UNIT-III
Relations and Their Properties

Relations

- Binary relations represent relationships between the elements of two sets.
- A binary relation R from set A to set B is defined by: $R \subseteq A \times B$
- If $(a,b) \in R$, we write:

  aRb (a is related to b by R)
- If $(a,b) \notin R$, we write:

  $a\not{R}b$ (a is not related to b by R)

Relations

- A relation is represented by a *set* of *ordered pairs*
- If A = {a, b} and B = {1, 2, 3}, then a relation R_1 from A to B might be, for example, $R_1 = \{(a, 2), (a, 3), (b, 2)\}$.
- The first element in each ordered pair comes from set A, and the second element in each ordered pair comes from set B.

- Example:

  $A = \{0,1,2\}$

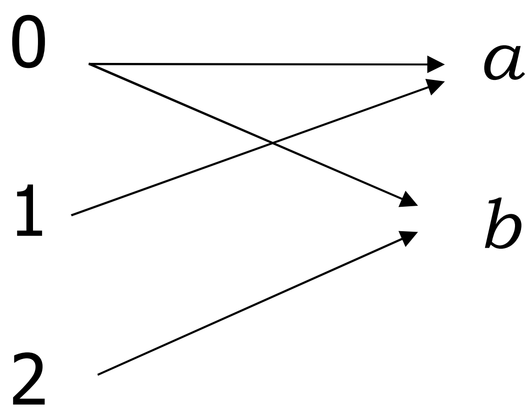
  $B = \{a,b\}$

  $A \times B = \{(0,a), (0,b), (1,a), (1,b), (2,a), (2,b)\}$

- Then $R = \{(0,a), (0,b), (1,a), (2,b)\}$ is a *relation* from A to B.
  - ✓Can we write $0Ra$?
  - ✓Can we write $2Rb$?
  - ✓Can we write $1Rb$?

Example

- A relation may be represented graphically or as a table:



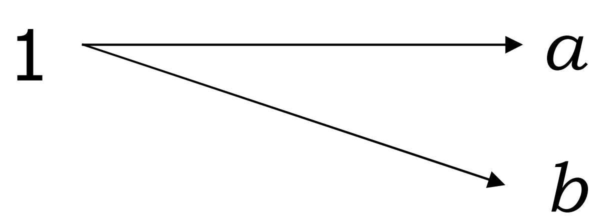
R	a	b
0	X	X
1	X	
2		X

We can <u>see</u> that $0Ra$ but $1 \not R b$.

Functions as Relations

- A function is a relation that has the restriction that each element of A can be related to exactly one element of B.

Relation



Function

Relations on a Set

- Relations can also be from a set to itself.
- A relation on the set A is a relation from set A to set A, i.e., $R \subseteq A \times A$
- Let $A = \{1, 2, 3, 4\}$
- Which ordered pairs are in the relation $R = \{(a,b) \mid a \text{ divides } b\}$?

;$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Relations on a Set

- Which of these relations (on the set of integers) contain each of the pairs (1,1), (1,2), (2,1), (1,-1), and (2,2)?

$$R_1 = \{(a,b) \mid a \leq b\}$$

$$R_2 = \{(a,b) \mid a > b\}$$

$$R_3 = \{(a,b) \mid a = b \; or \; a = -b\}$$

$$R_4 = \{(a,b) \mid a = b\}$$

$$R_5 = \{(a,b) \mid a = b + 1\}$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}$$

Relations on a Set

- The pair (1,1) is in R_1, R_3, R_4 and R_6
- The pair (1,2) is in R_1 and R_6
- The pair (2,1) is in R_2 , R_5 and R_6
- The pair (1,-1) is in R_2 , R_3 and R_6
- The pair (2,2) is in R_1 , R_3 and R_4

Relations on a Set

- How many relations are there on a set with n elements?

$$2^{n^2}$$

- If A has n elements, how many elements are there in $A \times A$?

$$n^2$$

Relations on a Set

- How many relations are there on set $S = \{a, b, c\}$?

- There are 3 elements in set S, so $S \times S$ has $3^2 = 9$ elements.

- Therefore, there are $2^9 = 512$ different relations on the set $S = \{a, b, c\}$.

Properties of Relations

- Let R be a relation on set A.
- R is *reflexive* if:

  $(a, a) \in R$ for every element $a \in A$.

Example

- Determine the properties of the following relations on {1, 2, 3, 4}
- Which of these is reflexive?

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{\mathbf{(1,1)}, (1,2), (1,4), (2,1), \mathbf{(2,2)}, \mathbf{(3,3)}, (4,1), \mathbf{(4,4)}\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{\mathbf{(1,1)}, (1,2), (1,3), (1,4), \mathbf{(2,2)}, (2,3), (2,4), \mathbf{(3,3)}, (3,4), \mathbf{(4,4)}\}$
$R_6 = \{(3,4)\}$

- The relations R_3 and R_5 are reflexive because they contain <u>all</u> pairs of the form (a,a); the other don’t [they are all missing (3,3)].

Properties of Relations

- Let R be a relation on set A.
- R is *symmetric* if:

  $(b, a) \in R$ whenever $(a, b) \in R$,

  where $a, b \in A$.

A relation is symmetric iff "a is related to b" implies that "b is related to a".

Example

- Which of these is symmetric?

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (\mathbf{1},\mathbf{2}), (\mathbf{2},\mathbf{1})\}$
$R_3 = \{(1,1), (\mathbf{1},\mathbf{2}), (\mathbf{1},\mathbf{4}), (\mathbf{2},\mathbf{1}), (2,2), (3,3), (\mathbf{4},\mathbf{1}), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

- The relations R_2 and R_3 are symmetric because in each case (b,a) belongs to the relation whenever (a,b) does.
- The other relations aren't symmetric.

Properties of Relations

- Let R be a relation on set A.
- R is *antisymmetric* if whenever $(a, b) \in R$ and $(b, a) \in R$, then $a = b$, where $a, b \in A$.

A relation is antisymmetric iff there are no pairs of distinct elements with a related to b and b related to a. That is, the only way to have a related to b and b related to a is for a and b to be the same element.

- Symmetric and antisymmetric are NOT exactly opposites.

Example

- Which of these is antisymmetric?

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

- The relations R_4, R_5 are antisymmetric because there is no pair of elements a and b with $a \neq b$ such that both (a,b) and (b,a) belong to the relation.
- The other relations aren't antisymmetric.

Properties of Relations

- Let R be a relation on set A.
- R is *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, where $a, b, c \in A$.

Example

- Which of these is transitive?

  $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
  $R_2 = \{(1,1), (1,2), (2,1)\}$
  $R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$
  $R_4 = \{\ \boldsymbol{(2,1)}, \mathbf{(3,1)}, \underline{\mathbf{(3,2)}}, (4,1), \underline{(4,2)}\ , \underline{(4,3)}\}$
  $R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
  $R_6 = \{(3,4)\}$

- The relations R_4, R_5 are transitive because if (a,b) and (b,c) belong to the relation, then (a,c) does also.
- The other relations aren't transitive.

Combining Relations

Relations from A to B are subsets of $A \times B$.

For example, if $A = \{1, 2\}$ and $B = \{a, b\}$, then
$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

Two relations from A to B can be combined in any way that two sets can be combined. Specifically, we can find the *union*, *intersection*, *exclusive-or*, and *difference* of the two relations.

Combining Relations

Let A = {1, 2, 3} and B= {1, 2, 3, 4}, and suppose we have the relations:

$R_1 = \{(1,1), (2,2), (3,3)\}$, and

$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$.

Then we can find the union, intersection, and difference of the relations:

$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

Composition of Relations

- If R_1 is a relation from A to B and R_2 is a relation from B to C, then the composition of R_1 with R_2 (denoted $R_1 \circ R_2$) is the relation from A to C.
  - If (a, b) is a member of R_1 and (b, c) is a member of R_2, then (a, c) is a member of $R_1 \circ R_2$, where $a \in A$, $b \in B$, $c \in C$.

Example

- Let

  $A=\{a,b,c\}$, $B=\{w,x,y,z\}$, $C=\{A,B,C,D\}$

  $R_1=\{(a,z),(b,w)\}$, $R_2=\{(w,B),(w,D),(x,A)\}$
- Find *R1o R2*
- { (b,B),(b,D)}

Example

- Given the following relations, find $S \circ R$:

$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

$S = \{(1,0),(2,0), (3,1), (3,2), (4,1)\}$

- Construct the ordered pairs in $S \circ R$:

$S \circ R = \{(3,1),(3,4),(3,3)(4,1),(4,4)\}$

The Powers of a Relation

- The powers of a relation R are recursively defined from the definition of a composite of two relations.

