Assignment 1

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(1) (a) (i) $\nabla \cdot (\vec{a} e^{i\vec{k}\cdot\vec{r}})$

identity: $\nabla \cdot (f\vec{A}) = (\nabla f)\cdot\vec{A} + f(\nabla\cdot\vec{A})$

where $f = e^{i\vec{k}\cdot\vec{r}}$, $\vec{A} = \vec{a}$

$$\Rightarrow \nabla\cdot(\vec{a}e^{i\vec{k}\cdot\vec{r}}) = (\nabla e^{i\vec{k}\cdot\vec{r}})\vec{a} + e^{i\vec{k}\cdot\vec{r}}\underbrace{(\nabla\cdot\vec{a})}_{0}$$

$$= \boxed{(i)(\vec{k}\cdot\vec{a})e^{i\vec{k}\cdot\vec{r}}}$$

(ii) $\nabla \times (\vec{a} e^{i\vec{k}\cdot\vec{r}})$

identity: $\nabla\times(f\vec{A}) = (\nabla f)\times\vec{A} + f(\nabla\times\vec{A})$

where $f = e^{i\vec{k}\cdot\vec{r}}$ $\vec{A} = \vec{a}$

$$\Rightarrow \nabla\times(\vec{a}e^{i\vec{k}\cdot\vec{r}}) = (\nabla e^{i\vec{k}\cdot\vec{r}})\times\vec{a} + e^{i\vec{k}\cdot\vec{r}}\underbrace{(\nabla\times\vec{a})}_{0}$$

$$= (i\vec{k}e^{i\vec{k}\vec{r}})\times\vec{a} = i(\vec{k}\times\vec{a})e^{i\vec{k}\vec{r}}$$

~~(b) (c) (ii)~~ $\cancel{\vec{A} = yz\hat{i} + xz\hat{j} + xy\hat{k}}$

~~(c)~~

(1) (c) (ii) $\nabla^2\left(\frac{e^{-\lambda r}}{r}\right)$

we know that $\nabla^2 f(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)$

where $f(r) = \frac{e^{-\lambda r}}{r}$

$$\frac{\partial f}{\partial r} = \frac{-\lambda e^{-\lambda r} r - e^{-\lambda r}}{r^2} = \frac{-e^{-\lambda r}(\lambda r + 1)}{r^2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) = \frac{-1}{r^2}\frac{\partial}{\partial r}\left(e^{-\lambda r}(\lambda r + 1)\right)$$

$$= \frac{+(\lambda r + 1)\lambda e^{-\lambda r} - e^{-\lambda r}\lambda}{r^2} = \boxed{\frac{+e^{-\lambda r}(\lambda^2)}{r}}$$

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(1)(d) Show that:

$$-\hbar^2 \nabla^2_{\vec{r}}\left[u(\vec{r}) e^{i\vec{k}\cdot\vec{r}}\right] = e^{i\vec{k}\cdot\vec{r}}\left[-i\hbar\nabla + \hbar\vec{k}\right]^2 u(\vec{r})$$

using product rule: $\nabla(u e^{i\vec{k}\cdot\vec{r}}) = e^{i\vec{k}\cdot\vec{r}}(\nabla u + i\vec{k} u)$

$$\Rightarrow \nabla^2\left[u(\vec{r}) e^{i\vec{k}\cdot\vec{r}}\right] = u\nabla^2(e^{i\vec{k}\cdot\vec{r}}) + e^{i\vec{k}\cdot\vec{r}}\nabla^2 u + 2(\nabla u)\cdot(\nabla e^{i\vec{k}\cdot\vec{r}})$$

$$\Rightarrow \text{LHS} = -\hbar^2\left[u(-k^2 e^{i\vec{k}\cdot\vec{r}}) + e^{i\vec{k}\cdot\vec{r}}\nabla^2 u + 2(\nabla u)\cdot(i\vec{k} e^{i\vec{k}\cdot\vec{r}})\right]$$

$$= -\hbar^2 e^{i\vec{k}\cdot\vec{r}}\left[-k^2 u + \nabla^2 u + 2i\vec{k}\cdot(\nabla u)\right]$$

$$= -e^{i\vec{k}\cdot\vec{r}}\left[\hbar^2\vec{k}^2 u - \hbar^2\nabla^2 u - 2i\hbar^2\vec{k}\cdot(\nabla u)\right]$$

$$= -e^{i\vec{k}\cdot\vec{r}}\left[\nabla + i\vec{k}\right]^2 u\,\hbar^2$$

$$= -e^{i\vec{k}\cdot\vec{r}}\left[-i\hbar\nabla + \hbar\vec{k}\right]^2 u(\vec{r}) \rightarrow \text{RHS}$$

QED

(2) (a) $A = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix}$$

$$= \hat{i}(x - x) + \hat{j}(y - y) + \hat{k}(z - z)$$

$$= 0$$

(b) $f(x, y, z)$ such that $\vec{\nabla} f(x, y, z) = \vec{A}$

$$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = yz\hat{i} + xz\hat{j} + xz\hat{j}$$

$$\Rightarrow \boxed{f(x, y, z) = xyz + C}$$

(c) $\vec{B}$ such that $\vec{\nabla}\times\vec{B} = \vec{A}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_y & B_z \end{vmatrix} = \vec{A} \Rightarrow \begin{aligned} \partial B_z/\partial y - \partial B_y/\partial z &= yz \\ \partial B_x/\partial z - \partial B_z/\partial x &= xz \\ \partial B_y/\partial x - \partial B_x/\partial y &= xy \end{aligned}$$

let $B_z = 0$

$\Rightarrow -\partial B_y/\partial z = yz \Rightarrow B_y = -yz^2/2 + C_{z_1}$ const in z

$-\partial B_x/\partial z = xz \Rightarrow B_x = -xz^2/2 + C_{z_2}$

$\partial B_y/\partial x - \partial B_x/\partial y = \partial C_{z_1}/\partial x - \partial C_{z_2}/\partial y = xy$

let $C_{z_2} = 0 \Rightarrow C_{z_1} = x^2 y/2$

$$\Rightarrow \boxed{B = \left(-xz^2/2\right)\hat{i} + \left(x^2y/2 - yz^2/2\right)\hat{j}}$$

