1. Random Variables and Probability Distribution

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Sample Space :-

The collection of all posible outcomes of a random experiment is called a sample space and it is denoted by S.

Ex:- Tossing a coin

$\therefore S = \{H, T\}$

Tossing two coins

$S = \{HH, HT, TH, TT\}$

Event :-

A non-empty subset of sample space is called event.

Ex:- Tossing a coin and E an event selecting head.

$E = \{H\}$

Mutually Exclusive Events :-

Two event E_1 and E_2 are said to be M.E.E if

$E_1 \cap E_2 = \phi$

Probability :-

In a random experiment 'S' be the sample space and 'E' be an event then the probability of the event 'E' is depend as P(E).

$$P(E) = \frac{\text{No. of outcome favorable to event } E}{\text{No. of all possible outcomes of an Exp}}$$

$$P(E) = \frac{n(E)}{n(S)}$$

Axioms of Probability:-

If 's' is the sample space and E be an event then

(i) $0 \leq P(E) \leq 1$

(ii) $P(S) = 1$

(iii) $P(E_1 \cap E_2) = P(E_1) + P(E_2)$

where E_1, E_2 are m.e.e

Q①:- In a simultaneous throw of two dice find the probability of getting a total of 7.

$n(S) = 36$

Let E be an event getting sum 7

$E = \{(1,6), (2,5), (3,4), (4,3) (5,2) (6,1)\}$

$n(E) = 6$

$P(E) = \frac{6}{36} = \frac{1}{6}$

② What is the probability of a leap year selected at random will contain 53 sundays.

$n(E) = 2$

$n(S) = 7$

$P(E) = \frac{2}{7}$

Random variables:- 4/9/24

A random variable 'X' whose values is determine by outcomes of random experiment is called random variable

i.e A random variable 'x' is a real value function defined on the sample space 's'. $X: S \to R$

Ex:- Tossing two coins the possible outcomes.

$S = \{HH, HT, TH, TT\}$

Let 'x' be random variable selecting number of heads

$HH = 2 \quad HT = 1 \quad TH = 1 \quad TT = 0$

$x = \{0, 1, 2\}$

Types of Random variables :-

① Discrete random variable

② Continuous random variable

1) Discrete random variables:-

A random variable 'x' which can take only finite number of discrete values in an interval of domain is called Discrete random variables.

i.e 'x' takes the values the set $\{0, 1, 2, 3 \ldots n\}$

Ex:. Tossing coin, Throwing a dice.

2) Continuous random variables:-

A random variable 'x' which takes all possible values in a given interval. is called continuous random variable.

Ex:- Height, age, weight of individuals.

Probability Distribution functions:-

There are 2 types of probability Distribution.

① Discrete Probability Distribution.

② Continuous " "

1) Discrete probability Distribution:-

Probability: Suppose 'X' is a discrete random variable with possible outcomes $x_1, x_2, x_3, x_4 \ldots x_n$ then the probability of each possible outcome x_i is $P_i = P(X = x_i) = P(x_i)$ is called discrete probability distribution.

Properties:—

(i) $P(x_i) \geq 0$ for all x_i

(ii) $\sum_{i=1}^{n} P(x_i) = 1$

(iii) $P(X < x_n) = P(x_1) + P(x_2) + \ldots + P(x_{n-1})$

$P(X \leq x_n) = P(x_1) + P(x_2) + \ldots + P(x_n)$

$P(X > x_n) = 1 - P(X \leq x_n)$

Cummulative Distribution of Discrete Random variable:

Let 'X' be a DRV then the Discrete probability distribution function or cummulative distribution function is denoted by $F(x)$

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} P(x_i)$$

Mean or Expection:—

Let 'X' assumes the values $x_1, x_2, x_3, \ldots x_n$ with respective probabilities $P_1, P_2, P_3 \ldots P_n$ then the mathematical expection or mean is denoted by $E(x)$ or μ. and defined as

$$E(x) = \mu = \sum_{i=1}^{n} x_i P(x_i)$$

$$||ly \quad E(x^2) = \sum_{i=1}^{n} x_i^2 P(x_i), \quad E(x^n) = \sum_{i=1}^{n} x_i^n P(x_i)$$

1) $E(X+K) = E(X) + E(K)$

2) $E(K) = K$ where K is const

3) $E(KX) = K\,E(X)$

4) $E(X+Y) = E(X) + E(Y)$

5) $E(XY) = E(X) \cdot E(Y)$

Variance :-

Variance of the probability distribution of a random variable X is the mathematical expectation of $(X - E(X))^2$ (or), $(x_i - \mu)^2$

$E(X - E(X))^2 = E(x^2) - [E(x)]^2$

and it is denoted by σ^2 (or) $V(X)$

i.e $\boxed{V(X) = \sigma^2 = E(x^2) - [E(x)]^2}$

$$E(x-\mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2 P_i$$
$$= \sum_{i=1}^{n} (x_i^2 + \mu^2 - 2\mu x) P_i$$
$$= E(x^2) + \mu^2 - 2\mu^2$$
$$= E(x^2) - \mu^2$$
$$= E(x^2) - [E(x)]^2$$

Note :-

1) $V(K) = 0$

2) $V(KX) = K^2 V(X)$

3) $V(X+K) = V(X)$

4) $V(aX+b) = a^2 V(X)$

5) $V(X \pm Y) = V(X) \pm V(Y)$

a, b, K are constants.

