BTP-1

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0.1 Introduction

introduction

Chapter 1

Mathematical Formulation

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The geometry considered is a thin film of Silicon in the domain $-L/2 \leq X \leq L/2$ and $0 \leq Y \leq H$. Consider a certain particle, initially located at the coordinate $\boldsymbol{X}$. During deformation, this particle follows a path

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) \tag{1.1}$$

Let $\boldsymbol{u}(\boldsymbol{X}, t)$ be the displacement of a material particle located at $\boldsymbol{X}$. Then

$$\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X} \tag{1.2}$$

The total deformation gradient and Green-Lagrange strain are denoted by $\mathbf{F}$ and $\mathbf{E}$, respectively. Therefore,

$$\mathbf{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{u} + \mathbf{I} \tag{1.3}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathsf{T}} \cdot \mathbf{F} - \mathbf{I}) \tag{1.4}$$

where, $\mathbf{I}$ is the second-order isotropic tensor.

Let $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ be the orthonormal basis in the reference configuration. Denoting the corresponding components of $\boldsymbol{X}$ by X, Y and Z and that of $\boldsymbol{u}$ by u, v and w, and assuming plane strain deformation the components of F are given by:

$$[\mathbf{F}] = \begin{bmatrix} 1 + \frac{\partial u}{\partial X} & \frac{\partial u}{\partial Y} & 0 \\ \frac{\partial v}{\partial X} & 1 + \frac{\partial v}{\partial Y} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix} \tag{1.5}$$

Both Inelastic and elastic deformation gradients are considered to be finite. Hence, a multiplicative decomposition of $\mathbf{F}$ into elastic and inelastic deformation is necessary. The body is first considered to reach an intermediate stress-free state and then it undergoes elastic deformation to reach the current configuration (Lee 1969).

$$\mathbf{F} = \mathbf{F}^{\mathrm{el}} \cdot \mathbf{F}^{\mathrm{inel}} \tag{1.6}$$

$$\text{in indicial notation, } F_{ij} = F^{\mathrm{el}}_{ik} F^{inel}_{kj} \tag{1.7}$$

where $\mathbf{F}^{\mathrm{el}}$ and $\mathbf{F}^{\mathrm{inel}}$ are the deformation gradients due to elastic deformation and inelastic deformation respectively.

The inelastic deformation gradient tensor, $\mathbf{F}^{\mathrm{inel}}$, has contribution from two sources - deformation due to concentration gradient, $\mathbf{F}^{\mathrm{c}}$, and viscoplastic deformation, $\mathbf{F}^{\mathrm{p}}$.

$$\mathbf{F}^{\mathrm{inel}} = \mathbf{F}^{\mathrm{c}} \cdot \mathbf{F}^{\mathrm{p}} \tag{1.8}$$

1.1 Viscoplastic Deformation

A viscoplastic constitutive relation of the following form is considered.

$$\mathbf{D}^{\mathrm{P}} = \frac{\partial G(\sigma_{\mathrm{eff}})}{\partial \boldsymbol{\tau}} \tag{1.9}$$

Where $\mathbf{D}^{\mathrm{P}}$ is the rate dependent plastic deformation tensor, $G(\sigma_{\mathrm{eff}})$ is the flow potential, σ_{eff} is the von Mises stress and $\boldsymbol{\tau}$ is the deviatoric part of Cauchy stress tensor. citation

$$G(\sigma_{\mathrm{eff}}) = \frac{\sigma_{\mathrm{f}} \dot{d}_0}{\mathrm{m}+1} \left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1 \right)^{\mathrm{m}+1} \mathrm{H} \left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1 \right) \tag{1.10}$$

$$\Rightarrow \mathbf{D}^{\mathrm{P}} = \frac{3\boldsymbol{\tau} \dot{d}_0}{2\sigma_{\mathrm{eff}}} \left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1 \right)^{\mathrm{m}} \mathrm{H} \left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1 \right) \tag{1.11}$$

where, H is the unit step function, σ_{f} is the yield strength of Silicon, m is the stress exponent for plastic flow and $\dot{d}_0$ is the strain rate for plastic flow.

$$\mathbf{D}^{\mathrm{P}} = \mathbf{F}^{\mathrm{el}} \mathbf{F}^{\mathrm{c}} \dot{\mathbf{F}}^{\mathrm{p}} (\mathbf{F}^{\mathrm{p}})^{-1} (\mathbf{F}^{\mathrm{c}})^{-1} (\mathbf{F}^{\mathrm{el}})^{-1} \tag{1.12}$$

$$\Rightarrow \dot{\mathbf{F}}^{\mathrm{p}} = (J)^{-1} \frac{3}{2} \frac{\mathbf{M}^{\mathrm{el}}_{\mathbf{0}} \mathbf{F}^{\mathrm{p}}}{\sigma_{\mathrm{eff}}} \dot{d}_0 \left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1 \right)^{\mathrm{m}} \mathrm{H} \left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1 \right) \tag{1.13}$$

$$\text{where, } \mathbf{M}^{\mathrm{el}}_{\mathbf{0}} = J (\mathbf{F}^{\mathrm{el}})^{\mathsf{T}} \boldsymbol{\tau} (\mathbf{F}^{\mathrm{el}})^{-\mathsf{T}} \tag{1.14}$$

$$J = \det(\mathbf{F}) \tag{1.15}$$

$\mathbf{M}^{\mathrm{el}}_{\mathbf{0}}$ is the deviatoric part of Mandel stress (). The expression for Mandel stress is $\mathbf{M}^{\mathrm{el}} = J(\mathbf{F}^{\mathrm{el}})^{\mathsf{T}}\boldsymbol{\sigma}(F^{\mathrm{el}})^{-\mathsf{T}}$. $\mathbf{F}^{\mathrm{p}}$ is assumed to be of the following form:

$$[\mathbf{F}^{\mathrm{p}}] = \begin{bmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{bmatrix} \tag{1.16}$$

Since, $\det(\mathbf{F}^{\mathrm{p}}) = 1$

$$\lambda_{33} = 1/(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}) \tag{1.17}$$

