Finite Length Stirling's Approximation

Stirling's Approximation

$$n! \sim n^n e^{-n}\sqrt{2\pi n} \tag{1}$$

Log Sum Inequality For two sets of nonnegative numbers $\{a_1, ...a_t\}$ and $\{b_1, ...b_t\}$. The log sum inequalty states that:

$$\sum_i a_i \log\left(\frac{a_i}{b_i}\right) \geq \left(\sum_i a_i\right) \times \log\left(\frac{\sum_i a_i}{\sum_i b_i}\right) \tag{2}$$

the inequality becomes an equality when $b_i = ca_i$ for all i.

Approximating the multinomial coefficient

To approximate the multinomial coefficient $\binom{n}{q_1 t..q_t t}$, we use the Stirling's appriximation (Equation 1):

$$\begin{aligned}\binom{n}{q_1 t, ..., q_t t} &= n^n e^{-n} \frac{\sqrt{2\pi n}}{\prod_i \left(q_i^{q_i n} n^{q_i n} e^{-n q_i} \sqrt{2\pi q_i n}\right)} \\ &\sim \cancel{n^n e^{-n}} \frac{\sqrt{2\pi n}}{\cancel{n^n e^{-n}} (2\pi n)^{\frac{t}{2}} \prod_i (q_i)^{n q_i} (q_i)^{\frac{1}{2}}} \\ &= (2\pi n)^{\frac{1}{2}-\frac{t}{2}} 2^{nH(q_i)} \times \left(\frac{1}{\prod_i q_i}\right)^{\frac{1}{2}}\end{aligned} \tag{3}$$

Using the log-sum inequality, we can say that

$$\begin{aligned}\log\left(\frac{1}{\prod_i q_i}\right) = \sum_{i=0}^{t} 1 \log\left(\frac{1}{q_i}\right) &\geq \left(\sum_{i=0}^{t} 1\right) \times \log\left(\frac{\sum_{i=0}^{t} 1}{\sum_{i=0}^{t} q_i}\right) \\ \log\left(\frac{1}{\prod_i q_i}\right) &\geq t \log\left(\frac{t}{1}\right) = \log(t^t)\end{aligned} \tag{4}$$

$$\frac{1}{\prod_i q_i} \geq t^t \tag{5}$$

going back to Equation 3, we then have

$$\binom{n}{q_1 t, ..., q_t t} = (2\pi n)^{\frac{1}{2}-\frac{t}{2}} 2^{nH(q_i)} \left(\frac{1}{\prod_i q_i}\right)^{\frac{1}{2}} \geq (2\pi n)^{\frac{1}{2}-\frac{t}{2}} 2^{nH(q_i)} t^{\frac{t}{2}} \tag{6}$$

The inequality becomes equaltity when $q_i^* = c \times 1$ where c is some constant and from the normalisation condition on $\sum_i^t q_i^* = 1$ implies that $c = q_i^* = \frac{1}{t}$.

$$\binom{n}{q_1 t, ..., q_t t} \approx (2\pi n)^{\frac{1}{2}-\frac{t}{2}} 2^{nH(q_i)} t^{\frac{t}{2}} \tag{7}$$