Let $Y = aX + b$

$\Rightarrow E(Y) = aE(X) + b$

$Y - E(Y) = a[X - E(X)]$

$E(Y - E(Y))^2 = a^2 E[X - E(X)]^2$

$V(Y) = a^2 V(X)$

$V(aX+b) = a^2 V(X)$

$a = 0$	$b = 0$
$V(b) = 0$	$V(aX) = a^2 V(X)$

Standard deviation(S.D):—

The square root of variance is called S.D.

It is denoted by σ

$$SD = \sqrt{V(x)} = \sqrt{\sigma^2}$$
$$= \sigma$$

problems:-

① ~~Where~~ A random variable X has the following probability distribution.

X	1	2	3	4	5	6	7	8
P(x)	k	2k	3k	4k	5k	6k	7k	8k

then find the values of (i) k (ii) $P(x \leq 2)$ (iii) $P(2 \leq X \leq 5)$

(i) we have $\sum_{x=1}^{8} P(x_i) = 1$

$$\Rightarrow k + 2k + 3k + 4k + 5k + 6k + 7k + 8k = 1$$

$$36k = 1$$

$$k = 1/36$$

(ii) $P(x \leq 2) = P(x=1) + P(x=2)$

$$= k + 2k$$
$$= 3k$$
$$= 3/36 = 1/12$$

(iii) $P(2 \leq X \leq 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= 2k + 3k + 4k + 5k$$
$$= 14k = \cancel{14}^{7} \cdot \frac{1}{\cancel{36}_{18}} = 7/18$$

**
(2) The probability of mass function of variable x is

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

then find the values of (i) K (ii) $P(X<6)$, $P(X\geq 6)$, $P(0<X<5)$ and $P(0\leq X\leq 4)$

(iii) $P(X\leq n) > \frac{1}{2}$ find the minimum value of 'n'.

(iv) Determine the distribution function of x.

(v) Mean (vi) Variance.

(i) $\sum_{x=0}^{7} p(x_i) = 1$

$0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

$9k+10k^2 = 1$

$10k^2+9k-1=0$

$10k^2+10k-k-1=0$

$10k(k+1)-1(k+1)=0$

$k=-1$ (or) $\boxed{k = 1/10}$

(ii) $P(X<6) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$= 0+k+2k+2k+3k+k^2$

$= 8k+k^2$

$= k(k+8)$

$= \frac{1}{10}\left(\frac{1}{10}+8\right)$

$= \frac{1}{10}\left(\frac{1+80}{10}\right) \Rightarrow \frac{1}{10}\left(\frac{81}{10}\right) = \frac{81}{100} = 0.81$

$P(X\geq 6) =$ ~~[illegible]~~ $P(X=6)+P(X=7)$ (or) $\Rightarrow 1-P(X<6)$

$= 2k^2+7k^2+k$

$= 9k^2+k \Rightarrow \frac{1}{10}\left(\frac{9}{10}+1\right) = \frac{19}{100} = 0.19$

$$P(0 < X < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$$
$$= k + 2k + 2k + 3k$$
$$= 8k$$
$$= \frac{\cancel{8}^{4}}{\cancel{10}_{5}} = \frac{4}{5} = 0.8$$

$$P(0 \le X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=4)$$
$$= 0 + 8k$$
$$= 8k \Rightarrow \frac{4}{5} = 0.8$$

(iii) $P(X \le n) > \frac{1}{2}$

$P(X \le 0) = 0 > \frac{1}{2}$ ×

$P(X \le 1) = P(X=0) + P(X=1) = 0 + k = \frac{1}{10} > \frac{1}{2}$ ×

$P(X \le 2) = 0 + k + 2k = 3k = \frac{3}{10} > \frac{1}{2}$ ×

$P(X \le 3) = 0 + k + 2k + 2k = \frac{5}{10} > \frac{1}{2}$ ×

$P(X \le 4) = 0 + k + 2k + 2k + 3k = \frac{8}{10} > \frac{1}{2}$ ✓

∴ Minimum value $n = 4$

iv) Distribution function of x

X	$F(x) = P(X \le x)$
0	$F(x) = P(X \le 0) = 0$
1	$F(x) = P(X \le 1) = 0 + k = \frac{1}{10}$
2	$F(x) = P(X \le 2) = k + 2k = 3k = \frac{3}{10}$
3	$F(x) = P(X \le 3) = 3k + 2k = \frac{5}{10}$
4	$F(x) = P(X \le 4) = 5k + 3k = \frac{8}{10}$
5	$F(x) = P(X \le 5) = 8k + k^2 = \frac{81}{100}$
6	$F(x) = P(X \le 6) = 8k + \cancel{3k}k^2 = k(8+3k) = \frac{83}{100}$
7	$F(x) = P(X \le 7) = 9k + 10k^2 = k(9 + 10k) = \frac{1}{10}(9 + \frac{10}{10}) = \frac{1}{10}(10) = 1$

v) Mean $= E(x) = \mu = \sum_{i=0}^{7} x_i P(x_i)$

$= 0 + k + 4k + 6k + 12k + 5k^2 + 12k^2 + 49k^2 + 7k$

$= 66k^2 + 30k$

$= k\left(\frac{66}{10} + 30\right)$

$= \frac{1}{10}(6.6 + 30)$

$= \frac{1}{10}(36.6)$

$= 3.66$

vi) Variance

$V(x) = \sigma^2 = E(x^2) - (E(x))^2$

$= \sum_{i=0}^{7} x_i^2 P(x_i) - (3.66)^2$

$= 0 + k + 8k + 18k + 48k + 25k^2 + 72k^2 + 343k^2 + 49k$

$= 440k^2 + 124k$

$= \frac{44\cancel{0}}{10\cancel{0}} + \frac{124}{10}$

$= 4.4 + 12.4$

$\sigma^2 = 16.8 - 13.39$

$= 3.41$

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3 A random variable 'X' has the following probability function.

