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UNIT-5

Recurrence Relation:

Recurrence relation is a formula which relates one or more preceding terms

eg: Fibonacci series

$F_0 = 0, \quad F_1 = 1$

$$\boxed{F_n = F_{n-1} + F_{n-2} \quad \text{where } n \geq 2.}$$

Q) a_n = no. of n-digit binary sequences with no consecutive zeroes.

Sol: case(i): $\underline{1}\ \underbrace{- - - - - -}_{a_{n-1} \text{ ways}}$

case(ii): $\underline{0}\ \underline{1}\ \underbrace{- - - - -}_{a_{n-2} \text{ ways}}$

$$\boxed{a_n = a_{n-1} + a_{n-2} \quad \text{where } n>2, \ \& \ a_1 = 2, \ a_2 = 3.}$$

a_1: 1, 0

a_2: 11, 10, 01

- 00, is not possible because given: no consecutive zeroes.

$a_4 = \overset{a_3 + a_2}{5+3} = 8 \quad (\because a_3 = a_2 + a_1 = 3+2 = 5)$

a_4:
1111
1110
1101
1011
0111
0101
1010
0110

Q) Find the recurrence relation of $\{a, ar, ar^2, ar^3, \ldots ar^n\}$.

Sol: $\boxed{a_n = a_{n-1} \cdot r; \text{ where } a_0 = a, \ n>0}$

Ⓠ $\{a, a+d, a+2d, \dots a+nd\}$

Sol: $\boxed{a_n = a_{n-1} + d \quad \text{where } n>0,\ a_0 = a}$

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Substitution of Recaurrence Relation

The so

(i) Forward substitution

(ii) Backward substitution.

Forward Substitution:

In this method the given recurrence relation is repeatedly used for $n=1,2,\dots$, and then solution is obtained by adding the 1st n-terms by appropriate formula.

eg. Solve the recurrence relation

(i) $a_n = n a_{n-1}$ $(n \geq 1)$ $a_0 = 1$.

Sol: $n=1 \Rightarrow a_1 = 1\cdot(a_0) = 1\cdot 1 = 1$

$n=2 \Rightarrow a_2 = 2\cdot(a_1) = 2\cdot 1 = 2$.

$n=3 \Rightarrow a_3 = 3\cdot(a_2) = 3\cdot 2 = 6$.

$n=4 \Rightarrow a_4 = 4\cdot(a_3) = 4\cdot 6 = 24$.

$\vdots$

$n=n \Rightarrow a_n = n\cdot a_{n-1} = n\cdot(n-1)\cdot(n-2)\cdot\dots 1 = n!$

$\boxed{a_n = n!}$.

(ii) $a_n = a_{n-1} + 3^{n-1}$ $n>1$ $a_1 = 2$.

Sol $n=2 \rightarrow a_2 = a_1 + 3^1 = 2+3 = 5 = 2+3$

$n=3 \Rightarrow a_3 = a_2 + 3^2 = 5+9 = 14 = 2+3^1+3^2$

$n=4 \Rightarrow a_4 = a_3 + 3^3 = 14+27 = 41 = 2+3^1+3^2+3^3$

$\vdots$

$n=n \Rightarrow a_n = a_{n-1} + 3^{n-1} = 2+3+3^2+3^3+\dots 3^{n-1}$

$= 1 + 3^0(1+3+3^2+3^3+\cdots+3^{n-1})$

$= 1 + \frac{3^{n}-1}{3-1}$

$= \frac{2+3^n-1}{2}$

constant < log n < n < n log n < n² ; $2^n < 3^n$

$$\boxed{a_n = \frac{3^n+1}{2}}$$

(iii) $a_n = a_{n-1} + n \ (n \geq 1)$ $a_0 = 1$.

Sol

$n=1 \Rightarrow a_1 = a_0 + 1 = 1+1 = 2$

$n=2 \Rightarrow a_2 = a_1 + 2 = 1+1+2 = 4$

$n=3 \Rightarrow a_3 = a_2 + 3 = 1+1+2+3 = 7$

$n=4 \Rightarrow a_4 = a_3 + 4 = 1+1+2+3+4$

$\vdots$

$n=n \Rightarrow a_n = a_{n-1} + n = 1+1+2+3+\cdots n$

$a_n = 1 + \frac{n(n+1)}{2}$

$= \frac{2+n(n+1)}{2}$

$$\boxed{a_n = \frac{n^2+n+2}{2}}$$

(iv) $a_n = a_{n-1} + n^2$ $a_0 = 5$ $(n \geq 1)$

$n=1 = a_0 + (1)^2 = 5 + 1^2$

$n=2 = a_1 + (2)^2 = 5 + 1^2 + 2^2$

$n=3 = a_2 + (3)^2 = 5 + 1^2 + 2^2 + 3^2$

$n=n \Rightarrow a_n = a_{n-1} + (n)^2 = 5 + 1^2 + 2^2 + \cdots n^2$

$a_n = 5 + \frac{n(n+1)(2n-1)}{2}$

$a_n = \frac{10 + [(n^2+n)(2n-1)]}{2} = \frac{10 + [2n^3 - n^2 + 2n^2 - n]}{2}$

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$$a_n = \frac{2n^3+n^2-n+10}{2}.$$