1.2 Deformation due to concentration gradient

The compound formed between Lithium and Silicon is of the form $\mathrm{Li}_{\chi}\mathrm{Si}$. Let the stoichiometric concentration and maximum concentration of Silicon atoms per atom of Lithium be denoted by χ_0 and $\chi_{\max}$. Defining a non-dimensional measure of the Li-ions concentration as $\tilde{c} = (\chi - \chi_0)/\chi_{\max}$. since, χ_0 is the stoichiometric ratio it signifies the stress free state of the particle and hence, $\tilde{c}$ is a measure of the deviation of the particle from undeformed state.

$$\mathbf{F}^{\mathrm{c}} = (J^{c})^{1/3}\mathbf{I} \tag{1.18}$$

$$\text{where } J^{c} = 1 + 3\eta\chi_{\max}\tilde{c} \tag{1.19}$$

η is a material parameter giving rate of change in volume w.r.t. $\tilde{c}$. It may be noted that as $\tilde{c}$ approaches 1, $\det(\mathbf{F}^{\mathrm{c}})$ approaches 4. Therefore the body undergoes a volumetric change of about 300% due to diffusion of Li-ions, justifying the use of large deformation analysis.

1.3 Momentum Conservation

From equations 1.6 and 1.8, $\mathbf{F}^{\mathrm{el}}$ can be expressed as,

$$\mathbf{F}^{\mathrm{el}} = \mathbf{F} \cdot (\mathbf{F}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{c}})^{-1} \tag{1.20}$$

The elastic Green-Lagrange strain, $\mathbf{E}^{\mathrm{el}} = \frac{1}{2}\left[(\mathbf{F}^{\mathrm{el}})^{\mathsf{T}} \cdot \mathbf{F}^{\mathrm{el}} - \mathbf{I}\right]$

The strain energy per unit volume in the reference configuration, $W(\mathbf{F}, \tilde{c})$, is expressed as $W(\mathbf{F}, \tilde{c}) = J^{\mathrm{inel}}\bar{w}(\mathbf{F}, \tilde{c})$, where $\bar{w}(\mathbf{F}, \tilde{c})$ is the strain energy per unit volume in the intermediate configuration and $J^{\text{inel}} = \det(\mathbf{F}^{\text{inel}}) = J^c$.

$$W(\mathbf{F}, \tilde{c}) = \frac{J^c}{2} \frac{E(\tilde{c})}{1+\nu} \left(\frac{\nu}{1-2\nu} (\text{tr}\mathbf{E}^{\text{el}})^2 + \text{tr}(\mathbf{E}^{\text{el}} \cdot \mathbf{E}^{\text{el}}) \right). \tag{1.21}$$

The elastic modulus of Silicon is concentration dependent with $E(\tilde{c}) = E_{\text{si}}(1 + \eta_{\text{E}} \chi_{\text{max}} \tilde{c})$. The elastic second Piola-Kirchhoff stress is denoted by $\mathbf{S}^{\text{el}}$. Differentiating W w.r.t $\mathbf{E}^{\text{el}}$ gives,

$$\mathbf{S}^{\text{el}} = J^c[2\mu_{\text{si}}(\tilde{c})\mathbf{E}^{\text{el}} + \lambda_{\text{si}}(\tilde{c})\text{tr}(\mathbf{E}^{\text{el}})\mathbf{I}] \tag{1.22}$$

Let $\mathbf{P}$ and $\mathbf{S}$ denote the first and second Piola-Kirchhoff stress, respectively. Thus

$$\mathbf{S} = (\mathbf{F}^{\text{c}})^{-1} \cdot (\mathbf{F}^{\text{p}})^{-1} \cdot \mathbf{S}^{\text{el}} \cdot (\mathbf{F}^{\text{p}})^{-\mathsf{T}} \cdot (\mathbf{F}^{\text{c}})^{-\mathsf{T}} \tag{1.23}$$

$$\mathbf{P} = \mathbf{F} \cdot \mathbf{S} \tag{1.24}$$

The Cauchy stress tensor, $\boldsymbol{\sigma}$ is given by $\boldsymbol{\sigma} = (J)^{-1}\mathbf{P} \cdot \mathbf{F}^{\mathsf{T}}$. And, the deviatoric part of Cauchy is $\boldsymbol{\tau} = \boldsymbol{\sigma} - (1/3)\text{tr}(\boldsymbol{\sigma})\mathbf{I}$. The von Mises stress is $\sigma_{\text{eff}} = \sqrt{\frac{3}{2}(\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2 + 2\tau_{12}^2)}$.

Conservation of momentum leads to

$$\boldsymbol{\nabla}_{\boldsymbol{X}} \cdot \mathbf{P} = 0. \tag{1.25}$$

1.4 Mass Conservation

Assuming flux to be negligible in the z direction, the conservation of mass is given by

$$\frac{\partial c}{\partial t} = -\boldsymbol{\nabla}_{\boldsymbol{X}} \cdot \boldsymbol{j} = -\left(\frac{\partial j_X}{\partial X} + \frac{\partial j_Y}{\partial Y} \right). \tag{1.26}$$

Where, $\boldsymbol{j}$ is the flux vector and c is a dimensional measure of Li-ions concentration, defined as $c = \chi_{\text{max}}/V_{\text{m}}^{\text{B}}\,\tilde{c}$.

1.5 Non-Dimensionalization

$$\tilde{j}_X, \tilde{j}_Y, \tilde{J}_0, \tilde{\boldsymbol{j}} = \frac{HV_{\mathrm{m}}^{\mathrm{B}}}{(\chi_{\max} D_0)}(j_X, j_y, J_0, \boldsymbol{j}) \tag{1.27}$$

$$\tilde{X}, \tilde{Y}, \tilde{u}, \tilde{v} = \frac{1}{H}(X, Y, u, v) \tag{1.28}$$

$$\tilde{t} = D_0 t / H^2 \tag{1.29}$$

$$\tilde{\mu}_{\mathrm{si}}, \tilde{\lambda}_{\mathrm{si}}, \tilde{E}_{\mathrm{si}} = \frac{1}{E_0}(\mu_{\mathrm{si}}, \lambda_{\mathrm{si}}, E_{\mathrm{si}}), \text{where } E_0 = \frac{R_g T}{V_{\mathrm{m}}^{\mathrm{B}}} \tag{1.30}$$

