AE675

Introduction to Finite Element Method

Project Report

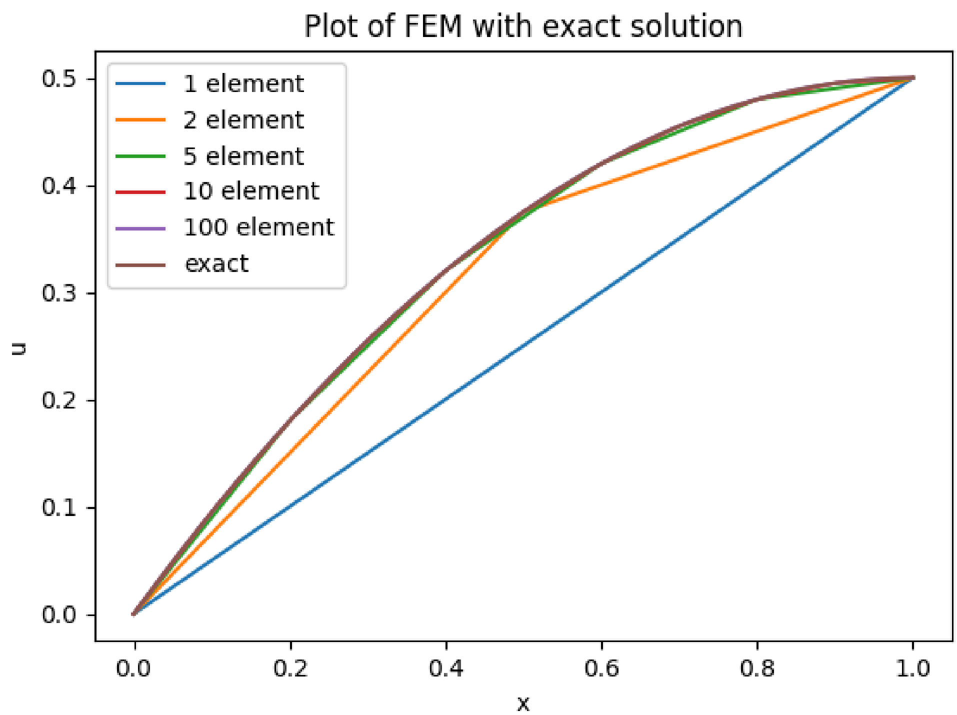
Graphs and results

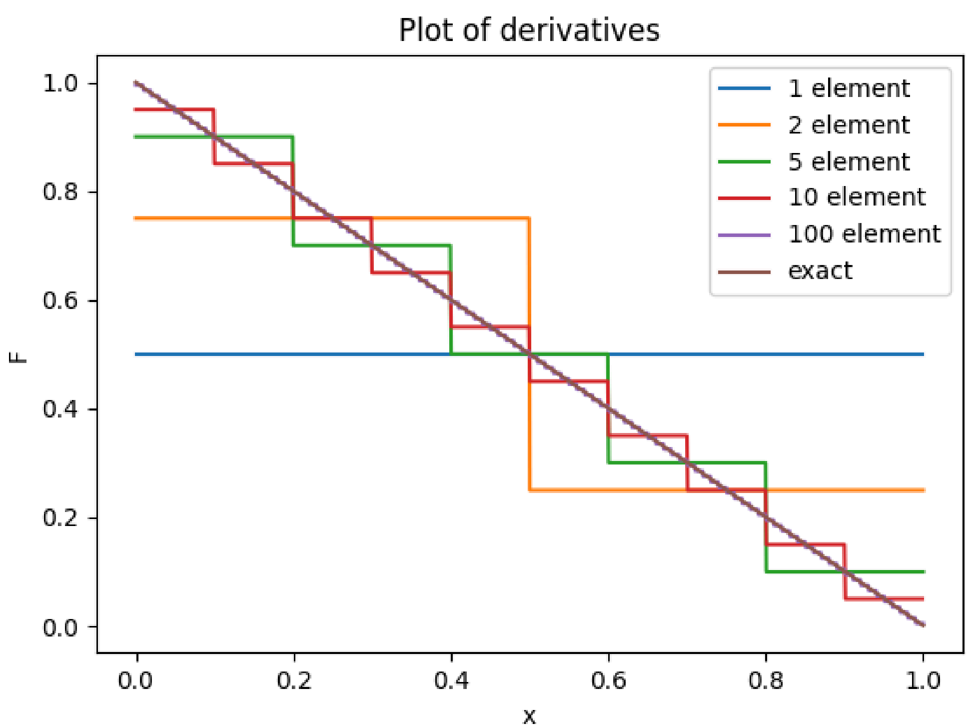
Name: Himanshu Mittal (210441)
Name: Suvarn Jaiswal (211088)

1D hp code

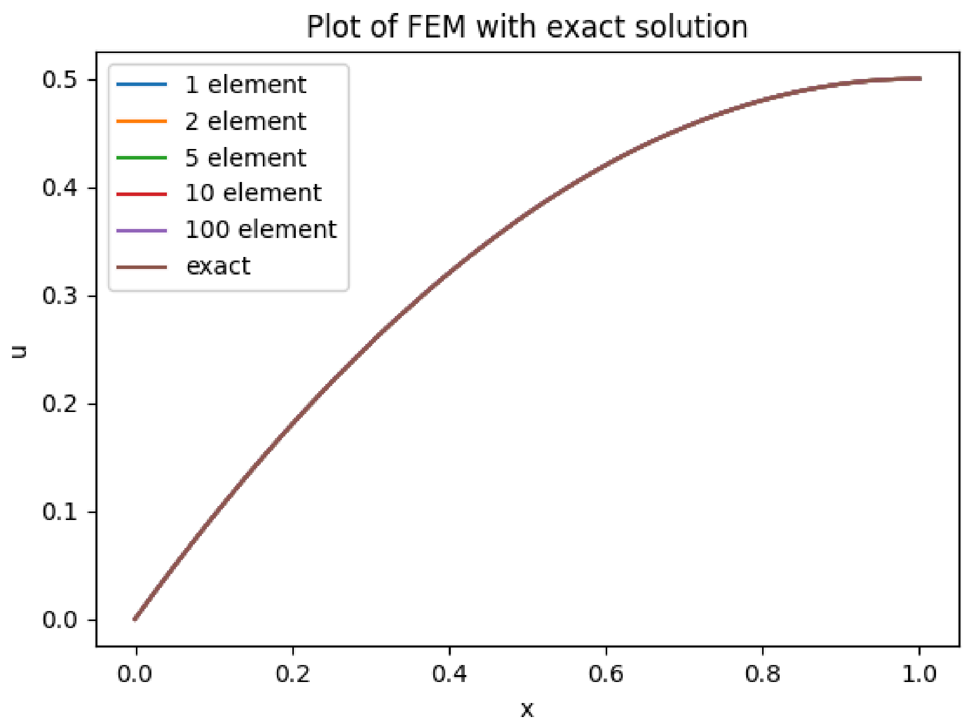
Q1

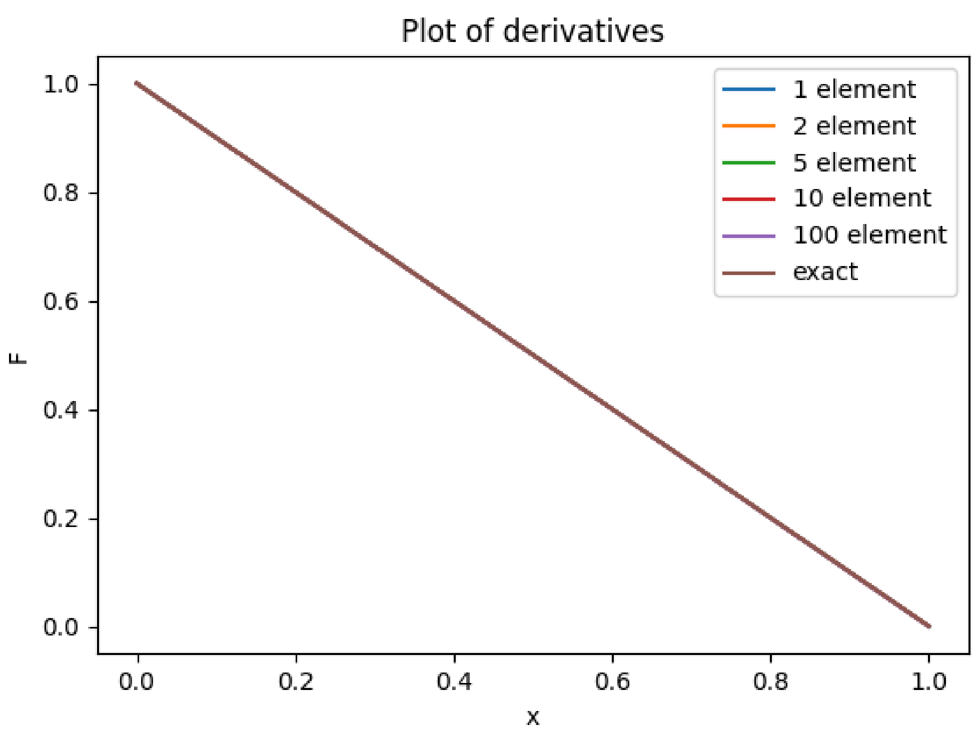
Linear (f(x)=1)



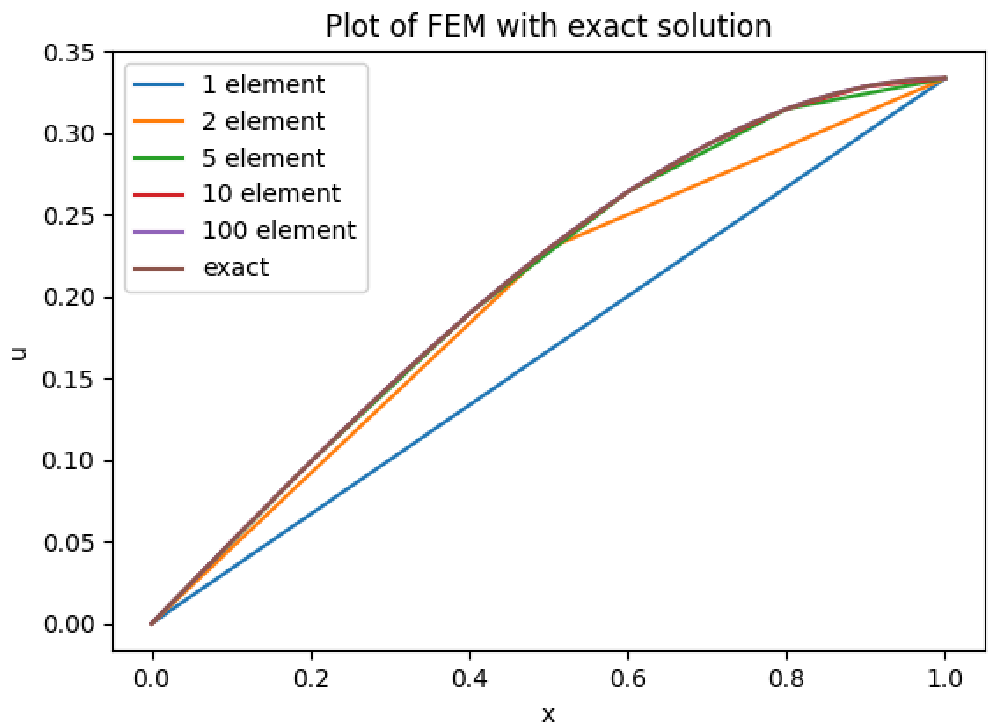


Quadratic(f(x)=1)

