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Homework-2

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1. Consider the following set of vectors:

$$|\eta_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; |\eta_2\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; |\eta_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

   (a) Show that $|\eta_k\rangle$ are linearly indepedent.

   (b) Construct orthonormal basis vectors $|v_k\rangle$ $(k = 1, 2, 3)$ from the three vectors $|\eta_k\rangle$ using the Gram-Schmidt orthogonalization procedure.

   (c) Construct the projection operators $P_k = |v_k\rangle\langle v_k|$.

   (d) Check that orthormal basis vectors satisfy the completeness relation.

   Answers:

$$|v_1\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad |v_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

2. In a real n-dimensional LVS, consider the vectors $|v_k\rangle$ $(k = 1, 2, 3...n)$ are given by

$$|v_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \; |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \; . . \; |v_n\rangle = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ .1 \end{pmatrix}.$$

   Does the $\{|v_k\rangle\}$ form a basis in the space?

   Construct a vector $|\psi\rangle$ in terms of $|v_k\rangle$ such that $\langle v_k|\psi\rangle = 1$ for all k.

3. Verify that

$$[\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}, \boldsymbol{\sigma}] = 2i\boldsymbol{\sigma} \times \hat{\mathbf{n}}, \quad (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) = 2\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma}$$

   Here $\hat{\mathbf{n}}$ is the radial unit vector and $\boldsymbol{\sigma}$ is the Pauli matrix vector.

4. (a)

$$\boldsymbol{\sigma} \times \boldsymbol{\sigma} = i\boldsymbol{\sigma}.$$

   (b) Prove that

$$(\boldsymbol{\sigma} \cdot \mathbf{A})(\boldsymbol{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma} \cdot (\mathbf{A} \times \mathbf{B})$$

   for any two vectors $\mathbf{A}$ and $\mathbf{B}$.

   (c) Evaluate (i) $(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})^2$ and (ii) $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})^l$ with $l = 2, 3, 4$ and $\hat{\mathbf{r}}$ being the unit radial vector.

(d) Show that the Pauli Hamiltonian for an electron (charge: $q = -e$ and mass m) in presence of the magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$,

$$H_P = \frac{[\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A})]^2}{2m},$$

is simplified to

$$H_P = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} - (\boldsymbol{\mu}_s \cdot \mathbf{B}).$$

Here $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$, $\boldsymbol{\mu}_s = (-e/m)\mathbf{S}$ is the spin magnetic moment of a free electron. Note that the first term of the above Hamiltonian is the kinetic energy of a particle with charge q in presence of a magnetic field $\mathbf{B}$ and the last term is the Zeeman coupling.

(e) Consider $V(\mathbf{r}) = 1/r$ and $\psi(\mathbf{r})$ is a wave function. Show that

$$[\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} V(\mathbf{r})][\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\psi(\mathbf{r})] = \frac{dV(\mathbf{r})}{dr}\frac{\partial\psi(\mathbf{r})}{\partial r} - \frac{2}{\hbar^2}\left(\frac{1}{r}\frac{dV(\mathbf{r})}{dr}\right)(\mathbf{L} \cdot \mathbf{S})\,\psi(\mathbf{r}).$$

Here $\mathbf{L} = -i\hbar(\mathbf{r} \times \boldsymbol{\nabla})$ and $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$.
Observe that the second term on the right hand side of the above expression is proportional to the spin-orbit coupling $\mathbf{L} \cdot \mathbf{S}$: a coupling between electrons's orbital angular momentum $\mathbf{L}$ with its own spin angular momentum $\mathbf{S}$.

5. (a) Show that

$$e^{-i\sigma_z \frac{\theta}{2}} \sigma_x e^{i\sigma_z \frac{\theta}{2}} = \sigma_x \cos\theta + \sigma_y \sin\theta,$$

using two different methods: (i) expanding the exponentials and (ii) Baker-Campbell-Hausdorff formula.

(b) Show that the spin rotation operator $U(\hat{\mathbf{n}}, \theta)$ around any arbitrary direction $\hat{\mathbf{n}}$ can be simplified as

$$U(\hat{\mathbf{n}}, \theta) = e^{\pm i(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\frac{\theta}{2}} = \cos\frac{\theta}{2} \pm i(\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})\sin\frac{\theta}{2}.$$

Replacing $\theta \to (-i\theta)$ in the above result, find the compact expression of $V(\hat{\mathbf{n}}, \theta) \equiv U(\hat{\mathbf{n}}, -i\theta) = e^{\pm(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\frac{\theta}{2}}$.

