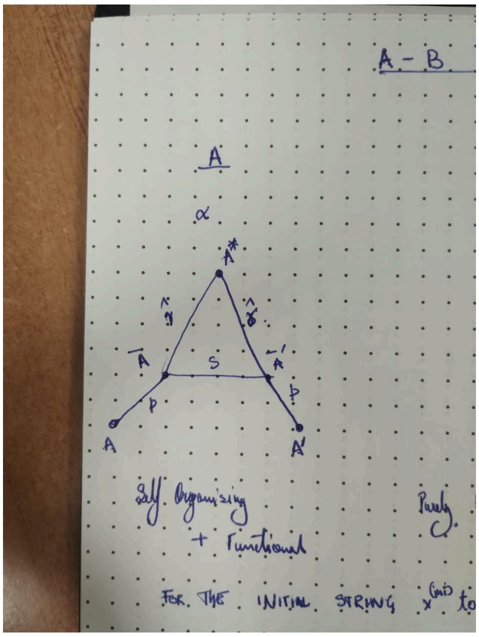
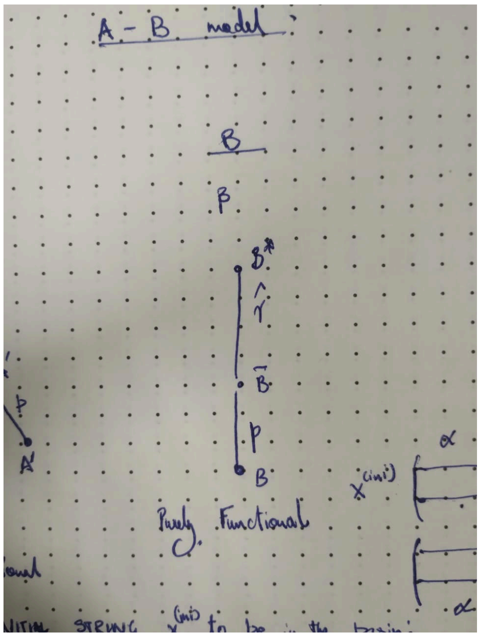
Self Organised and Function + Pure function

We assume the final string representing the stable FP as a combination of substring of the self organised and functional substring (denoted by A) plus the purely functional substring (denoted by B).

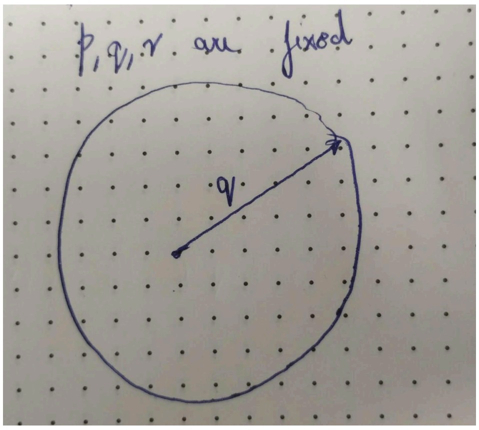
0.1 Some diagrams and introduction to the problem



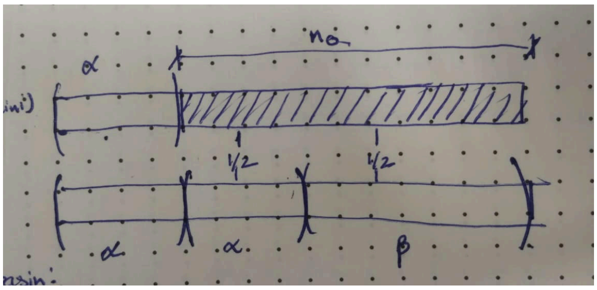
(a)



(b)



(c)



(d)

Figure 1:

These are the following fixed parameters:
- p, the developmental noise, specifically the relative distance due to the developmental noise.
- q, the complexity parameter related to the number of fixed points.
- $\hat{r}$, which the relative distance between some ideal *copy* to the noisefree copy.

Where both $0 \leq p, q \leq \frac{1}{2}$. We see a case (Section 0.5.2.1) where $\hat{r}$ and hence $r = \hat{r} \oplus p$, be constrained in the following region $0 \leq \hat{r} < \frac{1}{4}$, in order for the inqualities obtained by the positivity of the probabilities to be consistent.

