Q: a) $x > -d \Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left[|E| + V_0\,\delta(x)\right]\psi = 0$, $E<0$, $k = \sqrt{\frac{2m|E|}{\hbar^2}}$

$\psi(x) = A e^{kx} + B e^{-kx}$, $-d<x<0$

$= C e^{-kx}$ (put $C=1$) , $x>0$

* $\psi(x=-d) = 0 \Rightarrow A e^{-kd} + B e^{kd} = 0$
* ψ is continuous at $x=0 \Rightarrow A+B=1$

$\Big\}\ \boxed{A e^{-kd} + (1-A) e^{kd} = 0}$

* $\frac{d\psi}{dx}$ at $x=0$ is discontinuous $\Rightarrow$

$$\left.\frac{d\psi}{dx}\right|_{x=0+} - \left.\frac{d\psi}{dx}\right|_{x=0-} = -\frac{2mV_0}{\hbar^2}\psi(0)$$

$\Rightarrow -k - (Ak - Bk) = -\frac{2mV_0}{\hbar^2}$ $[\because \psi(0)=1]$

$\Rightarrow \boxed{A - B = \frac{2mV_0}{k\hbar^2} - 1}$

$A e^{-kd} + (1-A) e^{kd} = 0 \Rightarrow \boxed{A = \frac{-e^{2kd}}{1-e^{2kd}}}$

$\boxed{B = 1 - A = \frac{1}{1-e^{2kd}}}$

Again, $A + (1-B) = \frac{2mV_0}{k\hbar^2} \Rightarrow \boxed{A = \frac{mV_0}{k\hbar^2}}$

$\Rightarrow \frac{-e^{2kd}}{1-e^{2kd}} = \frac{mV_0}{k\hbar^2} \Rightarrow k = -\frac{mV_0}{\hbar^2}\frac{(1-e^{2kd})}{e^{2kd}}$

$\Rightarrow \boxed{k = \frac{mV_0}{\hbar^2}\left(1 - e^{-2kd}\right)}$

The infinite potential wall is far away $\Rightarrow kd \gg 1 \Rightarrow k \simeq \frac{mV_0}{\hbar^2}$

$k \simeq \frac{mV_0}{\hbar^2}\left(1 - \exp\left(-\frac{2mV_0 d}{\hbar^2}\right)\right)$

Bound state energy $E = -\frac{\hbar^2 k^2}{2m} \simeq -\frac{\hbar^2}{2m}\frac{m^2V_0^2}{\hbar^4}\left[1 - \exp\left(-\frac{2mV_0 d}{\hbar^2}\right)\right]^2$

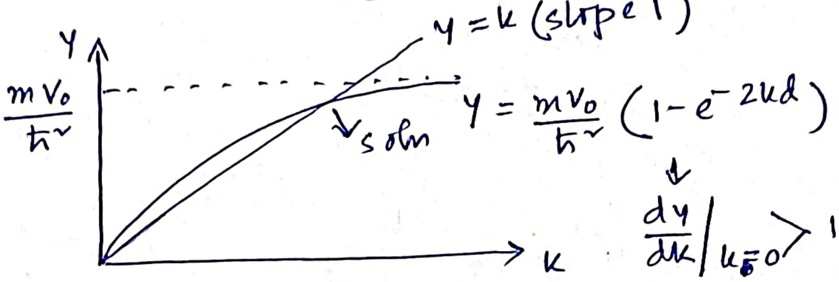
$\simeq -\frac{mV_0^2}{2\hbar^2}\left[1 - 2\exp\left(-\frac{2mV_0 d}{\hbar^2}\right)\right]$

$$\boxed{E \simeq \underbrace{-\frac{mV_0^2}{2\hbar^2}}_{\text{binding energy of bare delta potential}} + \underbrace{\frac{mV_0^2}{\hbar^2}\exp\left(-\frac{2mV_0 d}{\hbar^2}\right)}_{\text{+ve quantity}}}$$

The binding energy decreases.

Q2 b)

$$\left.\begin{aligned} y &= \frac{mV_0}{\hbar^2}\left(1-e^{-2kd}\right) \\ y &= k \end{aligned}\right\} \text{ solve graphically}$$



$$\left.\frac{dy}{dk}\right|_{k=0} > 1 \quad \text{for non-zero } k$$

$$\left.\frac{dy}{dk}\right|_{k=0} = -\frac{mV_0}{\hbar^2}\left(-2de^{-2kd}\right)_{k=0} = \frac{2mV_0 d}{\hbar^2}$$

condition for existence of bound state : $\frac{2mV_0 d}{\hbar^2} > 1$

$$\Rightarrow \boxed{V_0 d > \frac{\hbar^2}{2m}}$$

Q1 a) $\psi(x,0) = A\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi x}{a} = A\sin\frac{\pi x}{a} + \frac{A}{2}\sin\frac{2\pi x}{a}$

$$\int_0^a |\psi(x,0)|^2 dx = 1$$

$$\Rightarrow A^2\int_0^a \sin^2\frac{\pi x}{a}\,dx + \frac{A^2}{4}\int_0^a \sin^2\frac{2\pi x}{a}\,dx + \frac{A^2}{2}\int_0^a \sin\frac{\pi x}{a}\sin\frac{2\pi x}{a} = 1$$

$$\Rightarrow \boxed{A = \sqrt{\frac{8}{5a}}}$$

b) Infinite squar well : eigenfunctions $\psi_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$

Energy eigenvalues : $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$, $n = 1,2,\dots$

$$\begin{aligned} \psi(x,0) &= \sqrt{\frac{4}{5}}\cdot\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a} + \sqrt{\frac{1}{5}}\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a} \\ &= \sqrt{\frac{4}{5}}\,\psi_1(x,0) + \sqrt{\frac{1}{5}}\,\psi_2(x,0) \end{aligned}$$

$$\psi(x,t_0) = \sqrt{\frac{4}{5}}\,\psi_1(x,0)\,e^{-iE_1 t_0/\hbar} + \sqrt{\frac{1}{5}}\,\psi_2(x,0)\,e^{-iE_2 t_0/\hbar}$$

where $E_1 = \frac{\pi^2\hbar^2}{2ma^2}$, $E_2 = \frac{4\pi^2\hbar^2}{2ma^2}$

c) at $t=0$, $\langle E\rangle = \sum |c_n|^2 E_n = \frac{4}{5}E_1 + \frac{1}{5}E_2 = \frac{4\pi^2\hbar^2}{5ma^2}$

at $t=t_0$, $\langle E\rangle$ will remain the same.

d) $P(t) = \int_0^{a/2}|\psi(x,t_0)|^2 dx = \frac{4}{5}\int_0^{a/2}|\psi_1(x,0)|^2 dx + \frac{1}{5}\int_0^{a/2}|\psi_2(x,0)|^2 dx + \frac{4}{5}\int_0^{a/2}\psi_1(x,0)\psi_2(x,0)\,dx\,\cos\frac{3\pi^2\hbar t_0}{2ma^2}$

$$\Rightarrow \boxed{P(t) = \frac{1}{2} + \frac{16}{15\pi}\cos\left(\frac{3\pi^2\hbar}{2ma^2}t_0\right)}$$