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UNIT-4

COMBINATORICS

Sum Rule:

Let $E_1, E_2, \ldots E_n$ are mutually exclusive events which can happen $e_1, e_2, \ldots e_n$ ways respectively, then no of ways $n(E_1 \text{ or } E_2 \text{ or } E_3 \ldots E_n) = e_1 + e_2 + e_3 + \ldots e_n$

Product rule:

Let $E_1, E_2, \ldots E_n$ are independent events which can happen $e_1, e_2, \ldots e_n$ ways then

$\boxed{\text{no. of ways} = n(E_1 \cap E_2 \ldots \cap E_n) = e_1 e_2 e_3 \ldots e_n}$

Q) There are 6 English movie series CD's. 8 Hindi, and 10 Telugu movie CD's. How many ways we can chose 2 CD's from different languages

Sol No. of ways: $6C_1 8C_1 + 8C_1 10C_1 + 10C_1 6C_1$

$= 188.$

Q) How many bit strings of length eleven are possible?

Sol: 0/1

2 2 2 2 2 2 2 2 2 2 2

No. of bit strings: 2^{11}

Q) How many integers between 10^5 and 10^6 have no digits other than 0 2 5 8.

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$\frac{}{3}\ \frac{}{4}\ \frac{}{4}\ \frac{}{4}\ \frac{}{4}\ \frac{}{4}$

No.of integers $= 3\times 4^5$

$= 3072$.

(5) A pair of dice are tossed no.of ways in which sum of 7 or 8 occurs if the dice are distinguishable.

sol: (1,6) (2,5) (3,4) (4,3), (5,2), (6,1).

(2,6) (3,5) (5,3) (6,2), (4,4)

No.of ways = 11

(6) How many 4-digit even numbers are possible with all distinct digits.

$9\times 8\times 7\times 1$ $\quad 8\times 8\times 7\times 4$ $\quad \frac{}{9}\ \frac{}{9}\ \frac{}{8}\ \frac{}{7}$

No.of digits = 2296.

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Permutation:

Arrangement of ordered selection of objects.

(Q) Suppose selections are to be made from the 4 objects a, b, c, d. Two-word letter

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combinations without repetition and

Two permutation with ~~out~~ repetition.

3 letter combination, permutation

Sol: a, b, c, d.

Combinations:

2-letter ~~wo~~ - (a, b), (a c),

ad, bc, bd, ~~b~~ c, d

$\therefore$ No. of combinations of 2-letter word = 6.

3-letter - (a, b, c) (a, b, d) (a, c, d)

(b, c, d)

$\therefore$ No. of combinations of 3-letter word = 4.

Permutations:

2-letter - ab, ac, ad, bc, bd, cd, ba,

ca, da, cb, db, dc.

3-letter - abc, bcd, cda, dab, acb, bdc,

cad, dba, bca, cab, bac, bca,

cba, cbd, cdb, dbc, dcb,

~~cad~~, dac, dca, acd, adc,

bad, bda, abd, adb.

Permutation.

$${}^nP_r = \frac{n!}{(n-r)!} = n(n-1)\ldots(n-r+1)$$

$$= n(n-1)\ldots n-(r-1)$$

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(9) How many ways 6 persons sit in a room. — 6!

(8) 10 different books among 15 persons so that no person can take more than one book and all the books are distributed

Sol

$B_1, B_2, \ldots\ldots B_{10}$

$15 \cdot 14 \cdots\cdots 6 = {}^{15}P_{10}$

(8) In how many different ways can the letter of the word DETAIL can be arranged in such a way that vowels occupy odd positions

vowels - E, A, I.

Sol

$\underset{3}{\_}\ \underset{3}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{1}{\_}\ \underset{1}{\_} = 36$ ways

vowels

Vowels in 3! ways, consonants in 3!

(8) In how many different ways can the letter of the word MACHINE can be arranged vowels may occupy only odd positions

Sol Vowels - A, I, E

1 2 3 4 5 6 7

$3 \times 2 \times 1$

$4(3!)\ 4! = 576$.

1, 3, 5
3, 5, 7
1, 5, 7
1, 3, 7

4 places 3-vowels - ${}^4P_3$

remaining 4 in 4! ways.

$\therefore$ ~~4B~~ $4! \cdot {}^4P_3$.

(Q) How many ways 5 boys & 5 girls sit in a row so that boys and girls sit alternatively.

