}}\, p - \frac{i m\omega^2 x}{\sqrt{2m\hbar\omega}}$

$= \frac{\omega}{i} \frac{p(t)}{\sqrt{2m\hbar\omega}} + \frac{\omega(i)}{i^2} \frac{x(t)\sqrt{m\omega}}{\sqrt{2\hbar}}$

$\Rightarrow \boxed{\dot{z} = -i\omega z}$ equation of motion

sol^n of form $z(t) = C \exp(-i\omega t)$

~~$z(t)/\hbar$~~ $\Rightarrow \boxed{z(t) = \left(\sqrt{\frac{m\omega}{2\hbar}}\, x(0) + i \frac{p(0)}{\sqrt{2m\hbar\omega}} \right) e^{-i\omega t}}$

solution to the equation

$x(t) = (z(t) + z^*(t)) \sqrt{\frac{\hbar}{2m\omega}}$

$p(t) = (z(t) - z^*(t)) \frac{\sqrt{m\hbar\omega}}{i\sqrt{2}}$

let $\sqrt{\frac{m\omega}{2\hbar}}\, x(0) + i \frac{p(0)}{\sqrt{2m\hbar\omega}} = C \Rightarrow z^* = C^* e^{i\omega t}$

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$$\Rightarrow x(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(Ce^{-i\omega t} + C^{*}e^{i\omega t}\right)$$

$$p(t) = -i\sqrt{\frac{m\hbar\omega}{2}}\left(Ce^{-i\omega t} - C^{*}e^{+i\omega t}\right)$$

we can write $C = Ae^{i\phi} \Rightarrow C^{*} = Ae^{-i\phi}$

substituting and solving, we obtain:

$$x(t) = \frac{\sqrt{\hbar}}{\sqrt{2m\omega}}\left(2A\cos(\omega t-\phi)\right) = \boxed{\sqrt{\frac{2\hbar}{m\omega}}\,A\cos(\omega t-\phi)}$$

$$\boxed{p(t) = -\sqrt{2m\hbar\omega}\;A\sin(\omega t-\phi)}$$

Ans 2- $S_p = \sum_{n=0}^{\infty} p^n e^{in\theta} = \dfrac{1-pe^{-i\theta}}{1-2p\cos\theta+p^2}, \quad |p|<1$

plot $|S_p|^2$ v/s θ for $p = 0.01, 0.5, 0.99$

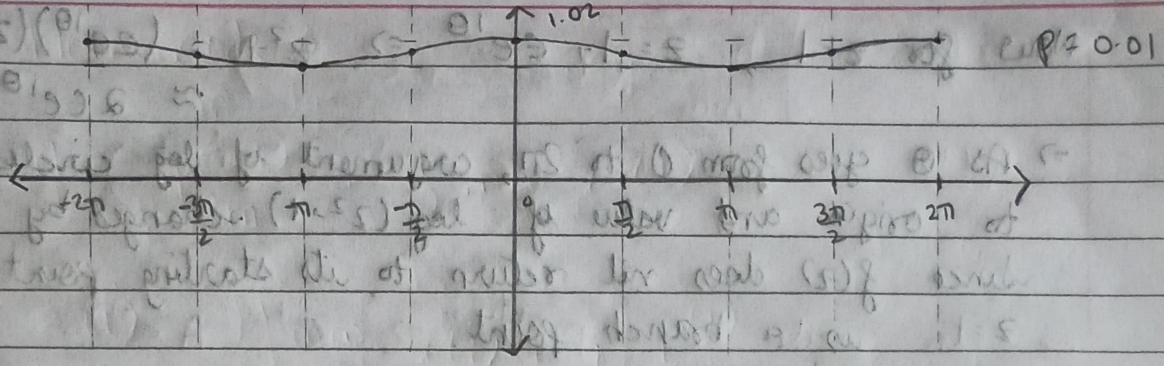
let $N = 1-pe^{i\theta} \Rightarrow |N|^2 = (1-pe^{-i\theta})(1-pe^{i\theta})$

