$$\begin{bmatrix} 0.8 & -0.4 & 0 \\ 0.4 & 0.8 & -0.4 \\ 0 & -0.4 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix}$$

(1) Jacobi vs Gauss Siedel

(a) Jacobi

$$x_i^k = b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}\, x_j^{(k-1)}$$

$$x_1^{(1)} = 0 \qquad x_2^{(1)} = 0 \qquad x_3^{(1)} = 0$$

$$x_1^{(2)} = \frac{b - \sum_{j=2}^{3} a_{ij}\, x_j^{(1)}}{a_{11}}$$

$$x_1^{(2)} = \frac{41 - 0}{0.8} = 51.25$$

$$x_2^{(2)} = \frac{b_2 - \sum_{j=1,3} a_{ij}\, b_j}{0.8}$$

$$x_2^{(2)} = \frac{25 - 0}{0.8} = 31.25$$

$$x_3^{(2)} = \frac{105-0}{0.8} = 131.25$$

$$\boxed{\begin{aligned} x_1 &= 51.25 \\ x_2 &= 31.25 \\ x_3 &= 131.25 \end{aligned}}$$

(b) Gauss Siedel

$$x_i^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}$$

$$x_1^{(2)} = \frac{b_1 - \sum_{j=2,3} a_{ij} x_j^{(1)}}{a_{11}}$$

$$x_1^{(2)} = \frac{41 - 0}{0.8} = 51.25$$

$$x_2^{(2)} = \frac{b_2 - a_{21} x_1^{(2)} - a_{13} x_3^{(1)}}{a_{22}}$$

$$x_2^{(2)} = \frac{25 - (-0.4)(51.25)}{0.8} - 0 = 56.875$$

$$x_3^{(2)} = \frac{105 - a_{31} x_1^{(2)} - a_{32} x_2^{(2)}}{a_{33}}$$

$$x_3^{(2)} = \frac{105 - (-0.4)(56.875)}{0.8} = 159.68$$

$$\boxed{\begin{aligned} x_1 &= 51.25 \\ x_2 &= 56.875 \\ x_3 &= 159.68 \end{aligned}}$$

(b) Jacobi

$$\underbrace{\begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}}_{D} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 41 \\ 125 \\ 105 \end{bmatrix} - \begin{bmatrix} 0 & -0.4 & 0 \\ -0.4 & 0 & -0.4 \\ 0 & -0.4 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \frac{1}{0.8} \begin{bmatrix} 41 \\ 125 \\ 105 \end{bmatrix} = \begin{bmatrix} 51.25 \\ 31.25 \\ 131.25 \end{bmatrix} \quad \text{--- } (1)$$

$$\begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix} - \begin{bmatrix} 0 & -0.4 & 0 \\ -0.4 & 0 & 0.4 \\ 0 & -0.4 & 0 \end{bmatrix} \begin{bmatrix} 51.25 \\ 31.25 \\ 131.25 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix} - \begin{bmatrix} -12.5 \\ -73 \\ -12.5 \end{bmatrix} = \begin{bmatrix} 53.5 \\ 98.0 \\ 117.5 \end{bmatrix} \quad \text{--- } (2)$$

Gauss Siedel

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & a_{n-1\,n} \\ 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0 & 0 \\ -0.4 & 0.8 & 0 \\ 0 & -0.4 & 0.8 \end{bmatrix} X^{(2)} = \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix} - \begin{bmatrix} 0 & -0.4 & 0 \\ 0 & 0 & -0.4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = (L+U)^{-1} B = \begin{bmatrix} 51.25 \\ 56.875 \\ 159.68 \end{bmatrix}$$

$$(L+U)(X^{(3)}) = B - UX^{(2)}$$

$$\begin{bmatrix} 0.8 & 0 & 0 \\ -0.4 & 0.8 & 0 \\ 0 & -0.4 & 0.8 \end{bmatrix} X^{(3)} = \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix} - \begin{bmatrix} 0 & -0.4 & 0 \\ 0 & 0 & -0.4 \\ 0 & 0 & 0 \end{bmatrix} X^{(2)}$$