- Let R be a relation on the set A. The powers R^n, for $n = 1, 2, 3, \ldots$ are defined recursively by:

  $$R^1 = R$$

  $$R^{n+1} = R^n \circ R$$

So:

$$R^2 = R \circ R$$

$$R^3 = R^2 \circ R = (R \circ R) \circ R)$$

etc.

The Powers of a Relation

- Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$
- Find the powers R^n , where $n = 1, 2, 3, 4, \ldots$

$$R^1 = R = \{(1,1), (2,1), (3,2), (4,3)\}$$

$$R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$

$$R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$

$$R^5 = R^4 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$

Representing Relations

Representing Relations Using Matrices

- Let R be a relation from A to B

  $A = \{a_1, a_2, \ldots , a_m\}$

  $B = \{b_1, b_2, \ldots , b_n\}$

- The zero-one matrix representing the relation R has a 1 as its (i, j) entry when a_i is related to b_j and a 0 in this position if a_i is not related to b_j.

Example

- Let R be a relation from A to B

  $A=\{a, b, c\}$

  $B=\{d, e\}$

  $R=\{(a, d), (b, e), (c, d)\}$
- Find the relation matrix for R

Relation Matrix

Let R be a relation from A to B

$A=\{a, b, c\}$

$B=\{d, e\}$

$R=\{(a, d), (b, e), (c, d)\}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note that A is represented by the rows and B by the columns in the matrix.

Cell_{ij} in the matrix contains a 1 iff a_i is related to b_j.

Relation Matrices and Properties

- Let R be a binary relation on a set A and let M be the zero-one matrix for R.
  - R is *reflexive* iff $M_{ii} = 1$ for all i
  - R is *symmetric* iff M is a symmetric matrix, i.e., $M = M^{\mathrm{T}}$
  - R is *antisymmetric* if $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$

Example

- Suppose that the relation R on a set is represented by the matrix M_R.

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- Is R reflexive, symmetric, and / or antisymmetric?

Example

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive?
Is R symmetric?
Is R antisymmetric?

- All the diagonal elements = 1, so R is reflexive.
- The lower left triangle of the matrix = the upper right triangle, so R is symmetric.
- To be antisymmetric, it must be the case that no more than one element in a symmetric position on either side of the diagonal = 1. But $M_{23} = M_{32} = 1$. So R is not antisymmetric.

Representing Relations Using Digraphs

- Represent:
  - each element of the set by a point
  - each ordered pair using an arc with its direction indicated by an arrow

Representing Relations Using Digraphs

- A *directed graph* (or *digraph*) consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*).
- The vertex *a* is called the *initial vertex* of the edge (*a*, *b*).
- The vertex b is called the *terminal vertex* of the edge (*a*, *b*).

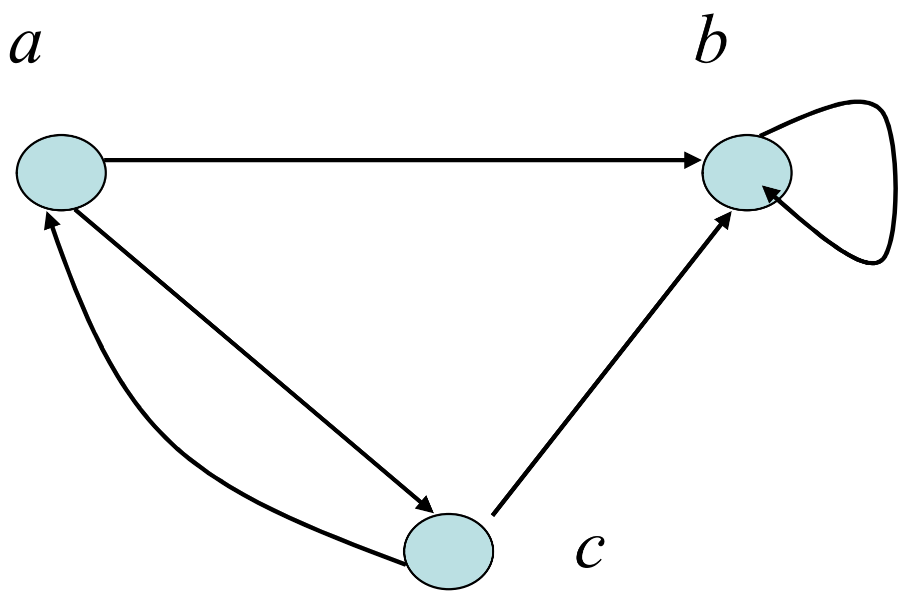
Example

- Let R be a relation on set A

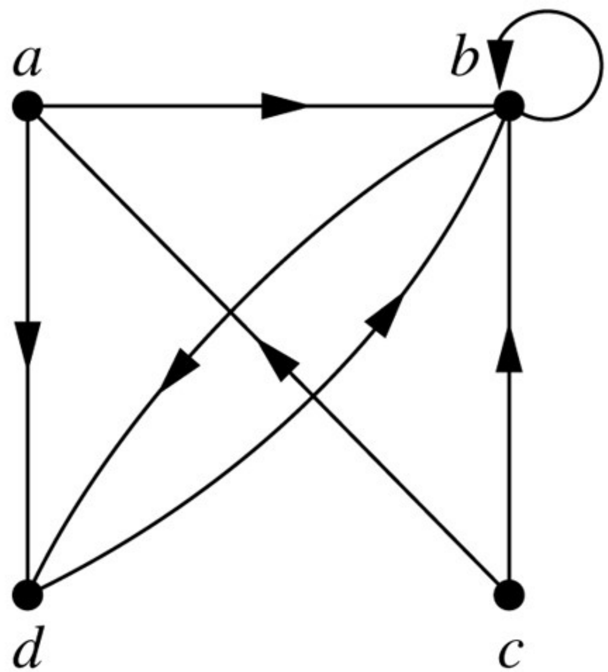
  $A=\{a, b, c\}$

  $R=\{(a, b), (a, c), (b, b), (c, a), (c, b)\}$.

- Draw the digraph that represents R



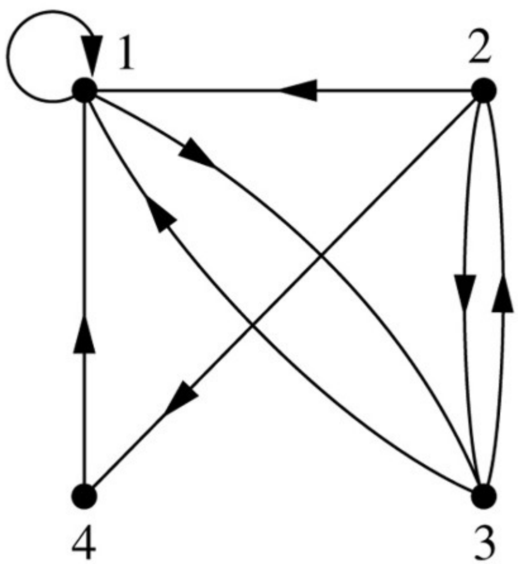
Representing Relations Using Digraphs



This is a digraph with:
V = {a, b, c}
E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}

Note that edge (*b*, *b*) is represented using an arc from vertex *b* back to itself. This kind of an edge is called a *loop*.

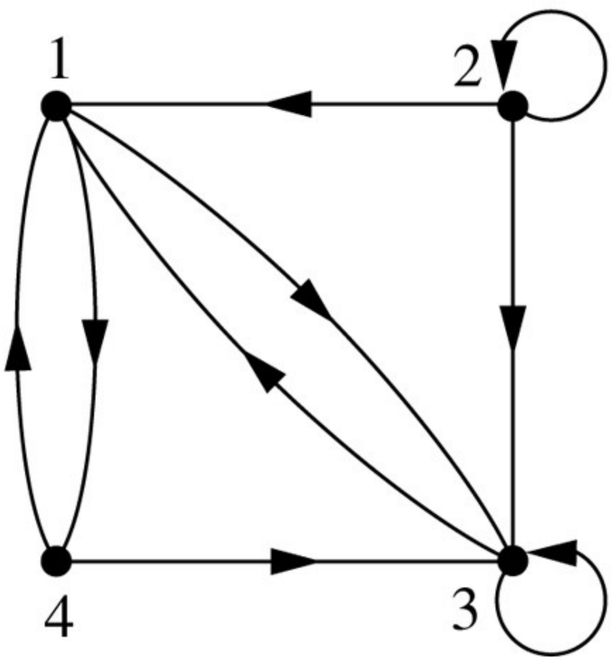
Example



What are the ordered pairs in the relation R represented by the directed graph to the left?

