Department of Physics, IIT-Kanpur

Instructor: Tarun Kanti KGhosh   Math Methods-I (PHY421)   AY 2025-26, SEM-I

Homework-4

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1. Consider the complex function $h(k) = h_x(k) + ih_y(k)$, where $h_x(k) = v + w\cos k$ and $h_y(k) = w\sin k$. Here k is a parameter varying from $-\pi$ to π (0 to 2π). Also, v and w are real positive constants.

   The tip of the "complex vector" $h(k)$ traces out a circle of radius w centered at $(v, 0)$ in the $h_x(k)$-$h_y(k)$ plane while varying k from $-\pi$ to π. Note that both $h_x(k)$ and $h_y(k)$ vanish at $k = \pm\pi$ for $v = w$.

   (a) Draw these circles on $h_x(k)$-$h_y(k)$ plane for the following three different cases: (i) $v < w$, (ii) $v = w$, and (iii) $v > w$.

   (b) Plot the argument of $h(k)$ vs k for (i) $w/v = 0.5$, (ii) $w/v = 1$ and (iii) $w/v = 1.5$.

   (c) Evaluate the following integral:

   $$I = \frac{1}{2\pi i}\oint_C \frac{dh(k)}{h(k)},$$

   for the three different cases: (i) $v < w$, (ii) $v = w$, and (iii) $v > w$. Here the closed contour C is the path traces out by the tip of $h(k)$ while k varing from $-\pi$ to π.

   This integral is also known as winding number. The discrete change in the winding number while changing the system parameters (v, w) continuously. This has been verified in laboratory in recent past. It is a topological phase transition.

2. (a) Show that $\frac{d}{dz}\sin z = \cos(z)$ and $\int \cos z dz = \sin z$.

   b) Verify that $\int_{0,0}^{1,1} z^* dz$ depends on the path by evaluating the integral for the two paths: i) $(0,0) \to (1,0) \to (1,1)$ and ii) $(0,0) \to (1,1)$.

   c) Evaluate the line integral

   $$\int_\Gamma (z^2+1)e^{-iz^2} dz,$$

   where Γ is the ray $\theta = -\pi/4$.

   d) Evaluate $\oint_C z^n e^{-|z|^2} dz$ where $n = 0, 1, 2, ...$ and the contour C is a circle with radius ϵ centered at $z = z_0 = 0$.

   ie) Show that

   $$\frac{1}{2\pi i}\oint_C z^{m-n-1} dz = \delta_{m,n}.$$

   Here, $\delta_{m,n}$ is the Kronecker delta and the contour C is a unit circle centered at origin.

   f) Consider the integral

   $$I = \oint_C \frac{1}{(az-b)(pz-q)} dz,$$

   where C describes the circle $|z| = R$. Here, a, b, p and q are real constants. Evaluate I for i) $\frac{b}{a} < R < \frac{q}{p}$, ii) $\frac{q}{p} < R < \frac{b}{a}$, iii) $R > \frac{q}{p} > \frac{b}{a}$, and iv) $R < \frac{q}{p} < \frac{b}{a}$.

3. i) Using the fact that $f(z) = e^{-iz^n}$ is an analytic function, show that

$$\int_o^{2\pi} e^{\sin n\theta} \cos(\theta - \cos n\theta) d\theta = \int_o^{2\pi} e^{\sin n\theta} \sin(\theta - \cos n\theta) d\theta = 0.$$

Here, n is a positive integer.

ii) Evaluate the integral $\oint_C \frac{dz}{z(R-z)}$, where C is a circle: $|z| = r < R$. Use this result to evaluate the following integral:

$$\int_0^{2\pi} \frac{(R - r\cos\theta)d\theta}{R^2 + r^2 - 2Rr\cos\theta}.$$

4. Consider $f(z)$ is an analytic function inside and on a unit circle C centered at z_0. Show that

$$f^{(n)}(z_0) = \frac{n!}{2\pi} \int_0^{2\pi} e^{-in\theta} f(z_0 + e^{i\theta}) d\theta.$$

5. (a) Consider the integral

$$I = \int_\Gamma e^{-z^2} dz,$$

where Γ is a line segment of length L running parallel to the imaginary axis from R to $R + iL$. Using the Darboux's theorem, show that $|I|$ tends to zero as $R \to \infty$.

(b) Following the proof of the Jordan's lemma, show that

$$\lim_{R\to\infty} \Big| \int_\Gamma e^{ikz^N} dz \Big| = 0,$$

where Γ is an arc of the circle $|z| = R$ lying in the sector $0 \le \theta \le \frac{\pi}{2N}$ and $N > 1$. Using this result, evaluate the following integral of the real variable x

$$I = \int_0^\infty e^{ix^N} dx.$$

(c) Prove that

$$I = \int_0^\infty e^{ipx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{p}} e^{i\pi/4}.$$

6. i) Using the Cauchy's integral theorem and Cauchy's integral formula, evaluate the following integral:

$$\oint_C \frac{e^{-z}\sin z}{z^2},$$

where C is any closed curve encircles $z = 0$ point.

ii) Using the Cauchy's integral formula, evaluate the following integral:

$$\oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-3)} dz,$$

where $C : |z - i| = 3$.