Introduction to Electrostatics

$$E(\bar{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\bar{x}')\,\frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3}\,d^3x'$$

Dirac Delta function

(i) $\int f(x)\,\delta(x-a)\,dx = f(a)$

(ii) $\int f(x)\,\delta'(x-a)\,dx = f'(a)$

(iii) $\delta(f(x)) = \sum_i \frac{1}{\left|\frac{df}{dx}(x_i)\right|}\,\delta(x-x_i)$

(iv) $\delta(\bar{x}-\bar{X}) = \delta(x_1-X_1)\,\delta(x_2-X_2)\,\delta(x_3-X_3)$

Gauss's Law

$$\bar{E}\cdot d\bar{a} = \frac{q\cos\theta}{4\pi\varepsilon_0 r^2}\,da \quad , \quad d\Omega = \frac{dA\cos\theta}{r^2}$$

$$\Rightarrow \bar{E}\cdot d\bar{a} = \frac{q\,d\Omega}{4\pi\varepsilon_0} \Rightarrow \oint_S \bar{E}\cdot\hat{n}\,da = \int \frac{q\,d\Omega}{4\pi\varepsilon}$$

$$\Rightarrow \oint_S \bar{E}\cdot\hat{n}\,da = \begin{cases} q/\varepsilon_0 , & q \text{ lies inside } S \\ 0 , & q \text{ lies outside } S \end{cases}$$

$$\Rightarrow \boxed{\oint_S \bar{E}\cdot\hat{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(x)\,d^3x}$$

Divergence Theorem: $\oint_S \bar{A}\cdot\bar{n}\,da = \int_V \bar{\nabla}\cdot\bar{A}\,d^3x$

$$\Rightarrow \boxed{\bar{\nabla}\cdot\bar{E} = \rho/\varepsilon_0}$$

Scalar Potential

$$\bar{E}(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(x') \frac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3} d^3x'$$

note: $\dfrac{(\bar{x}-\bar{x}')}{|\bar{x}-\bar{x}'|^3} = -\bar{\nabla}\cdot\left(\dfrac{1}{|\bar{x}-\bar{x}'|}\right)$

$$\Rightarrow \bar{E}(\bar{x}) = \frac{-1}{4\pi\varepsilon_0} \int \rho(x') \, \bar{\nabla}\cdot\left(\frac{1}{|\bar{x}-\bar{x}'|}\right) d^3x'$$

involves only $\bar{x}$

$$\Rightarrow \bar{E}(\bar{x}) = \frac{-1}{4\pi\varepsilon_0} \nabla \int \frac{\rho(x')}{|\bar{x}-\bar{x}'|} d^3x'$$

we know $\bar{\nabla}\times\bar{\nabla}\psi = 0$ for scalar ψ

$$\Rightarrow \boxed{\bar{\nabla}\times\bar{E}(\bar{x}) = 0}$$

Scalar Potential

$$\boxed{\bar{E} = -\bar{\nabla}\phi}, \quad \boxed{\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(x')}{|\bar{x}-\bar{x}'|} d^3x'}$$

$$W_{AB} = -\int_A^B F\cdot dl = -q\int_A^B \bar{E}\cdot d\bar{l} = q\int_A^B \bar{\nabla}\phi\cdot dl = q(\phi_B - \phi_A)$$

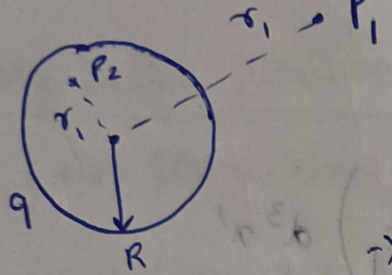
$$\Rightarrow \boxed{W_{AB} = q(\phi_B - \phi_A)}$$

when path is closed: $\boxed{\oint \bar{E}\cdot d\bar{l} = 0}$

Stoke's Theorem: $\oint_C \bar{A}\cdot d\bar{l} = \int_S (\bar{\nabla}\times\bar{A})\cdot\bar{n}\,da$