This digraph represents the relation
$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$
on the set $\{1, 2, 3, 4\}$.

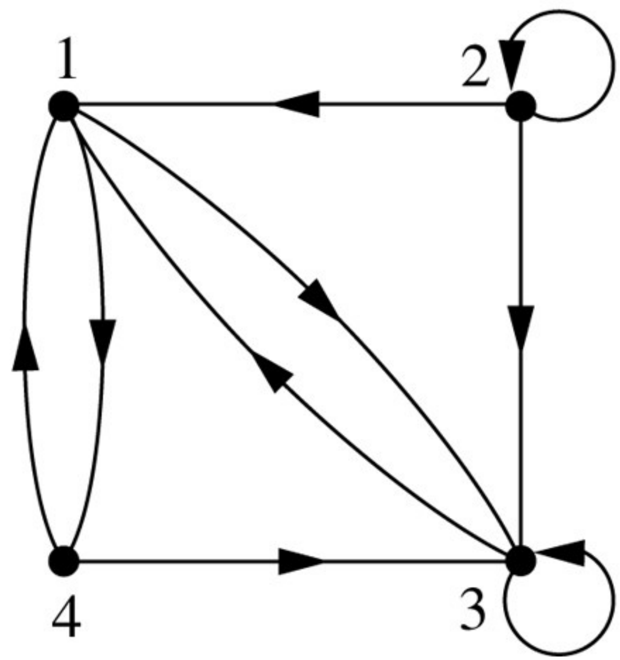
Example



What are the ordered pairs in the relation R represented by the directed graph to the left?

$R = \{(1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,3), (4,1), (4,3)\}$

Example



According to the digraph representing R:

- is (4,3) an ordered pair in R?
- is (3,4) an ordered pair in R?
- is (3,3) an ordered pair in R?

(4,3) is an ordered pair in R

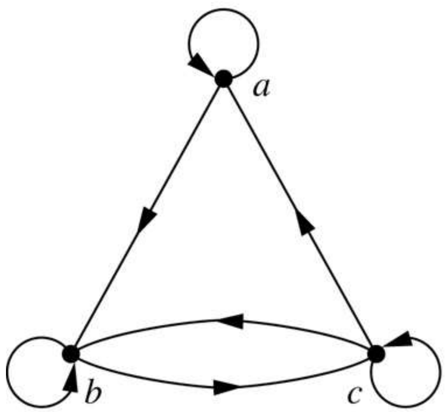
(3,4) is <u>not</u> an ordered pair in R – no arrowhead pointing from 3 to 4

(3,3) is an ordered pair in R – loop back to itself

Relation Digraphs and Properties

- A relation digraph can be used to determine whether the relation has various properties
  - *Reflexive* - must be a loop at every vertex.
  - *Symmetric* - for every edge between two distinct points there must be an edge in the opposite direction.
  - *Antisymmetric* - There are never two edges in opposite direction between two distinct points.
  - *Transitive* - If there is an edge from x to y and an edge from y to z, there must be an edge from x to z.

Example



(a) Directed graph of *R*

According to the digraph representing *R*:

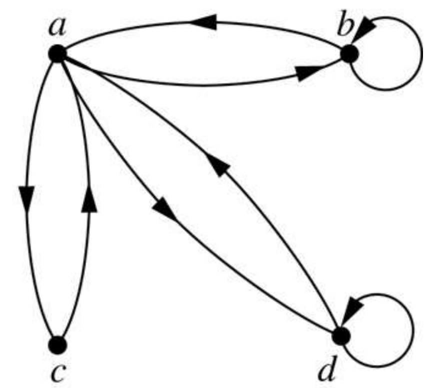
- is *R* reflexive?
- is *R* symmetric?
- is *R* antisymmetric?
- is *R* transitive?

- R is reflexive – there is a loop at every vertex
- R is not symmetric – there is an edge from *a* to *b* but not from *b* to *a*
- R is not antisymmetric – there are edges in both directions connecting *b* and *c*
- R is not transitive – there is an edge from *a* to *b* and an edge from *b* to *c*, but not from *a* to *c*

Example

According to the digraph representing *S*:

- is *S* reflexive?
- is *S* symmetric?
- is *S* antisymmetric?
- is *S* transitive?



(b) Directed graph of *S*

- S is not reflexive – there aren't loops at every vertex
- S is symmetric – for every edge from one distinct vertex to another, there is a matching edge in the opposite direction
- S is not antisymmetric – there are edges in both directions connecting *a* and *b*
- S is not transitive – there is an edge from *c* to *a* and an edge from *a* to *b*, but not from *c* to *b*

Equivalence Relations

Equivalence Relations

- A relation on set A is called an *equivalence relation* if it is:
  - reflexive,
  - symmetric, and
  - transitive

Equivalence Relations

- Two elements a and b that are related by an equivalence relation are said to be *equivalent*.
- We use the notation

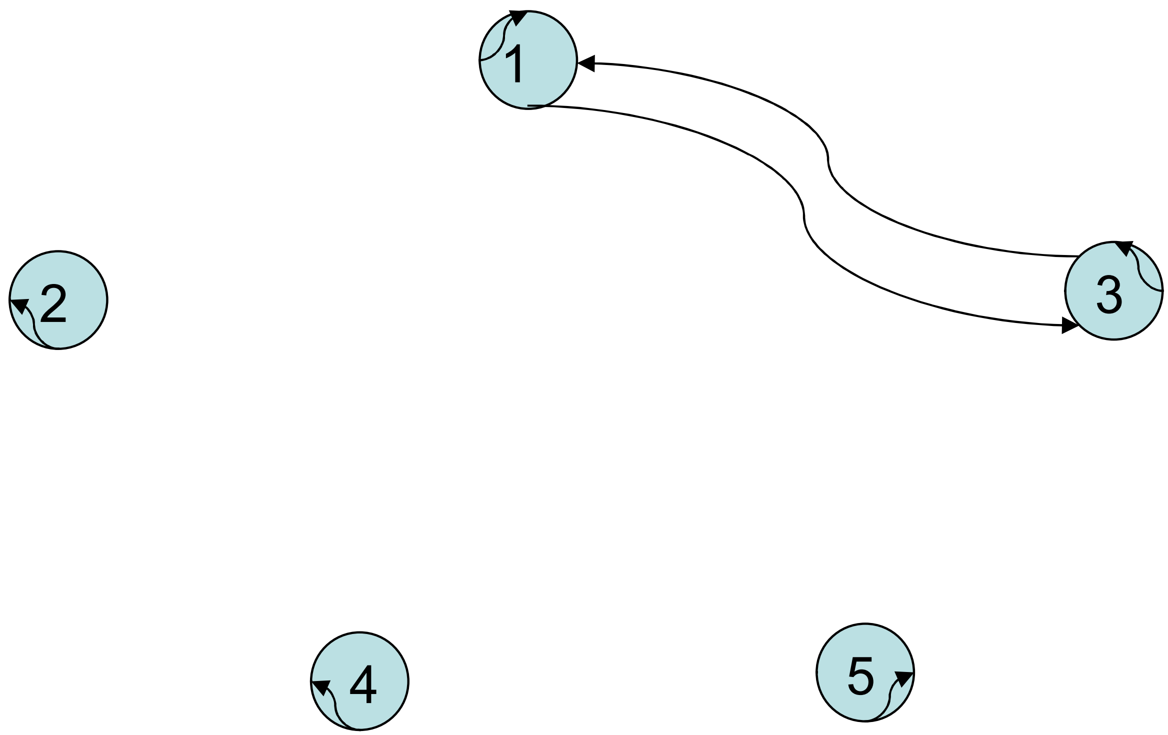
  $$a \sim b$$

  to denote that a and b are equivalent elements with repect to a particular equivalence relation.

Example

- Let R be a relation on set A, where $A = \{1, 2, 3, 4, 5\}$ and $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,1)\}$
- Is R an equivalence relation?
- We can solve this by drawing a relation digraph:
  - *Reflexive* – there must be a loop at every vertex.
  - *Symmetric* - for every edge between two distinct points there must be an edge in the opposite direction.
  - *Transitive* - if there is an edge from x to y and an edge from y to z, there must be an edge from x to z.