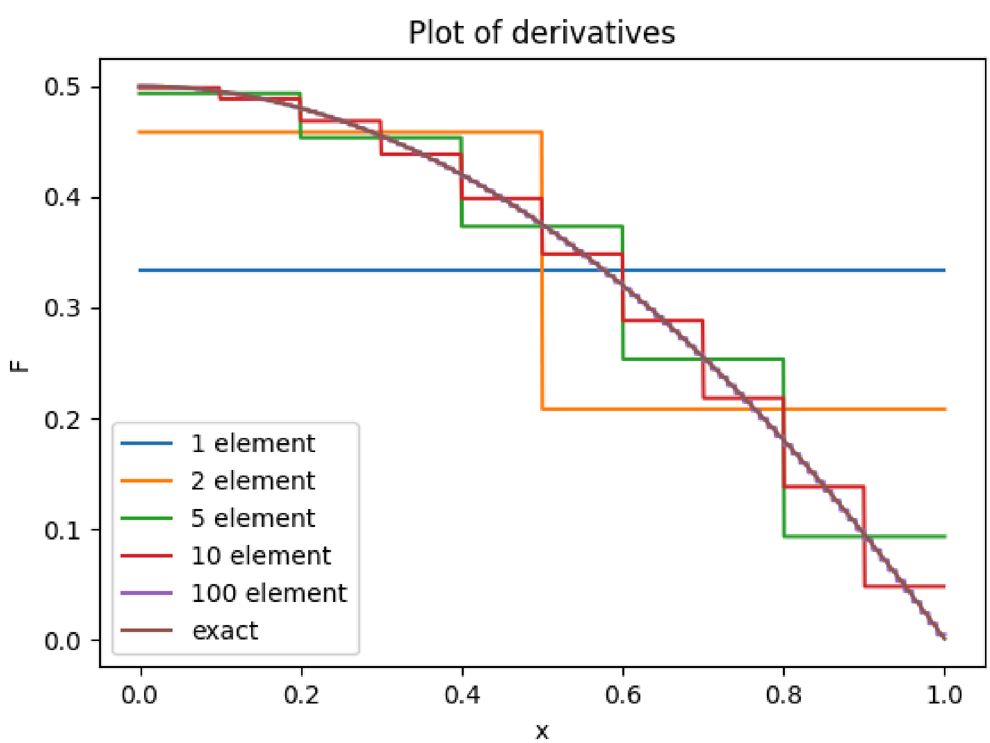




Linear(f(x)=x)

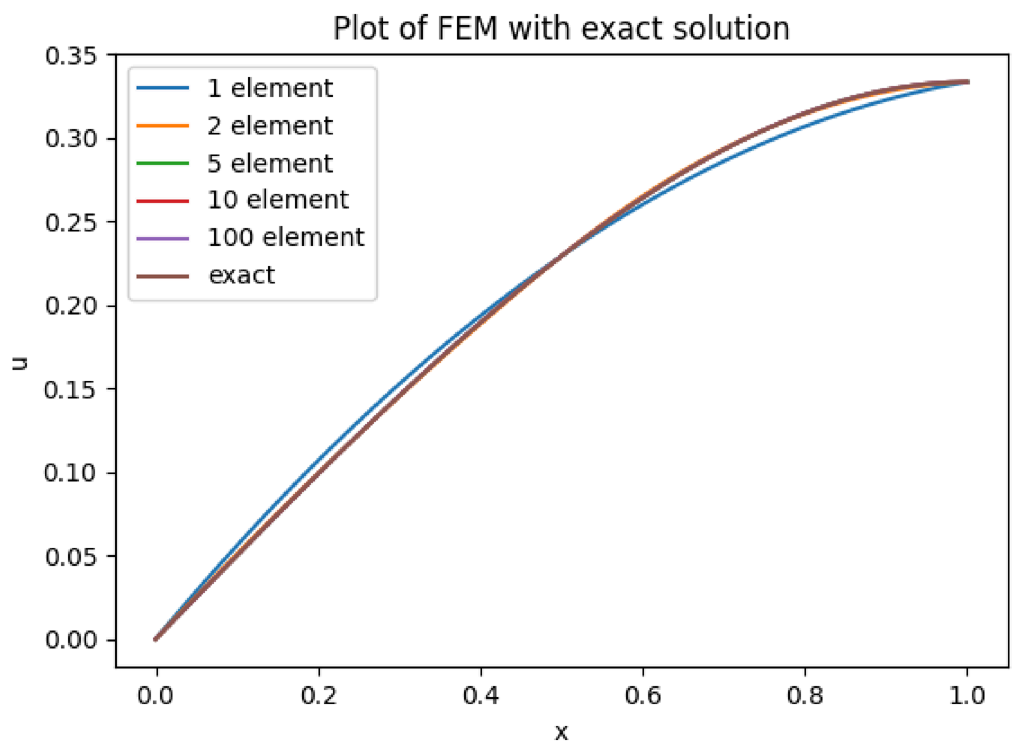


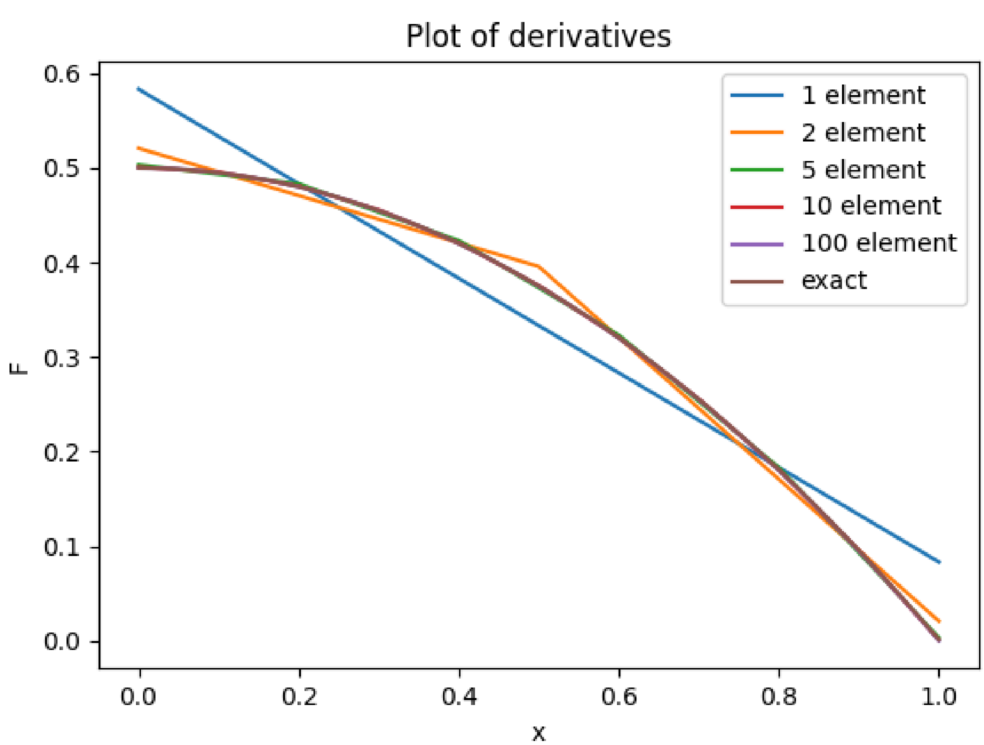
Plot of FEM with exact solution
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
u
0.0
0.2
0.4
0.6
0.8
1.0
x
1 element
2 element
5 element
10 element
100 element
exact



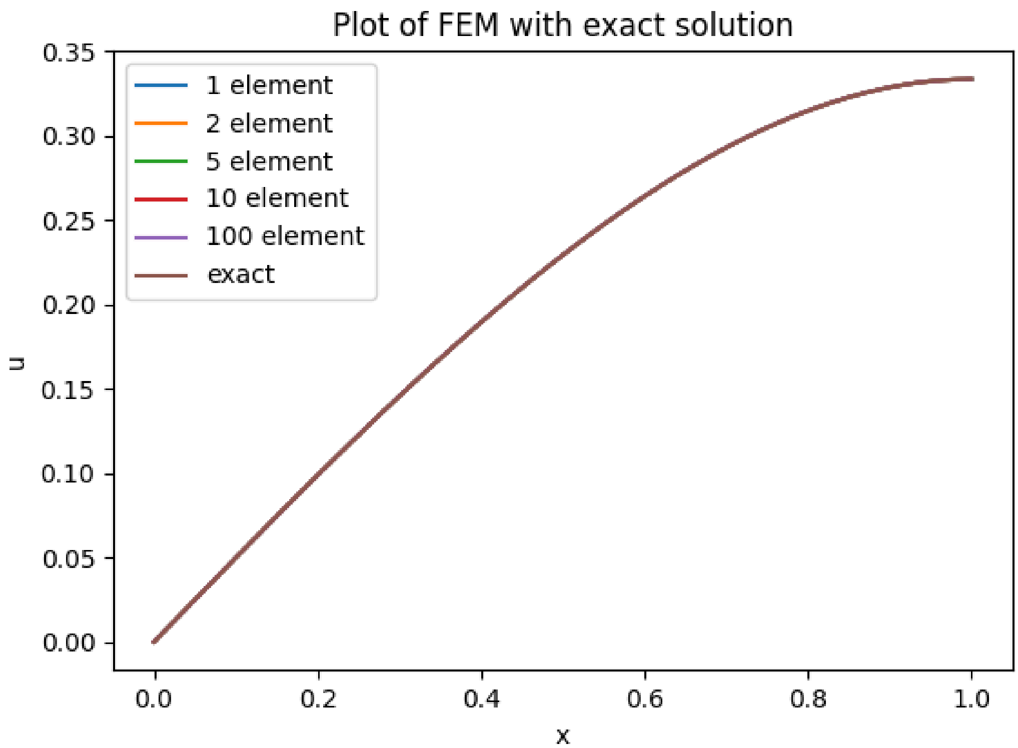
Plot of derivatives
0.5
0.4
0.3
0.2
0.1
0.0
F
0.0
0.2
0.4
0.6
0.8
1.0
x
1 element
2 element
5 element
10 element
100 element
exact