$$\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3 = \frac{1}{R_g T}(\mu_0, \mu_1, \mu_2, \mu_3) \tag{1.31}$$

$$\tilde{D} = \frac{D}{D_0} \tag{1.32}$$

$$\tilde{\dot{d}}_0 = \frac{\dot{d}_0 H^2}{D_0} \tag{1.33}$$

$$\tilde{\mathbf{S}}^{\mathrm{el}}, \tilde{\mathbf{S}}, \tilde{\mathbf{P}}, \tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\tau}}, \tilde{\mathbf{M}}_0^{\mathrm{el}}, \tilde{\sigma}_{\mathrm{eff}}, \tilde{\sigma}_{\mathrm{f}} = \frac{1}{E_0}(\mathbf{S}^{\mathrm{el}}, \mathbf{S}, \mathbf{P}, \boldsymbol{\sigma}, \boldsymbol{\tau}, \mathbf{M}_0^{\mathrm{el}}, \sigma_{\mathrm{eff}}, \sigma_{\mathrm{f}}) \tag{1.34}$$

1.6 Definition of the state of charge

state of charge is a measure of the degree of lithiation. It can be expressed as an average concentration over the domain as follows:

$$\mathrm{soc} = \frac{\int_{-L/2}^{L/2} \int_0^H \tilde{c} \mathrm{dydx}}{LH} \tag{1.35}$$

$$= H^2 \frac{\int_{-L/2H}^{L/2H} \int_0^1 \tilde{c}(\tilde{x}, \tilde{y}) \mathrm{d\tilde{y}d\tilde{x}}}{LH} \tag{1.36}$$

$$= H \frac{\int_{-L/2H}^{L/2H} \int_0^1 \tilde{c}(\tilde{x}, \tilde{y}) \mathrm{d\tilde{y}d\tilde{x}}}{L} \tag{1.37}$$

1.7 Non-Dimensional Equations in Component Form

$$[\mathbf{F}] = \begin{bmatrix} 1 + \frac{\partial \tilde{u}}{\partial \tilde{X}} & \frac{\partial \tilde{u}}{\partial \tilde{Y}} & 0 \\ \frac{\partial \tilde{v}}{\partial \tilde{X}} & 1 + \frac{\partial \tilde{v}}{\partial \tilde{Y}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix} \tag{1.38}$$

$$F^{el}_{11} = \frac{F_{11}\,\lambda_{22} - F_{12}\,\lambda_{21}}{{J^c}^{1/3}\,\left(\lambda_{11}\,\lambda_{22} - \lambda_{12}\,\lambda_{21}\right)} \tag{1.39}$$

$$F^{el}_{21} = \frac{F_{21}\,\lambda_{22} - F_{22}\,\lambda_{21}}{{J^c}^{1/3}\,\left(\lambda_{11}\,\lambda_{22} - \lambda_{12}\,\lambda_{21}\right)} \tag{1.40}$$

$$F^{el}_{12} = -\frac{F_{11}\,\lambda_{12} - F_{12}\,\lambda_{11}}{{J^c}^{1/3}\,\left(\lambda_{11}\,\lambda_{22} - \lambda_{12}\,\lambda_{21}\right)} \tag{1.41}$$

$$F^{el}_{22} = -\frac{F_{21}\,\lambda_{12} - F_{22}\,\lambda_{11}}{{J^c}^{1/3}\,\left(\lambda_{11}\,\lambda_{22} - \lambda_{12}\,\lambda_{21}\right)} \tag{1.42}$$

$$F^{el}_{33} = \frac{1}{{J^c}^{1/3}\,\lambda_{33}} \tag{1.43}$$

$$E^{\mathrm{el}}_{11} = \frac{{F^{\mathrm{el}}_{11}}^2}{2} + \frac{{F^{\mathrm{el}}_{21}}^2}{2} - \frac{1}{2} \tag{1.44}$$

$$E^{\mathrm{el}}_{21} = \frac{F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{12}}{2} + \frac{F^{\mathrm{el}}_{21}\,F^{\mathrm{el}}_{22}}{2} \tag{1.45}$$

$$E^{\mathrm{el}}_{12} = \frac{F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{12}}{2} + \frac{F^{\mathrm{el}}_{21}\,F^{\mathrm{el}}_{22}}{2} \tag{1.46}$$

$$E^{\mathrm{el}}_{22} = \frac{{F^{\mathrm{el}}_{12}}^2}{2} + \frac{{F^{\mathrm{el}}_{22}}^2}{2} - \frac{1}{2} \tag{1.47}$$

$$E^{\mathrm{el}}_{33} = \frac{{F^{\mathrm{el}}_{33}}^2}{2} - \frac{1}{2} \tag{1.48}$$

$$\tilde{S}^{\mathrm{el}}_{11} = J^c\,\left(2\,E^{\mathrm{el}}_{11}\,\tilde{\mu}_{\mathrm{si}} + \tilde{\lambda}_{\mathrm{si}}\,\left(E^{\mathrm{el}}_{11} + E^{\mathrm{el}}_{22} + E^{\mathrm{el}}_{33}\right)\right) \tag{1.49}$$

$$\tilde{S}^{\mathrm{el}}_{21} = 2\,E^{\mathrm{el}}_{21}\,J^c\,\tilde{\mu}_{\mathrm{si}} \tag{1.50}$$

$$\tilde{S}^{\mathrm{el}}_{12} = 2\,E^{\mathrm{el}}_{12}\,J^c\,\tilde{\mu}_{\mathrm{si}} \tag{1.51}$$

$$\tilde{S}^{\mathrm{el}}_{22} = J^c\,\left(2\,E^{\mathrm{el}}_{22}\,\tilde{\mu}_{\mathrm{si}} + \tilde{\lambda}_{\mathrm{si}}\,\left(E^{\mathrm{el}}_{11} + E^{\mathrm{el}}_{22} + E^{\mathrm{el}}_{33}\right)\right) \tag{1.52}$$

$$\tilde{S}^{\mathrm{el}}_{33} = J^c\,\left(2\,E^{\mathrm{el}}_{33}\,\tilde{\mu}_{\mathrm{si}} + \tilde{\lambda}_{\mathrm{si}}\,\left(E^{\mathrm{el}}_{11} + E^{\mathrm{el}}_{22} + E^{\mathrm{el}}_{33}\right)\right) \tag{1.53}$$