Problems

1. $\bar{E}$ and ϕ due to spherical shell



$$\oint_S E\cdot\hat{n}\,da = \frac{1}{\varepsilon_0}\int_0 \rho(x)\,d^3x$$

$$\cancel{\rho(x) = \delta^3(0)\,q}$$

$$\Rightarrow \oint_S E\cdot\hat{n}\,da = q/\varepsilon_0 \qquad q \equiv q_{enc}$$

$$\Rightarrow \bar{E}_1\cdot 4\pi r_1^2 = q/\varepsilon_0 \Rightarrow \boxed{\bar{E}_1 = \frac{q}{4\pi\varepsilon_0 r_1^2}\hat{r}_1}$$

$$\bar{E}_2\cdot 4\pi r_2^2 = 0/\varepsilon_0 \Rightarrow \boxed{\bar{E}_2 = 0}$$

$$\phi = \frac{1}{4\pi\varepsilon}\int \frac{\rho(x')}{|\bar{x}-\bar{x}'|}\,d^3x'$$

$$\phi(r) - \phi(\infty) = -\int_{+\infty}^{r} \bar{E}\cdot d\bar{r}$$

when $r \geq R$: $\phi(r) = \frac{-q}{4\pi\varepsilon}\int_\infty^r \frac{1}{r^2}\,dr = \boxed{\frac{q}{4\pi\varepsilon r}}$

when $r < R$: $\phi(r) = \int_\infty^R \frac{-q}{4\pi\varepsilon_0 r^2}\,dr + \int_R^r 0\,dr = \boxed{\frac{q}{4\pi\varepsilon_0 R}}$

Q. $\bar{E}$ and ϕ due to solid sphere

$$\rho = \frac{3q}{4\pi R^3} \qquad q_r = \frac{3q\,dr}{4\pi R^3} \cdot \cancel{\pi 4} r^2 = \frac{3qr^2}{R^3}\,dr$$

outside ($|\vec{r}| \geq R$) : $\bar{E} = \cancel{\tfrac{1}{4\pi\varepsilon}} \int_0^R \frac{q_{r'}}{4\pi\varepsilon r_1^2}\,\hat{r}_1$

$$\Rightarrow \bar{E}(\bar{r}) = \int^R \frac{1}{4\pi\varepsilon r_1^2} \frac{3qr^2}{R^3}\,dr\,\hat{r}_1$$

$$= \frac{\cancel{3}q}{4\pi\varepsilon r_1^2 \cancel{R^3}} \frac{\cancel{R^3}}{\cancel{3}}\,\hat{r}_1 = \frac{q}{4\pi\varepsilon_0 r_1^2}\,\hat{r}_1$$

inside ($|\vec{r}_2| < R$) : $\bar{E}(\bar{r}_2) = \int_0^{r_2} \frac{1}{4\pi\varepsilon_0 r_2^2} \frac{3qr^2}{R^3}\,dr$

$$\Rightarrow \bar{E}(\bar{r}_2) = \frac{\cancel{3}q}{4\pi\varepsilon_0 \cancel{r_2^2} R^3} \frac{r_2^{\cancel{3}}}{\cancel{3}} = \boxed{\frac{q r_2}{4\pi\varepsilon_0 R^3}}$$

$$\phi(r) = -\int_{+\infty}^{r} E \cdot d\bar{r} = \frac{q}{4\pi\varepsilon_0 r} \qquad (r > R)$$

$$\phi(r) = -\int_{+\infty}^{R} \bar{E}\cdot dr - \int_R^r \bar{E}\cdot dr = \frac{q}{4\pi\varepsilon R} - \int_R^r \frac{qr}{4\pi\varepsilon R^3}\,dr$$

$$= \cancel{\frac{q}{4\pi\varepsilon R}} - \cancel{\frac{q}{4\pi\varepsilon} \frac{r^2}{2R^3}} = \boxed{\cancel{\frac{q}{4\pi\varepsilon}\left(1 - \frac{r^2}{2R^3}\right)}}$$

$$= \frac{q}{4\pi\varepsilon R} - \left(\frac{r^2}{2R^3} - \frac{R^2}{2R^3}\right)\frac{q}{4\pi\varepsilon}$$

$$= \boxed{\frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)} \qquad r < R$$

3. Solid sphere with $\rho = kr$,

$$\vec{E}_{out}(\vec{r}) = \frac{\cancel{\pi} k R^4}{4\cancel{\pi} \varepsilon r^2} = \boxed{\frac{kR^4}{4\varepsilon_0 r^2}}$$

$$q = \int_0^R \int_0^{\pi} \int_0^{2\pi} k r^3 \, dr \, d\phi \, d\theta \sin\theta$$

$$= \frac{kR^4}{\cancel{4}} \cdot \cancel{4}\pi = \pi k R^4$$

$$\vec{E}_{in}(\vec{r}) = \frac{q_{in}}{4\pi r^2 \varepsilon} = \frac{\cancel{\pi} k r^{\cancel{4}\,2}}{4\cancel{\pi} \cancel{r^2} \varepsilon_0} = \boxed{\frac{k r^2}{4\varepsilon_0}}$$