(c) Show that for any vector $\mathbf{v}$

$$e^{\pm i\boldsymbol{\sigma}\cdot\mathbf{v}} = \cos v \pm i(\boldsymbol{\sigma} \cdot \hat{\mathbf{v}})\sin v.$$

Here $\hat{\mathbf{v}} = \mathbf{v}/v$ is the unit vector with $v = |\mathbf{v}|$.

(d) Show that the Pauli vector $\boldsymbol{\sigma}$ transfoms under the spin rotation operator $U(\hat{\mathbf{n}}, \theta)$ as

$$U^\dagger \boldsymbol{\sigma} U = \cos^2\frac{\theta}{2}\boldsymbol{\sigma} - i\cos\frac{\theta}{2}\sin\frac{\theta}{2}[\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}, \boldsymbol{\sigma}] + (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\sin^2\frac{\theta}{2}.$$

(e) Using the results of Problem 3, show that the above expression simplifies further as given by

$$U^\dagger \boldsymbol{\sigma} U = \cos\theta\boldsymbol{\sigma} + (1 - \cos\theta)(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\hat{\mathbf{n}} - \sin\theta(\hat{\mathbf{n}} \times \boldsymbol{\sigma}).$$

6. Simplify $U_x = e^{i\sigma_x\theta/2}$ without expanding the expontial of the matrix $A = i\sigma_x\theta/2$. Instead, use $e^A = Pe^D P^{-1}$, where D is the diagonal matrix contains eigenvalues of $A = i\sigma_x\theta/2$ and the matrix P diagonalizes the matrix A such that $P^{-1}P = PP^{-1} = I$. You will be able to recover the known result.

7. Show that under the unitary rotation operator

$$U = e^{-i\frac{\pi}{4}\sigma_z} e^{-i\frac{\pi}{2}\sigma_y},$$

the Pauli matrices transform as

$$\sigma_x \to -\sigma_y, \quad \sigma_y \to -\sigma_x, \quad \sigma_z \to -\sigma_z,$$

Note that the rotation operator U is a product of two rotation operators. First, rotation around y axis through an angle π and then followed by another rotation around z axis through an angle $\pi/2$.

8. Show that

$$\ln \sigma_j = \frac{i\pi}{2} \left[\mathbb{I} - \sigma_j\right],$$

where $\mathbb{I}$ is a 2×2 unit matrix and $j = x, y, z$.

9. The Hamiltonian for one-dimensional quantum harmonic oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2 y^2.$$

Introducing a dimensionless variable $x = y/a$ with $a = \sqrt{\hbar/(m\omega)}$. So the Hamiltonian can expressed as

$$H = \frac{\hbar\omega}{2}\left[-\frac{d^2}{dx^2} + x^2\right].$$

The differential forms of the annihilation and creation operators for the harmonic oscillator states are

$$a = \frac{1}{\sqrt{2}}\left[x + \frac{d}{dx}\right], a^\dagger = \frac{1}{\sqrt{2}}\left[x - \frac{d}{dx}\right].$$

Here the coordinate x is in the dimensionless form. Given that $\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$. Proof of this result will be shown after mid-sem. The operators are not self-adjoint but the product $a^\dagger a$ is self-adjoint.

The Hamiltonian for a quantum oscillator with the oscillation frequency ω is $H = \hbar\omega(a^\dagger a + 1/2)$.

(a) Verify that $[a, a^\dagger] = 1$.

(b) Show that

$$e^a a^\dagger e^{-a} = a^\dagger + 1, \qquad e^{a^\dagger} a e^{-a^\dagger} = a - 1.$$

(c) Verify that the operator $U \equiv U(\lambda, \lambda^*) = e^{\lambda a^\dagger - \lambda^* a}$ is unitary:

$$U^\dagger U = U U^\dagger = 1.$$

Here λ is a real/complex parameter.

(d) Show that

$$\begin{aligned} e^{\lambda^* a - \lambda a^\dagger} a e^{\lambda a^\dagger - \lambda^* a} &= a + \lambda \\ e^{\lambda^* a - \lambda a^\dagger} a^\dagger e^{\lambda a^\dagger - \lambda^* a} &= a^\dagger + \lambda^*. \end{aligned}$$

Can you obtain the second equality from the first one?

