PHY626 - Assignment 2

Date:

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15) ~~$H = \frac{1}{2M}$~~   Homework 1

Ans 15: Aharonov-Bohm Ring

a)

$$\vec{H} = \frac{1}{2M}(\vec{p} - q\vec{A})^2 \quad , \quad \vec{p} = -i\hbar\nabla = \frac{-i\hbar}{R}\frac{\partial}{\partial\phi}\hat{\phi}$$

$$A|_{r=R} = (BR/2)\,\hat{\phi}$$

$$\Rightarrow \vec{H}_0 = \frac{1}{2M}\left(\frac{-i\hbar}{R}\frac{\partial}{\partial\phi}\hat{\phi} - \frac{eBR}{2}\hat{\phi}\right)^2$$

$$= \frac{\hbar^2}{2MR^2}\left(-i\frac{\partial}{\partial\phi}\hat{\phi} - \frac{eBR^2}{2\hbar}\hat{\phi}\right)^2$$

$$\Phi = \pi R^2 B \qquad \phi_0 = h/e = 2\pi\hbar/e$$

$$\Rightarrow \frac{eBR^2}{2\hbar} = \frac{e\Phi}{e\phi_0} = \Phi/\phi_0$$

$$\Rightarrow \boxed{H_0 = \frac{\hbar^2}{2MR^2}\left(i\frac{\partial}{\partial\phi} + \frac{\Phi}{\phi_0}\right)^2}$$

b) For a solenoid : $\vec{A} = \frac{\Phi_s}{2\pi r}\hat{\phi} = \frac{\Phi_s}{2\pi R}\hat{\phi}$

$$H_0 = \frac{\hbar^2}{2MR^2}\left(i\frac{\partial}{\partial\phi} + \frac{e\Phi_s}{2\pi\hbar}\right)^2 = \boxed{\frac{\hbar^2}{2MR^2}\left(i\frac{\partial}{\partial\phi} + \frac{\Phi_s}{\phi_0}\right)^2}$$

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(c) $H_0 \psi(\phi) = E\psi(\phi)$

$$\Rightarrow \frac{\hbar^2}{2MR^2}\left(i\frac{\partial}{\partial\phi} + \frac{\Phi}{\phi_0}\right)^2 \psi(\phi) = \cancel{E} \, E\psi(\phi)$$

Consider ansatz: $\psi(\phi) = Ce^{im\phi}$

Periodic Boundary: $\psi(\phi + 2\pi) = \psi(\phi)$

$$\Rightarrow e^{2m\pi i} = 1 \Rightarrow m \in \mathbb{Z}$$

$$\int_0^{2\pi} |\psi(\phi)|^2 \, d\phi = 1 \Rightarrow |c|^2 \int_0^{2\pi} d\phi = 1 \Rightarrow |C| = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \boxed{\langle \phi | m \rangle = \frac{1}{\sqrt{2\pi}} e^{im\phi}}, \quad m \in \mathbb{Z}$$

$$\frac{\hbar^2}{2MR^2}\left(i\frac{\partial}{\partial\phi} + \frac{\Phi}{\phi_0}\right)^2 \frac{1}{\sqrt{2\pi}} e^{im\phi} = \frac{\hbar^2}{2\sqrt{2}MR^2\sqrt{\pi}}\left(i\frac{\partial}{\partial\phi} + \frac{\Phi}{\phi_0}\right) \times \left(i^2 m e^{im\phi} + \frac{\Phi}{\phi_0} e^{im\phi}\right)$$

$$= \frac{\hbar^2}{2\sqrt{2}MR^2\sqrt{\pi}}\left(i^2 m + \frac{\Phi}{\phi_0}\right)\left(i^2 m + \frac{\Phi}{\phi_0}\right) e^{im\phi}$$

$$\Rightarrow \boxed{E_m = \frac{\hbar^2}{2MR^2}\left(\frac{\Phi}{\phi_0} - m\right)^2}$$

let $E_m = y$

$\Phi / \phi_0 = x$

$\mathcal{E}_0 = \hbar^2 / 2MR^2$

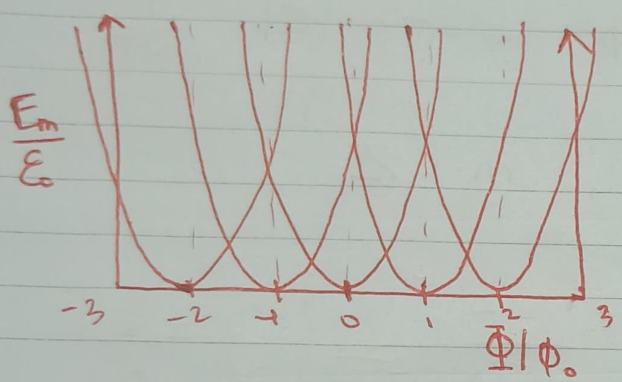
$$\Rightarrow y = \mathcal{E}_0 (x - m)^2$$

~~(d) Ground State : m_g = $\min\{|\lfloor\Phi/\phi_0\rfloor - \Phi/\phi_0|, |\lceil\Phi/\phi_0\rceil - \Phi/\phi_0|\}$~~

~~First excited : min {~~

(d) Ground state : $m_g \in \mathbb{Z}$ closest to Φ/ϕ_0

1^{st} excited : $m_e \in \mathbb{Z}$ for which $|\Phi/\phi_0 - m_e|$ is second lowest $\forall m$



(e) $E_m^{(0)} = \varepsilon_0 (m - \Phi/\phi_0)^2$

given : $\Phi/\phi_0 = (m + 1/2) + \delta$

$\Rightarrow E_m^{(0)} = \varepsilon_0 (\delta + 1/2)^2$, $|m\rangle = e^{im\phi}/\sqrt{2\pi}$

$E_{m+1}^{(0)} = \varepsilon_0 (\delta - 1/2)^2$, $|m+1\rangle = e^{i(m+1)\phi}/\sqrt{2\pi}$

