Department of Physics, IIT-Kanpur

Instructor: Tarun Kanti Ghosh   Math methods-I (PHY421)   AY 2024-25, I-SEM

Homework-1

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1. (a) Evaluate (i) $\boldsymbol{\nabla} e^{i\mathbf{k}\cdot\mathbf{r}}$, (ii) $\boldsymbol{\nabla}\left(\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{r}|}\right)$ and (iii) $\boldsymbol{\nabla}\cos(\mathbf{k}\cdot\mathbf{r})$.

   (b) Evaluate (i) $\boldsymbol{\nabla}\cdot(\mathbf{a}\, e^{i\mathbf{k}\cdot\mathbf{r}})$ and (ii) $\boldsymbol{\nabla}\times(\mathbf{a}\, e^{i\mathbf{k}\cdot\mathbf{r}})$.

   (c) Evaluate (i) $\nabla^2\left(\frac{1}{|\mathbf{r}|}\right)$ and (ii) $\nabla^2\left(\frac{e^{-\lambda r}}{|\mathbf{r}|}\right)$.

   Here $\mathbf{a}$ and $\mathbf{k}$ are constant vectors. Also, λ is a constant.

2. Show that the gradient operator in spherical coordinates is

$$\boldsymbol{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

   Calculate $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla} f) = \nabla^2 f$. Note: The partial derivatives $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\phi}$ in the left-hand $\boldsymbol{\nabla}$ operator act on the unit vectors $\hat{r},\hat{\theta},\hat{\phi}$ of right-hand $\boldsymbol{\nabla}$ operator.

3. Show that the angular momentum operator $\mathbf{L} = -i\hbar[\mathbf{r}\times\boldsymbol{\nabla}]$ in spherical coordinates is

$$\mathbf{L} = -i\hbar\Big[-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \hat{\phi}\frac{\partial}{\partial\theta}\Big].$$

   Obtain the form of $\mathbf{L}$ at the equator from the above expression.

4. The kinetic momentum of a particle with charge $q > 0$ and mass m in presence of the magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ is $\boldsymbol{\Pi} = (-i\hbar\boldsymbol{\nabla} - q\mathbf{A})$. Here $\mathbf{A}$ is the vector potential corresponding to the magnetic field $\mathbf{B}$.

   (a) Show that $\boldsymbol{\Pi}\times\boldsymbol{\Pi} = i\hbar q\mathbf{B}$ and the commutator $[\Pi_x,\Pi_y] = i\hbar qB$, without choosing any specific form of the vector potential.

   (b1) Find a symmetric form of $\mathbf{A}$ for $\mathbf{B} = B\hat{z}$. Check that $\mathbf{A}$ is divergenceless.

   (b2) Calculate the line integral $\int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{A}\cdot d\mathbf{r}$ along the straight line joining between $\mathbf{r}_1$ and $\mathbf{r}_2$ points and show that its value is the magnetic flux passing through the triangle spanned by the vectors $\mathbf{r}_1 = x_1\hat{i} + y_1\hat{j}$, $\mathbf{r}_2 = x_2\hat{i} + y_2\hat{j}$ and $\mathbf{r}_2 - \mathbf{r}_1$.

   (b3) Show that

$$H = \frac{\boldsymbol{\Pi}^2}{2m} = \frac{\boldsymbol{\Pi}\cdot\boldsymbol{\Pi}}{2m} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{qB}{2m}L_z + \frac{q^2B^2}{8m}(x^2+y^2)$$

   with $L_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$ is the angular momentum operator.

5. Sketch the following functons:

$$(i)\ \lim_{\epsilon\to 0}\tanh(x/\epsilon),\quad (ii)\ \lim_{\epsilon\to 0}\frac{\sin(x/\epsilon)}{\pi x},\ (iii)\ \lim_{\epsilon\to 0}\frac{\epsilon/\pi}{x^2+\epsilon^2}.$$

6. Sketch the Fermi-Dirac distribution function and its derivative:

$$f(E) = \lim_{T\to 0} \frac{1}{e^{(E-\mu)/T}+1},$$

where μ is a real constant. Express $f(E)$ in terms of the unit step function as $T \to 0$. Express $f'(E) = \frac{\partial f(E)}{\partial E}$ in terms of the Dirac delta function as $T \to 0$.

7. Evaluate the following integral:

$$D(E) = \int_V \frac{d^3k}{(2\pi)^3} \delta\left(E - \frac{\hbar^2 k^2}{2m}\right).$$

Here $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ is three-dimensional wavevector. The final result will give you the density of states of a free electron with mass m.

8. Evaluate the following integral:

$$D(E) = \int_S \frac{d^2k}{(2\pi)^2} \delta\left(E - \frac{\hbar^2 k^2}{2m} - \alpha k\right).$$

Here α is a constant having suitable dimension and $k = \sqrt{k_x^2 + k_y^2}$ is two-dimensional wavevector.

9. Find the points x_n, the range of values of the summation index n and the coefficients c_n in the following expansion

$$\delta(\sin x - \cos x) = \sum_n c_n \delta(x - x_n). \tag{1}$$

Evaluate the following integral:

$$\int_{-\infty}^{\infty} dx\, e^{-|x|}\, \delta(\sin x).$$

10. Show that

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\mathbf{r}).$$

Evaluate the following volume integral using (i) direct integration and (ii) using the above result:

$$J = \int_V e^{-r} \left[\boldsymbol{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right)\right] d\tau,$$

where V is a sphere of radius R, centered at origin. Try to undestand role of the auxiliary function e^{-r} in the integrand.

[Hints: In direct integration, use $\boldsymbol{\nabla}\cdot(f\mathbf{A}) = f\boldsymbol{\nabla}\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\nabla} f$ and Gauss divergence theorem. Here $f \equiv f(\mathbf{r})$ is a scaler funtion.]

11. Show that

$$\boldsymbol{\nabla} \times \left(\frac{\hat{\phi}}{\rho}\right) = 2\pi\delta^2(\boldsymbol{\rho}),$$

with $\boldsymbol{\rho} = x\hat{\imath} + y\hat{\jmath}$.

Evaluate the following surface integral using (i) direct integration and (ii) using the above reult:

$$J = \int_S e^{-\rho} \left[\boldsymbol{\nabla} \times \left(\frac{\hat{\phi}}{\rho} \right) \right] \cdot d\mathbf{a},$$

where S is a circular disk of radius R, centered at origin. Try to undestand role of the auxiliary function $e^{-\rho}$ in the integrand.

[Hints: In direct integration, use $\boldsymbol{\nabla} \times (f\mathbf{A}) = \boldsymbol{\nabla} f \cdot \mathbf{A} + f\boldsymbol{\nabla} \times \mathbf{A}$ and Stokes theorem.]

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Homework-2

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1. Consider the following set of vectors:

$$|\eta_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ |\eta_2\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ |\eta_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Show that $|\eta_k\rangle$ are linearly indepedent.

(b) Construct orthonormal basis vectors $|v_k\rangle$ $(k = 1, 2, 3)$ from the three vectors $|\eta_k\rangle$ using the Gram-Schmidt procedure.

(c) Check that orthormal basis vectors satisfy the completeness relation.

(d) Construct the projection operator $P_k = |v_k\rangle\langle v_k|$.

2. In a real n-dimensional LVS, consider the vectors $|v_k\rangle$ $(k = 1, 2, 3...n)$ are given by

$$|v_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \ |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \ . . |v_n\rangle = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ .1 \end{pmatrix}.$$

Does the $\{|v_k\rangle\}$ form a basis in the space?

Construct a vector $|\psi\rangle$ in terms of $|v_k\rangle$ such that $\langle v_k|\psi\rangle = 1$ for all k.

3. Verify that

$$[\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}, \boldsymbol{\sigma}] = 2i\boldsymbol{\sigma} \times \hat{\mathbf{n}}, \quad (\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) = 2\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) - \boldsymbol{\sigma}$$

Here $\hat{\mathbf{n}}$ is the radial unit vector.