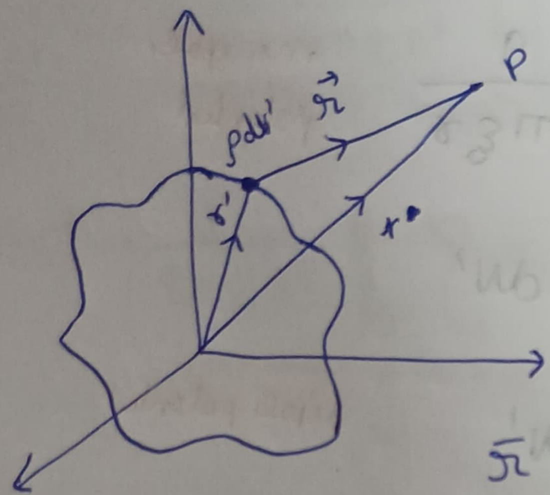
$$\phi_{out}(\vec{r}) = -\int_{\infty}^{r} \bar{E} \cdot d\bar{r} = \boxed{\frac{kR^4}{4\varepsilon_0 r}}$$

$$\phi_{in}(\vec{r}) = -\int_{\infty}^{R} \bar{E} \cdot d\bar{r} - \int_{R}^{r} \bar{E} \cdot d\bar{r}$$

$$= \frac{kR^4}{4\varepsilon_0 r} - \int_R^r \frac{k r^2}{4\varepsilon} dr = \frac{kR^{\cancel{4}\,3}}{4\varepsilon_0 \cancel{R}} - \frac{k}{4\varepsilon_0}\left(\frac{r^3}{3} - \frac{R^3}{3}\right)$$

$$= \frac{k}{12\varepsilon_0}\left(3R^3 - r^3 + R^3\right) = \boxed{\frac{k}{12\varepsilon_0}\left(4R^3 - r^3\right)}$$

Potential due to arbitrary charge distribution ($r >> r'$)



$$d\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{\rho(\bar{r}')\,dv'}{\mathscr{r}}\right)$$

$$\Rightarrow \phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\bar{r}')\,dv'}{\mathscr{r}}$$

$$\bar{\mathscr{r}} = \bar{r} - \bar{r}'$$

$$\Rightarrow \mathscr{r}^2 = \bar{r}^2 + \bar{r}'^2 - 2\bar{r}\bar{r}'$$

$$= \bar{r}^2 + \bar{r}'^2 - 2rr'\cos\alpha$$

$$\Rightarrow \mathscr{r} = r\left(1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\alpha\right)^{1/2} \qquad r >> r'$$

$$\Rightarrow \mathscr{r} = r(1+\varepsilon)^{1/2} \Rightarrow \frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2}$$

$$\Rightarrow \frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - \frac{\varepsilon}{2} + \frac{3\varepsilon^2}{8} - \frac{5\varepsilon^3}{16} + \cdots\right)$$

$$\Rightarrow \phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\bar{r}')\,dv'}{r}(1+\varepsilon)^{-1/2}$$

$$\Rightarrow \phi = \frac{1}{4\pi\varepsilon_0 r}\int \rho(r')\,dv'\,(1+\varepsilon)^{-1/2}$$

$$(1+\varepsilon)^{-1/2} = \left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \cdots\right]$$

$$\Rightarrow (1+\varepsilon)^{-1/2} = \sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\alpha)$$

legendre polynomial

$$\Rightarrow \boxed{\phi(\bar{r}) = \frac{1}{4\pi\varepsilon_0 r}\sum_{l=0}^{\infty}\int_{v'}\left(\frac{r'}{r}\right)^l \rho(\bar{r}')\,P_l(\cos\theta)\,dv'}$$

For $l = 0$:

$$\phi_0(\bar{r}) = \frac{1}{4\pi\varepsilon_0 r}\int_{V'} \rho(r')\, dV' = \frac{q}{4\pi\varepsilon_0 r}$$

monopole potential

$$\phi_1(\bar{r}) = \frac{1}{4\pi\varepsilon_0 r}\int_{V'} \rho(r')\left(\frac{r'}{r}\right)\cos\alpha\, dV'$$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\,\hat{r}\cdot\underbrace{\int_{V'} \rho(r')\,\bar{r}'\, dV'}_{\text{dipole moment}}$$

dipole potential

$$\bar{p} = \int_{V'} r' \rho(r')\, dV'$$

$$\phi_1(\bar{r}) = \frac{\hat{r}}{4\pi\varepsilon_0 r^2}\cdot\bar{p}$$

$$\phi_2(\bar{r}) = \frac{1}{4\pi\varepsilon_0 r}\int_{V'} \rho(r')\left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) dV'$$

$$= \frac{1}{8\pi\varepsilon_0 r^3}\int_{V'} \rho(r')(r')^2(3\cos^2\alpha - 1)\, dV'$$