$$\tilde{S}_{11} = \frac{\tilde{S}^{\rm el}_{11}\,{\lambda_{22}}^2 + \tilde{S}^{\rm el}_{22}\,{\lambda_{12}}^2 - \tilde{S}^{\rm el}_{12}\,\lambda_{12}\,\lambda_{22} - \tilde{S}^{\rm el}_{21}\,\lambda_{12}\,\lambda_{22}}{{J^c}^{2/3}\left(\lambda_{11}\,\lambda_{22} - \lambda_{12}\,\lambda_{21}\right)^2} \tag{1.54}$$

$$\tilde{S}_{21} = \frac{\tilde{S}^{\rm el}_{12}\,\lambda_{12}\,\lambda_{21} - \tilde{S}^{\rm el}_{22}\,\lambda_{11}\,\lambda_{12} - \tilde{S}^{\rm el}_{11}\,\lambda_{21}\,\lambda_{22} + \tilde{S}^{\rm el}_{21}\,\lambda_{11}\,\lambda_{22}}{{J^c}^{2/3}\left(\lambda_{11}\,\lambda_{22} - \lambda_{12}\,\lambda_{21}\right)^2} \tag{1.55}$$

$$\tilde{S}_{12} = \frac{\tilde{S}^{\rm el}_{12}\,\lambda_{11}\,\lambda_{22} - \tilde{S}^{\rm el}_{22}\,\lambda_{11}\,\lambda_{12} - \tilde{S}^{\rm el}_{11}\,\lambda_{21}\,\lambda_{22} + \tilde{S}^{\rm el}_{21}\,\lambda_{12}\,\lambda_{21}}{{J^c}^{2/3}\left(\lambda_{11}\,\lambda_{22} - \lambda_{12}\,\lambda_{21}\right)^2} \tag{1.56}$$

$$\tilde{S}_{22} = \frac{\tilde{S}^{\rm el}_{11}\,{\lambda_{21}}^2 + \tilde{S}^{\rm el}_{22}\,{\lambda_{11}}^2 - \tilde{S}^{\rm el}_{12}\,\lambda_{11}\,\lambda_{21} - \tilde{S}^{\rm el}_{21}\,\lambda_{11}\,\lambda_{21}}{{J^c}^{2/3}\left(\lambda_{11}\,\lambda_{22} - \lambda_{12}\,\lambda_{21}\right)^2} \tag{1.57}$$

$$\tilde{S}_{33} = \frac{\tilde{S}^{\rm el}_{33}}{{J^c}^{2/3}\,{\lambda_{33}}^2} \tag{1.58}$$

$$\tilde{P}_{11} = F_{11}\,\tilde{S}_{11} + F_{12}\,\tilde{S}_{21} \tag{1.59}$$

$$\tilde{P}_{21} = F_{21}\,\tilde{S}_{11} + F_{22}\,\tilde{S}_{21} \tag{1.60}$$

$$\tilde{P}_{12} = F_{11}\,\tilde{S}_{12} + F_{12}\,\tilde{S}_{22} \tag{1.61}$$

$$\tilde{P}_{22} = F_{21}\,\tilde{S}_{12} + F_{22}\,\tilde{S}_{22} \tag{1.62}$$

$$\tilde{P}_{33} = \tilde{S}_{33} \tag{1.63}$$

$$\tilde{\sigma}_{11} = \frac{F_{11}\,\tilde{P}_{11} + F_{12}\,\tilde{P}_{12}}{J} \tag{1.64}$$

$$\tilde{\sigma}_{21} = \frac{F_{11}\,\tilde{P}_{21} + F_{12}\,\tilde{P}_{22}}{J} \tag{1.65}$$

$$\tilde{\sigma}_{12} = \frac{F_{21}\,\tilde{P}_{11} + F_{22}\,\tilde{P}_{12}}{J} \tag{1.66}$$

$$\tilde{\sigma}_{22} = \frac{F_{21}\,\tilde{P}_{21} + F_{22}\,\tilde{P}_{22}}{J} \tag{1.67}$$

$$\tilde{\sigma}_{33} = \frac{\tilde{P}_{33}}{J} \tag{1.68}$$

$$\tilde{\tau}_{11} = \frac{2\,\tilde{\sigma}_{11}}{3} - \frac{\tilde{\sigma}_{22}}{3} - \frac{\tilde{\sigma}_{33}}{3} \tag{1.69}$$
$$\tilde{\tau}_{21} = \tilde{\sigma}_{21} \tag{1.70}$$
$$\tilde{\tau}_{12} = \tilde{\sigma}_{12} \tag{1.71}$$
$$\tilde{\tau}_{22} = \frac{2\,\tilde{\sigma}_{22}}{3} - \frac{\tilde{\sigma}_{11}}{3} - \frac{\tilde{\sigma}_{33}}{3} \tag{1.72}$$
$$\tilde{\tau}_{33} = \frac{2\,\tilde{\sigma}_{33}}{3} - \frac{\tilde{\sigma}_{22}}{3} - \frac{\tilde{\sigma}_{11}}{3} \tag{1.73}$$

$$\tilde{\sigma}_{\mathrm{eff}} = \sqrt{\frac{3}{2}(\tilde{\tau}_{11}^2 + \tilde{\tau}_{22}^2 + \tilde{\tau}_{33}^2 + 2\tilde{\tau}_{12}^2)} \tag{1.74}$$