(3) (a) $\vec{A} = \vec{a}\, e^{i\vec{k}\cdot\vec{r}}$

$$\oint_S A\cdot dS = \int_V (\nabla\cdot A)\, dV \quad \text{(Divergence Theorem)}$$

$$(\nabla\cdot\vec{A}) = \nabla\cdot(\vec{a}\, e^{i\vec{k}\cdot\vec{r}}) = i(\vec{k}\cdot\vec{a})\, e^{i\vec{k}\cdot\vec{r}}$$

$$\Rightarrow I = \int_V i(\vec{k}\cdot\vec{a})\, e^{i\vec{k}\cdot\vec{r}}\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} e^{ikr\cos\theta}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

we align our coordinate system such that $\vec{k}$ is parallel to the z-axis. We can do this because the volume we're integrating over is spherically symmetric.

$$= (2\pi)\int_0^{\pi}\int_0^{R} e^{ikr\cos\theta}\, r^2 \sin\theta\, dr\, d\theta$$

let $u = \cos\theta \rightarrow du = -\sin\theta\, d\theta$

$$= \left(2\pi \int_1^{-1}\int_0^{R} e^{ikru}\, r^2\, du\, dr\right) i(\vec{k}\cdot\vec{a})$$

$$= i(\vec{k}\cdot\vec{a})\, 2\pi \int_0^{R} r^2 \left[\frac{e^{ikru}}{ikr}\right]_{-1}^{1} = \left(2\pi \int_0^{R} r^2\, \frac{e^{ikr} - e^{-ikr}}{ikr}\right) i(\vec{k}\cdot\vec{a})$$

$$\Rightarrow I = \left(2\pi i \int_0^R r^2 \frac{2\sin(kr)}{kr}\right)(\bar{k}\cdot\bar{a})\, dr$$

$$= 4\pi i(\bar{k}\cdot\bar{a}) \int_0^R \frac{r\sin(kr)}{k}\, dr$$

using integration by parts:

$$\int_0^R r\sin(kr)\, dr = \frac{1}{k^2}\left(\sin(kR) - kR\cos(kR)\right)$$

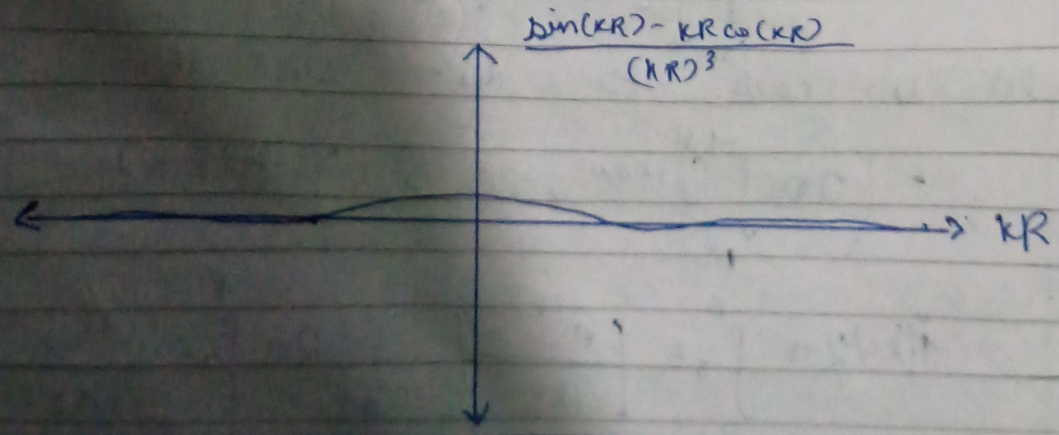
$$\Rightarrow I = \frac{4\pi i(\bar{k}\cdot\bar{a})}{k^3}\left[\sin(kR) - kR\cos(kR)\right]$$

$$= 4\pi R^3 (i\bar{k}\cdot\bar{a})\left[\frac{\sin(kR) - kR\cos(kR)}{k^3R^3}\right]$$

QED

(3)(b) plot $\dfrac{\sin(kR) - kR\cos(kR)}{(kR)^3}$ v/s kR

or, plot $\dfrac{\sin x - x\cos x}{x^3} = \dfrac{\sin x}{x^3} - \dfrac{\cos x}{x^2}$



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(3) (c) (i) $kR \to 0 \Rightarrow \lim_{x\to 0} \frac{\sin x - x\cos x}{x^3}$

$$= \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)}{x^3}$$

$$= -1/3! + 1/2! = 1/2 - 1/6 = \boxed{1/3}$$

(ii) $\lim_{kR\to\infty} \frac{\sin kR}{x^3} - \frac{\cos kR}{x^2} = \boxed{0}$

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(4)(b) $\vec{L} = \vec{r} \times \vec{p} = -i\hbar[\vec{r} \times \vec{\nabla}]$ where $\vec{p} = -i\hbar\vec{\nabla}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{\nabla} = \partial/\partial x\,\hat{i} + \partial/\partial y\,\hat{j} + \partial/\partial z\,\hat{k}$

$$\vec{L} = -i\hbar \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} = -i\hbar \begin{bmatrix} \hat{i}(y\,\partial/\partial z - z\,\partial/\partial y) + \\ \hat{j}(z\,\partial/\partial x - x\,\partial/\partial z) \\ \hat{k}(x\,\partial/\partial y - y\,\partial/\partial x) \end{bmatrix}$$

$$\Rightarrow L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

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(4)(c) (iii) $[L_x, \bar{r}\cdot\bar{p}] = 0$

$$\bar{r}\cdot\bar{p} = -i\hbar\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

$[A,B] = AB - BA$

$$[L_k, \vec{r}\cdot\vec{p}]f = \hbar^2\left(l\frac{\partial}{\partial m} - m\frac{\partial}{\partial l}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

$$- \hbar^2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(l\frac{\partial}{\partial m} - m\frac{\partial}{\partial l}\right)$$

$\Rightarrow$ ~~let $l = y$, $m = z$, $k = x$~~

$$[L_x\ \vec{r}\cdot\vec{p}] = \hbar^2\left(\left[l\frac{\partial}{\partial m} - m\frac{\partial}{\partial l}\right] - \left[l\frac{\partial}{\partial m} - m\frac{\partial}{\partial l}\right]\right)$$

$$\Rightarrow \boxed{[L_k, \vec{r}\cdot\vec{p}] = 0}$$

if $k = x$, $l = y$ and $m = z$ and so on...

