PSO201/PHY204: Assignment 1
Jan-Apr 2025

1. Suppose that at $t = 0$, a particle in an infinite well is in the state

$$\psi(x,0) = \sqrt{\tfrac{1}{2}}\psi_1(x) + \sqrt{\tfrac{1}{3}}\psi_2(x) + \sqrt{\tfrac{1}{6}}\psi_3(x)\,.$$

Here $\psi_n(x)$ $(n > 0)$ are the energy eigenfunctions.
(a) How does ψ evolve with time? Write down the expression for $\psi(x,t)$.
(b) Calculate the expectation value of the energy for the particle described by $\psi(x,t)$. Write your answer in terms of E_1, the eigen energy value corresponding to the ground state. Does this quantity change with time?
(c) What is the probability of measuring the energy to equal $< E >$ as a result of a single measurement at $t = 0$?
(d) What energy values will be observed as a result of a single measurement at $t = 0$ and with what probabilities? How do these probabilities change with time?

2. In the lecture we have discussed that `observables' such as position and momentum are represented by linear operators acting on the wavefunction, $\psi(x)$, as

$$\hat{p}\psi(x) = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x);\ \hat{x}\psi(x) = x\psi(x), \text{ respectively.}$$

a) Show that the position-momentum commutator takes the value, $[\hat{p},\hat{x}] = \frac{\hbar}{i}\,\mathbf{1}$
b) Suppose $\hat{A}$ and $\hat{B}$ are the quantum operators representing two observables of a physical system. What must be value the commutator so that the system will have definite values of $\hat{A}$ and $\hat{B}$ simultaneously?

3. Consider a particle of mass m that is in a superposition state of the first two eigenstates of an infinite potential well of width L,

$$\psi(x,t) = \frac{1}{\sqrt{L}}\sin\frac{\pi x}{L}\, e^{-i\omega_1 t} + \frac{1}{\sqrt{L}}\sin\frac{2\pi x}{L}\, e^{-i\omega_2 t};\ \text{for } 0 \le x \le L.$$

a) Verify that $\psi(x,t)$ is properly normalized and remains so for all time t.
b) Calculate the probability distribution $P(x,t) = |\psi(x,t)|^2$. What is the period T of this superposition (i.e., after what time T will the system return to its original configuration)? Plot the probability distribution $P(x,t)$ at time $t = \pi/[2(E_2 - E_1)]$. What fraction of T is t?
c) Find the probability that the particle be found in the left half of the well at time t.
d) Find the expectation value $\langle\hat{x}\rangle$ of the particle's position as a function of time.
e) Show that the probability density $P(x,t)$ at $x = L/2$ is independent of time.

4. A particle is in a state described by the wavefunction $\psi(x,t)$. Let $P_{ab}(t)$ be the probability of finding the particle in the range $a < x < b$ at time t.
a) Show that $\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)$
b) What is the unit of J ?

5. Suppose $\psi_0(x)$ is a properly-normalized wavefunction with $< x >_{\psi_o} = x_0$ and $< p >_{\psi_o} = p_0$, where x_0 and p_0 are constants. Define a new wavefunction $\psi_1(x) = e^{iqx/\hbar}\,\psi_0(x)$; where q is a real number with the appropriate dimensions.
   a) What is the expectation value $< x >_{\psi_1}$?
   b) What is the expectation value $< p >_{\psi_1}$?
   c) Based on your results, state the physical significance of introducing an overall phase factor $e^{iqx/\hbar}$ to the wavefunction.

6. Consider a particle with mass m in the state described by the complex wavefunction $\psi(x)$. One can always express the complex wavefunction as, $\psi(x) = A(x)e^{i\theta(x)}$, where the amplitude $A(x)$ and the phase $\theta(x)$ are real functions.
   a) Find the probability current density $J(x)$ and interpret the result.
   b) Will there be any current in a region where the wavefunction is real?

7. Consider the finite square well potential of depth V_0 and width $2L$, i.e. $V(x) = -V_0$ for $|x| \leq L$ and $V(x) = 0$ elsewhere; where V_0 and L are real, positive constants. We discussed the even bound state solutions in the lecture. Here, you are asked to analyse the odd bound state solutions.
   a) The width L sets a characteristic length scale for our finite well. Use dimensional analysis to identify a second characteristic length scale l_0 and a dimensionless measure of the overall size of the well, g_0.
   b) Derive a transcendental equation for the allowed energies and solve the resulting pair of equations graphically. Find the condition for a bound state to exist in terms of the dimensionless parameter g_0 or l_0.

8. At time t = 0, a free particle in 3d where $V(x_1, x_2, x_3) = 0$ is in the superposition state, $\psi(\vec{x}, 0) = A\sin(3x_1/L)\,\mathrm{e}^{i(5x_2+x_3)/L}$.
   a) If the energy of the particle is measured at $t = 0$, what is the obtained energy value?
   b) What are the possible outcomes of the measurement of momentum $\vec{p} = (p_1, p_2, p_3)$ in the state $\psi(\vec{x}, 0)$ at $t = 0$ and with what probabilities?
   c) Given the state $\psi(\vec{x}, 0)$ above, what is $\psi(\vec{x}, t)$?

9. Plot the first three eigenfunctions $\psi_n(x)$ for an infinite square well starting with the ground state. Plot the corresponding values of $\frac{d\psi_n(x)}{dx}$ as a function of x over all space. Are the slopes of the wavefunctions continuous everywhere? If not, why? Are the wavefunctions continuous everywhere? If yes, why?

10. Suppose somebody presents the following argument: If a particle is in an eigenstate of a one-dimensional infinite square well of width L, then we know its energy exactly. We also know that the energy in the box is purely kinetic. Hence, we know the particle's momentum exactly as well. This contradicts the Heisenberg uncertainty principle that the position and momentum of a quantum particle cannot be specified exactly and simultaneously specified since the uncertainty in the particle position is finite ($\Delta x < L$). What is the problem with such an argument, if any?