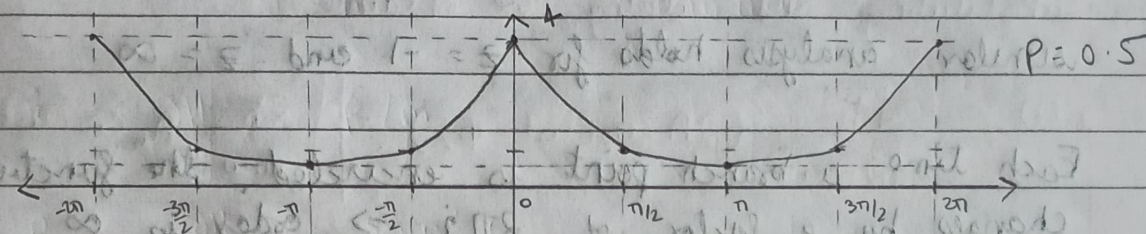
$$\Rightarrow |N|^2 = 1+p^2-p(e^{i\theta}+e^{-i\theta}) = 1+p^2-2p\cos\theta$$

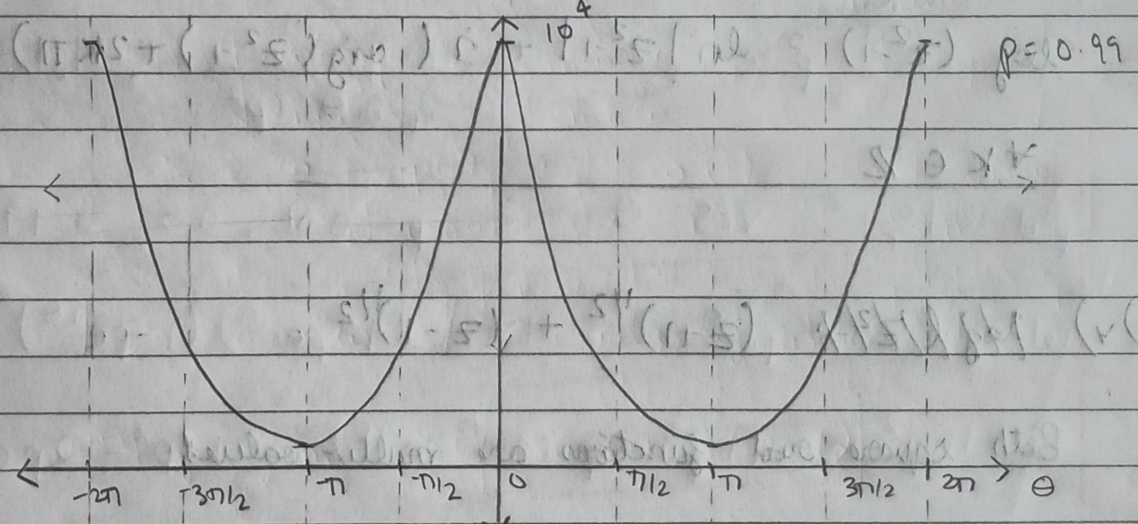
$$\Rightarrow |S_p|^2 = \frac{1}{1+p^2-2p\cos\theta}$$

$\lvert S_p\rvert^2$	$\frac{1}{(1-p)^2}$	$\frac{1}{1+p^2}$	$\frac{1}{(1+p)^2}$	$\frac{1}{1+p^2}$	$\frac{1}{(1-p)^2}$
θ	0	$\pi/2$	π	$3\pi/2$	2π

now plotting for various values of p:







Ans 3) iv) $\log(z^2-1)$ This is a multi valued function

Branch points exist where $(z^2-1) = 0$ or ∞

$\Rightarrow$ $z = 1, -1, \infty$ are branch points

To verify, Trace a small circle around each point.

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Now for $z=1$, $z = 1 + \epsilon e^{i\theta} \Rightarrow z^2 - 1 = (\epsilon e^{i\theta})(2 + \epsilon e^{i\theta})$
$\approx 2\epsilon e^{i\theta}$

$\Rightarrow$ As θ goes from 0 to 2π, argument of log circles to origin and value of $\log(z^2-1)$ changes by $2\pi i$. Since $f(z)$ does not return to its starting point, $z=1$ is a branch point.

similar analysis holds for $z=-1$ and $z=\infty$

Each time a branch point is encircled, the functions changes by a factor of $2\pi i$ $\Rightarrow$ order is ∞

There exist infinite number of brances as:

$$\log(z^2-1) = \ln|z^2-1| + i(\arg(z^2-1) + 2k\pi)$$

$\forall k \in \mathbb{Z}$

Ans3) v) ~~Log 1/z^2-1~~ $(z+1)^{1/2} + (z-1)^{1/2}$

Both square root functions are multi-valued.