4. (a) Prove that

$$(\boldsymbol{\sigma} \cdot \mathbf{A})(\boldsymbol{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma} \cdot (\mathbf{A} \times \mathbf{B}).$$

Evaluate (i) $(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})^2$ and (ii) $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})^l$ with $l = 2, 3, 4$ and $\hat{\mathbf{r}}$ being the unit radial vector.

(b) Show that the Pauli Hamiltonian

$$H_P = \frac{[\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A})]^2}{2m}$$

is simplified to

$$H_P = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} - \frac{q\hbar}{2m}\boldsymbol{\sigma} \cdot \mathbf{B}.$$

Here $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, charge q and mass m.

Observe that the last term of the above Hamiltonian is the Zeeman coupling.

(c) Consider $V(\mathbf{r}) = 1/r$ and $\psi(\mathbf{r})$ is a wave function. Show that

$$[\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} V(\mathbf{r})][\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \psi(\mathbf{r})] = \frac{dV(\mathbf{r})}{dr}\frac{\partial \psi(\mathbf{r})}{\partial r} - \frac{2}{\hbar^2}\left(\frac{1}{r}\frac{dV(\mathbf{r})}{dr}\right)(\mathbf{L}\cdot\mathbf{S})\,\psi(\mathbf{r}).$$

Here $\mathbf{L} = -i\hbar(\mathbf{r} \times \boldsymbol{\nabla})$ and $\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$.

Observe that the second term on the right hand side of the above expression is proportional to the spin-orbit coupling $\mathbf{L}\cdot\mathbf{S}$: a coupling between electrons's orbital angular momentum $\mathbf{L}$ with its own spin angular momentum $\mathbf{S}$.

5. (a) Show that the spin rotation operator $U(\hat{\mathbf{n}}, \theta)$ around any arbitrary direction $\hat{\mathbf{n}}$ can be simplified as

$$U(\hat{\mathbf{n}}, \theta) = e^{\pm i(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\frac{\theta}{2}} = \cos\frac{\theta}{2} \pm i(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\sin\frac{\theta}{2}.$$

Replacing $\theta \rightarrow (-i\theta)$ in the above result, find the compact expression of $V(\hat{\mathbf{n}}, \theta) \equiv U(\hat{\mathbf{n}}, -i\theta) = e^{\pm(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\frac{\theta}{2}}$.

(b) Show that the Pauli vector $\boldsymbol{\sigma}$ transfoms under the spin rotation operator $U(\hat{\mathbf{n}}, \theta)$ as

$$U^\dagger \boldsymbol{\sigma} U = \cos^2\frac{\theta}{2}\boldsymbol{\sigma} - i\cos\frac{\theta}{2}\sin\frac{\theta}{2}[\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}, \boldsymbol{\sigma}] + (\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})\sin^2\frac{\theta}{2}.$$

(c) Using the results of Problem 3, show that the above expression simplifies further as given by

$$U^\dagger \boldsymbol{\sigma} U = \cos\theta\boldsymbol{\sigma} + (1-\cos\theta)(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})\hat{\mathbf{n}} - \sin\theta(\hat{\mathbf{n}}\times\boldsymbol{\sigma}).$$

6. Simplify $U_x = e^{i\sigma_x\theta/2}$ without expanding the expontial of the matrix $A = i\sigma_x\theta/2$. Instead, use $e^A = Pe^D P^{-1}$, where D is the diagonal matrix contains eigenvalues of $A = i\sigma_x\theta/2$ and the matrix P diagonalizes the matrix A such that $P^{-1}P = PP^{-1} = I$. You will be able to recover the known result.

7. Show that under the unitary rotation operator

$$U = e^{-i\frac{\pi}{4}\sigma_z}e^{-i\frac{\pi}{2}\sigma_y},$$

the Pauli matrices transform as

$$\sigma_x \rightarrow -\sigma_y, \quad \sigma_y \rightarrow -\sigma_x, \quad \sigma_z \rightarrow -\sigma_z,$$

Note that the rotation operator U is a product of two rotation operators. First, rotation around y axis through an angle π and then followed by another rotation around z axis through an angle $\pi/2$.

8. **Markov matrix**: A square matrix with non-negative elements such that sum of elements of each column vector or row vector is always 1. This is also known as stochastic matrix. One of the eigenvalues of a Markov matrix is always 1 and rest of the eigenvalues (λ) will be $-1 \leq \lambda \leq +1$. A simple example is the Pauli matrix σ_x.

Consider the following Markov matrix:

$$M = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix},$$

with $0 \le a \le 1$ and $0 \le b \le 1$.

(a) Express M in terms of $\sigma_0, \sigma_x, \sigma_y$ and σ_z.

(b) Express M in terms of the four operators $|0\rangle\langle 0|, |1\rangle\langle 0|, |0\rangle\langle 1|$ and $|1\rangle\langle 1|$. Here, $|0\rangle = (1\ 0)^T$ and $|1\rangle = (0\ 1)^T$ with T being the transpose operation.

(c) Show that the eigenvalues of M are $\lambda_+ = (a-b)$ and $\lambda_- = 1$. It can be easily checked that $|\lambda_+| < 1$. Show that the corresponding eigenvectors are

$$|v_+\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad |v_-\rangle = \begin{pmatrix} \frac{b}{1-a} \\ 1 \end{pmatrix}.$$

(d) Are the vectors $|v_\pm\rangle$ linearly indepedent? If so, check if they are orthogonal to each other or not. If not, make them orthonormal vectors using Gram-Schmidt orthogonalization method.

9. The model Hamiltonian for a **generic two-level Dirac system** can be written as

$$H = d_0\sigma_0 + \boldsymbol{\sigma} \cdot \mathbf{d}.$$

Here σ_0 is the 2×2 identity matrix, $\boldsymbol{\sigma} = \sigma_x\hat{\mathbf{i}} + \sigma_y\hat{\mathbf{j}} + \sigma_z\hat{\mathbf{k}}$, and $\mathbf{d} = d_x\hat{\mathbf{i}} + d_y\hat{\mathbf{j}} + d_z\hat{\mathbf{k}}$ is a constant vector. Paramaterizing components of $\mathbf{d}$ as $d_x = d\sin\theta\cos\phi, d_y = d\sin\theta\sin\phi$ and $d_z = d\cos\theta$ with $d = |\mathbf{d}|$.

(a) Show that the eigenvalues are $E_\pm = d_0 \pm d$ and the corresponding eigenvectors are

$$|\chi_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}; \quad |\chi_-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}.$$

Observe that eigenvectors do not depend on d_0 and $|\mathbf{d}|$.

(b) Write down the projection operators $P_\pm = |\chi_\pm\rangle\langle\chi_\pm|$. Show that $(P_\pm)^2 = P_\pm$ and $P_+P_- = 0$. Check that the vectors satisfy the completeness relation i.e $P_+ + P_- = \sigma_0$.

(c) Defining $\mathbf{A}_\pm = i\langle\chi_\pm|\boldsymbol{\nabla}_\mathbf{d}|\chi_\pm\rangle$. Show that

$$\mathbf{A}_+ = -\frac{1}{2d}\tan\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}}, \quad \mathbf{A}_- = +\frac{1}{2d}\cot\left(\frac{\theta}{2}\right)\hat{\boldsymbol{\phi}}$$

(d) Defining $\boldsymbol{\Omega}_\pm = \boldsymbol{\nabla}_\mathbf{d} \times \mathbf{A}_\pm$. Show that

$$\boldsymbol{\Omega}_\pm = \mp\frac{\mathbf{d}}{2d^3}.$$

10. The **Hadamard matrix** is given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix};$$

This matrix reprsents a quantum logic gate, or simply quantum gate. This quantum gate is very popular in the study of quantum compuation and quantum information processing.

(a) Express H in terms of the Pauli matrices.

(b) Express H in terms of the four operators $|0\rangle\langle 0|$, $|1\rangle\langle 0|$, $|0\rangle\langle 1|$ and $|1\rangle\langle 1|$. Here, $|0\rangle = (1\ 0)^T$ and $|1\rangle = (0\ 1)^T$ with T being the transpose operation.