Sol: $\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ } = 5! \cdot 5!$

G B G B G B G B G B

$\underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{\ } = 5! \cdot 5!$

B G B G B G B G B G

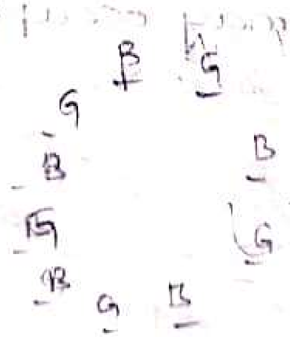
No.of ways $= 2(5!)(5!) = 28800$

(Q) How many ways 5 boys & 5 girls sit in a row such that no 2 boys are side by side.

Sol: $\_ G \_ G \_ G \_ G \_ G \_$

$5! \times {}^6P_5$

(Q) How many ways 5 boys & 5 girls sit around a table so that boys sit alterafively.



In a table if there are n persons then no. of ways $= (n-1)!$

$\therefore 4!\,5!$ ways.

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(6) Let there be n object that are not all distinct. Let there be q_1 objects of 1st kind, q_2 objects of the 2nd type then no. of no. of permutations of these n objects is given by the formula

$$\frac{n!}{q_1!\, q_2!}$$

If $r, b, 0 \leq$

* $^nC_r = {^nC_{n-r}}$

(Q) In a college there are 10 professors. How many ways committee of 3 professors can be formed so that atleast 1 from A or B professor is ~~com~~ included in the committee.

Sol : $^{10}C_3 - {^8C_3}$

both are not selected.

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Q In how many ways can a pack 5 cards is selected from a deck of 52 cards. ~~How~~

Sol No. of ways = $^{52}C_5$

Q How many 5 card hearts ~~hands~~ consists only of hearts.

Sol No. of ways = $^{13}C_5$

Q n-couples are attending a party if each person shakes hand with every person except his/her partner finding no. of handshakes possible in the party.

Sol Total no. of possible handshakes = $^{2n}C_2$

$\therefore$ No. of handshakes = $^{2n}C_2 - n$.

Combinations with repetitions:

Combinations with repetitions is represented as $v(n, r)$.

$v(n, r) \Rightarrow$ no. of r combinations of n distinct objects taken r at a time with unlimited repetitions.

$v(n, r) = c(n-1+r, r)$

$v(n, r) = {}^{n-1+r}C_r$.

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Application

(4) The no of non-negative integral sol'ns to $x_1,\ x_1+x_2+x_3+x_4+\dots+x_n = r$.

$\rightarrow$ No. of ways = n.

(5) No of ways of placing 10 similar balls in 6 numbered boxed.

$\rightarrow\ {}^{6-1+10}C_{10} \Rightarrow {}^{15}C_{10}$

(6) How many ways we can distribute 16 similar apples among 4 persons so that each persons ~~so~~ ~~the~~ gets atleast one apple.

$P_1\quad P_2\quad P_3\quad P_4$

$1\quad 1\quad 1\quad 1$

Sol $r = 16$

$n = 4$.

$\cancel{{}^{n+r-1}C_{r-1}} \Rightarrow \cancel{{}^{16+4-1}C_{16-1}}$.

$r = 12$

$n = 4$.

${}^{4-1+12}C_{12}$.

(8) How many non-negative integer solutions possibe for the equation

$x_1+x_2+x_3+x_4+x_5 = 20$

where $x_1 \geq 2,\ x_2 \geq 3,\ x_3 \geq 4,\ x_4 \geq 6\ x_5 \geq 0$

Sol $x_1 + x_2 + x_3 + x_4 + x_5 = 20$.

$2\quad 3\quad 4\quad 6\quad 0$

$V(5,5) = {}^{5-1+5}C_5 \Rightarrow {}^9C_5 = {}^9C_4$

$= 126$

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The no. of non-negative integer solutions.

Permutations with repetitions:

Let $v(n,r)$ denotes the no. of r permutation of n objects with unlimited repetitions

eg: $\{\infty \cdot a_1, \infty \cdot a_2 \cdots \infty \cdot a_n\}$

$v(n,r) = n^r$.

eg: There are 25 true and false questions in an examination. How many different ways can a ~~statement~~ student do the examination if he/she can also chose to leave the answer as blank.

$\rightarrow$ $r = 25$, $n = 3$ $\Rightarrow 3^{25}$.

