Recurrence Relations

Generating Functions of Sequences

Sequences

$A = \{a_r\}$, $r = 0 .. \infty$.

Examples:

1. $A = \{a_r\}$, $r = 0 .. \infty$, where $a_r = 2^r$.

$$= \{1, 2, 4, 8, 16, \ldots, 2^r, \ldots\}$$

2. $B = \{b_r\}$, $r = 0 .. \infty$, where

$$b_r = 0, \text{ if } 0 \leq r \leq 4$$
$$= 2, \text{ if } 5 \leq r \leq 9$$
$$= 3, \text{ if } r = 10$$
$$= 4, \text{ if } 11 \leq r$$

$$= \{0, 0, 0, 0, 0, 2, 2, 2, 2, 2, 3, 4, 4, \ldots\}$$

3. $C = \{c_r\}$, $r = 0 .. \infty$, where $c_r = r + 1$.

$= \{1, 2, 3, 4, 5, \ldots\}$

4. $D = \{d_r\}$, $r = 0 .. \infty$, where $d_r = r^2$.

$= \{0, 1, 4, 9, 16, 25, \ldots\}$

Generating function for the sequence $A = \{a_r\}$, $r = 0 .. \infty$.

$$A(X) = a_0 + a_1 X + a_2 X^2 + \dots + a_n X^n + \dots$$

$$= \sum a_r X^r, r = 0 .. \infty.$$

Examples:

1. Generating function for the sequence $A = \{a_r\}$, $r = 0 .. \infty$,

   where $a_r = 2^r$.

$$A(X) = 1 + 2X + 4X^2 + \dots + 2^n X^n + \dots$$

$$= \sum 2^r X^r, r = 0 .. \infty.$$

2. Generating function for the sequence $B = \{b_r\}$, $r = 0 .. \infty$, where

$$
\begin{aligned}
b_r &= 0, \text{ if } 0 \le r \le 4 \\
&= 2, \text{ if } 5 \le r \le 9 \\
&= 3, \text{ if } r = 10 \\
&= 4, \text{ if } 11 \le r
\end{aligned}
$$

$$
B(X) = 2X^5 + 2X^6 + 2X^7 + 2X^8 + 2X^9 + 3X^{10} + 4X^{11} + 4X^{12} + \ldots + 4X^n + \ldots
$$

3. Generating function for the sequence $C = \{c_r\}$, $r = 0 .. \infty$, where $c_r = r + 1$.

$$
\begin{aligned}
C(X) &= 1 + 2X + 3X^2 + \ldots + (n+1)X^n + \ldots \\
&= \sum (r+1)X^r, \ r = 0 .. \infty.
\end{aligned}
$$

4. Generating function for the sequence $D = \{d_r\}$, $r = 0 .. \infty$, where $d_r = r^2$.

$$
\begin{aligned}
D(X) &= X + 4X^2 + 9X^3 + 16X^4 + 25X^5 + \ldots + n^2 X^n + \ldots \\
&= \sum r^2 X^r, \ r = 0 .. \infty.
\end{aligned}
$$

Definitions

Let the Generating Functions / Formal Power Series be

$A(X) = \sum a_r X^r$, $r = 0 .. \infty$.

and $B(X) = \sum b_s X^s$, $s = 0 .. \infty$.

1. **Equality**

   $A(X) = B(X)$, iff $a_n = b_n$ for each $n \geq 0$.

2. **Multiplication by a scalar number C**

   $C\ A(X) = \sum (C\ a_r) X^r$, $r = 0 .. \infty$.

3. **Sum**

   $A(X) + B(X) = \sum (a_n + b_n) X^r$, $r = 0 .. \infty$.

4. **Product**

   $A(X)\ B(X) = \sum P_n X^n$, $n = 0 .. \infty$,

   where $P_n = \sum_{j+k-n} a_j b_k$.

