PHY 431 Quantum Mechanics I
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Model Question Paper 1
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1. Prove: $[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB.$

2. Suppose a 2×2 matrix X(not necessarily Hermitian, nor unitary) is written as

$$X = a_0 + \boldsymbol{\sigma}.\mathbf{a},$$

where the matrices $\boldsymbol{\sigma}$ are Pauli matrices and a_0 and $a_{1,2,3}$ are numbers.
a. How are a_0 and $a_k (k = 1, 2, 3)$ related to $\mathrm{tr(X)}$ and $\mathrm{tr(\sigma_k X)}$?
b. Obtain a_0 and a_k in terms of the matrix elements X_{ij}.

3. Using the rules of bra-ket algebra, prove or evaluate the following:
a. $\mathrm{tr(XY)} = \mathrm{tr(YX)}$, where X and Y are operators;
b. $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators;

4. Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\epsilon_{ijk}\hbar S_k, \quad \{S_i, S_j\} = \tfrac{\hbar^2}{2}\delta_{ij},$$

where

$$S_x = \tfrac{\hbar}{2}\big(|+\rangle\langle-| + |-\rangle\langle+|\big), \quad S_y = \tfrac{i\hbar}{2}\big(-|+\rangle\langle-| + |-\rangle\langle+|\big), \quad S_z = \tfrac{\hbar}{2}\big(|+\rangle\langle+| - |-\rangle\langle-|\big).$$

5. The Hamiltonian operator for a two-state system is given by

$$H = a\big(|1\rangle\langle1| - |2\rangle\langle2| + |1\rangle\langle2| + |2\rangle\langle1|\big),$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$)

6. A two-state system is characterized by the Hamiltonian

$$H = H_{11}|1\rangle\langle1| + H_{22}|2\rangle\langle2| + H_{12}\big[|1\rangle\langle2| + |2\rangle\langle1|\big],$$

where H_{11} , H_{22} , and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$.

7. A certain observable in quantum mechanics has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

a. Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?
b. Give a physical example where all this is relevant.

8. Two Hermitian operators anticommute:

$$\{A, B\} = AB + BA = 0.$$

Is it possible to have a simultaneous (that is, common) eigenket of A and B? Prove or illustrate your assertion.

9. Two observables A_1 and A_2 , which do not involve time explicitly, are known not to commute,

$$[A_1, A_2] \neq 0,$$

yet we also know that A_1 and A_2 both commute with the Hamiltonian:

$$[A_1, H] = 0, [A_2, H] = 0.$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-force problem $H = p^2/2m + V(r)$, with $A_1 \to L_z, A_2 \to L_x$.

10. a.

$$[x_i, G(p)] = i\hbar\frac{\partial G}{\partial p_i}, \quad [p_i, F(x)] = -i\hbar\frac{\partial F}{\partial x_i}$$

These two equations can be "easily derived" from the fundamental commutation relations for all functions of F and G that can be expressed as power series in their arguments. Verify this statement.

b. Evaluate $[x^2, p^2]$. Compare your result with the classical Poisson bracket $[x^2, p^2]_{\text{classical}}$.

11. The translation operator for a finite (spatial) displacement is given by

$$\mathcal{J}(\mathbf{I}) = \exp\left(\frac{-i\mathbf{p}.\mathbf{I}}{\hbar}\right),$$

where $\mathbf{p}$ is the momentum operator.
a. Evaluate

$$[x_i, \mathcal{J}(\mathbf{I})].$$

b. Using (a) (or otherwise), demonstrate how the expectation value $\langle \mathbf{x} \rangle$ changes under translation.

12. Prove the following:

$$(i)\, \langle p' | \, x \, |\alpha\rangle = i\hbar\frac{\partial}{\partial p'} \langle p' | \alpha\rangle\,, \quad (ii)\, \langle \beta | \, x \, |\alpha\rangle = \int dp' \phi^*_\beta(p') i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p'),$$

where $\phi_\alpha(p') = \langle p' | \alpha\rangle$ and $\phi_\beta(p') = \langle p' | \beta\rangle$ are momentum-space wave functions.