(5) (a) $[\Pi_j, \Pi_k] = [p_j - qA_j, p_k - qA_k]$

$$= [p_j, p_k] - q[p_j, A_k] - q[A_j, p_k] + q^2[A_j, A_k]$$

we know that $[p_j, p_k] = 0$, $[A_j, A_k] = 0$

$[p_j, A_k] = p_j A_k - A_k p_j = -i\hbar(\partial_j A_k)$,

$[A_j, p_k] = -[p_k, A_j] = i\hbar(\partial_k A_j)$

$\Rightarrow [\Pi_j, \Pi_k] = -q(-i\hbar\partial_j A_k) - q(i\hbar\partial_k A_j)$

$$= i\hbar q(\partial_j A_k - \partial_k A_j)$$

$\Rightarrow \boxed{[\Pi_j, \Pi_k] = i\hbar q\, \varepsilon_{jkl} B_l}$

now $\varepsilon_{ijk}\Pi_j\Pi_k = \frac{1}{2}\varepsilon_{ijk}(\Pi_j\Pi_k - \Pi_k\Pi_j)$

$$= \frac{1}{2}\varepsilon_{ijk}[\Pi_j, \Pi_k]$$

$(\Pi \times \Pi)_i = \varepsilon_{ijk}\Pi_j\Pi_k = \frac{1}{2}\varepsilon_{ijk}[\Pi_j, \Pi_k]$

$$= i\hbar q \cdot \frac{1}{2}(\varepsilon_{ijk}\varepsilon_{jkl})B_l$$

now $\varepsilon_{ijk}\varepsilon_{jkl} = 2\delta_{il}$

$\Rightarrow (\Pi \times \Pi)_i = i\hbar q\, \delta_{il} B_l \Rightarrow \boxed{\Pi \times \Pi = i\hbar q \vec{B}}$

(5)(b) $\vec{B} = B\hat{z}$

symmetric form: $\vec{A}_{sym} = \frac{1}{2}\vec{B}\times\vec{r} = \left(\frac{-B}{2}y, \frac{B}{2}x, 0\right)$

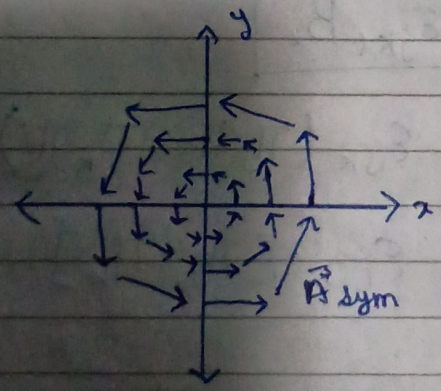
verification: $\nabla\cdot\vec{A}_{sym} = \partial_x(-By/2) + \partial_y(Bx/2) = 0$

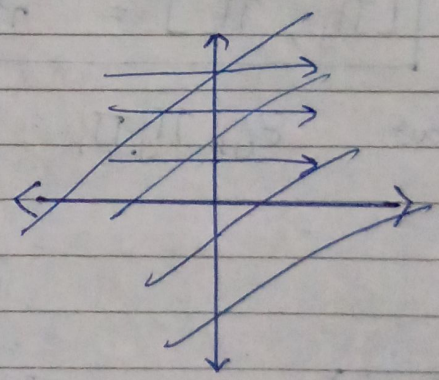
$\nabla\times\vec{A}_{sym} = B\hat{z} = \vec{B}$

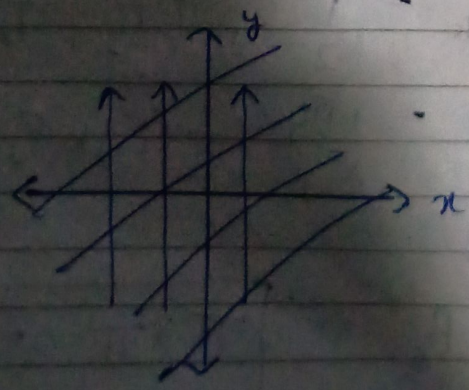
asymm form: $\vec{A}_{asym} = (0, Bx, 0)$

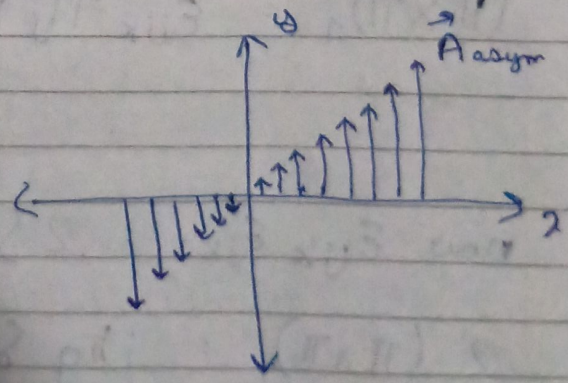
$\nabla\cdot\vec{A}_{asym} = 0$, $\nabla\times\vec{A}_{asym} = B\hat{z}$

another $\vec{A}_{asym} = -B_y\,\hat{\imath}$









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(5) (bi) $r_1 = (x_1, y_1, 0)$, $r_2 = (x_2, y_2, 0)$

$r(t) = r_1 + t(r_2 - r_1)$, $t \in [0,1]$, $\frac{dr}{dt} = (r_2 - r_1)$

let $\Delta x = (x_2 - x_1)$ $\Delta y = (y_2 - y_1)$

I) symmetric form of A

$$I_{sym} = \int_0^1 A_{sym}(r(t)) \cdot (r_2 - r_1)\, dt$$

$$= \frac{B}{2} \int_0^1 [-y(t)\Delta x + x(t)\Delta y]\, dt$$

$$= \frac{B}{2}(x_1 \Delta y - y_1 \Delta x) = \boxed{\frac{B}{2}(x_1 y_2 - x_2 y_1).}$$

$I_{sym} \rightarrow B \times$ (area of Δ formed by $0, r_1, r_2$) $= \phi_\Delta$

II) $\vec{A}_{asym} = (0, B_x, 0)$

$$I_{asym} = \int_0^1 B x(t) \Delta y\, dt = B\Delta y \left(x_1 + \frac{1}{2}\Delta x\right)$$

$$= \frac{B}{2}(x_1 + x_2)(y_2 - y_1)$$

now $I_{asym} - I_{sym} = \frac{B}{2}(x_2 y_2 - x_1 y_1)$

now $\oint_{\partial\Delta} \vec{A} \cdot d\vec{r} = \int_\Delta (\vec{\nabla} \times \vec{A}) \cdot ds = \vec{B}$ (area of Δ)

$$= \frac{B}{2}(x_1 y_2 - x_2 y_1)$$

This confirms stokes theorem

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6(a) $\vec{A} = \dfrac{(-y\hat{i} + x\hat{j})}{2}$

consider the circle : $r(\theta) = R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j}$