X	-3	-2	-1	0	1	2	3
P(X)	k	0.1	k	0.2	2k	0.4	2k

find (i) k (ii) Mean (iii) Variance

(i) $\sum_{i=-3}^{3} P(x_i) = 1$

$k+0.1+k+0.2+2k+0.4+2k=1$

$6k+0.7=1$

$6k=1-0.7$

$6k=0.3$

$k=\frac{\cancel{0.3}^{0.1}}{\cancel{6}_{2}}$

$k=0.05$

(ii) Mean

$$\mu=\sum_{i=-3}^{3} P(x_i)\cdot x_i$$

$$=-3(k)+(-2)(0.1)+(-1)k+0(0.2)+1(2k)+2(0.4)+3(2k)$$

$$=-3k-0.2-k+2k+0.8+6k$$

$$=4k+0.6$$

$$=4(0.05)+0.6$$

$$=0.2+0.6$$

$$\boxed{\mu=0.8}=E(x)$$

(iii) variance $=\sigma^2=E(x^2)-(E(x))^2$

$$=\sum_{i=-3}^{3} x_i^2 P(x)-(E(x))^2$$

$$=(-3)^2k+(-2)^2(0.1)+(-1)^2k+0^2(0.2)+1^2(2k)+2^2(0.4)+3^2(2k)$$

$$=9k+0.4+k+2k+1.6+18k$$

$$=2+30k$$

$$=2+30(0.05)$$

$$=2+1.5$$

$$=3.5-(E(x))^2$$

$$=3.5-0.64$$

$$\sigma^2=2.86$$

④ A variable 'X' has the following probability distribution

x	-3	6	9
$P(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

then find (i) $E(X)$ (ii) $E(X^2)$ (iii) $E(2X+1)^2$ (iv) $V(X)$

(i) $E(X) = -\cancel{3}\left(\frac{1}{\cancel{6}_2}\right) + \cancel{6}^3\left(\frac{1}{\cancel{2}}\right) + \cancel{9}^3\left(\frac{1}{\cancel{3}}\right)$

$= -\frac{1}{2} + 3 + 3$

$= -\frac{1}{2} + 6$

$= \frac{11}{2}$

(ii) $E(X^2) = \sum x_i^2 P(x_i)$

$= (-3)^2 \frac{1}{6} + (6)^2 \frac{1}{2} + 9^2 \frac{1}{3}$

$= \cancel{9}^3 \frac{1}{\cancel{6}_2} + \cancel{36}^{18} \frac{1}{\cancel{2}} + \cancel{81}^{27} \frac{1}{\cancel{3}}$

$= \frac{3}{2} + 18 + 27$

$= \frac{3}{2} + 45 = \frac{93}{2}$

$\begin{array}{r} 18 \\ 27 \\ \hline 45 \end{array}$

(iii) $E(2X+1)^2 = E(4X^2 + 1 + 4X)$

$= 4E(X^2) + E(1) + 4E(X)$

$= \cancel{4}^2 \frac{93}{\cancel{2}} + 1 + \cancel{4}^2 \frac{11}{\cancel{2}}$

$= 209$

(iv) $V(X) = E(X^2) - (E(X))^2$

$= \frac{93}{2} - \left(\frac{11}{2}\right)^2$

$= \frac{93}{2} - \frac{121}{4}$

$= \frac{186 - 121}{4} = \frac{65}{4}$

$\begin{array}{r} 93 \\ 93 \\ \hline 186 \\ 121 \\ \hline 65 \end{array}$

Continuous Probability Distributions :-

The function formed by continuous random variable is called probability density function.
It is denoted by $f(x)$

properties:-

(i) $f(x) \geq 0 \quad \forall x \in R$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a<x<b) = \int_{a}^{b} f(x)dx$

= Area under the curve $f(x)$ b/w $x=a$ and b

$P(a<x<b) = P(a \leq x \leq b) = P(a < x \leq b) = P(a \leq x < b)$

Cummulative distribution function of C.R.V :-

The cummulative distribution function or simply the distribution function of continuous random variable 'X', is denoted by $F(x)$

i.e $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$

properties:-

1) $0 \leq F(x) \leq 1$, $-\infty < x < \infty$

2) $F'(x) = f(x) \geq 0$

3) $F(-\infty) = 0$, $F(\infty) = 1$

4) $P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx = F(b) - F(a)$

i) **Mean :-**

Mean of distribution given by

$$\text{Mean} = E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

Suppose if x is denoted from a to b

$$E(x) = \mu = \int_a^b x f(x)\,dx.$$

2) Variance:—

$$V(x) = \sigma^2 = E(x^2) - (E(x))^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

3) Median:—

Median is the point which divides the entire distrib-ution into two equal parts.