Exercises:

1. Find a Generating function for the sequence

$$A = \{a_r\},\ r = 0\ ..\ \infty, \text{ where}$$

$$a_r = 1, \text{ if } 0 \le r \le 2$$

$$= 3, \text{ if } 3 \le r \le 5$$

$$= 0, \text{ if } r \ge$$

$$A(X) = 1 + X + X^2 + 3X^3 + 3X^4 + 3X^5$$

2. Build a generating function for a_r = no. of integral solutions to the equation $e_1 + e_2 + e_3 = r$, if $0 \le e_i \le 3$ for each i.

$$A(X) = (1 + X + X^2 + X^3)^3$$

3. Write a generating function for a_r = no. of ways of selecting r balls from 3 red balls, 5 blue balls, and 7 white balls.

$$A(X) = (1 + X + X^2 + X^3)(1 + X + X^2 + X^3 + X^4 + X^5)$$
$$(1 + X + X^2 + X^3 + X^4 + X^5 + X^6 + X^7)$$

4. Find the coefficient of X^{23} in $(1 + X^5 + X^{9)10}$.

$e_1 + e_2 + \ldots + e_{10} = 23$ where $e_i = 0, 5, 9.$

$1 \times 5 + 2 \times 9 + 7 \times 0 = 23$

$$\begin{aligned}\text{Coefficient of } X^{23} &= 10! / (1!\ 2!\ 7!) \\ &= 10 . 9 . 8 / (2) \\ &= 10. 9 . 4 \\ &= 360\end{aligned}$$

5. Find the coefficient of X^{32} in $(1 + X^5 + X^{9)10}$.

$e_1 + e_2 + \ldots + e_{10} = 32$ where $e_i = 0, 5, 9.$

$1 \times 5 + 3 \times 9 + 6 \times 0 = 32$

$$\begin{aligned}\text{Coefficient of } X^{32} &= 10! / (1!\ 3!\ 6!) \\ &= 10 . 9 . 8 . 7/ (3 . 2) \\ &= 10. 3 . 4 . 7 \\ &= 840\end{aligned}$$

6. Find a Generating function for the no. of r-combinations of {3.a, 5.b, 2.c}.

$$A(X) = (1 + X + X^2 + X^3)(1 + X + X^2 + X^3 + X^4 + X^5)$$
$$(1 + X + X^2)$$

Calculating Coefficient of generating function

If $A(X) = \sum a_r X^r$, $r = 0 .. \infty$, then $A(X)$ is said to have a multiplicative inverse if there is $B(X) = \sum b_k X^k$, $k = 0 .. \infty$ such that $A(X) B(X) = 1$.

$a_0 b_0 = 1$

$a_1 b_0 + a_0 b_1 = 0$

$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$

$a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 0$

…

$a_0 b_2 + a_0 b_2 + \ldots + a_0 b_2 = 0$

…

$b_0 = 1 / a_0$

$b_1 = -a_1 b_0 / a_0$

$b_2 = -a_1 b_1 - a_2 b_0 / a_0$

…

Geometric Series

$A(X) = 1 - X$

$a_0 = 1$, $a_1 = -1$.

$b_0 = 1 / a_0 = 1$

$b_1 = -a_1 b_0 / a_0 = -(-1)(1) / (1) = 1$

$b_2 = -a_1 b_1 - a_2 b_0 / a_0 = -(-1)(1) - (0)(1) / (1) = 1$

…

$\therefore b_i = 1$

$1 / (1 - X) = \sum X^r$, $r = 0 .. \infty$.

Replace X by aX, where a is a real no.

