Elementary Combinatorics

<u>Combinatorics</u>

- Deals with enumeration (counting) techniques and related algebra.
- Basis of counting

X	Set
$\|X\|$	No. of elements in X

Sum Rule / Principle of Disjunctive Counting

S1, S2, …, Sn   Disjoint nonempty subsets of set X.

- $X = S1 \cup S2 \cup \ldots \cup Sn$
- $|X| = |S1| + |S2| + \ldots + |Sn|$

Sum Rule for counting events

- E1, E2, …, En   Mutually Exclusive events
- E1 happens in e1 ways.
- E2 happens in e2 ways.
- …
- En happens in en ways.
- E1, E2, …, En happens in e1 + e2 + … + en ways.

Product Rule / Principle of Sequential Counting

S1, S2, …, Sn Disjoint nonempty subsets of set X.

No. of elements in the Cartesian Product S1 x S2 x … x Sn

= |S1 x S2 x … x Sn|

= $\prod$ |Si|, i = 1 .. n.

Example:

S1 = {a, b, c, d, e} and S2 = {1, 2, 3}

S1 x S2 = {(a, 1), (a, 2), (a, 3),
(b, 1), (b, 2), (b, 3),
(c, 1), (c, 2), (c, 3),
(d, 1), (d, 2), (d, 3),
(e, 1), (e, 2), (e, 3)}

|S1 x S2| = |S1| . |S2| = 5 . 3 = 15

Product Rule for counting events

- E1, E2, …, En Mutually Exclusive events
- E1 happens in e1 ways.
- E2 happens in e2 ways.
- …
- En happens in en ways.
- Sequence of events E1, E2, …, En
- happens in e1 . e2 . … . en ways.

Exercises

- How many possible telephone numbers are there when there are seven digits, the first two of which are between 2 and 9 inclusive, the third digit between 1 and 9 inclusive, and each of the remaining may be between 0 and 9 inclusive?

- Suppose that a state's license plates consist of three letters followed by three digits, how many different plates can be manufactured (repetitions are allowed)?

- A company produces combination locks; the combinations consist of three numbers from 0 to 39 inclusive. Because of the construction no number can occur twice in a combination. How many different combinations for locks can be attained?

- How many ways are there to pick a man and a woman who are not married from 30 married couples?

- A shoe store has 30 styles of shoes. If each style is available in 12 different lengths, 4 different widths, and 6 different colours, how many kinds of shoes must be kept in stock?

- How many 5–letter words are there where the first and last letters are consonants?

- How many 5–letter words are there where the first and last letters are vowels?

- How many 5–letter words are there where the first and last letters are vowels and the middle letters are consonants?

- How many ways are there to select the 2 cards (without replacement) from a deck of 52 such that neither card is an ace?

Combinations & Permutations

r-combination of n objects

Unordered selection of r of the objects.

Combination of n objects taken r at a time.

r-permutation of n objects

Ordered selection / arrangement of r of the objects.

Permutation of n objects taken r at a time.

∞ unlimited repetitions

Examples:

- 3-combinations of {3.a, 1.b, 1.c}.
  aaa, aab, aac, abc.

- 3-permutations of {3.a, 1.b, 1.c}.
  aaa, aab, aac, baa, caa, aba, aca, abc, acb, bac, bca, cab, cba.

- 3-combinations of {3.a, 2.b, 5.c}.
  aaa, aab, aac, abb, acc, abc, bbc, bcc, ccc.

- 3-combinations of {3.a, 2.b, 2.c, 1.d}
  aaa, aab, aac, aad, abb, acc, abc, abd, acd, bbc, bbd, bcc, bcd. ccd.

- 3-combinations of {∞.a, 2.b, ∞.c}.
  aaa, aab, aac, abb, acc, abc, bbc, bcc, ccc.