(e)

$$\begin{aligned} e^{\lambda a^\dagger a} a e^{-\lambda a^\dagger a} &= a e^{-\lambda} \\ e^{\lambda a^\dagger a} a^\dagger e^{-\lambda a^\dagger a} &= a^\dagger e^{\lambda}. \end{aligned}$$

10. Show that

(a) If $[A, B] = \alpha A$ with α is being a scaler, we have

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda \alpha A.$$

(b) If $[A, B] = \beta B$ with β is being a scaler, we have

$$e^{\lambda A} B e^{-\lambda A} = e^{\lambda \beta} B.$$

11. **Markov matrix**: A square matrix with non-negative elements such that sum of elements of each column vector or row vector is always 1. This is also known as stochastic matrix. One of the eigenvalues of a Markov matrix is always 1 and rest of the eigenvalues (λ) will be $-1 \leq \lambda \leq +1$. A simple example is the Pauli matrix σ_x.

Consider the following Markov matrix:

$$M = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix},$$

with $0 \leq a \leq 1$ and $0 \leq b \leq 1$.

(a) Express M in terms of $\sigma_0, \sigma_x, \sigma_y$ and σ_z.

(b) Express M in terms of the four operators $|0\rangle\langle 0|, |1\rangle\langle 0|, |0\rangle\langle 1|$ and $|1\rangle\langle 1|$. Here, $|0\rangle = (1\ 0)^T$ and $|1\rangle = (0\ 1)^T$ with T being the transpose operation.

(c) Show that the eigenvalues of M are $\lambda_+ = (a-b)$ and $\lambda_- = 1$. It can be easily checked that $|\lambda_+| < 1$. Show that the corresponding eigenvectors are

$$|v_+\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad |v_-\rangle = \begin{pmatrix} \frac{b}{1-a} \\ 1 \end{pmatrix}.$$

(d) Are the vectors $|v_\pm\rangle$ linearly indepedent? If so, check if they are orthogonal to each other or not. If not, make them orthonormal vectors using Gram-Schmidt orthogonalization method.

(e)

(f) Construct the unitary matrix $U(a, b)$ which diagonalizes the Markov matrix M.

12. The model Hamiltonian for a **generic two-level Dirac system** can be written as

$$H = d_0 \sigma_0 + \boldsymbol{\sigma} \cdot \mathbf{d}.$$

Here σ_0 is the 2×2 identity matrix, $\boldsymbol{\sigma} = \sigma_x \hat{\mathbf{i}} + \sigma_y \hat{\mathbf{j}} + \sigma_z \hat{\mathbf{k}}$, and $\mathbf{d} = d_x \hat{\mathbf{i}} + d_y \hat{\mathbf{j}} + d_z \hat{\mathbf{k}}$ is a constant vector. Paramaterizing components of $\mathbf{d}$ as $d_x = d \sin\theta \cos\phi$, $d_y = d \sin\theta \sin\phi$ and $d_z = d \cos\theta$ with $d = |\mathbf{d}|$.

(a) Show that the eigenvalues are $E_\pm = d_0 \pm d$ and the corresponding eigenvectors are

$$|\chi_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}; \quad |\chi_-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}.$$

Observe that eigenvectors do not depend on d_0 and $|\mathbf{d}|$.

(b) Construct the unitary matrix $U(\theta, \phi)$ which diagonalizes the Hamiltonian H.

(c) Write down the projection operators $P_\pm = |\chi_\pm\rangle\langle\chi_\pm|$. Show that $(P_\pm)^2 = P_\pm$ and $P_+P_- = 0$. Check that the vectors satisfy the completeness relation i.e $P_+ + P_- = \sigma_0$.

(d) Defining $\mathbf{A}_\pm = i\langle\chi_\pm|\boldsymbol{\nabla}_\mathbf{d}|\chi_\pm\rangle$. Show that

$$\mathbf{A}_+ = -\frac{1}{2d}\tan\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}}, \quad \mathbf{A}_- = +\frac{1}{2d}\cot\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}}$$

(e) Defining $\boldsymbol{\Omega}_\pm = \boldsymbol{\nabla}_\mathbf{d} \times \mathbf{A}_\pm$. Show that

$$\boldsymbol{\Omega}_\pm = \mp\frac{\mathbf{d}}{2d^3}.$$

13. The **Hadamard matrix** is given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix};$$

This matrix reprsents a quantum logic gate, or simply quantum gate. This quantum gate is very popular in the study of quantum compuation and quantum information processing.