$1 / (1 - aX) = \sum a_r X^r, r = 0 .. \infty.$

Let $a = -1$

$1 / (1 + X) = \sum (-1)^r X^r, r = 0 .. \infty.$

$1 / (1 + aX) = \sum (-1)^r a_r X^r, r = 0 .. \infty.$

$1 / (1 - X)^n = (\sum X^k)^n, k = 0 .. \infty.$

$= \sum C(n - 1 + r, r) X^r, r = 0 .. \infty.$

$$1 / (1 + X)^n = (\sum (-1)^r X^k)^n, k = 0 .. \infty.$$

$$= \sum C(n - 1 + r, r) (-1)^r X^r, r = 0 .. \infty.$$

$$1 / (1 - aX)^n = (\sum a^r X^k)^n, k = 0 .. \infty.$$

$$= \sum C(n - 1 + r, r) a^r X^r, r = 0 .. \infty.$$

$$1 / (1 - X^k) = \sum X^{kr}, k = 0 .. \infty.$$

$$1 / (1 + X^k) = \sum (-1)^r X^{kr}, k = 0 .. \infty.$$

$$1 / (a - X) = (1 / a) \sum X^r / a^r, r = 0 .. \infty.$$

$$1 / (X - a) = (-1 / a) \sum X^r / a^r, r = 0 .. \infty.$$

$$1 / (X + a) = (1 / a) \sum X^r / ((-1)^r a^r), r = 0 .. \infty.$$

$$1 + X + X^2 + \ldots + X^n = (1 - X^{n+1}) / (1 - x)$$

Special Cases of Binomial Theorem

$$(1 + X)^n = 1 + C(n, 1)\, X + C(n, 2)\, X^2 + \ldots + C(n, n)\, X^n$$

$$(1 + X^k)^{\,n} = 1 + C(n, 1)\, X^k + C(n, 2)\, X^{2k} + \ldots + C(n, n)\, X^{nk}$$

$$(1 - X) = 1 - C(n, 1)\, X + C(n, 2)\, X^2 + \ldots + (-1)^n\, C(n, n)\, X^n$$

$$(1 - X^k)^{\,n} = 1 - C(n, 1)\, X^k + C(n, 2)\, X^{2k} + \ldots + (-1)^n\, C(n, n)\, X^{nk}$$

Examples:

1. Calculate $A(X) = \Sigma\, a_r X^r$, $r = 0 .. \infty = 1 / (X^2 - 5X + 6)$.

$(X^2 - 5X + 6) = (X - 3)(X - 2)$

$1 / (X^2 - 5X + 6) = A / (X - 3) + B / (X - 2)$

$\therefore\ A(X - 2) + B(X - 3) = 1$

Let $X = 2$, Then $B = -1$

Let $X = 3$, Then $A = 1$

$1 / (X^2 - 5X + 6)$

$= 1 / (X - 3) - 1 / (X - 2)$

$= (-1 / 3) \sum X^r / 3^r - (-1 / 2) \sum X^r / 2^r$, $r = 0 .. \infty$.

$= (-1 / 3) \sum X^r / 3^r + (1 / 2) \sum X^r / 2^r$, $r = 0 .. \infty$.

2. Compute the coefficients of $A(X) = \Sigma\, a^r X^r$, $r = 0 .. \infty$

$$= (X^2 - 5X + 3) / (X^4 - 5X^2 + 4).$$

$$(X^4 - 5X^2 + 4) = (X^2 - 1)(X^2 - 4)$$

$$= (X - 1)(X + 1)(X - 2)(X + 2)$$

$$(X^2 - 5X + 3) / (X^4 - 5X^2 + 4)$$

$$= A / (X - 1) + B / (X + 1) + C / (X - 2) + D / (X + 2)$$

$$\therefore (X^2 - 5X + 3) = A (X + 1)(X - 2)(X + 2)$$

$$+ B (X - 1)(X - 2)(X + 2)$$

$$+ C (X - 1)(X + 1)(X + 2)$$

$$+ D (X - 1)(X + 1)(X - 2)$$

For $X = 1$, $A = 1 / 6$

For $X = -1$, $B = 3 / 2$

For $X = 2$, $C = -1 / 4$

For $X = -2$, $D = -17 / 12$

$\therefore (X^2 - 5X + 3) / (X^4 - 5X^2 + 4)$

$= 1/(6(X - 1)) + 3/(2(X + 1)) - 1/(4(X - 2)) - 17/(12(X + 2))$

$= (-1/6)\sum X^r + 3/2\sum(-1)^r X^r - 1/4(-1/2)\sum X^r /2^r$

$17/12(1/2)\sum X^r /((-1)^r 2^r)$, $r = 0 .. \infty$

$= \sum [(-1/6) + 3/2(-1)^r + 1/8(1/ 2^r) - 17/24(-1)^r / 2^r)] X^r$, $r = 0 .. \infty$

