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Homework-3

1. Consider a classical harmonic oscillator described by the Hamiltonian $H = \frac{p^2(t)}{2m} + \frac{1}{2}m\omega^2 x^2(t)$. Construct a dimensionless complex variable z as $z(t) = \sqrt{\frac{m\omega}{2\hbar}}x(t) + i\frac{p(t)}{\sqrt{2m\hbar\omega}}$. Find an equatin of motion in the complex variable $z(t)$. Solve the equation of motion and extract $x(t)$ and $p(t)$.

   [Note: The system is one dimensional but we are using momentum as a second dimension and constructing a complex variable. While solving harmonic oscillator quantum mechanically, we use bosonic ladder operators. The form of the ladder operators is exactly same as $z(t)$. In quantum case, we treat x and p as operators.]

2. Show that

   (a)
   $$S_N = \sum_{n=0}^{N-1} e^{in\theta} = \frac{\sin(N\theta/2)}{\sin(\theta/2)} e^{i(N-1)\theta/2}.$$
   Plot $|S_N|^2$ vs θ for $N = 10, 100$ and 1000.

   (b)
   $$S_p = \sum_{n=0}^{\infty} p^n e^{in\theta} = \frac{1 - pe^{-i\theta}}{1 - 2p\cos\theta + p^2},$$
   where $|p| < 1$. Plot $|S_p|^2$ vs θ for $p = 0.01, 0.5$ and 0.99. These results are very useful in optics.

   (c)
   $$S_p = \sum_{n=-\infty}^{\infty} p^{|n|} e^{in\theta} = \frac{1 - p^2}{1 - 2p\cos\theta + p^2} = \mathrm{Re}\left[\frac{1 + pe^{i\theta}}{1 - pe^{i\theta}}\right],$$
   where $0 \le |p| < 1$. This is known as Poisson kernel for the unit disc.

3. Determine number of branches, branch points and their order of the following complex functions:

   $$(i) z^{1/2}(z-1)^{1/3},\ (ii)(z^2+1)^{1/3},\ (iii)\sqrt{z(z-1)},\ (iv)\log(z^2-1)\ (v)(z+1)^{1/2} + (z-1)^{1/2}.$$

   Draw all possible branch cuts and domains where the function is single-valued.

4. Is $\log(i^2) = 2\log(i)$ for all branches?

5. Show that $f(z) = \cos^{-1} z$ is an infinitely many-valued function. Find the branch points.

6. Consider a function

   $$\begin{aligned} f(x,y) &= \frac{x^2 y}{x^4 + y^2}, \qquad z \neq 0 \\ &= 0, \qquad z = 0. \end{aligned}$$

   Is this function continuous at $z = 0$?

   [Hints: Along all the straight lines $y = mx$, it seems to be continuous. Consider a set of parabolic paths $y = ax^2$ and see what happens.]

7. (a) Is $f(z) = z^m e^{-|z|^2/4}$ an analytic function?

   (b) Consider $f(z) = u(x,y) + iv(x,y)$ is an analytic function. Check whether $F(x,y) = u(x,y)v(x,y)$ is a solution of the Laplace equation or not.

   (c) Consider the real part of an analytic function is given by $u(x,y) = -2xy + y/(x^2+y^2)$. Calculate its harmonic conjugate function $v(x,y)$.

8. Consider the following **matrix-differential** operator

$$H = \boldsymbol{\sigma} \cdot \boldsymbol{\nabla},$$

where $\boldsymbol{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$. Note that x and y are dimensionless here.

   (a) Show that the components of the operator $\boldsymbol{\nabla}$ can be expressed in terms of complex conjugate coordinates as

$$\frac{\partial}{\partial x} = \left[\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right]$$

   and

$$\frac{\partial}{\partial y} = i\left[\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right].$$

   (b) Express the operator H in complex coordinates.

   (c) Find the eigenvector $|\psi\rangle$ which satisfies $H|\psi\rangle = 0$.

9. The quantum mechanical Hamiltonian of a two-dimensional isotropic harmonic oscillator is given by

$$H = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + \frac{1}{2}M\omega^2(x^2+y^2).$$

Here M is the mass of the particle and ω is the oscillator frequency.

Note: The energy spectrum is

$$E_{n_r,m} = \hbar\omega(2n_r + |m| + 1) = \hbar\omega(n+1),$$

where $n = 2n_r + |m|$ with $n_r = 0, 1, 2....$ and $m = 0, \pm 1, \pm 2, ...$ For a given n, there are $(n+1)$ degenerate states.

   (a) Obtain the Hamiltonian H in complex coordinates as given below:

$$H = \hbar\omega\left[-2\frac{\partial^2}{\partial z \partial z^*} + \frac{zz^*}{2}\right].$$

   Note that here z is a dimensionless variable: $z = (x+iy)/a_0$ with the oscillator length $a_0 = \sqrt{\hbar/(M\omega)}$. Note that $H = H^*$ since the system is invariant under time-reversal operation.

   (b) Show that the complex function $\psi_l(z)$ (normalized to one)

$$\psi_l(z, z^*) = \frac{1}{\sqrt{\pi a_0^2 n!}} z^l e^{-|z|^2/2}, \qquad l = 0, 1, 2, ...$$

   and its conjugate $\psi_l^*(z)$ are the eigenfunctions of H with the same eigenvalues $E_l = \hbar\omega(l+1)$.

   Note: The eigenvalues $E_l = \hbar\omega(l+1)$ are for $n_r = 0$ and l correspond to m.

