Assignment 2

DATE
PAGE

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1. $|n_1\rangle = (1\ 1\ 1)^T \quad |n_2\rangle = (1\ 1\ 0)^T \quad |n_3\rangle = (1\ 0\ 1)^T$

(a) let $\sum_{i=1}^{3} a_i |n_i\rangle = 0$

$$\Rightarrow \quad a_1 + a_2 + a_3 = 0 \quad -①$$
$$a_1 + a_2 = 0 \quad -②$$
$$a_1 + a_3 = 0 \quad -③$$

From ① and ②: $a_3 = 0$
From ① and ③: $a_2 = 0$
$\Rightarrow a_1 = 0$

Since $(a_1, a_2, a_3) = (0, 0, 0)$ is the only solution, the set of vectors is linearly independent

(b) $\boxed{|v_1\rangle = \frac{1}{\sqrt{3}}(1\ 1\ 1)^T}$

$$|v_2'\rangle = (1\ 1\ 0)^T - \left[(1\ 1\ 1)\cdot(1\ 1\ 0)^T\right]\left(\frac{1}{\sqrt{3}}\right)^3 (1\ 1\ 1)^T$$

$$= (1\ 1\ 0)^T - \frac{2}{3}(1\ 1\ 1)^T$$

$$= \left(\left(1-\frac{2}{3}\right)\ \left(1-\frac{2}{3}\right)\ \frac{-2}{3}\right)^T$$

$$= \frac{1}{3}(1\ 1\ -2)^T$$

$$\boxed{|v_2\rangle = \frac{1}{\sqrt{6}}(1\ 1\ -2)^T}$$

$$|v_3'\rangle = |n_3\rangle - \langle v_1|n_3\rangle|v_1\rangle - \langle v_2|n_3\rangle|v_2\rangle$$
$$= (1\ 0\ 1)^T - \frac{2}{3}(1\ 1\ 1)^T + \frac{1}{6}(1\ 1\ -2)^T$$

$$\Rightarrow |v_3'\rangle = \frac{1}{6}(3 \quad -3 \quad 0)^T$$

$$\Rightarrow |v_3\rangle = \frac{1}{\sqrt{2}}(1 \quad -1 \quad 0)^T$$

$$\boxed{|v_3\rangle = \frac{1}{\sqrt{2}}(1 \quad -1 \quad 0)^T}$$

(c) $P_1 = |v_1\rangle\langle v_1| = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$P_2 = |v_2\rangle\langle v_2| = \frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}$

$P_3 = |v_3\rangle\langle v_3| = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

4) $\bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(a) $\bar{\sigma} \times \bar{\sigma} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sigma_x & \sigma_y & \sigma_z \\ \sigma_x & \sigma_y & \sigma_z \end{vmatrix}$

$= \left(\sigma_y\sigma_z - \sigma_z\sigma_y, \ [\sigma_z, \sigma_x], \ [\sigma_x, \sigma_y]\right)$

$= 2(i\sigma_x, \ i\sigma_y, \ i\sigma_z)$

$= \boxed{2i\bar{\sigma}}$ as $[\sigma_i, \sigma_j] = i\,\varepsilon_{ijk}\sigma_k$

DATE
PAGE

4(b) $(\bar{\sigma}\cdot\bar{A})(\bar{\sigma}\cdot\bar{B}) = A\cdot B + i\bar{\sigma}\cdot(\bar{A}\times\bar{B})$

let $\bar{A} = (A_x, A_y, A_z)$ $\quad \bar{B} = (B_x, B_y, B_z)$

$LHS = (A_x\sigma_x + A_y\sigma_y + A_z\sigma_z)(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z)$

// $A_x B_x$ now $\sigma_i\sigma_j = i\sigma_k\varepsilon_{ikj} + I\delta_{ij}$

$\Rightarrow LHS = A_xB_xI + A_xB_y i\sigma_z - A_xB_z i\sigma_y$
$- B_xA_y i\sigma_z + A_yB_yI + A_yB_z i\sigma_z$
$+ A_zB_x i\sigma_y - A_zB_y i\sigma_x + A_zB_zI$

