Department of Physics, IIT-Kanpur

Instructor: Tarun Kanti Ghosh Quantum Mechancs-II (PHY626) AY 2025-26, SEM-II

Homework-2: Zeeman effect

1. Consider a hydrogenic Hamiltonian (including electron's spin magnetic moment) in presence of a static and uniform magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$. Here $\mathbf{A} = (\mathbf{B}\times\mathbf{r})/2$ is the vector potential.

   (a) Show that the diamagnetic term can be neglected compared to the paramagnetic term if $B \ll Z^2 10^6/n^4$.

   (b) Now write down the full Hamiltonian by adding the three fine-structure correction terms.

2. **Weak-field Zeeman effect**: This is also known as Anomalous Zeeman effect.

   (a) Find out the critical magnetic field (B_c) below which the Zeeman interaction energy is less than the fine structure energy correction.

   (b) What are the "good" quantum numbers in this case?

   (c) The total magnetic moment of an electron in hydrogenic atoms is

   $$\boldsymbol{\mu} = -\frac{\mu_B}{\hbar}(\mathbf{L} + 2\mathbf{S}),$$

   where $\mu_B = e\hbar/2m_0$ is the Bohr magneton.
   The quantum projection theorem (or Wigner-Eckart theorem) is given by

   $$\langle jm_j|\mathbf{V}|jm_j\rangle = \frac{\langle jm_j|\mathbf{V}\cdot\mathbf{J}|jm_j\rangle\langle jm_j|\mathbf{J}|jm_j\rangle}{j(j+1)\hbar^2}.$$

   Here $\mathbf{V}$ is any vector operator and $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Using this theorem, show that the magnetic moment $\boldsymbol{\mu}$ can be expressed as $\boldsymbol{\mu} = -g_{\text{eff}}\mu_B\mathbf{J}/\hbar$, where the effective Lande-g factor is

   $$g_{\text{eff}} = 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}.$$

   Note that this expression of g_{eff} is valid for weak magnetic field as well as for low values of Z.
   [Hint: see Griffiths quantum mechanics book, page 289 (second edition)].

   (d) In the weak-field case, the perturbative Zeeman term can be written as

   $$H_z = g_e\mu_B B J_z.$$

   Calculate the Zeeman energy splitting of $1S_{1/2}$ and $2S_{1/2}$, $2P_{1/2}$ and $2P_{3/2}$ fine structure multiplets. Compare these results with that of obtained in class in different methods.

3. **Zeeman splitting for an arbitrary magnetic field** B: One can solve the Zeeman effect problem by diagonalizing the full Hamiltonian $H = H_0 + H_{\text{fs}} + H_Z$ for an arbitrary magnetic field B. Consider the hydrogen atom in the first excited state ($n = 2$). There are 8 degenerate states in this level. Use $|jm_j>$ as basis states for degenerate perturbation theory. Use the

Clebsch-Gordan co-efficients to express $|jm_j\rangle$ as a linear combination of $|lm_l\rangle$ and $|sm_s\rangle$ states:

$$|jm_j\rangle = \sum_{m_j=m_l+m_s} C_{l,s,j}^{m_l,m_s}|lsm_l,m_s\rangle.$$

For any l and $s = 1/2$, the $|jm_j\rangle$ states as a linear combination of $|lsm_lm_s\rangle$ are given by

$$\left|j = l\pm\frac{1}{2}, m_j\right\rangle = \pm\sqrt{\frac{l \pm m_j + \frac{1}{2}}{2l+1}}\left|l\frac{1}{2}m_l = m_j - \frac{1}{2}, +\frac{1}{2}\right\rangle + \sqrt{\frac{l \mp m_j + \frac{1}{2}}{2l+1}}\left|l\frac{1}{2}m_l = m_j + \frac{1}{2}, -\frac{1}{2}\right\rangle.$$

[Hint: In order to solve the following problems, see Griffiths quantum mechanics book, page 293 (second edition)].

(a) Calculate the Zeeman energy levels for any arbitrary magnetc field B.

(b) Plot the Zeeman energy levels as a function of the field B.