(f) $|m\rangle$, $|m+1\rangle$ are exact eigenstates

$\Rightarrow \langle m+1|H_0|m\rangle = \langle m|H_0|m+1\rangle = 0$

$$H_0 = \begin{bmatrix} \langle m|H_0|m\rangle & \langle m|H_0|m+1\rangle \\ \langle m+1|H_0|m\rangle & \langle m+1|H_0|m+1\rangle \end{bmatrix} = \begin{bmatrix} E_m^{(0)} & 0 \\ 0 & E_{m+1}^{(0)} \end{bmatrix}$$

$$\Rightarrow H_0 = \varepsilon_0 \begin{bmatrix} (\delta+1/2)^2 & 0 \\ 0 & (\delta-1/2)^2 \end{bmatrix}$$

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$\Rightarrow H_0 = \varepsilon_0 \left[(\delta^2 + 1/4)\sigma_0 + \delta \sigma_z \right]$ $\quad \delta << 1$

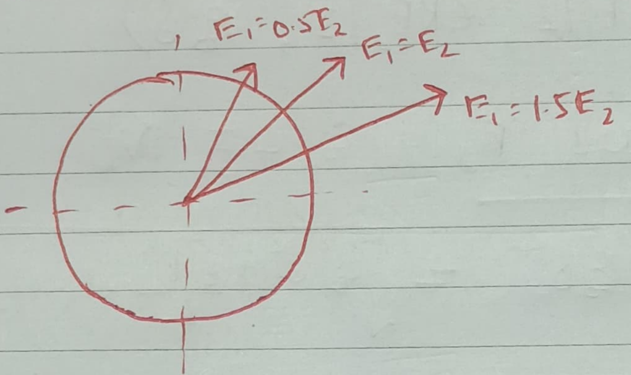
$\Rightarrow H_0 = \varepsilon_0 \left[\sigma_0/4 + \delta \sigma_z \right]$

(g) $\bar{E} = E_1 \hat{i} + E_2 \hat{j}$

(i) $E_1 = E_2 \Rightarrow \theta = \tan^{-1}(1) = 45^\circ$

(ii) $E_1 = 0.5 E_2 \Rightarrow \theta = \tan^{-1}(2)$

(iii) $E_1 = 1.5 E_2 \Rightarrow \theta = \tan^{-1}(2/3)$



$H_1 = -\bar{p} \cdot \bar{E}$, $\bar{p} = q\bar{r} = e\bar{r}$

$\Rightarrow H_1 = -(eR\cos\phi\,\hat{i} + eR\sin\phi\,\hat{j}) \cdot (E_1\hat{i} + E_2\hat{j})$

$= -eR(E_1 \cos\phi + E_2 \sin\phi)$

$\cos\phi = (e^{i\phi} + e^{-i\phi})/2$ $\quad \sin\phi = (e^{i\phi} - e^{-i\phi})/2i$

$\Rightarrow H_1 = -\frac{eR}{2} \left[e^{i\phi}(E_1 - iE_2) + e^{-i\phi}(E_1 + iE_2) \right]$

$\langle m | H_1 | m \rangle \propto \int_0^{2\pi} e^{-im\phi} (e^{\pm i\phi}) e^{im\phi} \, d\phi = 0$

$\langle m | H_1 | m+1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} H_1 e^{i\phi} \, d\phi = \frac{-eR}{2}(E_1 + iE_2)$

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$$\Rightarrow H_1 = \frac{-eR}{2}\begin{bmatrix} 0 & E_1 + iE_2 \\ E_1 - E_2 i & 0 \end{bmatrix} = \frac{-eR}{2}(\sigma_x E_1 - \sigma_y E_2)$$

(h) $H = H_0 + H_1 = \frac{\varepsilon_0}{4}\sigma_0 - \frac{eR}{2}E_1\sigma_x + \frac{eR}{2}E_2\sigma_y + \varepsilon_0\delta\sigma_z$

$$= \varepsilon_0\sigma_0/4 + \bar{V}\cdot\bar{\sigma}$$

where $\bar{V} = \cancel{(-eR/2)}\cdot(-eRE_1/2,\ eRE_2/2,\ \varepsilon_0\delta)$

eigenvalues of $a_0 I + \bar{V}\cdot\bar{\sigma}$ are $E_\pm = \varepsilon/4 \pm |V|$

$$|V| = \sqrt{\frac{e^2R^2E_1^2}{4} + \frac{e^2R^2E_2^2}{4} + \varepsilon_0^2\delta^2} = \sqrt{\varepsilon_0^2\delta^2 + \frac{e^2R^2}{4}(E_1^2 + E_2^2)}$$

$$\Rightarrow E_\pm = \boxed{\frac{\varepsilon_0}{4} \pm \sqrt{\varepsilon_0\delta^2 + \frac{e^2R^2E^2}{4}}}$$

$$|\psi_+\rangle = \cos(\theta/2)|m\rangle + \sin(\theta/2)e^{i\alpha}|m+1\rangle$$

$$|\psi_-\rangle = \sin(\theta/2)|m\rangle - \cos(\theta/2)e^{-i\alpha}|m+1\rangle$$

$$\theta = \arctan(eRE/2\varepsilon_0\delta)$$

$$e^{-i\alpha} = -(E_1 - iE_2)/E$$

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(ii) ~~Late~~ Level crossing happens when $\delta = 0$

$$\Rightarrow E_\pm = \frac{\varepsilon_0}{4} \pm \frac{eRE}{2} \Rightarrow \Delta E = eRE = \boxed{eR\sqrt{E_1^2 + E_2^2}}$$

independent of m.

