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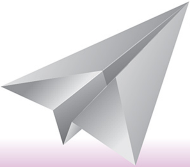

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PREFACE

Wiley Acing the GATE: Engineering Mathematics and General Aptitude is intended to be the complete book for those aspiring to compete in the *Graduate Aptitude Test in Engineering* (*GATE*) in various engineering disciplines, including *Electronics and Communication*, *Electrical*, *Mechanical*, *Computer Science*, *Civil, Chemical* and *Instrumentation,* comprehensively covering all topics as prescribed in the syllabus in terms of study material and an elaborate question bank. There are host of salient features offered by the book as compared to the content of the other books already published for the same purpose. Some of the important ones include the following.

One of the notable features of the book includes presentation of study material in ***simple and lucid language and in small sections while retaining focus on alignment of the material in accordance with the requirements of GATE examination***. While it is important for a book that has to cover a wide range of topics in *Engineering Mathematics*, *General Aptitude* including *Verbal Ability* and *Numerical Ability* to be precise in the treatment of different topics, the present book achieves that goal without compromising completeness. The study material and also the question bank have all the three important 'C' qualities, Conciseness, Completeness and Correctness, for communicating effectively with the examinees.

Another important feature of the book is its ***comprehensive question bank***. The question bank is organized in three different categories, namely the *Solved Examples*, *Practice Exercise* and *Solved GATE Previous Years' Questions. Solved Examples* contain a large number of questions of varying complexity. Each question in this category is followed by its solution. Under *Practice Exercise*, again there are a large number of multiple choice questions. The answers to these questions are given at the end of the section. Each of the answers is supported by an explanation unlike other books where solutions to only selected questions are given. The third category contains questions from previous GATE examinations from 2003 onwards. Each question is followed by a complete solution.

Briefly outlining the length and breadth of the material presented in the book, it is divided into two broad sections, namely ***Engineering Mathematics*** and ***General Aptitude***. Section on General Aptitude is further divided into two sections covering ***Verbal Ability*** and ***Numerical Ability***.

Engineering Mathematics is covered in eleven different chapters. These include *Linear Algebra* covering important topics such as matrix algebra, systems of linear equations, and eigenvalues and eigenvectors; *Calculus* covering important topics such as functions of single variable, limit, continuity and differentiability, mean value theorem, definite and improper integrals, partial derivatives, maxima and minima, gradient, divergence and curl, directional derivatives, line, surface and volume integrals, Stokes' theorem, Gauss theorem and Green's theorem, and Fourier series; *Differential Equations* covering linear and non-linear differential equations, Cauchy's and Euler's equations, Laplace transforms, partial differentiation, solutions of one-dimensional heat and wave equations, Laplace equation, and method of variation of parameters; *Complex Variables* covering analytic functions, Cauchy's integral theorem, and Taylor and Laurent series; *Probability and Statistics* covering conditional probability, random variables, discrete and continuous distributions, Poisson, normal, uniform, exponential and binomial distributions, correlation and regression analyses, residue theorem, and solution integrals; *Numerical Methods* covering numerical solutions of linear and non-linear algebraic equations, integration by trapezoidal and Simpson's rules, single and multi-step methods for differential equations, numerical solutions of non-linear algebraic equations by secant, bisection, Runge–Kutta and Newton–Raphson methods; *Mathematical Logic* including first-order logic and proportional logic; *Set Theory and Algebra* including sets, relations, functions and groups, partial orders, lattice, and Boolean algebra; *Combinatory* covering permutations and combinations, counting, summation, generating functions, recurrence relations, and asymptotics; *Graph Theory* covering connectivity and spanning trees, cut vertices and edges, covering and matching, independent sets, colouring, planarity, and isomorphism; and *Transform Theory* covering Fourier transform, Laplace transform and z-transform.

General Aptitude comprises two sub-sections namely ***Verbal Ability*** and ***Numerical Ability***. Important topics covered under *Verbal Ability* include English grammar, synonyms, antonyms, sentence completion, verbal analogies, word groups, and critical reasoning and verbal deduction. Under *Numerical Ability*, important topics covered include basic arithmetic, algebra, and reasoning and data interpretation. Under *Basic Arithmetic*, we have discussed number system; percentage; profit and loss; simple interest and compound interest; time and work; average, mixture and allegation; ratio, proportion and variation; and speed, distance and time. Under *Algebra*, we have discussed permutation and combination; progression; probability; set theory; and surds, indices and logarithm. The topics covered under *Reasoning and Interpretation* are cubes and dices, line graph, tables, blood relationship, bar diagram, pie chart, puzzles, and analytical reasoning.

The ***Graduate Admission Test in Engineering (GATE)*** is an All-India level competitive examination for engineering graduates aspiring to pursue Master's or Ph.D. programs in India. The examination evaluates the examinees in General Aptitude, Engineering Mathematics and the subject discipline. Though majority of questions is asked from the subject discipline; there are sizable number of questions set from *Engineering Mathematics* and *General Aptitude*. It is an examination where close to ten lakh students appear every year. The level of competition is therefore very fierce. While admission to a top institute for the Master's programme continues to be the most important reason for working hard to secure a good score in the GATE examination; another great reason to appear and handsomely qualify GATE examination is that many Public Sector Undertakings (PSUs) are and probably in future almost all will be recruiting through GATE examination. And it is quite likely that even big private sector companies may start considering GATE seriously for their recruitment as GATE score can give a bigger clue about who they are recruiting. The examination today is highly competitive and the GATE score plays an important role. This only reiterates the need to have a book that prepares examinees not only to qualify the GATE examination by getting a score just above the threshold but also enabling them to achieve a competitive score. In a competition that is as fierce as the GATE is, a high score in Engineering Mathematics and General Aptitude section can be a great asset.

The present book is written with the objective of fulfilling this requirement. The effort is intended to offer to the large section of GATE aspirants a *self-study* and *do-it-yourself* book providing comprehensive and step-by-step treatment of each and every aspect of the examination in terms of concise but complete study material and an exhaustive set of questions with solutions. The authors would eagerly look forward to the feedback from the readers through publishers to help them make the book better.

Dr ANIL KUMAR MAINI
VARSHA AGRAWAL
NAKUL MAINI

CONTENTS

ENGINEERING MATHEMATICS

CHAPTER 1

LINEAR ALGEBRA

MATRIX

A set of mn number (real or imaginary) arranged in the form of a rectangular array of m rows and n columns is called an $m \times n$ matrix. An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A = [a_{ij}]_{m \times n}$ or $A = [a_{ij}]$.

The numbers a_{11}, a_{12}, ..., a_{mn} are known as the elements of the matrix A. The element a_{ij} belongs to i^{th} row and j^{th} column and is called the $(ij)^{\text{th}}$ element of the matrix $A = \left[a_{ij}\right]$.

Types of Matrices

1. **Row matrix:** A matrix having only one row is called a row matrix or a row vector. Therefore, for a row matrix, $m = 1$.

   For example, $A = [1\ 2\ -1]$ is a row matrix with $m = 1$ and $n = 3$.

2. **Column matrix:** A matrix having only one column is called a column matrix or a column vector. Therefore, for a column matrix, $n = 1$.

   For example, $A = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is a column matrix with $m = 3$ and $n = 1$.

3. **Square matrix:** A matrix in which the number of rows is equal to the number of columns, say n, is called a square matrix of order n.

   For example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a square matrix of order 2.

4. **Diagonal matrix:** A square matrix is called a diagonal matrix if all the elements except those in the leading diagonal are zero, i.e. $a_{ij} = 0$ for all $i \neq j$.

For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$ is a diagonal matrix and is denoted by A = diag[1, 5, 10].

5. **Scalar matrix:** A matrix $A = [a_{ij}]_{n\times n}$ is called a scalar matrix if
(a) $a_{ij} = 0$, for all $i \neq j$.
(b) $a_{ii} = c$, for all i, where c $\neq$ 0.

For example, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix of order 5.

6. **Identity or unit matrix:** A square matrix $A = [a_{ij}]_{n\times n}$ is called an identity or unit matrix if
(a) $a_{ij} = 0$, for all $i \neq j$.
(b) $a_{ij} = 1$, for all i.

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix of order 2.

7. **Null matrix:** A matrix whose all the elements are zero is called a null matrix or a zero matrix.

For example, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a null matrix of order 2 × 2.

8. **Upper triangular matrix:** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

For example, $A = \begin{bmatrix} 1 & 2 & 6 & 3 \\ 0 & 5 & 7 & 4 \\ 0 & 0 & 9 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is an upper triangular matrix.

9. **Lower triangular matrix:** A square matrix $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0$ for $i < j$.

For example, $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 6 & 2 & 9 & 0 \\ 3 & 7 & 4 & 3 \end{bmatrix}$ is a lower triangular matrix.

Types of a Square Matrix

1. **Nilpotent matrix:** A square matrix A is called a nilpotent matrix if there exists a positive integer n such that $A^n = 0$. If n is least positive integer such that $A^n = 0$, then n is called index of the nilpotent matrix A.

2. **Symmetrical matrix:** It is a square matrix in which $a_{ij} = a_{ji}$ for all i and j. A symmetrical matrix is necessarily a square one. If A is symmetric, then $A^T = A$.

For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{bmatrix}$.

3. **Skew-symmetrical matrix:** It is a square matrix in which $a_{ij} = -a_{ji}$ for all i and j. In a skew-symmetrical matrix, all elements along the diagonal are zero.

For example, $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 0 \end{bmatrix}$.

4. **Hermitian matrix:** It is a square matrix A in which $(i, j)^{\text{th}}$ element is equal to complex conjugate of the $(j, i)^{\text{th}}$ element, i.e. $a_{ij} = \bar{a}_{ji}$ for all i and j. A necessary condition for a matrix A to be Hermitian is that $A = A^{\theta}$, where A^{θ} is transposed conjugate of A.

For example, $\begin{bmatrix} 1 & 1+4i & 2+3i \\ 1-4i & 2 & 5+i \\ 2-3i & 5-i & 4 \end{bmatrix}$.

5. **Skew-Hermitian matrix:** It is a square matrix $A = [a_{ij}]$ in which $a_{ij} = -\bar{a}_{ij}$ for all i and j.

The diagonal elements of a skew-Hermitian matrix must be pure imaginary numbers or zeroes. A necessary and sufficient condition for a matrix A to be skew-Hermitian is that

$$A^{\theta} = -A$$

6. **Orthogonal matrix:** A square matrix A is called orthogonal matrix if $AA^T = A^TA = I$.

For example, if

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

then $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Equality of a Matrix

Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{x\times y}$ are equal if

1. $m = x$, i.e. the number of rows in A equals the number of rows in B.
2. $n = y$, i.e. the number of columns in A equals the number of columns in B.
3. $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

Addition of Two Matrices

Let A and B be two matrices, each of order $m \times n$. Then their sum $(A + B)$ is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same order, their sum $(A + B)$ is defined to be the matrix of order $m \times n$ such that

$$(A + B)_{ij} = a_{ij} + b_{ij} \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

The sum of two matrices is defined only when they are of the same order.

For example, if $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 \\ 1 & 2 \end{bmatrix}$

Hence, $A + B = \begin{bmatrix} 9 & 9 \\ 4 & 7 \end{bmatrix}$

However, addition of $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ is not possible.

Some of the important properties of matrix addition are:

1. **Commutativity:** If A and B are two $m \times n$ matrices, then $A + B = B + A$, i.e. matrix addition is commutative.
2. **Associativity:** If A, B and C are three matrices of the same order, then
$$(A + B) + C = A + (B + C)$$
i.e.matrix addition is associative.
3. **Existence of identity:** The null matrix is the identity element for matrix addition. Thus, $A + O = A = O + A$
4. **Existence of inverse:** For every matrix $A = [a_{ij}]_{m\times n}$, there exists a matrix $[a_{ij}]_{m\times n}$, denoted by $-A$, such that $A + (-A) = O = (-A) + A$
5. **Cancellation laws:** If A, B and C are matrices of the same order, then
$$A + B = A + C \Rightarrow B = C$$
$$\mathrm{B} + A = C + A \Rightarrow B = C$$

Multiplication of Two Matrices

If we have two matrices A and B, such that

$$A = [a_{ij}]_{m\times n} \quad \text{and} \quad B = [b_{ij}]_{x\times y}, \text{ then}$$

$A \times B$ is possible only if $n = x$, i.e. the columns of the pre-multiplier is equal to the rows of the post multiplier. Also, the order of the matrix formed after multiplying will be $m \times y$.

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Then $A \times B = \begin{bmatrix} 1\times 4 + 2\times 2 & 1\times 3 + 2\times 1 \\ 3\times 4 + 4\times 2 & 3\times 3 + 4\times 1 \end{bmatrix}$

$C = A \times B = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}$

Some important properties of matrix multiplication are:

1. Matrix multiplication is not commutative.
2. Matrix multiplication is associative, i.e. $(AB)C = A(BC)$.
3. Matrix multiplication is distributive over matrix addition, i.e. $A(B + C) = AB + AC$.
4. If A is an $m\times n$ matrix, then $I_m A = A = AI_n$.
5. The product of two matrices can be the null matrix while neither of them is the null matrix.

Multiplication of a Matrix by a Scalar

If $A = [a_{ij}]$ be an $m \times n$ matrix and k be any scalar constant, then the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA.

For example, if $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 3 \\ 4 & 9 & 8 \end{bmatrix}$ and $k = 2$.

$$\Rightarrow kA = \begin{bmatrix} 2 & 6 & 2 \\ 2 & 14 & 6 \\ 3 & 18 & 16 \end{bmatrix}$$

Some of the important properties of scalar multiplication are:

1. $k(A + B) = kA + kB$
2. $(k + l) \cdot A = kA + lA$
3. $(kl) \cdot A = k(lA) = l(kA)$
4. $(-k) \cdot A = -(kA) = k(-A)$
5. $1 \cdot A = A$
6. $-1 \cdot A = -A$

Here A and B are two matrices of same order and k and l are constants.

If A is a matrix and $A^2 = A$, then A is called idempotent matrix. If A is a matrix and satisfies $A^2 = I$, then A is called involuntary matrix.

Transpose of a Matrix

Consider a matrix A, then the matrix obtained by interchanging the rows and columns of A is called its transpose and is represented by A^T.

For example, if $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 9 & 8 \\ 7 & 6 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 9 & 6 \\ 5 & 8 & 4 \end{bmatrix}$.

Some of the important properties of transpose of a matrix are:

1. For any matrix A, $(A^T)^T = A$
2. For any two matrices A and B of the same order
$$(A + B)^T = A^T + B^T$$
3. If A is a matrix and k is a scalar, then
$$(kA)^T = k(A^T)$$
4. If A and B are two matrices such that AB is defined, then
$$(AB)^T = B^T A^T$$

Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be the cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A.