$$\tilde{M}^{\mathrm{el}}_{11} = -\frac{J\,\left(F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{12}\,\tilde{\tau}_{12} - F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{22}\,\tilde{\tau}_{11} + F^{\mathrm{el}}_{12}\,F^{\mathrm{el}}_{21}\,\tilde{\tau}_{22} - F^{\mathrm{el}}_{21}\,F^{\mathrm{el}}_{22}\,\tilde{\tau}_{21}\right)}{F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{22} - F^{\mathrm{el}}_{12}\,F^{\mathrm{el}}_{21}} \tag{1.75}$$
$$\tilde{M}^{\mathrm{el}}_{21} = -\frac{J\,\left({F^{\mathrm{el}}_{12}}^2\,\tilde{\tau}_{12} - {F^{\mathrm{el}}_{22}}^2\,\tilde{\tau}_{21} - F^{\mathrm{el}}_{12}\,F^{\mathrm{el}}_{22}\,\tilde{\tau}_{11} + F^{\mathrm{el}}_{12}\,F^{\mathrm{el}}_{22}\,\tilde{\tau}_{22}\right)}{F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{22} - F^{\mathrm{el}}_{12}\,F^{\mathrm{el}}_{21}} \tag{1.76}$$
$$\tilde{M}^{\mathrm{el}}_{12} = \frac{J\,\left({F^{\mathrm{el}}_{11}}^2\,\tilde{\tau}_{12} - {F^{\mathrm{el}}_{21}}^2\,\tilde{\tau}_{21} - F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{21}\,\tilde{\tau}_{11} + F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{21}\,\tilde{\tau}_{22}\right)}{F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{22} - F^{\mathrm{el}}_{12}\,F^{\mathrm{el}}_{21}} \tag{1.77}$$
$$\tilde{M}^{\mathrm{el}}_{22} = \frac{J\,\left(F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{12}\,\tilde{\tau}_{12} - F^{\mathrm{el}}_{12}\,F^{\mathrm{el}}_{21}\,\tilde{\tau}_{11} + F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{22}\,\tilde{\tau}_{22} - F^{\mathrm{el}}_{21}\,F^{\mathrm{el}}_{22}\,\tilde{\tau}_{21}\right)}{F^{\mathrm{el}}_{11}\,F^{\mathrm{el}}_{22} - F^{\mathrm{el}}_{12}\,F^{\mathrm{el}}_{21}} \tag{1.78}$$
$$\tilde{M}^{\mathrm{el}}_{33} = J\,\tilde{\tau}_{33} \tag{1.79}$$

Viscoplastic rates: These equations need editing if Fpdot.mat.tex is changed

$$\dot{\tilde{F}}^{\mathrm{p}}_{11} = \dot{\tilde{d}}_0 \left(\frac{\tilde{\sigma}_{\mathrm{eff}}}{\tilde{\sigma}_f} - 1\right)^m \left(\frac{3\,\tilde{M}^{\mathrm{el}}_{11}\,\lambda_{11}}{2\,J\,\tilde{\sigma}_{\mathrm{eff}}} + \frac{3\,\tilde{M}^{\mathrm{el}}_{12}\,\lambda_{21}}{2\,J\,\tilde{\sigma}_{\mathrm{eff}}}\right) \mathrm{H}\left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1\right) \tag{1.80}$$
$$\dot{\tilde{F}}^{\mathrm{p}}_{21} = \dot{\tilde{d}}_0 \left(\frac{\tilde{\sigma}_{\mathrm{eff}}}{\tilde{\sigma}_f} - 1\right)^m \left(\frac{3\,\tilde{M}^{\mathrm{el}}_{21}\,\lambda_{11}}{2\,J\,\tilde{\sigma}_{\mathrm{eff}}} + \frac{3\,\tilde{M}^{\mathrm{el}}_{22}\,\lambda_{21}}{2\,J\,\tilde{\sigma}_{\mathrm{eff}}}\right) \mathrm{H}\left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1\right) \tag{1.81}$$
$$\dot{\tilde{F}}^{\mathrm{p}}_{12} = \dot{\tilde{d}}_0 \left(\frac{\tilde{\sigma}_{\mathrm{eff}}}{\tilde{\sigma}_f} - 1\right)^m \left(\frac{3\,\tilde{M}^{\mathrm{el}}_{11}\,\lambda_{12}}{2\,J\,\tilde{\sigma}_{\mathrm{eff}}} + \frac{3\,\tilde{M}^{\mathrm{el}}_{12}\,\lambda_{22}}{2\,J\,\tilde{\sigma}_{\mathrm{eff}}}\right) \mathrm{H}\left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1\right) \tag{1.82}$$
$$\dot{\tilde{F}}^{\mathrm{p}}_{22} = \dot{\tilde{d}}_0 \left(\frac{\tilde{\sigma}_{\mathrm{eff}}}{\tilde{\sigma}_f} - 1\right)^m \left(\frac{3\,\tilde{M}^{\mathrm{el}}_{21}\,\lambda_{12}}{2\,J\,\tilde{\sigma}_{\mathrm{eff}}} + \frac{3\,\tilde{M}^{\mathrm{el}}_{22}\,\lambda_{22}}{2\,J\,\tilde{\sigma}_{\mathrm{eff}}}\right) \mathrm{H}\left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1\right) \tag{1.83}$$
$$\dot{\tilde{F}}^{\mathrm{p}}_{33} = \frac{3\,\tilde{M}^{\mathrm{el}}_{33}\,\dot{\tilde{d}}_0\,\lambda_{33}\left(\frac{\tilde{\sigma}_{\mathrm{eff}}}{\tilde{\sigma}_f} - 1\right)^m}{2\,J\,\tilde{\sigma}_{\mathrm{eff}}} \mathrm{H}\left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1\right) \tag{1.84}$$

Viscoplastic rate equations in non-dimensional form:

$$\frac{d\lambda_{11}}{dt} = \frac{1}{t_{\mathrm{ref}}}\frac{d\lambda_{11}}{d\tilde{t}} = \frac{3\dot{d}_0}{2J\tilde{\sigma}_{\mathrm{eff}}}(\tilde{M}^{\mathrm{el}}_{11}\lambda_{11} + \tilde{M}^{\mathrm{el}}_{12}\lambda_{21})\left(\frac{\tilde{\sigma}_{\mathrm{eff}}}{\tilde{\sigma}_{\mathrm{f}}} - 1\right)^{\mathrm{m}} \mathrm{H}\left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1\right) \tag{1.85}$$