Ans16 - Jaynes Cummings model

$$H_{JC} = H_0 + H_{int}$$

$$H_0 = \hbar\omega_c(a^\dagger a + 1/2) + \hbar\omega_a\sigma_z/2$$

$$H_{int} = (\hbar g/2)(a\sigma_+ + a^\dagger\sigma_-)$$

(a) at resonance, $\omega_c = \omega_a = \omega \Rightarrow H_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2} + \frac{\sigma_z}{2}\right)$

$|e\rangle = (1\ 0)^\dagger \qquad |g\rangle = (0\ 1)^\dagger$

$$\sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad \sigma_+ = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \cancel{|g\rangle\langle e|}\ |e\rangle\langle g|$$

$$\sigma_- = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = \cancel{|e\rangle\langle g|}\ |g\rangle\langle e|$$

$$\sigma_+\sigma_- = \cancel{|g\rangle\langle g|}\ |e\rangle\langle e| = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \quad \sigma_-\sigma_+ = |g\rangle\langle g| = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$H_0 = \hbar\omega(a^\dagger a + \sigma_+\sigma_-) = \hbar\omega N, \quad N = a^\dagger a + \sigma_+\sigma_-$$

(b) $\cancel{[H_0, H_{int}] = \frac{\hbar^2\omega g}{2}\left[(a^\dagger a + \sigma_+\sigma_-)(a\sigma_+ + a^\dagger\sigma_-) - (a\sigma_+ + a^\dagger\sigma_-)(a^\dagger a + \sigma_+\sigma_-)\right]}$

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(b) $[H_0, H_{int}] = \frac{\hbar^2 \omega g}{2} [a^\dagger a + \sigma_+ \sigma_-, a\sigma_+ + a^\dagger \sigma_-]$

$\propto [a^\dagger a, a\sigma_+ + a^\dagger \sigma_-] + [\sigma_+\sigma_-, a\sigma_+ + a^\dagger\sigma_-]$
$= [a^\dagger a, a\sigma_+] + [a^\dagger a, a^\dagger \sigma_-] + [\sigma_+\sigma_-, a\sigma_+] + [\sigma_+\sigma_-, a^\dagger\sigma_-]$
$= [a^\dagger a, a]\sigma_+ + [a^\dagger a, a^\dagger]\sigma_- + a[\sigma_+\sigma_-, \sigma_+] + [\sigma_+\sigma_-, \sigma_-]a^\dagger$
$= -a\sigma_+ + a^\dagger\sigma_- + a\sigma_+ - a^\dagger\sigma_- = 0$

$\Rightarrow [H_0, H_{int}] = 0$

$[H_0, H_{JC}] = [H_0, H_0 + H_{int}] = [H_0, H_0] + [H_0, H_{int}] = 0$

(c) $[N, H_{JC}] = 0 \rightarrow [N, H_0] + [N, H_{int}] = 0$ (To prove)

$\Rightarrow [a^\dagger a + \sigma_+\sigma_-, H_0] + [a^\dagger a + \sigma_+\sigma_-, H_{int}] = 0$

(i) $[N, H_0] = \hbar\omega_c [a^\dagger a + \sigma_+\sigma_-, a^\dagger a] + \frac{\hbar\omega_a}{2}[a^\dagger a + \sigma_+\sigma_-, \sigma_z]$

$= \hbar\omega_c [\sigma_+\sigma_-, a^\dagger a] + \frac{\hbar\omega_a}{2}\{[a^\dagger a, \sigma_z] + [\sigma_+\sigma_-, \sigma_z]\}$

$= 0 + \frac{\hbar\omega_a}{2}\{0 + |e\rangle\langle e|(|e\rangle\langle e| - |g\rangle\langle g|) - (|e\rangle\langle e| - |g\rangle\langle g|)|e\rangle\langle e|\}$

$= |e\rangle\langle e| - 0 - |e\rangle\langle e| + 0 = \boxed{0}$

(ii) $[N, H_{int}] \propto [a^\dagger a + \sigma_+\sigma_-, a\sigma_+ + a^\dagger\sigma_-]$

$= [a^\dagger a, a]\sigma_+ + [a^\dagger a, a^\dagger]\sigma_- + \dots$

$= 0$ (proved in part (b))

QED

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(d) $H_0 = \hbar\omega\mathcal{N} = \hbar\omega(a^\dagger a + \sigma_+\sigma_-)$

(i) $|\chi_n^+\rangle = |n,e\rangle$

$H_0|n,e\rangle = \hbar\omega(a^\dagger a + \sigma_+\sigma_-)|n,e\rangle$

$= \hbar\omega(a^\dagger a|n\rangle \otimes |e\rangle + |n\rangle \otimes \sigma_+\sigma_-|e\rangle)$

$= \hbar\omega(n|n,e\rangle + |n,e\rangle)$

$= \hbar\omega(n+1)|n,e\rangle$

$\Rightarrow$ eigenvalue : $\hbar\omega(n+1)$
state is an eigenvector

(ii) $|\chi_n^-\rangle = |n+1,g\rangle$

$H_0|n+1,g\rangle = \hbar\omega(a^\dagger a + \sigma_+\sigma_-)|n,g\rangle$

$\cancel{= \hbar\omega((n+1)|n,e\rangle + 0)}$

$= \cancel{\hbar\omega(n+1)|n,e\rangle}\ \cancel{\hbar\omega(n+1)|n+1,e\rangle}$

~~not in eigenstate~~

$= \hbar\omega(n+1)|n+1,g\rangle$

$|\chi_n^-\rangle$ and $|\chi_n^+\rangle$ are doubley degenerat

(iii) $H_0|0,g\rangle = \hbar\omega(0+0) = 0 \Rightarrow E_0 = 0$

(e) $\delta = \omega_a - \omega_c \Rightarrow \omega_a = \omega_c + \delta$

$H_0 = \hbar\omega_c(a^\dagger a + 1/2) + \hbar(\omega_c + \delta)(\sigma_z/2)$

$= \hbar\omega_c(a^\dagger a + 1/2) + \hbar\omega_c\sigma_z/2 + \hbar\delta\sigma_z/2$

$$\boxed{H_0 = \hbar\omega_c\left(a^\dagger a + \frac{1}{2} + \frac{\sigma_z}{2}\right) + \frac{\hbar\delta\sigma_z}{2}}$$

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$|\chi_n^+\rangle = |n, e\rangle \qquad |\chi_n^-\rangle = \cancel{|n, g\rangle}\ |n+1, g\rangle$

$$H_0|\chi_n^+\rangle = \hbar\omega_c\left(n+\frac{1}{2}+\frac{1}{2}\right)|n,e\rangle + \frac{\hbar\delta}{2}|n,e\rangle$$

$$= \left[\hbar\omega_c(n+1) + \frac{\hbar\delta}{2}\right]|\chi_n^+\rangle$$

$$H_0|\chi_n^-\rangle = \left[\hbar\omega_c\left(n+1+\frac{1}{2}-\frac{1}{2}\right) - \frac{\hbar\delta}{2}\right]|n+1, g\rangle$$

$$= (\hbar\omega_c(n+1) - \hbar\delta/2)|\chi_n^-\rangle$$

$$E_\pm = (\hbar\omega_c(n+1) \pm \hbar\delta/2) \qquad \text{degeneracy removed}$$