- 3-combinations of a, b, c, d with unlimited repetitions / $\{\infty.a, \infty.b, \infty.c, \infty.d\}$.

  aaa, aab, aac, aad, abc, abd, acd,

  bbb, bba, bbc, bbd, bcd,

  ccc, cca, ccb, ccd,

  ddd, dda, ddb, ddc.
- 2-combinations of a, b, c, d with unlimited repetitions / $\{\infty.a, \infty.b, \infty.c, \infty.d\}$.

  aa, ab, ac, ad, bb, bc, bd, cc, cd, dd.
- 2-permutations of a, b, c, d with unlimited repetitions / $\{\infty.a, \infty.b, \infty.c, \infty.d\}$.

  aa, ab, ac, ad,
  ba, bb, bc, bd,
  ca, cb, cc, cd,

  da, db, dc, dd.

- 2-combinations of a, b, c, d without repetitions.
  ab, ac, ad, bc, bd, cd.
- 2-permutations of a, b, c, d without repetitions.
  ab, ac, ad, ba, bc, bd,
  ca, cb, cd, da, db, dc.
- 3-combinations of a, b, c, d without repetitions.
  abc, abd, acd, bcd.
- 3-permutations of a, b, c, d without repetitions.
  abc, acb, bac, bca, cab, cba,
  abd, adb, bad, bda, dab, dba,
  acd, adc, cad, cda, dac, dca,
  bcd, bdc, cbd, cdb, dbc, dcb.

Enumerating r-permutations without repetitions

(No. of r-permutations of n elements without repetitions)

$P(n, r) = n(n-1) \ldots (n-r+1)$

$= n! / (n-r)!$

Proof:

Filling first position = n ways

Filling second position = n – 1 ways

Filling third position = n – 2 ways

…

Filling rth position = n – r + 1 ways

Filling r positions= n (n – 1) (n – 2) … (n – r + 1) ways

For r = n,

$P(n, n) = n!$

There are n! permutations of n distinct objects.

There are (n – 1)! Permutations of n distinct objects in a circle.

Examples:

1. In how many ways can 7 women and 3 men be arranged in a row if the 3 men must always stand next to each other?

- No. of ways of arranging 3 men = 3!
- 3 men must always stand next to each other treated as Single entity X
- No. of ways of arranging X and 7 women = 8!
- Total ways 7 women and 3 men arranged in a row if the 3 men must always stand next to each other = (3!) (8!)

2. In how many ways can the letters of the English alphabet be arranged so that there are exactly 5 letters between the letters a and b?

- No. of ways to place a and b = 2
- No. of ways to arrange 5 letters between a and b = P(24, 5)
- No. of ways to arrange 7-letter word along with the remaining 19 letters = 20!
- Total ways the letters of the English alphabet arranged so that there are exactly 5 letters between the letters a and b = (2) (20!) P(24, 5)

Enumerating r-combinations without repetitions

- (No. of r-combinations of n elements without repetitions)

  $C(n, r) = P(n, r) / r!$

  $= n! / r! (n - r)!$

Examples:

1. In how many ways can a hand of 5 cards be selected from a deck of 52 cards?

   $n = 52, r = 5$

   $C(n, r) = n! / r! (n - r)!$

   $C(52, 5) = 52! / (5!\ 47!)$

   $= 52 . 51 . 10. 49 . 2$

   $= 25,98,960$

2. How many 5-card hands consist only of hearts?

$n = 13, r = 5$

$C(n, r) = n! / r! (n - r)!$

$C(13, 5) = 13! / (5! 8!)$

$= 13 . 11 . 9$

$= 1{,}287$

3. How many 5-card hands consist of cards from a single suit?

No. of suits = 4

In each suit, no. of 5-card hands

$= C(13, 5)$

$= 13! / (5! 8!)$

$= 13 . 11 . 9$

$= 1{,}287$

Total no. of 5-card hands = 4 x 1,287 = 5,148

4. How many 5-carde hands have 2 clubs and 3 hearts?

   $C(13, 2)\ C(13, 3)$

5. How many 5-card hands have 2 cards of one suit and 3 cards of a different suit?

   No. of ways to choose 2 suits = $C(4, 2)$
   No. of ways to choose 2 cards from one suit and 3 cards from the other = $2\ C(13, 2)\ C(13, 3)$
   Total no. of ways = $2\ C(13, 2)\ C(13, 3)\ C(4, 2)$

6. How many 5-card hands contain exactly 2 of one kind and 3 of another kind?

No. of ways to choose exactly 2 hands of one kind

$$= 13\ C(4, 2)$$

No. of ways to choose exactly 3 hands of another kind

$$= 12\ C(4, 3)$$

Total no. of ways = $12 \cdot 13 \cdot C(4, 2)\ C(4, 3)$

7. In how many ways can a committee of 5 be chosen from 9 people?

$$C(9, 5)$$

8. There are 21 consonants and 5 vowels in the English alphabet. Consider only 8–letter words with 3 different vowels and 5 different consonants.