Thus, adj $A = [C_{ij}]^T \Rightarrow (\text{adj } A)_{ij} = C_{ji}$ = cofactor of a_{ji} in A.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{then } \text{adj}(A) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Inverse of a Matrix

A square matrix of order n is invertible if there exists a square matrix B of the same order such that
$$AB = I_n = BA$$

In the above case, B is called the inverse of A and is denoted by A^{-1}.
$$A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Some of the important properties of inverse of a matrix are:

1. A^{-1} exists only when A is non-singular, i.e. $|A| \neq 0$.
2. The inverse of a matrix is unique.
3. Reversal laws: If A and B are invertible matrices of the same order, then
$$(AB)^{-1} = B^{-1}A^{-1}$$
4. If A is an invertible square matrix, then $(A^T)^{-1} = (A^{-1})^T$
5. The inverse of an invertible symmetric matrix is a symmetric matrix.
6. Let A be a non-singular square matrix of order n. Then
$$|\text{adj } A| = |A|^{n-1}$$
7. If A and B are non-singular square matrices of the same order, then
$$\text{adj } (AB) = (\text{adj } B)(\text{adj } A)$$
8. If A is an invertible square matrix, then
$$\text{adj } A^T = (\text{adj } A)^T$$
9. If A is a non-singular square matrix, then
$$\text{adj } (\text{adj } A) = |A|^{n-2} A$$
10. If A is a non-singular matrix, then
$$|A^{-1}| = |A|^{-1}, \text{ i.e. } |A^{-1}| = \frac{1}{|A|}$$
11. Let A, B and C be three square matrices of same type and A be a non-singular matrix. Then
$$AB = AC \Rightarrow B = C$$
and
$$BA = CA \Rightarrow B = C$$

Rank of a Matrix

The column rank of matrix A is the maximum number of linearly independent column vectors of A. The row rank of A is the maximum number of linearly independent row vectors of A.

In linear algebra, column rank and row rank are always equal. This number is simply called rank of a matrix.

The rank of a matrix A is commonly denoted by rank (A). Some of the important properties of rank of a matrix are:

1. The rank of a matrix is unique.
2. The rank of a null matrix is zero.
3. Every matrix has a rank.
4. If A is a matrix of order $m \times n$, then rank $(A) \leq m \times n$ (smaller of the two)
5. If rank $(A) = n$, then every minor of order $n + 1$, $n + 2$, etc., is zero.
6. If A is a matrix of order $n \times n$, then A is non-singular and rank $(A) = n$.
7. Rank of $I_A = n$.
8. A is a matrix of order $m \times n$. If every k^{th} order minor $(k < m, k < n)$ is zero, then
$$\text{rank } (A) < k$$
9. A is a matrix of order $m \times n$. If there is a minor of order $(k < m, k < n)$ which is not zero, then
$$\text{rank } (A) \geq k$$
10. If A is a non-zero column matrix and B is a non-zero row matrix, then rank $(AB) = 1$.
11. The rank of a matrix is greater than or equal to the rank of every sub-matrix.
12. If A is any n-rowed square matrix of rank, $n - 1$, then
$$\text{adj } A \neq 0$$
13. The rank of transpose of a matrix is equal to rank of the original matrix.
$$\text{rank } (A) = \text{rank } (A^T)$$

14. The rank of a matrix does not change by pre-multiplication or post-multiplication with a non-singular matrix.

15. If $A - B$, then rank (A) = rank (B).

16. The rank of a product of two matrices cannot exceed rank of either matrix.

$$\text{rank } (A \times B) \leq \text{rank } A$$

or $$\text{rank } (A \times B) \leq \text{rank } B$$

17. The rank of sum of two matrices cannot exceed sum of their ranks.

18. Elementary transformations do not change the rank of a matrix.

DETERMINANTS

Every square matrix can be associated to an expression or a number which is known as its determinant. If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det A or $|A|$. If $A = [a_{11}]$ is a square matrix of order 1, then determinant of A is defined as

$$|A| = a_{11}$$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{23} \end{bmatrix}$ is a square matrix of order 2, then determinant of A is defined as

$$|A| = a_{11}a_{23} - a_{12}a_{21}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then determinant of A is defined as

$$A| = a_{11}\ (a_{22}a_{33} - a_{23}a_{32}) - a_{21}\ (a_{12}a_{33} - a_{13}a_{32}) + a_{21}\ (a_{12}a_{23} - a_{13}a_{22})$$

or $$|A| = a_{11}\ (a_{22}a_{33} - a_{23}a_{32}) - a_{12}\ (a_{21}a_{33} - a_{23}a_{31}) + a_{13}\ (a_{21}a_{32} - a_{22}a_{31})$$

For example, determinants of the matrices [1], $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 7 & 1 \\ 2 & -7 & 3 \\ 5 & 9 & 8 \end{bmatrix}$ will be, respectively,

$$\Delta = 1$$
$$\Delta = (2 \times 5) - (1 \times 3) = 7$$
$$\begin{aligned} \Delta &= 1[(-7 \times 8) - (3 \times 9)] - 2\ [(7 \times 8) - (1 \times 9)] \\ &\quad + 5\ [(7 \times 3) - (-7 \times 1)] \\ &= -83 - 94 + 140 = -177 + 40 \\ &= -37 \end{aligned}$$

Minors

The minor M_{ij} of $A = [a_{ij}]$ is the determinant of the square sub-matrix of order $(n - 1)$ obtained by removing i^{th} row and j^{th} column of the matrix A.

For example, say $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, then minors of A will be

$$M_{11} = 4$$
$$M_{12} = 2$$
$$M_{21} = 3$$
$$M_{22} = 1$$

Say $$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ -4 & 4 & 7 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 5 & 1 \\ 4 & 7 \end{vmatrix} = 35 - 4 = 31$$

$$M_{12} = \begin{vmatrix} 3 & 1 \\ -4 & 7 \end{vmatrix} = 21 - (-4) = 25$$

$$M_{13} = \begin{vmatrix} 3 & 5 \\ -4 & 4 \end{vmatrix} = 12 - (-20) = 32$$

$$M_{21} = \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = 14 - 12 = 2$$

$$M_{22} = \begin{vmatrix} 1 & 3 \\ -4 & 7 \end{vmatrix} = 7 - (-12) = 19$$

$$M_{23} = \begin{vmatrix} 1 & 2 \\ -4 & 4 \end{vmatrix} = 4 - (-8) = 12$$

$$M_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = 2 - 15 = -13$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 1 - 9 = -8$$

$$M_{33} = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5 - 6 = -1$$

Cofactors

The cofactor C_{ij} of $A = [a_{ij}]$ is equal to $(-1)^{i+j}$ times the determinant of the sub-matrix of order $(n - 1)$ obtained by leaving i^{th} row and j^{th} column of A.

For example, say $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ -4 & 4 & 7 \end{bmatrix}$.

Cofactors of A will be

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 1 \\ 4 & 7 \end{vmatrix} = 31$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 1 \\ -4 & 7 \end{vmatrix} = -25$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 5 \\ -4 & 4 \end{vmatrix} = 32$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = -2$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ -4 & 7 \end{vmatrix} = 19$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ -4 & 4 \end{vmatrix} = -12$$

$$C_{31} = (-1)^{3+1}\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = -13$$

$$C_{32} = (-1)^{3+2}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = 8$$

$$C_{33} = (-1)^{3+3}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = -1$$

Some of the important properties of determinants are:

1. Sum of the product of elements of any row or column of a square matrix $A = [a_{ij}]$ of order n with their cofactors is always equal to its determinant.

$$\sum_{i=1}^{n} a_{ij}c_{ij} = |A| = \sum_{j=1}^{n} a_{ij}c_{ij}$$

2. Sum of the product of elements of any row or column of a square matrix $A = [a_{ij}]$ of order n with the cofactors of the corresponding elements of other row or column is zero.

$$\sum_{i=1}^{n} a_{ij}c_{ik} = 0 = \sum_{j=1}^{n} a_{ij}c_{kj}$$

3. For a square matrix $A = [a_{ij}]$ of order n, $|A| = |A^T|$.
4. Consider a square matrix $A = [a_{ij}]$ of order $n \geq 2$ and B obtained from A by interchanging any two rows or columns of A, then $|B| = -A$.
5. For a square matrix $A = [a_{ij}]$ of order $(n \geq 2)$, if any two rows or columns are identical, then its determinant is zero, i.e. $|A| = 0$.
6. If all the elements of a square matrix $A = [a_{ij}]$ of order n are multiplied by a scalar k, then the determinant of new matrix is equal to $k|A|$.
7. Let A be a square matrix such that each element of a row or column of A is expressed as the sum of two or more terms. Then $|A|$ can be expressed as the sum of the determinants of two or more matrices of the same order.
8. Let A be a square matrix and B be a matrix obtained from A by adding to a row or column of A a scalar multiple of another row or column of A, then $|B| = |A|$.
9. Let A be a square matrix of order n (≥ 2) and also a null matrix, then $|A| = 0$.
10. Consider $A = [a_{ij}]$ as a diagonal matrix of order n (≥ 2), then
$$|A| = a_{11} \times a_{22} \times a_{33} \times \cdots \times a_{nn}$$
11. Suppose A and B are square matrices of same order, then
$$|AB| = |A| \cdot |B|$$

SOLUTIONS OF SIMULTANEOUS LINEAR EQUATIONS

Suppose we have a system of 'i' linear equation in 'j' unknowns, such as

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{ij}x_j = b_1 \\ a_{221}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j = b_i \end{array}$$

The above system of equations can be written in the matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{ij} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \end{bmatrix}$$

The equation can be represented by the form $AX = B$.

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{ij} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} \end{bmatrix}$ is of the order of $i \times j$,

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \end{bmatrix}$ is of the order of $j \times 1$ and

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \end{bmatrix}$ is of the order of $i \times 1$.

$[A]_{i\times j}$ is called the coefficient matrix of system of linear equations.

For example,
$$\begin{aligned} 2x + 3y - z &= 2 \\ 3x + y + 2z &= 1 \\ 4x - 7y + 5z &= -7 \end{aligned}$$

The above equation can be expressed in the matrix form as

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ 4 & -7 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$$

A consistent system is a system of equations that has one or more solutions.

An inconsistent system is a system of equations that has no solutions.

For example, consider

$$\begin{aligned} x + y &= 4 \\ 2x + 3y &= 9 \end{aligned}$$

The above system of equation is consistent as it has the solution, $x = 3$ and $y = 1$.

Now, consider

$$x + y = 4$$
$$2x + 2y = 6$$

The above system of equation is inconsistent as it has no solution for both equations simultaneously.

A homogeneous system is a system when $B = 0$ in the equation $AX = B$.

A non-homogeneous system is a system when $B \neq 0$ in the equation $AX = B$.

For example, consider

$$3x + y = 0$$
$$6x + 2y = 0$$

Such a system is a homogeneous system.

Now, consider

$$3x + y = 1$$
$$6x + 2y = 2$$

Such a system is a non-homogeneous system.

Solution of Homogeneous System of Linear Equations

As already discussed, for a homogeneous system of linear equation with 'j' unknowns,

$$AX = B \text{ becomes}$$
$$AX = 0 \ (\because\ B = 0)$$

There are two cases that arise for homogeneous systems:

1. Matrix A is non-singular or $|A| \neq 0$.

   The solution of the homogeneous system in the above equation has a unique solution, $X = 0$, i.e. $x_1 = x_2 = \cdots = x_j = 0$.

2. Matrix A is singular or $|A| = 0$, then it has infinite many solutions. To find the solution when $|A| = 0$, put $z = k$ (where k is any real number) and solve any two equations for x and y using the matrix method. The values obtained for x and y with $z = k$ is the solution of the system.

   For example, given the following system of homogeneous equation

$$2x + y + 3z = 0$$
$$3x - y + z = 0$$
$$-x - 2y + 3z = 0$$

   This system can be rewritten as

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $AX = 0$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ -1 & -2 & 3 \end{bmatrix} = \big(2(-3+2) - 3(3+6) - (1+3)\big)$$
$$= -2 - 27 - 4 = -33 \neq 0$$

As $|A| \neq 0$, given system of equations has the solution $x = y = z = 0$.

Solution of Non-Homogeneous System of Simultaneous Linear Equations

Non-homogeneous equations have already been discussed in the previous sections. Also, the methods to solve homogeneous system of equations were also discussed in the previous section.

In this section, we will discuss the method to solve a non-homogeneous system of simultaneous linear equations. Please note the number of unknowns and the number of equations.

1. Given that A is a non-singular matrix, then a system of equations represented by $AX = B$ has the unique solution which can be calculated by $X = A^{-1} B$.
2. If $AX = B$ is a system with linear equations equal to the number of unknowns, then three cases arise:
   (a) If $|A| \neq 0$, system is consistent and has a unique solution given by $X = A^{-1} B$.
   (b) If $|A| = 0$ and $(\text{adj } A)B = 0$, system is consistent and has infinite solutions.
   (c) If $|A| = 0$ and $(\text{adj } A)B \neq 0$, system is inconsistent.

For example, consider the following system of equations:

$$x + 2y + z = 7$$
$$x + 3z = 11$$
$$2x - 3y = 1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$\Rightarrow AX = B, \text{ where } A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} = 1(0+9) - 1(0+3) + 2(6)$$
$$= 9 - 3 + 12 = 18 \neq 0$$

So, the given system of equations has a unique solution given by $X = A^{-1} B$. Now, calculate the adjoint of matrix.

$$\operatorname{adj} A = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{18} \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$X = A^{-1} B$$

$$X = \frac{1}{18} \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 63 - 33 + 6 \\ 42 - 22 - 2 \\ -21 + 77 - 2 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 2,\ y = 1,\ z = 3$ is the unique solution of the given set of equations.

Cramer's Rule

Suppose we have the following system of linear equations:

$$a_1 x + b_1 y + c_1 z = k_1$$
$$a_2 x + b_2 y + c_2 z = k_2$$
$$a_3 x + b_3 y + c_3 z = k_3$$

Now, if

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_1 = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix} \neq 0$$

Thus, the solution of the system of equations is given by

$$x = \frac{\Delta_1}{\Delta}$$
$$y = \frac{\Delta_2}{\Delta}$$
$$z = \frac{\Delta_3}{\Delta}$$

AUGMENTED MATRIX

Consider the following system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This system can be represented as $AX = B$.

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

The matrix $\left[A|B\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ is called augmented matrix.

GAUSS ELIMINATION METHOD

Consider the following system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0 \end{aligned}$$

In the matrix form, the above equations can be represented as $AX = B$. We consider two particular cases:

Case 1: Suppose the coefficient matrix A is such that all elements above the leading diagonal are zero. Then A is a lower triangular matrix and the system can be written as follows:

$$\begin{aligned} a_{11}x_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

The system given above can be solved using forward substitution.

$$\therefore x_1 = \frac{b_1}{a_{11}}; x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}; x_3 = \frac{b_3 - (a_{31}x_1 + a_{32}x_2)}{a_{33}}$$

and so on.

Case 2: Suppose the coefficient matrix A is such that all elements below the leading diagonal are zero. Then A is an upper triangular matrix and the system can be written in the following form:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\quad\vdots \\ &a_{nn}x_n = b_n \end{aligned}$$

The system given above can be solved using backward substitution.

$$\therefore x_3 = \frac{b_3}{a_{33}}; x_2 = \frac{b_2 - a_{23}x_3}{a_{22}}; x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3)}{a_{11}}$$

and so on.

Gauss elimination method is a standard elimination method for solving linear systems that proceeds systematically irrespective of particular features of the coefficients.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

The first equation of the above system is called the pivotal equation and the leading coefficient a_{11} is called first pivot.

Step 1: Form the augmented matrix of the above system.