$\rightarrow (z+1)^{1/2}$ has branch points at $z=-1, \infty$
$\rightarrow (z-1)^{1/2}$ has branch points at $z=1, \infty$

now trace a small closed loop around each point.

(a) For $z=1$:
As we circle $z=1$, the value of $(z-1)^{1/2}$ changes sigh. The function $(z+1)^{1/2}$ is analytic within this loop, so its value does not change. Hence the

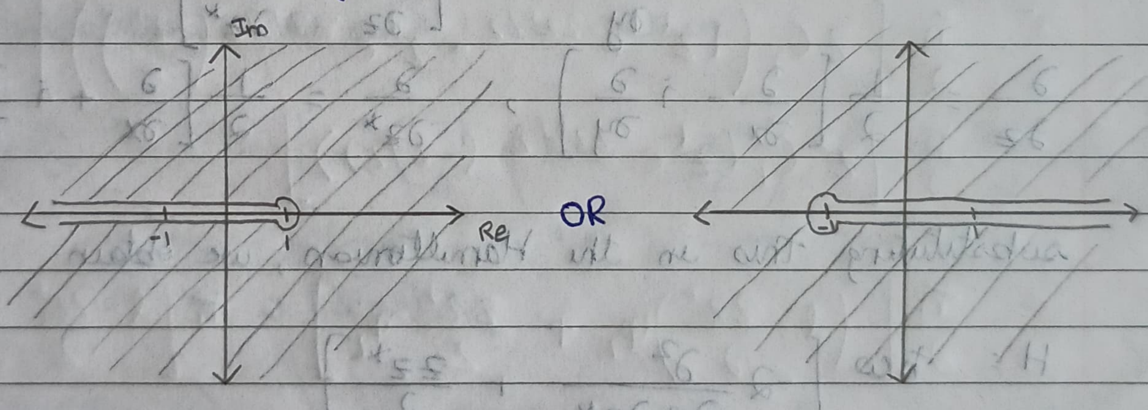
value of the function changes from $(z+1)^{1/2} + (z-1)^{1/2}$ to $(z+1)^{1/2} - (z-1)^{1/2}$.
Hence $z=1$ is a branch point

By symmetric analysis, $z=-1$ is also a branch point

At $z=\infty$, both functions change signs hence the complete function changes sign making $z=\infty$ one of the branch points.

$z = 1, -1, \infty$ are branch points, all of order 1

For each square root function there are two possible distinct values, hence $2\times 2 = 4$ branches are present in the function



Branch cuts and single-valued domains for 3(iv) & 3(v)

Ans 9 - $H = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + \frac{1}{2}M\omega^2(x^2+y^2)$

$E_{n_r, m} = \hbar\omega(2n_r + |m| + 1) = \hbar\omega(n+1)$

where $n = 2n_r + |m|$, $n_r = 0, 1, 2 \ldots$
$m = 0, \pm 1, \pm 2 \ldots$

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(a) $a_0 = \sqrt{\dfrac{\hbar}{M\omega}}$, $z = \dfrac{x+iy}{a_0}$

$\uparrow$ oscillator length

we map this problem to z, z^* coordinates

let $\hat{x} = x/a_0$ $\qquad \hat{y} = y/a_0$

$$\Rightarrow H = \hbar\omega\left[\frac{-1}{2}\left(\frac{\partial^2}{\partial\hat{x}^2} + \frac{\partial^2}{\partial\hat{y}^2}\right) + \frac{1}{2}(\hat{x}^2 + \hat{y}^2)\right]$$

now $\hat{x} \to x$, $\hat{y} \to y$

also $z = \hat{x} + i\hat{y} \longrightarrow x + iy$

we know that: $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial z^*}$

$$\frac{\partial}{\partial y} = i\left[\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right]$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right], \quad \frac{\partial}{\partial z^*} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$$

substituting this in the Hamiltonian, we obtain:

$$H = \hbar\omega\left[-2\frac{\partial^2}{\partial z\,\partial z^*} + \frac{zz^*}{2}\right]$$