$$r'\cos\alpha = r'\cdot\hat{r} = \sum_{i=1}^{3} r'_i\,\hat{r}_i \Rightarrow$$

$$\Rightarrow (r'\cos\alpha)^2 = \sum_i^3\sum_j^3 (r'_i\cdot\hat{r}_i)(r_j\,\hat{r}_j)$$

$$= \sum_i\sum_j \frac{r_i r_j r'_i r'_j}{r^2}$$

$$1 = \hat{r}\cdot\hat{r} = \sum_{ij}\frac{r_i r_j}{r^2}\,\delta_{ij}$$

$$\Rightarrow \phi_2(r) = \frac{1}{8\pi\varepsilon r^3}\int_{V'}\left[3\sum_{i,j}\frac{r_i r_j}{r^2} r_i' r_j' - (r')^2\sum_{i,j}\frac{r_i r_j}{r^2}\delta_{ij}\right]\rho(r')\,dV'$$

$$= \boxed{\frac{1}{8\pi\varepsilon_0 r^5}\sum_{i,j} r_i r_j \int_{V'}(3r_i' r_j' - r'^2\delta_{ij})\,\rho(r')\,dV'}$$

quadrapole expansion

$$Q_{ij} = \int_{V'}(3x_i' x_j' - r'^2\delta_{ij})\,\rho(r')\,dV'$$ quadrapole tensor

$\boxed{Q_{ij} = Q_{ji}}$, $\boxed{Q_{ii} = 0}$

$$Q_{xx} = \int (3x'^2 - r'^2)\,\rho(r')\,dV'$$

$$Q_{xy} = \int 3x'y'\,\rho(r')\,dv'$$

Problem → Dipole and Quadrapole Terms due to uniform spherical shell.

$\cancel{\sigma = \frac{q}{4\pi R^2}}$

(i) $\cancel{\bar{p} = \int_A R\sigma\, dA}$

$\cancel{\bar{p} = 4\pi\sigma R^3 = qR}$

(i) $$P = \int r'\rho(r')\,d^3r' = \sigma\int r'\,\delta(r'-R)\,d^3r'$$

$$= \sigma\int r'^3\,\delta(r'-R)\sin\theta\,dr'\,d\theta\,d\phi$$

$$= \sigma R^3\int_0^{\pi}\int_0^{2\pi}\sin\theta\,d\theta\,d\phi = 0$$

(ii) $$Q_{ij} = \int (3x_i' x_j' - r'^2 \delta_{ij}) \rho(r')\, dV'$$

$$= \sigma \int (3x_i' x_j' - r'^2 \delta_{ij})\, \delta(r' - R)\, dV'$$

$$x_i' x_j' = r'^2 \hat{r}_i' \hat{r}_j'$$

$$\Rightarrow Q_{ij} = \sigma \int_{V'} r'^2 (3\hat{r}_i \hat{r}_j - \delta_{ij})\, \delta(r' - R)\, dV'$$

$$= \underbrace{\sigma \int_0^\infty r'^4 \delta(r' - R)\, dr'}_{R^4} \int_0^\pi \int_0^{2\pi} (3\hat{r}_i' \hat{r}_j' - \delta_{ij}) \sin\theta\, d\theta\, d\phi$$

$$\cancel{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (3\hat{r}_i \hat{r}_j \ldots) \sin\theta\, d\theta\, d\phi}$$

$$\cancel{r_i r_j = \cos^2\theta,\ \sin^2\phi \cos^2\theta,\ \sin^2\phi \sin^2\theta,\ \sin\phi\cos\phi\cos\theta,\ \sin^2\phi\cos\theta\sin\theta,\ \sin\phi\cos\phi\sin\theta}$$

~~when $i \neq j$, integral $= 0$~~

$$\cancel{\Rightarrow \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \hat{r}_i \hat{r}_j \sin\theta\, d\theta\, d\phi = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \delta_{ij} \hat{r}_i \hat{r}_j \sin\theta\, d\theta\, d\phi}$$

$$\cancel{\int_{\phi=0}^{\pi} \cos^2\phi\, d\phi = \int_0^\pi (2\cos 2\phi + 1)\, d\phi}$$