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Step 2: Performing the following operations $R_2 \to R_2 - (a_{21}/a_{11})R_1$ and $R_3 \to R_3 - (a_{31}/a_{11})R_1$, we get

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

where

$$\begin{aligned} a'_{22} &= a_{22} - (a_{21}/a_{11})a_{12} \\ a'_{23} &= a_{23} - (a_{21}/a_{11})a_{13} \\ a'_{32} &= a_{32} - (a_{31}/a_{11})a_{12} \\ a'_{33} &= a_{33} - (a_{31}/a_{11})a_{13} \\ b'_2 &= b_2 - (a_{21}/a_{11})b_1 \\ b'_3 &= b_3 - (a_{31}/a_{11})b_1 \end{aligned}$$

Here $-a_{21}/a_{11}$ and $-a_{31}/a_{11}$ are called the multipliers for the first stage of elimination. It is clear that we have assumed $a_{11} \neq 0$.

Step 3: Performing $R_3 \to R_3 - (a'_{32}/a'_{22})R_2$, we get

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

where

$$\begin{aligned} a''_{33} &= a'_{33} - (a'_{32}/a'_{22}) \times a'_{23} \\ b''_{33} &= b'_{33} - (a'_{32}/a'_{22}) \times b'_2 \end{aligned}$$

This is the end of the forward elimination. From the system of equations obtained from step 3, the values of x_1, x_2 and x_3 can be obtained by back substitution. This procedure is called partial pivoting. If this is impossible, then the matrix is singular and the system has no solution.

CAYLEY–HAMILTON THEOREM

According to the Cayley—Hamilton theorem, every square matrix satisfies its own characteristic equations. Hence, if

$$|A - \lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n)$$

is the characteristic polynomial of a matrix A of order n, then the matrix equation

$$X^n + a_1X^{n-1} + a_2X^{n-2} + \cdots + a_nI = 0$$

is satisfied by $X = A$.

EIGENVALUES AND EIGENVECTORS

If $A = [a_{ij}]_{n\times n}$ is a square matrix of order n, then the vector equation $AX = \lambda X$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$ is an unknown vector and λ is an unknown scalar value, is called an eigenvalue problem. To solve the problem, we need to determine the value of X's and λ's to satisfy the above-mentioned vector. Note that the zero vector (i.e. $X = 0$) is not of our interest. A value of λ for which the above equation has a solution $X \neq 0$ is called an *eigenvalue* or *characteristic value* of the matrix A. The corresponding solutions $X \neq 0$ of the equation are called the *eigenvectors* or *characteristic vectors* of A corresponding to that eigenvalue, λ. The set of all the eigenvalues of A is called the *spectrum* of A. The largest of the absolute values of the eigenvalues of A is called the *spectral radius* of A. The sum of the elements of the principal diagonal of a matrix A is called the *trace* of A.

Properties of Eigenvalues and Eigenvectors

Some of the main characteristics of eigenvalues and eigenvectors are discussed in the following points:

1. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A, then $k\lambda_1, k\lambda_2, k\lambda_3, \ldots, k\lambda_n$ are eigenvalues of kA, where k is a constant scalar quantity.
2. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots, \frac{1}{\lambda_n}$ are the eigenvalues of A^{-1}.
3. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A, then $\lambda_1^k, \lambda_2^k, \lambda_3^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k.
4. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A, then $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots, \frac{|A|}{\lambda_n}$ are the eigenvalues of adj A.
5. The eigenvalues of a matrix A are equal to the eigenvalues of A^T.
6. The maximum number of distinct eigenvalues is n, where n is the size of the matrix A.
7. The trace of a matrix is equal to the sum of the eigenvalues of a matrix.
8. The product of the eigenvalues of a matrix is equal to the determinant of that matrix.
9. If A and B are similar matrices, i.e. $A = IB$, then A and B have the same eigenvalues.
10. If A and B are two matrices of same order, then the matrices AB and BA have the same eigenvalues.
11. The eigenvalues of a triangular matrix are equal to the diagonal elements of the matrix.

SOLVED EXAMPLES

1. If $A = \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & -7 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 9 \\ -8 & 2 \end{bmatrix}$, then find the value of $3A + 2B - 4C$.

Solution: We have

$$A = \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & -7 \\ 3 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 9 \\ -8 & 2 \end{bmatrix}$$

Now,

$$3A = \begin{bmatrix} 3 & 18 \\ 6 & 12 \end{bmatrix} \quad (1)$$

$$2B = \begin{bmatrix} 8 & -14 \\ 6 & 10 \end{bmatrix} \quad (2)$$

$$4C = \begin{bmatrix} 4 & 36 \\ -32 & 8 \end{bmatrix} \quad (3)$$

Now calculating Eqs. (1) + (2) − (3), we get

$$3A + 2B - 4C = \begin{bmatrix} 3 & 18 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} 8 & -14 \\ 6 & 10 \end{bmatrix} - \begin{bmatrix} 4 & 36 \\ -32 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3+8-4 & 18-14-36 \\ 6+6+32 & 12+10-8 \end{bmatrix} = \begin{bmatrix} 7 & -32 \\ 44 & 14 \end{bmatrix}$$

2. Find a, b, c and d such that $\begin{bmatrix} a-b & 2c+d \\ 2a-b & 2a+d \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}$.

Solution: We know that the corresponding elements of two equal matrices are equal.

Therefore,

$$a - b = 5 \quad (1)$$

$$2a - b = 12 \quad (2)$$

$$2c + d = 3 \quad (3)$$

$$2a + d = 15 \quad (4)$$

Subtracting Eq. (1) from Eq. (2), we get

$$a = 7$$

$$\Rightarrow 7 - b = 5 \Rightarrow b = 2$$

Substituting the value of a in Eq. (4), we get

$$14 + d = 15 \Rightarrow d = 1$$

Substituting the value of d in Eq. (3), we get

$$2c + 1 = 3 \Rightarrow c = 1$$

Hence, $a = 7, b = 2, c = 1$ and $d = 1$.

3. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find the value of A^2.

Solution: We have

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

Now,

$$A^2 = A \cdot A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

4. If $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$.

Solution: We have

$$A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 4 \\ -4 & -2 & -8 \\ -6 & -3 & -12 \end{bmatrix}$$

$$(AB)^{\mathrm{T}} = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix}$$

Also,

$$B^T A^T = \begin{bmatrix} -2 & -1 & -4 \end{bmatrix}^T \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix}$$

Hence, it can be seen that $(AB)^T = B^T A^T$.

5. For what value of x, the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$ is singular?

Solution: The matrix A is singular if

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow 1(-6-2) - 1(6-6) + x(-2-6) = 0$$

$$\Rightarrow -8 - 8x = 0$$

$$\Rightarrow x = -1$$

6. Solve the following system of equations using Cramer's rule:

$$5x - 7y + z = 11$$
$$6x - 8y - z = 15$$
$$3x + 2y - 6z = 7$$

Solution: The given set of equations is

$$5x - 7y + z = 11$$
$$6x - 8y - z = 15$$
$$3x + 2y - 6z = 7$$

Now,

$$D = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix}$$

$$= 5(48+2) + 7(-36+3) + 1(12+24)$$

$$= 250 - 231 + 36 = 55$$

$$D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix}$$

$$= 11(48+2) + 7(-90+7) + 1(30+56)$$

$$= 550 - 581 + 86 = 55$$

$$D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix}$$

$$= 5(-90+7) - 11(-36+3) + 1(42-45)$$

$$= -415 + 363 - 3 = -55$$

$$D_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix}$$

$$= 5(-56-30) + 7(42-45) + 11(12+24)$$

$$= -430 - 21 + 396 = -55$$

Now, using Cramer's rule,

$$x = \frac{D_1}{D} = \frac{55}{55} = 1,$$

$$y = \frac{D_2}{D} = \frac{-55}{55} = -1,$$

$$z = \frac{D_3}{D} = \frac{-55}{55} = -1$$

Hence, the solution of the given set of equations is $x = 1$, $y = -1$ and $z = -1$.

7. Find the adjoint of the matrix $A=\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}$.

Solution: Let C_{ij} be the cofactor of A. Then

$$C_{11}=(-1)^{1+1}\begin{vmatrix}\cos\phi & 0\\ 0 & 1\end{vmatrix}=\cos\phi,\ C_{12}$$
$$=(-1)^{1+2}\begin{vmatrix}\sin\phi & 0\\ 0 & 1\end{vmatrix}$$
$$=-\sin\phi,\ C_{13}=(-1)^{1+3}\begin{vmatrix}\sin\phi & \cos\phi\\ 0 & 0\end{vmatrix}=0$$
$$C_{21}=(-1)^{2+1}\begin{vmatrix}-\sin\phi & 0\\ 0 & 1\end{vmatrix}=\sin\phi,\ C_{22}$$
$$=(-1)^{2+2}\begin{vmatrix}\cos\phi & 0\\ 0 & 1\end{vmatrix}$$
$$=\cos\phi,\ C_{23}=(-1)^{2+3}\begin{vmatrix}\cos\phi & -\sin\phi\\ 0 & 0\end{vmatrix}=0$$
$$C_{31}=(-1)^{3+1}\begin{vmatrix}-\sin\phi & 0\\ \cos\phi & 0\end{vmatrix}=0,\ C_{32}$$
$$=(-1)^{3+2}\begin{vmatrix}\cos\phi & 0\\ \sin\phi & 0\end{vmatrix}$$
$$=0,\ C_{33}=(-1)^{3+3}\begin{vmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{vmatrix}=1$$

Therefore,

$$\text{adj } A=\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}^{\mathrm{T}}=\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}$$

8. If $A=\begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}$, then what is the value of A^{-1}.

Solution: We have

$$A=\begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}$$

Now,

$$|A|=\begin{vmatrix}3 & 2\\ 7 & 5\end{vmatrix}=15-14=1\neq 0$$

Hence, A is invertible.

The cofactors of A is given by

$$A_{11}=(-1)^{1+1}\cdot 5=5$$
$$A_{12}=(-1)^{1+2}\cdot 7=-7$$
$$A_{21}=(-1)^{2+1}\cdot 2=-2$$
$$A_{22}=(-1)^{2+2}\cdot 3=3$$

The adjoint of the matrix A is given by

$$\text{adj } A=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}^{T}=\begin{bmatrix}5 & -2\\ -7 & 3\end{bmatrix}$$

Hence, $A^{-1}=\dfrac{1}{1}\begin{bmatrix}5 & -2\\ -7 & 3\end{bmatrix}$.

9. Show that the homogeneous system of equations has a non-trivial solution and find the solution.

$$x-2y+z=0$$
$$x+y-z=0$$
$$3x+6y-5z=0$$

Solution: The given system of equations can be written in the matrix form as follows:

$$\begin{bmatrix}1 & -2 & 1\\ 1 & 1 & -1\\ 3 & 6 & -5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

which is similar to $AX=O$, where $A=\begin{bmatrix}1 & -2 & 1\\ 1 & 1 & -1\\ 3 & 6 & -5\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $O=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

Now,

$$|A|=\begin{vmatrix}1 & -2 & 1\\ 1 & 1 & -1\\ 3 & 6 & -5\end{vmatrix}=1(-5+6)-1(10-6)$$
$$+3(2-1)=0$$

Thus, $|A|=0$ and hence the given system of equations has a non-trivial solution.

Now, to find the solution, we put $z=k$ in the first two equations.

$$x-2y=-k$$
$$x+y=k$$
$$\begin{bmatrix}1 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-k\\ k\end{bmatrix}$$

which is similar to $AX=B$, where $A=\begin{bmatrix}1 & -2\\ 1 & 1\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}-k\\ k\end{bmatrix}$.

Now,

$$|A|=\begin{vmatrix}1 & -2\\ 1 & 1\end{vmatrix}=3\neq 0$$

Hence, A^{-1} exists.

Now, adj $A=\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix}$.

$$A^{-1}=\frac{1}{3}\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix}$$

Now, $X=A^{-1}B\Rightarrow\begin{bmatrix}x\\ y\end{bmatrix}=\dfrac{1}{3}\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix}\begin{bmatrix}-k\\ k\end{bmatrix}=\begin{bmatrix}k/3\\ 2k/3\end{bmatrix}$

$$\Rightarrow x=k/3, y=2k/3$$

Hence, $x=k/3, y=2k/3$ and $z=k$, where k is any real number that satisfies the given set of equations.

10. Check the following system of equations for consistency.

$$4x - 2y = 3$$
$$6x - 3y = 5$$

Solution: The given system of equations can be written as

$AX = B$, where $A = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix} = -12 + 12 = 0$

So, the given system of equations is inconsistent or it has infinitely many solutions according to $(\text{adj}\, A)B \neq 0$ or $(\text{adj}\, A)B = 0$, respectively.

The cofactors can be calculated as follows:

$$C_{11} = (-1)^{1+1}(-3) = -3$$
$$C_{12} = (-1)^{1+2}(6) = -6$$
$$C_{21} = (-1)^{2+1}(-2) = 2$$
$$C_{22} = (-1)^{2+2}(4) = 4$$
$$\text{adj}\, A = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$$

Thus,

$$(\text{adj}\, A)B = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9+10 \\ -18+20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq 0$$

Hence, the given system of equations is inconsistent.

PRACTICE EXERCISE

1. What is the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 5 & 1 \\ 2 & 4 & 8 & 0 \\ 3 & 1 & 7 & 5 \end{bmatrix}$?

(a) 1 (b) 2
(c) 3 (d) 4

2. What is the rank of the matrix $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$?

(a) 1 (b) 2
(c) 3 (d) 4

3. What is the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$?

(a) 1 (b) 2
(c) 3 (d) 4

4. What is the rank of the matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix}$?

(a) 0 (b) 1
(c) 2 (d) 3

5. If $\begin{bmatrix} a+b & 3 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 3 \end{bmatrix}$, then what are the values of a and b?

(a) (2, 1) or (1, 2) (b) (2, 4) or (4, 2)
(c) (0, 3) or (3, 0) (d) (1, 3) or (3, 1)

6. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 7 & 1 \end{bmatrix}$, then what is the value of $4A - 3B$?

(a) $\begin{bmatrix} -5 & -20 \\ -9 & 17 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 20 \\ 9 & -17 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & -2 \\ -4 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -5 & 20 \\ 9 & 17 \end{bmatrix}$

7. What is the value of

$$\sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} + \cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}?$$

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \sin\theta\cos\theta & \sin\theta + \cos\theta \\ \sin\theta - \cos\theta & \sin\theta\cos\theta \end{bmatrix}$

(d) $\begin{bmatrix} \sin\theta\cos\theta & 0 \\ 0 & \sin\theta + \cos\theta \end{bmatrix}$

8. If $B = \begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 1 \\ 6 & 8 \end{bmatrix}$, and $2A + 3B - 6C = 0$, then what is the value of A?