$0 \le \theta \le 2\pi$

$d\vec{r} = \dfrac{d\vec{r}}{d\theta}\,d\theta = (-R\sin\theta\,\hat{i} + R\cos\theta\,\hat{j})\,d\theta$

$A = \dfrac{1}{2}(-R\sin\theta\,\hat{i} + R\cos\theta\,\hat{j})$

$A \cdot dr = \dfrac{1}{2}(-R\sin\theta(-R\sin\theta) + (R\cos\theta)^2)\,d\theta$

$= \dfrac{1}{2}R^2\,d\theta$

$\Rightarrow \boxed{\oint_c A \cdot dr} = \int_0^{2\pi} \dfrac{1}{2}R^2\,d\theta = \boxed{\pi R^2} = LHS$

$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\dfrac{1}{2}(-y\hat{i} + x\hat{j})\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y/2 & x/2 & 0 \end{vmatrix}$

$= \hat{i}(0-0) + \hat{k}\left(\dfrac{\partial}{\partial x}\left(\dfrac{x}{2}\right) - \dfrac{\partial}{\partial y}\left(-\dfrac{y}{2}\right)\right) = \hat{k}$

now for the open surface of north hemisphere:

$(\nabla \times A) \cdot dS = \hat{k} \cdot dS = -R^2 \sin\theta\,d\theta\,d\phi\,(\hat{r} \cdot \hat{k})$

$= R^2 \sin\theta\cos\theta\,d\theta\,d\phi$

$\int_S (\nabla \times A) \cdot dS = \int_0^{2\pi}\int_0^{\pi/2} (\hat{k} \cdot \hat{r}) R^2 \sin\theta\,d\theta\,d\phi = R^2(2\pi)\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta$

$\Rightarrow \boxed{\int_S (\nabla \times A) \cdot dS} = 2\pi R^2 \int_0^1 u\,(du)$ where $u = \sin\theta$

$= \pi R^2 =$ line integral

Hence verified stokes law

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(6) (b) For a circular disk:

$(\nabla \times A) \cdot ds = (\hat{k}) \cdot (ds\,\hat{k}) = dS$

$\int_S dS$ = area of disk = πR^2 = line integral

Hence verified

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(9) $f(E) = \dfrac{1}{e^{\beta(E-\mu)}+1}$, $\beta = \dfrac{1}{k_B T}$, $\mu \in \mathbb{R}$

(9) (a) $f'(E) = -\beta \dfrac{e^{\beta(E-\mu)}}{(1+e^{\beta(E-\mu)})^2}$

$f(E)$ is monotone decreasing in E, $f(\mu) = 1/2$, $f(+\infty) \to 0$, $f(-\infty) \to 1$, $f'(E) < 0$ and peaks at $E = \mu$

as $T \to 0 \Rightarrow \beta \to \infty$:

(i) for $(E-\mu) < 0$, $\beta(E-\mu) \to -\infty \Rightarrow f \to 1$

(ii) for $E > \mu$, $\beta(E-\mu) \to \infty \Rightarrow f \to 0$

(iii) at $E = \mu$, $f = 1/2$

$\Rightarrow \lim_{T \to 0} f(E) = \Theta(\mu - E)$ unit step func

For $f'(E)$ when $T \to 0$, consider test func $\phi(E)$

$$\int_{-\infty}^{\infty} f'(E)\,\phi(E)\,dE = \int_{-\infty}^{\infty} -\beta \frac{e^{\beta(E-\mu)}}{1+e^{\beta(E-\mu)}}\,\phi(E)\,dE$$

let $\xi = E - \mu$, $u = \beta\xi$ $d\xi = du/\beta$

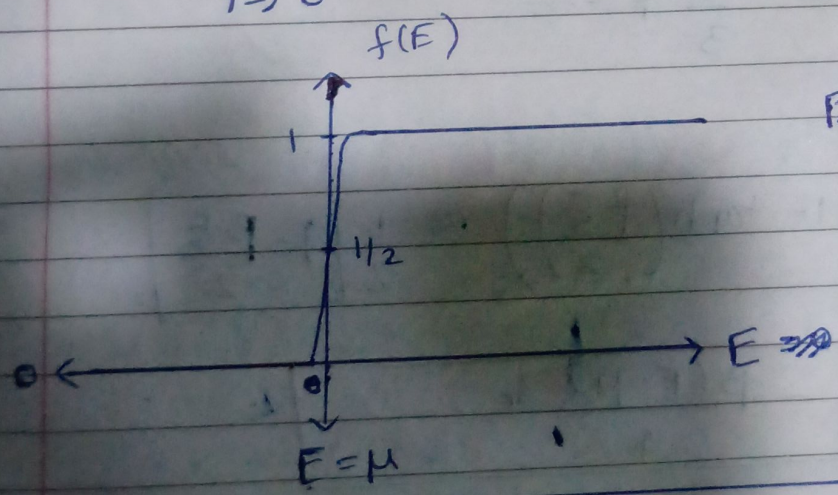
$$\Rightarrow \int_{-\infty}^{\infty} f'(E)\,\phi(E)\,dE = \int_{-\infty}^{\infty} \frac{-e^{u}}{(1+e^{u})^2}\,\phi\left(\mu + \frac{u}{\beta}\right) du$$

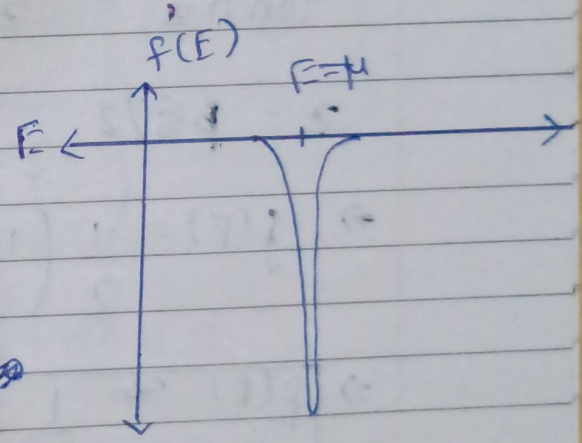
As $\beta \to \infty$, $\phi(\mu + u/\beta) \to \phi(\mu) \to$ integral tends to

$$\phi(\mu)\int_{-\infty}^{\infty} \frac{-e^{u}}{(1+e^{u})^2}\,du = -\phi(\mu)\cdot 1$$

because $\displaystyle\int_{-\infty}^{\infty} \frac{e^{u}}{(1+e^{u})^2}\,du = \left[\frac{-1}{1+e^{u}}\right]_{-\infty}^{\infty} = 1$