Thus 'x' is defined from a to b and M is the median then $\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$

4) Mode:—

Mode is the value of 'x' for which f(x) is maximum

i.e mode is given by $f'(x) = 0$ and $f''(x) < 0$ for $a < x < b$

5) Mean deviation:—

Mean deviation about the mean μ is given by

$$\int_{-\infty}^{\infty} |x - \mu| f(x)\,dx$$

① If a random variable has the probability density function $f(x) = \begin{cases} 2e^{-2x}, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}$

then find

(i) $P(1 < x < 3)$ (ii) $P(x > 0.5)$

(i) $P(1 < x < 3) = \int_1^3 f(x)\,dx = \int_1^3 2e^{-2x}\,dx$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_1$$

$$= -\left[e^{-6} - e^{-2}\right]$$

$$= -e^{-6} + e^{-2}$$

$$= e^{-2} - e^{-6}$$

(iii) $P(x > 0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{\infty} 2e^{-2x}\,dx$

$$= \left[\frac{2e^{-2x}}{-2}\right]_{0.5}^{\infty} = e^{-2(0.5)} = e^{-1}$$

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(2) A continuous random variable has the probability density function $f(x) = \begin{cases} kxe^{-\lambda x}, & \text{for } x \geq 0,\ \lambda > 0 \\ 0 & ; \text{otherwise.} \end{cases}$

find (i) K (ii) Mean (iii) Variance.

(i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{\infty} k\underset{u}{x}\,\underset{v}{e^{-\lambda x}}\,dx = 1$$

$$k\left[(x)\left(\frac{e^{-\lambda x}}{-\lambda}\right) - (1)\left(\frac{e^{-\lambda x}}{+\lambda^2}\right)\right]_0^{\infty} = 1$$

$$k\left[0 - \left(\frac{-1}{\lambda^2}\right)\right] = 1$$

$$\frac{k}{\lambda^2} = 1$$

$$k = \lambda^2$$

(ii) Mean $= E(x) = \mu = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{\infty} x f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} x\, k x e^{-\lambda x}\,dx$$

$$= k\int_{0}^{\infty} \underset{u}{x^2}\, \underset{v}{e^{-\lambda x}}\,dx$$

$$= k\left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2x\left(\frac{e^{-\lambda x}}{+\lambda^2}\right) + 2\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)\right]_0^{\infty}$$

$$= k\left[0 - \frac{2}{-\lambda^3}\right]$$

$$= \frac{-2k}{+\lambda^3}$$

$$= \frac{+2\lambda^2}{\lambda^3} = \frac{+2}{\lambda}$$

(iii) Variance $V(x) = \sigma^2 = E(x^2) - (E(x))^2$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$= \left[\int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{\infty} x^2 f(x)\,dx\right] - \mu^2$$

$$= \left[0 + \int_{0}^{\infty} x^2 k x e^{-\lambda x}\,dx\right] - \mu^2$$

$$= \left[k\int_{0}^{\infty} \underset{u}{x^3}\, \underset{v}{e^{-\lambda x}}\,dx\right] - \mu^2$$

$$= k\left[x^3\frac{e^{-\lambda x}}{-\lambda} - 3x^2\frac{e^{-\lambda x}}{+\lambda^2} + 6x\frac{e^{-\lambda x}}{-\lambda^3} - 6\frac{e^{-\lambda x}}{+\lambda^4}\right]_0^{\infty} - \mu^2$$

$$= k\left[0 - \frac{6}{-\lambda^4}\right] - \frac{4}{\lambda^2}$$

$$= \frac{6k}{\lambda^4} - \frac{4}{\lambda^2} \Rightarrow \frac{6\lambda^2}{\lambda^4} - \frac{4}{\lambda^2} \Rightarrow \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$$

3) The probability density function of a random variable X is $f(x) = \begin{cases} \frac{1}{2}\sin x & \text{for } 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$

then find

(i) Mean (ii) Mode (iii) Median (iv) $P(0 \le x \le \pi/2)$

(i) Mean

$$E(x) = \mu = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{0} x f(x) dx + \int_{0}^{\pi} x f(x) dx + \int_{\pi}^{\infty} x f(x) dx$$

$$= 0 + \int_{0}^{\pi} x \frac{1}{2} \sin x \, dx + 0$$

$$= \frac{1}{2} \int_{0}^{\pi} \underset{u}{x} \underset{v}{\sin x} \, dx \Rightarrow \frac{1}{2} \left[x(-\cos x) - (1)(-\sin x) \right]_{0}^{\pi}$$

$$= \frac{1}{2} \left[-\pi \cos \pi + \sin \pi - 0 \right]$$

$$= \frac{1}{2} \left[-\pi(-1) \right] = \frac{\pi}{2}$$

(ii) Mode

$f'(x) = 0$

$\frac{\partial}{\partial x} \frac{1}{2} \sin x = 0$

$\frac{1}{2} \cos x = 0$

$\cos x = 0$

$x = \pi/2$

$f''(x) < 0$

$\frac{\partial}{\partial x} \left(\frac{1}{2} \cos x \right) < 0$

$\frac{1}{2}(-\sin x) < 0$

$f''(x) = \frac{1}{2}(-\sin x)$

$f''(\pi/2) = \frac{-1}{2} < 0$

(iii) Median

$$\int_{-\infty}^{M} f(x) dx = \int_{M}^{\infty} f(x) dx = \frac{1}{2}$$

$$\int_{-\infty}^{M} f(x) dx = \frac{1}{2} \Rightarrow \int_{0}^{M} \frac{1}{2} \sin x \, dx = \frac{1}{2}$$

$$\cancel{\tfrac{1}{2}} \left[-\cos x \right]_{0}^{M} = \cancel{\tfrac{1}{2}}$$

$= (\cos 0 - \cos M) = 1$

$= 1 - \cos M = 1$

$\cos M = 0$

$M = \pi/2$

(iii) $P(0 \le x \le \pi/2)$

$$\int_0^{\pi/2} f(x)\,dx \Rightarrow \int_0^{\pi/2} \frac{1}{2}\sin x\,dx \Rightarrow \frac{1}{2}\left(-\cos x\right)_0^{\pi/2}$$

$$= -\frac{1}{2}\left[0 - 1\right]$$

$$= 1/2$$

(4) For a continuous probability function $f(x) = kx^2e^{-x}$, when $x \ge 0$ then find (i) k (ii) $E(x)$ (iii) σ^2