Example



Is R an equivalence relation? *yes*

Example – Congruence modulo m

- Let $R = \{(a, b) \mid a \equiv b \pmod{m}\}$ be a relation on the set of integers and m be a positive integer > 1.

  Is R an equivalence relation?
  - *Reflexive* – is it true that $a \equiv a \pmod{m}\}$? *yes*
  - *Symmetric* – is it true that if $a \equiv b \pmod{m}$ then $b \equiv a \pmod{m}$? *yes*
  - *Transitive* - is it true that if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $a \equiv c \pmod{m}$? *yes*

Example

- Suppose that R is the relation on the set of strings of English letters such that aRb iff $l(a) = l(b)$, where $l(x)$ is the length of the string x.
- Is R an equivalence relation?

Example

- Since $l(a) = l(a)$, then aRa for any string a. So R is reflexive.
- Suppose aRb, so that $l(a) = l(b)$. Then it is also true that $l(b) = l(a)$, which means that bRa. Consequently, R is symmetric.
- Suppose aRb and bRc. Then $l(a) = l(b)$ and $l(b) = l(c)$. Therefore, $l(a) = l(c)$ and so aRc. Hence, R is transitive.
- Therefore, R is an equivalence relation.

Equivalence Class

- Let R be a equivalence relation on set A.
- The set of all elements that are related to an element a of A is called the *equivalence class* of a.
- The equivalence class of a with respect to R is:

$$[a]_R = \{s \mid (a,s) \in R\}$$

  – When only one relation is under consideration, we will just write $[a]$.

Equivalence Class

- If R is a equivalence relation on a set A, the *equivalence class* of the element a is:

$$[a]_R = \{s \mid (s, a) \in R\}$$

If $b \in [a]_R$, then b is called a *representative* of this equivalence class.

Equivalence Class

- Let R be the relation on the set of integers such that aRb iff $a = b$ or $a = -b$. We can show that this is an equivalence relation.
- The equivalence class of element a is

  $[a] = \{a, -a\}$
- Examples:

  $[7] = \{7, -7\}$ $[-5] = \{5, -5\}$

  $[0] = \{0\}$

Equivalence Example

- Consider the equivalence relation R on set A. What are the equivalence classes?

  $A = \{1, 2, 3, 4, 5\}$

  $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,1)\}$

- Just look at the aRb relationships. Which elements are related to which?

  $[1] = \{1, 3\}$

  $[2] = \{2\}$

  $[3] = \{3, 1\}$

  $[4] = \{4\}$

  $[5] = \{5\}$

A useful theorem about classes

- Let R be an equivalence relation on a set A. These statements for a and b of A are equivalent:

$$aRb$$

$$[a] = [b]$$

$$[a] \cap [b] \neq \varnothing$$

A useful theorem about classes

- More importantly:

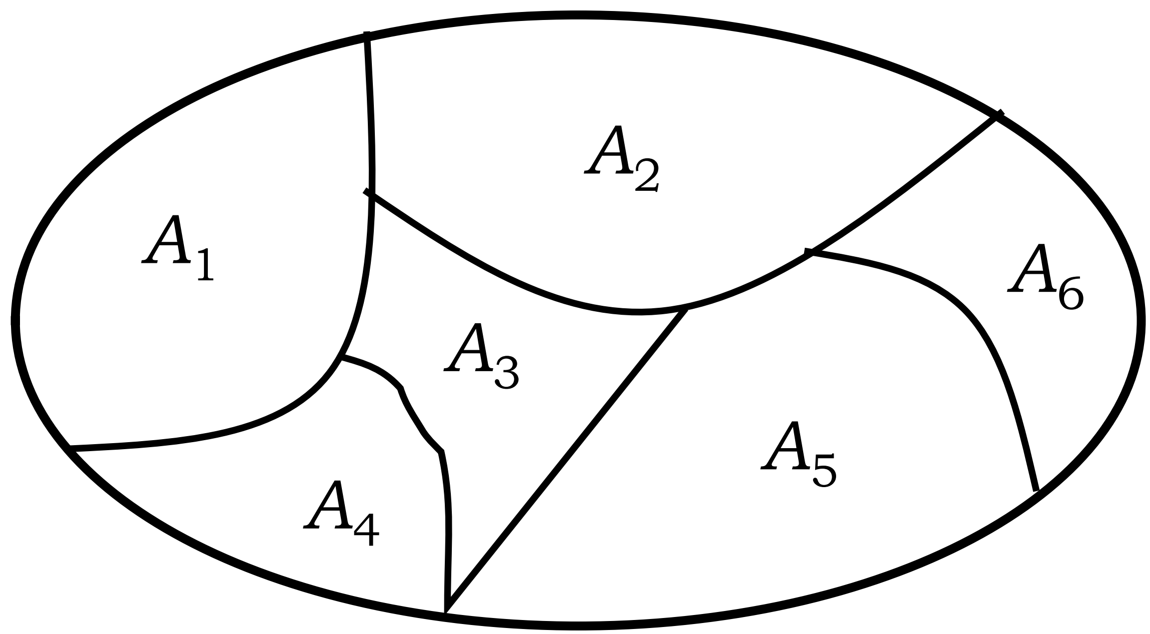
  Equivalence classes are EITHER

  - **equal** or
  - **disjoint**

Partitions

- A *partition* of a set A divides A into non-overlapping subsets:
  - A partition of a set A is a collection of disjoint nonempty subsets of A that have A as their union.

Example 1



Set A

Partitions

- Let S be a given set and A={A1,A2,….Am} where each Ai,i-1,2… is a subset of S
- Set A is called covering of S and setsA1,A2,… are said to cover S
  - A partition of a set A is a collection of disjoint nonempty subsets of A that have A as their union. Then the subsets are called blocks of the partition.
- <u>Example 2</u>

  $S = \{a, b, c, d, e, f\}$

  $S_1 = \{a, d, e\}$

  $S_2 = \{b\}$

  $S_3 = \{c, f\}$

  $P = \{S_1, S_2, S_3\}$

  P is a partition of set S

Example

If $S = \{1, 2, 3, 4, 5, 6\}$, then

$A_1 = \{1, 3, 4\}$

$A_2 = \{2, 5\}$

$A_3 = \{6\}$

form a partition of S, because:

- these sets are disjoint
- the union of these sets is S.

Example

If $S = \{1, 2, 3, 4, 5, 6\}$, then

$A_1 = \{1, 3, 4, 5\}$

$A_2 = \{2, 5\}$

$A_3 = \{6\}$

do not form a partition of S, because:

- these sets are not disjoint (5 occurs in two different sets)

Example

If $S = \{1, 2, 3, 4, 5, 6\}$, then

$A_1 = \{1, 3\}$

$A_2 = \{2, 5\}$

$A_3 = \{6\}$

do not form a partition of S, because:

– the union of these sets is not S (since 4 is not a member of any of the subsets, but is a member of S).

Example

If $S = \{1, 2, 3, 4, 5, 6\}$, then

$A_1 = \{1, 3, 4\}$

$A_2 = \{2, 5\}$

$A_3 = \{6, 7\}$

do not form a partition of S, because:

- the union of these sets is not S (since 7 is a member of set A_3 but is not a member of S).

Constructing an Equivalence Relation from a Partition

Given set S = {1, 2, 3, 4, 5, 6} and a partition of S,

$A_1 = \{1, 2, 3\}$

$A_2 = \{4, 5\}$

$A_3 = \{6\}$

then we can find the ordered pairs that make up the equivalence relation R produced by that partition.

Constructing an Equivalence Relation from a Partition

The subsets in the partition of S,

$A_1 = \{1, 2, 3\}$

$A_2 = \{4, 5\}$

$A_3 = \{6\}$

are the equivalence classes of R. This means that the pair $(a,b) \in R$ iff a and b are in the same subset of the partition.

Let's find the ordered pairs that are in R:

$A_1 = \{1, 2, 3\} \rightarrow (1,1), (1,2), (1,3), (2,1),$
$(2,2), (2,3), (3,1), (3,2), (3,3)$

$A_2 = \{4, 5\} \rightarrow (4,4), (4,5), (5,4), (5,5)$

$A_3 = \{6\} \rightarrow (6,6)$

So R is just the set consisting of all these ordered pairs:

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}$

Compatibility Relation

- A relation R in set A is said to be a compatibility relation if it is reflexive and symmetric.