3. Find the coefficient of X^{20} in $(X^3 + X^4 + X^5 + \ldots)^5$.

$(X^3 + X^4 + X^5 + \ldots)^5$

$= [X^3 (1 + X + X^2 + \ldots)]^5$

$= X^{15} (\Sigma X^r)5, r = 0 .. \infty$

$= X^{15} \Sigma C(5 - 1 + r, r) X^r, r = 0 .. \infty$

$= X^{15} \Sigma C(4 + r, r) X^r, r = 0 .. \infty$

Coefficient of X^{20} in $(X^3 + X^4 + X^5 + \ldots)5$

= Coefficient of X^5 in $\Sigma C(4 + r, r) X^r, r = 0 .. \infty$

$\therefore r = 5$

$C(4 + r, r)$

$= C(9, 5)$

$= 9! / (5!\ 4!)$

$= 9 . 8 . 7 . 6 / (4 . 3 . 2)$

$= 9 . 7 . 2$

$= 126$

Recurrence relations

Recurrence relation

Formula that relates for any integer $n \geq 1$, the nth term of a sequence $A = \{a_r\}$, $r = 0 .. \infty$ to one or more of the terms $a_0, a_1, \ldots, a_{n-1}$.

Examples

$a_n + 5a_{n-1} + 6a_{n-2} = 0.$

$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1.$

Linear recurrence relation

A recurrence relation of the form

$c_0(n) + c_1(n)a_n + \ldots + c_k(n)\, a_{n-k} = f(n)$ for $n \geq k$,

where $c_0(n)$, $c_1(n)$, …, $c_k(n)$, and $f(n)$ are functions of n.

Example

$a_n - (n - 1)a_{n-1} - (n - 1)a_{n-2} = 5n$.

Linear recurrence relation of degree k

$c_0(n)$ and $c_k(n)$ are not identically zero.

Example

$a_n + 5a_{n-1} + 6a_{n-2} = 0$.

Linear recurrence relation with constant coefficients

$c_0(n)$, $c_1(n)$, …, $c_k(n)$ are constants.

Example

$a_n + 5a_{n-1} + 6a_{n-2} = 0.$

Homogeneous recurrence relation

$f(n)$ is identically zero.

Example

$a_n + 5a_{n-1} + 6a_{n-2} = 0.$

Inhomogeneous recurrence relation

$f(n)$ is not identically zero.

Example

$a_n + 5a_{n-1} + 6a_{n-2} = 5n.$

Solving recurrence relation by substitution and Generating functions

Solving recurrence relation by substitution / Backtracking

Technique for finding an explicit formula for the sequence defined by a recurrence relation.

Backtrack the value of an by substituting the definition of a_{n-1}, a_{n-2}, … until a pattern is clear.

Examples

1. Use the technique of *backtracking*, to find an explicit formula for the sequence defined by the recurrence relation and initial condition for

   $a_n = a_{n-1} + 3$, a1=2.

   $a_n = a_{n-1} + 3$

   $= (a_{n-2} + 3) + 3$

   $= ((a_{n-3} + 3) + 3) + 3 = a_{n-3} + 3.3$

   …

   or $a_n = a_{n-1} + 1.3$

   $= a_{n-2} + 2.3$

   …

   $= a_{n-(n-1)} + (n–1).3$

   $= a_1 + (n–1).3$

   $= 2 + (n–1).3$

   ∴ The explicit formula for the sequence is

   $a_n = 2 + (n-1).3$

2. Use the technique of *backtracking*, to find an explicit formula for the sequence defined by the recurrence relation and initial condition for $a_n = 2.5a_{n-1}$, $a_1 = 4$.

$$
\begin{aligned}
a_n &= 2.5a_{n-1} \\
&= 2.5(2.5a_{n-2}) \\
&= (2.5)^2 a_{n-2} \\
&= (2.5)^3 a_{n-3} \\
&\quad \cdots \\
&= (2.5)^{n-1} a_{n-(n-1)} \\
&= (2.5)^{n-1} a_1 \\
&= 4(2.5)^{n-1}
\end{aligned}
$$

$\therefore$ Explicit formula is an $= 4(2.5)^{n-1}$

3. Use the technique of *backtracking*, to find an explicit formula for the sequence defined by the recurrence relation and initial condition for $a_n = 5a_{n-1} + 3$, $a_1 = 3$.