(i) $k \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx = 1$$

$$\int_0^{\infty} kx^2e^{-x}\,dx = 1$$

$$k\left[\frac{x^2e^{-x}}{-1} - 2x\frac{e^{-x}}{1} + \frac{2e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$k\left[+2\right] = 1$$

$$\boxed{k = 1/2}$$

(ii) $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= k\int_0^{\infty} x^3e^{-x}\,dx$$

$$= k\left[\frac{x^3e^{-x}}{-1} - \frac{3x^2e^{-x}}{1} + \frac{6xe^{-x}}{-1} - \frac{6e^{-x}}{1}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[6\right]$$

$$= 3$$

(iii) $\sigma^2 = E(x^2) - (E(x))^2$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

$$= k\int_0^{\infty} x^4 e^{-x} dx$$

$$= k\left[\frac{x^4e^{-x}}{-1} - \frac{4x^3e^{-x}}{1} + \frac{12x^2e^{-x}}{-1} - \frac{24xe^{-x}}{1} + \frac{24e^{-x}}{-1}\right]_0^{\infty}$$

$$= \frac{1}{2}[24] - \mu^2$$

$$= 12 - 3^2 = 3$$

Bernoulion trial:— 18/9/24

Bernoulion trial is a random experiment having only two possible outcomes success (p) & failure (q) such that $p+q=1$

Ex:- Tossing a coin

getting Head is success & getting tail is failure

we have the following distributions.

1) Binomial distribution
2) Possion distribution
3) Normal distribution

Note:- In above 1 & 2 are discrete theory distribution and 3 is continuous theoritical distribution.

1) Binomial distribution:-

Binomial distribution is a discrete probability distributions this distribution was discovered by James Bernouli. Hence, it is known as Binomial distribution

Conditions of BD:-

1) An experiment is conducted N number of times
2) There are only two possible outcomes (success or failure).

3) The probability of success in each trail remains constant and doesnot change from trial to trail

4) The trial are independent.

Consider a random experiment having n trails. Let it succeed x times, probability of getting x successes $= p^x$

→ The probability of getting x number of success out of n trails is given by

$$P(X=x) = b(x,n,P) = {}^nC_x\, p^x q^{n-x}$$

Ex:- The no. of defective items in a box containing n items.

Ex The no. of machines lying ideal in a factory having n machines.

The Binomial distribution function / cummulative:-

$$F(x) = P(X \leq x)$$

① A coin is tossed 6 times find the probability of getting a) Exactly 2 heads
b) Atleast 4 heads
c) No heads
d) Atleast 1 head

Sol: $n=6$

The probability of getting head $p = 1/2$

$\therefore q = 1-p = 1/2$

we have BD

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$

$$P(X=2) = {}^6C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{6-2}$$

$$= 15 \times \tfrac{1}{4} \times \tfrac{1}{16}$$

$$= \tfrac{15}{64}$$

$$\frac{6!}{2!\,4!} \Rightarrow \frac{\overset{3}{\cancel{6}} \times 5 \times \cancel{4!}}{\cancel{2} \times 1 \times \cancel{4!}}$$

2) $P(x \geq 4) = P(x=4) + P(x=5) + P(x=6)$

$= {}^6C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 + {}^6C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^1 + {}^6C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^0$

$= 15 \times \frac{1}{16} \times \frac{1}{4} + 6 \times \frac{1}{64} + \frac{1}{64}$

$= \frac{22}{64} = \frac{11}{32}$

3) No heads

$P(x=0) = {}^6C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^6$

$= \frac{1}{64}$

4) $P(x \geq 1) = 1 - P(x \leq 1)$

$= 1 - P(x=0)$

$= 1 - \frac{1}{64}$

$= \frac{63}{64}$

3) A fair die is trowed 7 times determine the probability 5 (or) 6 appear 1) Exactly 3 times 2) Never occurs.

→ Number of trails $n=7$

for 1 time 5 (or) 6 getting is $P = \frac{2}{6} = \frac{1}{3}$

$q = \frac{2}{3}$

1) $P(x=3) = {}^7C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^4$

$= 35 \times \frac{1}{27} \times \frac{16}{81}$

$= \frac{560}{2187}$

2) $P(x=0) = {}^7C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^7$

$= \frac{128}{2187}$

3) The probability of a man hitting a target is $1/3$.

a) If he tries 5 times what is the probability of he's hitting the target atleast twice?

b) How many times he must fires so that the probability of he's hitting the target atleast once is more than 90%?

Sol:- $p = \frac{1}{3}$ $q = \frac{2}{3}$

a) $n = 5$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$= 1 - {}^5C_0 \left(\tfrac{1}{3}\right)^0 \left(\tfrac{2}{3}\right)^5 - {}^5C_1 \left(\tfrac{1}{3}\right)^1 \left(\tfrac{2}{3}\right)^4$$

$$= 1 - \frac{32}{243} - \frac{80}{243}$$

$$= 1 - \frac{112}{243}$$

$$= \frac{131}{243}$$

Mean of binomial distribution:-

By B.D we have $P(X=x) = {}^nC_x p^x q^{n-x}$

$$\text{Mean} = E(X) = \mu = \sum_{x=0}^{n} x\,P(x)$$

$$= \sum_{x=0}^{n} x\,{}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \frac{n!}{(n-x)!\,\cancel{x}(x-1)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(n-x)!\,(x-1)!} p^x q^{n-x}$$