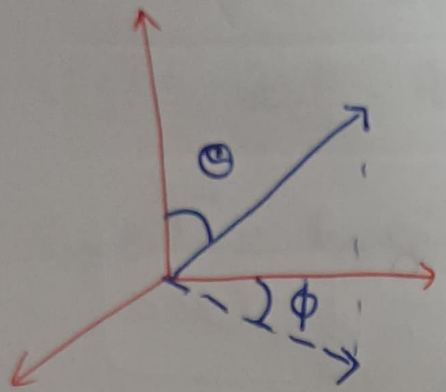
$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \hat{r}_i \hat{r}_j \sin\theta\, d\theta\, d\phi = \int_0^{2\pi} \int_0^{\pi} \delta_{ij} \hat{r}_i \hat{r}_j \sin\theta\, d\theta\, d\phi$$

$$= 4\pi/3\ \delta_{ij}$$

$$\int_0^\pi \int_0^{2\pi} (3r_i r_j - \delta_{ij}) \sin\theta\, d\theta\, d\phi = \left(\frac{4\pi}{3} \cdot 3 - 4\pi\right) \delta_{ij} = 0$$

$$\Rightarrow \boxed{Q_{ij} = 0}$$

Radial Notations



$\phi : 0 \to 2\pi$

$\theta : 0 \to \pi$

$dV = r^2 \sin\theta \, d\theta \, d\phi$

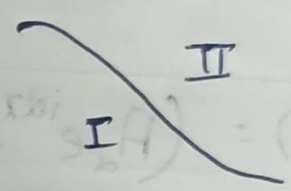
$x = \sin\theta \cos\phi$

$y = \sin\theta \sin\phi$

$z = \cos\theta$

Electrostatic Boundary Conditions

① Dirichlet Boundary Condition



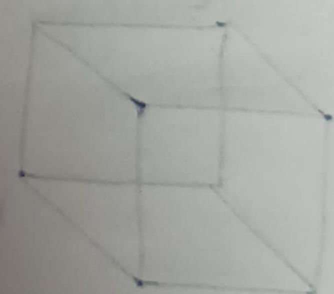
$$\phi_1 = \phi_2 \Big|_{I \to II}$$

potential is continuous as we move from region I to II

② Neumann Boundary Condition

$$(\bar{E}_2 - \bar{E}_1) \cdot \hat{n} = \sigma / \varepsilon_0$$, σ = surface charge density

Laplace's eq^n solutions



$$\bar{\nabla} \cdot \bar{E} = \rho / \varepsilon_0$$

$$\bar{\nabla} \times \bar{E} = 0 \quad \Rightarrow \quad \bar{E} = -\bar{\nabla}\phi$$

$$\Rightarrow \boxed{\nabla^2 \phi = -\rho/\varepsilon_0}$$ poisson's equation

where $\rho = 0$

$$\boxed{\nabla^2 \phi = 0}$$ Laplace's equation

where $\phi(x) = \displaystyle\int \frac{\rho(x')}{|x - x'|} d^3x'$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

let $\phi(x, y, z) = X(x) Y(y) Z(z)$

$$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

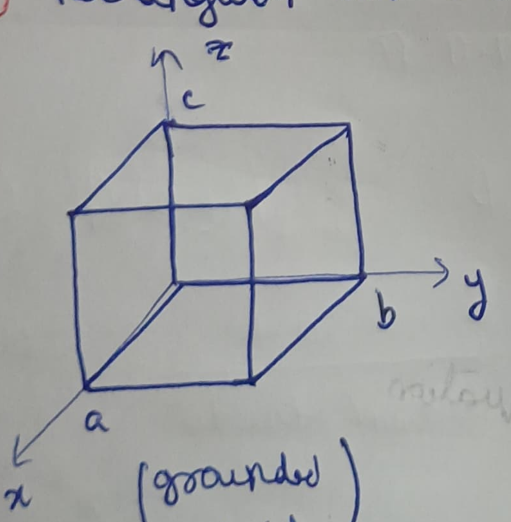
let $\frac{\partial^2 X}{\partial x^2} = -\alpha^2 X \quad \frac{\partial^2 Y}{\partial y^2} = -\beta^2 Y \quad \frac{\partial^2 Z}{\partial z^2} = +\gamma^2 Z$

where $\gamma = \sqrt{\alpha^2 + \beta^2}$

$$\Rightarrow \boxed{\phi(x, y, z) = \left(A_\alpha e^{i\alpha x} + B_\alpha e^{-i\alpha x}\right)\left(C_\beta e^{i\beta y} + D_\beta e^{-i\beta y}\right)\left(F_{\alpha\beta} e^{\gamma z} + G_{\alpha\beta} e^{-\gamma z}\right)}$$

Problems

① Rectangular Box (a, b, c)



(grounded cuboid)