<u>Example:</u> Let a={ball, bed, dog, let, egg} and the relation R be given by

R = {(x, y) / x, y ϵ A & x R y if x and y contain some common letter}

Maximal Compatibility Relation

- Let X be a set and R is a compatibility relation on X.A is a subset of X is called a maximal compatibility block if any element of A is compatible to every other element of A and no element of X –A is compatible to all the elements of A.

Partial ordering

- A relation *R* on a set *S* is called a partial ordering or *partial order* if it is:
  - reflexive
  - antisymmetric
  - transitive
- A set *S* together with a partial ordering *R* is called a *partially ordered set*, or *poset*, and is denoted by (*S*, *R*).

Example

- Let R be a relation on set A. Is R a partial order?

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

Example

- Is R a partial order?

  $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

To be a partial order, R must be reflexive, antisymmetric, and transitive.

R is reflexive because R includes (1,1), (2,2), (3,3) and (4,4).

R is antisymmetric because for every pair (a,b) in R, (b,a) is not in R (unless $a = b$).

R is transitive because for every pair (a,b) in R, if (b,c) is in R then (a,c) is also in R.

Example

So, given

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

R <u>is</u> a partial order, and

(A, R) is a poset.

Example

- Is the "$\geq$" relation a partial ordering on the set of integers?
  - Since $a \geq a$ for every integer a, $\geq$ is reflexive
  - If $a \geq b$ and $b \geq a$, then $a = b$. Hence $\geq$ is anti-symmetric.
  - Since $a \geq b$ and $b \geq c$ implies $a \geq c$, $\geq$ is transitive.
  - Therefore "$\geq$" is a partial ordering on the set of integers and $(Z, \geq)$ is a poset.

Comparable / Incomparable

- In a poset the notation $a \preccurlyeq b$ denotes $(a, b) \in \mathrm{R}$
  - The "less than or equal to" ($\leq$)is just an example of partial ordering
- The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
- The elements a and b of a poset $(S, \preccurlyeq)$ are called *incomparable* if neither $a \preccurlyeq b$ nor $b \preccurlyeq a$.
- In the poset $(\mathbf{Z}^+, |)$:
  - Are 3 and 9 comparable? *Yes; 3 divides 9*
  - Are 5 and 7 comparable? *No; neither divides the other*

Total Order

- We said: “Partial ordering” because pairs of elements may be incomparable.
- If every two elements of a poset $(S, \preccurlyeq)$ are comparable, then S is called a *totally ordered* or *linearly ordered* set and $\preccurlyeq$ is called a *total order* or *linear order*.
- A totally ordered set is also called a *chain*.

Total Order

- The poset $(\mathbf{Z}, \leq)$ is totally ordered. Why?

  Every two elements of $\mathbf{Z}$ are comparable; that is, $a \leq b$ or $b \leq a$ for all integers.

- The poset $(\mathbf{Z}^+, |)$ is not totally ordered. Why?
- It contains elements that are incomparable; for example $5 \nmid 7$.

Hasse Diagram

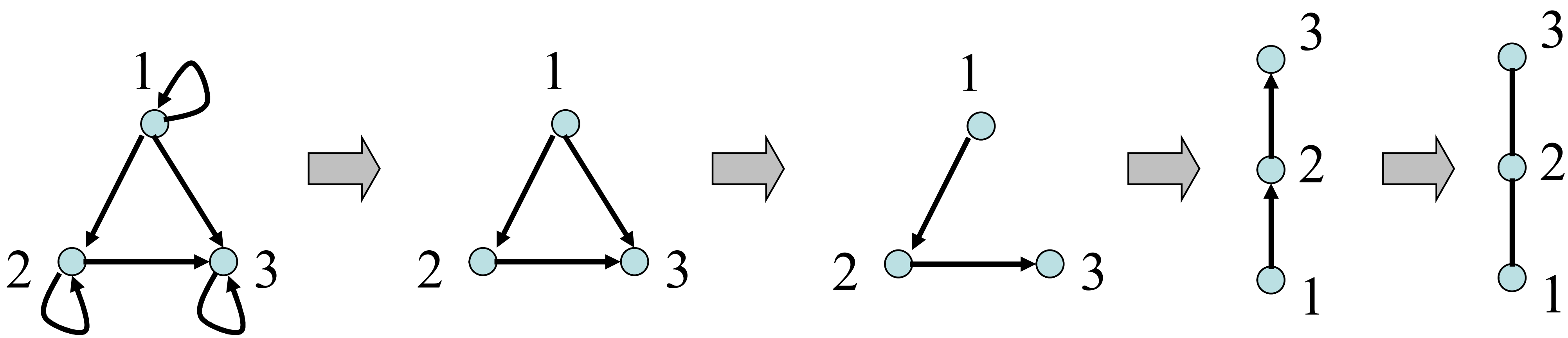
- A Hasse diagram is a graphical representation of a poset.
- Since a poset is by definition reflexive and transitive (and antisymmetric), the graphical representation for a poset can be compacted.
- For example, why do we need to include loops at every vertex? Since it's a poset, it *must* have loops there.

Constructing a Hasse Diagram

- Start with the digraph of the partial order.
- Remove the loops at each vertex.
- Remove all edges that *must* be present because of the transitivity.
- Arrange each edge so that all arrows point up.
- Remove all arrowheads.

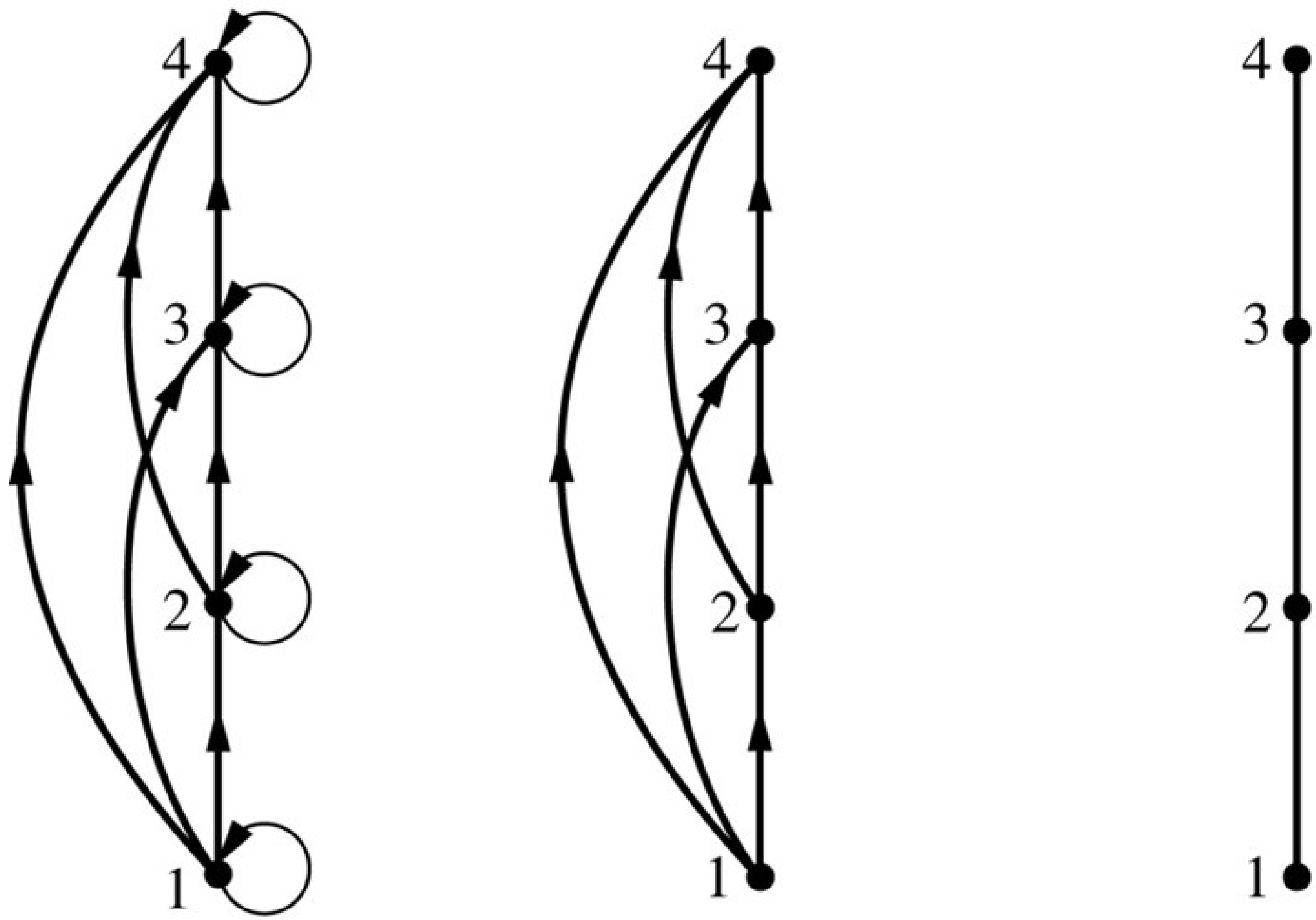
Example

- Construct the Hasse diagram for ({1, 2, 3}, ≤)