put $x-1=y \Rightarrow x=1+y$

$x=1 \Rightarrow y=0$

$x=n \Rightarrow y=n-1$

$$= \sum_{y=0}^{n-1} \frac{n(n-1)!}{(n-1-y)!\,y!} p^{1+y} q^{(n-1-y)}$$

$$= np \sum_{y=0}^{n-1} {}^{n-1}C_y\, p^y q^{(n-1)-y}$$

$$= np\,(p+q)^{n-1}$$

$$= np\,(1)^{n-1}$$

$$= np$$

Variance of binomial distribution:-

By BD we have $P(X=x) = {}^nC_x p^x q^{n-x}$

$$\sigma^2 = V(X) = E(X^2) - (E(X))^2$$

$$= E(X^2) - \mu^2$$

$$= \sum_{x=0}^{n} x^2 P(x) - (np)^2$$

$$= \sum_{x=0}^{n} (x^2 - x + x) P(x) - n^2p^2$$

$$= \sum_{x=0}^{n} (x^2 - x) P(x) + \sum_{x=0}^{n} x P(x) - n^2p^2$$

$$= \sum_{x=0}^{n} x(x-1)\, {}^nC_x\, p^x q^{n-x} + np - n^2p^2$$

$$= \sum_{\substack{x=0 \\ x=2}}^{n} x(x-1) \frac{n!}{(n-x)!\, x(x-1)(x-2)!} p^x q^{n-x} + np - n^2p^2$$

put $x-2 = y \rightarrow x = 2+y$

$x=2 \Rightarrow y=0$

$x=n \Rightarrow y=n-2$

$$= \sum_{y=0}^{n-2} \frac{n(n-1)(n-2)!}{(n-2-y)!\, y!} p^{2+y} q^{(n-2)-y} + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{y=0}^{n-2} \frac{(n-2)!}{(n-2-y)!\, y!} p^y q^{(n-2)-y} + np - n^2p^2$$

$$= n(n-1)p^2 \sum_{y=0}^{n-1} {}^{(n-2)}C_y\, p^y q^{(n-2)-y} + np - n^2p^2$$

$$= n(n-1)p^2 (p+q)^{n-2} + np - n^2p^2$$

$$= n(n-1)p^2 (1)^{n-2} + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$= npq$$

Standard deviation:-

$$S.D = \sqrt{V(x)}$$
$$= \sqrt{npq}$$

Mode of Binomial distribution:-

$$\text{Mode} = \begin{cases} \text{Integral part of } (n+1)p \text{ ; if } (n+1)p \text{ is not an integer.} \\ (n+1)p \text{ and } (n+1)p-1 \text{ ; if } (n+1)p \text{ is an integer.} \end{cases}$$

① The mean and variance of B.D are 4 and 4/3 then find (i) B.D (ii) P(X=1) (iii) P(X>1)

$np = 4 \rightarrow ①$ $\quad npq = 4/3 \rightarrow ②$

$$\frac{②}{①} \Rightarrow \frac{\cancel{np}q}{\cancel{np}} = \frac{\cancel{4}/3}{\cancel{4}} \Rightarrow q = \frac{1}{3}$$

$$p = \frac{2}{3}$$

$$np = 4$$
$$n\left(\frac{2}{3}\right) = 4$$
$$n = 6$$

(i) $P(x) = b(x:n,p) = {}^nC_x \, p^x q^{n-x}$

$P(X=x) = b(x, 6, 2/3) = {}^6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}$

(ii) $P(X=1) = {}^6C_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{6-1}$

$$= \cancel{6}^{\,2} \times \frac{2}{\cancel{3}} \times \frac{1}{3^5}$$

$$= \frac{2^2}{3^5} = \frac{4}{243}$$

(iii) $P(X>1) = 1 - P(X \le 1)$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[{}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{6-0} + \frac{4}{243}\right]$$

$= 1 - \left[1 \times 1 \times \frac{1}{36} + \frac{4}{243}\right]$

$= 1 - \left\{\frac{1}{729} + \frac{4}{243}\right\}$

$= \frac{716}{729}$

$243 \times 3 = 729$

② In a BD continuo consists of 5 independent trials probability of 1 and 2 are 0.4096 & 0.2048 respectively. find the parameter p of the distribution.

and $n=5$

$P(x=1) = 0.4096$ and $P(x=2) = 0.2048$

$nC_1 p^1 q^{n-1} = 0.4096$

$5C_1 p q^4 = 0.4096$ → ①

$nC_2 p^2 q^{n-2} = 0.2048$

$5C_2 p^2 q^3 = 0.2048$ → ②

$\frac{②}{①}$

$\frac{5C_2 \cancel{p^2} \cancel{q^3}}{5C_1 \cancel{p} \cancel{q^3} q} = \frac{0.2048}{0.4096}$

$\frac{5!}{2!\,3!} = \frac{5 \times 4}{2}$

$\frac{\cancel{10}^{2} p}{\cancel{5} q} = \frac{\cancel{0.2048}^{1}}{\cancel{0.4096}_{2}}$

$4p = q$

$p + q = 1$

$p + 4p = 1$

$5p = 1$

$p = 1/5, \; q = \frac{4}{5}$

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Poisson distribution:-

The poisson distribution can be obtained from BD with the following conditions.