(f) $\langle\chi_n^+|H_{JC}|\chi_n^+\rangle = (n+1)\hbar\omega_c + \hbar\delta/2$

$\langle\chi_n^-|H_{JC}|\chi_n^-\rangle = (n+1)\hbar\omega_c - \hbar\delta/2$

$\langle\chi_n^-|H_{JC}|\chi_n^+\rangle = \langle\chi_n^-|H_{int}|\chi_n^+\rangle = \frac{\hbar g}{2}\sqrt{n+1}$

$$\Rightarrow [H_{JC}]_n = \hbar\begin{bmatrix}(n+1)\omega_c + \delta/2 & g/2\sqrt{n+1} \\ g/2\sqrt{n+1} & (n+1)\omega_c - \delta/2\end{bmatrix}$$

$$\det(H - \lambda I) = 0$$

$$\Rightarrow \boxed{E_n^\pm = (n+1)\hbar\omega_c \pm \frac{1}{2}\hbar\sqrt{\delta^2 + (n+1)g^2}}$$

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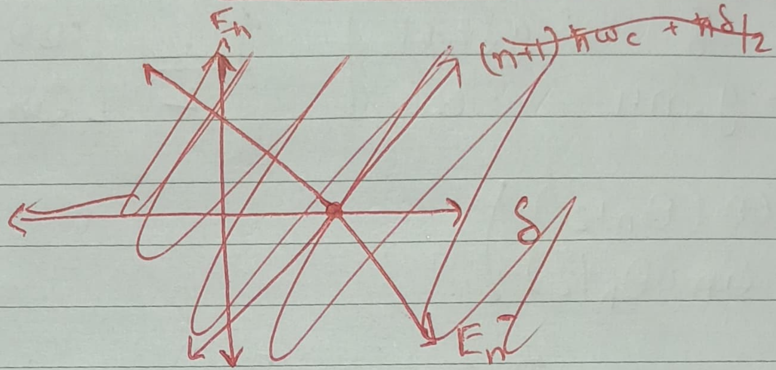
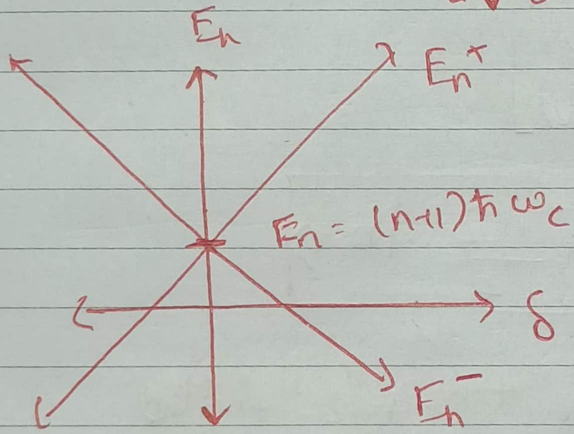
$$\boxed{\Delta E_n = \hbar g\sqrt{n+1}}$$

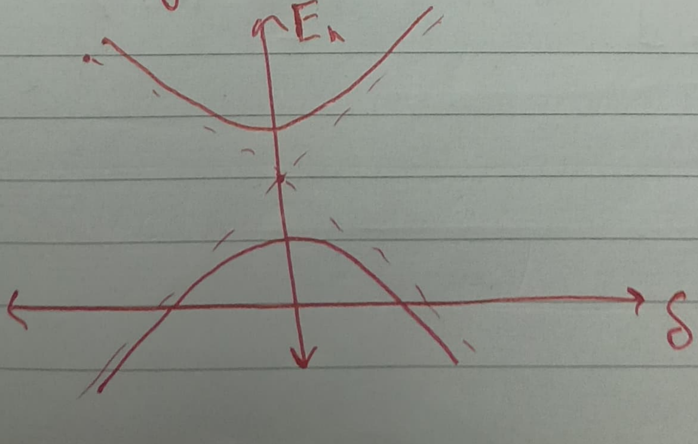
(g) (i) $g = 0 \Rightarrow$ $$\boxed{E_n^{\pm} = (n+1)\hbar\omega_c \pm \frac{\hbar\delta}{2}}$$





(ii) $g \neq 0$, $\underbrace{E_n^{\pm}}_{y} = \underbrace{(n+1)\hbar\omega_c}_{A} \pm \frac{1}{2}\hbar\sqrt{\underbrace{\delta^2}_{x} + \underbrace{(n+1)g^2}_{B}}$

$\Rightarrow 4(y-A)^2 = \hbar(x^2 + B)$



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(h) $\theta_n = \tan^{-1}(g\sqrt{n+1} / \delta)$

$\Rightarrow \sin\theta_n = \dfrac{g\sqrt{n+1}}{\Omega_n} \qquad \cos\theta_n = \dfrac{\delta}{\Omega_n}$

$$H_{sub} = \frac{\hbar}{2}\begin{bmatrix} \delta & g\sqrt{n+1} \\ g\sqrt{n+1} & -\delta \end{bmatrix} = \frac{\hbar\Omega_n}{2}\begin{bmatrix} \cos\theta_n & \sin\theta_n \\ \sin\theta_n & -\cos\theta_n \end{bmatrix}$$

$$|n,+\rangle = \begin{pmatrix} \cos(\theta_n/2) \\ \sin(\theta_n/2) \end{pmatrix}$$

$$\frac{\hbar\Omega_n}{2}\begin{bmatrix} \cos\theta_n & \sin\theta_n \\ \sin\theta_n & -\cos\theta_n \end{bmatrix}\begin{bmatrix} \cos(\theta_n/2) \\ \sin(\theta_n/2) \end{bmatrix}$$

$$= \frac{\hbar\Omega_n}{2}\begin{bmatrix} \cos(\theta_n/2) \\ \sin(\theta_n/2) \end{bmatrix} = \boxed{+\frac{\hbar\Omega}{2}|n,+\rangle}$$