Euler Function:

If n is a +ve integer then $\phi(n)$ = the no. of integers 'x' such that $1 \le x \le n$ and n & x are relatively prime.

eg: Two +ve integers a & b relatively prime if GCD of a & b is 1.

eg: ~~sol~~: $\phi(6) = \{1, 2, 3, 4, 5, \cancel{6}\}$.

$\phi(6) = \{2\}$.

$\phi(10) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\phi(10) = \{1, 3, 7, 9\}$

$\phi(10) = \{4\}$

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Note:
1. If n is a prime number then Euler function of $\phi(n)$ is $n-1$

eg : $\phi(7) = \{1, 2, 3, 4, 5, 6\}$.

$\phi\{7\} = \{6\}$.

2. If n is +ve integer then $\phi(n)$

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$$

$$= \frac{n \times (p_1-1)(p_2-1)\cdots(p_k-1)}{p_1 \cdot p_2 \cdots p_k}.$$

where $p_1, p_2, \cdots p_k$ are distinct prime factors of n

Q. No. of +ve integers which are less than 110 and relatively prime to 110 :

$$\begin{array}{r|l} 2 & 110 \\ \hline 5 & 55 \\ \hline & 11 \end{array}$$

Sol $\phi(n) = \dfrac{110(2-1)(5-1)(11-1)}{2 \cdot 5 \cdot 11}$

$= 4 \times 10 = 40$

→ $n = p^2q^3$ where p & q are prime numbers +ve integer m that m is $1 \le m \le n$ gcd $\{m, n\} = 1$ is

$$\phi(n) = \frac{p^2q^3(p-1)(q-1)}{p \cdot q} = pq^2(p-1)(q-1)$$

$$= p^2q^3 - p^2q^2 - pq^3 + pq^2.$$

Q. $n = 323$

$$\begin{array}{r|l} 19 & 323 \\ \hline & 17 \end{array}$$

$$\phi(n) = \frac{323(17-1)(19-1)}{17 \cdot 19}$$

$$= \frac{18 \times 16 \times 323}{323}$$

$= 288$

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Pegion hole Principle:

If A is average no. of pegions per pegion hole then

(i) some pegion hole atleast $\lceil A \rceil$ ceiling of A pegions.

(ii) some pegion hole contain atmost floor of A. $\lfloor A \rfloor$ pegions.

eg: If 10 is the no. of pegions then 4 is no. of pegion holes then

$\frac{10}{4} = 2.5$

There is atleast 3

Atmost - 2.

Q → If n+1 pegions are kept in n pegion hole then what is the average no. of pegions per pegion hole.

$A = \frac{n+1}{n} = 1 + \frac{1}{n}$

Some pegion hole contain atleast 2 pegions and some pegion hole contain atmost 1 pegion.

→ If 2n+1 pegions are kept in n pegion hole then what is the average no. of pegions per pegion hole.

$A = \frac{2n+1}{n} = 2 + \frac{1}{n}$

Some pegion hole contain atleast 3 pegions & some pegion hole contain

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atmost 2.

→ If kn+1 pegions are kept in n pegions hole then what is the average no. of pegions per pegion hole.

$A = \frac{kn+1}{n} = k + \frac{1}{n}$

some pegion contain atleast k+1 no. of pegion & some pegions contains atmost k

Note:

Minimum no. of pegions required to ensured that some pegion hole contain atleast k+1 pegion is kn+1 where n is no. of pegion holes & k is any +ve integer

Q. Minimum no. of persons we have to choose randomly 9 persons are both in the same month.

→ No. of months = 12 ⇒ n = 12.

$k+1 = 9 \Rightarrow k = 8$.

Minimum no. of persons = (8)(12) + 1

= 97.

Q. In a group 73 persons, then which of the following statements is true

(i) Atleast 7 persons are born in same month.

(ii) Atmost 6 persons born in same month.

(iii) ~~Atmost~~ least 8 persons born in same month.

(iv) Atmost 5 persons born in same month.

Sol

Sol : $\left[\frac{kn+1}{n} = k + \frac{1}{n} = 73\right]$ ×

$A = \frac{73}{12} = 6.\dots$

$\lceil A \rceil = 7$

$\lfloor A \rfloor = 6.$

Q) If 410 letters are distributed in 50 apartments which of the following statement is true.

(i) some apartment receive atleast 9 letters

(ii) Atmost 8 letters.

(iii) Atmost 7 letters

(iv) Atleast 6 letters.