(c) Show that the **angular momentum** operator in terms of the complex coordinates can be expressed as

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \hbar\left[z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right].$$

(d) Show that

$$L_z\psi_l(z,z^*) = l\hbar\psi_l(z,z^*), \quad L_z\psi_l^*(z,z^*) = -l\hbar\psi_l^*(z,z^*).$$

(e) Show that

$$\sum_{l=0}^{\infty}|\psi_l(z)|^2 = \frac{1}{\pi a_0^2}.$$

10. You have already derived the following Hamiltonian for a charge ($q > 0$) particle in presence of the magnetic field $\mathbf{B} = B\hat{z}$ in the symmetric gauge:

$$H = -\frac{\hbar^2}{2M}\nabla^2 - \frac{qB}{2M}(xp_y - yp_x) + \frac{q^2B^2}{8M}\left(x^2+y^2\right).$$

Introducing **dimensionless** and **independent** complex conjugate variables $z = (x+iy)/l_0$ and $z^* = (x-iy)/l_0$, where $l_0 = \sqrt{\hbar/(qB)}$ is the magnetic length scale.

(a) Show that the components of the canonical momentum operator can be expressed in terms of complex conjugate coordinates as

$$p_x = -i\hbar\frac{\partial}{\partial x} = -\frac{i\hbar}{l_0}\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right]$$

and

$$p_y = -i\hbar\frac{\partial}{\partial y} = \frac{\hbar}{l_0}\left[\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right].$$

(b) You have already derived **canonical angular momentum** operator in terms of the complex coordinates in the previous problem:

$$L_z = (\mathbf{r}\times\mathbf{p})_z = +\hbar\left[z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right].$$

The **mechanical angular momentum** operator is given by $L_z^{\text{mech}} = (\mathbf{r}\times\mathbf{\Pi})_z = (x\Pi_y - y\Pi_x)$. Show that the mechanical angular momentum L_z^{mech} can be written in terms of the complex coordinates as

$$L_z^{\text{mech}} = +\hbar\left[z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*} - \frac{|z|^2}{2}\right]$$

(c) Show that the above Hamiltonian H can be re-written as

$$H = \hbar\omega_c\left[-2\frac{\partial^2}{\partial z\partial z^*} - \frac{1}{2}\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right) + \frac{zz^*}{8}\right].$$

Here, $\omega_c = qB/M$ is the cyclotron frequency.

(d) You have already derived the commutator $[\Pi_x, \Pi_y] = i\hbar qB$, which is similar to $[x, p_x] = i\hbar$. We know that two ladder operators (or, raising and lowering operators) in terms of x and p_x are constructed in order to quantize the Hamiltonian of a quantum mechanical harmonic oscillator.

Similarly, constructing the following ladder operators:

$$a = \frac{i}{\sqrt{2\hbar qB}}[\Pi_x + i\Pi_y], \quad a^\dagger = \frac{-i}{\sqrt{2\hbar qB}}[\Pi_x - i\Pi_y].$$

Show that the ladder operators can be expressed in complex coordinates as

$$a = \frac{1}{\sqrt{2}}\left[\frac{z}{2} + 2\frac{\partial}{\partial z^*}\right], \quad a^\dagger = \frac{1}{\sqrt{2}}\left[\frac{z^*}{2} - 2\frac{\partial}{\partial z}\right].$$

Show that $[a, a^\dagger] = 1$.

(e) Show that

$$a^\dagger a + \frac{1}{2} = \left[-2\frac{\partial^2}{\partial z \partial z^*} - \frac{1}{2}\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right) + \frac{|z|^2}{8}\right].$$

Therefore, the Hamiltonian can be re-written as

$$H = \hbar\omega_c\left(a^\dagger a + \frac{1}{2}\right).$$

Here, $\hat{N} = a^\dagger a$ is the number operator.

[This is the Hamiltonian of a simple harmonic oscillator with the quantized energy levels $E_n = (n + 1/2)\hbar\omega_c$ with $n = 0, 1, 2.....$ Each discrete energy is called Landau level.]

(f) Show that

$$a^\dagger a \psi_m(z, z^*) = 0,$$

where $\psi_m(z, z^*) = \phi_m(z) e^{-\frac{|z|^2}{4}}$. Here $\phi_m(z)$ is a set of analytic functions given as

$$\phi_m(z) = \frac{1}{\sqrt{2\pi l_0^2 2^m m!}} z^m,$$

with $m = 0, 1, 2.......$

(g) Show that

$$\sum_{m=0}^{\infty} |\psi_m(z, z^*)|^2 = \frac{qB}{2\pi\hbar} = \frac{1}{2\pi l_0^2}.$$

(h) Show that

$$L_z \psi_m(z, z^*) = m\hbar\, \psi_m(z, z^*).$$

Show that

$$\langle \psi_m(z, z^*) | L_z^{\text{mech}} | \psi_m(z, z^*) \rangle = -\hbar.$$

Note that $\psi_m(z, z^*)$ are eigenfunctions of L_z, but not the eigenfunctions of L_z^{mech}.

11. Consider a set of analytic functions

$$\phi_m(z) = \frac{1}{\sqrt{2\pi 2^m m!}} z^m,$$

where $m = 0, 1, 2.......$ One can use $\phi_m(z)$ as basis states with the measure or weight factor $\mu[z, z^*] = e^{-|z|^2/2}$ to construct an infinite dimensional complex space with finite norm. Defining an inner product as

$$\langle \phi_n | \phi_m \rangle = \int \phi_n^*(z) \phi_m(z) \; \mu[z, z^*] \; dxdy.$$

(a) Prove that $\langle \phi_n | \phi_m \rangle = \delta_{mn}$.

(b) Show that $b^\dagger = \frac{z}{\sqrt{2}}$ and $b = \sqrt{2} \frac{d}{dz}$ act like raising and lowering operators, respectively, in this infinite dimensional complex space.

(c) Check that $b^\dagger$ and b satisfy the bosonic commutation relation: $[b, b^\dagger] = 1$.