(v) $a_n = a_{n-1} + a^3 \ (n \geq 1)$ where $a_0 = 7$.

$$a_1 = a_0 + (1)^3 = 7 + 1^3$$

$$a_2 = a_1 + (2)^3 = 7 + 1^3 + 2^3$$

$$\vdots$$

$$a_n = a_{n-1} + (n)^3 = 7 + 1^3 + 2^3 + \cdots n^3.$$

$$a_n = 7 + \left(\frac{n^2(n+1)^2}{4}\right)$$

(vi) $a_n = a_{n-1} + \dfrac{1}{n(n+1)} \quad (n \geq 1) \quad a_0 = 1$

$$a_1 = a_0 + \frac{1}{1(1+1)} = 1 + \frac{1}{1(2)}$$

$$a_2 = a_1 + \frac{1}{2(2+1)} = 1 + \frac{1}{1(2)} + \frac{1}{2(3)}$$

$$a_3 = a_2 + \frac{1}{3(3+1)} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12}$$

$$= 1 + \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)}$$

$$a_n = 1 + \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \cdots + \frac{1}{n(n+1)}$$

$$a_n = 1 + \frac{1}{1(2)} + \frac{1}{2(3)} + \cdots \frac{1}{n} - \frac{1}{n+1}$$

$$a_n = 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \frac{1}{n} - \frac{1}{n+1}$$

$$a_n = 1 + 1 - \frac{1}{n+1} = 2 - \frac{1}{n+1} = \frac{2n+2-1}{n+1}$$

$$a_n = \frac{2n+1}{\ldots}$$

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Solving reccurrence relations using characteristic roots:

Shift operator E.

$E(a_n) = a_{n+1}$

$E^2(a_n) = a_{n+2}$

$E^3(a_n) = a_{n+3}$.

For any position integer k

$E^k(a_n) = a_{n+k}$.

Q. The solution at the recurrence relation. $a_n - 2 \cdot a_{n-1} = 0$ where $a_0 = 1$ is —

Sol. Given $a_n - 2a_{n-1} = 0$

It is a homogeneous recurrence relation

Minimum suffix must be n, so

replace n with $n+1$.

$a_{n+1} - 2a_n = 0$

$E(a_n) - 2a_n = 0$

$(E-2)a_n = 0$

∴ The characteristic equation is $(E-2)a_n = 0$

Replace E with t.

$(t-2) = 0$ (characteristic)

$\boxed{t = 2}$ → Root

The solution is $a_n = c_1 \cdot 2^n$ —①.

Since c_1 is a constant

to get c_1 put $n = 0$

$a_0 = c_1 2^0$

$$\boxed{1 = c_1}$$

$$\therefore \boxed{a_n = 2^n}.$$

Method of characteristic roots:

Shift operator E.

$E(a_n) = a_{n+1}$

$E^2(a_n) = a_{n+2}$

$E^3(a_n) = a_{n+3}$.

For any positive integer k,

$E^k(a_n) = a_{n+k}$.

Consider the linear recurrence relation with constant coefficients λ

$\lambda_0 a_n + \lambda_1 a_{n-1} + \dots \lambda_k a_{n-k} = f(n)$ —①

Minimum suffix should be n, so replace n by (n+k)

$\lambda_0 a_{(n+k)} + \lambda_1 a_{n+k-1} + \lambda_2 a_{n+k-2} + \dots + \lambda_k a_n = f(n+k)$

$\lambda_0 E^k a_n + \lambda_1 E^{k-1} a_n + \lambda_2 E^{k-2} a_n + \dots + \lambda_k E^k a_n = f(n+k)$

$a_n(\lambda_0 E^k + \lambda_1 E^{k-1} + \lambda_2 E^{k-2} + \dots + \lambda_k) = f(n+k) F(n)$

$\phi(E) a_n = F(n)$

The characteristic equation is $\phi(t) = 0$.

The roots at this equation becomes characteristic roots.

Let $t_1, t_2, \dots t_k$ be the characteristic roots

Complementary function (C.F)

This is the solution for eq-①.

Rules for complementary function.

characteristic roots of C.F :

1. Roots are real and distinct say $t_1, t_2, \ldots t_k$.

$$\boxed{c_1 t_1^n + c_2 t_2^n + \ldots c_k \cdot t_k^n}$$

2. Roots are real and two roots are equal. say $t_1, t_1, t_3, \ldots t_k$

$$\boxed{(c_1 + c_2 n) t_1^n + c_3 t_3^n + \ldots + c_k t_k^n}$$

3. Roots are real & 3 roots are equal say ~~$t_1, t_2, t_3, t_4, \ldots t_k$~~ $t_1, t_1, t_1, t_4, \ldots t_k$.

$$\boxed{(c_1 + c_2 n + c_3 n^2) t_1^n + \ldots + c_k t_k^n}$$

4. Roots are complex say $(\alpha \pm i\beta), t_3 \ldots$

$$\boxed{r\{c_1 \cos(n\theta) + c_2 \sin(n\theta)\}}$$

where $r = \sqrt{\alpha^2 + \beta^2}$ $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

5. Particular solution.

Q) The solution of the recurrence relation $x_n = 2 \cdot x_{n-1} - 1$ $(n > 1)$, where $a_1 = 2$ is ____