$\Rightarrow |n,+\rangle$ is eigenvector for $+\hbar\Omega_n/2$ shifted energy state

$$|n,-\rangle = \begin{pmatrix} \sin(\theta_n/2) \\ -\cos(\theta_n/2) \end{pmatrix}$$

$$\frac{\hbar\Omega_n}{2}\begin{bmatrix} \cos\theta_n & \sin\theta_n \\ \sin\theta_n & -\cos\theta_n \end{bmatrix}|n,-\rangle = \frac{-\hbar\Omega}{2}|n,-\rangle$$

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(i) unperturbed Hamiltonian : $H_0 = \hbar\omega_c N + \hbar\delta\sigma_z/2$

$$E^{(0)}_{n,\pm} = (n+1)\hbar\omega_c \pm \hbar\delta/2$$

$$E^{(1)}_{n,\pm} = \langle \chi_n^{\pm} | H_{int} | \chi_n^{\pm} \rangle = 0$$

First order correction

$$E^{(2)}_{n,\pm} = \frac{|\langle \chi_n^{-} | H_{int} | \chi_n^{+} \rangle|^2}{E^{(0)}_{n,+} - E^{(0)}_{n,-}} = \frac{\left(\frac{\hbar g}{2}\sqrt{n+1}\right)^2}{\hbar\delta} = \frac{\hbar g^2 (n+1)}{4\delta}$$

$$E_{n,\pm} \approx (n+1)\hbar\omega_c \pm \frac{\hbar\delta}{2} + \frac{\hbar g^2 (n+1)}{4\delta}$$

$$E_n^{+} = (n+1)\hbar\omega_c + \frac{\hbar}{2}\sqrt{\delta^2 + g^2(n+1)}$$

$$= (n+1)\hbar\omega_c + \frac{\hbar\delta}{2}\sqrt{1 + \frac{g^2(n+1)}{\delta^2}}$$

$$\approx \boxed{(n+1)\hbar\omega_c + \frac{\hbar\delta}{2}\left(1 + \frac{g^2(n+1)}{2\delta^2}\right)}$$

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Homework -2

Ans 1

a) $H = \frac{1}{2m_e}(\bar{p}+e\bar{A})^2 + V(r) - \bar{\mu}_s \cdot \bar{B}$

$V(r) = -Ze^2 / 4\pi\varepsilon_0 r$

$(\bar{p}+e\bar{A})^2 = p^2 + e(\bar{p}\cdot\bar{A} + \bar{A}\cdot\bar{p}) + e^2A^2$

$\bar{\nabla}\cdot\bar{A} = 0 \quad , \quad \bar{p} = -i\hbar\nabla$

$\Rightarrow \bar{p}\cdot\bar{A}\psi = -i\hbar\nabla\cdot(\bar{A}\psi) = -i\hbar(\bar{\nabla}\cdot\bar{A})\psi - i\hbar\bar{A}\cdot(\nabla\psi)$

$\Rightarrow \bar{p}\cdot\bar{A}\psi = \bar{A}\cdot(-i\hbar\nabla\cdot\psi) \Rightarrow [\bar{p},\bar{A}] = 0$

$\frac{(\bar{p}+e\bar{A})^2}{2m_e} = \frac{p^2}{2m_e} + \frac{e}{m_e}\bar{A}\cdot\bar{p} + \frac{e^2A^2}{2m_e}$

$\bar{A} = \frac{1}{2}(\bar{B}\times\bar{r}) \Rightarrow \frac{eA\cdot p}{m_e} = \frac{e\bar{B}\cdot\bar{L}}{2m_e}$

$\bar{\mu}_s = -g_s\frac{e}{2m_e}S \quad , \quad g_s \approx 2$

$\Rightarrow -\bar{\mu}_s\cdot\bar{B} = \frac{e}{m_e}\bar{S}\cdot\bar{B}$

$\Rightarrow \boxed{H_{para} = \frac{e}{2m_e}(\bar{L}+2\bar{S})\cdot\bar{B}}$

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$$\bar{H}_{dia} = \frac{e^2 A^2}{2m_e} = \frac{e^2}{8m_e}(\bar{B}\times\bar{r})^2$$

$$\langle \bar{H}_{para} \rangle \sim \frac{e\hbar B}{m_e} \qquad \langle \bar{H}_{dia} \rangle \sim \frac{e^2 B^2}{8m_e}\langle r^2 \rangle$$

$$\langle r^2 \rangle \sim \frac{a_0^2 n^4}{Z^2} \quad \Rightarrow \quad \langle \bar{H}_{dia} \rangle \sim \frac{e^2 B^2 a_0^2 n^4}{m_e Z^2}$$

we require $\langle H_{dia} \rangle << \langle H_{para} \rangle$

$$\Rightarrow \frac{e^2 B^2 a_0^2 n^4}{m_e Z^2} << \frac{e\hbar B}{m_e}$$

$$\Rightarrow B << \frac{\hbar Z^2}{e a_0^2 n^4} = \frac{Z^2}{n^4} 10^6 T$$

$$\Rightarrow \boxed{B << (Z^2/n^4)\, 10^6 T}$$

(b) $$H_0 + H_B = \frac{p^2}{2m_e} - \frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{e}{2m_e}(\bar{L}+2\bar{S})\cdot\bar{B} + \frac{e^2}{8m_e}(\bar{B}\times\bar{r})^2$$

$$H_{rel} = \frac{-p^4}{8m_e^3 c^2}$$ via first order Taylor expansion

$$H_{SO} = \frac{Ze^2}{8\pi\varepsilon_0 m_e^2 c^2 r^3} \bar{L}\cdot\bar{S}$$

$$H_{Darwin} = \frac{\pi\hbar^2 Ze^2}{2\varepsilon_0 m_e^2 c^2} \delta^3(r)$$

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$$H = \frac{p^2}{2m_e} - \frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{e}{2m_e}(\bar{L}+2\bar{S})\cdot\bar{B} + \frac{e^2}{8m_e}(\bar{B}\times\bar{r})^2$$
$$- \frac{p^4}{8m_e^3c^2} + \frac{Ze^2}{8\pi\varepsilon_0 m_e^2c^2r^3}\bar{L}\cdot\bar{S} + \frac{\pi\hbar^2 Ze^2}{2\varepsilon_0 m_e^2c^2}\delta^3(r)$$

Ans 2 → Weak Field Zeeman effect

(a) Zeeman interaction energy, $E_Z \sim \mu_B B$

Fine structure energy, $E_{FS} \sim \alpha^2\left(\frac{Z^2 Ry}{n^2}\right) \approx \frac{Z^4\alpha^2 Ry}{n^3}$

~~$E_Z \sim E_{FS} \Rightarrow B_c$~~ $\qquad E_Z << E_{FS}$

let B for which $E_Z \sim E_{FS}$ be B_c

$\Rightarrow B_c \approx E_{FS}/\mu_B \Rightarrow \boxed{B << B_c \approx 1T}$

(b) "good" quantum numbers that describe the eigenstates of the unperturbed atom + fine structure are n, l, j, m_j.