The information (in bits) given by the initial condition is given as:

$$k = n(\alpha)(1 - H(q)) \tag{1}$$

Now going back to Figure 1(c) and Figure 1(d) , we may easily write the following inquality on the initial string:

$$(1-\alpha)\frac{1}{2} + \alpha(p \oplus s \oplus p) \leq q \tag{2}$$

$$\begin{gathered} p \oplus \left(s\alpha + \frac{1}{2}(1-\alpha)\right) \oplus p \leq q \\ p \oplus \overline{s} \oplus p \leq q \end{gathered} \tag{3}$$

where,

$$\overline{s} = \left(s\alpha + \frac{1}{2}(1-\alpha)\right)$$

Going back to Equation 3, we may now write:

$$\overline{s} \leq \frac{1}{2}\left(\frac{q - p \oplus p}{\frac{1}{2} - p \oplus p}\right) \tag{4}$$

$$s \leq \frac{1}{2\alpha}\left[\left(\frac{q - p \oplus p}{\frac{1}{2} - p \oplus p}\right) - 1\right] + \frac{1}{2} = s_{max} \tag{5}$$

for ease we can write

$$1 - \left(\frac{q - p \oplus p}{\frac{1}{2} - p \oplus p}\right) = D(q,p)$$

since $0 \leq q \leq \frac{1}{2}$, $D(q,p) \geq 0$. From Equation 5 we have the following:

$$s \leq -\frac{D(q,p)}{2\alpha} + \frac{1}{2} = s_{max} \tag{6}$$

Let us now take a moment to rewrite all the constraints and inqualities that we have so far:

$$0 \leq p \leq \frac{1}{2},$$

$$0 \leq q \leq \frac{1}{2},$$

$$s \leq -\frac{D(q,p)}{2\alpha} + \frac{1}{2} = s_{max},$$

$$0 \leq \alpha \leq 1,$$

$$\beta = 1 - 2\alpha,$$

$$r = \hat{r} \oplus p$$

0.2 Contribution of the A term

Without going into details we just write the probability of the strings is:

$$P = 2^{-l_s D_{KL}(\{\tilde{s}_i \| p_i\})} \tag{7}$$

in our case $l_s = \alpha n$. We have the observed probabilities as:

$$\tilde{s}_1 = 1 - \frac{1}{2}(2\hat{r} + s) \tag{8}$$

$$\tilde{s}_2 = \tilde{s}_3 = \frac{s}{2} \tag{9}$$

$$\tilde{s}_4 = \left(\hat{r} - \frac{s}{2}\right) \tag{10}$$

And assuming that the generation of the strings is random ($p_i = \frac{1}{4}$), we have:

$$D_{KL}(\{\tilde{s}_i \| p_i\}) = 2 - H(\{\tilde{s}_i\}) \tag{11}$$

and we can use that to get,

$$P = 2^{-\alpha n(2 - H(\tilde{s}_i))} \tag{12}$$

and then we can compute $\log\left(\frac{1}{P}\right)$ as,

$$\log\left(\frac{1}{P}\right) = \alpha n(2 - H(\tilde{s}_i)) \tag{13}$$

0.3 Contribution of the B part

The contribution of the B part is straightforward to write:

$$\log\left(\frac{1}{P_B}\right) = \beta n(1 - H(r)) \tag{14}$$

0.4 A and B string together

From Equation 1, Equation 13 and Equation 14, we write:

$$k + l = \alpha n(\alpha)(2 - H(\tilde{s}_i)) + \beta n(\alpha)(1 - H(r)) \tag{15}$$

Plugging in the value of k from Equation 1 and subtracting it from the right-hand side:

$$l = \alpha n(\alpha)(2 - H(\{\tilde{s}_i\})) + \beta n(\alpha)(1 - H(r)) - n(\alpha)(1 - H(q))$$

$$l = n(\alpha)[\alpha(2 - H(\{\tilde{s}_i\})) + \beta(1 - H(r)) - (1 - H(q))] \tag{16}$$

remembering that $\beta = 1 - 2\alpha$,

$$l = n(\alpha)[\alpha(2 - H(\{\tilde{s}_i\})) + (1 - 2\alpha)(1 - H(r)) - (1 - H(q))] \tag{17}$$

where,

$$n(\alpha) = \frac{n_0}{1 - \alpha} \tag{18}$$

Let us rename the following:

$$A(\{\tilde{s}_i\}) = (2 - H(\{\tilde{s}_i\})) \tag{19}$$

$$B(r) = (1 - H(r)) \tag{20}$$

$$C(q) = (1 - H(q)) \tag{21}$$

Now we may rewrite Equation 17 as

$$l = n(\alpha)[\alpha A(\{\tilde{s}_i\}) + (1 - 2\alpha)B(r) - C(q)] \tag{22}$$

Now again since $n(\alpha) \geq 0$, the positivity constraint on l implies that

$$[\alpha A(\{\tilde{s}_i\}) + (1-2\alpha)B(r) - C(q)] \geq 0 \tag{23}$$

for now I have not looked much into this constraint but it should be kept in mind moving forward, since it is a much general constraint that is true irrespective of the domains we shall talk about. (In Mathematica we simply assert $l \geq 0$). If this is positivity constraint is not kept in mind the optimizer fails, citing imaginary values of the function.