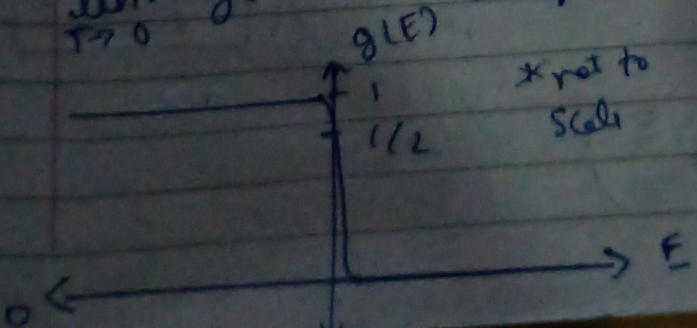
$$\Rightarrow \lim_{T\to 0} f'(E) = -\delta(E-\mu)$$

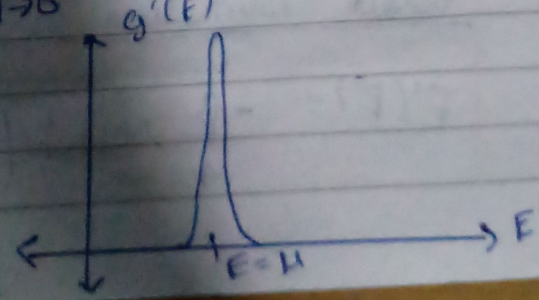




(b) $g(E) = 1 - f(E)$, $g'(E) = -f'(E)$

$\lim_{T\to 0} g(E) = \Theta(E-\mu)$ $\lim_{T\to 0} g'(E) = \delta(E-\mu)$





(c) let $x = \beta\xi$ where $\xi = E-\mu$

$$\tanh(x/2) = \frac{e^x-1}{e^x+1} \Rightarrow \frac{1}{e^x+1} = \frac{1}{2}\left(1-\tanh\frac{x}{2}\right)$$

$$\Rightarrow f(E) = \frac{1}{2}\left(1-\tanh\left(\frac{\beta\xi}{2}\right)\right)$$

$$f'(E) = \frac{-\beta}{4}\operatorname{sech}^2\left(\frac{\beta\xi}{2}\right)$$

as $\frac{d}{dE}\left(\tanh\left(\frac{\beta E}{2}\right)\right) = \frac{\beta}{2}\operatorname{sech}^2\left(\frac{\beta\xi}{2}\right)$

(d) let $|E-\mu| \ll k_B T$

$$\tanh z = z - \frac{z^3}{3} + \cdots$$

$z = \beta\xi/2$

$$\Rightarrow f(E) = \frac{1}{2}\left(1-\tanh\left(\frac{\beta\xi}{2}\right)\right) \approx \frac{1}{2}\left[1-\frac{\beta\xi}{2}\right]$$

$$\Rightarrow \boxed{f(E) \approx \frac{1}{2} - \frac{\beta(E-\mu)}{4} + \cdots}$$

linear slope at fermi energy: $f(E)\big|_{E=\mu} = -\beta/4$

$$= \boxed{-1/4k_BT}$$

width estimation

$$f'(E) = \frac{-\beta}{4}\operatorname{sech}^2\left(\frac{\beta\xi}{2}\right)$$

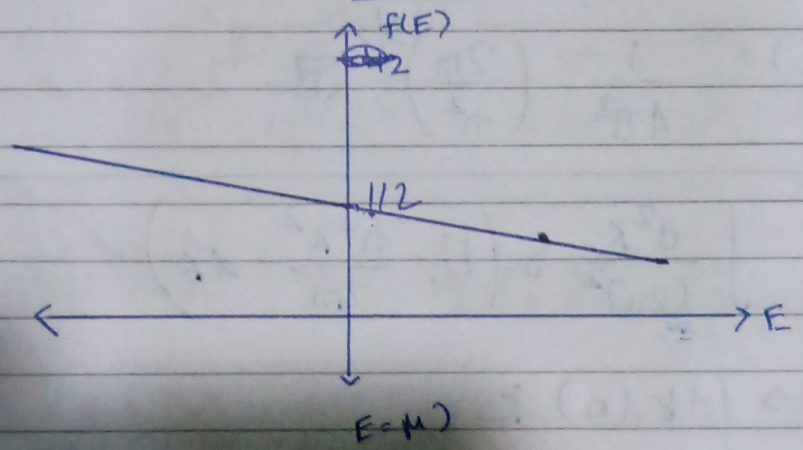
we calculate the full width at half maximum

$\operatorname{sech}^2(\beta\xi/2) = 1/2 \Rightarrow \frac{\beta\xi_{1/2}}{2} = \cosh^{-1}(\sqrt{2})$

$= \ln(\sqrt{2}+1)$

$\Rightarrow \boxed{\xi_{1/2} = 1.76/\beta}$

$FWHM = 2\xi_{1/2} = \boxed{3.52\, k_B T}$



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11) a) $D_3(E) = \int_V \frac{d^3k}{(2\pi)^3}\, \delta\left(E - \frac{\hbar^2 k^2}{2m}\right)$

$d^3k = k^2 \sin\theta\, dk\, d\theta\, d\phi = k^2\, dk\, d\Omega$

$\int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\, d\theta = 4\pi$

$D_3(E) = \frac{4\pi}{(2\pi)^3} \int_0^{\infty} k^2\, \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) dk$

$= \frac{1}{2\pi^2} \int_0^{\infty} k^2\, \delta\left(E - \frac{\hbar^2 k^2}{2m}\right) dk$

let $\mu = \hbar^2 k^2/2m$

$$\Rightarrow D_3(E) = \frac{1}{2\pi^2}\int_0^\infty \sqrt{\frac{2mu}{\hbar^2}}\left(\frac{m}{\hbar^2}\,du\right)\delta(E-u)$$

$$= \frac{1}{2\pi^2}\,\frac{\sqrt{2m}\;m}{\hbar^3}\int_0^\infty \sqrt{u}\;\delta(E-u)\,du$$

$$= \frac{m\sqrt{2m}}{2\pi^2\hbar^3}\sqrt{E}$$

$$\Rightarrow D_3(E) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}$$

(ii)(b) $$D_2(E) = \int \frac{d^2k}{(2\pi)^2}\,\delta\left(E - \frac{\hbar^2k^2}{2m} - \alpha k\right)$$

$$d^2k \longrightarrow (dk\,d\theta)\,k$$

$$\Rightarrow D_2(E) = \frac{2\pi}{(2\pi)^2}\int_0^\infty k\,\delta\left(E - \frac{\hbar^2k^2}{2m} - \alpha k\right)dk$$