Hasse Diagram Example

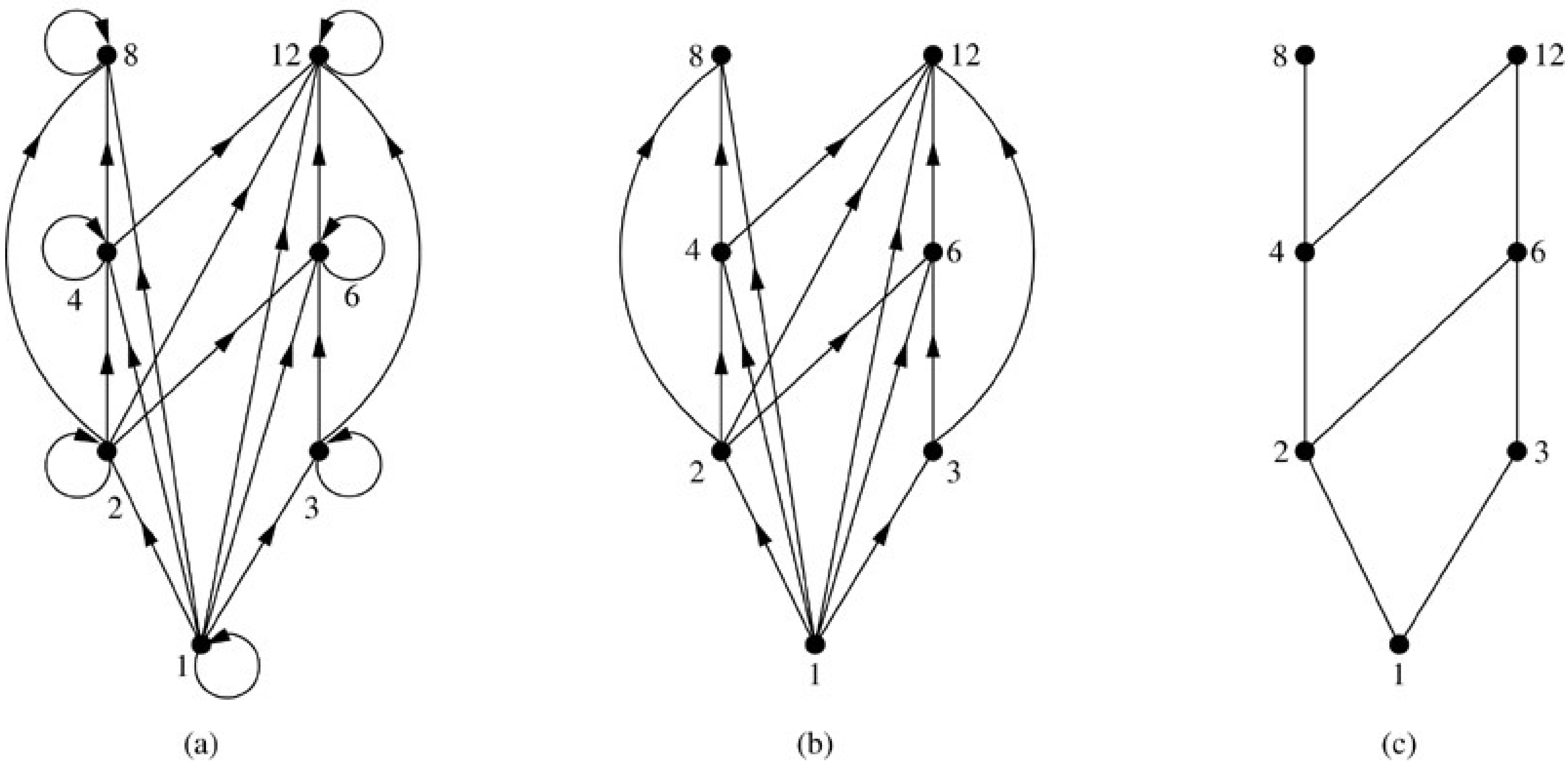
Steps in the construction of the Hasse diagram for $(\{1, 2, 3, 4\}, \leq)$



(a) (b) (c)

Hasse Diagram Example

Steps in the construction of the Hasse diagram for ({1, 2, 3, 4, 6, 8, 12}, |)

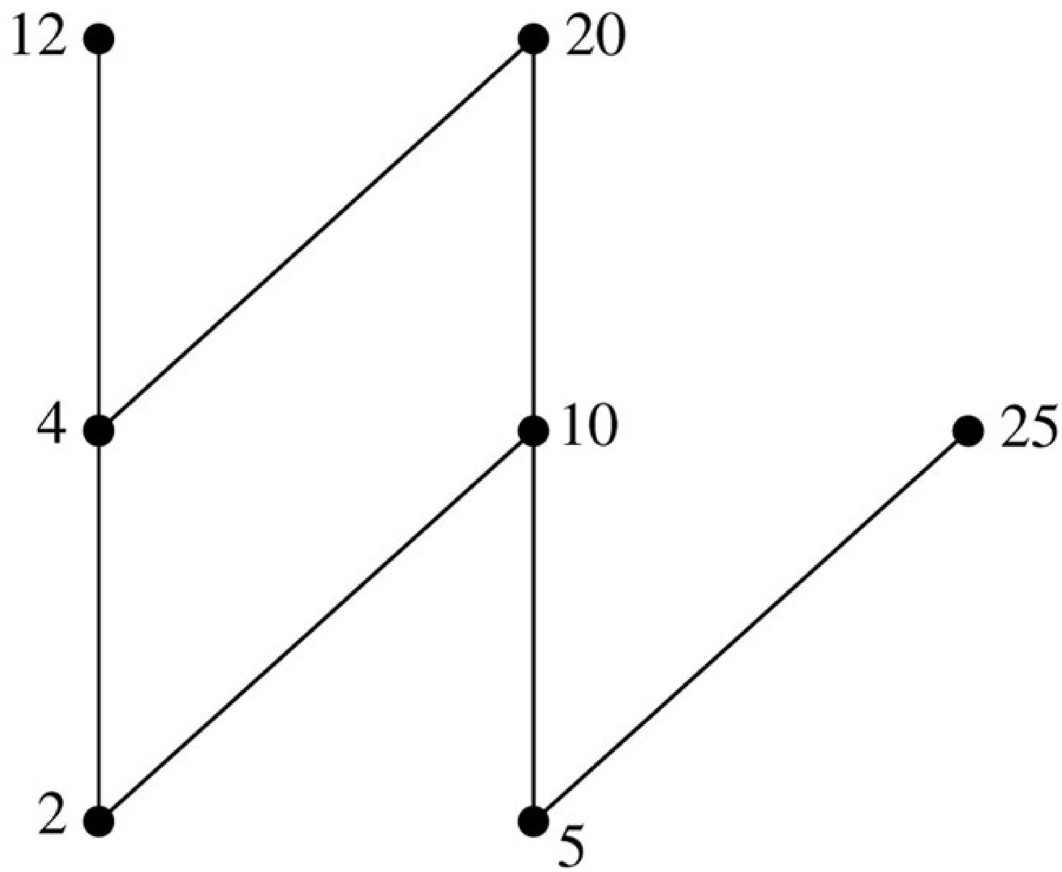


Hasse Diagram Terminology

- Let $(S, \preccurlyeq)$ be a poset.
- a is *maximal* in $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \preccurlyeq b$. (top of the Hasse diagram)
- a is *minimal* in $(S, \preccurlyeq)$ if there is no $b \in S$ such that $b \preccurlyeq a$. (bottom of the Hasse diagram)

Hasse Diagram Terminology

Which elements of the poset ({, 2, 4, 5, 10, 12, 20, 25}, |) are maximal? Which are minimal?