$\Rightarrow I(A_xB_x + A_yB_y + A_zB_z) +$
$i\sigma_x(A_yB_z - A_zB_y) + i\sigma_y(A_zB_x - A_xB_z) +$
$i\sigma_z(A_xB_y - A_yB_x)$

$= \bar{A}\cdot\bar{B}I + i\bar{\sigma}\cdot(\bar{A}\times\bar{B}) =$ RHS $\therefore$ QED

5(c) To prove: $e^{\pm i\bar{\sigma}\cdot\bar{v}} = \cos v \pm i(\bar{\sigma}\cdot\hat{v})\sin v$

$\hat{v} = \bar{v}/v$, $v = |v|$

$\exp(\pm i\bar{\sigma}\cdot\bar{v}) = 1 \pm \frac{i\bar{\sigma}\cdot\bar{v}}{1!} + \frac{(i\bar{\sigma}\cdot\bar{v})^2}{2!} \pm \frac{(i\bar{\sigma}\cdot\bar{v})^3}{3!} + \ldots$

$\bar{\sigma}\cdot\bar{v} = \sigma_x v_x + \sigma_y v_y + \sigma_z v_z$
$(\bar{\sigma}\cdot\bar{v})^2 = (v_x^2 + v_y^2 + v_z^2)I$
$(\bar{\sigma}\cdot\bar{v})^3 = (v_x^2 + v_y^2 + v_z^2)(\sigma_x v_x + \sigma_y v_y + \sigma_z v_z)$
$(\bar{\sigma}\cdot\bar{v})^4 = (v_x^2 + v_y^2 + v_z^2)^2$ and so on

① - $\cos v = 1 + \frac{(iv)^2}{2!} + \frac{(iv)^4}{4!} + \ldots = 1 + \frac{(i\bar{\sigma}\cdot\bar{v})^2}{2!} + \frac{(i\bar{\sigma}\cdot\bar{v})^4}{4!} + \ldots$

Page 4

$$\sin v = \frac{v^1}{1!} + \frac{v^3}{3!} + \frac{v^5}{5!} + \cdots$$

$$\pm\left(\frac{i\bar{\sigma}\cdot\bar{v}}{1!} \pm \frac{(i\bar{\sigma}\cdot\bar{v})^3}{3} + \frac{(i\bar{\sigma}\cdot\bar{v})^5}{5!} + \cdots\right)$$

$$= \pm(i\bar{\sigma}\cdot\hat{v})\left(\frac{v}{1!} + \frac{v^2 v}{3!} + \frac{v^4 v}{5!} + \cdots\right)$$

$$= \pm(i\bar{\sigma}\cdot\hat{v})\sin v \quad \text{— (2)}$$

combining (1) and (2):

$$\exp(\pm i\,\bar{\sigma}\cdot\bar{v}) = \cos v \pm i(\bar{\sigma}\cdot\hat{v})\sin v \quad \therefore \text{QED}$$

(8) $\ln \sigma_j = \frac{i\pi}{2}[\mathbb{I} - \sigma_j]$

exponentiating both sides

$$\sigma_j = \exp\left(\frac{i\pi}{2}[\mathbb{I}-\sigma_j]\right)$$

solving the RHS

$\left(\begin{array}{l}\exp(i\theta M) = \mathbb{I}(\cos\theta) + (i\sin\theta)M \\ \text{for any matrix that satisfies } M^2 = \mathbb{I}\end{array}\right)$

~~Here, $(\mathbb{I}-\sigma_j)^2 = \mathbb{I}^2 - 2\sigma_j + \sigma_j^2$~~

$$\text{RHS} = \exp\left(\frac{i\pi}{2}\mathbb{I}\right)\exp\left(\frac{-i\pi}{2}\sigma_j\right)$$

$$= \left(\mathbb{I}\cos\pi/2 + i\,\mathbb{I}\sin\pi/2\right)\left(\mathbb{I}\cos\pi/2 + i\sigma_j\sin(-\pi/2)\right)$$

$$= (i^2\,\mathbb{I})(-1)\sigma_j = \boxed{\sigma_j} = \text{LHS}$$

QED

DATE
PAGE

(9) $H = \frac{-\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2 y^2$

$x = y/a \quad a = \sqrt{\hbar/(m\omega)}$

$H = \frac{\hbar\omega}{2}\left[-\frac{d^2}{dx^2} + x^2\right]$

$a = \frac{1}{\sqrt{2}}\left[x + \frac{d}{dx}\right], \quad a^\dagger = \frac{1}{\sqrt{2}}\left[x - \frac{d}{dx}\right]$

where $\left(\frac{d}{dx}\right)^\dagger = \frac{-d}{dx}, \quad H = \hbar\omega\left(a^\dagger a + 1/2\right)$