$$\begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = (L+U)^{-1} \begin{bmatrix} 63.75 \\ 88.75 \\ 105 \end{bmatrix} = \begin{bmatrix} 79.6875 \\ 150.9375 \\ 206.7188 \end{bmatrix}$$

```
A = [ 0.8 ,-0.4, 0;
     -0.4 ,0.8 ,-0.4;
      0, -0.4, 0.8 ];
operations = 0;
B = [41; 25; 105];

n = length(A);

Xi = zeros(n,1);
Xf = zeros(n,1);

error = 10^(-6);

while 1>0
    for i = 1:n
        sum = 0;
        for j = 1:i-1
            sum = sum + (A(i,j)*Xf(j));
            operations = operations + 2;
        end
        for j = i+1:n
            sum = sum + (A(i,j)*Xi(j));
            operations = operations + 2;
        end

        Xi(i) = Xf(i);
        Xf(i) = (B(i) - sum)/A(i,i);
        operations = operations + 2;
    end

    isConverging = 1; % like boolean value for checking convergence

    % checking convergence for every element
    for k = 1:n
        if abs((Xf(k) - Xi(k))/Xf(k)) > error
            isConverging = 0;
        end
    end

    if(isConverging == 1)
        break;
    end
end

linsolve(A, B)
Xf
```

```
A = [ 0.8 ,-0.4, 0;
     -0.4 ,0.8 ,-0.4;
      0, -0.4, 0.8 ];
operations = 0;
B = [41; 25; 105];

n = length(A);

Xi = zeros(n,1);
Xf = zeros(n,1);

error = 10^(-6);

while 1>0
    for i = 1:n

        sum = 0;
        for j = 1:n
            if(i~=j)
                sum = sum + (A(i,j)*Xi(j));
                operations = operations + 2;
            end
        end

        Xi(i) = Xf(i);
        Xf(i) = (B(i) - sum)/A(i,i);
        operations = operations + 2;
    end

    isConverging = 1; % like boolean value for checking convergence

    % checking convergence for every element
    for k = 1:n
        if abs((Xf(k) - Xi(k))/Xf(k)) > error
            isConverging = 0;
        end
    end

    if(isConverging == 1)
        break;
    end
end

linsolve(A, B)
Xf
```

(a) Gauss Siedel

$$x_i^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}$$

$$x_1^{(2)} = \frac{b_1 - \sum_{j=2,3} a_{ij} x_j^{(1)}}{a_{11}}$$

$$x_1^{(2)} = \frac{41 - 0}{0.8} = 51.25$$

$$x_2^{(2)} = \frac{b_2 - a_{21} x_1^{(2)} - a_{13} x_3^{(1)}}{a_{22}}$$

$$x_2^{(2)} = \frac{25 - (-0.4)(51.25)}{0.8} - 0 = 56.875$$

$$x_3^{(2)} = \frac{105 - a_{31} x_1^{(2)} - a_{32} x_2^{(2)}}{a_{33}}$$

$$x_3^{(2)} = \frac{105 - (-0.4)(56.875)}{0.8} = 159.68$$

$$\boxed{\begin{aligned} x_1 &= 51.25 \\ x_2 &= 56.875 \\ x_3 &= 159.68 \end{aligned}}$$

SOR $(\omega = 1.2)$

$$x_i^{(k)} = (1-\omega) x_i^{(k-1)} + \omega x_{i,GS}^{k}$$

$$x_1^{(2)} = (1-\omega) x_1^{(1)} + \omega x_{1,GS}^{(2)}$$

$$x_1^{(2)} = (1-1.2)(0) + (1.2) 51.25$$

$$x_1 = 61.5$$

$$x_2^{(2)} = (1-\omega) x_2^{(1)} + \omega \left(\frac{b_2 - a_{21} x_1^{(2)} - a_{23} x_3^{(1)}}{a_{22}} \right)$$

$$x_2^{(2)} = 1.2 \left(\frac{25 - (-0.4)(61.5) - (-0.4)0}{0.8} \right)$$

$$x_2^{(2)} = 74.4$$

$$x_3^{(2)} = (1-\omega) x_2^{(1)} + \omega \left(\frac{b_3 - a_{31} x_1 - a_{32} x_2}{a_{33}} \right)$$

$$x_3^{(2)} = (1-\omega) 0 + 1.2 \left(\frac{105 - (0) - (-0.4)(74.4)}{0.8} \right)$$

$$x_3^{(2)} = 202.14$$

$$\boxed{\begin{aligned} x_1^{(2)} &= 61.5 \\ x_2^{(2)} &= 74.4 \\ x_3^{(2)} &= 202.14 \end{aligned}}$$