$$
\begin{aligned}
a_n &= 5a_{n-1} + 3 \\
&= 5(5a_{n-2} + 3) + 3 \\
&= 5^2 a_{n-2} + (5 + 1)3 \\
&= 5^2 (5a_{n-3} + 3) + (5 + 1)3 \\
&= 5^3 a_{n-3} + (5^2 + 5 + 1)3 \\
&\qquad\qquad \ldots \\
&= 5^{n-1} a_{n-(n-1)} + (5^{n-2} + \ldots + 5^2 + 5 + 1)3 \\
&= 5^{n-1} a_1 + (5^{n-2} + \ldots + 5^2 + 5 + 1)3 \\
&= 5^{n-1} 3 + (5^{n-2} + \ldots + 5^2 + 5 + 1)3 \\
&= (5^{n-1} + 5^{n-2} + \ldots + 5^2 + 5 + 1)3 \\
&= 3(5^n - 1) / 4
\end{aligned}
$$

$\therefore$ Explicit formula is $a_n = 3(5^n - 1) / 4$

4. Use the technique of *backtracking*, to find an explicit formula for the sequence defined by the recurrence relation and initial condition for

$a_n = a_{n-1} + n$, $a_1 = 4$.

$$
\begin{aligned}
a_n &= a_{n-1} + n \\
&= a_{n-2} + (n-1) + n \\
&= a_{n-3} + (n-2) + (n-1) + n \\
&\quad \dots \\
&= a_{n-(n-1)} + [n-(n-1) + 1] + \dots + (n-1) + n \\
&= a_1 + 2 + \dots + (n-1) + n \\
&= a_1 - 1 + [1 + 2 + \dots + (n-1) + n] \\
&= 4 - 1 + n(n+1)/2 \\
&= 3 + n(n+1)/2
\end{aligned}
$$

$\therefore$ Explicit formula is an = 3 + n(n+1)/2

Solving recurrence relations by Generating functions

Shifting properties of generating functions

$X^k A(X) = X^k \sum a_n X^n$, $n = 0 .. \infty$

$= \sum a_n X^{n+k}$, $n = 0 .. \infty$

Replacing n+k by r, we get

$\sum a_{r-k} X^r$, $r = k .. \infty$

Equivalent expressions for generating functions

If $A(X) = \sum a_n X^n$, $n = 0 \,..\, \infty$, then

$\sum a_n X^n$, $n = k \,..\, \infty = A(X) - a_0 - a_1 X - \ldots - a_{k-1} X^{k-1}$.

$\sum a_{n-1} X^n$, $n = k \,..\, \infty = X(A(X) - a_0 - a_1 X - \ldots - a_{k-2} X^{k-2})$.

$\sum a_{n-2} X^n$, $n = k \,..\, \infty = X^2(A(X) - a_0 - a_1 X - \ldots - a_{k-3} X^{k-3})$.

$\sum a_{n-3} X^n$, $n = k \,..\, \infty = X^3(A(X) - a_0 - a_1 X - \ldots - a_{k-4} X^{k-4})$.

…

$\sum a_{n-k} X^n$, $n = k \,..\, \infty = X^k(A(X))$.

Examples

- 1. Solve the *recurrence relation* $a_n - 7\,a_{n-1} + 10\,a_{n-2} = 0,\ n \geq 0$,

  $a_0 = 10,\ a_1 = 41$, using *generating functions*.

1. Let $A(X) = \sum a_n X^n,\ n = 0 \,..\, \infty$.
2. Multiply each term in the recurrence relation by X^n and sum from 2 to ∞.

   $$\sum a_n X^n - 7\sum a_{n-1} X^n + 10\sum a_{n-2} X^n = 0,\ n = 2 \,..\, \infty.$$
3. Replace each infinite sum by an equivalent expression.

   $$[A(X) - a_0 - a_1X] - 7X[A(X) - a_0] + 10X^2[A(X)] = 0.$$
4. Simplify.

   $$A(X)(1 - 7X + 10X^2) = a_0 + a_1X - 7\,a_0\,X.$$

   $$A(X) = [a_0 + (a_1 - 7\,a_0)X] \,/\, (1 - 7X + 10X^2)$$

   $$= [a_0 + (a_1 - 7\,a_0)X] \,/\, [(1 - 2X)\,(1 - 5X)]$$

5. Decompose $A(X)$ as a sum of partial fractions.

$$A(X) = C_1 / (1 - 2X) + C_2 / (1 - 5X)$$

6. Express $A(X)$ as a sum of familiar series.

$$A(X) = C_1 \sum 2^n X^n + C_2 \sum 5^n X^n , n = 0 .. \infty.$$
$$= \sum (C_1 2^n + C_2 5^n) X^n , n = 0 .. \infty.$$