i) n is very large i.e $n \to \infty$

ii) The success p in each trial is very neglible

iii) $\lambda = np$ (finite valu)

under the above conditions the BD tends to posion distribution and its density function is given by

$$f(x,\lambda) = P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

Mean and variance of PD

i) Mean $E(X) = \mu = \sum_{x=0}^{n} x P(x)$

from P.P $P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \cancel{x} \frac{e^{-\lambda}\lambda^x}{\cancel{x}(x-1)!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

$$= e^{-\lambda}\left[\frac{\lambda}{0!} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \cdots\right]$$

$$= e^{-\lambda}\lambda\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= e^{-\lambda}\lambda e^{\lambda}$$

$$\boxed{\text{Mean} = \lambda}$$

Variance $= \sigma^2 = V(x) = E(x^2) - (E(x))^2$

$$= \sum_{x=0}^{n} x^2 P(x) - \lambda^2$$

$$= \sum_{x=0}^{n} (x^2 - x + x) P(x) - \lambda^2$$

$$= \sum_{x=0}^{n} (x^2 - x) P(x) + \sum_{x=0}^{n} x P(x) - \lambda^2$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} + \lambda - \lambda^2$$

$$= \sum_{x=2}^{\infty} \frac{\cancel{x(x-1)} e^{-\lambda}\lambda^x}{\cancel{x(x-1)}(x-2)!} + \lambda - \lambda^2$$

$$= e^{-\lambda}\left(\frac{\lambda^2}{0!} + \frac{\lambda^3}{1!} + \frac{\lambda^4}{2!} + \cdots\right) + \lambda - \lambda^2$$

$$= e^{-\lambda}\lambda^2\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right) + \lambda - \lambda^2$$

$$= \cancel{e^{-\lambda}}\cancel{\lambda^2}\cancel{e^{\lambda}} + \lambda - \cancel{\lambda^2}$$

$$\boxed{V(x) = \lambda}$$

$S.D = \sqrt{V(x)}$

$\boxed{S.D = \sqrt{\lambda}}$

Examples:- of PD:-

1) The no. of effective electric bulbs manufactured by deputed company (n is large)

2) The no. of persons born blind per year in a large city.

properties:-

1) Range of variable is 0 to ∞

2) Mean and variance are equal.

Mode:-

$$\text{Mode} = \begin{cases} \text{Integral part of } \lambda & \text{if } \lambda \text{ is not an integer} \\ \lambda \ \& \ (\lambda - 1) & \text{if } \lambda \text{ is integer} \end{cases}$$

① If the probability that an individual suffers a bad reaction from a certain injection is 0.001. Determine the probability that out of 2000 individuals suffers a bad reaction. i) Exactly 3 ii) more than two individuals.
(iii) none iv) More than 1 individual.

Sol:- Given $P = 0.001$ $n = 2000$

$\lambda = np$

$\lambda = 2000 \times 0.001$

$\boxed{\lambda = 2}$

(i) $P(x=3) = \dfrac{e^{-2}2^3}{3!} = \dfrac{e^{-2}\cancel{8}^{4}}{\cancel{6}\,3}$

$P(x=3) = \frac{4}{3}e^{-2} = 0.1804$

(ii) $P(x>2) = 1 - P(x \leq 2) \Rightarrow 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - \left[\dfrac{e^{-2}2^0}{0!} + \dfrac{e^{-2}2^1}{1!} + \dfrac{e^{-2}2^2}{2!}\right]$

$= 1 - e^{-2}[1 + 2 + 2]$

$= 1 - 5e^{-2} = 0.3233$

(iii) None $= P(x=0) = \frac{e^{-2}2^0}{0!}$

$= e^{-2} = 0.135$

iv) $P(x>1) = 1 - [P(x=0) + P(x=1)]$

$= 1 - e^{-2}[1+2]$

$= 1 - 3e^{-2} = 0.594$

2) If a random variable has a PD such that $P(1)=P(2)$ then find i) Mean of the distribution ii) $P(x=4)$ (iii) $P(x \geq 1)$ (iv) $P(1<x<4)$

(i) $P(1) = P(2)$

$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$ | $\lambda^2 = 2\lambda$

$\lambda(2-\lambda) = 0$

$\boxed{\lambda=2}$ $\boxed{\lambda=0}$

$\therefore \boxed{\text{Mean} = 2}$

ii) $P(x=4) = \frac{e^{-2}2^4}{4!} = \frac{\cancel{16}\, e^{-2}}{\cancel{4}\times\cancel{6}\,3} = \frac{2}{3}e^{-2} = 0.09$

iii) $P(x \geq 1) = 1 - P(x=0)$

$= 1 - \frac{e^{-2}2^0}{0!} = 1 - e^{-2} = 0.865$

iv) $P(1<x<4) = P(x=2) + P(x=3)$

$= \frac{e^{-2}2^2}{2!} + \frac{e^{-2}2^3}{3!} = 4e^{-2}\left[\frac{1}{2} + \frac{1}{3}\right]$

$= \cancel{4}\,2e^{-2}\left[\frac{5}{\cancel{6}\,3}\right]$

$= \frac{10}{3}e^{-2} = 0.45$

③ The average no. of accidents on any day on a notational highway is 1.8. Determine the probability that the no. of accidents are i) atleast 1 ii) atmost 1.