Sol Replace n with $n+1$ (Since minimum suffix must be n)

$$x_{n+1} = 2 \cdot x_n - 1$$

$$x_{n+1} - 2x_n = -1$$

$$E(x_n) - 2(x_n) = -1$$

$$(E-2) x_n = -1$$

The characteristic equation is $(E-2)x_n = -1$

$t - 2 = 0$

$\boxed{t = 2}$

$a_n = c_1 t_1^n$

$a_n = c_1 2^n$ — Complementary Function

$$x_n = \frac{F(n)}{\phi(E)} = \frac{b^n}{\phi(b)}$$

Particular solution $= \dfrac{b^n}{\phi(b)} = \dfrac{-1}{E-2} = \dfrac{-1(1)^n}{(1-2)}$

$= 1$

To know the value of C1 put $n = 1$

$\cancel{a_1 = c_1 (2)^1}$

$\cancel{2 = c_1 (2)}$

$\boxed{c_1 = 1} \Rightarrow 2^n$

Final solution = complementary sol^n + particular

$\cancel{\boxed{x_n = 2^n + 1}}$

$x_n = c_1 2^n + 1$

put $n = 1$

$x_1 = c_1 2^1 + 1$

$2 = c_1 2 + 1$

$1 = 2c_1$

$\boxed{c_1 = \frac{1}{2}}$

$x_n = \frac{1}{2} 2^n + 1$

$$\boxed{x_n = 2^{n-1} + 1}$$

(Q) $T(2^k) = 3 \cdot T(2^{k-1}) + 1$ where $T(1) = 1$

Sol. $T(2^k) = 3 \cdot T(2^{k-1}) + 1$

$T(2^k) - 3 \cdot T(2^{k-1}) = 1$

$T(2^k) = x_n$

Let $T(2^k) = x_n$.

$2^k = 1$

$x_n - 3x_{n-1} = 1$

$\boxed{k = 0}$

substitute $n = n+1$

(2^0)

$x_{n+1} - 3x_n = 1$.

$\boxed{n = 0}$

$E(x_n) - 3x_n = 1$

$(E-3)x_n = 1$

The characteristic equation is $(E-3)x_n = 1$

$\phi(E) = 0$

$E - 3 = 0$

$\boxed{t = 3}$.

$x_n = c_1 3^n$ — Complementary Function

Particular solution $= \frac{1}{E-3} = \frac{(1)^n}{1-3}$

$= \frac{1}{-2}$.

Final solution $= c_1 3^n + \left(-\frac{1}{2}\right)$

$x_n = c_1 3^n - \frac{1}{2}$.

put $x = 0$

$x_0 = c_1 (3)^0 - \frac{1}{2}$.

$1 = c_1 - \frac{1}{2}$

$\boxed{c_1 = \frac{3}{2}}$

$\Rightarrow T(2^k) = x_n = \frac{3}{2}(3)^n - \frac{1}{2} \Rightarrow x_n = \frac{1}{2}\left(3^{n+1} - 1\right)$

$$T(2^k) = \frac{1}{2}(3^{k+1} - 1)$$

Q. $a_n - 7a_{n-1} + 12a_{n-2} = 0$ where $a_0 = 2, a_1 = 5$

Sol. put $n = n+2$.

$a_{n+2} - 7a_{n+1} + 12a_n = 0$

$E^2(a_n) - 7E(a_n) + 12a_n = 0$

$a_n[E^2 - 7E + 12] = 0$

$\phi(E) = E^2 - 7E + 12$

$\phi(t) = 0$

$t^2 - 7t + 12 = 0$

$t^2 - 3t - 4t + 12 = 0$

$t(t-3) - 4(t-3) = 0$

$(t-3)(t-4) = 0$

$\boxed{t = 3, 4}$

$a_n = c_1 t_1^n + c_2 t_2^n$

$a_n = c_1(3)^n + c_2(4)^n$ is the complementary function.

put $n = 0$

$a_0 = c_1(3)^0 + c_2(4)^0$

$2 = c_1 + c_2$ ---- (1) $\Rightarrow c_2 = 2 - c_1$

put $n = 1$

$a_1 = c_1(3)^1 + c_2(4)^1$

$5 = 3c_1 + 4c_2$.

$5 = 3c_1 + 4(2 - c_1)$

$5 = 3c_1 + 8 - 4c_1$

$-3 = -c_1$

$\boxed{c_1 = 3}$

$c_2 = 2 - c_1 = 2 - 3$

$\boxed{c_2 = -1}$

$\therefore a_n = 3(3)^n - 1(4)^n$

$\boxed{a_n = 3^{n+1} - 4^n}$

⑤ $a_n - 2a_{n-1} = 2^n$ where $a_0 = 1$

sol $a_n - 2a_{n-1} = 2^n$

put $n = n+1$

$a_{n+1} - 2a_n = 2^{n+1}$

$E(a_n) - 2a_n = 2^{n+1}$

$(E-2)a_n = 2^{n+1}$

$\phi(E) = E - 2$

$\phi(t) = t - 2$.

$t - 2 = 0$

$\boxed{t = 2}$

$a_n = c_1 2^n$.