7. Express a_n as the coefficient of X^n in $A(X)$ and in the sum of the other series.

$$a_n = C_1 2^n + C_2 5^n.$$

8. Determine the values of C_1 and C_2 .

For $n = 0$, $a_0 = C_1 + C_2 = 10$ … (1)

For $n = 1$, $a_1 = 2 C_1 + 5 C_2 = 41$ … (2)

Solving (1) and (2), we get

$C_1 = 3$

$C_2 = 7$

$\therefore a_n = (3) 2^n + (7) 5^n.$

- 2. Solve the *recurrence relation* $a_n - 9\,a_{n-1} + 26\,a_{n-2} - 24\,a_{n-3} = 0$, $n \geq 3$, $a_0 = 0$, $a_1 = 1$, and $a_2 = 10$ using *generating functions*.

1. Let $A(X) = \sum a_n X^n$, $n = 0 .. \infty$.
2. Multiply each term in the recurrence relation by X^n and sum from 3 to ∞.

$$\sum a_n X^n - 9 \sum a_{n-1} X^n + 26 \sum a_{n-2} X^n - 24 \sum a_{n-3} X^n = 0,$$
$$n = 3 .. \infty.$$

3. Replace each infinite sum by an equivalent expression.

$$[A(X) - a_0 - a_1 X - a_2 X^2] - 9X\,[A(X) - a_0 - a_1 X] -$$
$$26X^2\,[A(X) - a_0] - 24X^3[A(X)] = 0.$$

4. Simplify.

$$A(X)(1 - 9X + 26X^2 - 24X^3)$$
$$= a_0 + a_1 X + a_2 X^2 - 9 a_0 X - 9 a_1 X^2 + 26 a_0 X^2.$$
$$A(X) = [a_0 + (a_1 - 9 a_0) X + (a_2 - 9 a_1 + 26 a_0) X^2] / (1 - 9X + 26X^2 - 24X^3)$$
$$= [a_0 + (a_1 - 9 a_0) X + (a_2 - 9 a_1 + 26 a_0) X^2] / [(1 - 2X) (1 - 3X) (1 - 4X)]$$

5. Decompose A(X) as a sum of partial fractions.

$$A(X) = C_1 / (1 - 2X) + C_2 / (1 - 3X) + C_3 / (1 - 4X)$$

6. Express A(X) as a sum of familiar series.

$$A(X) = C_1 \sum 2^n X^n + C_2 \sum 3^n X^n + C_3 \sum 4^n X^n, \; n = 0 \,..\, \infty.$$
$$= \sum(C_1 2^n + C_2 3^n + C_2 3^n + C_3 4^n) X^n, \; n = 0 \,..\, \infty.$$

7. Express an as the coefficient of X^n in A(X) and in the sum of the other series.

   $a_n = C_1 2^n + C_2 3^n + C_3 4^n$.

8. Determine the values of C_1, C_2 and C_3.

   Substituting a0 = 0, a1 = 1, and a2 = 10 in step 4, we get

   $$A(X) = [X + X2] / [(1 – 2X) (1 – 3X) (1 – 4X)]$$
   $$= C_1 / (1 – 2X) + C_2 / (1 – 3X) + C_3 / (1 – 4X)$$

   i.e., $C_1(1 – 3X) (1 – 4X) + C_2 (1 – 2X) (1 – 4X) + C_3(1 – 2X) (1 – 3X) = X + X^2$

   for X = 1/2, $C_1 = 3/2$

   for X = 1/3, $C_2 = –4$

   for X = 1/4, $C_3 = 5/2$

   $\therefore a_n = (3/2) 2^n – (4) 3^n + (5/2) 4^n$.

