PSO201/PHY204: Assignment 2
Jan-Apr 2025

1) Consider the wavefunction $\psi(x) = N$ for $-\frac{a}{2} \leq x \leq \frac{a}{2}$ and zero otherwise. This is just a step function centred on the origin.
    a) Determine the normalization constant N.
    b) Calculate $< x >$ and $< x^2 >$ and the uncertainty Δx.
    c) Calculate $< p >$ and $< p^2 >$ and the uncertainty Δp .
    d) Interpret your result. The wavefunction is normalisable and yet there is a problem!

2) Prove the following statements regarding energy eigenfunctions:
    a) We can always choose the energy eigenstates $\varphi_E(x)$ we work with to be purely real functions (unlike the physical wavefunction $\psi(x,t)$, which is necessarily complex).
    b) If $V(x)$ is an even function [i.e. $V(-x) = V(x)$], then the energy eigenfunctions $\varphi_E(x)$ can always be taken to be either even or odd.
    c) The lowest energy eigenvalue, E_0, corresponding to a normalizable eigenfunction is strictly greater then the minimum value V_{min} of the potential, $V(x)$.

3) Consider a particle of mass m in one dimension due to the short-range potential $V = -V_0\delta(x)$.
    a) Write down the boundary conditions on $\psi(x)$ and $\frac{d\psi}{dx}$ with justifications.
    b) How many bound states are possible?
    c) Obtain the binding energy of the particle.
    d) Find the corresponding wave function.
    e) Find the value of x_0 such that the probability of finding the particle with $|x| < x_0$ is exactly equal to $1/2$.

4) Consider the following yet to be normalized wavefunction for a particle moving in one dimension: $\psi(x) = Ne^{-(x-x_0)^2/a^2}e^{ik_0x}$.
    a) Determine the normalization constant N up to an overall phase.
    b) Sketch the real part of the wavefunction and the corresponding probability distribution.
    c) Compute the Fourier transform $\phi(k)$. Sketch the function and corresponding probability distribution.
    d) Where is the particle most likely be found? What is the most likely momentum?
    e) Calculate the expectation values $< x >$ and $< p >$ and the uncertainties Δx and Δp in this state. Do these values satisfy the uncertainty principle? If so, how efficiently? Comment on what this says about wavefunctions of this form.
    f) What happens to the shape of the wavefunction as $a \to 0$? What happens to $\phi(k)$? What happens to both in the limit $a \to \infty$? How do Δx and Δp behave in these limits? Does this make sense?

5) A particle of mass m moves in one dimension under the influence of a potential $V(x)$**.** Suppose it is in an energy eigenstate $\psi(x) = (\alpha^2/\pi)^{1/4}\, exp\,(-\alpha^2x^2/2)$ with energy $E = \frac{\hbar^2\alpha^2}{2m}$
    a) Find the mean position of the particle.
    b) Find the mean momentum of the particle.
    c) Find $V(x)$.
    d) Find the probability that the particle's momentum is between p and $p + dp$.

6) A wave packet at $t = 0$ is given by, $\psi(x,0) = C\exp\left(-\frac{x^2}{4\alpha^2} + ik_0x\right)$. Assume the relation $\omega(k) = \frac{\hbar k^2}{2m}$.
    a) Find the wave function in k-space.
    b) Calculate $\psi(x,t)$.

c) Plot $|\psi(x,t)|^2$ as a function of x for different values of $t_1 < t_2 < t_3$.

7) Consider a wave packet localized in k-space about k_0 with $\Phi(k) = e^{-\alpha(k-k_0)^2}$, α being a constant. Discuss the motion of the wave packet in a non-dispersive and a dispersive medium. Calculate $|\psi(x,t)|^2$ in each case.

8) A particle of mass m moves in a one-dimensional infinite square well of length L ($V = 0$ for $0 < x < L$. At a certain instant, say $t = 0$, the wave function of this particle is given by $\psi = Ax(1-x)$, for $0 < x < L$ and zero elsewhere. Write down an expression for $\psi(x, t > 0)$ as a series along with expressions for the coefficients in the series.

9) Consider the one dimensional motion of a particle of mass m in a potential, $V = \infty$ for $x \leq 0$; $V = 0$ for $0 < x < L$, and $V = V_0$ for $x \geq L$.
a) Find the condition that the bound state energies are obtained ($E < V_0$).
b) What is the minimum value of V_0 so that at least one bound state is obtained.
c) Sketch the ground state wave function using physical reasoning.

10) A particle of mass m undergoes one dimensional motion in the potential $V = -V_0[\delta(x-a) + \delta(x+a)]$ where $V_0 > 0$ is a constant. Find the ground state wavefunction. Sketch the ground state wave function primarily using physical reasoning as well.