(c) Show that the eigenvalues are $\epsilon_{\pm} = \pm 1$ and the corresponding eigenvectors can be expressed as

$$|\chi_+\rangle = \begin{pmatrix} \cos\frac{\pi}{8} \\ \sin\frac{\pi}{8} \end{pmatrix}; \quad |\chi_-\rangle = \begin{pmatrix} -\sin\frac{\pi}{8} \\ \cos\frac{\pi}{8} \end{pmatrix}.$$

(d) Find $e^{\phi H}$ using two different meothods: (i) Expand $e^{\phi H}$ and sum the series, and (ii) $e^A = Pe^D P^{-1}$, where P is the transformation matrix which diagonalizes the matrix $A = \phi H$ and D is diagonal matrix contains the eigenvalues of A.

(e) Replace $\phi \to i\phi$ in the above result and evalute $e^{i\phi H}$.

11. (a) If M is the $(n \times n)$ matrix with each element $M_{jk} = 1$ ($1 \le j \le n$ and $1 \le k \le n$). Find e^M and e^{iM}. [Hints: Calculate M^2 and relate with M.]

(b) A matrix M is given as

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

Find e^M and its eigenvalues. [Hints: Calculate M^2, M^3, M^4 etc.]

12. The finite-angle rotation matrices around x, y and z axes, respectively, are given as

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}, \; R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}, \quad R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Consider $R_j(\phi)$ with $k = x, y, z$ and obtain the corresponding matrices for an infinitesimal rotation $\delta\phi$ about x, y and z axes, respectively. Writing these as $R_k(\delta\phi) = [I + i\delta\phi J_k]$ and identify the generators J_x, J_y and J_z.

(b) A finite-angle rotation ϕ can be generated by considering n succesive infinitesimal rotation $\delta\phi$ such that $\delta\phi = \phi/n$ with $n \to \infty$. Show that

$$R_k(\phi) = \lim_{n\to\infty} [R_k(\delta\phi)]^n = e^{i\phi J_k}.$$

(c) Show that the generators satisfy the following commutation relation:

$$[J_k, J_l] = i\epsilon_{klm} J_m.$$

(d) Expand $e^{i\phi J_k}$ and sum the exponential series. Verify that they correctly reproduce the finite-angle rotation operators $R_k(\phi)$ as given above.

(e) Find the eigenvalues and the correspodning eigenvectors of $R_j(\phi)$ with $j = x, y, z$.

13. The angular momentum operators in quantum mechanics are the differential operators acting on some function $\psi(\mathbf{r})$:

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right); L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right); L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Show that

$$[L_k, L_l] = i\epsilon_{klm} L_m.$$

The differential form of the angular momentum operators in quantum mecahnics are the generators of the group of rotations in three dimensions.

14. **Non-Hermitian matrix and its physical realization**: There are many physical systems described by non-Hermitian Hamiltonians.

    Consider a model non-Hermitian matrix:

    $$H = a\sigma_0 + b\sigma_x + ig\sigma_z.$$

    Here a, b and g are three real parameters.

    (a) Show that the eigenvalues are $\epsilon_\pm = +a \pm \sqrt{b^2 - g^2}$. and the corresponding eigenvectors are

    $$|\chi_\pm\rangle = \begin{pmatrix} ig \pm \sqrt{b^2 - g^2} \\ b \end{pmatrix}.$$

    Observe that $\epsilon_\pm$ are purely real for $b > g$, $\epsilon_\pm$ are purely imaginary for $b < g$. Both the eigenvalues collapse to $\epsilon_\pm = a$ and the eigenvectors become parallel for $b = g$.

    (c) Check that the eigenvectors are not orthogonal. Make them orthonormal using Gram-Schmidt orthogonalization method for $b \neq g$.

15. **Generators of the Lorentz boost**: Inverse Lorentz boost, along x axis, transformation can be obtained from

    $$ct = \gamma(ct' + \beta x'),\ x = \gamma(x' + vt'),\ y = y',\ z = z'.$$

    Here $\beta = v/c$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

    Introducing rapidity λ as $\tanh\lambda = \beta$, so $-1 < \beta < 1$ and $-\infty < \lambda < \infty$.

    (a) Show that the above transformation equations can be written in terms of λ as

    $$ct = (ct'\cosh\lambda + x'\sinh\lambda),\ x = (x'\cosh\lambda + ct'\sinh\lambda),\ y = y',\ z = z'.$$

    This set of equations can be expressed as

    $$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = B_x(\lambda) \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

    Therefore the matrix B_x is the boost operator along x axis. Similarly, the boost operator along y and z axes can easily be obtained as

    $$B_y = \begin{pmatrix} \cosh\lambda & 0 & \sinh\lambda & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\lambda & 0 & \cosh\lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ B_z = \begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}.$$

    (b) Obtain the generators K_j for an infinitesimal boost $(\delta\lambda)$ along j-th axes. Here $j = x, y, z$.

    (c) Show that the boost operators B_j can be written as

    $$B_j = e^{i\lambda K_j}.$$

16. The spin-1 matrices in one of the representations are given by

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Show that they satisfy the following commutation relation

$$[S_j, S_k] = i\epsilon_{jkl} S_l.$$

(b) Simplify the exponential of the operator M_x i.e. $e^{i\theta S_x}$.

(c) Consider the following matrix:

$$M = \mathbf{S} \cdot \hat{\mathbf{n}},$$

where $\mathbf{S} = S_x \hat{\mathbf{i}} + S_y \hat{\mathbf{j}} + S_z \hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$ is the radial unit vector.

(i) Show that the eigenvalues are $\epsilon_{\pm 1} = \pm 1$ and $\epsilon_0 = 0$.

(ii) Show that the corresponding eigenvectors are

$$|n_+\rangle = \begin{pmatrix} \cos^2 \frac{\theta}{2} e^{-i\phi} \\ \sqrt{2} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \\ \sin^2 \frac{\theta}{2} e^{i\phi} \end{pmatrix}, \quad |n_0\rangle = \begin{pmatrix} -\sin\theta e^{-i\phi} \\ \sqrt{2}\cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}, \quad |n_-\rangle = \begin{pmatrix} -\sin^2 \frac{\theta}{2} e^{-i\phi} \\ \sqrt{2} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \\ \cos^2 \frac{\theta}{2} e^{i\phi} \end{pmatrix}.$$

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Homework-3

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1. Consider a classical harmonic oscillator described by the Hamiltonian $H = \frac{p^2(t)}{2m} + \frac{1}{2}m\omega^2 x^2(t)$. Consider a dimensionless complex variable z as $z(t) = \sqrt{\frac{m\omega}{2\hbar}}x(t) + i\frac{p(t)}{\sqrt{2m\hbar\omega}}$. Find an equatin of motion in the complex variable $z(t)$. Solve the equation of motion and extract $x(t)$ and $p(t)$.

   [Note: The system is one dimensional but we are using momentum as a second dimension and constructing a complex variable. While solving harmonic oscillator quantum mechanically, we use bosonic ladder operators. The form of the ladder operators is exactly same as $z(t)$. In quantum case, we treat x and p as operators.]

2. The rotation matrix $R_z(\phi)$ (as given in class) changes a point (x, y) to (x', y'). Define complex cooridnates as $\xi = x + iy$ and $\xi' = x' + iy'$ and their conjugates accordingly, so that the transformation in the complex plane can be written as

$$\begin{pmatrix} \xi' \\ (\xi')^* \end{pmatrix} = R_z(\phi) \begin{pmatrix} \xi \\ \xi^* \end{pmatrix}.$$

   Find the rotation matrix in the complex plane ξ.

3. Show that

   a)

$$S_N = \sum_{n=0}^{N-1} e^{in\theta} = \frac{\sin(N\theta/2)}{\sin(\theta/2)} e^{i(N-1)\theta/2}.$$

   Plot $|S_N|^2$ vs θ for $N = 10, 100$ and 1000.

   b)

$$S_p = \sum_{n=0}^{\infty} p^n e^{in\theta} = \frac{1 - pe^{-i\theta}}{1 - 2p\cos\theta + p^2},$$

   where $|p| < 1$. Plot $|S_p|^2$ vs θ for $p = 0.01, 0.5$ and 0.99. These results are very useful in optics.

   c)

$$S_p = \sum_{n=-\infty}^{\infty} p^{|n|} e^{in\theta} = \frac{1 - p^2}{1 - 2p\cos\theta + p^2} = \mathrm{Re}\left[\frac{1 + pe^{i\theta}}{1 - pe^{i\theta}}\right],$$

   where $0 \leq |p| < 1$. This is known as Poisson kernel for the unit disc.