(a) $[a, a^\dagger]\psi = \frac{1}{2}\left[x^2\psi - x\frac{d\psi}{dx} + \frac{d}{dx}(x\psi) - \frac{d^2}{dx^2}(\psi)\right] -$
$\frac{1}{2}\left[x^2\psi + x\frac{d\psi}{dx} - \frac{d}{dx}(x\psi) - \frac{d^2}{dx^2}\psi\right]$

$= \frac{1}{2}\left[-x\frac{d\psi}{dx} + \psi + x\frac{d\psi}{dx} - x\frac{d\psi}{dx} + \psi + x\frac{d\psi}{dx}\right]$

$= \psi \Rightarrow \boxed{[a, a^\dagger] = 1}$

(b) $e^a a^\dagger e^{-a} = a^\dagger + 1, \quad e^{a^\dagger} a e^{-a^\dagger} = a - 1$

By Hadamard lemma:

$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots$

$\Rightarrow e^a a^\dagger e^{-a} = a^\dagger + [a, a^\dagger] + \frac{1}{2!}[a, [a, a^\dagger]] + \cdots$

$= a^\dagger + 1 + 0 + 0 \ldots = \boxed{a^\dagger + 1}$

DATE
PAGE

now:

$$e^{a^\dagger} a e^{-a^\dagger} = \cancel{a-1}$$

$$= a + [a^\dagger, a] + \frac{1}{2!}[a^\dagger, [a^\dagger, a]] + \dots$$

$$= a - 1 + 0 + 0$$

$$= \boxed{a-1} \qquad \text{QED}$$

(c) $U \equiv U(\lambda, \lambda^*) = e^{\lambda a^\dagger - \lambda^* a}$

To ~~prove~~ prove: $U^\dagger U = UU^\dagger = 1$

$U = e^{\lambda a^\dagger} e^{-\lambda^* a}$ let $X = \lambda a^\dagger - \lambda^* a$

now $X^\dagger = \lambda^* a - \lambda a^\dagger = -X$

$$U^\dagger U = e^{X^\dagger} e^{X} = e^{-X} e^{X} = 1$$

$$U U^\dagger = e^{X} e^{X^\dagger} = e^{X} e^{-X} = 1$$

$$\Rightarrow U^\dagger U = UU^\dagger = 1$$

(d) To show: $e^{\lambda^* a - \lambda a^\dagger} a e^{\lambda a^\dagger - \lambda^* a} = a + \lambda$

again, we use the Hadamard lemma:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots$$

let $X = \lambda^* a - \lambda a^\dagger$ now $\lambda a^\dagger - \lambda^* a = -X = X^\dagger$

$$\text{LHS} = a + [\lambda^* a - \lambda a^\dagger, a] + \frac{1}{2!}[X, [X, a]] + \dots$$

$$= a + \lambda^*[a, a] - \lambda[a^\dagger, a] + \dots$$

$$= a - \lambda(-1) + \dots$$

$$= a + \lambda + \underbrace{\frac{1}{2!}[X, \lambda] + \dots}_{\to 0}$$

$$\Rightarrow e^{\lambda^* a - \lambda a^\dagger} a e^{\lambda a^\dagger - \lambda^* a} = a + \lambda$$

similarly:

$$e^{\lambda^* a - \lambda a^\dagger} a^\dagger e^{\lambda a^\dagger - \lambda^* a} = a^\dagger + \left(\lambda^* [a, a^\dagger] - \lambda [a^\dagger, a^\dagger]\right) + 0 + 0 + \ldots$$

$$= \boxed{a^\dagger + \lambda^*}$$

obtaining second exp from first, take a conj. on both sides:

$$\left(e^{\lambda^* a - \lambda a^\dagger} a e^{\lambda a^\dagger - \lambda^* a}\right)^\dagger = (a + \lambda)^\dagger$$

$$\Rightarrow e^{\lambda^* a - \lambda a^\dagger} a^\dagger e^{\lambda a^\dagger - \lambda^* a} = a^\dagger + \lambda^*$$