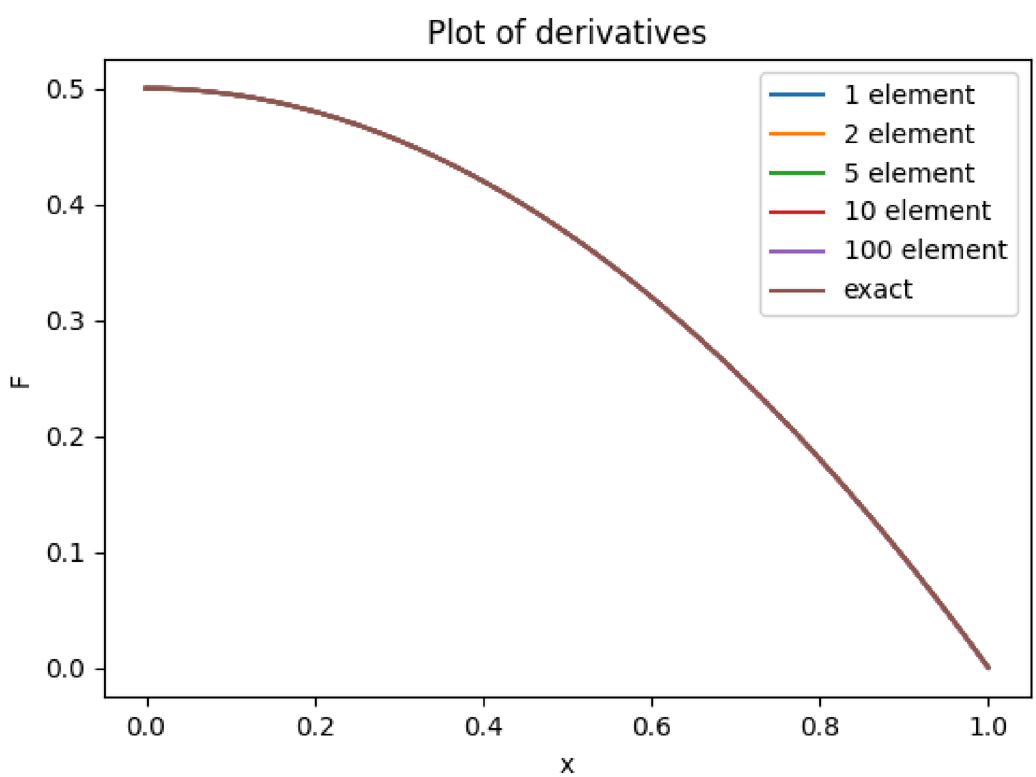
Quadratic(f(x)=x)



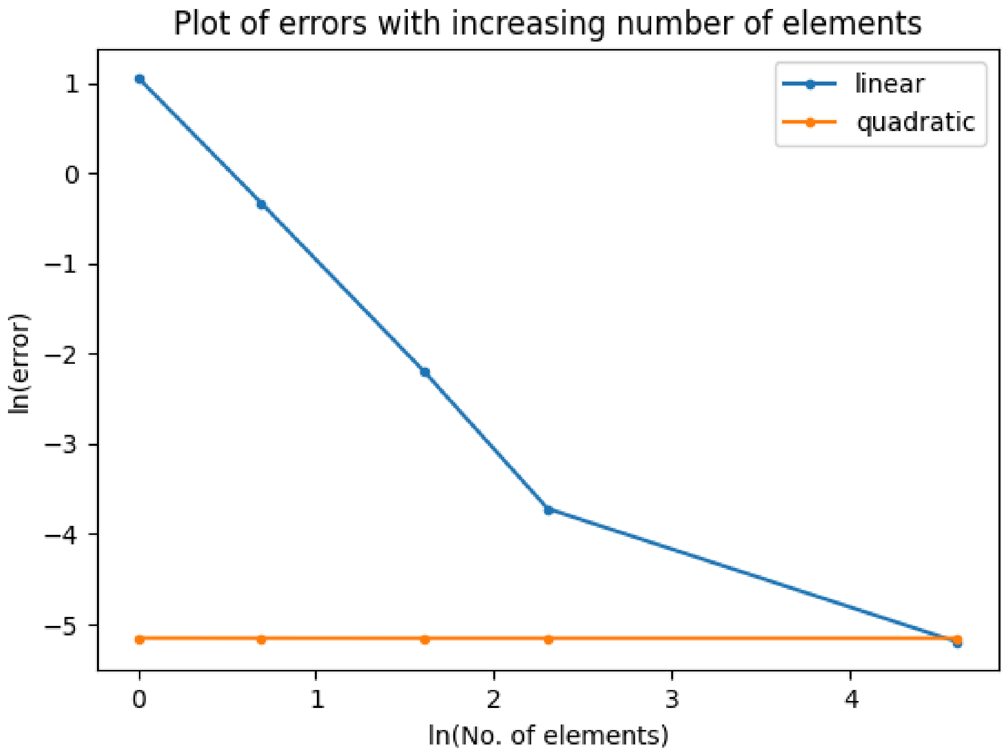


Cubic(f(x)=x)

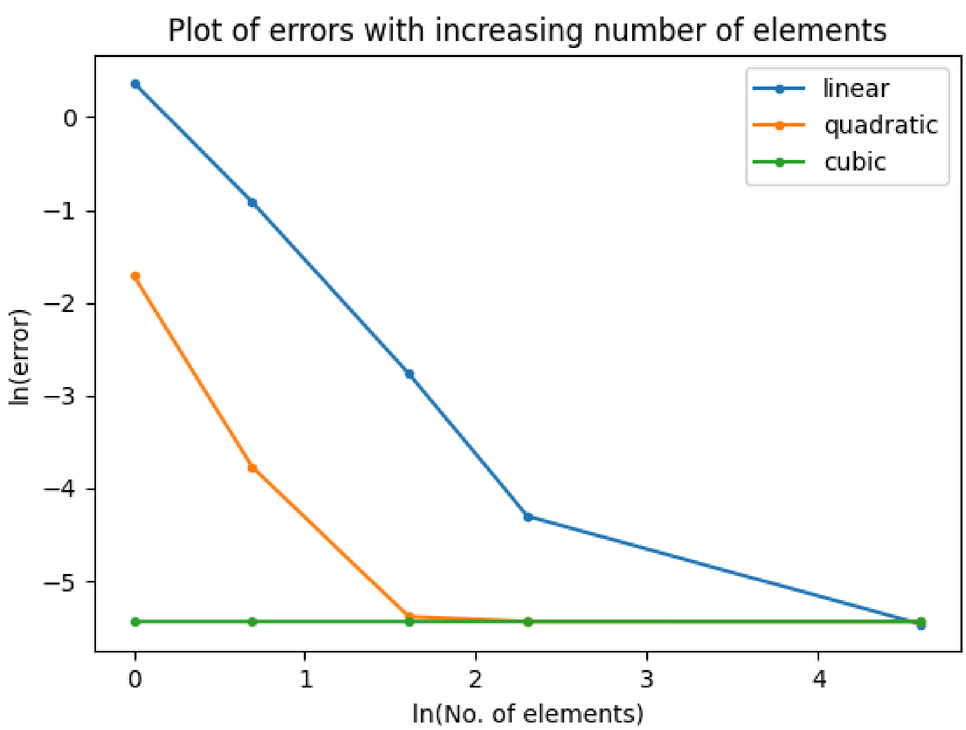

Plot of FEM with exact solution
1 element
2 element
5 element
10 element
100 element
exact
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
u
0.0
0.2
0.4
0.6
0.8
1.0
x


Plot of derivatives
1 element
2 element
5 element
10 element
100 element
exact
0.5
0.4
0.3
0.2
0.1
0.0
F
0.0
0.2
0.4
0.6
0.8
1.0
x

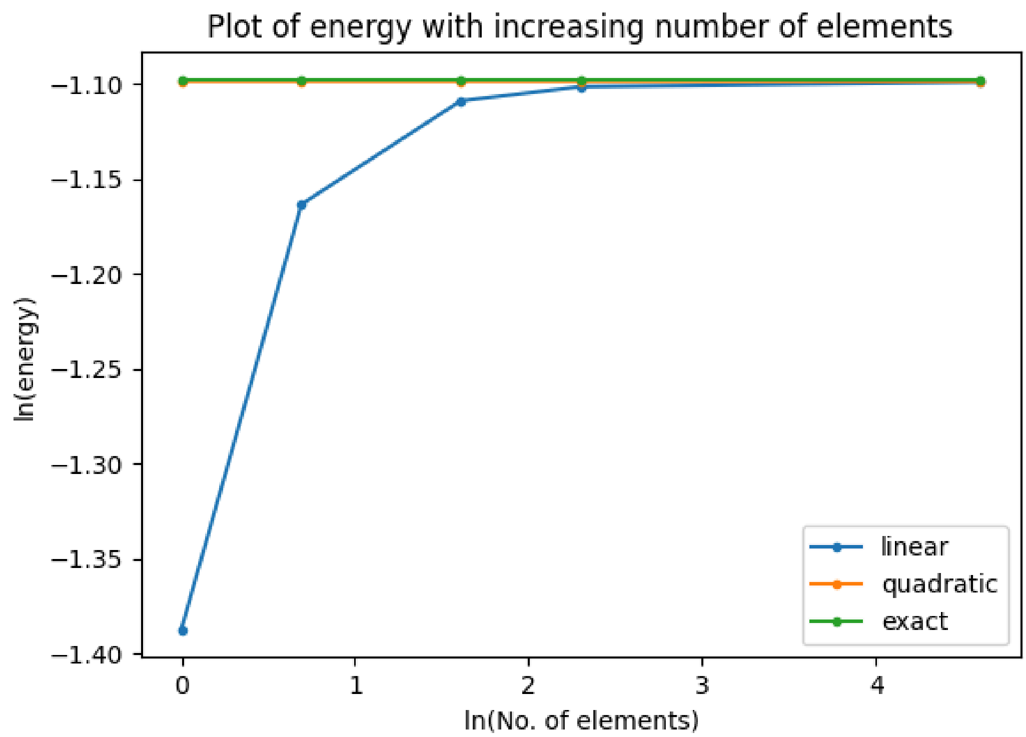
Error(f(x)=1)



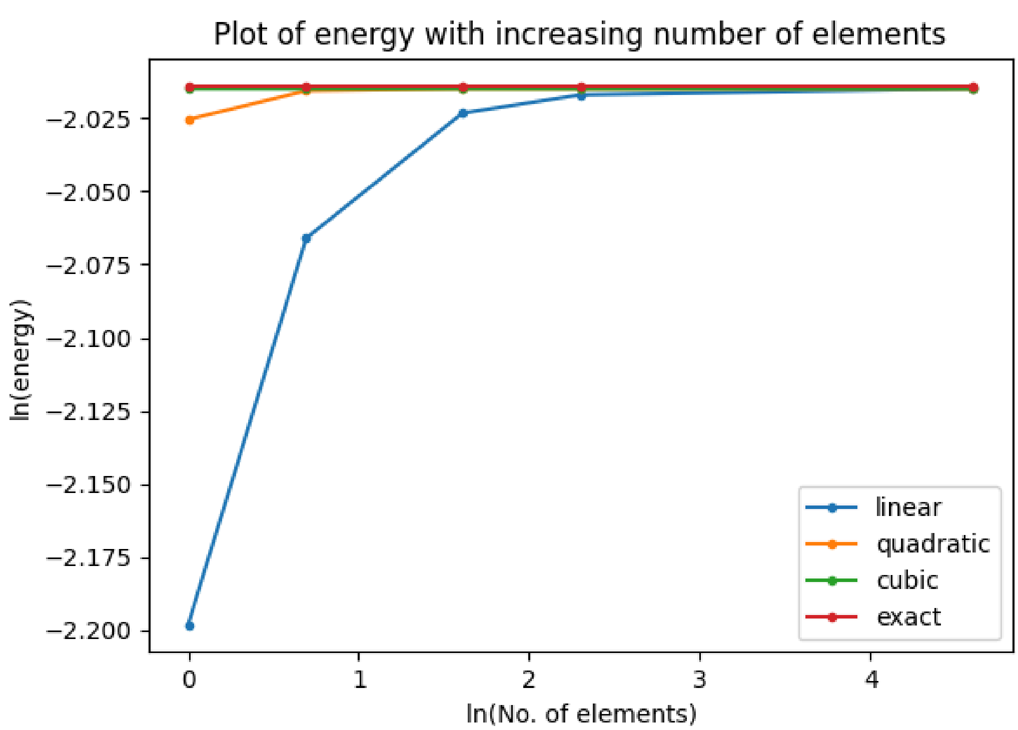
Error(f(x)=x)