   (d) Evaluate

$$F(\phi) = \sum_{l=-\infty}^{\infty} t_l e^{il\phi} = 2e^{-i\phi/2} \sum_{l=0}^{\infty} t_l \cos[(l + 1/2)\phi]$$

   with the condition $t_l = t_{-(l+1)}$. For what value of ϕ, $F(\phi)$ vanishes, irrespective of l and t_l values.

   This result is useful in the study of quantum scattering in two-dimensions.

4. Determine number of branches, branch points and their order of the following complex functions:

$$(i) z^{1/2}(z-1)^{1/3},\ (ii)(z^2+1)^{1/3},\ (iii)\sqrt{z(z-1)},\ (iv)\log(z^2-1)\ (v)(z+1)^{1/2}+(z-1)^{1/2}.$$

Draw all possible branch cuts and domains where the function is single-valued.

5. (a) **Trigonometric mapping function** $w(z) = \sin z$.

Show that straight lines $x = c_1 (c_1 \neq 0, \pm\pi/2)$, are mapped onto hyperbolas in the w plane. Here, c_1 is a real constant.

(b) **Logarithmic mapping function** $w(z) = \ln z$.

Show that circles centered at the origin in the z-plane are mapped onto lines parallel to the v axis.

(c) **Joukowski mapping function** $w(z) = z + 1/z$.

Show that a circle with radius $r_0 \neq 1$ is centered at $z = 0$ is mapped onto ellipses in w-plane.

Show that a circle with radius $r_0 = 1$ is centered at $z = 0$ is mapped onto the segment $-2 \leq u \leq 2$ of u axis.

6. Consider a function

$$\begin{aligned} f(x,y) &= \frac{x^2 y}{x^4+y^2}, \quad & z \neq 0 \\ &= 0, & z = 0. \end{aligned}$$

Is this function continuous at $z = 0$?

[Hints: Along all the straight lines $y = mx$, it seems to be continuous. Consider a set of parabolic paths $y = ax^2$ and see what happens.]

7. (a) Is $f(z) = z^m e^{-|z|^2/4}$ an analytic function?

(b) Consider $f(z) = u(x,y) + iv(x,y)$ is an analytic function. Check whether $F(x,y) = u(x,y)v(x,y)$ is a solution of the Laplace equation or not.

(c) Consider the real part of an analytic function is given by $u(x,y) = -2xy + y/(x^2+y^2)$. Calculate its harmonic conjugate function $v(x,y)$.

8. Consider the following matrix operator

$$H = \boldsymbol{\sigma} \cdot \boldsymbol{\nabla},$$

where $\boldsymbol{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$. Note that x and y are dimensionless here.

(a) Show that the components of the operator $\boldsymbol{\nabla}$ can be expressed in terms of complex conjugate coordinates as

$$\frac{\partial}{\partial x} = \left[\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right]$$

and

$$\frac{\partial}{\partial y} = \left[\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right].$$

(a) Express the operator H in complex coordinates.

(b) Find the eigenvector $|\psi\rangle$ which satisfies $H|\psi\rangle = 0$.

9. The Hamiltonian of a two-dimensional isotropic harmonic oscillator is given by

$$H = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + \frac{1}{2}M\omega^2(x^2 + y^2).$$

(a) Obtain the Hamiltonian H in complex coordinates as given below:

$$H = \hbar\omega\left[-2\frac{\partial^2}{\partial z \partial z^*} + \frac{zz^*}{2}\right].$$

Note that here z is a dimensionless variable: $z = (x + iy)/a$ with the oscillator length $a = \sqrt{\hbar/(M\omega)}$.

(b) Show that the complex function $\psi_l(z)$ (normalized to one)

$$\psi_l(z) = \frac{1}{\sqrt{\pi a^2 n!}} z^l e^{-|z|^2/2}, \qquad l = 0, 1, 2, ...$$

and its conjugate $\psi_l^*(z)$ are the eigenfunctions of H with the same eigenvalues $E_l = \hbar\omega(l+1)$.

(c) Show that the **angular momentum** operator in terms of the complex coordinates can be expressed as

$$L_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) = \hbar\left[z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right].$$

(d) Show that

$$L_z\psi_l(z) = l\hbar\psi_l(z), \quad L_z\psi_l^*(z) = -l\hbar\psi_l^*(z).$$

(e) Show that

$$\sum_{l=0}^{\infty} |\psi_l(z)|^2 = \frac{1}{\pi a^2}.$$

10. You have already derived the following Hamiltonian for a charge ($q > 0$) particle in presence of the magnetic field $\mathbf{B} = B\hat{z}$ in the symmetric gauge:

$$H = -\frac{\hbar^2}{2M}\nabla^2 - \frac{qB}{2M}(xp_y - yp_x) + \frac{q^2B^2}{8M}(x^2 + y^2).$$

Introducing **dimensionless** and **independent** complex conjugate variables $z = (x + iy)/l$ and $z^* = (x - iy)/l$, where $l = \sqrt{\hbar/(qB)}$ is the magnetic length scale.

(a) Show that the components of the canonical momentum operator can be expressed in terms of complex conjugate coordinates as

$$p_x = -i\hbar\frac{\partial}{\partial x} = -\frac{i\hbar}{l}\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right]$$

and

$$p_y = -i\hbar\frac{\partial}{\partial y} = \frac{\hbar}{l}\left[\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right].$$

(b) You have already derived **canonical angular momentum** operator in terms of the complex coordinates in the previous problem:

$$L_z = (\mathbf{r} \times \mathbf{p})_z = +\hbar\left[z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right].$$

The **mechanical angular momentum** operator is given by $L_z^{\text{mech}} = (\mathbf{r} \times \mathbf{\Pi})_z = (x\Pi_y - y\Pi_x)$. Show that the mechanical angular momentum L_z^{mech} in terms of the complex coordinates can be written as

$$L_z^{\text{mech}} = +\hbar \left[z\frac{\partial}{\partial z} - z^* \frac{\partial}{\partial z^*} - \frac{|z|^2}{2} \right]$$

(c) Show that the above Hamiltonian H can be re-written as

$$H = \hbar\omega_c \left[-2\frac{\partial^2}{\partial z \partial z^*} - \frac{1}{2}\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*} \right) + \frac{zz^*}{8} \right].$$

Here, $\omega_c = qB/M$ is the cyclotron frequency.

(d) You have already derived the commutator $[\Pi_x, \Pi_y] = i\hbar qB$, which is similar to $[x, p_x] = i\hbar$. We know that two ladder operators (or, raising and lowering operators) in terms of x and p_x are constructed in order to quantize the Hamiltonian of a quantum mechanical harmonic oscillator.

Similarly, constructing the following ladder operators:

$$a = \frac{i}{\sqrt{2\hbar qB}}[\Pi_x + i\Pi_y], \quad a^\dagger = \frac{-i}{\sqrt{2\hbar qB}}[\Pi_x - i\Pi_y].$$

Show that the ladder operators can be expressed in complex coordinates as

$$a = \frac{1}{\sqrt{2}}\left[\frac{z}{2} + 2\frac{\partial}{\partial z^*} \right], \quad a^\dagger = \frac{1}{\sqrt{2}}\left[\frac{z^*}{2} - 2\frac{\partial}{\partial z} \right].$$

Show that $[a, a^\dagger] = 1$.

(e) Show that

$$a^\dagger a + \frac{1}{2} = \left[-2\frac{\partial^2}{\partial z \partial z^*} - \frac{1}{2}\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*} \right) + \frac{|z|^2}{8} \right].$$

Therefore, the Hamiltonian can be re-written as

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right).$$

Here, $\hat{N} = a^\dagger a$ is the number operator.

[This is the Hamiltonian of a simple harmonic oscillator with the quantized energy levels $E_n = (n + 1/2)\hbar\omega_c$ with $n = 0, 1, 2.....$ Each discrete energy is called Landau level.]