The Hasse diagram for this poset shows that the maximal elements are: 12, 20, 25

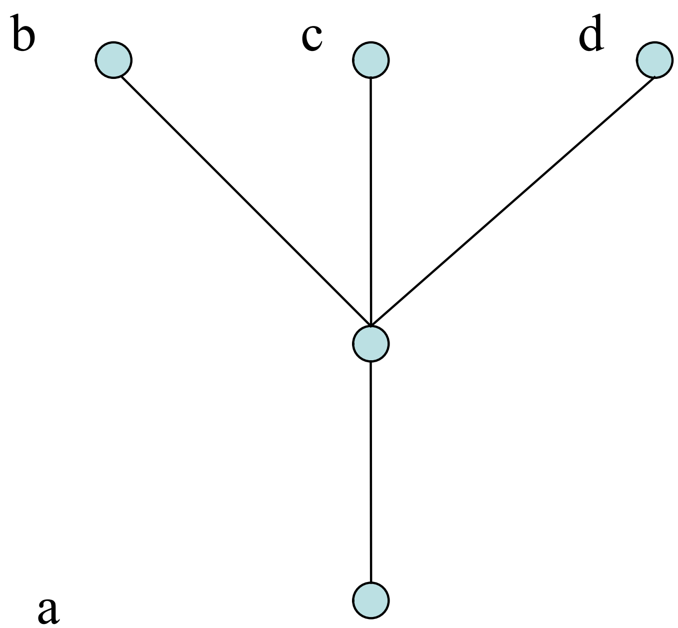
The minimal elements are: 2, 5

Hasse Diagram Terminology

- Let $(S, \preccurlyeq)$ be a poset.
- a is the *greatest element* of $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$…
  - It must be unique
- a is the *least element* of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.
  - It must be unique

Hasse Diagram Terminology

- Does the poset represented by this Hasse diagram have a *greatest element*? If so, what is it? Does it have a *least element*? If so, what is it?

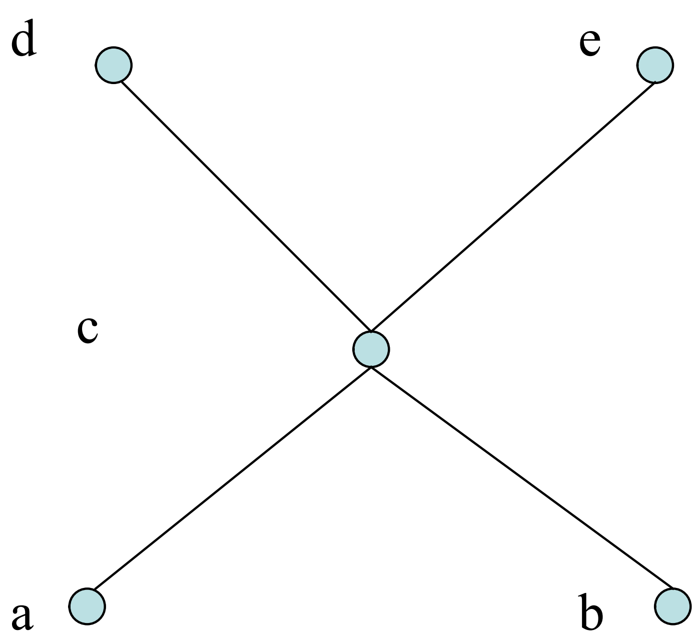


The poset represented by this Hasse diagram does not have a *greatest element*, because the greatest element must be unique.

It does have a *least element*, *a*.

Hasse Diagram Terminology

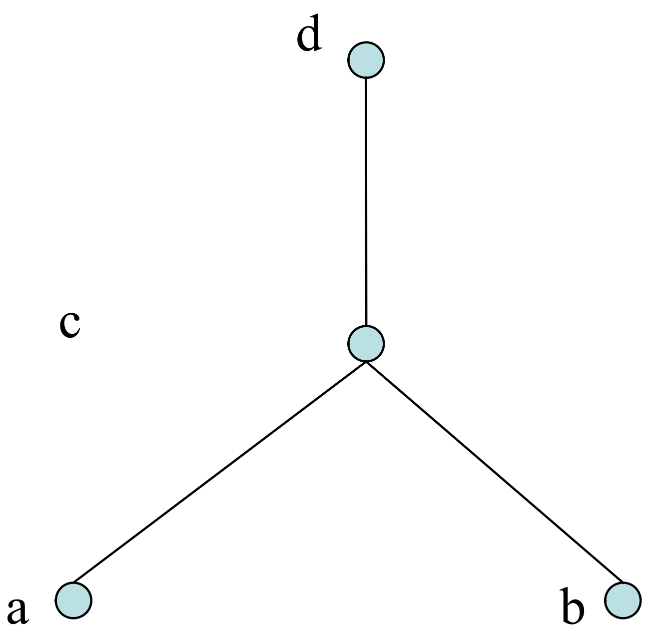
- Does the poset represented by this Hasse diagram have a *greatest element*? If so, what is it? Does it have a *least element*? If so, what is it?



The poset represented by this Hasse diagram has neither a *greatest element* nor a *least element*, because they must be unique.

Hasse Diagram Terminology

- Does the poset represented by this Hasse diagram have a *greatest element*? If so, what is it? Does it have a *least element*? If so, what is it?

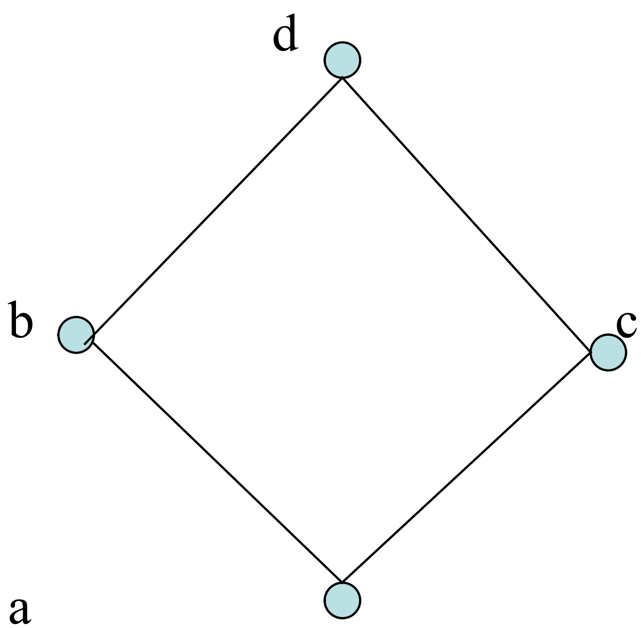


The poset represented by this Hasse diagram does not have a *least element*, because the least element must be unique.

It does have a *greatest element*, *d*.

Hasse Diagram Terminology

- Does the poset represented by this Hasse diagram have a *greatest element*? If so, what is it? Does it have a *least element*? If so, what is it?



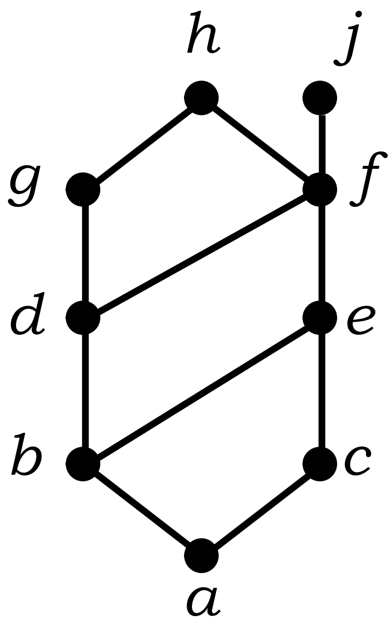
The poset represented by this Hasse diagram has a *greatest element*, *d*.

It also has a *least element*, *a*.