(b) Gauss Siedel

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & a_{n-1\,n} \\ 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0 & 0 \\ -0.4 & 0.8 & 0 \\ 0 & -0.4 & 0.8 \end{bmatrix} X^{(2)} = \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix} - \begin{bmatrix} 0 & -0.4 & 0 \\ 0 & 0 & -0.4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = (L+U)^{-1} B = \begin{bmatrix} 51.25 \\ 56.875 \\ 159.68 \end{bmatrix}$$

$$(L+U)(X^{(3)}) = B - UX^{(2)}$$

$$\begin{bmatrix} 0.8 & 0 & 0 \\ -0.4 & 0.8 & 0 \\ 0 & -0.4 & 0.8 \end{bmatrix} X^{(3)} = \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix} - \begin{bmatrix} 0 & -0.4 & 0 \\ 0 & 0 & -0.4 \\ 0 & 0 & 0 \end{bmatrix} X^{(2)}$$

$$\begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix} = (L+U)^{-1} \begin{bmatrix} 63.75 \\ 88.75 \\ 105 \end{bmatrix} = \begin{bmatrix} 79.6875 \\ 150.9375 \\ 206.7188 \end{bmatrix}$$

SOR

$$(D + \omega L) X^{(k)} = ((1-\omega) D - \omega U) X^{(k-1)} + \omega b$$

$$D = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 \\ -0.4 & 0 & 0 \\ 0 & -0.4 & 0 \end{bmatrix} \quad \omega = 1.2$$

$$U = \begin{bmatrix} 0 & -0.4 & 0 \\ 0 & 0 & -0.4 \\ 0 & 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix}$$

$$D + \omega L = \begin{bmatrix} 0.8 & 0 & 0 \\ -0.48 & 0.8 & 0 \\ 0 & -0.48 & 0.8 \end{bmatrix}$$

$$(D + \omega L) X^{(2)} = ((1-1.2) D - 1.2 U) X^{(1)} + \omega b$$

$$(D + \omega L) X^{(2)} = (-0.2 D - 1.2 U) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \omega b$$

$$(D + \omega L) X^{(2)} = 1.2 \times \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix}$$

$$X^{(2)} = (D + \omega L)^{-1} \cdot 1.2 \cdot \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix}$$

$$X^{(2)} = \begin{bmatrix} 61.5 \\ 74.4 \\ 202.14 \end{bmatrix}$$

$$(D+\omega L)\,x^{(3)} = (-0.2D - 1.2U)\,x^{(2)} + \omega b$$

$$(D+\omega L)\,x^{(3)} = \begin{bmatrix} -0.16 & 0.48 & 0 \\ 0 & -0.16 & 0.48 \\ 0 & 0 & -0.16 \end{bmatrix} \begin{bmatrix} 61.5 \\ 74.4 \\ 202.14 \end{bmatrix} + 1.2 \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix}$$

$$= \begin{bmatrix} 25.872 \\ 85.1232 \\ -32.3424 \end{bmatrix} + 1.2 \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix}$$

$$x^{(3)} = (D+\omega L)^{-1} \begin{bmatrix} 25.872 \\ 85.1232 \\ -32.3424 \end{bmatrix}$$

$$x^{(3)} = \begin{bmatrix} 93.84 \\ 200.208 \\ 237.1968 \end{bmatrix}$$

```
A = [0.8 ,-0.4, 0;
     -0.4 ,0.8 ,-0.4;
      0, -0.4, 0.8 ];
B = [41;25;105];

n = length(A);

Xf = zeros(n,1);
Xg = zeros(n,1);

error = 0.000001;
w = 1.2;

while 1>0
    for i = 1:n
        sum = 0;
        for j = 1: n
            if ( i ~= j)
                sum = sum + (A(i,j)*Xf(j));
            end
        end

        Xf(i) = Xg(i);
        Xg(i) = (B(i) - sum)/A(i,i) ;
        Xf(i) = Xf(i)  + w*(Xg(i) - Xf(i));
    end
     if abs((Xg - Xf)/Xf) < error
        break;
     end
end

Xf
```