(a) $\begin{bmatrix} 21/2 & 27/2 \\ -15/2 & 45/2 \end{bmatrix}$ (b) $\begin{bmatrix} 21/4 & 27/4 \\ -15/4 & 45/4 \end{bmatrix}$

(c) $\begin{bmatrix} 21/2 & -15/2 \\ 27/2 & 45/2 \end{bmatrix}$ (d) $\begin{bmatrix} 21/4 & -15/4 \\ 27/4 & 45/4 \end{bmatrix}$

9. Find A and B, if

$$A + B = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}.$$

(a) $A = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}$

(c) $A = \begin{bmatrix} 5 & 2 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 3 & 3 \\ 3 & 7 \end{bmatrix}$

(d) $A = \begin{bmatrix} 10 & 5 \\ 5 & 8 \end{bmatrix}, B = \begin{bmatrix} 4 & 4 \\ 3 & 5 \end{bmatrix}$

10. For what values of λ, the given set of equations has a unique solution?

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = 9$$

(a) $\lambda = 15$
(b) $\lambda = 5$
(c) For all values except $\lambda = 15$
(d) For all values except $\lambda = 5$

11. For what values of λ, the given set of equations has a unique solution?

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + \lambda z = 6$$

(a) 5 (b) 7
(c) 9 (d) 0

12. How many solutions does the following system of equations have?

$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$

(a) One solution
(b) Infinite solutions
(c) No solutions
(d) None of the above

13. If $A = \begin{bmatrix} 1 & 2 & -7 \\ 3 & 1 & 5 \\ 4 & 7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 & 1 \\ 4 & 8 & 5 \\ 1 & 2 & 6 \end{bmatrix}$, then what is the value of $(A \times B)$?

(a) $\begin{bmatrix} 4 & -3 & -31 \\ 18 & 3 & 38 \\ 41 & 38 & 45 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -3 & -38 \\ -3 & 3 & 38 \\ -31 & 31 & 45 \end{bmatrix}$

(c) $\begin{bmatrix} 11 & -3 & -31 \\ 18 & 9 & 18 \\ 41 & 38 & 35 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 3 & -31 \\ 18 & -3 & 38 \\ 45 & 38 & 41 \end{bmatrix}$

14. If $A = \begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 6 & 16 \end{bmatrix}$, then what is the value of k for which $A^2 = B$?

(a) -1 (b) -2
(c) 1 (d) 2

15. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then what is the value of k for which $A^2 = 8A + kI$?

(a) 7 (b) -7
(c) 10 (d) 8

16. If $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, then what is the value of A?

(a) $\begin{bmatrix} 3 & 4 & 0 \\ 1 & 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 & 0 \\ 1 & -2 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 4 & 0 \end{bmatrix}$

17. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix such that $AA^T = 9I_3$, then what are the values of a and b?

(a) $a = -1,\ b = -2$
(b) $a = -2,\ b = -1$
(c) $a = 1,\ b = 2$
(d) $a = 2,\ b = 1$

18. If $A = \begin{bmatrix} 8 & 4 & 6 \\ 4 & 0 & 2 \\ x & 6 & 0 \end{bmatrix}$ is singular, then what is the value of x?

(a) 12 (b) 8
(c) 4 (d) 1

19. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then what is the value of A^{-1}?

(a) $\frac{1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$ (b) $\frac{1}{29}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$

(c) $\frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ (d) $\frac{1}{29}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

20. What is the value of I^T, where I is an identity matrix of order 3?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

21. If $A = \begin{bmatrix} 1 & 7 & -1 \\ 3 & 2 & 2 \\ 4 & 5 & 1 \end{bmatrix}$, then what is the first row of A^T?

(a) [1 7 −1] (b) [1 3 4]
(c) [3 2 2] (d) [4 5 1]

22. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then what is the value of A^{-1}?

(a) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -1 & -1 \\ -3 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & -3 & 7 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

23. Calculate the adjoint of the matrix $A = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 9 & 4 \\ 1 & 2 & 8 \end{bmatrix}$.

(a) $\begin{bmatrix} 64 & 28 & 2 \\ 12 & 62 & 18 \\ 6 & 36 & -64 \end{bmatrix}$ (b) $\begin{bmatrix} 62 & -28 & -2 \\ -12 & 62 & -38 \\ -2 & -12 & -62 \end{bmatrix}$

(c) $\begin{bmatrix} 64 & -28 & -2 \\ -12 & 62 & -28 \\ -5 & -12 & 64 \end{bmatrix}$ (d) $\begin{bmatrix} 64 & 28 & 24 \\ 10 & 62 & 48 \\ 6 & 36 & -64 \end{bmatrix}$

24. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then what is the value of B such that $AB = C$?

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$

25. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$ and if $ABC = I_2$, then what is the value of C?

(a) $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

26. What is the value of $(AB)^{-1}B$?

(a) A^{-1} (b) B
(c) A (d) $A^{-1}B^{-1}$

27. If $A = \begin{bmatrix} x & 2 & 0 \\ 2 & 0 & 1 \\ 6 & 3 & 0 \end{bmatrix}$ is singular, then what is the value of x?

(a) 0 (b) 2
(c) 4 (d) 6

28. What is the nullity of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$?

(a) 0 (b) 1
(c) 2 (d) 3

29. What are the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$?

(a) 1, 4 (b) 2, 3
(c) 0, 5 (d) 1, 5

30. What are the eigenvalues of $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$?

(a) 1, 4, 4 (b) 1, 4, − 4
(c) 3, 3, 3 (d) 1, 2, 6

31. What is the eigenvector of the matrix $A = \begin{bmatrix} 5 & -4 \\ -1 & 2 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

32. What are the eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

33. What are the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$?

(a) $\sqrt{2}, -\sqrt{2}, 1$ (b) $i, -i, 1$

(c) $2, -2, 1$ (d) $0, \frac{1}{2}, \frac{1}{2}$

34. What are the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$?

(a) 1, 2, 3 (b) 4, 4, 5
(c) 3, 5, 6 (d) 3, 3, 7

35. Which one of the following options is not the eigenvector of matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

36. What are the eigenvectors of the matrix $A = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} k \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$

(c) $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2k \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2k \\ k \end{bmatrix}$

37. What is the sum of eigenvalues of $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 4 \\ 1 & 6 & 2 \end{bmatrix}$?

(a) 8 (b) 10
(c) 4 (d) 5

38. What is the value of x and y if $A = \begin{bmatrix} x & y \\ -4 & 10 \end{bmatrix}$ and eigenvalues of A are 4 and 8?

(a) $x = 3, y = 2$ (b) $x = 2, y = 4$
(c) $x = 4, y = 2$ (d) $x = 2, y = 3$

39. What are the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$?

(a) $1, -1$ (b) $1, i$
(c) $i, -i$ (d) $0, 1$

40. What are the values of x and y, if $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & x & 0 \\ 3 & 6 & y \end{bmatrix}$ and eigenvalues of A are 3, 4 and 1?

(a) $x = 2, y = 2$ (b) $x = 3, y = 2$
(c) $x = 2, y = 3$ (d) $x = 4, y = 1$

ANSWERS

1. (c)	**8.** (c)	**15.** (b)	**22.** (a)	**29.** (c)	**36.** (b)
2. (b)	**9.** (a)	**16.** (c)	**23.** (c)	**30.** (a)	**37.** (a)
3. (b)	**10.** (d)	**17.** (b)	**24.** (b)	**31.** (b)	**38.** (d)
4. (d)	**11.** (b)	**18.** (a)	**25.** (d)	**32.** (c)	**39.** (c)
5. (d)	**12.** (c)	**19.** (c)	**26.** (a)	**33.** (a)	**40.** (d)
6. (a)	**13.** (a)	**20.** (d)	**27.** (c)	**34.** (b)	
7. (b)	**14.** (d)	**21.** (b)	**28.** (b)	**35.** (d)	

EXPLANATIONS AND HINTS

1. (c) Matrix $A = \begin{vmatrix} 1 & 3 & 5 & 1 \\ 2 & 4 & 8 & 0 \\ 3 & 1 & 7 & 5 \end{vmatrix}$

Maximum possible rank = 3

Now, consider 3 × 3 minors

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 4 & 8 \\ 3 & 1 & 7 \end{vmatrix} = (28-8) - 2\ (21-5) + 3\ (24-20)$$

$$= 20 - 32 + 12 = 0$$

$$\begin{vmatrix} 3 & 5 & 1 \\ 4 & 8 & 0 \\ 1 & 7 & 5 \end{vmatrix} = 3(40 - 0) - 4(25 - 7) + 1(0 - 8)$$

$$= 120 - 72 - 8 = 40 \neq 0$$

Hence, rank of $A = 3$.

2. (b) Matrix $A = \begin{vmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{vmatrix}$

Maximum possible rank = 3

Now, consider 3 × 3 minors

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 3 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

and
$$\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Now, because all 3 × 3 minors are zero, let us consider 2 × 2 minors

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

Hence, rank of A is 2.

3. (b) Matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Maximum rank = 3

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = (-1-0) - 1(1-1) + 1(0+1)$$

$$= -1 + 1 = 0$$

Hence, rank ≠ 3.

Now, let us consider 2 × 2 minors

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1-1) = -2 \neq 0$$

Hence, rank of A is 2.

4. (d) Matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix}$

Maximum rank = 3

$$\begin{bmatrix} 4 & 2 & 3 \\ 1 & 0 & 0 \\ 4 & 0 & 3 \end{bmatrix} = 4(0) - 1(6-0) + 4(0) = -6 \neq 0$$

Hence, rank of $A = 3$.

5. (d) Since both the matrices are equal

$$\begin{bmatrix} a+b & 3 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 3 \end{bmatrix}$$

$$\Rightarrow a + b = 4 \qquad (1)$$

$$ab = 3 \qquad (2)$$

From Eq. (1), we have

$$a = 4 - b$$

Substituting the value of a in Eq. (2), we get

$$(4 - b)\ b = 3$$

$$\Rightarrow 4b - b^2 = 3$$

$$\Rightarrow b^2 - 4b + 3 = 0$$

$$\Rightarrow (b - 3)\ (b - 1) = 0$$

$$\Rightarrow b = 3, 1$$

For values of b, $a = 4 - 3, 4 - 1 = 1, 3$

Therefore, the values of a and $b = (1, 3)$ or $(3, 1)$

6. (a) $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 7 & 1 \end{bmatrix}$

$$4A = \begin{bmatrix} 4 & -8 \\ 12 & 20 \end{bmatrix}, \quad 3B = \begin{bmatrix} 9 & 12 \\ 21 & 3 \end{bmatrix}$$

$$4A - 3B = \begin{bmatrix} 4-9 & -8-12 \\ 12-21 & 20-3 \end{bmatrix} = \begin{bmatrix} -5 & -20 \\ -9 & 17 \end{bmatrix}$$

7. (b) We have

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

8. (c) We have

$$2A + 3B - 6C = 0$$

$$A = \frac{1}{2}(6C - 3B)$$

Also, $B = \begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 \\ 6 & 8 \end{bmatrix}$

$$A = \frac{1}{2}\left(6\begin{bmatrix} 4 & 1 \\ 6 & 8 \end{bmatrix} - 3\begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 24 & 6 \\ 36 & 48 \end{bmatrix} - \begin{bmatrix} 3 & 21 \\ 9 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 21 & -15 \\ 27 & 45 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 21/2 & -15/2 \\ 27/2 & 45/2 \end{bmatrix}$$

9. (a) We have

$$A + B = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\therefore (A + B) + (A - B) = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} + \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$$

$$2A = \begin{bmatrix} 14 & 6 \\ 10 & 16 \end{bmatrix} \Rightarrow A = \frac{1}{2}\begin{bmatrix} 14 & 6 \\ 10 & 16 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}$$

Also, $(A + B) - (A - B) = \begin{bmatrix} 8 & 5 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$

$$2B = \begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix} \Rightarrow B = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Thus, $A = \begin{bmatrix} 7 & 3 \\ 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

10. (d) The given set of equations can be written as

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 9 \end{bmatrix}$$

The system will have a unique solution if the rank of coefficient matrix is 3.

Thus,

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 2(3\lambda + 6) - 7(3\lambda - 15) + 2(-6 - 15)$$

$$= 6\lambda + 12 - 21\lambda + 105 - 12 - 30$$

$$= -15\lambda + 75$$

$$= 15(5 - \lambda)$$

For rank = 3,

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} \neq 0$$

$\therefore 15(5 - \lambda) \neq 0$

$\lambda \neq 5$

11. (b) The given set of equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Thus, $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} = 1(2k - 12) - 1(k - 4) + 1(3 - 2)$

$$= 2k - 12 - k + 4 + 1$$

$$= k - 7$$

Now, for a system to be unique

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} \neq 0$$

$$\Rightarrow k - 7 \neq 0$$

$$\Rightarrow k \neq 7$$

Thus, the value of k for which the given set of equations does not have a unique solution is 7.

12. (c) The given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$$

Now, $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1(1 - 2) - 2\ (2 - 1) + 1(4 - 1)$

$$= -1 - 2 + 3 = 0$$

Hence, rank of matrix is not 3.

Now, taking a minor from the matrix

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3 \neq 0$$

Hence, rank of matrix = 2.

Now, rank of matrix is less than the number of variables.

Hence, the system is inconsistent or has no solution.

13. (a) We have

$$A = \begin{bmatrix} 1 & 2 & -7 \\ 3 & 1 & 5 \\ 4 & 7 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -5 & 1 \\ 4 & 8 & 5 \\ 1 & 2 & 6 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 3 + 8 + (-7) & -5 + 16 - 14 & 1 + 10 - 42 \\ 9 + 4 + 5 & -15 + 8 + 10 & 3 + 5 + 30 \\ 12 + 28 + 1 & -20 + 56 + 2 & 4 + 35 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & -31 \\ 18 & 3 & 38 \\ 41 & 38 & 45 \end{bmatrix}$$

14. (d) We have

$$A = \begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 0 \\ 6 & 16 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} k & 0 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} k^2+0 & 0+0 \\ k+4 & 0+16 \end{bmatrix} = \begin{bmatrix} k^2 & 0 \\ k+4 & 16 \end{bmatrix}$$

Now, since $A^2 = B$

$$\begin{bmatrix} k^2 & 0 \\ k+4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & 16 \end{bmatrix}$$

$$k^2 = 4 \text{ and } k + 4 = 6$$

$$k = \pm 2 \text{ and } k = 2$$

Hence, value of $k = 2$.

15. (b) We have

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0 \\ -1-7 & 0+49 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

$$= 8A = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix}$$

$$kI = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Now,

$$A^2 = 8A + kI$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

or $\quad 8 + k = 1 \quad \text{and} \quad 56 + k = 49$

$$\Rightarrow k = -7$$

16. (c) As product is a 3 × 3 matrix and one of the matrix is 3 × 2, the order of A is 2 × 3.

Consider $A = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$, then

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 - x_2 & 2y_1 - y_2 & 2z_1 - z_2 \\ x_1 & y_1 & z_1 \\ -3x_1 + 4x_2 & -3y_1 + 4y_2 & -3z_1 + 4z_2 \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

Now, $2x_1 - x_2 = -1$, $x_1 = 1$

$$\Rightarrow 2 - x_2 = -1$$

$$\Rightarrow x_2 = 3$$

Also, $2y_1 - y_2 = -8$, $y_1 = -2$

$$\Rightarrow -4 - y_2 = -8$$

$$\Rightarrow y_2 = 4$$

Also, $2z_1 - z_2 = -10$, $z_1 = -5$

$$\Rightarrow z_2 = 0$$

Thus, $A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$.

17. (b) We have

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$AA^T = 9I_3$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a+2-2b \\ a+2b+4 & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$a + 2b + 4 = 0 \quad \Rightarrow a + 2b = -4$$

$$2a + 2 - 2b = 0 \quad \Rightarrow a - b = -1$$

Solving above equations,

$$3b = -3 \Rightarrow b = -1$$

$$a = -4 + 2 = -2$$

Hence, $a = -2$ and $b = -1$.