Q2 Energy(f(x)=1)



Energy(f(x)=x)

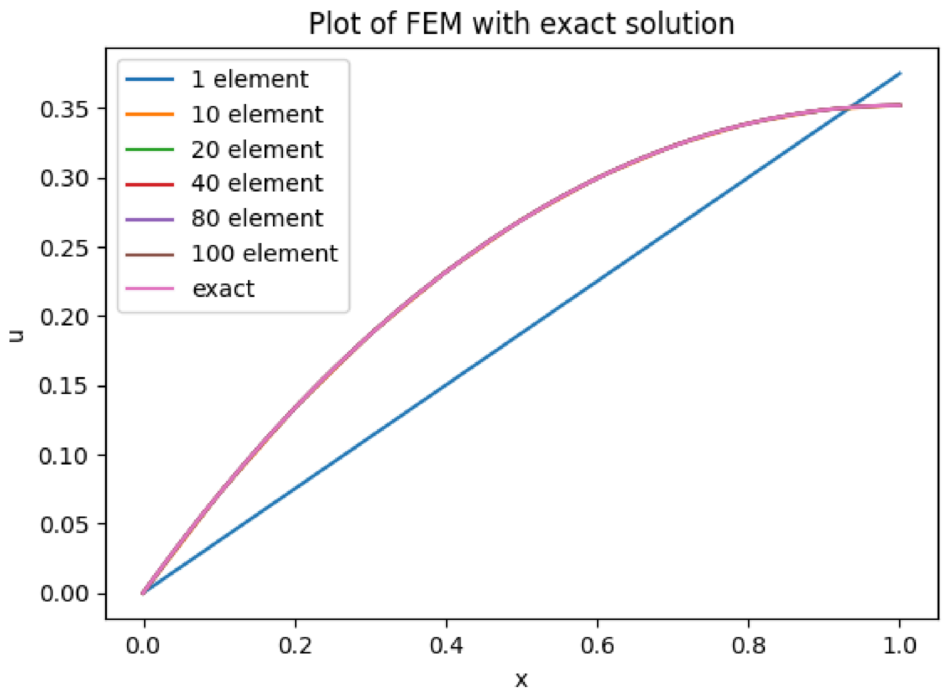


By the graphs of the solution and its derivatives, we can clearly verify that the FEM code passes the patch tests. The results show that for linear elements, the error in the approximation keeps on decreasing exponentially when the number of elements are increased. This is directly in coherence with the theory. When the exact solution is of the same order as the approximating functions, the error goes to a minimum and the solution matches the exact one very closely but it keeps on increasing with the number of elements because of numerical integration approximations and discretisation error.

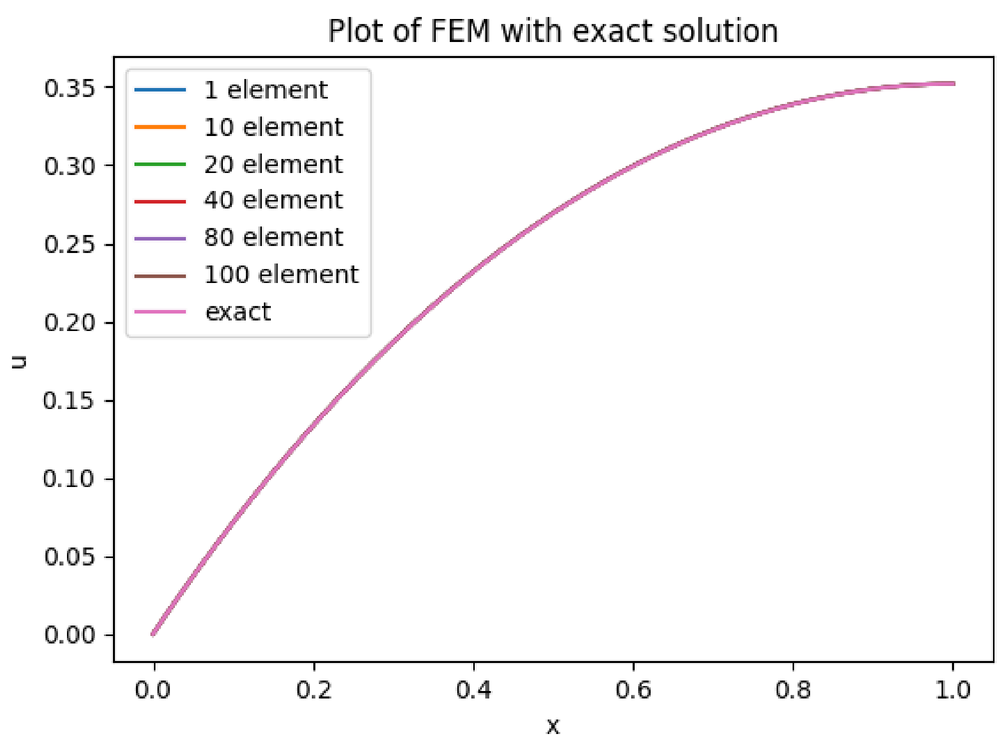
As for the strain energy, we see that as we increase the order of the approximating functions, the energy keeps on increasing until we reach the exact solution's energy. This is because higher order functions are able to capture more variation in detail and thus they make the bar less stiff than in lower order solution. Following that, as a result of integration approximations and discretization error, the solution with a greater number of elements exhibits lower energy compared to the solution with fewer elements.