Particular solution $= \dfrac{2^{n+1}}{\cancel{b-2}} = \cancel{\dfrac{2(2^n)}{2-2}}$

When $\phi(b) = 0 \Rightarrow \boxed{\dfrac{b^n}{(E-2)^k} = {}^nC_k (b)^{n-k}}$.

$\therefore 2^{n+1} \Rightarrow 2 \cdot 2^n \Rightarrow \dfrac{2(2^n)}{(E-2)^1}$

$\Rightarrow 2\left[{}^nC_1 (2)^{n-1}\right] = 2 \cdot \left[n \cdot 2^{n-1}\right]$

~~$a_n = 2k$~~

$\Rightarrow n \cdot 2^n$

Final solution = $c_1 2^n + n 2^n$

$$a_n = c_1 2^n + n 2^n$$

put $n = 0$

$$a_0 = c_1 2^0 + 0$$

$$\boxed{c_1 = 1}$$

$$\therefore a_n = 2^n + n 2^n$$

$$\boxed{a_n = 2^n(n+1)}$$

(8) Recurrence Relations Reduces to Linear.

(Q) $x_n^2 - 2x_{n-1}^2 = 1 \cdot\ x_0 = 2$

Sol: $x_n^2 - 2x_{n-1}^2 = 1$

put $x_n^2 = a_n$

$$a_n - 2a_{n-1} = 1$$

put $n = n+1$

$$a_{n+1} - 2a_n = 1$$

$$E(a_n) - 2(a_n) = 1$$

$$(E-2)a_n = 1$$

$$\phi(E) = E-2$$

$$\phi(t) = t-2 = 0$$

$\boxed{t=2}$. $\Rightarrow a_n = c_1 2^n$

Particular solution = $\dfrac{F(n)}{\phi(E)} = \dfrac{(1)^n}{1-2} = -1$

Final solution = $c_1 2^n - 1$

$$x_n^2 = a_n = c_1 2^n - 1$$

$n=0 \Rightarrow (2)^2 = c_1 2^0 - 1$

$2^2 = c_1 - 1$

$\boxed{c_1 = 5}$.

$\therefore x_n^2 = 5 2^n - 1$

$\boxed{x_n = \sqrt{5(2^n)-1}}$

(9) $\sqrt{a_n} - \sqrt{a_{n-1}} - 2\sqrt{a_{n-2}} = 0 \quad a_0 = a_1 = 1$

Sol: $a_n = x_n^2$

$x_n - x_{n-1} - 2x_{n-2} = 0$

put $n = n-2$

$x_{n+2} - x_{n+1} - 2x_n = 0$

$E^2(x_n) - E(x_n) - 2x_n = 0$

$x_n[E^2 - E - 2] = 0$

$\phi(E) = E^2 - E - 2$

$\phi(t) = 0$

$t^2 - t - 2 = 0$

$t^2 - 2t + t - 2 = 0$

$t(t-2) + 1(t-2) = 0$

$\boxed{t = -1, 2}$

$x_n = c_1(-1)^n + c_2(2)^n$

put $n = 0$

$x_0 = c_1(-1)^0 + c_2(2)^0$

$1 = c_1 + c_2 \Rightarrow \boxed{c_2 = 1 - c_2}$

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put $n=1$.

$$x_1 = c_1(-1)^1 + c_2(2)^1$$

$$1 = -c_1 + 2c_2$$

$$1 = -c_1 + 2(1-c_1)$$

$$1 = -3c_1 + 2$$

$$-1 = -3c_1$$

$$\boxed{c_1 = 1/3}$$

$$c_2 = 1 - 1/3 = \frac{2}{3}$$

$$\boxed{c_2 = \frac{2}{3}}$$

$$x_n = \frac{1}{3}(-1)^n + \frac{2}{3}(2)^n$$

$$a_n = \left[\frac{1}{3}(-1)^n + \frac{2}{3}(2)^n\right]$$

(b) $T(n) = 2T(n/2) + (n-1)$ where $T(1)=0$.

$[T(n) = a_n$ $\quad n=1$

$$a_n = 2a_{n/2} + (n-1)$$

put $n = 2n$ $\quad a_{n+1} = E(a_n)$

$$a_{2n} = 2a_n + (n-1) \qquad a_{n+n} = E^n(a_n)$$

$$E^n(a_n) = 2(a_n) + (n-1)$$

put $n = 2^k$. $\Rightarrow$ $\quad 1 = 2^k$

$$T(2^k) = 2T(2^{k-1}) + (2^k - 1) \qquad \Rightarrow k=0$$

put $T(2^k) = x_k$.

$$x_k - 2x_{k-1} - 2^k + 1 = 0$$

$x_k - 2x_{k-1} = 1^k \cdot 2^{k-1}$

put $k = k+1$

$x_{k+1} - 2x_k = 1^k \cdot 2^{k+1-1}$

$E(x_k) - 2x_k = 1^k \cdot 2^{k+1-1}$

$(E-2)x_k = 1^k \cdot 2^{k+1-1}$

$\phi(E) = E-2$

$\phi(t) = 0$

$\boxed{t=2}$.

$n = k$

$k = 1$

$x_k = c_1 2^k$.