18. (a) We know that $A = \begin{bmatrix} 8 & 4 & 6 \\ 4 & 0 & 2 \\ x & 6 & 0 \end{bmatrix}$ is singular.

Hence, $\begin{vmatrix} 8 & 4 & 0 \\ 4 & 0 & 2 \\ x & 6 & 0 \end{vmatrix} = 0$

$$8(0 - 12) - 4(0) + x(8 - 0) = 0$$

$$-96 + 8x = 0$$

$$8x = 96$$

$$x = 12$$

19. (c) We have

$$A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$|A| = \{[2 \times (-2)] - (3 \times 5)\} = -4 - 15 = -19 \neq 0$$

Hence, A is invertible.

Now, cofactors of the matrix A are

$$C_{11} = -2$$

$$C_{12} = -5$$

$$C_{21} = -3$$

$$C_{22} = 2$$

$$\text{adj}\, A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj A} = \frac{-1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2/19 & 3/19 \\ 5/19 & -2/19 \end{bmatrix}$$

20. (d) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

21. (b) $A = \begin{bmatrix} 1 & 7 & -1 \\ 3 & 2 & 2 \\ 4 & 5 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 5 \\ -1 & 2 & 1 \end{bmatrix}$$

The first row of transport of $A = [1 \quad 3 \quad 4]$.

22. (a) $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$|A| = 1(16 - 9) - 1(12 - 9) + 1(9 - 12)$$
$$= 7 - 3 - 3 = 1 \neq 0$$

Hence, A is invertible.

Now, cofactors of the matrix A are given as

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$$

$$C_{12} = (1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$$

Thus, $\text{adj } A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|}\text{adj}\, A$

$$= \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

23. (c) We have

$$A = \begin{bmatrix} 8 & 4 & 2 \\ 2 & 9 & 4 \\ 1 & 2 & 8 \end{bmatrix}$$

Now, cofactors of the matrix A are given as

$$C_{11} = \begin{vmatrix} 9 & 4 \\ 2 & 8 \end{vmatrix}(-1)^{1+1} = 64$$

$$C_{12} = \begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix}(-1)^{1+2} = -12$$

$$C_{13} = \begin{vmatrix} 2 & 9 \\ 1 & 2 \end{vmatrix}(-1)^{1+3} = -5$$

$$C_{21} = \begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix}(-1)^{2+1} = -28$$

$$C_{22} = \begin{vmatrix} 8 & 2 \\ 1 & 8 \end{vmatrix}(-1)^{2+2} = 62$$

$$C_{23} = \begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix}(-1)^{2+3} = -12$$

$$C_{31} = \begin{vmatrix} 4 & 2 \\ 9 & 4 \end{vmatrix}(-1)^{3+1} = -2$$

$$C_{32} = \begin{vmatrix} 8 & 2 \\ 2 & 4 \end{vmatrix}(-1)^{3+2} = -28$$

$$C_{33} = \begin{vmatrix} 8 & 4 \\ 2 & 9 \end{vmatrix}(-1)^{3+3} = 64$$

Thus,

$$\text{adj } A = \begin{bmatrix} 64 & -12 & -5 \\ -28 & 62 & -12 \\ -2 & -28 & 64 \end{bmatrix}^T = \begin{bmatrix} 64 & -28 & -2 \\ -12 & 62 & -28 \\ -5 & -12 & 64 \end{bmatrix}$$

24. (b) We know

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

Now, $\quad AB = C$

$$B = CA^{-1}$$

Now, we can calculate A^{-1} as follows:

$$|A| = 4 + 2 = 6 \neq 0$$

Hence, A is invertible. Now, cofactor of the matrix A are given as

$$C_{11} = (-1)^{1+1}\ (1) = 1$$
$$C_{12} = (-1)^{2+1}\ (-1) = 1$$
$$C_{21} = (-1)^{2+1}\ (2) = -2$$
$$C_{22} = (-1)^{2+2}\ (4) = 4$$

$$\text{adj}\, A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \times \frac{1}{6}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$$

25. (d) We know that

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

Now, $|A| = (4 - 3) = 1 \neq 0$

$$|B| = (9 - 10) = -1 \neq 0$$

Hence, both A and B are invertible.

Now, $ABC = I$

$$C = A^{-1}B^{-1}I$$

or $C = A^{-1}B^{-1}$

Calculating cofactors of A,

$$C_{A_{11}} = (-1)^{1+1}(2) = 2$$

$$C_{A_{12}} = (-1)^{1+2}(3) = -3$$

$$C_{A_{21}} = (-1)^{2+1}(1) = -1$$

$$C_{A_{22}} = (-1)^{2+2}(2) = 2$$

$$\text{adj}\, A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Calculating cofactors of B,

$$C_{B_{11}} = (-1)^{1+1}(-3) = -3$$

$$C_{B_{12}} = (-1)^{1+2}(5) = -5$$

$$C_{B_{21}} = (-1)^{2+1}(2) = -2$$

$$C_{B_{22}} = (-1)^{2+2}(-3) = -3$$

$$\text{adj}\, B = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|}\text{adj}\, B = \frac{1}{(-1)}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Now,

$$C = A^{-1}B^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & -9+10 \\ 4-3 & -6+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

26. (a) $(AB)^{-1} = (A^{-1}B^{-1})\ B$

$$= A^{-1}[(B^{-1}) \cdot B]$$

$$= \mathrm{A}^{-1}I = A^{-1}$$

27. (c) We know

$$A = \begin{bmatrix} x & 2 & 0 \\ 2 & 0 & 1 \\ 6 & 3 & 0 \end{bmatrix}$$

For A to be singular,

$$|A| = 0$$

$$x(0-3) - 2(0-0) + 6(2) = 0$$

$$-3x + 12 = 0 \Rightarrow x = \frac{-12}{-3}$$

$$\Rightarrow x = 4$$

28. (b) We know

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$|A| = 0(1-0) - 1(-1-0) + (-1)\ (0+1)$$

$$= 0 + 1 - 1 = 0$$

Hence, rank is not 3.

Choosing a minor from A, we get

$$\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0 - 1 = -1 \neq 0$$

Therefore, rank $(A) = 2$

Now, nullity = Number of columns − Rank of matrix

$$= 3 - 2$$

$$= 1$$

29. (c) We know

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

Now,

$$[A - \lambda I] = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - (-2)(-2) = 0$$

$$\lambda^2 - 5\lambda + 4 - 4 = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0, 5$$

Hence, eigenvalues are 0 and 5.

30. (a) We know

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Now,

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(3-\lambda)^2 - 1\right] - (-1)[-(3-\lambda)-1] + (-1)[1+(3-\lambda)] = 0$$

$$[(3-\lambda)+1]\{(3-\lambda)[(3-\lambda)-1] - (1) - 1\} = 0$$

$$(4-\lambda)\left[\lambda^2 - 5\lambda + 4\right] = 0 \Rightarrow (4-\lambda)(\lambda-1)(\lambda-4) = 0$$

$$\lambda = 1, 4, 4$$

31. (b) We have

$$A = \begin{bmatrix} 5 & -4 \\ -1 & 2 \end{bmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & -4 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 6)(\lambda - 1) = 0$$

$$\lambda = 6, 1$$

$\therefore$ Eigenvalues of $A = 1, 6$

Now, using $|A - \lambda|\widehat{X} = 0$

and substituting $\lambda = 1$, we get

$$\begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4X_1 - 4X_2 = 0$$

$$-X_1 + X_2 = 0$$

or $\qquad X_1 = X_2$

Now, the solution is $X_1 = X_2 = k$.

Hence, from the given options, the solution is $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

32. (c) We have

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 1 = 0$$

$$[(1-\lambda)+1][(1-\lambda)-1] = 0$$

$$\Rightarrow (2-\lambda)(-\lambda) = 0$$

$$\lambda = 2, 0$$

Hence, eigenvalues of $A = 2, 0$

Now, using $|A - \lambda I|\widehat{X} = 0$

Putting $\lambda = 0$, we have

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 - x_2 = 0$$

$$-x_1 + x_2 = 0$$

$$x_1 = x_2$$

Putting $\lambda = 2$, we have

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-x_1 - x_2 = 0$$

$$x_1 = -x_2$$

Hence, from the given options, eigenvectors for the corresponding eigenvalues can be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

33. (a) We have

$$A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now,

$$|A - \lambda I| = \begin{vmatrix} 0-\lambda & -1 & 1 \\ -1 & 0-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (0-\lambda)[(0-\lambda)(1-\lambda) - 0] + 1[(\lambda-1) - 1]$$
$$+ 1[0 - (0-\lambda)] = 0$$

$$\Rightarrow -\lambda\left[-\lambda + \lambda^2\right] + 1[\lambda - 2] + 1[\lambda] = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2 + \lambda - 2 + \lambda = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda^2(\lambda-1) - 2(\lambda-1) = 0$$

$$\Rightarrow (\lambda^2 - 2)(\lambda - 1) = 0$$

$$\Rightarrow \left(\lambda + \sqrt{2}\right)\left(\lambda - \sqrt{2}\right)(\lambda - 1) = 0$$

Hence, eigenvalues are $\lambda = \sqrt{2}, -\sqrt{2}, 1$.

34. (b) We have

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Now,

$$(A-\lambda I)=\begin{vmatrix}4-\lambda & 0 & 0\\ 1 & 4-\lambda & 0\\ 0 & 0 & 5-\lambda\end{vmatrix}=0$$

$$\Rightarrow (4-\lambda)[(4-\lambda)(5-\lambda)-0]-1[0-0]+0[0-0]=0$$

$$\Rightarrow (4-\lambda)(4-\lambda)(5-\lambda)=0$$

$\therefore$ Eigenvalues of the matrix are $\lambda = 4, 4, 5$.

35. (d) We have

$$A=\begin{bmatrix}1 & 1 & 0\\ 0 & 2 & 2\\ 0 & 6 & 3\end{bmatrix}$$

The characteristic equation is given by

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda & 1 & 0\\ 0 & 2-\lambda & 2\\ 0 & 0 & 3-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-0]-0[(3-\lambda)(2-\lambda)-0]$$
$$+0[2-0]=0 \Rightarrow (1-\lambda)(2-\lambda)(3-\lambda)=0$$

$$\lambda = 1,\ 2,\ 3$$

Using $|A-\lambda I|\widehat{X}=0$

and putting $\lambda = 1$, we get

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 1 & 2\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$y=0$$
$$y-2z=0$$
$$2z=y$$
$$2z=0$$

Hence, $x=k$, $y=0$, $z=0$.

Therefore, the eigenvector can be of the form $\begin{bmatrix}k\\ 0\\ 0\end{bmatrix}$.

Now putting $\lambda = 2$, we get

$$\begin{bmatrix}-1 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-x+y=0 \qquad \Rightarrow x=y$$
$$2z=0 \qquad \Rightarrow z=0$$
$$z=0$$

Hence, $x=k$, $y=k$, $z=0$.

Therefore, the eigenvector can be of the form $\begin{bmatrix}k\\ k\\ 0\end{bmatrix}$.

Now putting $\lambda = 3$, we get

$$\begin{bmatrix}-2 & 1 & 0\\ 0 & -1 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-2x+y=0 \Rightarrow x=y/2$$
$$-y+2z=0 \Rightarrow y=2z$$

Hence, $x=2k$, $y=k$, $z=2k$.

Therefore, the eigenvector can be of the form $\begin{bmatrix}2k\\ k\\ 2k\end{bmatrix}$.

Hence, the eigenvector which is not of the matrix A is $\begin{bmatrix}1\\ 2\\ 0\end{bmatrix}$.

36. (b) We have

$$A=\begin{bmatrix}2 & 5 & 0\\ 0 & 3 & 0\\ 0 & 1 & 1\end{bmatrix}$$

The characteristic equation is given by

$$|A-\lambda I|=0$$

$$\begin{vmatrix}2-\lambda & 5 & 0\\ 0 & 3-\lambda & 0\\ 0 & 1 & 1-\lambda\end{vmatrix}$$

$$\Rightarrow (2-\lambda)(3-\lambda)(1-\lambda)=0$$

$$\lambda = 1, 2, 3$$

Now, using $|A-\lambda I|\widehat{X}=0$

Putting $\lambda = 1$, we get

$$\begin{bmatrix}1 & 5 & 0\\ 0 & 2 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$x+5y=0$$
$$2y=0$$
$$y=0$$

Thus, the eigenvector can be of the form $\begin{bmatrix}0\\ 0\\ k\end{bmatrix}$.

Putting $\lambda = 2$, we get

$$\begin{bmatrix}0 & 5 & 0\\ 0 & 1 & 0\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$5y=0$$
$$y=0$$
$$y-z=0 \Rightarrow y=z$$

Thus, the eigenvector can be of the form $\begin{bmatrix}k\\ 0\\ 0\end{bmatrix}$.

Putting $\lambda = 3$, we get

$$\begin{bmatrix} -1 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x = 0 \Rightarrow x = 0$$

$$y - 2z = 0 \Rightarrow y = 2z$$

Thus, the eigenvector can be of the form $\begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$.

Hence, the eigenvectors for the corresponding eigenvalues are given as

$$\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ k \\ 2k \end{bmatrix}$$

37. (a) We have

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 4 \\ 1 & 6 & 2 \end{bmatrix}$$

Sum of eigenvalues = sum of diagonal elements of matrix A

$$= 1 + 5 + 2$$

$$= 8$$

38. (d) We know that

$$A = \begin{bmatrix} x & y \\ -4 & 10 \end{bmatrix}$$

Also, we know that the eigenvalues of A are 4 and 8.

Now, sum of eigenvalues = sum of diagonal elements

$$\therefore 4 + 8 = x + 10$$

$$\Rightarrow x = 12 - 10 = 2$$

Also, product of eigenvalues = $|A|$

$$\therefore 4 \times 8 = 10x + 4y$$

$$\Rightarrow 4 \times 8 = 10 \times 2 + 4y$$

$$\Rightarrow 4y = 32 - 20$$

$$\Rightarrow y = \frac{12}{4} = 3$$

Hence, $x = 2$ and $y = 3$.

39. (c) We have

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

$$\Rightarrow \lambda = \pm\, i$$

40. (d) We know

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & x & 0 \\ 3 & 6 & y \end{bmatrix}$$

Now, eigenvalues of A are 1, 3 and 4.

Sum of eigenvalues = Sum of diagonal

$$\therefore 3 + x + y = 1 + 3 + 4$$

$$\Rightarrow x + y = 5$$

Also, product of eigenvalues = |A|

$$\therefore 3(xy - 0) - 5(0 - 0) + 3(0 - 0) = 1 \times 3 \times 4$$

$$\Rightarrow xy = 4$$

Now, $y = 5 - x \Rightarrow (5 - x)x = 4$

$$\Rightarrow 5x - x^2 = 4$$

$$\Rightarrow x^2 - 5x + 4 = 0$$

$$\Rightarrow (x - 4)(x - 1) = 0$$

$$\Rightarrow x = 1, 4$$

$$\therefore y = 4, 1$$

Hence, from the given options

$$x = 4,\ y = 1$$

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. Given a matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, the rank of the matrix is

(a) 4 (b) 3
(c) 2 (d) 1

(GATE 2003, 1 Mark)

Solution: Consider four 3×3 minors because maximum possible rank is 3.

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 4 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

As all 3×3 minors are zero, rank cannot be 3.

