PHY 431 Quantum Mechanics I
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Model Question Paper 2
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1. Consider the states $\psi = 3i\,|\phi_1\rangle - 7i\,|\phi_2\rangle$ and $|\chi\rangle = -|\phi_1\rangle + 2i\,|\phi_2\rangle$, where $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal.
(a) Calculate $|\psi + \chi\rangle$ and $\langle\psi + \chi|$.
(b) Calculate the scalar products $\langle\psi|\chi\rangle$ and $\langle\chi|\psi\rangle$. Are they equal?
(c) Show that the states ψ and χ satisfy the Schwarz inequality.
(d) Show that the states ψ and χ satisfy the triangle inequality.

2. (a) Discuss the hermiticity of the operators $\left(\hat{A} + \hat{A}^\dagger\right)$, $i\left(\hat{A} + \hat{A}^\dagger\right)$, and $i\left(\hat{A} - \hat{A}^\dagger\right)$.
(b) Find the Hermitian adjoint of $f(\hat{A}) = (1 + i\hat{A} + 3\hat{A}^2)(1 - 2i\hat{A} - 9\hat{A}^2)/(5 + 7\hat{A})$.
(c) Show that the expectation value of a Hermitian operator is real and that of an anti-Hermitian operator is imaginary.

3. Show that the operator $|\psi\rangle\langle\psi|$ is a projection operator only when $|\psi\rangle$ is normalized.

4. Consider a matrix A (which represents an operator $\hat{A}$), a ket $|\psi\rangle$, and a bra $\langle\phi|$:

$$A = \begin{pmatrix} 5 & 3+2i & 3i \\ -i & 3i & 8 \\ 1-i & 1 & 4 \end{pmatrix}, \quad |\psi\rangle = \begin{pmatrix} -1+i \\ 3 \\ 2+3i \end{pmatrix}, \quad \langle\phi| = \begin{pmatrix} 6 & -i & 5 \end{pmatrix}$$

(a) Calculate the quantities $A\,|\psi\rangle$, $\langle\phi|\,A$, $\langle\phi|\,A\,|\psi\rangle$, and $|\psi\rangle\langle\phi|$.
(b) Find the complex conjugate, the transpose, and the Hermitian conjugate of $A, |\psi\rangle$, and $\langle\phi|$.
(c) Calculate $\langle\phi|\psi\rangle$ and $\langle\psi|\phi\rangle$; are they equal? Comment on the differences between the complex conjugate, Hermitian conjugate, and transpose of kets and bras.

5. Consider a two-dimensional space where a Hermitian operator $\hat{A}$ is defined by $\hat{A}\,|\phi_1\rangle = |\phi_1\rangle$ and $\hat{A}\,|\phi_2\rangle = -|\phi_2\rangle$; $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal.
(a) Do the states $|\phi_1\rangle$ and $|\phi_2\rangle$ form a basis?
(b) Consider the operator $\hat{B} = |\phi_1\rangle\langle\phi_2|$. Is $\hat{B}$ Hermitian? Show that $\hat{B}^2 = 0$.
(c) Show that the products $\hat{B}\hat{B}^\dagger$ and $\hat{B}^\dagger\hat{B}$ are projection operators.
(d) Show that the operator $\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B}$ is unitary.
(e) Consider $\hat{C} = \hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B}$. Show that $\hat{C}\,|\phi_1\rangle = |\phi_1\rangle$ and $\hat{C}\,|\phi_2\rangle = |\phi_2\rangle$.

6. Consider an operator $\hat{A}$ so that $\left[\hat{A}, \hat{A}^\dagger\right] = 1$.
(a) Evaluate the commutators $\left[\hat{A}^\dagger\hat{A}, \hat{A}\right]$ and $\left[\hat{A}^\dagger\hat{A}, \hat{A}^\dagger\right]$.
(b) If the actions of $\hat{A}$ and $\hat{A}^\dagger$ on the states $\left\{\,|a\rangle\,\right\}$ are given by $\hat{A}\,|a\rangle = \sqrt{a}\,|a-1\rangle$ and $\hat{A}^\dagger\,|a\rangle = \sqrt{a+1}\,|a+1\rangle$ and if $\langle a'|a\rangle = \delta_{a'a}$, calculate $\langle a|\,\hat{A}\,|a+1\rangle$, $\langle a+1|\,\hat{A}^\dagger\,|a\rangle$, $\langle a|\,\hat{A}^\dagger\hat{A}\,|a\rangle$, and $\langle a|\,\hat{A}\hat{A}^\dagger\,|a\rangle$.
(c) Calculate $\langle a|\,(\hat{A} + \hat{A}^\dagger)^2\,|a\rangle$ and $\langle a|\,(\hat{A} - \hat{A}^\dagger)^2\,|a\rangle$.

7. Consider an operator

$$\hat{A} = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3| - i\,|\phi_1\rangle\langle\phi_2| - |\phi_1\rangle\langle\phi_3| + i\,|\phi_2\rangle\langle\phi_1| - |\phi_3\rangle\langle\phi_1|$$

where $|\phi_1\rangle, |\phi_2\rangle$, and $|\phi_3\rangle$ form a complete and orthonormal basis.
(a) Is $\hat{A}$ Hermitian? Calculate $\hat{A}^2$; is it a projection operator?
(b) Find the 3×3 matrix representing $\hat{A}$ in the $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$ basis.
(c) Find the eigenvalues and the eigenvectors of the matrix.

8. The Hamiltonian of a two-state system is given by

$$\hat{H} = E\big(|\phi_1\rangle \langle\phi_1| - |\phi_2\rangle \langle\phi_2| - i\,|\phi_1\rangle \langle\phi_2| + i\,|\phi_2\rangle \langle\phi_1| \big),$$

where $|\phi_1\rangle$, $|\phi_2\rangle$ form a complete and orthonormal basis; E is a real constant having the dimensions of energy.
(a) Is $\hat{H}$ Hermitian? Calculate the trace of $\hat{H}$.
(b) Find the matrix representing $\hat{H}$ in the $|\phi_1\rangle$, $|\phi_2\rangle$ basis and calculate the eigenvalues and the eigenvectors of the matrix. Calculate the trace of the matrix and compare it with the result you obtained in (a).
(c) Calculate $\big[\hat{H}, |\phi_1\rangle \langle\phi_1|\big]$, $\big[\hat{H}, |\phi_2\rangle \langle\phi_2|\big]$, and $\big[\hat{H}, |\phi_1\rangle \langle\phi_2|\big]$.