(a) Express H in terms of the Pauli matrices.

(b) Express H in terms of the four operators $|0\rangle\langle 0|, |1\rangle\langle 0|, |0\rangle\langle 1|$ and $|1\rangle\langle 1|$. Here, $|0\rangle = (1\ 0)^T$ and $|1\rangle = (0\ 1)^T$ with T being the transpose operation.

(c) Show that the eigenvalues are $\epsilon_\pm = \pm 1$ and the corresponding eigenvectors can be expressed as

$$|\chi_+\rangle = \begin{pmatrix} \cos\frac{\pi}{8} \\ \sin\frac{\pi}{8} \end{pmatrix}; \quad |\chi_-\rangle = \begin{pmatrix} -\sin\frac{\pi}{8} \\ \cos\frac{\pi}{8} \end{pmatrix}.$$

(d) Find $e^{\phi H}$ using two different meothods: (i) Expand $e^{\phi H}$ and sum the series, and (ii) $e^A = Pe^DP^{-1}$, where P is the transformation matrix which diagonalizes the matrix $A = \phi H$ and D is diagonal matrix contains the eigenvalues of A.

(e) Replace $\phi \to i\phi$ in the above result and evalute $e^{i\phi H}$.

14. (a) If M is the $(n \times n)$ matrix with each element $M_{jk} = 1$ $(1 \le j \le n$ and $1 \le k \le n$). Find e^M and e^{iM}. [Hints: Calculate M^2 and relate with M.]

(b) A matrix M is given as

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find e^M and its eigenvalues. [Hints: Calculate M^2, M^3, M^4 etc.]

15. **Finite-angle rotation matrices and their generators**:

    (a) Obtain the following finite-angle rotation matrices around x, y and z axes, respectively:

    $$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}, \; R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}, \; R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

    (b) Consider $R_k(\phi)$ with $k = x, y, z$ and obtain the corresponding matrices for an infinitesimal rotation $\delta\phi$ about x, y and z axes, respectively. Writing these as $R_k(\delta\phi) = [I + i\delta\phi J_k]$ and identify the following generators J_x, J_y and J_z:

    $$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

    (c) A finite-angle rotation ϕ can be generated by considering n succesive infinitesimal rotation $\delta\phi$ such that $\delta\phi = \phi/n$ with $n \to \infty$. Show that

    $$R_k(\phi) = \lim_{n\to\infty} [R_k(\delta\phi)]^n = e^{i\phi J_k}.$$

    (d) Show that the generators satisfy the following commutation relation:

    $$[J_k, J_l] = i\epsilon_{klm} J_m.$$

    (e) Using the Baker-Campbell-Hausdorff formula, evaluate $e^{-i\phi J_z} J_x e^{i\phi J_z}$.

    (f) Expand $e^{i\phi J_k}$ and sum the exponential series. Verify that they correctly reproduce the finite-angle rotation operators $R_k(\phi)$ as given above.

    (g) Find the eigenvalues and the correspodning eigenvectors of $R_k(\phi)$ with $k = x, y, z$.

16. Show that the Laplacian ∇^2 is invariant under rotation around any axis. Show that $2xy$ is transformed to $(x^2 - y^2)$ and vice-versa under the rotation around z axis.

17. You have already derived the commutator relation $[L_k, L_l] = i\hbar\epsilon_{klm} L_m$, where L_k $(k : x, y, z)$ are the Cartesian components of the angular momentum operators in quantum mechanics. The differential form of the angular momentum operators in quantum mecahnics are the generators of the group of rotations in three dimensions. Show that L_z is the generator of rotation around z axis by an angle β:

    $$e^{-i\beta L_z/\hbar}\psi(r,\theta,\phi) = \psi(r,\theta,\phi+\beta).$$

    Here $\psi(r,\theta,\phi)$ is a well-behaved differentiable wave function with ϕ being azimuthal angle and β is a fixed angle.

18. **Non-Hermitian matrix and its physical realization**: There are many physical systems described by non-Hermitian Hamiltonians.