$\phi(a, y, z) = \phi(x, b, z) = \phi(x, y, c) = 0$

and

$\phi(0, y, z) = \phi(x, 0, z) = \phi(x, y, 0) = 0$

$A_\alpha + B_\alpha = 0$

$C_\beta + D_\beta = 0$

$F_{\alpha\beta} + G_{\alpha\beta} = 0$

$A_\alpha e^{i\alpha a} + B_\alpha e^{-i\alpha a} = 2i \sin(\alpha a) = 0$

$\Rightarrow \alpha = n\pi / a$

$$X(x) = \sin\left(\frac{n\pi x}{a}\right) (2iA_\alpha)$$

$$Y(y) = 2iC_\beta \sin\left(\frac{m\pi x}{b}\right)$$

$$F_{\alpha\beta} e^{\gamma c} - F_{\alpha\beta} e^{-\gamma c} = 0 \Rightarrow 2F_{\alpha\beta} \sinh(\gamma c) = 0$$

$$\Rightarrow Z(z) = 2F_{\alpha\beta} \sinh\left(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\, z\right)$$

$$\boxed{\phi(x,y,z) = \sum_{m,n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \sinh(\lambda_{mn} z)}$$

$$\boxed{\lambda_{mn} = \pi\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}}$$

case ①: $\phi(x,y)\big|_{z=0} = 0 \Rightarrow A_{mn} = 0 \;\forall m,n$

case ②: If $(x,y)\big|_{z=c} = f(x,y)$

$$\rightarrow \phi(x,y,z) = \sum_{m,n} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \sinh(\lambda_{mn} c)$$

<u>Orthogonality Condition</u>

$$\boxed{\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \frac{a}{2}\delta_{mn}}$$

$$\int_0^b\int_0^a \phi(x,y,z) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dx\, dy$$

$$= \int_0^b \int_0^a \sum_{mn} A_{nm} \underbrace{\sin^2\left(\frac{n\pi x}{a}\right)}_{a/2} \underbrace{\sin^2\left(\frac{m\pi y}{b}\right)}_{b/2} \sinh(\lambda_{mn} c)\, dx\, dy$$

$$\Rightarrow \boxed{A_{nm} = \frac{4}{ab} \frac{\int_0^a \int_0^b f(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dx\,dy}{\sinh(\lambda_{nm} c)}}$$

Problems

① $f(x,y) = \Phi_0$

$A_{nm} = 0$ if n is even or m is even

when both m, n are odd

$\cancel{A_{nm} = \frac{4\Phi_0}{ab \sinh(\lambda_{nm} c)}}$

$$A_{nm} = \frac{4\Phi_0}{ab \sin(\lambda_{nm} c)} \frac{ab}{\pi^2 mn} \times 4$$

$$= \frac{16\Phi_0}{\pi^2 mn \sin(\lambda_{mn} c)}$$

② $\Phi(x,y)\big|_{z=c} = \Phi_0 \sin(\pi x/a) \sin(\pi y/b)$

$$A_{nm} = \frac{4\Phi_0}{ab \sinh(\lambda_{nm} c)} \int_0^a \int_0^b \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \sin\left(\frac{\pi m y}{b}\right) dx\,dy$$

$$= \frac{4\Phi_0}{ab \sinh(\lambda_{nm} c)} \frac{a}{2} \cdot \frac{b}{2}, \quad m = n = 1$$

$$\boxed{A_{11} = \frac{\Phi_0}{\pi \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{1/2}}}$$

③ $\phi(x,y)\big|_{z=c} = \phi_0 \cos^2(\pi x/a)$

$$\int_0^a\int_0^b \cos^2\left(\frac{\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx\,dy$$

for even m, $A_{nm} = 0$

for odd m:

$$\frac{-b}{m\pi}\cos\left(\frac{m\pi y}{b}\right)\Bigg|_0^b \int_0^a \frac{1}{2}\left(1+\cos(2\pi x/a)\right)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2b}{m\pi}\,\frac{1}{2}\int_0^a\left[\sin\left(\frac{n\pi x}{a}\right)+\underbrace{\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{2\pi x}{a}\right)}\right]dx$$

$$\sin A\cos B = \left(\sin(A+B)+\sin(A-B)\right)/2$$

$$= \frac{1}{2}\int_0^a \sin\left(\frac{(n+2)\pi x}{a}\right)+\sin\left(\frac{(n-2)\pi x}{a}\right)dx$$

for even n, $A_{nm} = 0$

for odd n:

$$= \frac{b}{m\pi}\left[\frac{a}{n\pi}(2)+\frac{2a/2}{(n+2)\pi}+\frac{2a/2}{(n-2)\pi}\right]$$

$$\boxed{= \frac{ab}{m\pi^2}\left[\frac{2}{n}+\frac{1}{n+2}+\frac{1}{n-2}\right]}$$

for odd m, n

$$\boxed{A_{mn} = \frac{4}{m\pi^2}\left[\frac{2}{n}+\frac{1}{n+2}+\frac{1}{n-2}\right]\frac{1}{\sinh(\lambda_{nm}c)}}$$