We now try 2×2 minors.

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

Hence, rank = 2.

Ans. (c)

2. For the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$, the eigenvalues are

(a) 3 and −3 (b) −3 and −5
(c) 3 and 5 (d) 5 and 0

(GATE 2003, 1 Mark)

Solution: We have

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Now, the characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (4-\lambda)^2 - 1 = 0 \Rightarrow (4-\lambda)^2 - (1)^2 = 0$$

$$\Rightarrow (5-\lambda)(3-\lambda) = 0$$

Therefore, the eigenvalues are given by $\lambda = 3$ and $\lambda = 5$.

Ans. (c)

3. Consider the following system of simultaneous equations:

$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$

This system has

(a) a unique solution
(b) infinite number of solutions
(c) no solution
(d) exactly two solutions

(GATE 2003, 2 Marks)

Solution: The given equations are

$$x + 2y + z = 6$$
$$2x + y + 2z = 6$$
$$x + y + z = 5$$

The given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$$

The augmented matrix is given by $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right]$.

Now, rank of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ can be calculated as

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$
$$= (1)(1-2) - (2)(2-1) + (1)(4-1) = 0$$

Hence, rank (A) is not 3. Taking a minor from the matrix A, we get

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$
$$= 1 - 4 = -3 \neq 0$$

Hence, rank (A) is 2.
Now taking a minor from the augmented matrix, we get

$$\begin{vmatrix} 1 & 2 & 6 \\ 2 & 1 & 6 \\ 1 & 1 & 5 \end{vmatrix}$$
$$= (1)(5-6) - (2)(10-6) + (1)(12-6)$$
$$= -1 - 8 + 6 = -3$$

Hence, the rank of augment matrix is 3.

As rank of coefficient matrix $\neq$ rank of augmented matrix, the system has no solution, i.e. system is inconsistent.

Ans. (c)

4. Consider the following system of linear equations:

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and third columns of the coefficient matrix are linearly dependent.

For how many values of α, does this system of equations have infinitely many solutions?

(a) 0 (b) 1
(c) 2 (d) Infinitely many

(GATE 2003, 2 Marks)

Solution: The given system of equations is

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

The augmented matrix for the given system is

$$\left[\begin{array}{ccc|c} 2 & 1 & -4 & \alpha \\ 4 & 3 & -12 & 5 \\ 1 & 2 & -8 & 7 \end{array}\right]$$

For infinite solutions to exist, the rank of the augmented matrix should be less than the total unknown variables, i.e. 3. Therefore,

$$\begin{vmatrix} 2 & 1 & \alpha \\ 4 & 3 & 5 \\ 1 & 2 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \alpha(8-3) - 5(4-1) + 7(6-4) = 0$$

$$\Rightarrow \alpha = \frac{1}{5}$$

Therefore, there is only one value of α for which infinite solutions exist.

Ans. (b)

5. Real matrices $[A]_{3\times1}$, $[B]_{3\times3}$, $[C]_{3\times5}$, $[D]_{5\times3}$, $[E]_{5\times5}$ and $[F]_{5\times1}$ are given. Matrices $[B]$ and $[E]$ are symmetric.

The following statements are made with respect to these matrices.

1. Matrix product $[F]^T [C]^T [B] [C] [F]$ is a scalar.

2. Matrix product $[D]^T [F] [D]$ is always symmetric.

With reference to the above statements, which one of the following options applies?

(a) Statement 1 is true but statement 2 is false
(b) Statement 1 is false but statement 2 is true
(c) Both the statements are true
(d) Both the statements are false

(GATE 2004, 1 Mark)

Solution: Statement 1 is true as shown below.

$[F]^T$ has a size 1×5

$[C]^T$ has a size 5×3

$[B]$ has a size 3×3

$[C]$ has a size 3×5

$[F]$ has a size 5×1

Thus, $[F]^T [C]^T [B] [C] [F]$ has a size 1×1. Therefore, it is a scalar.

Hence, statement 1 is true.

Consider statement 2: $[D]^T [F] [D]$ is always symmetric.

Now, $[D]^T [F] [D]$ does not exist because $D^T_{3\times5}$, $F_{5\times1}$ and $D_{5\times3}$ are not compatible for multiplication as $D^T_{3\times5}F_{5\times1} = X_{3\times1}$ and $X_{3\times1}\ D_{5\times3}$ do not exist.

Hence, statement 2 is false.

Ans. (b)

6. Let A, B, C and D be $n \times n$ matrices, each with non-zero determinant. If $ABCD = I$, then B^{-1}

(a) is $D^{-1}C^{-1}A^{-1}$
(b) is CDA
(c) is ADC
(d) does not necessarily exist

(GATE 2004, 1 Mark)

Solution: Given that A, B, C and D are $n \times n$ matrices.
Also, $ABCD = I$

$$-ABCDD^{-1}C^{-1} = D^{-1}C^{-1}$$
$$-AB = D^{-1}C^{-1}$$
$$-A^{-1}AB = A^{-1}D^{-1}C^{-1}$$
$$-B = A^{-1}D^{-1}C^{-1}$$

Now,

$$B^{-1} = (A^{-1}D^{-1}C^{-1})^{-1} = (C^{-1})^{-1}(D^{-1})^{-1}(A^{-1})^{-1}$$
$$= CDA$$

Ans. (b)

7. The sum of the eigenvalues of the matrix given below is $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(a) 5 (b) 7
(c) 9 (d) 18

(GATE 2004, 1 Mark)

Solution: Sum of eigenvalues of given matrix = sum of diagonal elements of given matrix = 1 + 5 + 1 = 7.

Ans. (b)

8. For what value of x will the matrix given below become singular?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

(a) 4 (b) 6
(c) 8 (d) 12

(GATE 2004, 2 Marks)

Solution: For singularity of matrix, $\begin{vmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{vmatrix} = 0$

$\Rightarrow 8(0-12) - x(0-2\times 12) = 0 \ \Rightarrow -96 + 24x = 0$

$\Rightarrow x = 4$

Therefore, $x = 4$.

Ans. (a)

9. How many solutions does the following system of linear equations have?

$$-x + 5y = -1$$
$$x - y = 2$$
$$x + 3y = 3$$

(a) infinitely many
(b) two distinct solutions
(c) unique solution
(d) no solution

(GATE 2004, 2 Marks)

Solution: The given equations are

$$-x + 5y = -1$$
$$x - y = 2$$
$$x + 3y = 3$$

The system of equations can be represented as

$$\begin{bmatrix} -1 & 5 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

The augmented matrix is given by $\left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right]$.

Using Gauss elimination on above matrix, we get

$$\left[\begin{array}{cc|c} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+R_1} \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{cc|c} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Rank$[A|B] = 2$(number of non-zero rows in $[A|B]$)

Rank$[A] = 2$(number of non-zero rows in $[A]$)

Rank$[A|B] =$ Rank $[A] = 2 =$ number of variables

Therefore, the system has a unique solution.

Ans. (c)

10. The eigenvalues of the matrix $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$

(a) are 1 and 4
(b) are -1 and 2
(c) are 0 and 5
(d) cannot be determined

(GATE 2004, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

Characteristic equation is given by

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow [(4-\lambda)\times(1-\lambda)] - [(-2)\times(-2)] = 0$$

$$\Rightarrow \lambda^2 - 5\lambda = 0$$

$$\Rightarrow \lambda(\lambda - 5) = 0$$

Hence, $\lambda = 0$ and 5 are the eigenvalues.

Ans. (c)

11. Consider the matrices $X_{(4\times 3)}, Y_{(4\times 3)}$ and $P_{(2\times 3)}$. The order of $[P(X^TY)^{-1}\,P^T]^T$ will be

(a) (2×2) (b) (3×3)
(c) (4×3) (d) (3×4)

(GATE 2005, 1 Mark)

Solution: We know that the order of

$$X \rightarrow 4\times 3, Y \rightarrow 4\times 3 \text{ and } P \rightarrow 2\times 3.$$

Hence, we can calculate the following orders:

$X^T \rightarrow 3\times 4$, $X^TY \rightarrow 3\times 3$ and $(X^TY)^{-1} \rightarrow 3\times 3$

Also, $P^T \rightarrow 3\times 2$

Therefore, $(P(X^TY)^{-1}P^T)^T \rightarrow 2\times 2$ and $P(X^TY)^{-1}$ $P^T \rightarrow (2\times 3)(3\times 3)(3\times 2) \rightarrow 2\times 2$

Ans. (a)

12. Consider a non-homogeneous system of linear equation representing mathematically an over-determined system. Such a system will be

(a) consistent having a unique solution
(b) consistent having many solutions
(c) inconsistent having a unique solution
(d) inconsistent having no solution

(GATE 2005, 1 Mark)

Solution: In an over-determined system having more equations than variables, the three possibilities that still exist are consistent unique, consistent infinite and inconsistent with no solution. Therefore, options (a), (b) and (d) are correct.

Ans. (a), (b), (d)

13. A is a 3×4 real matrix and $Ax = b$ is an inconsistent system of equations. This highest possible rank of A is

(a) 1 (b) 2 (c) 3 (d) 4

(GATE 2005, 1 Mark)

Solution: We know that

$$\text{rank }(A_{m\times n}) \le \min(m, n)$$

Therefore, the highest possible rank is 3.

If the rank of A is 3, then the rank of $[A \mid B]$ would also be 3, which means the system would become consistent. However, it is given that the system is inconsistent. Hence, the maximum rank of A is only 2.

Ans. (b)

14. In the matrix equation $Px = q$, which one of the following options is a necessary condition for the existence of at least one solution for the unknown vector x?

(a) Augmented matrix $[Pq]$ must have the same rank as matrix P
(b) Vector q must have only non-zero elements
(c) Matrix P must be singular
(d) Matrix P must be square

(GATE 2005, 1 Mark)

Solution: Rank $[Pq]$ = Rank $[P]$ is necessary for existence of at least one solution to $Px = q$.

Ans. (a)

15. Given an orthogonal matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

What is the value of $\left[AA^T\right]^{-1}$?

(a) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$

(GATE 2005, 2 Marks)

Solution: For orthogonal matrix, we know that $A \cdot A^T = I$, i.e. identity matrix

Therefore, $(A \cdot A^T)^{-1} = I^{-1} = 1$.

Ans. (c)

16. If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, then top row of R^{-1} is

(a) [5 6 4] (b) [5 −3 1]
(c) [2 0 −1] (d) [2 −1 1/2]

(GATE 2005, 2 Marks)

Solution: We have

$$R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$R^{-1} = \frac{\text{adj}(R)}{|R|}$$

$$\Rightarrow |R| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix} = 1(2+3) - 0(4+2) - 1(6-2)$$

$$= 5 - 4 = 1$$

As we need only the top row of R^{-1}, we need to find only the first row of adj (R).

$$\text{adj}(1, 1) = +\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 2 + 3 = 5$$

$$\text{adj}(1, 2) = +\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = -3$$

$$\text{adj}(1, 3) = +\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = +1$$

Dividing by $|R| = 1$ gives the top row of R^{-1} = [5 −3 1].

Ans. (b)

17. Let $A=\begin{bmatrix}2 & -0.1\\0 & 3\end{bmatrix}$ and $A^{-1}=\begin{bmatrix}\frac{1}{2} & a\\0 & b\end{bmatrix}$, then $(a+b)$ is

(a) $\frac{7}{20}$ (b) $\frac{3}{20}$

(c) $\frac{19}{60}$ (d) $\frac{11}{20}$

(GATE 2005, 2 Marks)

Solution: We know that $[AA^{-1}]=1$

$$\Rightarrow\begin{bmatrix}2 & -0.1\\0 & 3\end{bmatrix}\begin{bmatrix}1/2 & a\\0 & b\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1 & 2a-0.1b\\0 & 3b\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\Rightarrow 2a-0.1b=0\Rightarrow a=0.1b/2 \qquad (1)$$

$$2b=1\Rightarrow b=\frac{1}{3}$$

Now substituting b in equation (1), we get

$$a=\frac{1}{60}$$

So, $a+b=\frac{1}{60}+\frac{1}{30}=\frac{1+20}{60}=\frac{21}{60}=\frac{7}{20}$

Ans. (a)

18. Consider the following system of equations in three real variables x_1, x_2 and x_3:

$$2x_1-x_2+3x_3=1$$
$$3x_1-2x_2+5x_3=2$$
$$-x_1-4x_2+x_3=3$$

This system of equations has

(a) no solution
(b) a unique solution
(c) more than one but a finite number of solutions
(d) infinite number of solutions

(GATE 2005, 2 Marks)

Solution: The equations can be represented as follows:

$$\begin{bmatrix}2 & -1 & 3\\3 & -2 & 5\\-1 & -4 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

The augmented matrix for the given system is

$$\left[\begin{array}{ccc|c}2 & -1 & 3 & 1\\3 & -2 & 5 & 2\\-1 & -4 & 1 & 3\end{array}\right]$$

Now,

$$\begin{vmatrix}2 & -1 & 3\\3 & -2 & 5\\-1 & -4 & 1\end{vmatrix}=(2)(-2+20)-(3)(-1+12)$$
$$+(-1)(-5+6)=36-33-1=2\neq 0$$

Hence, rank $(A)=3$ and rank $(A|B)=3$.

As rank $[A|B]=$ rank $[A]=$ number of variables, the system has a unique solution.

Ans. (b)

19. Which one of the following options is an eigenvector of the matrix $\begin{bmatrix}5 & 0 & 0 & 0\\0 & 5 & 5 & 0\\0 & 0 & 2 & 1\\0 & 0 & 3 & 1\end{bmatrix}$?

(a) $\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}$ (b) $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$

(c) $\begin{bmatrix}1\\0\\0\\-2\end{bmatrix}$ (d) $\begin{bmatrix}1\\-1\\2\\1\end{bmatrix}$

(GATE 2005, 2 Marks)

Solution: We first calculate the eigenvalues by solving characteristic equation $|\mathrm{A}-\lambda I|=0$.

$$|\mathrm{A}-\lambda I|=\begin{vmatrix}5-\lambda & 0 & 0 & 0\\0 & 5-\lambda & 5 & 0\\0 & 0 & 2-\lambda & 1\\0 & 0 & 3 & 1-\lambda\end{vmatrix}=0$$

$$\Rightarrow(5-\lambda)(5-\lambda)[(2-\lambda)(1-\lambda)-3]=0$$

$$\Rightarrow(5-\lambda)(5-\lambda)(\lambda^2-3\lambda-1)=0$$

$$\lambda=5,5,\frac{3\pm\sqrt{13}}{2}$$

Putting $\lambda=5$ in $[A-\lambda I]\widehat{X}=0$, we get

$$\begin{bmatrix}5-5 & 0 & 0 & 0\\0 & 5-5 & 5 & 0\\0 & 0 & 2-5 & 1\\0 & 0 & 3 & 1-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 5 & 0\\0 & 0 & -3 & 1\\0 & 0 & 3 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

which gives the equations

$$5x_3 = 0 \quad (1)$$
$$-3x_3 + x_4 = 0 \quad (2)$$
$$3x_3 - 4x_4 = 0 \quad (3)$$

Solving Eqs. (1)–(3), we get $x_3 = 0$, $x_4 = 0$, and x_1 and x_2 may take any value.