$$\frac{d\lambda_{11}}{d\tilde{t}} = \frac{3\tilde{\dot{d}}_0}{2J\tilde{\sigma}_{\mathrm{eff}}}(\tilde{M}^{\mathrm{el}}_{11}\lambda_{11} + \tilde{M}^{\mathrm{el}}_{12}\lambda_{21})\left(\frac{\tilde{\sigma}_{\mathrm{eff}}}{\tilde{\sigma}_{\mathrm{f}}} - 1\right)^{\mathrm{m}} \mathrm{H}\left(\frac{\sigma_{\mathrm{eff}}}{\sigma_{\mathrm{f}}} - 1\right) \tag{1.86}$$

$$\text{where, } \tilde{\dot{d}}_0 = \dot{d}_0 t_{\mathrm{ref}} \tag{1.87}$$

Momentum Conservation equations in non-dimensional form:

$$\frac{\partial \tilde{P}_{11}}{\partial \tilde{X}} + \frac{\partial \tilde{P}_{12}}{\partial \tilde{Y}} = 0 \tag{1.88}$$

$$\text{and, } \frac{\partial \tilde{P}_{21}}{\partial \tilde{X}} + \frac{\partial \tilde{P}_{22}}{\partial \tilde{Y}} = 0. \tag{1.89}$$

Mass conservation Equation in non-dimensional form:

$$\frac{\partial c}{\partial t} = -\boldsymbol{\nabla}_{\boldsymbol{X}} \cdot \boldsymbol{j} = -(\frac{\partial j_x}{\partial X} + \frac{\partial j_y}{\partial Y}) \tag{1.90}$$

$$\frac{\chi_{\mathrm{max}}}{V^{\mathrm{B}}_{\mathrm{m}}}\frac{D_0}{H^2}\frac{\partial \tilde{c}}{\partial \tilde{t}} = -(\frac{1}{H}\frac{\partial \tilde{j}_x}{\partial \tilde{X}} + \frac{1}{H}\frac{\partial \tilde{j}_y}{\partial \tilde{Y}})\frac{\chi_{\mathrm{max}}}{V^{\mathrm{B}}_{\mathrm{m}}}\frac{D_0}{H} \tag{1.91}$$

$$\implies \frac{\partial \tilde{c}}{\partial \tilde{t}} = -(\frac{\partial \tilde{j}_x}{\partial \tilde{X}} + \frac{\partial \tilde{j}_y}{\partial \tilde{Y}}) \tag{1.92}$$

For one way coupling:

$$\boldsymbol{j} = -D_0\boldsymbol{\nabla}_{\boldsymbol{X}} c \tag{1.93}$$

$$\tilde{j}_x = j_x H/(\frac{\chi_{\mathrm{max}}}{V^{\mathrm{B}}_{\mathrm{m}}}D_0) \tag{1.94}$$

$$= -D_0\frac{\partial c}{\partial X}H/(\frac{\chi_{\mathrm{max}}}{V^{\mathrm{B}}_{\mathrm{m}}}D_0) \tag{1.95}$$

$$= -\frac{\partial \tilde{c}}{\partial \tilde{X}} \tag{1.96}$$

$$\tilde{j}_y = j_y H/(\frac{\chi_{\mathrm{max}}}{V^{\mathrm{B}}_{\mathrm{m}}}D_0) \tag{1.97}$$

$$= -\frac{\partial \tilde{c}}{\partial \tilde{Y}} \tag{1.98}$$

For two way coupling:

$$\boldsymbol{j} = -\frac{1}{R_g T}\frac{D\chi_{\max}\tilde{c}}{V_m^b}(\mathbf{F})^{-1}(\mathbf{F})^{-\mathsf{T}}\boldsymbol{\nabla}_{\boldsymbol{X}}\mu \tag{1.99}$$

$$\tilde{\boldsymbol{j}} = \frac{\boldsymbol{j}H}{\frac{\chi_{\max}}{V_{\mathrm{m}}^{\mathrm{B}}}D_0} = -\frac{1}{R_g T}\tilde{D}\tilde{c}(\mathbf{F})^{-1}(\mathbf{F})^{-\mathsf{T}}\boldsymbol{\nabla}_{\boldsymbol{X}}\mu \tag{1.100}$$

$$D = D_0\mathrm{exp}(\frac{\alpha \mathrm{S_h}}{\mathrm{E_0}}) = \mathrm{D_0}\mathrm{exp}(\alpha\tilde{\mathrm{S}}_{\mathrm{h}}) = \mathrm{D_0}\mathrm{exp}\left(\alpha\frac{\tilde{\mathrm{S}}_{11}+\tilde{\mathrm{S}}_{33}}{2}\right) \tag{1.101}$$

$$\mu = \mu_0 + \mu_s; \tilde{\mu} = \frac{\mu}{R_g T} \tag{1.102}$$

$$\mu_0 = R_g T\mathrm{log}(\gamma\tilde{c}); \tilde{\mu_0} = \mathrm{log}(\gamma\tilde{c}) \tag{1.103}$$

$$\gamma = \frac{1}{1-\tilde{c}}\mathrm{exp}(\frac{1}{R_g T}[2(A_0-2B_0)\tilde{c} - 3(A_0-B_0)(\tilde{c}^2)]) \tag{1.104}$$

$$\mu_s = \frac{V_m^b}{\chi_{\max}}\left[-\frac{1}{3}\frac{\partial J^c}{\partial\tilde{c}}\tilde{F}^{\mathrm{el}}_{im}\tilde{F}^{\mathrm{el}}_{in}C_{mnkl}\tilde{E}^{\mathrm{el}}_{kl} + \frac{1}{2}\left(J^c\frac{\partial C_{ijkl}}{\partial\tilde{c}} + \frac{\partial J^c}{\partial\tilde{c}}C_{ijkl}\right)\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl}\right] \tag{1.105}$$

$$\tilde{\mu}_s = \frac{1}{R_g T}\mu_s \tag{1.106}$$

$$= \frac{V_m^b}{R_g T\chi_{\max}}\left[-\frac{1}{3}\frac{\partial J^c}{\partial\tilde{c}}\tilde{F}^{\mathrm{el}}_{im}\tilde{F}^{\mathrm{el}}_{in}C_{mnkl}\tilde{E}^{\mathrm{el}}_{kl} + \frac{1}{2}\left(J^c\frac{\partial C_{ijkl}}{\partial\tilde{c}} + \frac{\partial J^c}{\partial\tilde{c}}C_{ijkl}\right)\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl}\right] \tag{1.107}$$