How many such words can be formed?

How many such words contain the letter a?

How many contain the letters a and b?

How many contain the letters b and c?

How many contain the letters a, b, and c?

How many begin with a and end with b?

How many begin with b and end with c?

9. How many ways are there to distribute 10 different books among 15 people if no person is to receive more than 1 book?

10. How many ways are there to seat 10 boys and 10 girls around a circular table?

11. How many ways are there to seat 10 boys and 10 girls, if boys and girls alternate**?**

12. A multiple-choice test has 20 questions and 4 choices for each answer. How many ways can the 20 questions be answered so that exactly 5 answers are correct?

13. Find the number of ways in which 5 different English books, 6 French books, 3 German books, and 7 Russian books can be arranged on a shelf so that all books of the same language are together.

14. How many ways can 5 days be chosen from each of the 12 months of an ordinary year of 365 days?

15. A committee is to be chosen form a set of 9 women and 5 men. How many ways are there to form the committee if the committee has,

   6 people, 3 women, and 3 men?

   Any number of people but equal numbers of women and men?

   6 people and at least 3 are women?

   6 people including Mr. A?

Enumerating Combinations & Permutations with repetitions

Enumerating r–permutations with unlimited repetitions

$U(n, r) = n^r$

Examples:

1. There are 25 true or false questions on an examination. How many different ways can a student do the examination if he or she can also choose to leave the answer blank?

   No. of ways a student do the examination = 3^{25}

2. The results of 50 football games (win, lose or tie) are to be predicted. How many different forecasts can contain exactly 28 correct results?

   No. of ways to choose 28 correct results = C(50, 28)

   No. of ways to choose remaining 22 wrong forecasts = 2^{22}

   Total no. of different forecasts contain exactly 28 correct results

   = C(50, 28) 2^{22}

Enumerating r–combinations with unlimited repetitions

n objects with unlimited repetitions

$\{\infty.a_1, \infty.a_2, \ldots, \infty.a_n\}$,

$a_1, a_2, \ldots, a_n$ are distinct objects.

Any sequence of nonnegative integers $x_1, x_2, \ldots, x_n$, where $x_1 + x_2 + \ldots + x_n$ = r corresponds to an r-combination $\{x_1.a_1, x_2.a_2, \ldots, x_n.a_n\}$.

V(n, r)

= No. of r-combinations of n distinct objects with unlimited repetitions.

= No. of nonnegative integral solutions to $x_1 + x_2 + \ldots + x_n = r$.

= No. of ways of distributing r similar balls into n numbered boxes.

= No. of binary numbers with n – 1 one's and r zeros.

= C(n – 1 + r, r)

= C(n – 1 +r, n – 1)

= (n + r – 1)! / [r! (n – 1)!]

Examples:

1. Find the no. of 4-combinations of {∞.a1, ∞.a2, ∞.a3, ∞.a4, ∞.a5}.

   $n = 5, r = 4$

   No. of 4-combinations of {∞.a1, ∞.a2, ∞.a3, ∞.a4, ∞.a5}

   $= C(n - 1 + r, r)$
   $= C(5 - 1 + 4, 4)$
   $= C(8, 4)$
   $= 8! / (4!\ 4!)$
   $= 8 . 7. 6 . 5 / (4 . 3 . 2)$
   $= 7 . 2 . 5$
   $= 70$

2. Find the no. of 3-combinations of 5 objects with unlimited repetitions.

   $n = 5, r = 3$

   No. of 3-combinations of 5 objects with unlimited repetitions

   $= C(n - 1 + r, r)$
   $= C(5 - 1 + 3, 3)$
   $= C(7, 3)$
   $= 7! / (3!\ 4!)$
   $= 7. 6 . 5 / (3 . 2)$
   $= 7 . 5$
   $= 35$