(II) $M = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$, $a \in [0,1]$, $b \in [0,1]$

(a) $M = \eta_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \eta_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \eta_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \eta_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{aligned} a &= \eta_3 + \eta_0 \\ b &= \eta_1 - i\eta_2 \\ 1-a &= \eta_1 + i\eta_2 \\ 1-b &= \cancel{\eta_1 + i\eta_2}\ \eta_0 - \eta_3 \end{aligned}$$

$\cancel{\Rightarrow a = 1 - \eta_1 - i\eta_2}$

$\cancel{\Rightarrow b = 1 - \eta_0 + \eta_3}$

$$\Rightarrow \boxed{\frac{1+b-a}{2} = \eta_1}, \quad \boxed{\frac{1-a-b}{2i} = \eta_2}$$

$$\boxed{\frac{1+a-b}{2} = \eta_0}, \quad \boxed{\frac{a-1+b}{2} = \eta_3}$$

(b) $M = a|0\rangle\langle 0| + b|0\rangle\langle 1| + (1-a)|1\rangle\langle 0| + (1-b)|1\rangle\langle 1|$

(c) let $M|v\rangle = \lambda|v\rangle$ where $|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

~~now, from (b):~~

$$\cancel{M|v\rangle = a|0\rangle\langle 0|v\rangle + b|0\rangle\langle 1|v\rangle + (1-a)|1\rangle\langle 0|v\rangle + (1-b)|1\rangle\langle 1|v\rangle}$$

$$\cancel{\Rightarrow \begin{pmatrix} av_1 + bv_2 \\ (1-a)v_1 + (1-b)v_2 \end{pmatrix} =}$$

eigenvalue eqn: $|M - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ 1-a & 1-b-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)(1-b-\lambda) - (b)(1-a) = 0$$

$$\Rightarrow a - \cancel{ab} - \lambda a - \lambda + b\lambda + \lambda^2 - b + \cancel{ab} = 0$$

$$\Rightarrow (a-b) + \lambda^2(1) + \lambda(b-a-1) = 0$$

$$\Rightarrow \lambda = \frac{(a+1-b) \pm \sqrt{a^2+b^2+1+2a-2b-2ab-4a+4b}}{2}$$

$$\Rightarrow \lambda = \frac{(a+1-b) \pm \sqrt{(b-a+1)^2}}{2}$$

$$\Rightarrow \lambda_{\pm} = 1, (a-b)$$

$$\begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}\begin{pmatrix} v_+^{(1)} \\ v_+^{(2)} \end{pmatrix} = \begin{pmatrix} (a-b)\, v_+^{(1)} \\ (a-b)\, v_+^{(2)} \end{pmatrix}$$

$$\Rightarrow a v_+^{(1)} + b v_+^{(2)} = a v_+^{(1)} - b v_+^{(1)}$$

$$\Rightarrow v_+^{(1)} = - v_+^{(2)}$$

~~$t/\langle\lambda\rangle$~~ also, $(1-a)\, v_+^{(1)} + (1-b)\, v_+^{(2)} = a v_+^{(2)} - b v_+^{(2)}$

$\Rightarrow (v_+^{(1)} + v_+^{(2)}) - a v_+^{(1)} - b v_+^{(2)} = a v_+^{(2)} - b v_+^{(2)}$

$\Rightarrow -a v_+^{(2)} + b v_+^{(2)} - b v_+^{(2)} - a v_+^{(2)} = 0$

which is true

$\Rightarrow |v_+\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

now for $\lambda = 1$

$\begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow ax + by = x \qquad \Rightarrow y = \frac{1-a}{b} x$