$$= \frac{1}{\chi_{\max}}\left[-\frac{1}{3}\frac{\partial J^c}{\partial\tilde{c}}\tilde{F}^{\mathrm{el}}_{im}\tilde{F}^{\mathrm{el}}_{in}\tilde{C}_{mnkl}\tilde{E}^{\mathrm{el}}_{kl} + \frac{1}{2}\left(J^c\frac{\partial \tilde{C}_{ijkl}}{\partial\tilde{c}} + \frac{\partial J^c}{\partial\tilde{c}}\tilde{C}_{ijkl}\right)\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl}\right] \tag{1.108}$$

$$\tilde{C}_{ijkl}\tilde{E}^{\mathrm{el}}_{kl} = \tilde{P}^{\mathrm{el}}_{ij} = \tilde{S}^{\mathrm{el}}_{ij}/J^c \Longrightarrow \tilde{C}_{ijkl} = \tilde{\lambda}_{si}(\tilde{c})\delta_{ij}\delta_{kl} + \tilde{\mu}_{si}(\tilde{c})(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \tag{1.109}$$

$$\tilde{\mu}_s = \frac{1}{\chi_{\max}}\left[-\frac{1}{3}\frac{\partial J^c}{\partial\tilde{c}}\tilde{F}^{\mathrm{el}}_{im}\tilde{F}^{\mathrm{el}}_{in}\tilde{P}^{\mathrm{el}}_{mn} + \frac{1}{2}J^c\frac{\partial\tilde{C}_{ijkl}}{\partial\tilde{c}}\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl} + \frac{1}{2}\frac{\partial J^c}{\partial\tilde{c}}\tilde{C}_{ijkl}\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl}\right] \tag{1.110}$$

$$= \frac{1}{\chi_{\max}}\left[-\frac{1}{3}\frac{\partial J^c}{\partial\tilde{c}}\tilde{F}^{\mathrm{el}}_{im}\tilde{F}^{\mathrm{el}}_{in}\tilde{P}^{\mathrm{el}}_{mn} + \frac{1}{2}J^c\frac{\partial\tilde{C}_{ijkl}}{\partial\tilde{c}}\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl} + \frac{1}{2}\frac{\partial J^c}{\partial\tilde{c}}\tilde{E}^{\mathrm{el}}_{ij}\tilde{P}^{\mathrm{el}}_{kl}\right] \tag{1.111}$$

$$= \frac{1}{\chi_{\max}}\left[\frac{\partial J^c}{\partial\tilde{c}}\tilde{P}^{\mathrm{el}}_{mn}\left(-\frac{1}{3}\tilde{F}^{\mathrm{el}}_{im}\tilde{F}^{\mathrm{el}}_{in} + \frac{1}{2}\tilde{E}^{\mathrm{el}}_{mn}\right) + \frac{1}{2}J^c\frac{\partial\tilde{C}_{ijkl}}{\partial\tilde{c}}\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl}\right] \tag{1.112}$$

$$= \frac{1}{\chi_{\max}}\left[\frac{\partial J^c}{\partial\tilde{c}}\tilde{P}^{\mathrm{el}}_{mn}\left(-\frac{1}{3}(2\tilde{E}^{\mathrm{el}}_{mn} + \delta_{mn}) + \frac{1}{2}\tilde{E}^{\mathrm{el}}_{mn}\right) + \frac{1}{2}J^c\frac{\partial\tilde{C}_{ijkl}}{\partial\tilde{c}}\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl}\right] \tag{1.113}$$

$$= \frac{1}{\chi_{\max}}\left[-\frac{1}{6}\frac{\partial J^c}{\partial\tilde{c}}\tilde{P}^{\mathrm{el}}_{mn}\tilde{E}^{\mathrm{el}}_{mn} - \frac{1}{3}\frac{\partial J^c}{\partial\tilde{c}}\tilde{P}^{\mathrm{el}}_{mn}\delta_{mn} + \frac{1}{2}J^c\frac{\partial\tilde{C}_{ijkl}}{\partial\tilde{c}}\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl}\right] \tag{1.114}$$

$$= \frac{1}{\chi_{\max}}\left[-\frac{1}{6}\frac{\partial J^c}{\partial\tilde{c}}\tilde{P}^{\mathrm{el}}_{mn}\tilde{E}^{\mathrm{el}}_{mn} - \frac{1}{3}\frac{\partial J^c}{\partial\tilde{c}}\tilde{P}^{\mathrm{el}}_{mm} + \frac{1}{2}J^c\frac{\partial\tilde{C}_{ijkl}}{\partial\tilde{c}}\tilde{E}^{\mathrm{el}}_{ij}\tilde{E}^{\mathrm{el}}_{kl}\right] \tag{1.115}$$

$$= \frac{1}{\chi_{\max}}(\tilde{\mu}_1 + \tilde{\mu}_2 + \tilde{\mu}_3) \tag{1.116}$$

$$\tilde{\mu}_0 = \log(\gamma\tilde{c}) \tag{1.117}$$

$$\tilde{\mu}_1 = -\frac{1}{6}\frac{\partial J^c}{\partial \tilde{c}}[\tilde{P}^{\text{el}}_{11}\tilde{E}^{\text{el}}_{11} + \tilde{P}^{\text{el}}_{22}\tilde{E}^{\text{el}}_{22} + 2\tilde{P}^{\text{el}}_{12}\tilde{E}^{\text{el}}_{12} + \tilde{P}^{\text{el}}_{33}\tilde{E}^{\text{el}}_{33}] \tag{1.118}$$

$$\tilde{\mu}_2 = -\frac{1}{3}\frac{\partial J^c}{\partial \tilde{c}}[\tilde{P}^{\text{el}}_{11} + \tilde{P}^{\text{el}}_{22} + \tilde{P}^{\text{el}}_{33}] \tag{1.119}$$

$$\tilde{\mu}_3 = \frac{1}{2}J^c\frac{\partial \tilde{C}_{ijkl}}{\partial \tilde{c}}\tilde{E}^{\text{el}}_{kl}\tilde{E}^{\text{el}}_{ij} \tag{1.120}$$