Laplace Equation in spherical polar coordinates

$$\nabla^2 \Phi(r, \theta, \phi) = 0$$

$$\Phi(r, \theta, \phi) = R(r)\, \Theta(\theta)\, \phi(\phi)$$

General Sol^n:

$$\boxed{\Phi(r,\theta,\phi) = \sum_{l,m} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l^m(\cos\theta) \left[C_m e^{im\phi} + D_m e^{-im\phi}\right]}$$

Azimuthal Symmetry (ϕ independent sol^n)

$$\boxed{\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l^m(\cos\theta)}$$

consider a spherical shell

(i) for $r > R$, as $r \to \infty$, $\phi(r,\theta) \not\to \infty$

$$\Rightarrow \Phi(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l^m(\cos\theta)$$

(ii) for $r < R$, as $r \to 0$, $\phi \not\to \infty$

$$\Rightarrow \phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l^m(\cos\theta)$$

For continuity:

$$\Phi(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l^m(\cos\theta) = \sum_{l=0} B_l \frac{1}{R^{l+1}} P_l^m(\cos\theta)$$

$$\Rightarrow A_l R^l = B_l / R^{l+1} \Rightarrow \boxed{B_l = A_l R^{2l+1}}$$

Orthogonality condition of legendre polynomials :

$$\boxed{\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = \frac{2}{2l+1} \delta_{l,l'}}$$

Apply this to $\Phi(R,\theta)$

$$\int_0^{\pi} \Phi(R,\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = \sum_{l=0}^{\infty} \int_0^{\infty} A_l R^l P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta d\theta$$

$$\Rightarrow \quad = \sum_{l=0}^{\infty} A_l R^l \frac{2}{2l'+1} \delta_{ll'} = A_{l'} R^{l'} \frac{2}{2l'+1}$$

$$\Rightarrow \boxed{\left[\int_0^{\pi} \Phi(R,\theta) P_l(\cos\theta) \sin\theta \, d\theta\right] \frac{(2l+1)}{R^l 2} = A_l}$$

$$B_l = A_l R^{2l+1}$$

Problems

① $\Phi(R,\theta) = 0 \Rightarrow A_l = B_l = 0$

② $\Phi(R,\theta) = \Phi_0$

$$\Rightarrow A_l = \frac{\Phi_0 (2l+1)}{R^l . 2} \int_0^{\pi} P_l(\cos\theta) \sin\theta d\theta$$

$$A_0 = \Phi_0 \qquad B_0 = \Phi_0 R^{2l+1} = \Phi_0 R$$

$$A_1 = A_2 = \cdots = A_n = 0$$

③ $\Phi(R,\theta) = \Phi_0 (1 - \cos\theta)$

$$\Rightarrow \phi(R,\theta) = \Phi_0 (P_0(\cos\theta) - P_1(\cos\theta))$$

$$A_l = \frac{(2l+1)\,\phi_0}{2\,R^l} \int_0^{\pi} \left[P_0(\cos\theta) - P_1(\cos\theta) \right] P_l(\cos\theta) \sin\theta \, d\theta$$

$$A_0 = \frac{\phi_0}{2} \cdot 2 = \phi_0 \quad , \quad B_0 = R\phi_0$$

$$A_1 = -\frac{3\phi_0}{2R} \cdot \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$$

$$= +\frac{3\phi_0}{2R} \Big|_0^{\pi} \frac{\cos^3\theta}{3} = -\frac{\phi_0}{R}$$

$$A_1 = -\phi_0 / R \qquad B_1 = -R^2 \phi_0$$

$$\phi(r<R, \phi) = \phi_0 - \frac{\phi_0}{R} r \cos\theta$$

$$\phi(r>R, \phi) = \frac{R\phi_0}{r} - \frac{R^2\phi_0}{r^2} \cos\theta$$

Tutorial (15/1/2026)

① $\nabla^2 \Phi = 0$ (2D, cartesian)

$\Phi(0,y) = \Phi(a,y) = \Phi(x,0) = 0$

$\Phi(x,b) = \Phi(x)$

(i) find general solution

(ii) solve for $\Phi(x) = \Phi_0 \sin(\pi x/a)$

and $\Phi(x) = \Phi_0 (x/a)$

$\Phi(x) = \Phi_0$

Ans - $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ $\phi(x,y) = X(x) Y(y)$,

$\frac{\partial^2 X}{\partial x^2} = -\alpha^2 X$, $\frac{\partial^2 Y}{\partial y^2} = +\alpha^2 Y$

$X = A_\alpha e^{i\alpha x} + B_\alpha e^{-i\alpha x}$, $Y = C_\alpha e^{\alpha y} + D_\alpha e^{-\alpha y}$