(v) Atleast 10 letters

Sol:

$\frac{410}{50} = \frac{41}{5} = 8.1..$

$\lceil A \rceil = 9$

$\lfloor A \rfloor = 8.$

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Q) There are 6 red balls, 8 blue, 10 green 15 white, 20 yellow balls. Minimum no. of balls selected randomly to ensure we get atleast 6 balls of same color.

Sol

$n = 5$

$k+1 = 6$

$k = 5$

$kn+1 = 5(5)+1 = 26.$

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Principle of Mutual inclusion & Exclu.

If A, B, C are any sets that $n(A \cup B)$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

(Q) In a class of 100 students, 54 can speak french, 38 students can speak German, 19 students can speak both French & German. How many students can speak:

(i) Atleast one of the 2-languages.

(ii) None of the 2 languages.

Sol (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(F \cup G) = n(F) + n(G) - n(F \cap G)$

$= 54 + 38 - 19$

$= 73$.

(ii) $n(\overline{F \cup G}) = 100 - 73$

$= 27$.

(Q) (iii) Only French or only German or only one of the two languages.

Sol: $n(F \cap \bar{G}) + n(G \cap \bar{F})$

$= (54 - 19) + (38 - 19)$

$= 54$.

(Q) In a class, 100 students. 40 failed in Maths, 30 failed in physics,

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35 failed in chemistry, 15 failed in maths & physics, 20 failed in physics + chemistry, 10 failed in maths & chemistry, 5 failed in maths, physics, chemistry.

How many failed in (i) atleast one of the 3 subjects

(ii) None of the 3 subjects.

(iii) Only one of the 3 subjects.

(iv) Atleast 2 of the 3 subjects.

(v) Exactly 2 subjects.

Sol: (i) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) - n(A \cap B \cap C)$

$= 40 + 30 + 25 - 15 - 20 - 10 + 5$

$= 55$.

(ii) $n(\overline{A \cup B \cup C}) = 100 - 55 = 45$

(iii) $n(M \cap \bar{P} \cap \bar{C}) + n(\bar{M} \cap P \cap C) + n(\bar{M} \cap P \cap \bar{C})$

one of the 3 subjects $= 20 + 0 + 0$

$= 20$.

(iv) Atleast 2 of the 3 subjects $= 10 + 5 + 15 + 3$ (atleast)

(v) Exactly 2 subjects $= 10 + 15 + 5$

$= 30$

Dearrangements:

- A permutation of object in which no of object appears in its correct place is called "dearrangements".
- No. of dearrangements of n distinct objects $= D_n = n! \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots + \frac{(-1)^n}{n!}\right.$

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eg: $D_2 = 2!\left[\frac{1}{2!}\right]$; $D_3 = 3!\left[\frac{1}{2!} - \frac{1}{3!}\right] = 2$

$D_4 = 4!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$

$= 9$

$$\boxed{D_n = n D_{n-1} + (-1)^n}$$

Q How many ways letter l_1, l_2, l_3, l_4, l_5 in 5 envelopes e_1, e_2, e_3, e_4, e_5

(i) No letter is correctly placed

$D_0 = 1$
$D_1 = 0$
$D_2 = 1$

sol $D_5 = 5(D_4) + (-1)^5 = 44$.

Recurrence equation: $\boxed{D_n = (n-1)\{D_{n-1} + D_{n-2}\} \geq 3}$

(ii) Exactly 2 letters are correctly placed

sol: $5C_2 \cdot D_3 = 5C_2 \cdot 2 = 20$

(iii) Exactly one letter is wrongly placed

sol: Not possible.

(iv) Atleast one letter is correctly placed

sol: $5! - 1 = 119$

Q 5 different books are distributed among 5 students, the books were returned & distributed again. How many way? (condition: no book can be taken twice.

(i) All books can be distributed in $5!$ ways since no book can be taken twice $D_5 = 44$

$5! \times 44 = 5280$.

(ii) Atmost one letter is correctly placed

$D_5 + (5C_1\ D_4) = 44 + (45) = 89$.