TODO: Implement a numerical solver to find minimum l given the bounds on the α and $\hat{r}$. Keeping q and p fixed. Also need to make sure all these bounds that we pass would be consistent.

On taking the derivative of Equation 22 with respect to α, we have

$$\frac{dl}{d\alpha} = \left(\frac{n(\alpha)}{1-\alpha}\right)\left[A(\{\tilde{s}_i\}) - B(r) - C(q) + \alpha(1-\alpha)\sum_i \frac{dA}{d\tilde{s}_i}\frac{d\tilde{s}_i}{d\alpha}\right] \tag{24}$$

I might omit the function arguments of A, B and C sometimes, it is implied that they are functions of $\{\tilde{s}_i\}$, r and q respectively.

0.5 The Two Regimes $\hat{r} \oplus \hat{r} < s_{max}$ and $\hat{r} \oplus \hat{r} \geq s_{max}$

There are two regimes,
- if $\hat{r} \oplus \hat{r} < s_{max}$, then $s = \hat{r} \oplus \hat{r}$ is a *natural* distance.
- if $\hat{r} \oplus \hat{r} \geq s_{max}$, we use $s = s_{max}$, and in that case $s = s(\alpha)$ and $\tilde{s}_i = \tilde{s}_i(\alpha)$

0.5.1 Regime 1: $\hat{r} \oplus \hat{r} < s_{max}$

In this regime, $\frac{dl}{d\alpha} = 0$, then we have:

$$\frac{dl}{d\alpha} = \left(\frac{n(\alpha)}{1-\alpha}\right)[A - B - C] \tag{25}$$

$$A - B - C = H(r) + H(q) - H(\{\tilde{s}_i\}) \tag{26}$$

$$\frac{dl}{d\alpha} = \frac{n(\alpha)}{1-\alpha}[H(r) + H(q) - H(\{\tilde{s}_i\})] \tag{27}$$

0.5.2 Regime 2: $\hat{r} \oplus \hat{r} \geq s_{max}$

In the unexpanded form, (leaving all the $\tilde{s}_i$'s as is) the derivative of l in this regime is given by

$$\frac{dl}{d\alpha} = \frac{n(\alpha)}{1-\alpha}\left[A(\alpha) - B - C + \alpha(1-\alpha)\frac{D(q,p)}{4\alpha^2}\log_2\left(\frac{\tilde{s}_1(\alpha)\tilde{s}_4(\alpha)}{s_2^2(\alpha)}\right)\right] \tag{28}$$

0.5.2.1 Positivity constraint on probabilities will give a bound on $\hat{r}$

Now in this regime we put $s = s_{max}$ and then the probabilies $\{\tilde{s}_i\}$ are a function of α. There is a positivity constraint on the probability.

$$\tilde{s}_1 = 1 - \frac{1}{2}(2\hat{r} + s(\alpha)) \geq 0$$
$$\tilde{s}_2 = \tilde{s}_3 = s(\alpha) \geq 0$$
$$\tilde{s}_4 = \hat{r} - \frac{s(\alpha)}{2} \geq 0$$

from these constraints and plugging $s = s_{max}$ from Equation 6, we get the following equations

$$s_{max} \leq 2(1 - \hat{r})$$

$$s_{max} \leq 2\hat{r}$$

and since $\hat{r} \leq \frac{1}{2}$, these will be redundant and we can just write:

$$s_{max} \leq 2\hat{r} \tag{29}$$

and then we have additional constraint from positivity of $\tilde{s}_2 = \tilde{s}_3 = \frac{s}{2}$

$$s_{max} \geq 0 \tag{30}$$

After plugging in the value of s_{max}, we get the two inequalities:

$$D(q, p) \leq \alpha \tag{31}$$

and an upper bound from Equation 29

$$\alpha \leq \frac{D(q, p)}{1 - 4\hat{r}} \tag{32}$$

therefore, for the bounds given by Equation 32 and Equation 31 to be consistent, recalling that $D(q, p) \geq 0$, we require:

$$\boxed{\hat{r} < \frac{1}{4}} \tag{33}$$

so we cannot set $\hat{r}$ to be any value in 0 to $\frac{1}{2}$ like we do with p and q.