(c) Now take two limiting cases: weak field and strong field. Compare your results of the limiting cases with that of obtained in class.

[Hint: see Griffiths quantum mechanics book, page 293 (second edition)].

4. **Hyperfine states and Zeeman effect**:

(a) Hyperfine interaction is related to the interaction between nuclear spin $\mathbf{I}$ and electron spin $\mathbf{S}$, which is described as

$$H_{\rm hf} = \frac{A}{\hbar^2}\mathbf{I}\cdot\mathbf{S},$$

where $A = (4\mu_0/3)g_p\mu_B\mu_N|\psi_{100}(r=0)|^2 > 0$ for the ground state of a hydrogen atom. Also, $\mathbf{I} = \mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$. This is an useful example of two spin-1/2 particles.

Treat $H_{\rm hf}$ as a perturbation to the unperturbed Hamiltonian H_0 of a hydrogen atom. Obtain the splitting energy levels and the corresponding eigenstates of the ground state of a hydrogen atom.

[Note that $\mathbf{I}\cdot\mathbf{S}$ can be written in terms of the raising and lowering ladder operators: $\mathbf{I}\cdot\mathbf{S} = I_zS_z + (I_+S_- + I_-S_+)/2$.]

(b) Consider the ground state of a hydrogen atom in presence of a static and uniform magnetic field $\mathbf{B} = B\hat{z}$. The effective Hamiltonian can be written as

$$H = \frac{A}{\hbar^2}\mathbf{I}\cdot\mathbf{S} + \frac{a}{\hbar}S_z - \frac{b}{\hbar}I_z, \tag{1}$$

where $a = g_e\mu_BB$ and $b = g_p\mu_NB$. Also, $g_e = 2, g_p = 5.56$. This is an useful example of two spin-1/2 particles in presence of magnetic field.

We choose the following states as a basis states to diagonalize the above Hamiltonian: $|M_IM_S\rangle : |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$. The most general wavefunction is

$$\Psi = d_1|\uparrow\uparrow\rangle + d_2|\downarrow\downarrow\rangle + d_3|\uparrow\downarrow\rangle + d_4|\downarrow\uparrow\rangle. \tag{2}$$

The spatial part of the wave function does not matter here because the Hamiltonian is space independent.

(i) Show that the matrix representation of H in the above basis states (maintain the order) can be written as

$$M = \begin{pmatrix} \frac{A}{4} + \frac{(a-b)}{2} & 0 & 0 & 0 \\ 0 & \frac{A}{4} - \frac{(a-b)}{2} & 0 & 0 \\ 0 & 0 & -\frac{A}{4} - \frac{(a+b)}{2} & \frac{A}{2} \\ 0 & 0 & \frac{A}{2} & -\frac{A}{4} + \frac{(a+b)}{2} \end{pmatrix}.$$

(ii) Show that the eigenvalues are

$$E_\pm = \frac{A}{4} \pm \frac{(a-b)}{2} \tag{3}$$

$$\epsilon_\pm = -\frac{A}{4} \pm \sqrt{\frac{A^2}{4} + \frac{(a+b)^2}{4}}. \tag{4}$$

Obtain the corresponding eigenvectors.

(iii) Plot the Zeeman energy levels as a function of B. Indicate each level by the quantum numbers.

(iv) Obtain the energy levels in the two limiting cases: (i) weak-field and (ii) strong field.

(v) Introduce the total spin angular momentum $\mathbf{F} = \mathbf{S} + \mathbf{I}$.
Construct the coupled basis states $|FM_F\rangle$: three triplet states for $F = 1, M_F = 0, \pm 1$ and one singlet state for $F = 0, M_F = 0$ in terms of the uncoupled states $|M_I M_S\rangle$
Obtain the matrix representation of H in the $|F, M_f\rangle$ basis.
Show that the Zeeman energy levels are the same as we obtain from the uncoupled basis.
[This problem is important if you want to understand how to trap alkali atoms magnetically. The wavelength (21 cm) corresponding to the hyperfine transition in the ground state of hydrogen atom can be measured with very high accuracy. An atomic clock and MASER can be made based on the hyperfine transition in the ground state of atomic hydrogen.]