Q3

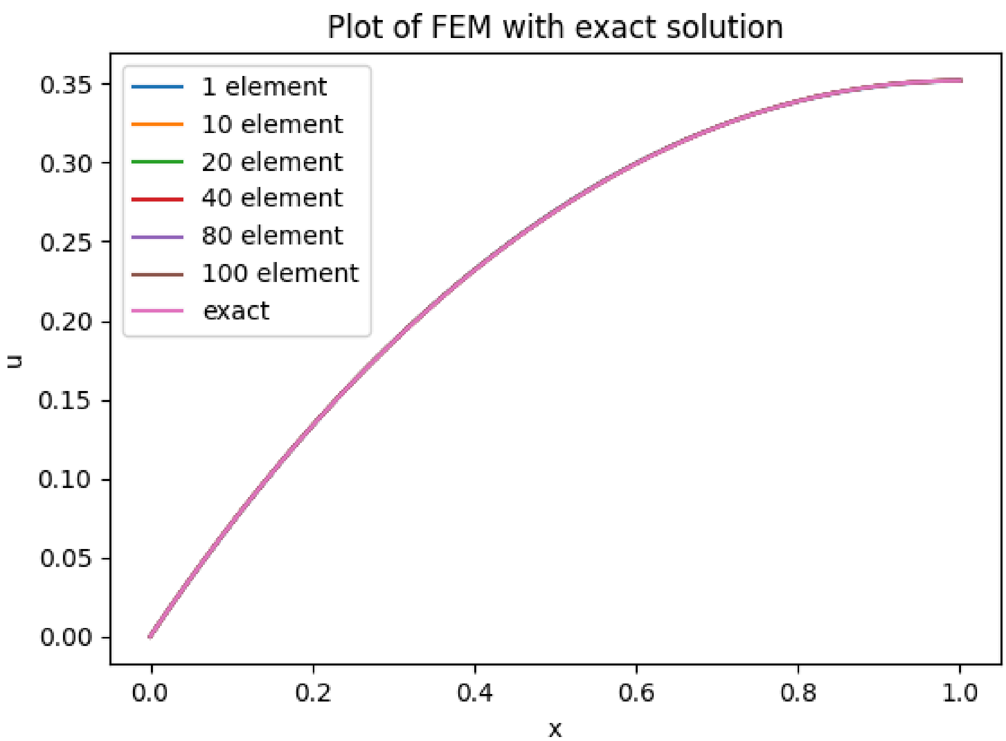
Linear



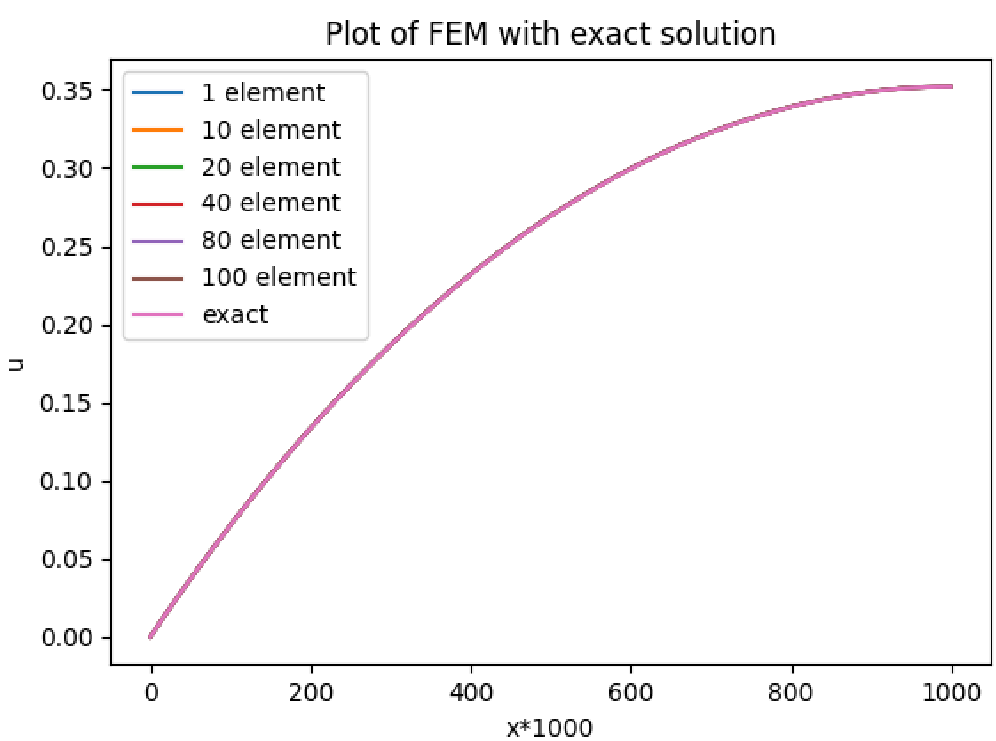
Quadratic



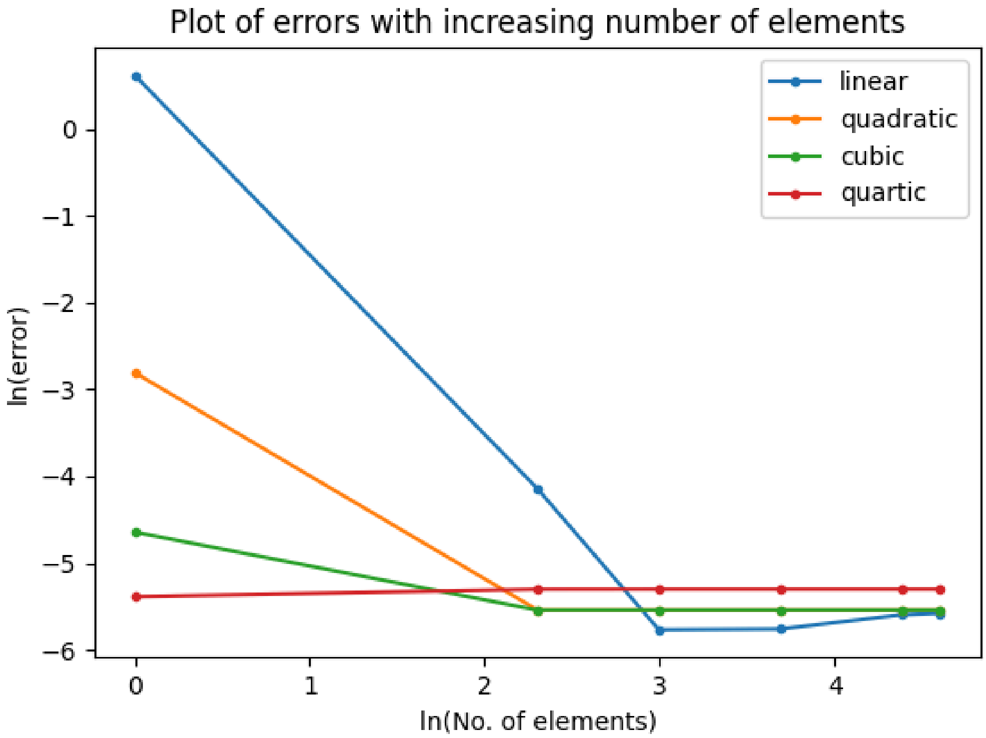
Cubic



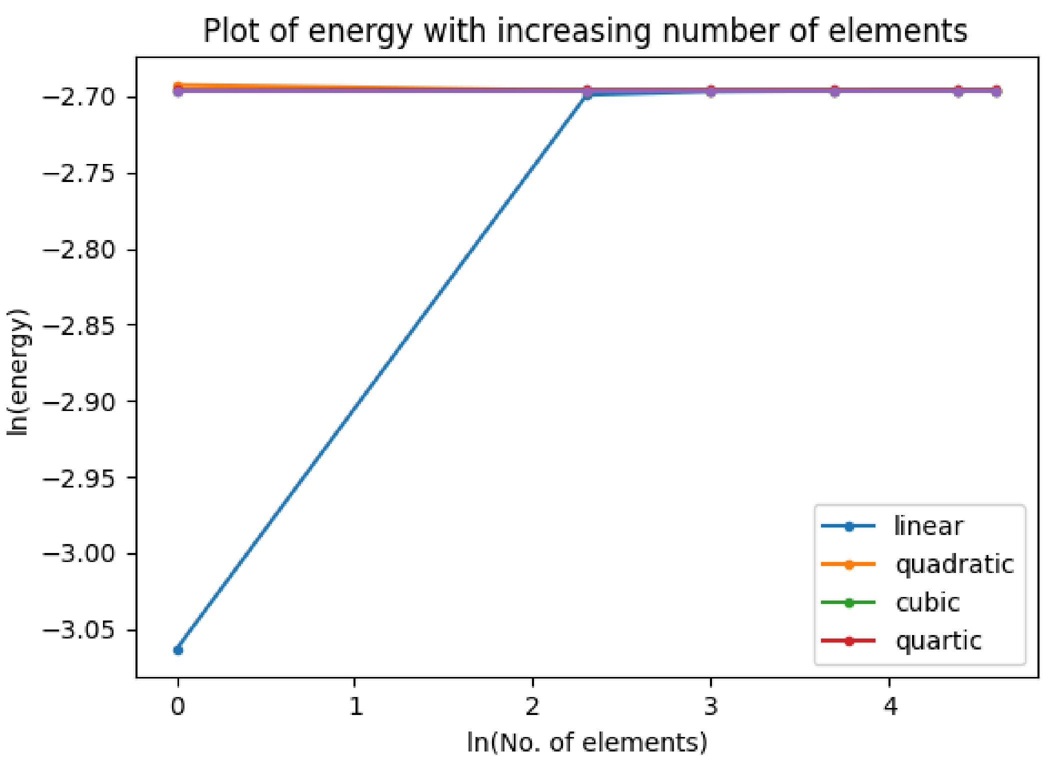
Quartic



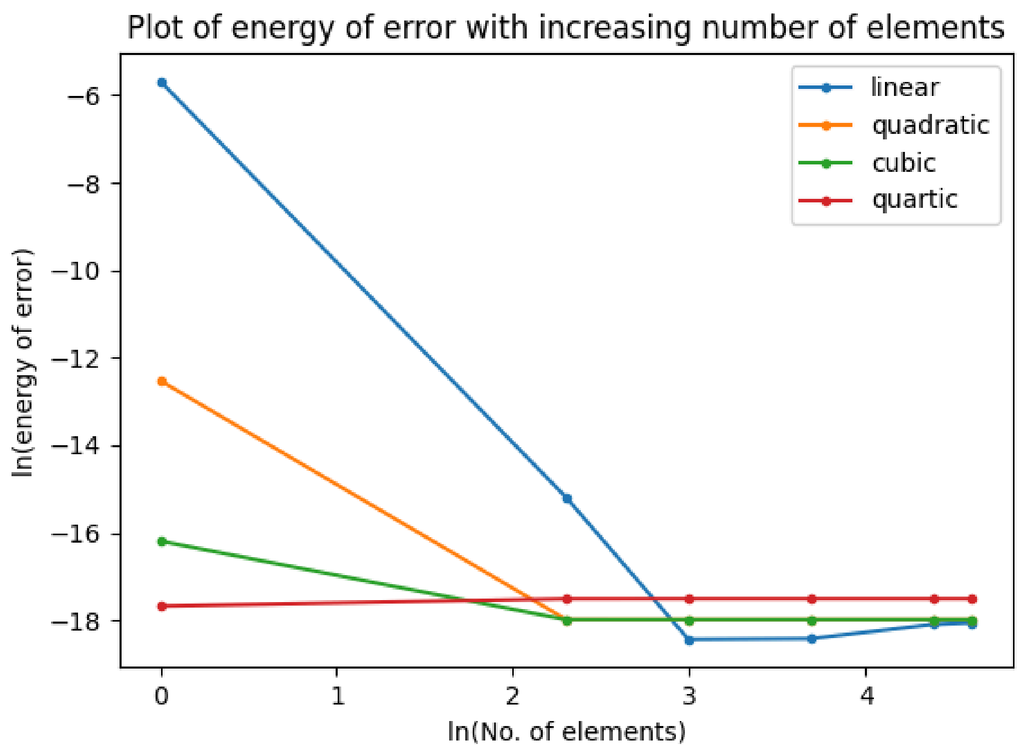
Error



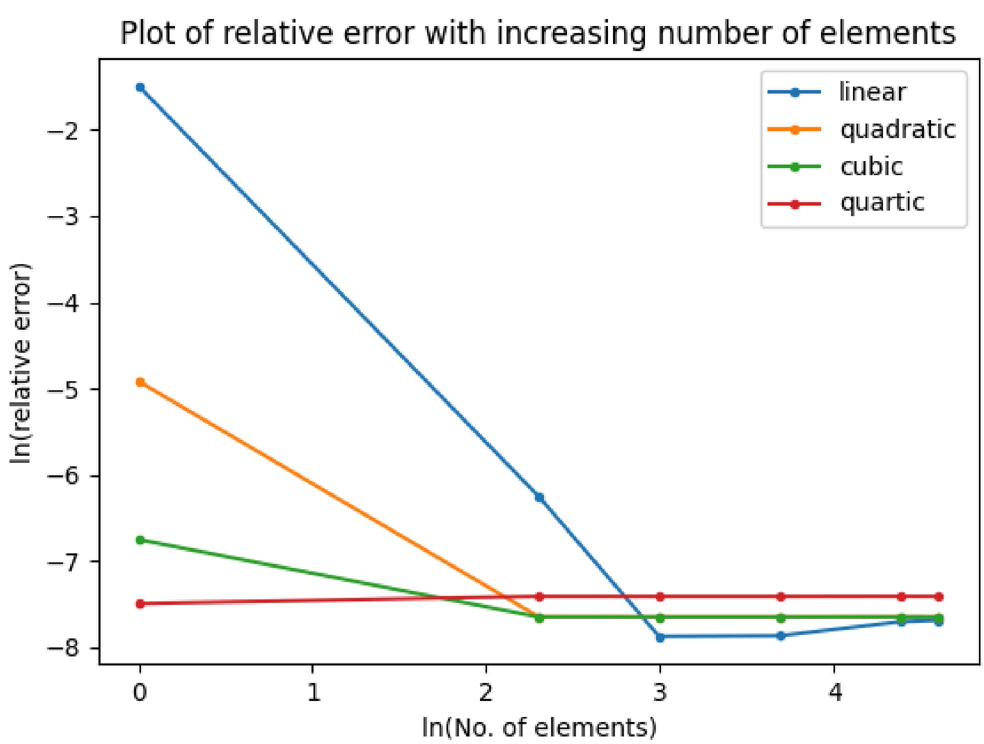
Energy



Energy of error



Relative error



As the number of elements in the discretized domain increases, the FEM solution approaches the exact solution. This is because a finer mesh allows for a more accurate representation of the exact solution. Each element contributes to capturing local variations, and as the mesh is refined, these combine to yield a more faithful representation of the overall solution.
Moreover, the order of approximation functions employed within each element also plays a crucial role in the accuracy of the FEM solution. Higher-order approximation functions can better capture complex variations in the solution, allowing for more accurate representation of phenomena such as steep gradients or sharp changes in behavior. By increasing the order of approximation functions, the FEM can better approximate the exact solution.
When we do reach quartic functions for approximating, the 6 point integration errors also play a dominant role in the approximating solution. This is why it becomes more and more visible as we keep increasing the number of elements.