(f) Show that

$$a^\dagger a \psi_m(z) = 0,$$

where $\psi_m(z) = \phi_m(z) e^{-\frac{|z|^2}{4}}$. Here $\phi_m(z)$ is a set of analytic functions given as

$$\phi_m(z) = \frac{1}{\sqrt{2\pi l^2 2^m m!}} z^m,$$

with $m = 0, 1, 2.......$

(g) Show that

$$\sum_{m=0}^{\infty} |\psi_m(z)|^2 = \frac{qB}{2\pi\hbar} = \frac{1}{2\pi l^2}.$$

(h) Show that

$$L_z\psi_m(z) = m\hbar\ \psi_m(z).$$

Show that

$$\langle\psi_m(z)|L_z^{\rm mech}|\psi_m(z)\rangle = -\hbar.$$

Note that $\psi_m(z)$ are eigenfunctions of L_z, but not the eigenfunctions of $L_z^{\rm mech}$.

11. Consider a set of analytic functions

$$\phi_m(z) = \frac{1}{\sqrt{2\pi 2^m m!}} z^m,$$

where $m = 0, 1, 2......$ One can use $\phi_m(z)$ as basis states with the measure or weight factor $\mu[z] = e^{-|z|^2/2}$ to construct an infinite dimensional complex space with finite norm. Defining an inner product as

$$\langle\phi_n|\phi_m\rangle = \int \phi_n^*(z)\phi_m(z)\ \mu[z]\ dxdy.$$

i) Prove that $\langle\phi_n|\phi_m\rangle = \delta_{mn}$.

ii) Show that $b^\dagger = \frac{z}{\sqrt{2}}$ and $b = \sqrt{2}\frac{d}{dz}$ act like raising and lowering operators, respectively, in this infinite dimensional complex space.

iii) Check that $b^\dagger$ and b satisfy the bosonic commutation relation: $[b, b^\dagger] = 1$.

Department of Physics, IIT-Kanpur

Instructor: Tarun Kanti KGhosh Math Methods-I (PHY421) AY 2024-25, I-SEM

Homework-5

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1. Determine locations and order of all the poles of the function

$$f(z) = \frac{e^{pz}}{\cosh^2 z},$$

with p is a real constant. Evaluate sum of all the residues in the upper-half plane.

2. Using the "indented" contour and "$i\epsilon$" method, show that

$$\text{P.V.} \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}. \tag{1}$$

3. Using the Dirac identity (real-axis version of Sokhotski–Plemelj theorem), show that

(a)

$$\int_{-\infty}^{\infty} \frac{e^{ik_x x}}{k_x^2 - k_0^2} dk_x = -\frac{\pi}{k_0} \sin(k_0 x).$$

Note that the same result is obtained in class using two different methods: "indented" contour and "$i\epsilon$" method.

(b)

$$\int_{-\infty}^{\infty} dE \int_0^\infty dt f(E) e^{-iEt} = \pi f(0) - i\text{P.V.} \int_{-\infty}^{\infty} \frac{f(E)}{E} dE.$$

(c)

$$\int_0^\infty e^{-i(\omega \pm \omega_0)t} dt = i\text{P.V.} \left[\frac{1}{\omega \pm \omega_0}\right] + \pi\delta(\omega \pm \omega_0).$$

(d)

$$\delta(x - x_0) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \left[\frac{1}{x - x_0 - i\epsilon} - \frac{1}{x - x_0 + i\epsilon}\right].$$

4. Solve the following integral using (i) argument principle and (ii) residue theorem:

$$\oint_C \frac{f'(z)}{f(z)} dz,$$

for $f(z) = \cos(z)/z$ and C describes a circle of radius 2π centered at $z = 0$.

5. Considering the following integral

$$\oint_{C_N} \frac{\csc z}{z^2} dz,$$

where the contour C_N is a positively oriented square whose sides are described by $x = \pm(N + 1/2)\pi$ and $y = \pm(N + 1/2)\pi$. Also, N is an integer. Using the result of this integral and taking the limit $N \to \infty$, we are going to evaluate sum of the following infinite series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

(a) On the horizontal lines: $z = z \pm i(N + 1/2)\pi$: Show that

$$\frac{1}{|\sin z|} \leq |\mathrm{csch(y)}|.$$

Note that horizontal lines are alwyas at $|y| \geq \pi/2$. Therefore, $1/|\sin(z)| \leq \mathrm{csch}(\pi/2)$. Hence, it goes to zero as $y \to \infty$ (Or, $N \to \infty$).

(b) On the vertical sides, $z = \pm(N + 1/2)\pi + iy$. Show that

$$\sin(z) \quad = \quad \pm(-1)^N \cosh(y).$$

Hence

$$\frac{1}{|\sin(z)|}\Big|_{\text{vertical}} = \mathrm{sech(y)} \leq 1.$$

(c) Using Darboux's theorem, show that

$$\Big|\oint \frac{1}{z^2 \sin(z)} dz\Big| \leq \frac{(8N + 4)}{N^2} \to 0;\ N \to \infty.$$

(d) In class, we have calculated residues of the complex integrand at all the singular points. Using Cauchy's residue theorem,

$$\begin{aligned} \frac{1}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} &= 0 \\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= \frac{\pi^2}{12}. \end{aligned}$$

6. (a) Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - b^2} = \frac{\pi}{b} \cot(\pi b).$$

Here b is a real constant.

(b) By taking $b \to 0$ limit, show that the Riemann zeta function

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

7. In class, we have derived

$$\int_{-\infty}^{\infty} d\epsilon H(\epsilon) \left[-\frac{\partial f}{\partial \epsilon} \right] \quad = \quad H(\mu) + \frac{\pi^2}{6}(k_B T)^2 \frac{d^2 H(\epsilon)}{d\epsilon^2}\Big|_{\epsilon=\mu} + \frac{7\pi^4}{360}(k_B T)^4 \frac{d^4 H(\epsilon)}{d\epsilon^4}\Big|_{\epsilon=\mu} + ..., \quad (2)$$

where $H(\epsilon)$ is a well-behaved differentiable function.

By integrating left hand side integral of the above Sommerfeld expansion by parts, using the fact that $f(\epsilon)H(\epsilon) \to 0$ as $\epsilon \to \pm\infty$ and defining $F(\epsilon) = \frac{dH}{d\epsilon}$, show that the above mentioned Sommerfeld expansion [Eq. (2)] can be written in terms of the function $F(\epsilon)$ as

$$\int_{-\infty}^{\infty} d\epsilon F(\epsilon) f(\epsilon) = \int_{-\infty}^{\mu} F(\epsilon) d\epsilon + \frac{\pi^2}{6}(k_B T)^2 \frac{dF(\epsilon)}{d\epsilon}\Big|_{\epsilon=\mu} + \frac{7\pi^4}{360}(k_B T)^4 \frac{d^3 F(\epsilon)}{d\epsilon^3}\Big|_{\epsilon=\mu} + ... \quad (3)$$

Department of Physics, IIT-Kanpur

Instructor: Tarun Kanti KGhosh Math Methods-I (PHY421) AY 2024-25, I-SEM

Homework-4

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1. Consider the complex function $h(k) = h_x(k) + ih_y(k)$, where $h_x(k) = v + w\cos k$ and $h_y(k) = w\sin k$. Here k is a parameter varying from $-\pi$ to π (0 to 2π). Also, v and w are real positive constants.

   The tip of the "complex vector" $h(k)$ traces out a circle of radius w centered at $(v, 0)$ in the $h_x(k)$-$h_y(k)$ plane while varying k from $-\pi$ to π. Note that $h_x(k = \pm\pi) = 0$ and $h_y(k = \pm\pi) = 0$ for $v = w$.

   (a) Draw these circles on $h_x(k)$-$h_y(k)$ plane for the following three different cases: (i) $v < w$, (ii) $v = w$, and (iii) $v > w$.

   (b) Plot the argument of $h(k)$ vs k for (i)$w/v = 0.5$, (ii) $w/v = 1$ and (iii) $w/v = 1.5$.