3. Find the no. of nonnegative integral solutions to $x1 + x2 + x3 + x4 + x5 = 50$.

$n = 5, r = 50$

No. of nonnegative integral solutions to $x1 + x2 + x3 + x4 + x5 = 50$

$= C(n - 1 + r, r)$

$= C(5 - 1 + 50, 50)$

$= C(54, 50)$

$= 54! / (50!\ 4!)$

$= 54 . 53 . 52 . 51 / (4 . 3. 2)$

$= 27 . 53 . 17 . 13$

$= 3,16,251$

4. Find the no. of ways of placing 10 similar balls in 6 numbered boxes.

$n = 6, r = 10$

No. of ways of placing 10 similar balls in 6 numbered boxes

$= C(n - 1 + r, r)$

$= C(6 - 1 + 10, 10)$

$= C(15, 10)$

$= 15! / (10!\ 5!)$

$= 15 . 14 . 13 . 12 . 11 / (5 . 4 . 3. 2)$

$= 7 . 13 . 6 . 11$

$= 3,003$

5. Find the no. of binary numbers with ten 1's and five 0's.

$n - 1 = 10, r = 5$

No. of binary numbers with ten 1's and five 0's

$= C(n - 1 + r, r)$

$= C(10 + 5, 5)$

$= C(15, 5)$

$= 15! / (10!\ 5!)$

$= 15 . 14 . 13 . 12 . 11 / (5 . 4 . 3. 2)$

$= 7 . 13 . 6 . 11$

$= 3,003$

No. of integral solutions of $x_1 + x_2 + \ldots + x_n = r$ where each $x_i \geq 0$

= No. of ways of distributing r similar balls into n numbered boxes with at least one ball in each box

= $C(n - 1 + (r - n), r - n)$

= $C(r - 1, r - n)$

= $C(r - 1, n - 1)$

No. of integral solutions of $x_1 + x_2 + \ldots + x_n = r$ where $x_1 \geq r_1$, $x_2 \geq r_2$, …, $x_n \geq r_n$, and $r_1, r_2, \ldots, r_n$ are integers.

= No. of ways of distributing r similar balls into n numbered boxes with at least r_1 balls in the first box, at least r_2 balls in the second box, …, at least rn balls in the nth box.

= $C(n - 1 + r - r_1 - r_2 - \ldots - r_n, r - r_1 - r_2 - \ldots - r_n)$

= $C(n - 1 + r - r_1 - r_2 - \ldots - r_n, n - 1)$

Examples:

1. Enumerate the no. of ways of placing 20 indistinguishable balls into 5 boxes where each box is nonempty?

2. How many integral solutions are there to $x1 + x2 + x3 + x4 + x5 = 20$ where each $xi \geq 2$?

3. How many integral solutions are there to $x1 + x2 + x3 + x4 + x5 = 20$ where $x1 \geq 3$, $x2 \geq 2$, $x3 \geq 4$ $x4 \geq 6$ $x5 \geq 0$?

4. How many integral solutions are there to $x1 + x2 + x3 + x4 + x5 = 20$ where $x1 \geq -3$, $x2 \geq 0$, $x3 \geq 4$ $x4 \geq 2$ $x5 \geq 2$?

5. Find the no. of integral solutions are there to $x1 + x2 + x3 + x4 = 50$ where $x1 \geq -4$, $x2 \geq 7$, $x3 \geq -14$ $x4 \geq 10$?

Combinations & Permutations with Constrained repetitions

Let $q_1, q_2, \ldots, q_t$ be non-negative integers

and $n = q_1 + q_2 + \ldots + q_t$.

Enumerating n-permutations with constrained repetitions / ordered partitions of a set

$P(n; q_1, q_2, \ldots, q_t)$

$= n! / (q_1!\, q_2! \ldots q_t!)$

$= C(n, q_1)\, C(n - q_1, q_2)\, C(n - q_1 - q_2, q_3) \ldots C(n - q_1 - q_2 \ldots$

$- q_{t-1}, q_t)$

Examples:

1. **Find the number of arrangements of letters in the word**

T A L L A H A S S E E.

n = 11

q1 = No. of T's = 1 q2 = No. of A's = 3

q3 = No. of L's = 2 q4 = No. of H's = 1

q5 = No. of S's = 2 q6 = No. of E's = 2

P(n; q1, q2, q3, q4, q5, q6) = n! / (q1! q2! q3! q4! q5! q6!)