$\Rightarrow \boxed{|v_-\rangle = \begin{pmatrix} b/(1-a) \\ 1 \end{pmatrix}}$

(d) let $x \begin{pmatrix} -1 \\ 1 \end{pmatrix} + y \begin{pmatrix} b/(1-a) \\ 1 \end{pmatrix} = 0$

$\Rightarrow \frac{by}{1-a} - x = 0, \qquad y + x = 0$

$\Rightarrow \frac{by}{1-a} + y = 0 \qquad \Rightarrow y = -x$

$\Rightarrow \frac{(1-a+b)\, y}{1-a} = 0$

$\Rightarrow y = 0 \qquad \Rightarrow x = 0 \quad \forall a, b$

Since $(x, y) = (0, 0)$ is the only solution, $|v_\pm\rangle$ are linearly independent

DATE
PAGE

$\langle v_+ | v_- \rangle = \frac{b}{a-1} + 1 \neq 0$, hence they are not orthogonal

$$\boxed{|v_+'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$|v_-''\rangle = \begin{pmatrix} b/(1-a) \\ 1 \end{pmatrix} - \left(\frac{b+a-1}{a-1}\right) \cdot \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2b}{2(1-a)} + \frac{b+a-1}{2(a-1)} \\ 1 + \frac{b+a-1}{2(1-a)} \end{pmatrix} = \begin{matrix} \frac{b-a+1}{2(1-a)} \\ \frac{b-a+1}{2(1-a)} \end{matrix}$$

$= \cancel{3b+a-1}$

$$\Rightarrow \boxed{|v_-'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

the orthogonal vectors are $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(f) $M = U D U^\dagger$

$\lambda_+ = 1$

$\lambda_- = (a-b)$

$$D \equiv \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

$$U = (|v_+'\rangle \; |v_-'\rangle) = \boxed{\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}$$

For M to be normalizable, $M^\dagger M = M M^\dagger$

$$\Rightarrow \begin{pmatrix} a^2+b^2 & a-a^2+b-b^2 \\ a-a^2+b-b^2 & 2+a^2+b^2-2a-2b \end{pmatrix} = \begin{pmatrix} 2a^2+1-2a & 2ab+1-2a-2b \\ 2ab+1-2a-2b & 2b^2+1-2b \end{pmatrix}$$

$$\Rightarrow \boxed{a+b=1}$$

15) $R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$

$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$

$R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(a) Rotation about x axis:

$x' = x$
$y' = y\cos\phi + z\sin\phi$
$z' = -y\sin\phi + z\cos\phi$
$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Rotation about y axis:

$x' = x\cos\phi - z\sin\phi$
$y' = y$
$z' = x\sin\phi + z\cos\phi$
$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Rotation about z axis:

$x' = x\cos\phi + y\sin\phi$
$y' = -x\sin\phi + y\cos\phi$
$z' = z$
$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) we use $\cos\delta\phi \to 1$, $\sin\delta\phi \to \delta\phi$

now $R_x(\delta\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\phi \\ 0 & -\delta\phi & 1 \end{pmatrix} = I + i\delta\phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$= I + i\delta\phi J_x$

DATE
PAGE

we get similar results for $R_y(\delta\phi)$ and $R_z(\delta\phi)$ as $\cos\delta\phi$ terms lie on the diagonal. simplifying, we get:

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) let $\delta\phi = \phi/n$ as $n \to \infty$

now $R_k(\phi)$ is a result of application of 'n' $\delta\phi$

$\Rightarrow R_k(\phi) = (R_k(\delta\phi))^n$ as $n \to \infty$

$\Rightarrow R_x(\phi) = \lim_{n\to\infty} [R_k(\delta\phi)]^n = \lim_{n\to\infty}\left[I + \frac{i\phi J_k}{n}\right]^n$

This is the identity for the exponential

$\Rightarrow \boxed{R_x(\phi) = \exp(i\phi J_k)}$

(d) $[J_x, J_y] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iJ_z$

similarly $[J_y, J_z] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = iJ_x$

$[J_z, J_x] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = iJ_y$

now since $[A, B] = -[B, A]$ $\Rightarrow$ $\boxed{[J_k, J_l] = i\epsilon_{klm} J_m}$