Particular solution $= \dfrac{1^k \cdot 2^k}{E-2} = \dfrac{(1)^k}{1-2} \cdot \dfrac{2^k}{2-2}$

$= \dfrac{2^{k+1}-1}{E-2} = \dfrac{2^{k+1}}{E-2} - \dfrac{(1)^n}{(1)-2}$

$= \dfrac{2^{k+1}}{(E-2)} + 1$

$= 2\left(\dfrac{2^k}{(E-2)}\right) + 1 = \cancel{(k+1)c_1 (2^{k+1-1}) + 1}$

$= \cancel{(k+1)c_1 (2^k) + 1}$

$= 2\left({}^kc_1 (2^{k-1})\right) + 1 = (k+1)2^k + 1$

$= k(2^k) + 1$

Final solution $= k \cdot (2^k) + 1 + c_1 2^k$

$x_k = k(2^k) + 1 + c_1 2^k$

put $k = 0 \Rightarrow x_0 = 0 + 1 + c_1(2^0)$

$x_0 = 1 + c_1$

$0 = 1 + c_1 \Rightarrow \boxed{c_1 = -1}$

$x_k = k(2^k) + 1 - 1 \, 2^k$

$T(2^k) = x_k = 2^k(k-1) + 1$

$2^k = n$

Appy log on both sides

$k \log 2 = \log n$

$\boxed{k = \log_2 n}$

$\boxed{T(n) = n(\log_2 n - 1) + 1}$

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F(n)	Charasteristic eq^n	Particular sol^n
n^k (where $k=1,2,\ldots$)	$\phi(1) \neq 0$ i.e characteristic roots is not 1	$(A_0 n^k + A_1 n^{k-1} + \ldots + A_k)$
n^k (where $k=1,2,\ldots$)	$\phi(1) = 0$ '1' is characteristic root with multiplicity 'm'	$(A_0 n^k + A_1 n^{k-1} + \ldots + A_k) n^m$
$b^n n^k$ (where $k=1,2,\ldots$)	$\phi(b) \neq 0$ b is not a charac-teristec root.	$b^n (A_0 n^k + A_1 n^{k-1} + \ldots + A_k)$
$b^n n^k$ (where $k=1,2,3,\ldots$)	$\phi(b) = 0$ b is a characteristic root with multiplicity 'm'	$b^n (A_0 n^k + A_1 n^{k-1} + \ldots + A_k) n^m$

(6) $a_n - 3a_{n-1} = n+2$ $\quad a_0 = 2$.

Sol: $a_n - 3a_{n-1} = n+2$ —— (1)

put $n = n+1$

$a_{n+1} - 3a_n = n+1+2$.

$E(a_n) - 3(a_n) = n+3$

$(E-3)\,a_n = n+3$.

$\phi(E) = E-3$

$\phi(t) = 0$ $\quad F(n) = n+1$

$t-3=0$ $\quad k=1$

$\boxed{t=3}$.

complementary ~~characteristic~~ function $= c_1 3^n$.

Particular solution $\Rightarrow \phi(1) \neq 0$.

$= A_0 n + A_1 (n)^0$

$PS = A_0 n + A_1 = a_n$

Substitute in eq-(1)

$(A_0 n + A_1) - 3(A_0(n-1) + A_1) = n+2$.

$A_0 n + A_1 - 3nA_0 + 3A_0 - 3A_1 = n+2$.

$(-2A_0)n + (-2A_1 + 3A_0) = n+2$.

$-2A_0 = 1$ $\quad -2A_1 + 3A_0 = 2$.

$\boxed{A_0 = -1/2}$ $\quad -2A_1 - \frac{3}{2} = 2$.

$-2A_1 = \frac{7}{2}$.

$\boxed{A_1 = \frac{-7}{4}}$

Particular solution $= \frac{-1}{2}n - \frac{7}{4}$

Final solution = Particular sol^n + complementary function

$a_n = \frac{-n}{2} - \frac{7}{4} + c_1 3^n$

put $n=0$.

$$a_0 = 0 - \frac{7}{4} + c_1 3^0$$

$$2 = -\frac{7}{4} + c_1$$

$$2 + \frac{7}{4} = c_1 \Rightarrow \boxed{c_1 = \frac{15}{4}}$$

Final solution

$$\boxed{a_n = \frac{15}{4} 3^n - \frac{n}{2} - \frac{7}{4}}.$$

(Q) $a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - 2$.

Sol $a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - 2$.

$$E^2(a_n) - 10E(a_n) + 21a_n = 3n^2 - 2.$$

$$a_n(E^2 - 10E + 21) = 3n^2 + 2$$

$$\phi(E) = E^2 - 10E + 21$$

$$\phi(t) = 0$$

$$t^2 - 10t + 21 = 0$$

$$t^2 - 3t - 7t + 21 = 0$$

$$t(t-3) - 7(t-3) = 0$$

$$\boxed{t = 7, 3}. \longrightarrow a_n = c_1 7^n + c_2 3^n$$

$$\phi(t) \neq 0 : \quad F(n) = 3n^2 - 2. \quad k = 2.$$

Particular soln $= A_0 n^k + A_1 n^{k-1} + \cdots$

$$a_n = A_0(n)^2 + A_1(n) + A_2(1) \quad -②$$

Substitute eq-② in eq-①

$$[A_0(n+2)^2 + A_1(n+2) + A_2] - 10[A_0(n+1)^2 + A_1(n+1) + A_2]$$
$$+ 21[A_0 n^2 + A_1 n + A_2] = 3n^2 - 2$$

Put $n=0$.

$$a_0 = 0 - \frac{7}{4} + c_1 3^0$$

$$2 = -\frac{7}{4} + c_1$$

$$2 + \frac{7}{4} = c_1 \Rightarrow \boxed{c_1 = \frac{15}{4}}$$

Final solution

$$\boxed{a_n = \frac{15}{4} 3^n - \frac{n}{2} - \frac{7}{4}}$$

Q $a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - 2$.