Hasse Diagram Terminology

- Let A be a subset of $(S, \preccurlyeq)$.
- If $u \in S$ such that $a \preccurlyeq u$ for all $a \in A$, then u is called an *upper bound* of A.
- If $l \in S$ such that $l \preccurlyeq a$ for all $a \in A$, then l is called an *lower bound* of A.
- If x is an upper bound of A and $x \preccurlyeq z$ whenever z is an upper bound of A, then x is called the *least upper bound* of A.
  - It must be unique
- If y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A, then y is called the *greatest lower bound* of A.
  - It must be unique

Example



Maximal: h, j

Minimal: a

Greatest element: None

Least element: a

Upper bound of $\{a,b,c\}$: e, f, j, h

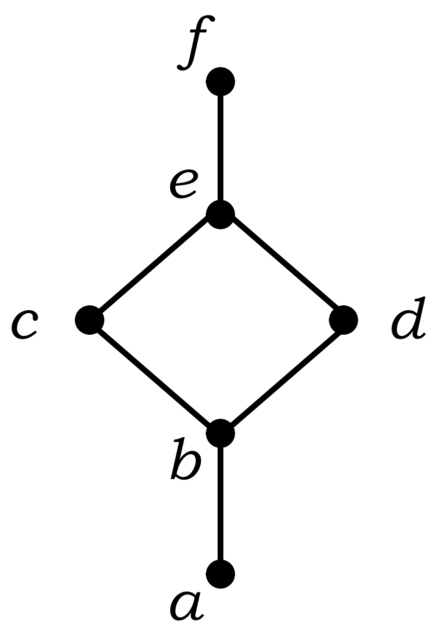
Least upper bound of $\{a,b,c\}$: e

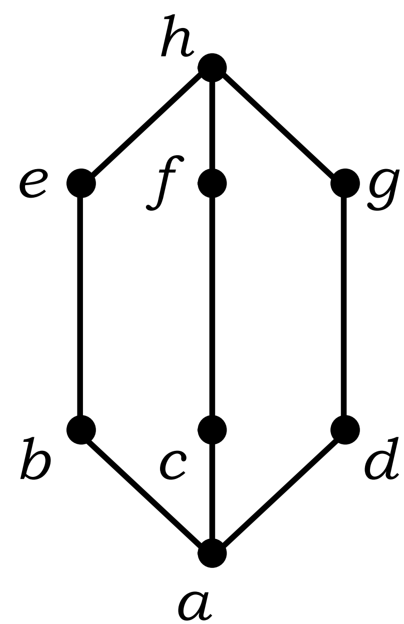
Lower bound of $\{a,b,c\}$: a

Greatest lower bound of $\{a,b,c\}$: a

Lattices

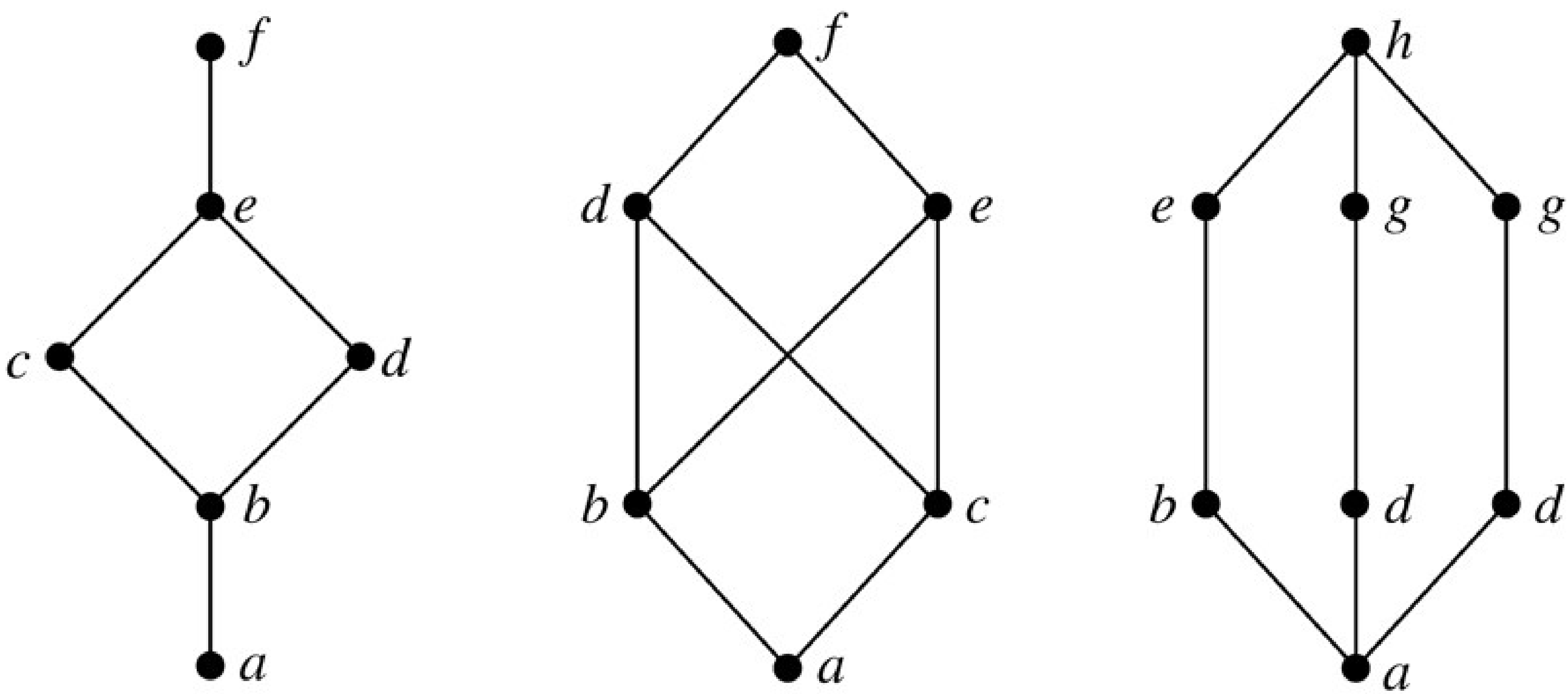
- A *lattice* is a partially ordered set in which every pair of elements has both a *least upper bound* and *greatest lower bound*.





Lattice example

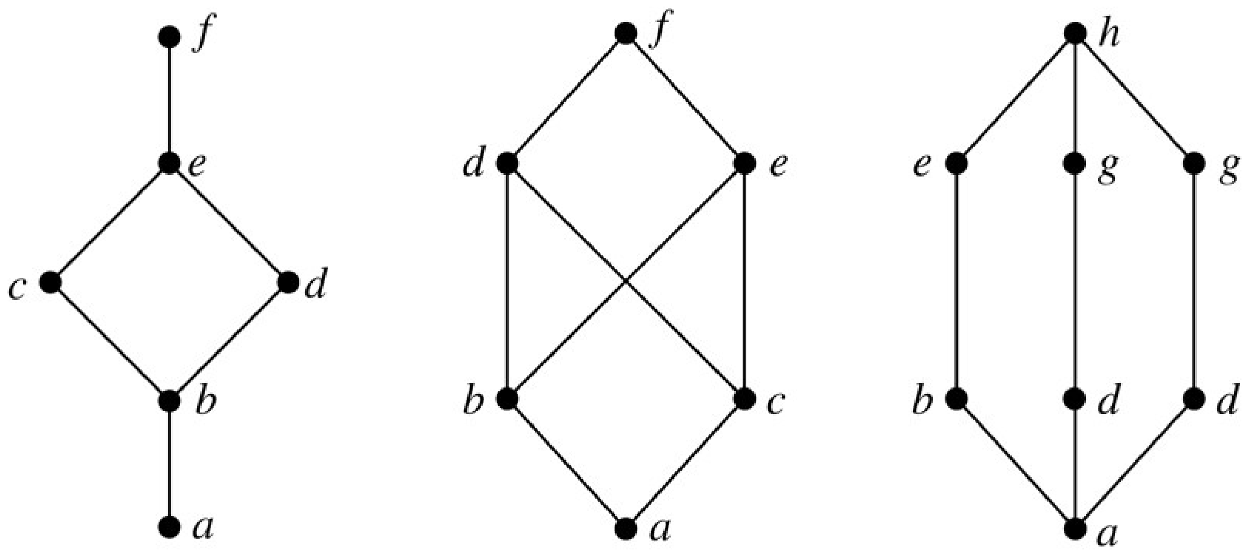
- Are the following three posets *lattices?*



(a) (b) (c)

Lattice example

- Are the following three posets *lattices?*



(a) (b) (c)

(a) Yes

(b) No; elements *b* and *c* have no least upper bound.

(c) Yes

Conclusion

In this chapter we have studied:

- Relations and their properties
- How to represent relations
- Closures of relations
- Equivalence relations
- Partial orderings