The eigenvector corresponding to $\lambda = 5$ may be written as

$$\widehat{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{bmatrix}$$

where k_1 and k_2 may be any real number. As only option (a) has x_3 and x_4 equal to 0, this is our desired answer.

Ans. (a)

20. For the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigenvalues is equal to -2. Which one of the following options is an eigenvector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

(GATE 2005, 2 Marks)

Solution: As matrix is triangular, the eigenvalues are the diagonal elements themselves, namely $\lambda = 3$, -2 and 1. Corresponding to eigenvalue $\lambda = -2$, let us find the eigenvector using the following equation:

$$[A - \lambda I]\widehat{X} = 0$$

$$\begin{bmatrix} 3-\lambda & -2 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = -2$ in the above equation, we get

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives the equations

$$5x_1 - 2x_2 + 2x_3 = 0 \quad (1)$$
$$x_3 = 0 \quad (2)$$
$$3x_3 = 0 \quad (3)$$

Putting $x_2 = k$ and $x_3 = 0$ in Eq. (1), we get

$$5x_1 - 2(k) + 2 \times 0 = 0$$
$$\Rightarrow x_1 = 2/5k$$

Therefore, eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/5k \\ k \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 : x_2 : x_3 = 2/5k : k : 0 = 2/5 : 1 : 0 = 2 : 5 : 0.$

Therefore, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$ is an eigenvector of matrix A.

Ans. (d)

21. Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$, the eigenvector is

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(GATE 2005, 2 Marks)

Solution: First, find the eigenvalues of $A = \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$.

The characteristic equation is given by $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (-4-\lambda)(3-\lambda) - 8 = 0$$
$$\Rightarrow \lambda^2 + \lambda - 20 = 0$$
$$\Rightarrow (\lambda + 5)(\lambda - 4) = 0$$
$$\Rightarrow \lambda_1 = -5 \text{ and } \lambda_2 = 4$$

Corresponding to $\lambda_1 = -5$, we need to find the eigenvector.

The eigenvalue problem is $[A - \lambda I]\widehat{X} = 0$.

$$\Rightarrow \begin{vmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

Putting $\lambda = -5$, we get

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad (1)$$
$$4x_1 + 8x_2 = 0 \quad (2)$$

As Eqs. (1) and (2) are same, we take

$$x_1 + 2x_2 = 0 \qquad (3)$$

$$x_1 = -2x_2$$

$$x_1 : x_2 = -2 : 1$$

$$\Rightarrow \frac{x_1}{x_2} = -2$$

Now from the given options, we look for the eigenvector having the ratio $-2 : 1$ and find that option (c) is the appropriate one.
So, option (c) is an eigenvector corresponding to $\lambda = -5$.

Ans. (c)

22. What are the eigenvalues of the following 2×2 matrix?

$$\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

(a) -1 and 1 (b) 1 and 6
(c) 2 and 5 (d) 4 and -1

(GATE 2005, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

The characteristic equation of this matrix is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -4 & 5-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) - 4 = 0$$

$$\lambda = 1, 6$$

Therefore, the eigenvalues of A are 1 and 6.

Ans. (b)

23. Consider the system of equations $A_{(n \times n)} x_{(n \times t)} = \lambda_{(n \times I)}$, where λ is a scalar. Let (λ_i, x_i) be an eigen-pair of an eigenvalue and its corresponding eigenvector for real matrix A. Let I be a $(n \times n)$ unit matrix. Which one of the following statements is not correct?

(a) For a homogeneous $n \times n$ system of linear equations, $(A - \lambda I)x = 0$ has a non-trivial solution, the rank of $(A - \lambda I)$ is less than n.
(b) For matrix A^m, m being a positive integer, $(\lambda_i^m, \lambda_i^m)$ will be the eigen-pair for all i.
(c) If $A^T = A^{-1}$, then $-\lambda_i- = 1$ for all i.
(d) If $A^T = A$, then λ_i is real for all i.

(GATE 2005, 2 Marks)

Solution: Although λ_i^m will be the corresponding eigenvalues of A^m, x_i^m need not be corresponding eigenvectors.

Ans. (b)

24. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

(GATE 2006, 1 Mark)

Solution: We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1(-1-0) - 1(1-1) + 1(0+1) = 0$$

Hence, rank is not 3.

Now considering a minor of matrix, we have

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-1-1) = -2 \neq 0$$

Therefore, its rank is the number of non-zero rows in this form.

Hence, rank = 2.

Ans. (c)

25. Solution for the system defined by the set of equations $4y + 3z = 8$, $2x - z = 2$ and $3x + 2y = 5$ is

(a) $x = 0,\ y = 1,\ z = 4/3$
(b) $x = 0,\ y = 1/2,\ z = 2$
(c) $x = 1,\ y = 1/2,\ z = 2$
(d) non-existent

(GATE 2006, 1 Mark)

Solution: The augmented matrix for given system is

$$\left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right] \xrightarrow{\text{Exchange 1st and 2nd row}} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

Now, by Gauss elimination procedure

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right] \xrightarrow{R_3 - \frac{3}{2}R_1} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 2 & 3/2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

Now, considering A.

$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Taking a minor of matrix A, we get

$$\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8 - 0 \neq 0$$

Hence, rank $(A) = 2$.
Now, taking a minor of matrix $(A|B)$.

$$\begin{vmatrix} 0 & -1 & 2 \\ 4 & 3 & 8 \\ 0 & 0 & -2 \end{vmatrix} = -2(0+4) = -8 \neq 0$$

Hence, rank $(A|B) = 3$. Since rank $(A) \neq$ rank $(A|B)$, the system of equations is inconsistent. Therefore, solution is non-existent for above system.

Ans. (d)

26. The multiplication of matrices E and F is G. Matrices E and G are

$$E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What is the matrix F?

(a) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(GATE 2006, 2 Marks)

Solution: We have

$$E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, it is given that

$E \times F = G$

$$\Rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that product of E and F is a unit matrix, so F has to be E^{-1}.

$$F = E^{-1} = \frac{\text{adj}(E)}{|E|} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans. (c)

27. Match List I with List II and select the correct answer using the codes given below the lists:

List I	**List II**
A. Singular matrix	1. Determinant is not defined
B. Non-square matrix	2. Determinant is always one
C. Real symmetric	3. Determinant is zero
D. Orthogonal matrix	4. Eigenvalues are always real
	5. Eigenvalues are not defined

Codes:

	A	B	C	D
(a)	3	1	4	2
(b)	2	3	4	1
(c)	3	2	5	4
(d)	3	4	2	1

(GATE 2006, 2 Marks)

Solution:

A. Singular matrix $\rightarrow$ Determinant is zero

B. Non-square matrix $\rightarrow$ Determinant is not defined

C. Real symmetric $\rightarrow$ Eigenvalues are always real

D. Orthogonal matrix $\rightarrow$ Determinant is always one

Ans. (a)

Common Data for Questions 28 and 29

$$P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T, \ Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T \text{ and } R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T \text{ are three vectors.}$$

28. An orthogonal set of vectors having a span that contains P, Q and R is

(a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

(GATE 2006, 2 Marks)

Solution: We are looking for orthogonal vectors having a span that contains P, Q and R.

Considering option (a), $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

Firstly, these are orthogonal because their dot product is given by

$$(-6 \times 4) + (-3 \times -2) + (6 \times 3) = 0$$

The space spanned by these two vectors is

$$k_1\begin{bmatrix}-6\\-3\\6\end{bmatrix}+k_2\begin{bmatrix}4\\-2\\3\end{bmatrix} \quad (1)$$

The span of $\begin{bmatrix}-6\\-3\\6\end{bmatrix}$ and $\begin{bmatrix}4\\-2\\3\end{bmatrix}$ contains P, Q and R.

We can show this by successively setting Eq. (1) to P, Q and R one by one and solving for k_1 and k_2 uniquely.

Also, the options (b), (c) and (d) are wrong because none of them are orthogonal as can be seen by taking pairwise dot products.

Ans. (a)

29. The following vector is linearly dependent upon the solution to the previous problem:

(a) $\begin{bmatrix}8\\9\\3\end{bmatrix}$ (b) $\begin{bmatrix}-2\\-17\\30\end{bmatrix}$

(c) $\begin{bmatrix}4\\4\\5\end{bmatrix}$ (d) $\begin{bmatrix}13\\2\\-3\end{bmatrix}$

(GATE 2006, 2 Marks)

Solution: We find the linear dependency by finding the determinant of the vectors taken as a matrix.

Considering option (a) and finding the determinant of $\begin{bmatrix}-6&-3&6\\4&-2&3\\8&9&3\end{bmatrix}$, we get

$$\begin{vmatrix}-6&-3&6\\4&-2&3\\8&9&3\end{vmatrix}=-6(-6-27)+3(12-24)$$
$$+6(36+16)\neq 0$$

Considering option (b) and finding the determinant of $\begin{bmatrix}-6&-3&6\\4&-2&3\\-2&-17&30\end{bmatrix}$, we get

$$\begin{vmatrix}-6&-3&6\\4&-2&3\\-2&-17&30\end{vmatrix}=-6(-60+51)+3(120+6)$$
$$+6(-68-4)=0$$

Hence, it is linearly dependent.

Ans. (b)

30. For the matrix $\begin{bmatrix}4&2\\2&4\end{bmatrix}$, the eigenvalue corresponding to the eigenvector $\begin{bmatrix}101\\101\end{bmatrix}$ is

(a) 2 (b) 4 (c) 6 (d) 8

(GATE 2006, 2 Marks)

Solution: We have

$$M=\begin{bmatrix}4&2\\2&4\end{bmatrix}$$

The characteristic equation is given by

$$[M-\lambda I]=\begin{bmatrix}4-\lambda&2\\2&4-\lambda\end{bmatrix}$$

The given eigenvector is $\begin{bmatrix}101\\101\end{bmatrix}$.

$$[M-\lambda I]\widehat{X}=0$$

$$\Rightarrow\begin{bmatrix}4-\lambda&2\\2&4-\lambda\end{bmatrix}\begin{bmatrix}101\\101\end{bmatrix}=0$$

$$\Rightarrow(4-\lambda)(101)+2\times101=0\Rightarrow 404-101\lambda+202=0$$

$$\Rightarrow\lambda=6$$

Ans. (c)

31. For a given matrix $A=\begin{bmatrix}2&-2&3\\-2&-1&6\\1&2&0\end{bmatrix}$, one of the eigenvalues is 3. The other two eigenvalues are

(a) 2, −5 (b) 3, −5
(c) 2, 5 (d) 3, 5

(GATE 2006, 2 Marks)

Solution: We know that

$$\sum\lambda_i=\text{Trace }(A)$$

$$\lambda_1+\lambda_2+\lambda_3=\text{Trace }(A)=2+(-1)+0=1$$

Now $\lambda_1=3$

$$\therefore\ 3+\lambda_2+\lambda_3=1$$
$$\Rightarrow\lambda_2+\lambda_3=-2$$

Only option (b) satisfies the condition.

Ans. (b)

32. The eigenvalues of a matrix $S=\begin{bmatrix}3&2\\2&3\end{bmatrix}$ are 5 and 1. What are the eigenvalues of the matrix $S^2=SS$?

(a) 1 and 25 (b) 6 and 4
(c) 5 and 1 (d) 2 and 10

(GATE 2006, 2 Marks)

Solution: If $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the eigenvalues of A. Then the eigenvalues of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots$. Here, S has eigenvalues 1 and 5.

So, S^2 has eigenvalues 1^2 and 5^2, i.e. 1 and 25.

Ans. (a)

33. The eigenvalues and the corresponding eigenvectors or a 2×2 matrix are given by eigenvalues

$$\lambda_1 = 8 \text{ and } \lambda_2 = 4$$

and eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The matrix is

(a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

(GATE 2006, 2 Marks)

Solution: According to the property of eigenvalues, sum of diagonal elements should be equal to sum of values of λ.

So, $\sum \lambda_i = \lambda_1 + \lambda_2 = 8 + 4 = 12 = \text{Trace } (A)$

Hence, trace $(A) = 12$. Only option (a) satisfies the property.

Ans. (a)

34. $[A]$ is a square matrix which is neither symmetric nor skew-symmetric and $[A]^T$ is its transpose. The sum and difference of these matrices are defined as $[S] = [A] + [A]^T$ and $[D] = [A] - [A]^T$, respectively. Which one of the following statements is true?

(a) Both $[S]$ and $[D]$ are symmetric
(b) Both $[S]$ and $[D]$ are skew-symmetric
(c) $[S]$ is skew-symmetric and $[D]$ is symmetric
(d) $[S]$ is symmetric and $[D]$ is skew-symmetric

(GATE 2007, 1 Mark)

Solution: Since

$$S^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = S$$

$$\Rightarrow S^T = S$$

Therefore, $[S]$ is symmetric.

Since

$$D^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A$$
$$= -(A - A^T) = -D$$

$$\Rightarrow D^T = -D$$

Therefore, $[D]$ is skew-symmetric.

Ans. (d)

35. $X = [x_1, x_2, \ldots, x_n]^T$ is an n-tuple non-zero vector. The $n \times n$ matrix $V = XX^T$

(a) has rank zero
(b) has rank 1
(c) is orthogonal
(d) has rank n

(GATE 2007, 1 Mark)

Solution: If $X = [x_1, x_2, \ldots, x_n]^T$

Now, it is clear that rank $(X) = 1$.

$$\text{rank } (X^T) = \text{rank } (X) = 1$$

Now, $\text{rank } (X^T) \leq (\text{rank } X, \text{ rank } X^T)$

$$\Rightarrow \text{rank}(XX^T) \leq 1$$

So, XX^T has a rank of either 0 or 1.

However, as both X and X^T are non-zero vectors, neither of their ranks can be zero.

Therefore, XX^T has a rank 1.

Ans. (b)

36. The minimum and maximum given values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are -2 and 6, respectively. What is the other eigenvalue?

(a) 5 (b) 3
(c) 1 (d) -1

(GATE 2007, 1 Mark)

Solution: We know that

$$\sum \lambda_1 = \text{Trace}(A)$$
$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1 = 7$$

Now, given that $\lambda_1 = -2$ and $\lambda_2 = 6$.

$$\therefore -2 + 6 + \lambda_3 = 7 \Rightarrow \lambda_3 = 3$$

Ans. (b)

37. If a square matrix A is real and symmetric, then the eigenvalues

(a) are always real
(b) are always real and positive
(c) are always real and non-negative
(d) occur in complex conjugate pairs

(GATE 2007, 1 Mark)

Solution: The eigenvalues of any symmetric matrix are always real.

Ans. (a)

38. The inverse of the 2 × 2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ is

(a) $\frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$ (b) $\frac{1}{3}\begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$ (d) $\frac{1}{3}\begin{bmatrix} -7 & -2 \\ -5 & -1 \end{bmatrix}$

(GATE 2007, 2 Marks)

Solution: Inverse of $A = \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ is given by

$$\frac{1}{|A|}(\text{adj}(A))$$

$$\Rightarrow \frac{1}{(7-10)}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$$

Ans. (a)

39. It is given that $X_1, X_2, \ldots, X_M$ are M non-zero, orthogonal vectors. The dimension of the vector space spanned by the $2M$ vectors $X_1, X_2, \ldots, X_M, -X_1, -X_2, \ldots, -X_M$ is

(a) $2M$
(b) $M+1$
(c) M
(d) dependent on the choice of $X_1, X_2, \ldots, X_M$

(GATE 2007, 2 Marks)

Solution: Since $(X_1, X_2, \ldots, X_M)$ are orthogonal, they span a vector space of dimension M.