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Binomial Theorem:

Let n be a positive integer then for all x and y

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_r x^{n-r} y^r + \dots {}^nC_n y^n$$

eg: $(x+y)^8 = {}^8C_0 (x)^8 + {}^8C_1 x^7 y + {}^8C_2 x^6 y^2 + {}^8C_3 (x)^5 y^3 + {}^8C_4 x^4 y^4 + {}^8C_5 x^3 y^5 + {}^8C_6 x^2 y^6 + {}^8C_7 x^1 y^7 + {}^8C_8 x^0 y^8$

~~$= {}^8C_0 (x^8 + y^8)$~~

Multinomial Theorem:

Let n be a positive integer then $\forall x_1, x_2, \dots x_t$ we have-

$$\boxed{(x_1+x_2+\dots+x_t)^n = \Sigma p(n; q_1, \dots q_t) x_1^{q_1} x_2^{q_2} \dots x_t^{q_t}}$$

where Σ extends overall sets of non-negative integers, $q_1, q_2, \dots q_t$

$$\boxed{q_1+q_2+\dots+q_t = n}$$

* There are $c(n+t-1, n)$ terms in the expansion of $(x_1+x_2+\dots+x_t)^n$

Q) What is the co-efficient of x^3y^7 in $(x+y)^{10}$ and in $(2x-9y)^{10}$

$(x+y)^{10}$

$\downarrow$

${}^{10}C_7$

$(2x-9y)^{10}$

$\downarrow$

${}^{10}C_7 (2x)^3 (-9y)^7$

${}^{10}C_7 (8)(-9)^7$

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(8) use the multinomial theorem to expand $(x_1+x_2+x_3+x_4)^4$.

(3) $(2x-3y+5z)^8$ coefficient of $x^3 y^3 z^2$

$\Rightarrow {}^8C_3 \cdot {}^5C_3 \cdot {}^2C_2 \times (2)^3(-3)^3(5)^2$

(or)

$\Rightarrow \dfrac{8!}{3!\,3!\,2!} \times (2)^3(-3)^3(5)^2$.

(3) Determine the no. of terms in the expansion $(x_1+x_2+x_3+x_4)^6$

Sol $\quad {}^{n+r-1}C_r \Rightarrow {}^{6+4-1}C_6 = {}^9C_6 = 84$

(3) Determine the no. of terms in the expansion

$(x-7y+3z-w)^{25}$ $\qquad n=4 \quad r=25$

Sol $\quad {}^{4+25-1}C_{25} = {}^{28}C_{25} = \dfrac{28\times 27\times 26}{3\times 2}$

$= 3276$.

(3) Expansion of $(x-2y+z)^3$

Sol: $(x-2y+z)^3 = \dfrac{3!}{3!0!0!}x^3 + \dfrac{3!}{0!3!0!}(-2y)^3 + \dfrac{3!}{0!0!3!}(z^3) +$

$\dfrac{3!}{2!1!0!}(x)^2(-2y)(z)^0 + \dfrac{3!}{2!0!1!}(x)^2(z) +$

$\dfrac{3!}{0!2!1!}(-2y)^2(z) + \dfrac{3!}{1!0!2!}(x)(z)^2 + \dfrac{3!}{1!2!0!}x(-2y)^2$

$+ \dfrac{3!}{0!1!2!}(-2y)(z)^2 + \dfrac{3!}{1!1!1!}(x)(-2y)(z)$

No. of terms = 10.

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$x^3 - 8y^3 + z^3 + 3x^2z - 6x^2y + 12xy^2 + 3xz^2 +$

$12yz^2 - 12xyz$.

Ⓠ Coefficient of $x_1^2 x_3 x_4^3 x_5^4$ in $(x_1+x_2+x_3+x_4+x_5)$

Sol: $\frac{10!}{2!\,1!\,3!\,4!}(1)^2(1)^1(1)^3(1)^4$

No. of terms: $C(10+5-1, 10)$

$= {}^{14}C_{10} = 1001$

Ⓠ Determine the coefficient of $x^5 y^{10} z^5 w^5$ in $(x - 7y + 3z - w)^{25}$

Sol Coefficient: $\frac{25!}{5!\,10!\,5!\,5!}(1)^5(-7)^{10}(3)^5(-1)^5$

No. of terms: $C(25+4-1, 25) = {}^{28}C_{25}$

$= 3276$.

Ⓠ $x_1 + x_2 + x_3 + x_4 + x_5 \le 10$. So how many non-ve integer sol'n's are possible

Sol: $x_1 + x_2 + x_3 + x_4 + x_5 \le 10$

Let $x_6 \ge 0$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$.

${}^{n+r-1}C_r \Rightarrow {}^{5+10-1}C_{10} = {}^{14}C_{10}$

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