(c) $\bar{\mu} = \frac{-\mu_B}{\hbar}(\bar{L}+2\bar{S})$, $\bar{L}+\bar{S}+\bar{S} = \bar{J}-\bar{S}+2\bar{S} = (\bar{J}+\bar{S})$

let $\bar{V} = \bar{J}+\bar{S}$

$$\langle jm_j|\bar{V}|jm_j\rangle = \frac{\langle jm_j|(\bar{J}+\bar{S})\cdot\bar{J}|jm_j\rangle}{j(j+1)\hbar^2}\langle jm_j|\bar{J}|jm_j\rangle$$

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now, $(\bar{J}+\bar{S})\cdot\bar{J} = \bar{J}^2 + \bar{S}\cdot\bar{J}$

$\bar{L} = \bar{J} - \bar{S} \Rightarrow L^2 = J^2 + S^2 - 2\bar{S}\cdot\bar{J} \Rightarrow \bar{S}\cdot\bar{J} = \dfrac{J^2+S^2-L^2}{2}$

$\Rightarrow (\bar{J}+\bar{S})\cdot\bar{J} = \dfrac{3J^2+S^2-L^2}{2}$

$\langle(\bar{J}+\bar{S})\cdot\bar{J}\rangle = \hbar^2\left[j(j+1) + \frac{1}{2}\left(j(j+1)+s(s+1)-l(l+1)\right)\right]$

$g_{eff} = \dfrac{j(j+1) + \frac{1}{2}\left(j(j+1)+s(s+1)-l(l+1)\right)}{j(j+1)}$

$= 1 + \dfrac{j(j+1)+s(s+1)-l(l+1)}{2j(j+1)}$

$\Rightarrow \boxed{\bar{\mu} = -g_{eff}\dfrac{\mu_B}{\hbar}\bar{J}}$

(d) $\Delta E = g_{eff}\,\mu_B B m_j$, $s = 1/2$

(i) $1S_{1/2}$: $l=0$, $j=1/2$, $m_j = \pm 1/2$

$g_{eff} = 1 + \dfrac{3/4+3/4}{3/2} = 2$

$\Delta E = 2\mu_B B m_j$

$\hookrightarrow m_j = +1/2 \Rightarrow \Delta E = \mu_B B$

$m_j = -1/2 \Rightarrow \Delta E = -\mu_B B$

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(ii) $2S_{1/2}$: $g_{eff} = 2 \Rightarrow \Delta E = \pm \mu_B B$

(iii) $2P_{1/2}$: $l = 1$, $j = 1/2$, $m_j = \pm 1/2$

$g_{eff} = 2/3 \Rightarrow \Delta E = \pm \mu_B B/3$

(iv) $2P_{3/2}$: $l = 1$, $j = 3/2$, $m_j = \pm 3/2, \pm 1/2$

$g_{eff} = 4/3$

$m_j = \pm 3/2$, $\Delta E = \pm 2\mu_B B$
$m_j = \pm 1/2$, $\Delta E = \pm \mu_B B\,(2/3)$

Answer 3 → Zeeman Splitting for arbitrary B

(a) $H = H_0 + H_{fs} + H_z$

$$H_z \rightarrow \frac{\mu_B B}{\hbar}(L_z + 2S_z)$$

Fine structure splits 8 degenerate $n=2$ states into two energy levels based on j:

(i) $E_{1/2}$: lower energy level with $2S_{1/2}$ $(l=0)$ and $2P_{1/2}$ $(l=1)$ states

(ii) $E_{3/2}$: higher energy level with $2P_{3/2}$ $(l=1)$

$D = E_{3/2} - E_{1/2}$

Date:

For uncoupled states: (1x1 blocks)

$$E_Z = \mu_B B (m_\ell + 2m_s)$$

→ $2S_{1/2}$: g-factor = 2

$$E(2S_{1/2}, m_j) = E_{1/2} + 2\mu_B B m_j$$

→ $2P_{3/2}$: $m_\ell = \pm 1$, $m_s = \pm 1/2$

$$E(2P_{3/2}, m_j = \pm 3/2) = E_{3/2} \pm 2\mu_B B$$

For coupled states (2x2 blocks):

$$|3/2, 1/2\rangle = \sqrt{\frac{2}{3}}\,|0, 1/2\rangle + \sqrt{\frac{1}{3}}\,|1, -1/2\rangle$$

$$|1/2, 1/2\rangle = \sqrt{\frac{1}{3}}\,|0, 1/2\rangle + \sqrt{\frac{2}{3}}\,|1, -1/2\rangle$$

$$H_{sub}^{(1/2)} = \begin{pmatrix} E_{3/2} + \frac{2}{3}\mu_B B & -\frac{\sqrt{2}}{3}\mu_B B \\ -\frac{\sqrt{2}}{3}\mu_B B & E_{1/2} + \frac{\mu_B B}{3} \end{pmatrix}$$

$$\det(H - I\lambda) = 0$$

$$\Rightarrow \boxed{E_{\pm}^{(m_j)} = E_{1/2} + \frac{\Delta}{2} + \mu_B B m_j \pm \frac{1}{2}\sqrt{\Delta^2 + \frac{4}{3} m_j \Delta \mu_B B + (\mu_B B)^2}}$$