we know, $$\int f(x)\,\delta(g(x))\,dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

now let $$g(k) = E - \frac{\hbar^2k^2}{2m} - \alpha k$$

solving for roots (positive as $k>0$):

$$k_0 = \frac{m}{\hbar^2}\left(-\alpha + \sqrt{\alpha^2 + \frac{2\hbar^2E}{m}}\right)$$

$$g'(k) = \frac{-\hbar^2k}{m} - \alpha$$

$$|g'(k_0)| = |g'(k_0)| = \left| -\frac{\hbar^2 k_0}{m} - \alpha \right| = \frac{\hbar^2 k_0}{m} + \alpha$$

$$|g'(k_0)| = \frac{\hbar^2}{m} \cdot \left[\frac{m}{\hbar^2} \left(-\alpha + \sqrt{\alpha^2 + \frac{2\hbar^2 E}{m}} \right) \right] + \alpha$$

$$\Rightarrow |g'(k_0)| = \sqrt{\frac{\alpha^2 + 2\hbar^2 E}{m}}$$

$$D_2(E) = \frac{1}{2\pi} \frac{k_0}{|g'(k_0)|} = \frac{m}{2\pi\hbar^2} \left[1 - \frac{\alpha}{\sqrt{\alpha^2 + \frac{2\hbar^2 E}{m}}} \right]$$

$$E_\alpha = m\alpha^2 / 2\hbar^2 \Rightarrow D_2(E) = \frac{m}{2\pi\hbar^2} \left[1 - \sqrt{\frac{E_\alpha}{E + E_\alpha}} \right]$$

when $\alpha = 0$: $\boxed{D_2(E) = \frac{m}{2\pi\hbar^2}}$

(ii) (c) $D_g(E) = \int \frac{d^2k}{(2\pi)^2} \delta(E - \beta k)$

using polar coordinates

$$D_g(E) = \frac{2\pi}{(2\pi)^2} \int_0^\infty k\,\delta(E - \beta k)\,dk = \frac{1}{2\pi} \int_0^\infty k\,\delta(E - \beta k)\,dk$$

$$\delta(ax) = 1/|a|\ \delta(x)$$

$$\Rightarrow \delta(E - \beta k) = 1/\beta\ \delta(k - E/\beta)$$

$$D_g(E) = \boxed{\frac{1}{2\pi\beta} \int_0^\infty k\,\delta(k - E/\beta)\,dk}$$

now, at $k = E/\beta$:

$$\boxed{D_g(E) = \frac{E}{2\pi\beta^2}}$$

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(12) (a) $\delta(\sin x - \cos x) = \sum_n C_n \delta(x - x_n)$

we know: $\delta(g(x)) = \sum_n \frac{\delta(x - x_n)}{|g'(x_n)|}$

here, $C_n = 1/|g'(x_n)|$.

solving for roots: $\sin x_n - \cos x_n = 0$

$\Rightarrow \tan x_n = 1$

$\Rightarrow \boxed{x_n = \frac{\pi}{4} + n\pi \quad , \quad n = 0, \pm 1, \pm 2 \ldots}$

$g'(x) = \frac{d}{dx}(\sin x - \cos x) = \cos x + \sin x$

$g'(x_n) = \cos\left(\frac{\pi}{4} + n\pi\right) + \sin\left(\frac{\pi}{4} + n\pi\right)$

when n = odd, $|g'(x_n)| = -\sqrt{2}$

even, $|g'(x_n)| = \sqrt{2}$

$\Rightarrow |g'(x_n)| = (-1)^n \sqrt{2} \Rightarrow |g'(x_n)| = \sqrt{2}$

$\Rightarrow \boxed{C_n = 1/\sqrt{2}}$

(b) $I = \int_{-\infty}^{\infty} dx\, e^{-a|x|} \delta(\sin x) \quad a > 0$

$\sin x = 0$ at $x_n = n\pi$, $n = 0, 1 \ldots$

$d(\sin x)/dx = \cos x$

$|\cos(n\pi)| = 1$

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$$\delta(\sin x) = \sum_{n=-\infty}^{\infty} \frac{\delta(x-n\pi)}{1} = \sum_{n=-\infty}^{\infty} \delta(x-n\pi)$$

$$I = \int_{-\infty}^{\infty} dx\, e^{-a|x|} \left(\sum_{n=-\infty}^{\infty} \delta(x-n\pi) \right)$$

$$= \sum_{n=-\infty}^{\infty} e^{-a\pi|n|} = 2\sum_{n=\phi}^{\infty} e^{-a\pi n} + 1$$

$$\Rightarrow I = 1 + 2\left(\frac{e^{-a\pi}}{1-e^{-a\pi}}\right) = \frac{1+e^{-a\pi}}{1-e^{-a\pi}}$$

$$\Rightarrow I = \frac{e^{a\pi/2} + e^{-a\pi/2}}{e^{a\pi/2} - e^{-a\pi/2}} = \boxed{\coth(a\pi/2)}$$

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(7)(b) $\vec{L} = -i\hbar[\vec{r} \times \vec{\nabla}]$

$$r = r\hat{r}, \quad \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\vec{r} \times \vec{\nabla} = (r\hat{r}) \times \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right)$$

$$= \hat{\phi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}$$

$$\vec{L} = -i\hbar(\vec{r} \times \vec{\nabla}) = -i\hbar\left[\hat{\phi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right]$$

at equator, $\theta = \pi/2$

$$\Rightarrow \boxed{\vec{L} = -i\hbar\left[\hat{\phi}\frac{\partial}{\partial \theta} - \hat{\theta}\frac{\partial}{\partial \phi}\right]_{\theta=\pi/2}}$$

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(7)(c) $L = -i\hbar \left[\hat{\phi} \frac{\partial}{\partial \theta} - \hat{\theta} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right]$

$\hat{\theta} = \hat{i} \cos\theta \cos\phi + \hat{j} \cos\theta \sin\phi - \hat{k} \sin\theta$

$\hat{\phi} = -\hat{i} \sin\phi + \hat{j} \cos\phi$

substituting the values:

$\vec{L} = -i\hbar \left[(-\hat{i} \sin\phi + \hat{j} \cos\phi) \frac{\partial}{\partial \theta} - \frac{1}{\sin\theta} (\hat{i} \cos\theta \cos\phi + \hat{j} \cos\theta \sin\phi - \hat{k} \sin\theta) \partial/\partial\phi \right]$

from this, we get:

$L_x = -i\hbar \left[-\sin\phi \frac{\partial}{\partial \theta} - \cos\phi \cot\theta \frac{\partial}{\partial \phi} \right]$

$L_y = -i\hbar \left[\cos\phi \frac{\partial}{\partial \theta} - \sin\phi \cot\theta \frac{\partial}{\partial \phi} \right]$

$L_z = -i\hbar \left[\frac{\partial}{\partial \phi} \right]$

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(7)(d) $\vec{L} \times \vec{L} = \hat{i} (L_y L_z - L_z L_y) + \hat{j} (L_z L_x - L_x L_z) + \hat{k} (L_x L_y - L_y L_x)$

$= \hat{i} [L_y, L_z] + \hat{j} [L_z, L_x] + \hat{k} [L_x, L_y]$

~~$(yp_z - zp_y)\hat{i} + (z$~~

we know that $[L_i, L_j] = i\hbar L_k$ in cyclic manner

$$\Rightarrow \vec{L} \times \vec{L} = \hat{i}(i\hbar L_x) + \hat{j}(i\hbar L_y) + \hat{k}(i\hbar L_z)$$

$$\Rightarrow \boxed{\vec{L} \times \vec{L} = i\hbar \vec{L}}$$

---

(7)(e) To prove:

$$-\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$= -\hbar^2 \left[r^2\nabla^2 - \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) \right]$$

we know that:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

In RHS:

$$r^2\nabla^2 = \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

substituting it:

$$RHS = -\hbar^2 \left[\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \cdots \right) - \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \right]$$

$$= -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]$$

$$= LHS \qquad QED$$

At the equator, $\theta = \pi/2 \Rightarrow \sin\theta = 1$

$$\boxed{L^2_{eq} = -\hbar^2 \left[\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\phi^2} \right]}$$

(7)(f) To prove: $[L_k, L^2] = 0$ for $k = x, y, z$

let $k = x$

$[L_x, L^2] = [L_x, L_x^2 + L_y^2 + L_z^2] = [L_x, L_x^2] + [L_x, L_y^2] + [L_x, L_z^2]$

$\Rightarrow [L_x, L^2] = 0 + [L_x, L_y] L_y + L_y [L_x, L_y] +$

$[L_x, L_z] L_z + L_z [L_x, L_z]$

$\Rightarrow [L_x, L^2] = i\hbar L_z L_y + i\hbar L_y L_z - i\hbar (L_y L_z + L_z L_y)$

$= 0$

$\Rightarrow \boxed{[L_x, L^2] = 0}$

in a cyclic manner, same result can be proved for $k = y$ and z.

(16)

(7)(g) To prove: $L_\pm = \hbar e^{\pm i\phi} \left[\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right]$.

we know:

$L_x = i\hbar \left(+\sin\phi \frac{\partial}{\partial\theta} + \cos\phi \cot\theta \frac{\partial}{\partial\phi} \right)$

$L_y = i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \sin\phi \cot\theta \frac{\partial}{\partial\phi} \right)$

$L_\pm = L_x \pm i L_y$

$\Rightarrow L_\pm = i\hbar\left(\sin\phi \frac{\partial}{\partial\theta} + \cos\phi \cot\theta \frac{\partial}{\partial\phi}\right) \mp \hbar\left(-\cos\phi \frac{\partial}{\partial\theta} + \sin\phi \cot\theta \frac{\partial}{\partial\phi}\right)$

$= \hbar\left[(i\sin\phi \pm \cos\phi)\frac{\partial}{\partial\theta} + (i\cos\phi \mp \sin\phi)\cot\theta \frac{\partial}{\partial\phi}\right]$

$= \left[\hbar e^{\pm i\phi}(i\sin\phi \pm \cos\phi)\frac{\partial}{\partial\theta} + e^{\mp i\phi}(i\cos\phi \mp \sin\phi)\cot\theta \frac{\partial}{\partial\phi}\right]$

let $A_\pm = e^{\mp i\phi}(i\sin\phi \pm \cos\phi) = \pm 1$

let $B_\pm = e^{\mp i\phi}(i\cos\phi \mp \sin\phi) = i$

substituting the values of coefficients

$$\boxed{L_\pm = \hbar e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\frac{\partial}{\partial\phi}\right]}$$

---

(16)(a) $\nabla_r^4 e^{-kr} = \left(\frac{-4k^3}{r} + k^4\right)e^{-kr} + 8\pi k\delta^3(\vec{r})$

$\nabla^2 f = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right)$ as its spherically symmetric

$\nabla^2 f = \frac{1}{r^2}\left(\frac{d}{dr}\left(r^2(-ke^{-kr})\right)\right)$

$= (k^2 - 2k/r)e^{-kr}$

$$\nabla^2(\nabla^2 e^{-kr}) = \nabla^2\left[\left(k^2 - \frac{2k}{r}\right)e^{-kr}\right]$$

$$= \nabla^2(k^2 e^{-kr}) + 2k(\nabla^2(e^{-kr}/r))$$

$$= \left(k^4 - \frac{2k^3}{r}\right)e^{-kr} - 2k\left(k^2\frac{e^{-kr}}{r} - 4\pi\delta^3(r)\right)$$

$$\Rightarrow \boxed{\nabla^4 e^{-kr} = \left(k^4 - \frac{4k^3}{r}\right)e^{-kr} + 8\pi k\,\delta^3(r)}$$

16 (b) $[\nabla_r^2 + k^2]\left(\frac{e^{\pm ikr}}{r}\right) = -4\pi\delta^3(\vec{r})$

let $f(r) = e^{\pm ikr}/r$

$$\nabla^2 f(r) = -k^2\frac{e^{\pm ikr}}{r} = -k^2 f(r)$$

now at $r=0$: $\nabla^2(1/r) = -4\pi\delta^3(r)$

$$f(r) = \frac{e^{\pm ikr} - 1}{r} + \frac{1}{r}$$

$(\nabla^2 + k^2)f(r) = 0$ everywhere but at origin

where ∇^2 operator acting on $1/r$ produces delta function

$$\Rightarrow \left[\nabla_r^2 + k^2\right]\left(\frac{e^{\pm ikr}}{r}\right) = -4\pi\delta^3(r)$$