(b) ~~For $n=0$, $|m| = l$, consider:~~

~~$H\,\psi(z, z^*) = E\psi(z, z^*)$~~

~~The sol^n is : $\psi_l = z^l \exp(-|z|^2/2)$~~

(b) To show: $H\psi_l = E_l \psi_l$ for $\psi_l(z, z^*) = C_l z^l e^{-|z|^2/2}$

where $C_l = \dfrac{1}{\sqrt{\pi a_0^2 \, l!}}$

$$H\psi_l = \hbar\omega \left[-2 \frac{\partial^2}{\partial z \, \partial z^*} + \frac{z z^*}{2} \right] \left(C_l z^l e^{-|z|^2/2} \right)$$

$$\frac{\partial \psi_l}{\partial z^*} = \frac{-C_l}{2} z^{l+1} e^{-|z|^2/2}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \psi_l}{\partial z^*} \right) = \frac{-C_l}{2} \left[(l+1) - \frac{zz^*}{2} \right] z^l e^{-zz^*/2}$$

$$= \frac{-1}{2} \left(l+1 - \frac{zz^*}{2} \right) \psi_l$$

substituting this back, we obtain:

$$H\psi_l = \hbar\omega \left[-2\left(\frac{-1}{2} \left(l+1 - \frac{zz^*}{2} \right) \psi_l \right) + \frac{zz^*}{2} \psi_l \right]$$

$$= \hbar\omega \left[(l+1 - zz^*/2) + zz^*/2 \right] \psi_l$$

$$= \hbar\omega (l+1) \psi_l$$

$\Rightarrow \psi_l$ is an eigenfunction with $E_l = \hbar\omega(l+1)$

Now we know that $H = H^*$ since the system is invariant under time-reversal operation

we have $H\psi_l = E_l \psi_l$

$\Rightarrow H^* \psi_l^* = E_l^* \psi_l^*$, $E_l = \hbar\omega(l+1) \in \mathbb{R}$

$\Rightarrow \boxed{H \psi_l^* = E_l \psi_l^*}$ hence ψ_l^* is also a eigen f^n with same eigen value E_l

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(c) $L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$

$$= -i\hbar\left[\left(\frac{z+z^*}{2}\right)\left[i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)\right] - \left(\frac{z-z^*}{2i}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right)\right]$$

$$= \frac{-i\hbar}{2}\left[zi\frac{\partial}{\partial z} - zi\frac{\partial}{\partial z^*} + iz^*\frac{\partial}{\partial z} - iz^*\frac{\partial}{\partial z^*} - \frac{z}{i}\frac{\partial}{\partial z} - \frac{z}{i}\frac{\partial}{\partial z^*} + \frac{z}{i}\frac{\partial}{\partial z} + \frac{z}{i}\frac{\partial}{\partial z^*}\right]$$

$$= \boxed{\frac{\hbar}{2}\left[z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right]}$$ QED

(d)(i) $L_z \psi_l(z, z^*)$

$$= \frac{\hbar}{2}\left[z\frac{\partial\psi_l}{\partial z} - z^*\frac{\partial\psi_l}{\partial z^*}\right]$$

$$= \frac{\hbar}{2}\left[zC_l\left(lz^{l-1}e^{-zz^*/2} + z^l e^{-zz^*/2}\left(\frac{-z^*}{2}\right)\right)\right]$$

$$- \frac{\hbar}{2}\left[z^* C_l\left(z^l e^{-zz^*/2}\right)(-z/2)\right]$$

$$= \frac{\hbar}{2}\left[\left(l - \frac{zz^*}{2}\right)\psi_l + \frac{zz^*}{2}\psi_l\right] = \hbar\left[l - \frac{zz^*}{2} + \frac{zz^*}{2}\right]\psi_l$$

$$\Rightarrow \boxed{L_z\psi_l = l\hbar\psi_l}$$

(ii) we know $L_z\psi_l(z, z^*) = l\hbar\psi_l(z, z^*)$

$$L_z^* = \frac{\hbar}{2}\left[z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right] = -L_z$$

now $L_z^*\psi_l^* = l\hbar\psi_l^*$

$$\Rightarrow \boxed{L_z\psi_l^* = -l\hbar\psi_l^*}$$ QED

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(e) $\Psi_l(z, z^*) = \frac{1}{(\pi a_0^2 l!)^{1/2}} z^l e^{-|z|^2/2}$