Sol $a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - 2$.

$$E^2(a_n) - 10E(a_n) + 21a_n = 3n^2 - 2$$

$$a_n(E^2 - 10E + 21) = 3n^2 + 2$$

$$\phi(E) = E^2 - 10E + 21$$

$$\phi(t) = 0$$

$$t^2 - 10t + 21 = 0$$

$$t^2 - 3t - 7t + 21 = 0$$

$$t(t-3) - 7(t-3) = 0$$

$$\boxed{t = 7, 3} \longrightarrow a_n = c_1 7^n + c_2 3^n$$

$\phi(t) \neq 0$. $F(n) = 3n^2 - 2$.

$k = 2$.

Particular $\text{sol}^n = A_0 n^k + A_1 n^{k-1} + \cdots$

$$a_n = A_0(n)^2 + A_1(n) + A_2(1) \quad \text{--- (2)}$$

Substitute eq-(2) in eq-(1).

$$[A_0(n+2)^2 + A_1(n+2) + A_2] - 10[A_0(n+1)^2 + A_1(n+1) + A_2]$$

$$+ 21[A_0 n^2 + A_1 n + A_2] = 3n^2 - 2$$

$A_0[n^2+4n+4-10(n^2+2n+1)+21n^2]$

$+A_1[n+2-10(n+1)+21n]+A_2-10A_2$
$+21A_2=3n^2-2$

$A_0[+12n^2-16n-6]+A_1[+12n-8]+12A_2=3n^2-2.$

$12A_0=3$ $\qquad -16A_0+12A_1=0$

$\boxed{A_0=\frac{1}{4}}$ $\qquad -16\times\frac{1}{4}+12A_1=0 \qquad -4+12A_1=0$

$-4-9A_1=0 \qquad \boxed{A_1=\frac{1}{3}}$

$\cancel{\boxed{A_1=-\frac{4}{9}}}$

$-6A_0-8A_1+12A_2=-2.$

$-6\left(\frac{1}{4}\right)-8\left(\frac{1}{3}\right)+12A_2=-2.$

$-\frac{3}{2}-\frac{8}{3}+12A_2=-2$

$\frac{-9-16}{6}+2+12A_2=0$

$\frac{-25+12}{6}+12A_2=0$

$12A_2=\frac{13}{6}$

$\boxed{A_2=\frac{13}{72}}$

$P_s=\frac{1}{4}n^2+\frac{1}{3}n+\frac{13}{72}.$

Final sol^n = $\frac{n^2}{4}+\frac{n}{3}+\frac{13}{72}+C_1 7^n+C_2 3^n$

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(9) $a_n-2a_{n-1}+a_{n-2}=3n-5 \qquad a_0=1, a_1=2.$

Sol $a_n-2a_{n-1}+a_{n-2}=3n-5$ —(1)

put $n=n+2$

$a_{n+2}-2a_{n+1}+a_n=3(n+2)-5$

$E^2(a_n) - 2E(a_n) + a_n = 3n+1$

$a_n[E^2 - 2E + 1] = 3n+1$

$\phi(E) = E^2 - 2E + 1$

$\phi(t) = 0$

$t^2 - 2t + 1 = 0$

$\boxed{t = 1, 1}$ $\quad$ $\cancel{a_n = C_1(1)^n + C_2(1)^n}$

$a_n = (C_1 + C_2 n)(1)^n$

$\phi(1) = 0$

$F(n) = 3n+1$

$P.S = (A_0 n^1 + A_1 (n)^0)\, n^2 \qquad k = 2$

$a_n = (A_0 n + A_1) n^2$

Substitute in eq-①

$n+2 = (A_0 n + A_1) n^2 - 2[(A_0(n-1) + A_1)(n-1)^2] + [(A_0(n-2) + A_1](n$

$n+2 = A_0 n^3 + A_1 n^2 - [(2nA_0 - 2A_0 + 2A_1)(n-1)^2]$
$\quad + [A_0(n-2) + A_1][n^2 - 4n + 4]$

$n+2 = A_0 n^3 + A_1 n^2 - [(2nA_0 - 2A_0 + 2A_1)(n^2 - 2n + 1)]$
$\quad + [nA_0 - 2A_0 + A_1][n^2 - 4n + 4]$

$n+2 = A_0 n^3 + A_1 n^2 - [2n^3 A_0 - 4n^2 A_0 + 2nA_0 - 2n^2 A_0 +$
$\quad 4nA_0 - 2A_0 + 2n^2 A_1 - 4nA_1 + 2A$
$\quad + [n^3 A_0 - 4n^2 A_0 + 4nA_0 - 2n^2 A_0 + 8nA_0 - 8A_0 +$
$\quad n^2 A_1 + 4nA_1 + 4A_1]$