   (c) Evaluate the following integral:

   $$I = \frac{1}{2\pi i}\oint_C \frac{dh(k)}{h(k)},$$

   for the three different cases: (i) $v < w$, (ii) $v = w$, and (iii) $v > w$. Here the closed contour C is the path traces out by the tip of $h(k)$ while k varing from $-\pi$ to π.

   This integral is also known as winding number. The discrete change in the winding number while changing the system parameters (v, w) continuously. This has been verified in laboratory in recent past. It is a topological phase transition.

2. (a) Show that $\frac{d}{dz}\sin z = \cos(z)$ and $\int \cos z dz = \sin z$.

   b) Verify that $\int_{0,0}^{1,1} z^* dz$ depends on the path by evaluating the integral for the two paths: i) $(0, 0) \to (1, 0) \to (1, 1)$ and ii) $(0, 0) \to (1, 1)$.

   c) Evaluate the line integral

   $$\int_\Gamma (z^2 + 1)e^{-iz^2} dz,$$

   where Γ is the ray $\theta = -\pi/4$.

   d) Evaluate $\oint_C z^n e^{-|z|^2} dz$ where $n = 0, 1, 2, ...$ and the contour C is a circle with radius ϵ centered at $z = z_0 = 0$.

   ie) Show that

   $$\frac{1}{2\pi i}\oint_C z^{m-n-1} dz = \delta_{m,n}.$$

   Here, $\delta_{m,n}$ is the Kronecker delta and the contour C is a unit circle centered at origin.

   f) Consider the integral

   $$I = \oint_C \frac{1}{(az - b)(pz - q)} dz,$$

   where C describes the circle $|z| = R$. Here, a, b, p and q are real constants. Evaluate I for i) $\frac{b}{a} < R < \frac{q}{p}$, ii) $\frac{q}{p} < R < \frac{b}{a}$, iii) $R > \frac{q}{p} > \frac{b}{a}$, and iv) $R < \frac{q}{p} < \frac{b}{a}$.

3. i) Using the fact that $f(z) = e^{-iz^n}$ is an analytic function, show that

$$\int_o^{2\pi} e^{\sin n\theta} \cos(\theta - \cos n\theta) d\theta = \int_o^{2\pi} e^{\sin n\theta} \sin(\theta - \cos n\theta) d\theta = 0.$$

Here, n is a positive integer.

ii) Evaluate the integral $\oint_C \frac{dz}{z(R-z)}$, where C is a circle: $|z| = r < R$. Use this result to evaluate the following integral:

$$\int_0^{2\pi} \frac{(R - r\cos\theta)d\theta}{R^2 + r^2 - 2Rr\cos\theta}.$$

4. Consider $f(z)$ is an analytic function inside and on a unit circle C centered at z_0. Show that

$$f^{(n)}(z_0) = \frac{n!}{2\pi} \int_0^{2\pi} e^{-in\theta} f(z_0 + e^{i\theta}) d\theta.$$

5. i) Consider the integral

$$I = \int_\Gamma e^{-z^2} dz,$$

where Γ is a line segment of length L running parallel to the imaginary axis from R to $R + iL$. Using the Darboux's theorem, show that $|I|$ tends to zero as $R \to \infty$.

ii) Following the proof of the Jordan's lemma, show that

$$\lim_{R\to\infty} \left| \int_\Gamma e^{ikz^N} dz \right| = 0,$$

where Γ is an arc of the circle $|z| = R$ lying in the sector $0 \le \theta \le \frac{\pi}{2N}$ and $N > 1$.

Using this result, evaluate the following integral of the real variable x

$$I = \int_0^\infty e^{ix^N} dx.$$

iii) Prove that

$$I = \int_{-\infty}^{\infty} e^{-px^2} \cos(2bx) dx = \sqrt{\frac{\pi}{p}} e^{-b^2/p}.$$

Here, $b > 0$ and $p > 0$.

iv) Prove that

$$I = \int_0^\infty e^{ipx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{p}} e^{i\pi/4}.$$

6. i) Using the Cauchy's integral theorem and Cauchy's integral formula, evaluate the following integral:

$$\oint_C \frac{e^{-z}\sin z}{z^2},$$

where C is any closed curve encircles $z = 0$ point.

ii) Using the Cauchy's integral formula, evaluate the following integral:

$$\oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-3)} dz,$$

where $C : |z - i| = 3$.

7. Using the entire function $f(z) = e^z$ in the Poisson integral formula, show that

$$\int_0^{2\pi} \frac{e^{2\cos\theta}\cos(2\sin\theta)}{5 - 4\cos(\theta - \phi)} d\theta = \frac{2\pi}{3} e^{\cos\phi} \cos(\sin\phi)$$

and

$$\int_0^{2\pi} \frac{e^{2\cos\theta}\sin(2\sin\theta)}{5 - 4\cos(\theta - \phi)} d\theta = \frac{2\pi}{3} e^{\cos\phi} \sin(\sin\phi).$$

Department of Physics, IIT-Kanpur

Instructor: Tarun Kanti KGhosh   Math Methods-I (PHY421)   AY 2024-25, I-SEM

Homework-6

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1. Obtain the Fourier series expansion of a step edge defined as

$$f(x) = +C, \; x > 0 \quad (1)$$
$$= -C, \; \text{otherwise}, \quad (2)$$

where C is a real constant. Using any standard mathematical software, plot the Fourier series expansion of $f(x)$ (in units of C) vs x over $-5 \leq x \leq +5$ for three different values of the upper limit of the series summation: $n_{\max} = 50, 100, 200$. Analyse the plots around the point of discontinuity and then you will see the Gibb's phenomenon.

2. (a) Show that the Fourier series expansion of $f(t) = \cos(\omega_0 t)$ (ω_0 is not an integer) is

$$\cos(\omega_0 t) = \frac{\sin(\pi\omega_0)}{\pi\omega_0}\left[1 + 2\omega_0^2 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(nt)}{\omega_0^2 - n^2}\right].$$

(b) Obtain Fourier transform of $f(t)$ (here ω_0 can be integer also):

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

(c) Plot the Fourier series expansion of $f(t)$ and sketch Fourier transform of $f(t)$ i.e. $F(\omega)$.

3. The Fourier series expansion of a periodic function $f(x) = x$ with a period $2L$ is given by

$$f(x) = L\left[1 - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)}{n}\right].$$

Using any standard mathematical software, plot the Fourier series expansion of $f(x)$ (in units of L) vs x/L over $-6 \leq x/L \leq +6$ for three different values of the upper limit of the series summation: $n_{\max} = 10, 50, 100$. Analyse the plots around the point of discontinuity and then you will see the Gibb's phenomenon.

4. (a) Defining $f(x) = e^{\pm x}$ are periodic functions with the fundamental interval $(-\pi < x < \pi)$. Show that the Fourier series of the periodic function $f(x)$ $(-\pi < x < \pi)$ is given by

$$e^{\pm x} = \frac{\sinh(\pi)}{\pi}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{\cos(nx) \mp n\sin(nx)}{1+n^2}\right].$$

(b) Defining $\sinh(x)$ and $\cosh(x)$ are periodic functions in the interval $-\pi < x < \pi$). Obtain the Fourier series of $\sinh(x)$ and $\cosh(x)$.

5. A function $f(x)$ is expanded in an exponential Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \text{with} \qquad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}.$$

(a) If $f(x)$ is real, what restriction is imposed on the coefficients c_n?

(b) If $f(-x) = f(x)$, what restriction is imposed on the coefficients c_n?

(c) If $f(-x) = -f(x)$, what restriction is imposed on the coefficients c_n?

(d) Assuming that the Fourier expansion of a periodic function $f(x)$ $(0 < x < 2\pi)$ is convergent, show that the Parseval's identity has the form given by

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = c_0^2 + 2\sum_{n=1}^{\infty}|c_n|^2.$$

6. Obtain the Fourier series form of the periodic function $f(x) = x^4$ $(-\pi < x < \pi)$. Using this result, show that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

7. Solve the following inhomogeneous differential equation:

$$\omega\frac{df(\theta)}{d\theta} + f(\theta) = a\sin\theta + b\cos(2\theta).$$

Here ω, a, b are real constant and $f(\theta)$ is a periodic function with period 2π.