Sol:- mean $\lambda = 1.8$

i) $P(x \geq 1) = 1 - P(x=0)$

$= 1 - \frac{e^{-1.8}(1.8)^0}{0!} \Rightarrow 1 - e^{-1.8} = 0.834$

ii) $P(x \le 1) = P(x=0) + P(x=1)$

$$= \frac{e^{-1.8}(1.8)^0}{0!} + \frac{e^{-1.8}(1.8)^1}{1!}$$

$$= e^{-1.8}[1+1.8] = 2.8 \times e^{-1.8}$$

$$= 0.4628$$

Recurrence formula for PD:-

we have from P.D

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{—①}$$

put $x = x+1$

$$P(x+1) = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} \quad \text{—②}$$

$$\frac{②}{①} \Rightarrow \frac{P(x+1)}{P(x)} \Rightarrow \frac{\cancel{e^{-\lambda}}\cancel{\lambda^{x+1}}}{(x+1)!} \times \frac{x!}{\cancel{e^{-\lambda}}\cancel{\lambda^{x}}}$$

$$\Rightarrow \frac{P(x+1)}{P(x)} = \frac{\lambda \times \cancel{x!}}{(x+1)\cancel{x!}}$$

$$\boxed{P(x+1) = \frac{\lambda}{(x+1)} P(x)}$$

Normal distribution:-

ND is a continuous distribution in which the variable 'x' can take all the values with in a given range. ND is given by the mathematician Karlfredrich guass.

∴ It is called gaussian distribution and its density function is given by $\boxed{f(x,\mu,\sigma) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$

where $\left.\begin{array}{l}\mu = \text{mean} \\ \sigma = S.D\end{array}\right\}$ are two parameters

where $-\infty < x < \infty$

$-\infty < \mu < \infty$

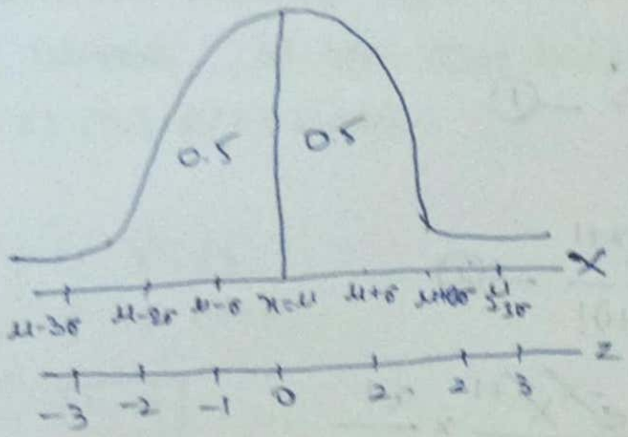
$\sigma > 0$

Note:- N.D Can be obtained from the B.D by the following 2 conditions.

i) n is very large ($n \to \infty$)

ii) neither p nor q is very small

Properties of normal probability curve:- 11/10/24



$$z = \frac{x-\mu}{\sigma}$$

1) The curve is well shaped and symmetrical about the line $x=\mu$

2) The curve ~~is~~ Mean, Median and Mode of its distribution are coincide

3) If x is increases numerically then the value of f(x) decreases and the maximum probability is occurring at the point $x=\mu$

4) Since f(x) being the probability can never be negative. so no portion of the curve lies below the x-axis

5) x-axis is an asymptote

6) Area under normal curve is unity

To find probability of Normal Curve:—

The probability that the normal variable "x" with mean "μ" standard deviation "σ" lies b/w two specific values x_1 & x_2 with $x_1 \le x_2$ can be obtained using area under the normal curve as follows.

Step-1:- Perform the change of scale using

$z = \frac{x-\mu}{\sigma}$ to get z_1 and z_2.

Step-2:-

a) To find $P(x_1 \le x \le x_2) = P(z_1 \le z \le z_2)$

i) If z_1 and z_2 are +ve (or) -ve

$P(z_1 \le z \le z_2) = |A(z_2) - A(z_1)|$

ii) If $z_1 < 0$ and $z_2 > 0$

$P(z_1 \le z \le z_2) = A(z_1) + A(z_2)$

b) To find $P(z > z_1)$

(i) If $z_1 < 0$

$P(z > z_1) = 0.5 + A(z_1)$

(ii) If $z_1 > 0$

$P(z > z_1) = 0.5 - A(z_1)$

c) To find $P(z < z_1)$

(i) If $z_1 < 0$

$P(z < z_1) = 0.5 - A(z_1)$

(ii) If $z_1 > 0$

$P(z < z_1) = 0.5 + A(z_1)$

① For a normally distributed variable with mean 1 and S.D 3 then find (i) $P(3.43 \le x \le 6.19)$
(ii) $P(-1.43 \le x \le 6.19)$

Sol:- Given $\mu = 1$, $\sigma = 3$

(i) $P(3.43 \le x \le 6.19)$

Let $x_1 = 3.43$, $x_2 = 6.19$

we have standard normal variable

$$z = \frac{x-\mu}{\sigma}$$

$$z_1 = \frac{x_1-\mu}{\sigma} \Rightarrow z_1 = \frac{3.43-1}{3} = 0.81$$

$$z_2 = \frac{x_2-\mu}{\sigma} \Rightarrow \frac{6.19-1}{3} = 1.73$$

$$P(3.43 \le x \le 6.19) = P(0.81 \le z \le 1.73)$$
$$= A(1.73) - A(0.81)$$
$$= 0.458 - 0.2910$$
$$= 0.1672$$

(ii) $P(-1.43 \le x \le 6.19)$

Let $x_1 = -1.43$ $x_2 = 6.19$

we have S.N.V

$$z_1 = \frac{x_1-\mu}{\sigma} = \frac{-1.43-1}{3} = -0.81$$

$$z_2 = \frac{x_2-\mu}{\sigma} = \frac{6.19-1}{3} = 1.73$$

$$P(-1.43 \le x \le 6.19) = P(-0.81 \le z \le 1.73)$$
$$= A(1.73) + A(-0.81)$$
$$= 0.4582 + 0.2910$$
$$= 0.7492$$