= 11! / (3! 2! 2! 2! 1! 1!)

= 11.10.9.8.7.6.5.4 / (2.2.2)

= 11.10.9.7.6.5.4

= 8,31,600

2. In how many ways can 23 different books be given to 5 students so that 2 of the students will have 4 books each and the other 3 will have 5 books each?

No. of ways to choose 2 students from 5 students = C(5, 2)

n = 23, q1 = q2 = 4, q3 = q4 = q5 = 5

P(n; q1, q2, q3, q4, q5)

= n! / (q1! q2! q3! q4! q5!)

= 23! / (4! 4! 5! 5! 5!)

= 23! / (4!2 5!2)

No. of ways 23 different books be given to 5 students

= C(5, 2) 23! / (4!2 5!2)

3. **In the game of bridge, four players (usually called North, East, South, West) seated in a specified order are each dealt a hand of 13 cards.**

- How many ways can the 52 cards be dealt to the four players?

  $52! / 13!^4$

- In how many ways will one player be dealt all four kings?

  $4(48! / 9!\ 13!^3) = 4\ C(48,9)\ C(39,13)\ C(26,13)\ C(13,3)$

- In how many deals will North be dealt 7 hearts and South the other 6 hearts?

  $C(13,7)\ 39! / 6!\ 7!\ 13!^2$

Enumerating unordered partitions of equal cell size

Let S be a set with n elements where $n = q . t$.

No. of unordered partitions of S of type $(q, q, \ldots, q)$

$$= 1 / (t! (n! / (q^{!)t))}$$

Examples:

1. In how many ways can 14 men are partitioned into 6 teams where the first team has 3 members, the second team has 2 members, the third team has 3 members, and the fourth, fifth, and sixth teams each have 2 members?

   $P(14,3,2,3,2,2,2)$

2. In how many ways can 12 of the 14 people be distributed into 3 teams where the first team has 3 members, the second has 5, and the third has 4 members?

3. In how many ways can 12 of the 14 people be distributed into 3 teams of 4 each?
4. In how many ways can 14 people be partitioned into 6 teams when the first and second teams have 3 members each and the third, fourth, fifth and sixth teams have 2 members each?
5. In how many ways can 14 people be partitioned into 6 teams where two teams have 3 each and 4 teams have 2 each?

End Exam questions

1 a) *How many ways* are there to place 20 identical balls into 6 different boxes in which exactly 2 boxes are empty?

b) In *how many ways* can we *partition* 12 similar coins into 5 numbered non-empty batches? (2006/1, 2007S/1) [16]

2. a) Using the digits 1, 3, 4, 5 6, 8 and 9 *how many* 3-digit numbers can be formed?

b) *How many* 3-digit numbers can be formed if *no digit is repeated*?

c) *How many* 3-digit numbers can be formed if 3 and 4 are *adjacent to each other*?

d) *How many* 3-digit numbers can be formed if 3 and 4 are *not adjacent to each other*? (2006/2)

[4+4+4+4]

3. a) *How many ways* are there to seat 10 boys and 10 girls around a *circular table*, if boys and girls seat *alternatively*?

(2006/3) [16]

b) In *how many ways* can the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 be arranged so that 0 and 1 are *adjacent* and in the order of 01?

4. a) Find the arrangements of letters of M I S S I S S I P P I.

b) In how many ways can 3 boys share 15 different sized apples if each takes 5? (2006/4, 2007S/2) [16]

5. a) A chain letter is sent to 10 people in the first week of the year. The next week each person who received a letter sends letters to 10 new people and so on. *How many people* have received the letters at the end of the year?

b) *How many* integers between 105 and 106 have no digits other than 2, 5 or 8? (2007S/3) [16]

6. a) Find total number of positive integers that can be formed from the digits 1, 2, 3, 4 and 5, if *no digit is repeated* in any integer.

b) A chain letter is sent to 10 people in the first week of the year. The next week each person who received a letter sends letters to 10 new people and so on. *How many people* have received the letters at the end of the year?

(2007S/4) [16]