8. **Dirac comb**: It is a periodic, infinite array of δ-functions with the fundamental interval $(-L/2, L/2)$ i.e. $\delta(x) = \sum_{n=-\infty}^{\infty}\delta(x - nL)$. The Dirac comb model potential is popularly used to study band structure in solid state systems.

Show that the Fourier series representation of $\delta(x)$ is

$$\delta(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty}e^{\pm i2\pi nx/L} = \frac{1}{L}\left[1 + 2\sum_{n=0}^{\infty}\cos(2\pi nx/L)\right].$$

Plot this function using any standard mathematical software.

9. a) Show that the Dirac delta function $\delta(x - a)$ expanded in a Fourier sine series in the interval $(0, L)$ $(0 < a < L)$ is given by

$$\delta(x - a) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

b) A string is clamped at both ends $x = 0$ and $x = L$. Assuming small-amplitude vibrations, the amplitude $u(x,t)$ satisfies the wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c_s^2}\frac{\partial^2 u(x,t)}{\partial t^2},$$

where c_s is the velocity of the wave propagation along the string. The string is set in vibration by a sharp blow at $x = a$. Hence the initial conditions are

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = Lv_0\delta(x - a).$$

Solve the wave equation subject to these initial conditions.

10. A string, clamped at $x = 0$ and $x = L$, is vibrating freely. Its motion is described by the wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c_s^2}\frac{\partial^2 u(x,t)}{\partial t^2},$$

where c_s is the velocity of the wave propagation along the string. Assume a Fourier series of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

The initial conditions are $u(x,0) = f(x)$ and $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = g(x)$. Determine the coefficients $b_n(t)$.

11. The one-dimensional heat equation is given by

$$D\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t},$$

where D is the diffusivity. The boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$. Show that the most general solution of the heat equation with the given boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 Dt/L}.$$

Determine the Fourier coefficients a_n if the initial condition is $u(x,0) = f(x)$.

12. The position of an underdamped oscillator is given by

$$x(t) = e^{-\gamma t}\cos(\omega_0 t)\theta(t),$$

where γ is the damping factor, ω_0 is the undamped oscillator frequency and $\theta(t)$ is the unit step function.

(a) Show that the Fourier transform of $x(t)$ is

$$x(\omega) = \frac{1}{i\sqrt{2\pi}}\left[\frac{\omega - i\gamma}{(\omega - i\gamma)^2 - \omega_0^2}\right].$$

(b) Plot $x(t)$ vs t and $|x(\omega)|^2$ vs ω for $\omega_0 = 10$ and $\gamma = 1$. Compare the plots.

13. Obtain Fourier transform of the following functions:

a) The ground and first excited state wave functions of a hydrogen atom

$$\psi_{100}(\mathbf{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}, \quad \psi_{200}(\mathbf{r}) = \frac{1}{\sqrt{32\pi a_0^3}}\left(2 - \frac{r}{a_0}\right) e^{-r/(2a_0)}.$$

Here a_0 is the Bohr radius and $r = \sqrt{x^2 + y^2 + z^2}$.

b) The ground and first excited state wave functions of 1D harmonic oscillator

$$\phi_0(x) = \frac{1}{\sqrt{\sqrt{\pi} a_h}} e^{-x^2/(2a_h^2)}, \quad \phi_1(x) = \sqrt{\frac{2}{\sqrt{\pi} a_h^3}}\, x e^{-x^2/(2a_h^2)},$$

where $a_h = \sqrt{\hbar/(m\omega)}$ is the oscillator length.

c) The ground state wave function of a shifted harmonic oscillator

$$\phi_0(x) = \frac{1}{\sqrt{\sqrt{\pi} a_h}} e^{-(x-x_0)^2/(2a_h^2)},$$

where x_0 is a constant.

d) The Coulomb potential in 3D:

$$V(\mathbf{r}) = \frac{C}{r},$$

where $C = 1/(4\pi\epsilon_0)$.

14. Calculate the following inverse Fourier transformation of the Fermi occupation number $n_{\mathbf{k}} = \Theta(k_f - |\mathbf{k}|)$ at $T = 0$:

$$n(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int_{V_{\mathbf{k}}} e^{i\mathbf{k}\cdot\mathbf{r}} n_{\mathbf{k}} \, d^3k,$$

where $V_{\mathbf{k}}$ is the volume of a sphere with radius k_f in $\mathbf{k}$-space. Here $\mathbf{k}$ and $\mathbf{r}$ are the three-dimensional vectors. Plot $n(\mathbf{r})$ vs $k_f r$.

Obtain approximate expressions of $n(\mathbf{r})$ for $r \to 0$ and $r \to \infty$ limits and sketch them.

Repeat the same calculations for 2D systems.

Answer for 3D case:

$$n(\mathbf{r}) \propto \frac{\sin(k_f r) - (k_f r)\cos(k_f r)}{r^3}.$$

For small r: $n(y) \simeq 1/3 - y^2/(30)...$ and for large r: $n(y) \simeq \frac{\cos y}{y^2}$ with $y = k_f r$.

This kind of integrals appear while calculating Hatree-Fock energy.

15. The drift-diffusion equation is given by

$$\frac{\partial n(x,t)}{\partial t} = D\frac{\partial^2 n(x,t)}{\partial x^2} - v_d \frac{\partial n(x,t)}{\partial x},$$

where $n(x,t)$ is the particle density, D is the diffusivity or diffusion constant and v_d is the drift velocity.

(a) Using the Fourier transform method, solve for $n(x,t)$ of the drift-diffusion equation for a given initial condition $n(x,0) = C_0\delta(x)$ with C_0 being a constant. Sketch $n(x,t)$ at different times.

16. The diffusion equation is given by

$$\frac{\partial n(x,t)}{\partial t} = D\frac{\partial^2 n(x,t)}{\partial x^2},$$

where $n(x,t)$ is the particle density and D is the diffusivity or diffusion constant. One can construct a typical length scale of a diffusive system as $l_{\mathrm{D}} = \sqrt{2Dt} \sim \sqrt{t}$. Keep in mind that the length scale goes as $\sqrt{t}$ and it is not a constant. This equation can be solved by assuming the following ansatz:

$$n(x,t) = \frac{N_0}{l_{\mathrm{D}}} f\left(\frac{x}{l_{\mathrm{D}}}\right).$$

Here, N_0 is the total particle number in the system and it is conserved i.e. at any given time t, $\int_{-\infty}^{\infty} n(x,t)dx = N_0$. Introducing a dimensionless variable $\xi = x/\sqrt{2Dt} = x/l_{\mathrm{D}}$. Therefore, $f(\xi)$ is a function of one variable ξ.
a) Show that $\int_{-\infty}^{\infty} f(\xi)d\xi = 1$.

b) Show that the partial diffusion equation reduces to the following ordinary differential equation:

$$\left[\frac{d^2}{d\xi^2} + \xi\frac{d}{d\xi} + 1\right] f(\xi) = 0.$$

Assuming initial condition is $f(x,t=0) = N_0\delta(x)$. First solve the ordinary differential equation for $f(\xi)$ and then show that the solution of the partial differential equation is

$$n(x,t) = \frac{N_0}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}.$$

Guess what will be the solution of the drift-diffusion equation

$$\frac{\partial n(x,t)}{\partial t} = D\frac{\partial^2 n(x,t)}{\partial x^2} - v_{\mathrm{d}}\frac{\partial n(x,t)}{\partial x},$$

where v_{d} is the drift velocity.

17. Using Fourier transform method, solve the following inhomogeneous differential equation

$$\frac{d^2 f(x)}{dx^2} = -e^{-x^2},$$

where $f(x)$ vanishes as $|x| \to \infty$.

Answer:

$$f(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-k^2/4}}{k^2} e^{ikx} dk.$$

18. Using the Parseval's identity, show that

$$\int_{-\infty}^{\infty} \frac{d\omega}{(w^2+a^2)^2} = \frac{\pi}{2a^3}.$$

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Homework-7

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1. Using the Frobenius method, obtain two linearly independent solutions of the following differential equations:

   (i)
   $$\left[x^2\frac{d^2}{dx^2} + x\frac{d}{dx} - n^2\right] y(x) = 0,$$
   where n is any number. This equation is obtained from two-dimensional Laplace's equation.