16(c) let $f(x) = e^{\pm ik|x-x'|}$

$$\frac{d}{dx}|u(x)| = u'(x)\,\mathrm{Sgn}(u(x)) \qquad \text{where } u = x-x', \; u' = 1$$

$$\Rightarrow \frac{df}{dx} = e^{\pm ik|x-x'|} \cdot (\pm ik)\,\mathrm{sgn}(x-x')$$

$$\frac{\partial^2 f}{\partial x^2} = (\pm ik)\left[\left(\frac{d}{dx}e^{\pm ik|x-x'|}\right)\mathrm{sgn}(x-x') + e^{\pm ik|x-x'|}\left(\frac{d}{dx}\mathrm{sgn}(x-x')\right)\right]$$

$$= (\pm ik)^2 e^{\pm ik|x-x'|} + (\pm ik)e^{\pm ik|x-x'|}\left(2\delta(x-x')\right)$$

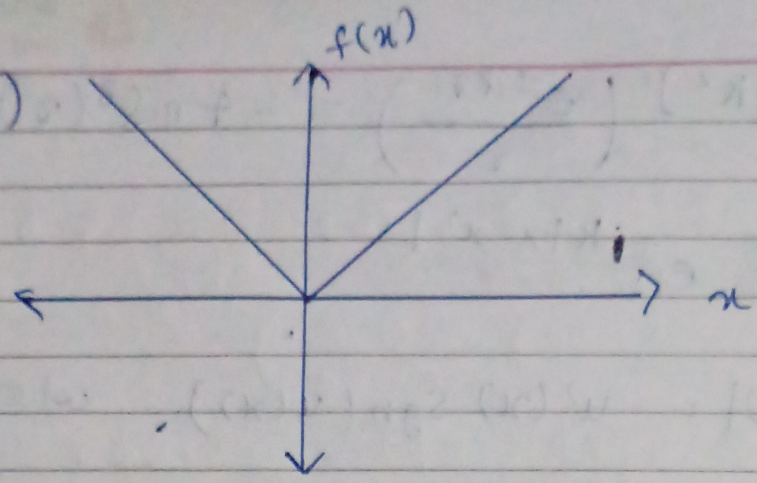
$$= -k^2 e^{\pm ik|x-x'|} + (\pm 2ik)e^0\delta(x-x')$$

$$= -k^2 e^{\pm ik|x-x'|} \pm 2ik\,\delta(x-x')$$

now, applying full op:

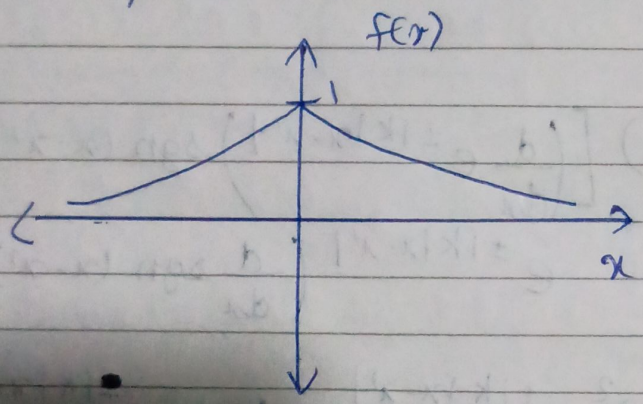
$$\left[\frac{d^2}{dx^2} + k^2\right]f(x) = \boxed{\pm 2ik\,\delta(x-x')}$$

(d) (i)



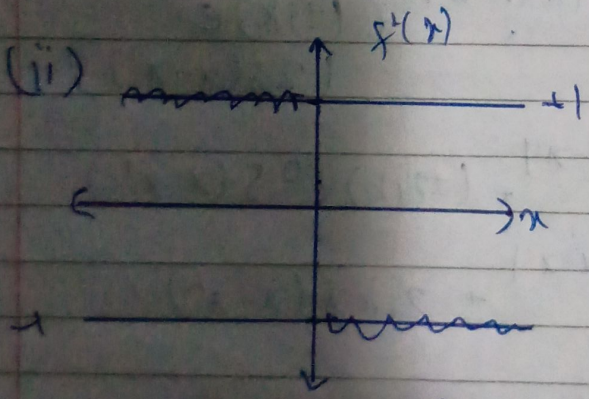
$f(x) = |x|$

at $x=0$, derivative does not exist for bot $f(x)$

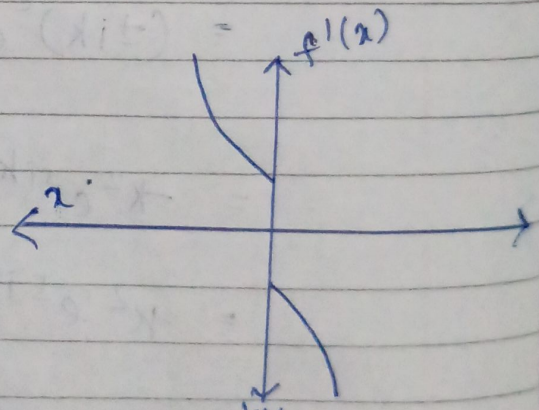


$\therefore f(x) = e^{-|x|}$

(ii)

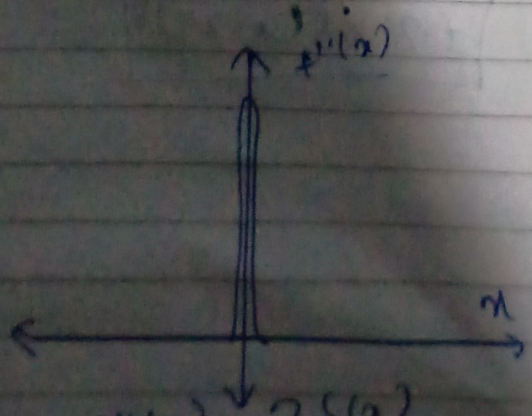


$f'(x) = +\text{Sgn}(x)$

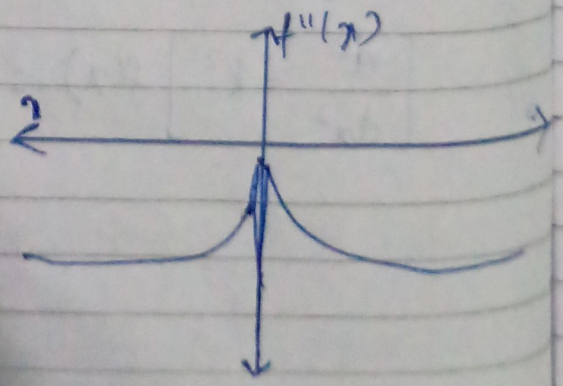


$f'(x) = -e^{|x|}\,\text{sgn}(x)$

(iii)



$f''(x) = 2\delta(x)$



$f'''(x) = e^{-|x|} - 2\delta(x)$