Convergence rate of Energy of error solution:

$log(U(error)) = \; -\, 2\beta.\, log(n) \,+\, log(c)$
where β is the convergence rate and c is a constant

Linear: 2.13	Quadratic: 1.17
Cubic: 0.38	Quartic: -0.04

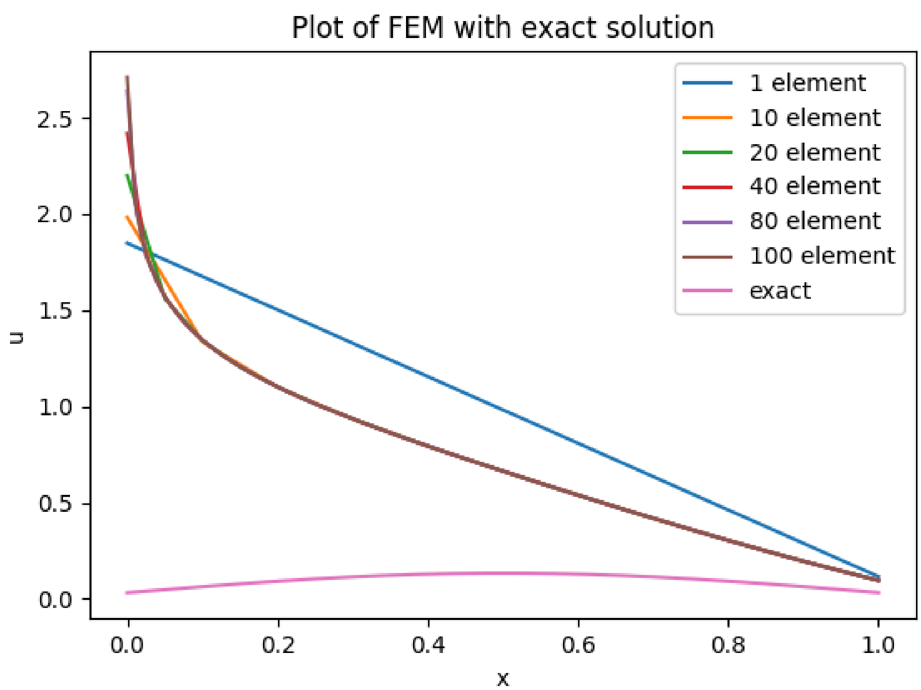
$log(||u_h - u_{ex}||) = \; -\, \beta.\, log(n) \,+\, log(c)$

Linear: 2.13	Quadratic: 1.17
Cubic: 0.38	Quartic: -0.04

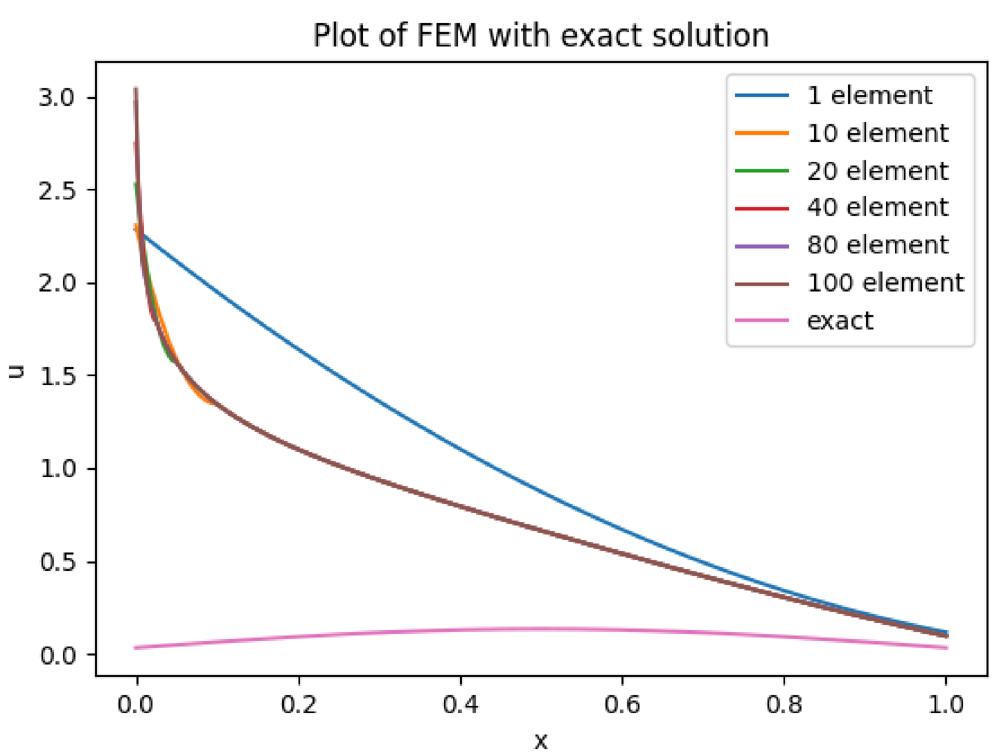
We see that the convergence rate decreases with the order of the approximating functions. This is because the rate of decrement of error is lesser as the order increases. This observation is in alignment with the theory. We see that when a larger number of elements are used, the error of integration becomes a major component in the total error and thus the graph is no longer linear.