   (ii)
   $$\left[x^2\frac{d^2}{dx^2} + C_1 x\frac{d}{dx} + x^2 + C_2\right] y(x) = 0,$$
   for two different cases: (a) $C_1 = 4, C_2 = 2$ and (b) $C_1 = 2, C_2 = -2$.

   (iii)
   $$\left[\frac{d^2}{dx^2} + x\frac{d}{dx} + 1\right] y(x) = 0.$$

   (iv)
   $$\left[\frac{d^2}{dx^2} - \frac{n(n+1)}{x^2}\right] y(x) = 0,$$
   where n is non-negative real number.

2. In the class, we have derived $J_1(x)$ and $J_{1/2}(x) = \sqrt{2/(\pi x)}\sin x$. Using Wronskian method, obtain the second independent solution $N_1(x)$ (at least first three terms in the series) and $N_{1/2}(x)$.

3. A quantum particle of mass M is confined in a cylinder of radius R and height H. The potential is described as
   $$\begin{aligned} V(\rho, \phi, z) &= 0, \quad 0 < \rho < R, 0 < z < H && (1)\\ &= \infty \quad \text{otherwise.} && (2) \end{aligned}$$
   Obtain the discrete energy spectrum and the corresponding wave functions.

4. In class we have seen that two-diemensional plane wave can be expressed as a series of cylindrical Bessel functions:
   $$e^{\pm ix\cos\phi} = \sum_{n=-\infty}^{\infty} (-i)^n J_n(x) e^{\pm in\phi}. \tag{3}$$
   From one of the above relations, show that
   $$\cos(x\cos\phi) = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)\cos(2n\phi) \tag{4}$$
   $$\sin(x\cos\phi) = 2\sum_{n=1}^{\infty}(-1)^{n+1} J_{2n-1}(x)\cos[(2n-1)\phi]. \tag{5}$$
   From these results, express $\sin x$ and $\cos x$ as a series of $J_n(x)$.

5. **Arfken & Weber: 11.1.13**: The scattering amplitude of a quantum scattering problem is given as

$$f(\theta) = \frac{k}{2\pi} \int_0^{2\pi} \int_0^{R} e^{ik\rho \sin\theta \sin\phi} \rho d\rho d\phi, \tag{6}$$

where θ is the angle through which the particle is scattered and R is the radius of the scatterer. Show that

$$|f(\theta)|^2 \sim \left[\frac{J_1(kr\sin\theta)}{\sin\theta}\right]^2. \tag{7}$$

6. The Helmholtz equation in spherical polar coordinates:

$$\left[\nabla_{\mathbf{r}}^2 + k^2\right] \psi(r, \theta, \phi) = 0, \tag{8}$$

where k is a constant. For a free particle of mass M and energy E, $k^2 = 2ME/\hbar^2$. Assume $\psi(r, \theta, \phi) = R(r)P(\theta)\Phi(\phi)$.

(a) Using the separation of variables, show that

$$\left[\frac{d^2}{d\phi^2} + m^2\right] \Phi(\phi) = 0, \tag{9}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{m^2}{\sin^2\theta} + Q\right] P(\theta) = 0 \tag{10}$$

and

$$\left[\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + k^2r^2 - Q\right] R(r) = 0. \tag{11}$$

Here, m^2 and Q are two separation constants.

(b) We have seen in the lectures that m must be an integer to have single-valued wave function and Q must have form like $Q = n(n+1)$ with n is non-negative integer to get converging series solution. In Eq. (9), why do we choose m^2 as a separation constant, instead of m?

Note: In physical systems, m is identified as azimuthal quantum number and n is identified as orbital angular momentum quantum number.

(c) Equation (11) is not the Bessel's differential equation. Obtain the following Bessel's differential equation by introducing a new dimensionless variable $x = kr$ and substitute $R(x) = Z(x)/\sqrt{x}$ and $Q = n(n+1)$ into Eq. (11):

$$\left[x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + x^2 - (n+1/2)^2\right] Z(x) = 0. \tag{12}$$

$Z(x)$ is the Bessel function of order $n + 1/2$: $Z(x) = J_{n+1/2}(x)$. Therefore, the original solution is

$$R(x) = \frac{Z(x)}{\sqrt{x}} = \frac{J_{n+1/2}(x)}{\sqrt{x}}. \tag{13}$$

The spherical Bessel function of first kind of order n is given as

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x). \tag{14}$$

It is convention to introduce $\sqrt{\frac{\pi}{2}}$ in $j_n(x)$. Similarly, the spherical Bessel function of second kind of order n is given as

$$n_n(x) = (-1)^{n+1}\sqrt{\frac{\pi}{2x}} J_{-(n+1/2)}(x) = \sqrt{\frac{\pi}{2x}} N_{n+1/2}(x). \tag{15}$$

The zeroth-order spherical Bessel function is $j_0(x) = \frac{\sin x}{x}$. Evaluate $n_0(x)$ by using the Wronskian approach.

7. Prove the following recurrence relations for the spherical Bessel function $j_n(x)$:

(a)

$$j_{n-1}(x) + j_{n+1}(x) = \frac{2n+1}{x} j_n(x). \tag{16}$$

(Hint: use $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$.]

(b)

$$j_n'(x) = \frac{n}{2n+1} j_{n-1} - \frac{n+1}{2n+1} j_{n+1}(x). \tag{17}$$

[Hint: use $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$ and $2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$.]

(c)

$$j_{n\pm1}(x) = \frac{(n + \frac{1}{2} \mp \frac{1}{2})}{x} j_n(x) \mp j_n'(x). \tag{18}$$

Using $j_0(x) = \frac{\sin x}{x}$, show that

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{19}$$

$$j_2(x) = \frac{3\sin x}{x^3} - \frac{3\cos x}{x^2} - \frac{\sin x}{x}. \tag{20}$$

Plot these three functions.

(d)

$$\frac{d}{dx}\Big[x^{n+1} j_n(x)\Big] = x^{n+1} j_{n-1}(x). \tag{21}$$

(e)

$$\frac{d}{dx}\Big[x^{-n} j_n(x)\Big] = -x^{-n} j_{n+1}(x). \tag{22}$$

Note: All the recurrence relations mentioned here are valid for $n_n(x)$ also.

8. a) Substitute $x = \cos\theta$ into Eq. (10) and obtain the associated Legendre differential equation:

$$\left[(1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \frac{m^2}{1-x^2} + n(n+1)\right]y(x) = 0. \tag{23}$$

Show that $x = 0$ is an ordinary point, $x = \pm 1$ and $x = \infty$ are regular singular points.

b) Consider the Legendre differential equation and assuming the ansatz for the series solution as

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+k}. \tag{24}$$

Obtain the following recurrence relation:

$$a_{j+2} = \frac{(j+k)(j+k+1) - n(n+1)}{(j+k+1)(j+k+2)} a_j. \tag{25}$$

c) Two roots of the indicial equation are $k = 0$ and $k = 1$. For $k = 0$, a_1 may or may not be zero. For $k = 1$, a_1 must be zero. For $k = 0$ case, we assume $a_1 = 0$.

For $k = 0$, show that

$$y_1(x) = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - ...\right] \tag{26}$$

and for $k = 1$, show that

$$y_2(x) = a_0\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - ...\right]. \tag{27}$$

9. Construct an orthonormal set from the set of functions $u_n(x) = x^n, (n = 0, 1, 2...)$ in the interval $-1 \le x \le +1$ by using Gram-Schmidt orthogonalization method. You will see n-th such function is proportional to $P_n(x)$.

10. Express the Dirac-Delta function $\delta(x \pm 1)$ in a series of $P_n(x)$. Assume that the entire delta function is covered when integrating over $[-1, +1]$.

11. Arfken: 12.3.11: The amplitude of a scattered wave is given by

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta). \tag{28}$$

Here, θ is the scattering angle, l is orbital angular momentum quantum number, $\hbar k$ is momentum of the incident particle and δ_l is the phase shift produced by the scattering potential. Calculate the total cross-section

$$\sigma_{\text{tot}} = \int |f(\theta)|^2 d\Omega, \tag{29}$$

where Ω is the solid angle.