    Consider a model non-Hermitian matrix:

    $$H = a\sigma_0 + b\sigma_x + ig\sigma_z.$$

    Here a, b and g are three real parameters.

(a) Show that the eigenvalues are $\epsilon_\pm = +a \pm \sqrt{b^2 - g^2}$. and the corresponding eigenvectors are

$$|\chi^R_\pm\rangle = \begin{pmatrix} ig \pm \sqrt{b^2 - g^2} \\ b \end{pmatrix}.$$

Observe that $\epsilon_\pm$ are purely real for $b > g$, $\epsilon_\pm$ are purely imaginary for $b < g$. Both the eigenvalues collapse to $\epsilon_\pm = a$ and the eigenvectors become parallel for $b = g$.

(b) Are the eigenvectors linearly independent? Check that the eigenvectors are not orthogonal. Make them orthonormal using Gram-Schmidt orthogonalization method for $b \neq g$.

(c) Show that the eigenvalues of $H^\dagger$ are $\tilde{\epsilon}_\pm = +a \pm \sqrt{b^2 - g^2}$ and the corresponding eigenvectors are

$$|\chi^L_\pm\rangle = \begin{pmatrix} -ig \pm \sqrt{b^2 - g^2} \\ b \end{pmatrix}.$$

(d) Calculate

$$\langle\chi^L_\pm|\chi^R_\pm\rangle, \quad \text{and} \quad \langle\chi^L_\pm|\chi^R_\mp\rangle.$$

19. **Generators of the Lorentz boost**: Inverse Lorentz boost, along x axis, transformation can be obtained from

$$ct = \gamma(ct' + \beta x'),\ x = \gamma(x' + vt'),\ y = y',\ z = z'.$$

Here $\beta = v/c$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Introducing rapidity λ as $\tanh\lambda = \beta$, so $-1 < \beta < 1$ and $-\infty < \lambda < \infty$.

(a) Show that the above transformation equations can be written in terms of λ as

$$ct = (ct'\cosh\lambda + x'\sinh\lambda),\ x = (x'\cosh\lambda + ct'\sinh\lambda),\ y = y',\ z = z'.$$

This set of equations can be expressed as

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = B_x(\lambda) \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

Therefore the matrix B_x is the boost operator along x axis. Similarly, the boost operator along y and z axes can easily be obtained as

$$B_y = \begin{pmatrix} \cosh\lambda & 0 & \sinh\lambda & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\lambda & 0 & \cosh\lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ B_z = \begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}.$$

(b) Obtain the generators K_j for an infinitesimal boost ($\delta\lambda$) along j-th axes. Here $j = x, y, z$.

(c) Show that the boost operators B_j can be written as

$$B_j = e^{i\lambda K_j},$$

where $j = x, y, z$ and K_j are the generators for the boost along the j-th axes. Identify the generators K_j.

Answers:

$$K_x = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_y = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_z = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(d) Evaluate the commutator $[K_x, K_y]$ and relate with the J_z.

20. The spin-1 matrices in one of the representations are given by

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Show that they satisfy the following commutation relation

$$[S_j, S_k] = i\epsilon_{jkl} S_l.$$

(b) Simplify the exponential of the operator M_x i.e. $e^{i\theta S_x}$.

(c) Consider the following matrix:

$$M = \mathbf{S} \cdot \hat{\mathbf{n}},$$

where $\mathbf{S} = S_x \hat{\mathbf{i}} + S_y \hat{\mathbf{j}} + S_z \hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$ is the radial unit vector.

(i) Show that the eigenvalues are $\epsilon_\pm = \pm 1$ and $\epsilon_0 = 0$.

(ii) Show that the corresponding eigenvectors are

$$|n_+\rangle = \begin{pmatrix} \cos^2 \frac{\theta}{2} e^{-i\phi} \\ \sqrt{2} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \\ \sin^2 \frac{\theta}{2} e^{i\phi} \end{pmatrix}, \quad |n_0\rangle = \begin{pmatrix} -\sin\theta e^{-i\phi} \\ \sqrt{2} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}, \quad |n_-\rangle = \begin{pmatrix} -\sin^2 \frac{\theta}{2} e^{-i\phi} \\ \sqrt{2} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \\ \cos^2 \frac{\theta}{2} e^{i\phi} \end{pmatrix}.$$