Q4

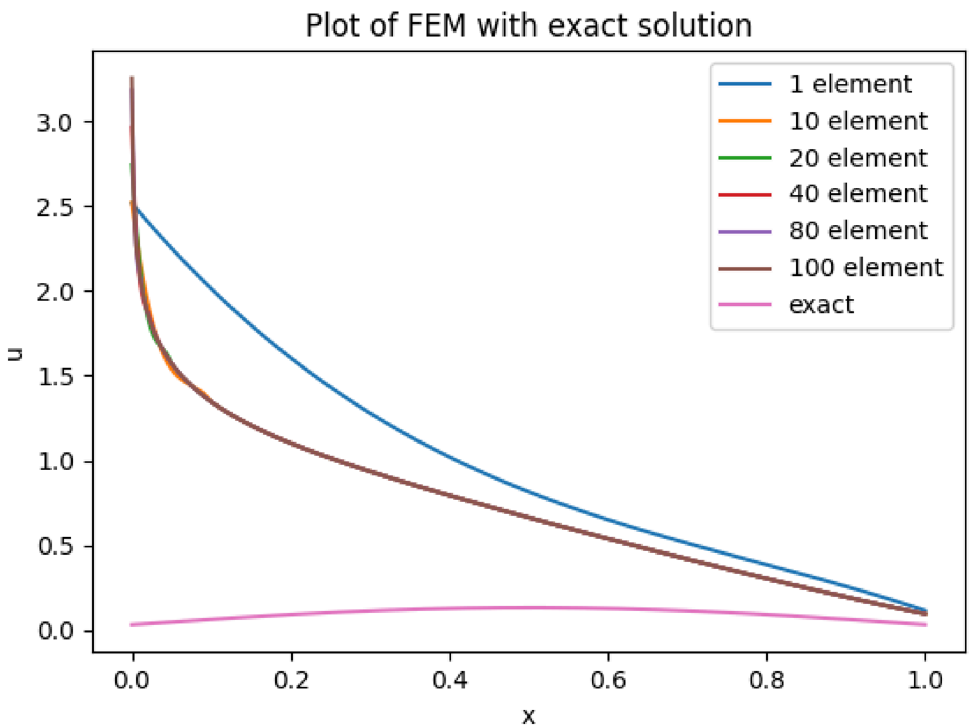
Linear



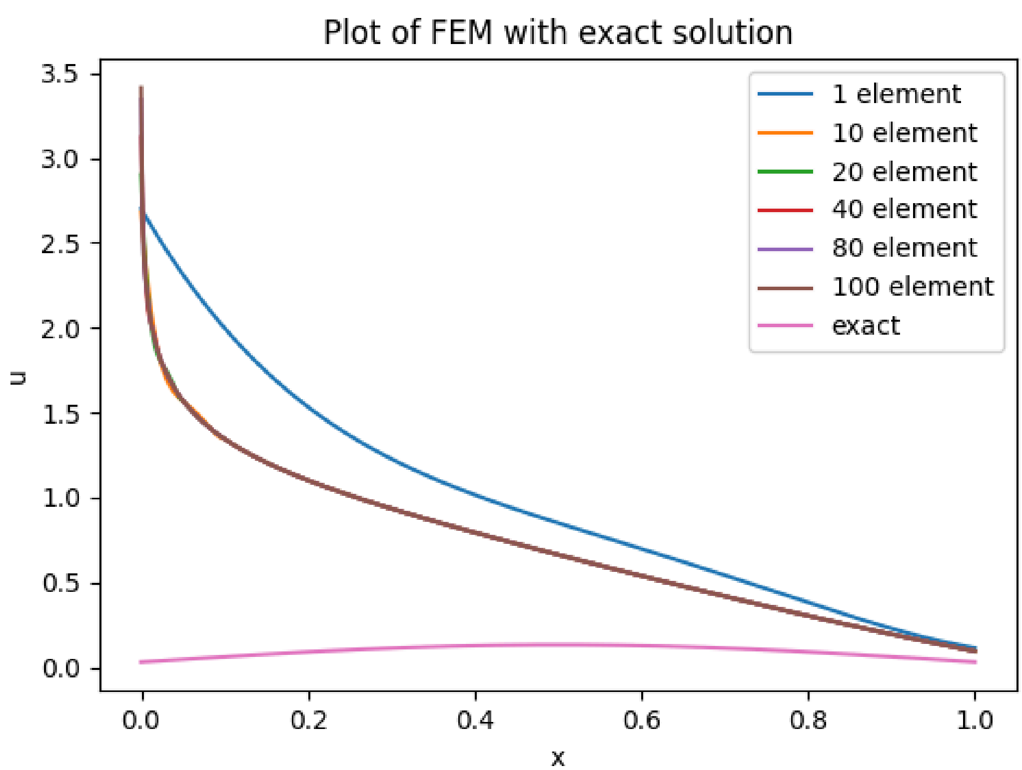
Quadratic



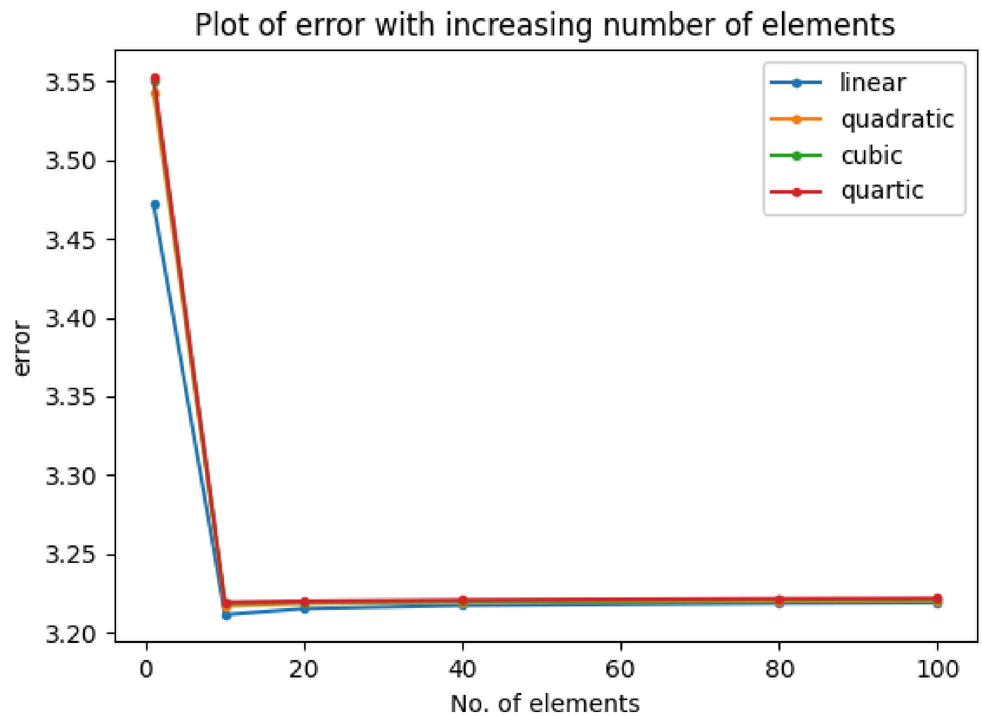
Cubic



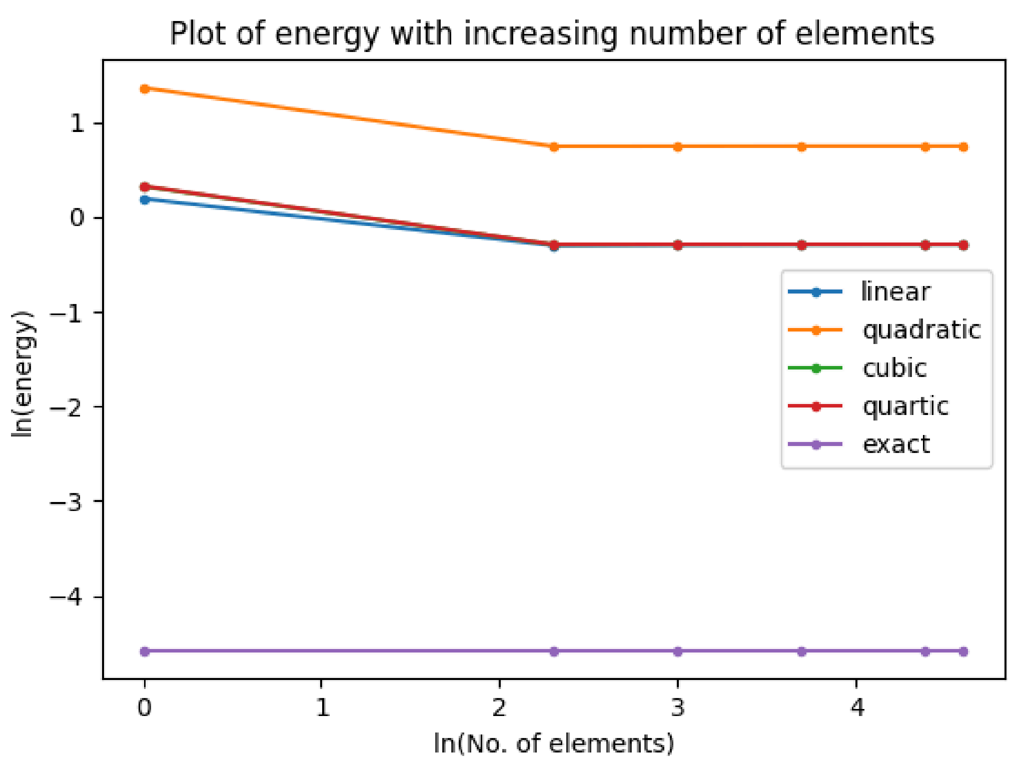
Quartic



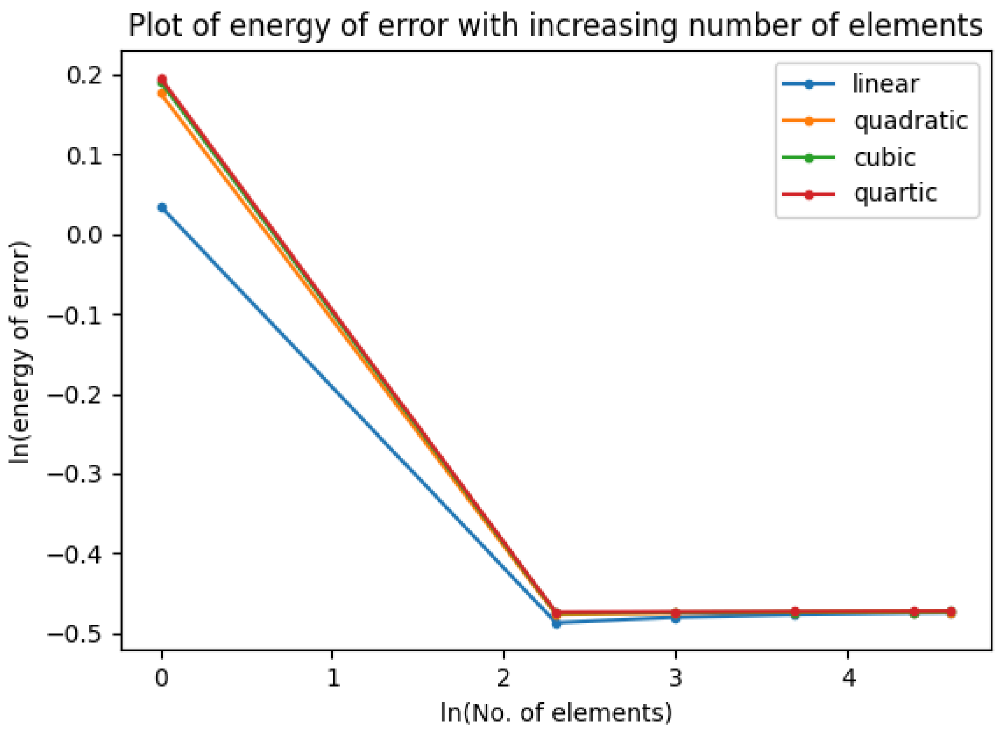
Error



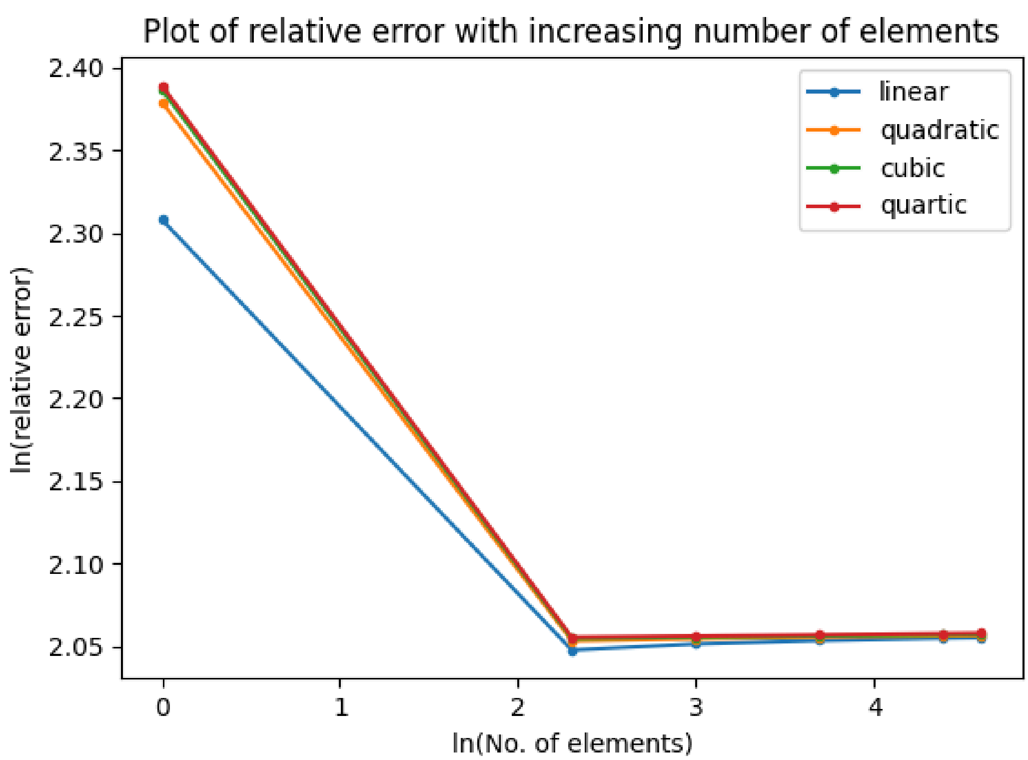
Energy



Energy of error



Relative error



Here, we have the force in form of sin(πx) so the approximation in the form of taylor series must be upto at least 9th power so as to minimize the error in approximation. If we take the series till the power of 3 then, the boundary conditions are not matched properly and this results in a lot of error in interpolation itself. This is because our code is not designed to handle sinusoidal forces. Maybe if our approximating functions were sinusoidal, then the solution could have been very close to the exact solution.

If we use 9th order polynomials for approximating the forcing, the error in boundary conditions is highly reduced but since we are using 6 point integration over the master element, our error keeps on increasing from linear to quartic to higher orders. This error cannot be avoided and thus, the solutions do not appear to be matching the exact solution.

Convergence rate of Energy of error solution:

$$log(U(error)) = \ - 2\beta. log(n) \ + \ log(c)$$

where β is the convergence rate and c is a constant

Linear: 0.113 Quadratic: 0.141

Cubic: 0.144 Quartic: 0.146

$$log(||u_h - u_{ex}||) = \ - \ \beta. log(n) \ + \ log(c)$$

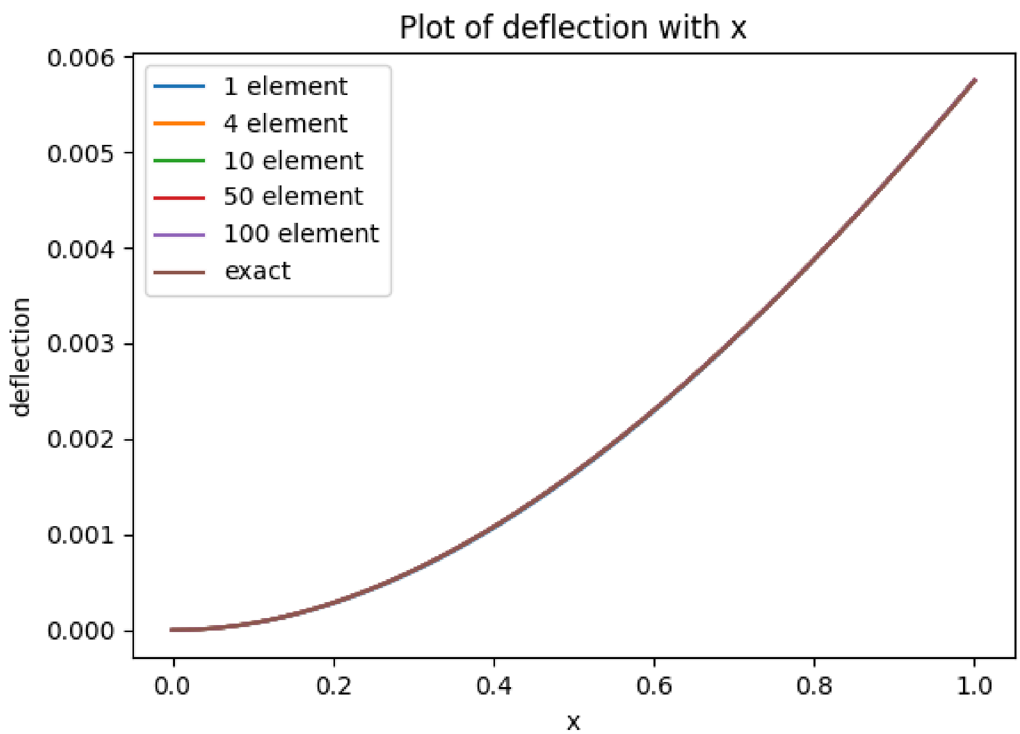
Linear: 0.113 Quadratic: 0.141

Cubic: 0.144 Quartic: 0.146

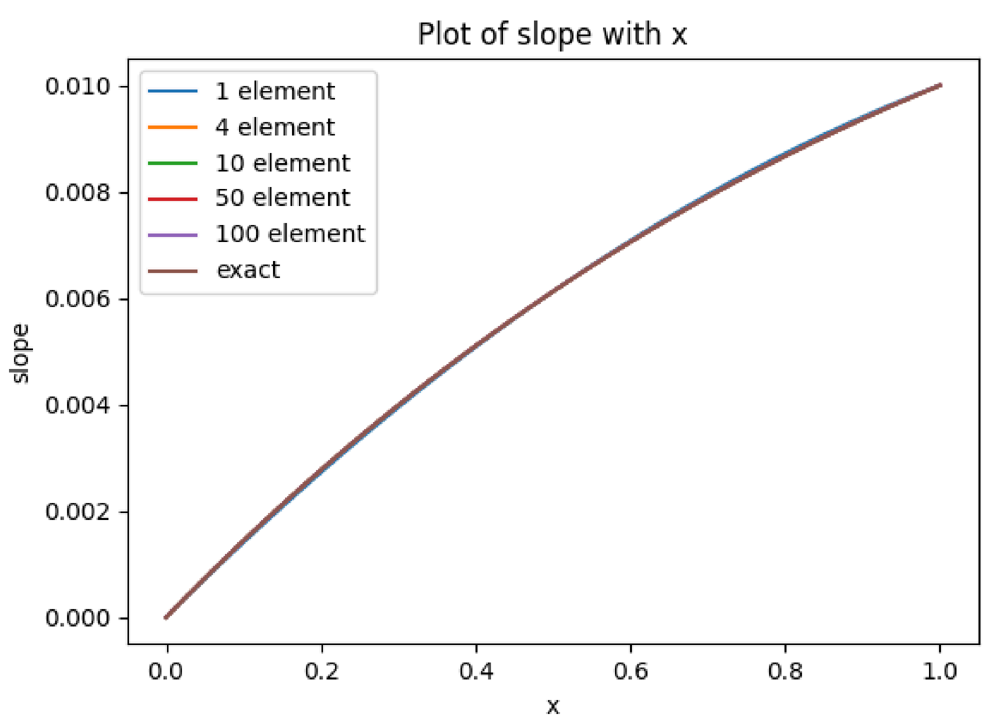
We see that the convergence rate increases with the order of the approximating functions. In theory the rate of decrement of error is greater as the order increases. This observation is not in alignment with the theory. Here, the approximating error and integration error are too great for the solution to be correct so the errors are very large compared to the actual solution. Here, we also see that the linear error is least in value so the amount of reduction in error is lesser over the number of elements. The error increases with the order of the elements, since the error of integration increases with the order of the approximating functions.

Beam bending

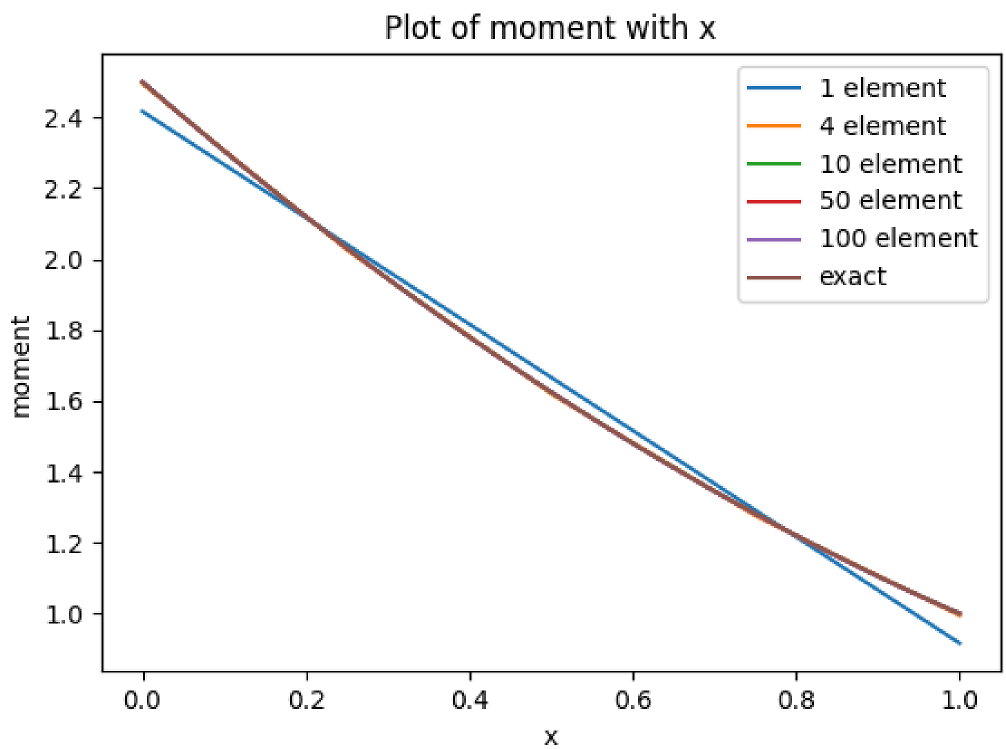
Deflection



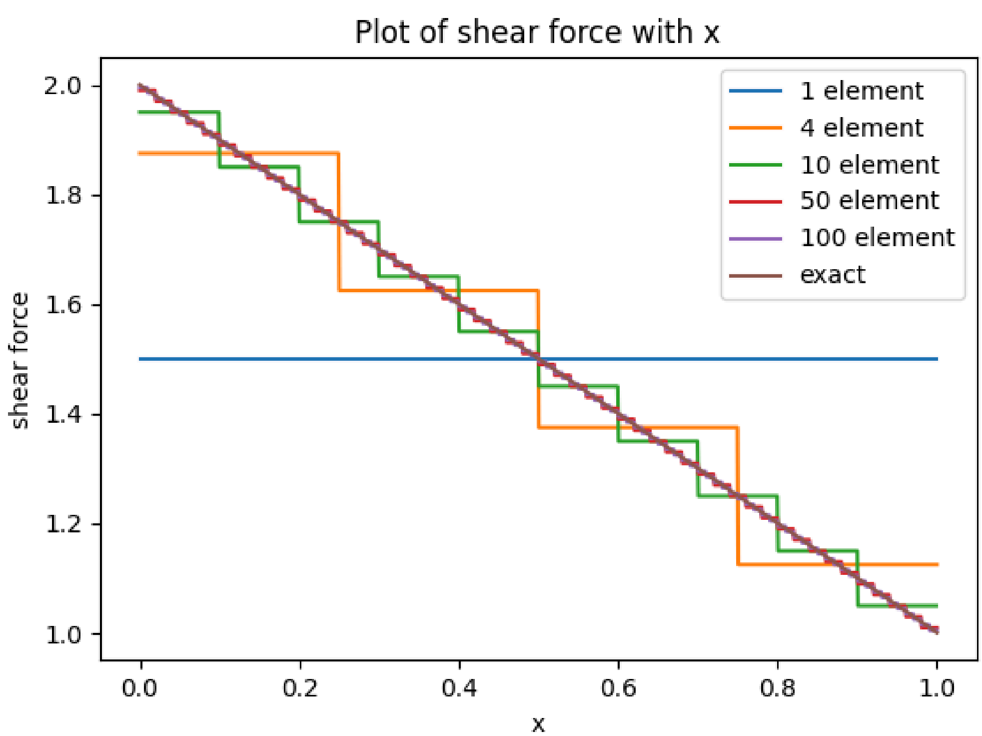
Slope



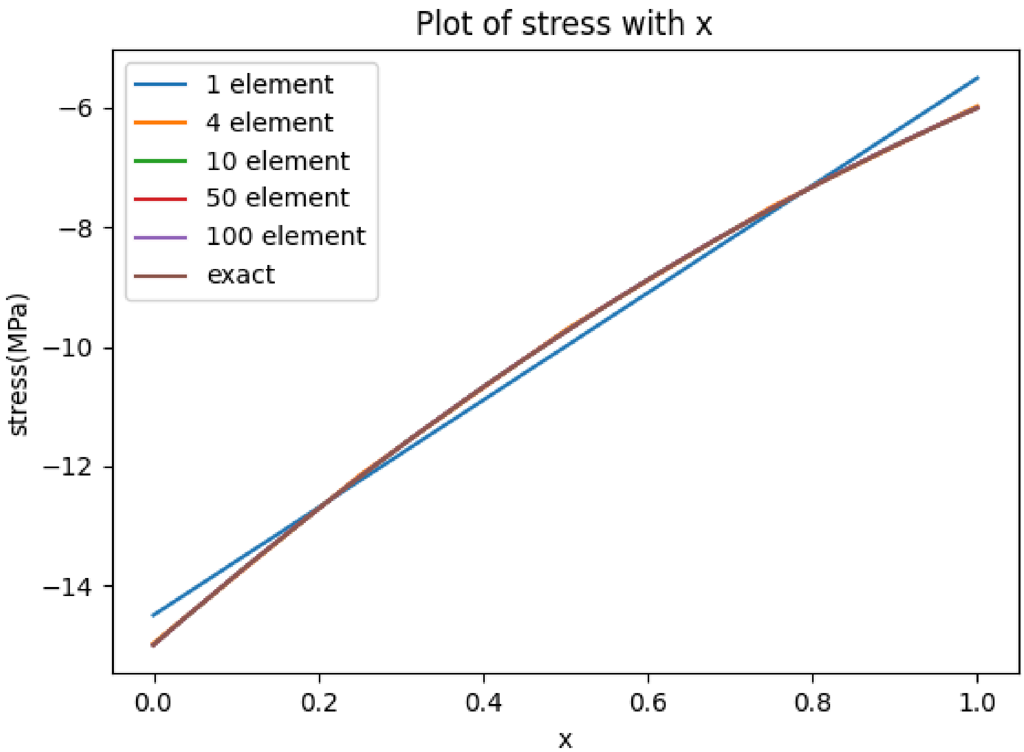
Moment



Force



Stress



We see that if we give a constant force distribution to the beam and then plot out the exact solution, then we see that we get a quartic exact solution from the Euler-Bernoulli beam bending theory. Therefore, if we try to approximate it using cubic hermite polynomials, we will see some error in approximation but as we keep increasing the number of elements, the approximation gets very close to the exact solution.

As we move to the slope, cubic polynomials are being approximated by quadratic polynomials, so the same thing happens as it happened with the deflection.

The clear difference in the order of approximation function is visible in moment curves, where the quadratic curves are approximated by linear curves, where the RMS error is minimum. But after 10 elements, the difference is not that much visible.

The shear force which is linear is approximated using constant functions so the average value at the boundary of the element is the value of that constant. Thus, we see that as we keep on increasing the number of elements, it almost approximates a sloped line.

The stress at the topmost point of the beam's cross section behaves like the moment plot itself as derived by the Euler-Bernoulli beam bending theory because we are using a beam of uniform square cross-section.