$\phi(x,y) = (A_\alpha e^{i\alpha x} + B_\alpha e^{-i\alpha x})(C_\alpha e^{\alpha y} + D_\alpha e^{-\alpha y})$

$A_\alpha = -B_\alpha$ $A_\alpha e^{i\alpha a} - A_\alpha e^{-i\alpha a} = 0$

$\Rightarrow$ ~~$A_\alpha \sinh(\alpha a) = 0$~~

$\Rightarrow 2A_\alpha i \sin(\alpha a) = 0$

$\Rightarrow \boxed{\alpha = \pi n/a}$ $Y = \sinh(n\pi y/a)$

$\Rightarrow \boxed{X = 2A_\alpha i \sin(n\pi x/a)}$

$C_\alpha = -D_\alpha$, $C_\alpha (e^{\alpha b} + e^{-\alpha b}) = 1$

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

orthogonality

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \frac{a}{2} \delta_{mn}$$

$$\Rightarrow \int_0^a \phi(x,y) \sin\left(\frac{m\pi x}{a}\right) dx = \sum_{n=1}^{\infty} \int_0^a A_n \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) dx$$

$$\Rightarrow \int_0^a \phi(x,y) \sin\left(\frac{m\pi x}{a}\right) dx = \frac{A_m a}{2} \sinh\left(\frac{n\pi y}{a}\right)$$

$$\Rightarrow \boxed{A_n = \frac{2}{a} \frac{\int_0^a \phi(x) \sin\left(\frac{n\pi x}{a}\right) dx}{\sinh(n\pi b/a)}}$$

(i) $\phi(x) = \phi_0 \sin(\pi x/a)$

$$A_n = \frac{2\phi_0}{a \sinh(n\pi b/a)} \int_0^a \sin(\pi x/a) \sin(n\pi x/a)\, dx$$

$$\Rightarrow A_1 = \frac{\phi_0}{\sinh(\pi b/a)}$$

$$\boxed{\phi(x,y) = \phi_0 \frac{\sinh(\pi y/a)}{\sinh(\pi b/a)} \sin\left(\frac{\pi x}{a}\right)}$$

② consider a shell : $\nabla^2 \phi = 0$

$\phi(R, \theta \in (0, \pi/2)) = +\Phi_0$

$\phi(R, \theta \in (\pi/2, \pi)) = -\Phi_0$

then find :

(i) $\phi(r,\theta)\big|_{r<R}$ (ii) $\phi(r,\theta)\big|_{r>R}$

$$\phi(r,\theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$$A_l = \frac{(2l+1)}{2R^l}\int_0^{\pi} \phi(R,\theta)\, P_l(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{(2l+1)\Phi_0}{2R^l}\left[\int_0^{\pi/2} P_l(\cos\theta)\sin\theta - \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\right]$$

$\cancel{A_0 = \frac{\Phi_0}{2}\times 2 = \Phi_0}$ $\cancel{A_0 = \frac{\Phi_0}{2}((0-1)-(-1)-0) = 0}$

$\cancel{A_1 = \frac{3\Phi_0}{2R\times 2}(-\cos\pi + \cos 0)}$

$$= \frac{(2l+1)\Phi_0}{2R^l}\int_0^{\pi/2} \sin\theta\,[P_l(\cos\theta) \mp P_l(-\cos\theta)]\, d\theta$$

when l is odd, $A_l \neq 0$, but when even, $A_l = 0$

$$\boxed{A_l = \frac{(2l+1)\Phi_0}{R^l}\int_0^{\pi/2} \sin\theta . P_l(\cos\theta)\, d\theta}$$

③ Potential due to uncharged spherical conductor placed in uniform field $\bar{E} = E_0 \hat{z}$ in region outside sphere :

Ans - $\phi(r,\theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$

when $r \to \infty$, $\phi(r,\theta) = -\int_\infty^{\vec{r}} E \cdot dl = -E\cos\theta\, r$

$\cancel{\phi(r\to\infty,\theta) = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta) = -E\cos\theta\, r}$

$\Rightarrow$

$A_l r^l P_l(\cos\theta) = -E_0 r \cos\theta$

$\Rightarrow A_1 = -E_0$

$\Rightarrow \phi(r,\theta) = \left(-E_0 r + \frac{B}{r^2} \right) \cos\theta$

at $r = R$, $\phi(r,\theta) = $ const

$\Rightarrow -E_0 R + \frac{B}{R^2} = 0 \Rightarrow B = E_0 R^3$

$$\Rightarrow \boxed{\phi(r,\theta) = \left(-E_0 r + \frac{E_0 R^3}{r^2} \right) \cos\theta}$$