$$\frac{\partial \tilde{C}_{ijkl}}{\partial \tilde{c}} = \frac{\partial \tilde{\lambda}_{si}(\tilde{c})}{\partial \tilde{c}}\delta_{ij}\delta_{kl} + \frac{\partial \tilde{\mu}_{si}(\tilde{c})}{\partial \tilde{c}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.121}$$

$$\tilde{\mu}_3 = \frac{1}{2}J^c\frac{\partial \tilde{\lambda}_{si}(\tilde{c})}{\partial \tilde{c}}\delta_{ij}\delta_{kl}\tilde{E}^{\text{el}}_{kl}\tilde{E}^{\text{el}}_{ij} + \frac{1}{2}J^c\frac{\partial \tilde{\mu}_{si}(\tilde{c})}{\partial \tilde{c}}(\delta_{ik}\delta_{jl}\tilde{E}^{\text{el}}_{kl}\tilde{E}^{\text{el}}_{ij} + \delta_{il}\delta_{jk}\tilde{E}^{\text{el}}_{kl}\tilde{E}^{\text{el}}_{ij}) \tag{1.122}$$

$$= \frac{1}{2}J^c\frac{\partial \tilde{\lambda}_{si}(\tilde{c})}{\partial \tilde{c}}\tilde{E}^{\text{el}}_{kk}\tilde{E}^{\text{el}}_{ii} + \frac{1}{2}J^c\frac{\partial \tilde{\mu}_{si}(\tilde{c})}{\partial \tilde{c}}(\tilde{E}^{\text{el}}_{ij}\tilde{E}^{\text{el}}_{ij} + \tilde{E}^{\text{el}}_{ji}\tilde{E}^{\text{el}}_{ij}) \tag{1.123}$$

$$= \frac{1}{2}J^c\frac{\partial \tilde{\lambda}_{si}(\tilde{c})}{\partial \tilde{c}}(\text{tr}(\mathbf{E}^{\text{el}}))^2 + J^c\frac{\partial \tilde{\mu}_{si}(\tilde{c})}{\partial \tilde{c}}\tilde{E}^{\text{el}}_{ij}\tilde{E}^{\text{el}}_{ij} \tag{1.124}$$

$$= \frac{1}{2}J^c[\tilde{\lambda}'_{\text{si}}(c)(\tilde{E}^{\text{el}}_{11} + \tilde{E}^{\text{el}}_{22} + \tilde{E}^{\text{el}}_{33})^2 + 2\tilde{\mu}'_{\text{si}}(\tilde{c})((\tilde{E}^{\text{el}}_{11})^2 + (\tilde{E}^{\text{el}}_{22})^2 + (\tilde{E}^{\text{el}}_{33})^2 + 2(\tilde{E}^{\text{el}}_{12})^2)] \tag{1.125}$$

$$\boldsymbol{j} = -\frac{D\chi_{\max}\tilde{c}}{V^b_m}\tilde{\mathbf{F}}^{-1}(\tilde{\mathbf{F}}^{-1})^{\mathsf{T}}\nabla_{\boldsymbol{X}}\tilde{\mu} \tag{1.126}$$

$$\tilde{\boldsymbol{j}} = \boldsymbol{j}HV^b_m/(\chi_{\max}D_0) \tag{1.127}$$

$$= -\frac{D}{D_0}H\tilde{c}\tilde{\mathbf{F}}^{-1}(\tilde{\mathbf{F}}^{-1})^{\mathsf{T}}\nabla_{\boldsymbol{X}}\tilde{\mu} \tag{1.128}$$

$$\tilde{j}_x = \frac{-\bar{D}H\tilde{c}}{J^2}\left(\frac{\partial\tilde{\mu}}{\partial X}(\tilde{F}^2_{12} + \tilde{F}^2_{22}) - \frac{\partial\tilde{\mu}}{\partial Y}(\tilde{F}_{11}\tilde{F}_{12} + \tilde{F}_{21}\tilde{F}_{22})\right) \tag{1.129}$$

$$= \frac{-\bar{D}\tilde{c}}{J^2}\left(\frac{\partial\tilde{\mu}}{\partial \tilde{X}}(\tilde{F}^2_{12} + \tilde{F}^2_{22}) - \frac{\partial\tilde{\mu}}{\partial \tilde{Y}}(\tilde{F}_{11}\tilde{F}_{12} + \tilde{F}_{21}\tilde{F}_{22})\right) \tag{1.130}$$

$$\tilde{j}_y = \frac{-\bar{D}H\tilde{c}}{J^2}\left(\frac{\partial\tilde{\mu}}{\partial Y}(\tilde{F}^2_{11} + \tilde{F}^2_{21}) - \frac{\partial\tilde{\mu}}{\partial X}(\tilde{F}_{11}\tilde{F}_{12} + \tilde{F}_{21}\tilde{F}_{22})\right) \tag{1.131}$$

$$= \frac{-\bar{D}\tilde{c}}{J^2}\left(\frac{\partial\tilde{\mu}}{\partial \tilde{Y}}(\tilde{F}^2_{11} + \tilde{F}^2_{21}) - \frac{\partial\tilde{\mu}}{\partial \tilde{X}}(\tilde{F}_{11}\tilde{F}_{12} + \tilde{F}_{21}\tilde{F}_{22})\right) \tag{1.132}$$

1.8 Boundary and Initial Conditions

$$\tilde{c}(\tilde{X},\tilde{Y},0)=0 \tag{1.133}$$

$$\tilde{u}(\tilde{X},\tilde{Y},0)=0 \tag{1.134}$$

$$\tilde{v}(\tilde{X},\tilde{Y},0)=0 \tag{1.135}$$

$$\tilde{u}(\tilde{X},0,\tilde{t})=\tilde{v}(\tilde{X},0,\tilde{t})=0 \tag{1.136}$$

$$\tilde{u}(-1/2,\tilde{Y},\tilde{t})=\tilde{u}(1/2,\tilde{Y},\tilde{t})=0 \tag{1.137}$$

$$\tilde{j}_x(\tilde{X},1,\tilde{t})=\tilde{j}_0(1-\tilde{c}(\tilde{X},1,\tilde{t})) \tag{1.138}$$

Bibliography

Lee, E. H. (1969), 'Elastic-plastic deformation at finite strains'.