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$\mu_B B << \Delta$: $(1+x)^{1/2} \approx 1 + x/2$

$$E_{\pm}^{(m_j)} \approx E_{1/2} + \frac{\Delta}{2} + \mu_B B m_j + \frac{\Delta}{2}\left(1 + \frac{2}{3} m_j \frac{\mu_B B}{\Delta}\right)$$

$$E_{+} \approx E_{3/2} + \frac{4}{3} \mu_B B m_j$$

$$E_{-} \approx E_{1/2} + \frac{2}{3} \mu_B B m_j$$

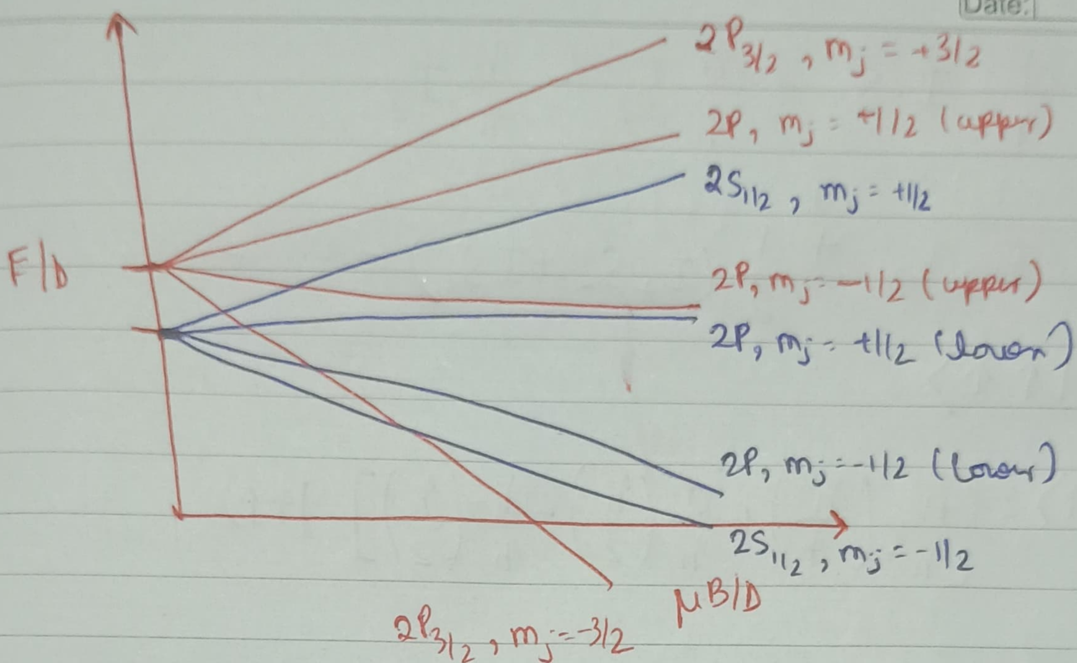
$\mu_B B >> \Delta$:

$$E_{\pm}^{(m_j)} \approx E_{1/2} + \Delta/2 + \mu_B B m_j \pm \frac{\mu_B B}{2}\left(1 + \frac{2m_j}{3} \frac{\Delta}{\mu_B B}\right)$$

now for $m_j = 1/2$:

upper state : $\mu_B B(1/2 + 1/2) = + \mu_B B$ $\quad |m_l = 0, m_s = +1/2\rangle$

lower state : $\mu_B B(1/2 - 1/2) = 0$ $\quad |m_l = 1, m_s = -1/2\rangle$



Answer 4

a) $H_{hf} = \frac{A}{\hbar^2} \bar{I} \cdot \bar{S}$, $\bar{F} = \bar{I} + \bar{S}$

$\bar{F}^2 = (\bar{I} + \bar{S})^2 = \bar{I}^2 + \bar{S}^2 + 2\bar{I}\cdot\bar{S}$

$\Rightarrow \bar{I}\cdot\bar{S} = (F^2 - I^2 - S^2)\, 0.5$

$= \frac{\hbar^2}{2}\left[F(F+1) - \frac{3}{4} - \frac{3}{4}\right] = \frac{\hbar^2}{2}\left[F(F+1) - \frac{3}{2}\right]$

Triplet state $(F=1)$: $E_{F=1} = A/4$

Singlet state $(F=0)$: $E_{F=0} = -3A/4$

Triplet state eigenstates: $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$
Singlet state eigenstates: $|0,0\rangle$

Date:

(b)(i) $H = \frac{A}{\hbar^2}\bar{I}\cdot\bar{S} + \frac{a}{\hbar}S_z - \frac{b}{\hbar}I_z$

$\bar{I}\cdot\bar{S} = I_z S_z + \frac{1}{2}(I_+S_- + I_-S_+)$

(i) $|\uparrow\uparrow\rangle$

$$H|\uparrow\uparrow\rangle = \left[\frac{A}{\hbar^2}\left(\frac{\hbar^2}{4}\right) + \frac{a}{\hbar}\left(\frac{\hbar}{2}\right) - \frac{b}{\hbar}\left(\frac{\hbar}{2}\right)\right]|\uparrow\uparrow\rangle$$

$$= \left[\frac{A}{4} + \frac{a-b}{2}\right]|\uparrow\uparrow\rangle$$

(ii) $H|\downarrow\downarrow\rangle = \left[\frac{A}{4} - \frac{a-b}{2}\right]|\downarrow\downarrow\rangle$

(iii) $H|\uparrow\downarrow\rangle = \left(-\frac{A}{4} - \frac{a+b}{2}\right)|\uparrow\downarrow\rangle + \frac{A}{2}|\downarrow\uparrow\rangle$

(iv) $H|\downarrow\uparrow\rangle = \left(-\frac{A}{4} + \frac{a+b}{2}\right)|\uparrow\downarrow\rangle + \frac{A}{2}|\uparrow\downarrow\rangle$

constructing $\langle i|H|j\rangle$ from above:

$$\begin{pmatrix} A/4 + (a-b)/2 & 0 & 0 & 0 \\ 0 & A/4 - (a-b)/2 & 0 & 0 \\ 0 & 0 & -A/4 - (a+b)/2 & A/2 \\ 0 & 0 & A/2 & -A/4 + \frac{(a+b)}{2} \end{pmatrix}$$

Date:

(b)(ii) $E_+ = \frac{A}{4} + \frac{a-b}{4}$ ev: $|\uparrow\uparrow\rangle$

$E_- = \frac{A}{4} - \frac{a-b}{4}$ ev: $|\downarrow\downarrow\rangle$

$$\begin{vmatrix} -A/2 - a+b/2 - \epsilon & A/2 \\ A/2 & -A/4 + a+b/2 - \varepsilon \end{vmatrix} = 0$$