$n+2 = \cancel{A_0 n^3} + \cancel{A_1 n^2} - 2n^3 A_0 + \cancel{4n^2 A_0} - 2nA_0 + \cancel{2n^2 A_0} - 4nA$
$\quad + 2A_0 - \cancel{2n^2 A_1} + 4nA_1 - 2A_1 + \cancel{n^3 A_0} - \cancel{4n^2 A_0} + 4nA_0$
$\quad - \cancel{2n^2 A_0} + 8nA_0 - 8A_0 + \cancel{n^2 A_1} - 4nA_1 + 4A_1$

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$$\cancel{-4nA_0} - 2nA_0 + 2A_0 + 4nA_1 - 2A_1 + 4nA_0 + 8nA_0$$
$$+ 4A_1 - 4nA_1 - 8A_0 = n+2$$

$\cancel{10A_0}$ $\cancel{16nA_0}$

$$n[-4A_0 - 2A_0 + 4A_1 + 4A_0 + 8A_0 - 4A_1] +$$
$$2A_0 - 2A_1 + 4A_1 - 8A_0 = n+2$$

$$n[-6A_0 + 8A_0] + [2A_1 - 6A_0] = n+2$$

$$n[2A_0] + [2A_1 - 6A_0] = n+2$$

$2A_0 = 1$ $\qquad$ $2A_1 - 3 = 2$

$\boxed{A_0 = 1/2}$ $\qquad$ $\cancel{\boxed{A_1 = \frac{5}{2}}}$ $\qquad$ $\boxed{A_1 = -1}$

$$a_n = \left(\frac{1}{2}n - 1\right)n^2$$

Final solution $= \left(\frac{1}{2}n^3 - n^2\right) + (C_1 + C_2 n)(1)^n$

$$a_n = \frac{1}{2}n^3 - n^2 + (C_1 + C_2 n)(1)^n$$

$n = 0$

$$a_0 = 0 - 0 + C_1 + \cancel{C_2}$$

$$\boxed{C_1 + \cancel{C_2} = 1}$$

$n = 1$

$$a_1 = \frac{1}{2} - 1 + (C_1 + C_2(1))$$

$$2 = -\frac{1}{2} + C_1 + C_2(1)$$

$$\frac{5}{2} = C_2 + 1$$

$$\boxed{C_2 = \frac{3}{2}}$$

$$a_n = \frac{1}{2}n^3 - n^2 + \frac{3}{2}n + 1$$

$$a_n = \frac{1}{2}(n^3 - 2n^2 + 3n + 2)$$

⑨ $a_n - 2a_{n-1} = 3^n \cdot n$

Sol $a_n - 2a_{n-1} = 3^n \cdot n$, —— ①

put $n = n+1$

$$a_{n+1} - 2a_n = 3^{(n+1)}(n+1)$$

$$E(a_n) - 2a_n = 3^{n+1}(n+1)$$

$$(E-2)\,a_n = 3^{n+1}(n+1)$$

$$\phi(E) = E - 2$$

$$\phi(t) = 0$$

$$\boxed{t = 2}$$

$$a_n = C_1 2^n$$

Particular solution $= 3^{n+1}(A_0(n+1)^1 + A_1(n+1)^0)$

$$a_n = 3^{n+1}(A_0(n+1) + A_1)$$

substitute in eq - ①

$$3^{n+1}[A_0(n+1) + A_1] - 2[(A_0(n) + A_1)3^n] = 3^{n+1}(n+1)$$

$$3^n[3A_0 n + 3A_0 + 3A_1 - 2nA_0 - 2A_1] = 3^n[3n+3]$$

$$3^n[n(3A_0 - 2A_0) + (3A_0 + A_1)] = 3^n[3n+3]$$

$$3^n[nA_0 + (3(A_0) + A_1)] = 3^n[3n+3]$$

$$\boxed{A_0 = 3}$$

$$3A_0 + A_1 = 3$$

$$9 + A_1 = 3$$

$$\boxed{A_1 = -6}$$

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$$a_n = 3^{n+1}\left[3(n+1)+\overset{-6}{}\right]$$

$$a_n = 3^{n+1}\left[3n-3\right]$$

$$a_n = 3^{n+2}\left[n-1\right]$$

~~Final solutl~~

Final solution $= 3^{n+2}\left[n-1\right] + c_1 2^n$

(8) $a_n - 3a_{n-1} = 3^n(n+2)$

Sol $a_n - 3a_{n-1} = 3^n(n+2)$

put $n = n+1$

$$a_{n+1} - 3a_n = 3^{n+1}(n+3)$$

$$E(a_n) - 3a_n = 3^{n+1}(n+3)$$

$3^n(A_0 n + A_1)\cdot n$

$$(E-3)a_n = 3^{n+1}(n+3)$$

$$\phi(E) = E-3$$

$$\phi(t) = 0$$

$$\boxed{E=3}$$

$$a_n = c_1 3^n$$

Particular solution $= \left[3^{n+1}\left(A_0(n+1)^1 + A_1(n+1)^0\right)\right.$ $(n+1)^1$

$$+ \cancel{6}\left[\cancel{\frac{3^n}{E}}\right] 6\left[n c_1 (3)^{n-1}\right]$$