Exercises

1. Solve the *recurrence relation* $a_n - a_{n-1} - 9\ a_{n-2} + 9\ a_{n-3} = 0$, $n \geq 3$, $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$ using *generating functions*.
2. Solve the *recurrence relation* $a_n - 3\ a_{n-2} + 2\ a_{n-3} = 0$, $n \geq 3$, $a_0 = 1$, $a_1 = 0$, and $a_2 = 0$ using *generating functions*

Method of Characteristics roots

Characteristic equation for a linear homogeneous recurrence relation of degree k, $a_n = r_1a_{n-1} + \ldots + r_ka_{n-k}$ is

$x^k = r_1x^{k-1} + r_2x^{k-2} + \ldots + r_k$.

1. Characteristic equation $x^2 - r_1x - r_2 = 0$ of the recurrence relation $a_n = r_1a_{n-1} + r_2a_{n-2}$, having two distinct roots s_1 and s_2.

   Explicit formula for the sequence is $a_n = us_1^n + vs_2^n$ and u and v depend on the initial conditions.

2. Characteristic equation $x^2 - r_1x - r_2 = 0$ of the recurrence relation $a_n = r_1a_{n-1} + r_2a_{n-2}$ having a single root s.

   Explicit formula for the sequence is $a_n = us^n + vns^n$ and u and v depend on the initial conditions.

Examples

1. Solve the *recurrence relation* $a_n = 4a_{n-1} + 5a_{n-2}$, $a_1 = 2$, $a_2 = 6$.

The associated equation is $x^2 - 4x - 5 = 0$

i.e. $(x - 5)(x + 1) = 0$

$\therefore$ The different roots are $s_1 = 5$ and $s_2 = -1$.

Explicit formula is $a_n = us_1^n + vs_2^n$

$a_1 = u(5) + v(-1) = 5u - v$

Given $a_1 = 2$

$\therefore$ $5u - v = 2$ (1)

$a_2 = u(5)^2 + v(-1)^2 = 25u + v$

Given $a_2 = 6$

$\therefore$ $25u + v = 6$ (2)

Solving the equations (1) and (2), we get

$u = 4/15$ and $v = -2/3$

$\therefore$ Explicit formula is $a_n = us_1^n + vs_2^n$

$= 4/15(5)n - 2/3(-1)n$

2. Solve the *recurrence relation* $a_n = -6a_{n-1} - 9a_{n-2}$,
$a_1 = 2.5$, $a_2 = 4.7$.

The associated equation is $x^2 + 6x + 9 = 0$

i.e. $(x + 3)^2 = 0$

$\therefore$ The multiple root is $s = -3$.

Explicit formula is $a_n = us^n + vs^n$

$a_1 = u(-3) + vn(-3) = -3u + 3v$

Given $a_1 = 2.5$

$\therefore$ $-3u + 3v = 2.5$ (1)

$a_2 = u(-3)^2 + vn(-3)^2 = 9u + 18v$

Given $a_2 = 4.7$

$\therefore$ $9u + 18v = 4.7$ (2)

Solving the equations (1) and (2), we get

$u = -19.7/9$ and $v = 12.2/9$

$\therefore$ Explicit formula is $a_n = us^n + vns^n$

$= (-19.7/9)(-3)n + (12.2/9)n(-3)n$

3. Solve the *recurrence relation* $a_n = 2a_{n-2}$, $a_1 = \sqrt{2}$, $a_2 = 6$.

The associated equation is $x^2 - 2 = 0$

i.e. $(x - \sqrt{2})(x + \sqrt{2}) = 0$

$\therefore$ The different roots are $s_1 = \sqrt{2}$ and $s_2 = -\sqrt{2}$.

Explicit formula is $a_n = us_1^n + vs_2^n$

$a_1 = u(\sqrt{2}) + v(-\sqrt{2}) = \sqrt{2}u - \sqrt{2}v$

Given $a_1 = \sqrt{2}$

$\therefore \sqrt{2}u - \sqrt{2}v = \sqrt{2}$

$u - v = 1$ (1)

$a_2 = u(\sqrt{2})^2 + v(-\sqrt{2})^2 = 2u + 2v$

Given $a_2 = 6$

$\therefore 2u + 2v = 6$

$u + v = 3$ (2)

Solving the equations (1) and (2), we get

$u = 2$ and $v = 1$

$\therefore$ Explicit formula is $a_n = us_1^n + vs_2^n$

$= 2(\sqrt{2})n + (-\sqrt{2})n$