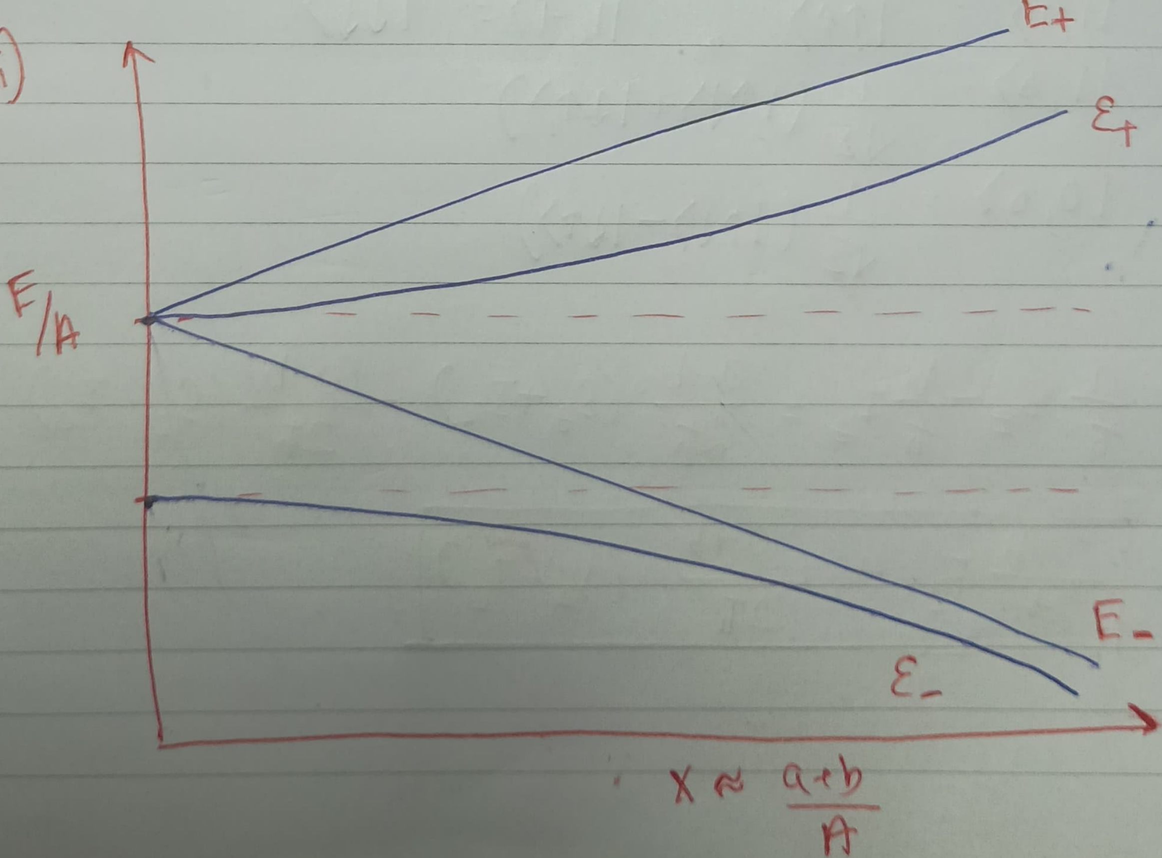
$$\Rightarrow \epsilon_\pm = \frac{-A}{4} \pm \sqrt{\frac{A^2}{4} + \frac{(a+b)^2}{4}}$$

$|\psi_+\rangle = \sin\theta\,|\uparrow\downarrow\rangle + \cos\theta\,|\downarrow\uparrow\rangle$ $\quad \varepsilon_+$

$|\psi_-\rangle = -\sin\theta\,|\downarrow\uparrow\rangle + \cos\theta\,|\uparrow\downarrow\rangle$ $\quad \varepsilon_-$

where $\theta = \tan^{-1}(A/2(a+b))$

(b)(iii)



Date:

(b)(iv) (A) weak field $(a, b << A)$:

$$\sqrt{\frac{A^2}{4}+\frac{(a+b)^2}{4}} = \frac{A}{2}\sqrt{1+\left(\frac{a+b}{A}\right)^2} \approx \frac{A}{2}\left[1+\frac{1}{2}\left(\frac{a+b}{2}\right)^2\right]$$

$$\approx \frac{A}{2}$$

$\Rightarrow \quad \mathcal{E}_+ = A/4 \qquad \mathcal{E}_- = -3A/4$

(B) strong field $(a, b >> A)$

$$\boxed{\mathcal{E}_\pm \approx \frac{-A}{4} \pm \frac{(a+b)}{2}}$$

(b)(v) $|F, M_F\rangle$:

$|1,1\rangle = |\uparrow\uparrow\rangle, \quad |1,-1\rangle = |\downarrow\downarrow\rangle$

$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

$$H_z = \frac{a}{\hbar}S_z - \frac{b}{\hbar}I_z = \frac{a-b}{2\hbar}(S_z+I_z) + \frac{(a+b)}{2\hbar}(S_z - I_z)$$

$$= \boxed{\frac{a-b}{2\hbar}F_z + \frac{a+b}{2\hbar}(S_z - I_z)}$$

Date:

$$M_{coupled} = \begin{pmatrix} A/4 + (a-b)/2 & 0 & 0 & 0 \\ 0 & A/4 - (a-b)/2 & 0 & 0 \\ 0 & 0 & A/4 & -(a+b)/2 \\ 0 & 0 & -\frac{(a+b)}{2} & -3A/4 \end{pmatrix}$$

Basis ordering: $|1,1\rangle$, $|1,-1\rangle$, $|1,0\rangle$, $|0,0\rangle$

$$\begin{vmatrix} A/4 - \lambda & -(a+b)/2 \\ -(a+b)/2 & -3A/4 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \boxed{\lambda_{\pm} = \frac{-A}{4} \pm \sqrt{\frac{A^2}{4} + \frac{(a+b)^2}{4}}}$$

identical to the $\varepsilon_{\pm}$ obtained