$$= 3^{n+1}\left[A_0(n+1) + A_1\right](n+1) + 6\left[n(3)^{n-1}\right]$$

$$a_n = 3^{n+1}\left[A_0(n+1)+A_1\right](n+1) + 2\left[n(3)^n\right] \times$$

$$a_n: P.S. = 3^n\left[A_0 n + A_1\right] n$$

$$\left[3^n(A_0 n + A_1)n\right] - 3\left[3^{n-1}(A_0(n-1)+A_1)(n-1)\right]$$

$$= 3^n\left[n+2\right]$$

$$n \cdot 3^n [A_0 n + A_1] - 3^n (n-1) [A_0 (n-1) + A_1]$$
$$= 3^n [n + 2]$$

$$3^n [A_0 n^2 + A_1 n - (n^2 - 2n + 1) A_0 - (n-1) A_1]$$
$$= 3^n [n+2]$$

$$3^n [\cancel{n^2 A_0} + A_1 n - \cancel{n^2 A_0} + 2n A_0 - A_0 - n A_1 + A_1]$$
$$= 3^n [n+2]$$

$2 n A_0 = 1$

$\boxed{A_0 = \frac{1}{2}}$

$A_1 - A_0 = 2$

$A_1 - \frac{1}{2} = 2$

$\boxed{A_1 = \frac{5}{2}}$

Final solution $= 3^n \left[\frac{1}{2}(n) + \frac{5}{2}\right] n + c_1 3^n$

$$= 3^n \left[\frac{n^2 + 5n}{2} + c_1\right]$$

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Generating Functions

Let $\{a_0, a_1, a_2, \dots a_n\}$ be a sequence of real numbers

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots$

is called generating function.

It contains infinitely many terms.

i.e $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$\rightarrow \sum_{n=0}^{\infty} x^n = (1-x)^{-1}$

(3) solve the recurrence relation by using the generation function

$a_{n+1} - a_n = 3^n \quad n \geq 0, \; \cancel{a_0 =} \; a_0 = 1.$

$a_{n+1} - a_n = 3^n.$

Sol: $a_{n+1} - a_n = 3^n$ $\qquad a_{n+1} = c a_n + \phi(n)$

$$f(x) = \frac{a_0 + x \cdot g(x)}{1 - cx}$$

$$g(x) = \sum_{n=0}^{\infty} f(n) x^n$$

$$= \sum_{n=0}^{\infty} 3^n x^n$$

$$= \sum_{n=0}^{\infty} (3x)^n$$

$$g(x) = (1-3x)^{-1} \qquad \left(\because \sum_{n=0}^{\infty} x^n = (1-x)^{-1} \right).$$

$$f(x) = \frac{1 + x(1-3x)^{-1}}{1 - x} \quad \text{i.e. Cofficient of } a_n$$

$$= \frac{1 - 3x + x}{(1-x)(1-3x)}$$

$$f(x) = \frac{1-2x}{(1-x)(1-3x)}$$

$$f(x) = \frac{A}{1-x} + \frac{B}{1-3x}$$

Solving we get $A = \frac{1}{2}, \; B = \frac{1}{2}.$

$$f(x) = \frac{1}{2(1-x)} + \frac{1}{2(1-3x)}$$

$$f(x) = \frac{1}{2}\left[\frac{1}{(1-x)} + \frac{1}{(1-3x)}\right]$$

$$f(x) = \frac{1}{2}\left[(1-x)^{-1} + (1-3x)^{-1}\right]$$

$$f(x) = \frac{1}{2}\left[\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (3x)^n\right]$$

$$f(x) = \frac{1}{2}\left[\sum_{n=0}^{n} (1+3^n) x^n\right]$$

$\therefore$ Comparing with $\sum_{n=0}^{\infty} a_n x^n$

$$\boxed{a_n = \frac{1}{2}(1+3^n)}$$

Case(ii): Method of generating functions for 2nd order recurrence relations

$$a_{n+2} + A a_{n+1} + B a_n = F(n)$$

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x + x^2 g(x)}{1 + Ax + Bx^2}$$

$$g(x) = \sum_{n=0}^{\infty} F(x) \cdot x^n$$

(Q) Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$ $n \geq 0$ & $a_0 = 1, a_1 = 2$ by the method of generating function.

Sol

$$g(x) = \sum_{n=0}^{\infty} f(x) x^n$$

$$= \sum_{n=0}^{\infty} 2^n x^n$$

$$g(x) = \sum_{n=0}^{\infty} (2x)^n$$

$$g(x) = (1-2x)^{-1}$$

$$f(x) = \frac{1 + (2 + 1(-2))x + x^2(1-2x)^{-1}}{1 + (-2)x + 1x^2}$$

$$= \frac{1 + (2-2)x + x^2(1-2x)^{-1}}{x^2 - 2x + 1}$$

$$= \frac{1 + \frac{x^2}{1-2x}}{(x-1)^2} = \frac{(x-1)^2}{(x-1)^2(1-2x)}$$

$$f(x) = \frac{1}{(1-2x)}$$

$$f(x) = (1-2x)^{-1}$$

$$f(x) = \sum_{n=0}^{\infty} (2x)^n$$

$$\boxed{a_n = 2^n}.$$

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