$\Psi_l(z) = \frac{1}{\sqrt{\pi a_0^2}} \frac{1}{(l!)^{1/2}} z^l e^{-|z|^2/2}$

$|\Psi_l(z)|^2 = \frac{1}{\pi a_0^2 l!} (|z|^2)^l e^{-|z|^2}$

$\Rightarrow S = \frac{e^{-|z|^2}}{\pi a_0^2} \sum_{l=0}^{\infty} \frac{(|z|^2)^l}{l!} = \frac{e^{-|z|^2} e^{|z|^2}}{\pi a_0^2}$

$\Rightarrow \boxed{S = 1/\pi a_0^2}$ QED

Ans 11) $\phi_m(z) = \frac{1}{\sqrt{2\pi 2^m m!}} z^m \quad m = 0, 1, 2 \ldots$

$\mu[z, z^*] = \exp(-|z|^2/2)$

$\langle \phi_n | \phi_m \rangle = \int \phi_n^*(z) \phi_m(z) \mu[z, z^*] dx dy$

(a) $\langle \phi_n | \phi_m \rangle = \int \frac{(z^*)^n z^m}{2\pi \sqrt{2^{n+m} n! m!}} \exp(-|z|^2/2) dx dy$

now convert this problem to polar coordinates

$dx dy \rightarrow r dr d\theta \quad , \quad |z|^2 = r^2$

$z = re^{i\theta} \Rightarrow \langle \phi_n | \phi_m \rangle = C_{m,n} \int r^{n+m} e^{(m-n) i\theta} e^{-r^2/2} r dr d\theta$

$\rightarrow \langle \phi_n | \phi_m \rangle = C_{m,n} \int e^{i(n+m)\theta} d\theta \int r^{n+m+1} e^{-r^2/2} dr$

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Now if $n \neq m$:

$$\int_0^{2\pi} \exp[i(m-n)\theta]\, d\theta = \left[\frac{e^{i(m-n)\theta}}{i(m-n)} \right]_0^{2\pi} = \frac{1-1}{i(m-n)} = 0$$

$\Rightarrow \langle \phi_n | \phi_m \rangle = 0$ if $n \neq m$

If $m = n$:

$$\int_0^{2\pi} e^0\, d\theta = 2\pi$$

now $\int_0^{\infty} r^{n+m+1} \exp(-r^2/2)\, dr$ let $r^2/2 = u$ $\Rightarrow r\,dr = du$ $\Rightarrow (2u)^{1/2} = r$

$$= \int_0^{\infty} r^{m+n} \exp(-u)\, du$$

$$= \int_0^{\infty} 2^{\frac{m+n}{2}} u^{\frac{m+n}{2}} \exp(-u)\, du$$

$$= \sqrt{2^{m+n}} \underbrace{\int_0^{\infty} u^m \exp(-u)\, du}_{\text{gamma function}} = 2^m\, m!$$

substituting : $\langle \phi_m | \phi_m \rangle = \dfrac{1}{2\pi\, m!\, 2^m} (2\pi)(m!\, 2^m) = 1$

$\Rightarrow \boxed{\langle \phi_m | \phi_n \rangle = \delta_{mn}}$ QED

(b) $b^{\dagger} = z/\sqrt{2}$ $\quad b = \sqrt{2}\, d/dz$

To show $b^{\dagger}\phi_m \propto \phi_{m+1}$
$b\,\phi_m \propto \phi_{m-1}$

$$b^{\dagger}\phi_m = \frac{1}{\sqrt{2\pi\, 2^{m+1}\, m!}} z^{m+1} = \phi_{m+1}\sqrt{(m+1)}$$

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Hence $b^\dagger$ raises the state from ϕ_m to ϕ_{m+1}

$$b\phi_m = \sqrt{2}\frac{d}{dz}\phi_m = \frac{m z^{m-1}}{\sqrt{2\pi\, 2^{m-1}(m-1)!\, m}} = \sqrt{m}\,\phi_{m-1}$$

Hence b lowers the state from ϕ_m to ϕ_{m-1}

(c) $[b, b^\dagger] = \left[\sqrt{2}\frac{d}{dz}, \frac{z}{\sqrt{2}}\right]$

$$[b, b^\dagger]\psi = \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}\frac{d}{dz}(z\psi) - \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} z\frac{d\psi}{dz}$$

$$= z\frac{d\psi}{dz} + \psi - z\frac{d\psi}{dz} = \psi$$

$\Rightarrow \boxed{[b, b^\dagger] = 1}$ QED