Since $(-X_1, -X_2, \ldots, X_M)$ are linearly dependent on $X_1, X_2, \ldots, X_M$, the set $X_1, X_2, \ldots, X_M, -X_1, -X_2, \ldots, -X_M$ will also span a vector space of dimension M only.

Ans. (c)

40. Consider the set of (column) vector defined by $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0$, where $x^T = (x_1, x_2, x_3)^T\}$. Which one of the following options is true?

(a) $\{(1,-1,0)^T, (1,0,-1)^T\}$ is a basis for the subspace X.
(b) $\{(1,-1,0)^T, (1,0,-1)^T\}$ is a linearly independent set, but it does not span X and therefore is not a basis of X.
(c) X is a subspace for R^3
(d) None of the above

(GATE 2007, 2 Marks)

Solution: To be basis for subspace X, two conditions are to be satisfied.

1. The vectors have to be linearly independent.

2. They must span X.

Here, $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0\}$

$$x^T = \{x_1, x_2, x_3\}^T$$

Now, $\{[1, \ -1, \ 0]^T, [1, \ 0, \ -1]^T\}$ is a linearly independent set because it cannot be obtained from another by scalar multiplication. The fact that it is independent can also be established by seeing that the rank of $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ is 2.

Next, we need to check if the set spans X.

Here, $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0\}$

The general infinite solution of X is $\begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$.

Choosing k_1, k_2 as $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix}$ and $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$, we get two linearly independent solutions, for X,

$$X = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} \text{ or } \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix}$$

Now, since both of these can be generated by linear combinations of $[1, \ -1, \ 0]^T$ and $[1, \ 0, \ -1]^T$, the set spans X. Also we have shown that the set is not only linearly independent but also spans X, therefore by definition it is for the subspace X.

Ans. (a)

41. For what values of α and β, the following simultaneous equations have infinite number of solutions?

$$x + y + z = 5; \quad x + 3y + 3z = 9; \quad x + 2y + \alpha z = \beta$$

(a) 2, 7 (b) 3, 8
(c) 8, 3 (d) 7, 2

(GATE 2007, 2 Marks)

Solution: The augmented matrix for this system is given by

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right]$$

Now, for the system to have infinite solutions, rank of the augmented matrix should be less than the total number of unknown variables, i.e. 3.

Hence,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 2 & \alpha \end{vmatrix} = 0$$

$$\Rightarrow (1)(3\alpha - 6) - (1)(\alpha - 2) + (1)(3 - 3) = 0$$
$$\Rightarrow 3\alpha - 6 - \alpha + 2 = 0$$
$$\Rightarrow 2\alpha = 4 \Rightarrow \alpha = 2$$

Also,

$$\begin{vmatrix} 1 & 1 & 5 \\ 1 & 3 & 9 \\ 1 & 2 & \beta \end{vmatrix} = 0$$

$$\Rightarrow (1)(3\beta - 18) - (1)(\beta - 10) + (1)(9 - 15) = 0$$
$$\Rightarrow 3\beta - 18 - \beta + 10 - 6 = 0$$
$$\Rightarrow 2\beta = 14 \Rightarrow \beta = 7$$

Hence, for the system to have infinite solutions, $\alpha = 2$ and $\beta = 7$.

Ans. (a)

42. The number of linearly independent eigenvectors of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) infinite

(GATE 2007, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Eigenvalues can be calculated using the following characteristic equation:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (2-\lambda)^2 = 0$$

$$\Rightarrow \lambda = 2$$

Now, consider the eigenvalue problem

$$|A - \lambda I|\widehat{X} = 0$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 2$, we get

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0 \tag{1}$$

The solution is therefore $x_2 = 0$ and x_1 can take any value

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

As there is only one parameter in infinite solutions, there is only one linearly independent eigenvector for this problem which may be written as $\begin{bmatrix} k \\ 0 \end{bmatrix}$ or as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Ans. (b)

43. The linear operation $L(x)$ is defined by the cross product $L(x) = b \times X$, where $b = [0\ 1\ 0]^T$ and $X = [x_1\ x_2\ x_3]^T$ are three-dimensional vectors. The 3×3 matrix M of this operation satisfies

$$L(x) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then the eigenvalues of M are

(a) 0, + 1, −1 (b) 1, −1, 1
(c) i, $-i$, 1 (d) i, $-i$, 0

(GATE 2007, 2 Marks)

Solution: We have $b = [0\ 1\ 0]^T$ and $X = [x_1\ x_2\ x_3]^T$. The cross product of b and X can be written as

$$b \times x = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix} = x_3\hat{i} + 0\hat{j} - x_1\hat{k}$$
$$= [x_3 \quad 0 \quad x_1]$$

Now, let $L(x) = b \times x = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, where A is a 3×3 matrix.

Let $A = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$. Now, $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b \times x$

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

Comparing L.H.S. and R.H.S., we get

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

So, $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Now, eigenvalues of A are calculated using characteristic equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 0) - 1(0 + \lambda) = 0$$

$$\Rightarrow \lambda^3 + \lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 + \lambda) = 0$$

$$\Rightarrow \lambda = 0, \lambda = \pm 1$$

So, the eigenvalues of A are i, $-i$ and 0.

Ans. (d)

Common Data for Questions 44 and 45 Cayley–Hamilton theorem states that a square matrix satisfies its own characteristic equation. Consider a matrix

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

44. A satisfies the relation

(a) $A + 3I + 2\ A^{-1} = 0$
(b) $A^2 + 2A + 2I = 0$
(c) $(A + 1)\ (A + 2)$
(d) $\exp\ (A) = 0$

(GATE 2007, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

Now,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -3-\lambda & 2 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-3-\lambda)(-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$

A will satisfy this equation according to Cayley–Hamilton theorem,

i.e. $A^2 + 3A + 2I = 0$

Multiplying by A^{-1} on both sides, we get

$$A^{-1}A^2 + 3A^{-1}A + 2A^{-1}I = 0$$

$$\Rightarrow A + 3I + 2A^{-1} = 0$$

Ans. (a)

45. A^9 equals

(a) 511 A + 510 I (b) 309 A + 104 I
(c) 154 A + 155 I (d) exp $(9A)$

(GATE 2007, 2 Marks)

Solution: In the previous question, we derived $A^2 + 3A + 2I = 0$.

$$\Rightarrow A^2 = -3A - 21$$

$$\begin{aligned} A^4 &= A^2 \times A^2 \\ &= (-3A - 2I)(-3A - 2I) \\ &= 9A^2 + 12A + 4I = 9(-3A - 2I) + 12A + 4I \\ &= -15A - 14I \end{aligned}$$

$$\begin{aligned} A^8 &= A^4 \times A^4 \\ &= (-15A - 141)(-15A - 141) \\ &= 225A^2 + 420A + 156I = 225(-3A - 21) + 420A \\ &\quad + 156I \\ &= -225A - 254I \end{aligned}$$

$$\begin{aligned} A^9 &= A \times A^8 \\ &= A(-255A - 254I) \\ &= -255A^2 - 254A = -255(-3A - 2I) - 25 \\ &= 511A + 510I \end{aligned}$$

Hence, $A^9 = 511\text{A} + 510I$.

Ans. (a)

46. The product of matrices $(PQ)^{-1}\ P$ is

(a) P^{-1} (b) Q^{-1}
(c) $P^{-1}\ Q^{-1}\ P$ (d) $PQ\ P^{-1}$

(GATE 2008, 1 Mark)

Solution: We have

$$\begin{aligned} (PQ)^{-1}P &= (Q^{-1}P^{-1})P = (Q^{-1})(P^{-1}P) \\ &= (Q^{-1})(1) = Q^{-1} \end{aligned}$$

Ans. (b)

47. All the four entries of the 2×2 matrix $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ are non-zero, and one of its eigenvalues is zero. Which one of the following statements is true?

(a) $p_{11}p_{22} - p_{12}p_{21} = 1$ (b) $p_{11}p_{22} - p_{12}p_{21} = -1$
(c) $p_{11}p_{22} - p_{12}p_{21} = 0$ (d) $p_{11}p_{22} + p_{12}p_{21} = 0$

(GATE 2008, 1 Mark)

Solution: Since $\Pi\lambda_i = |A|$ and if one of the eigenvalues is zero, then

$$\Pi\lambda_i = |A| = 0$$

Now,

$$|A| = \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} = 0$$

$$\Rightarrow p_{11}p_{22} - p_{12}p_{21} = 0$$

Ans. (c)

48. The characteristic equation of a (3×3) matrix P is defined as

$$a(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$$

If I denotes identity matrix, then the inverse of matrix P will be

(a) $(P^2 + P + 2I)$ (b) $(P^2 + P + 1)$

(c) $(-P^2 + P + 1)$ (d) $-(P^2 + P + 2I)$

(GATE 2008, 1 Mark)

Solution: If characteristic equation is

$$\lambda^3 + \lambda^2 + 2\lambda + 1 = 0$$

By Cayley–Hamilton theorem,

$$P^3 + P^2 + 2P + I = 0$$
$$I = -P^3 - P^2 - 2P$$

Multiplying by P^{-1} on both sides, we get

$$P^{-1} = -P^2 - P - 2I = -(P^2 + P + 2I)$$

Ans. (d)

49. If the rank of a (5×6) matrix Q is 4, then which one of the following statements is correct?

(a) Q will have four linearly independent rows and four linearly independent columns
(b) Q will have four linearly independent rows and five linearly independent columns
(c) QQ^T will be invertible
(d) Q^TQ will be invertible

(GATE 2008, 1 Mark)

Solution: If rank of (5×6) matrix is 4, then surely it must have exactly 4 linearly independent rows as well as 4 linearly independent columns, since rank = row rank = column rank.

Ans. (a)

50. The system of linear equations

$$4x + 2y = 7$$
$$2x + y = 6$$

has

(a) a unique solution
(b) no solution
(c) infinite number of solutions
(d) exactly two distinct solutions

(GATE 2008, 1 Mark)

Solution: The system can be written in the matrix form as

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

The augmented matrix $[A|B]$ is given by $\left[\begin{array}{cc|c} 4 & 2 & 7 \\ 2 & 1 & 6 \end{array}\right]$

Rank of the matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ is given by

$$\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 4 - 4 = 0$$

Hence, the rank of the matrix A is 1.

The rank of the augmented matrix $[A|B]$ is calculated by considering the minor

$$\begin{vmatrix} 2 & 7 \\ 1 & 6 \end{vmatrix} = 12 - 7 = 5 \neq 0$$

Hence, the rank of the augmented matrix is 2.

As rank $[A|B] \neq$ rank$[A]$, the system has no solution.

Ans. (b)

51. The following system of equations:

$$x_1 + x_2 + 2x_3 = 1$$
$$x_1 + 2x_3 + 3x_3 = 2$$
$$x_1 + 4x_2 + ax_3 = 4$$

has a unique solution. The only possible value(s) for a is/are

(a) 0
(b) either 0 or 1
(c) one of 0, 1 or -1
(d) any real number other than 5

(GATE 2008, 1 Mark)

Solution: The equations can be represented as follows:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 4 & a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

The augmented matrix for the above system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 4 & a & 4 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & a-2 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & 0 \end{array}\right]$$

Now as long as $a - 5 \neq 0$, rank(A) = rank$(A|B) = 3$.

Therefore, a can take any real value except 5.

Ans. (d)

52. The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{bmatrix}$ has one eigenvalue equal to 3.

The sum of the two eigenvalue is

(a) p
(b) $p-1$
(c) $p-2$
(d) $p-3$

(GATE 2008, 1 Mark)

Solution: We know that
Sum of the eigenvalues of the matrix = Sum of diagonal values present in the matrix = Trace of matrix

$$\therefore 1+0+p = 3+\lambda_2+\lambda_3$$
$$\Rightarrow p+1 = 3+\lambda_2+\lambda_3$$
$$\Rightarrow \lambda_2+\lambda_3 = p+1-3 = p-2$$

Ans. (c)

53. A is $m \times n$ full rank matrix with $m > n$ and I is the identity matrix. Let matrix $A' = (A^T A)^{-1} A^T$, then which one of the following statements is *true*?

(a) $AA'A = A$ (b) $(AA')^2$
(c) $AA'A = 1$ (d) $AA'A = A'$

(GATE 2008, 2 Marks)

Solution: Since

$$AA'A = A[(A^T A)^{-1} A^T]A = A\left[(A^T A)^{-1} A^T A\right]$$

Let $A^T A = X$

Then $A[X^{-1}X] = A \cdot I = A$

Hence, the correct option is (a).

Ans. (a)

54. The following simultaneous equations:

$$x+y+z=3$$
$$x+2y+3z=4$$
$$x+4y+kz=6$$

will not have a unique solution for k equal to

(a) 0 (b) 5
(c) 6 (d) 7

(GATE 2008, 2 Marks)

Solution: The augmented matrix for given system is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & k & 6 \end{array}\right]$

For the system to have a unique solution, rank of the augmented matrix should be equal to the number of unknown variables. Hence, system will not have a unique solution if rank of augmented matrix is less than 3.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{vmatrix} = 0$$
$$\Rightarrow (1)(2k-12)-(1)(k-4)+(1)(3-2)=0$$
$$\Rightarrow k-12+4+1=0 \ \Rightarrow k=7$$

Therefore, when $k=7$, unique solution is not possible and only infinite solutions are possible.

Ans. (d)

55. For what value of a, if any, will the following system of equations in x, y and z have a solution?

$$2x+3y=4$$
$$x+y+z=4$$
$$x+2y-z=a$$

(a) Any real number
(b) 0
(c) 1
(d) There is no such value

(GATE 2008, 2 Marks)

Solution: The system can be represented as

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ a \end{bmatrix}$$

Augmented matrix is given by $\left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right]$

Performing Gauss elimination on this matrix, we get

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right] \xrightarrow[R_3-\frac{1}{2}R_1]{R_2-\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 0 & -1/2 & 1 & 2 \\ 0 & 1/2 & -1 & a-2 \end{array}\right]$$
$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 0 & -1/2 & 1 & 2 \\ 0 & 0 & 0 & a \end{array}\right]$$

If $a \neq 0$, $r(A)=2$ and $r(A|B)=3$, then the system will have no solutions.

If $a=0$, $r(A)=r(A|B)=2$, then the system will be consistent and will have infinite solutions.

Ans. (b)

56. The eigenvalues of the matrix $P = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$ are

(a) -7 and 8 (b) -6 and 5
(c) 3 and 4 (d) 1 and 2

(GATE 2008, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix}$$

Characteristic equation of A is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 5 \\ 2 & -5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(-5-\lambda) - (2 \times 5) = 0$$

$$\Rightarrow \lambda^2 + \lambda - 30 = 0$$

The eigenvalues are $\lambda = 5$ and -6.

Ans. (b)

57. The eigenvectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the from $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is $a + b$?

(a) 0 (b) 1/2
(c) 1 (d) 2

(GATE 2008, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

Eigenvalues can be calculated using the following characteristic equation:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} (1-\lambda) & 2 \\ 0 & (2-\lambda) \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \lambda = 1 \text{ and } 2$$

Now, since the eigenvalue problem is

$$[A - \lambda I]\widehat{X} = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix}\widehat{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting the value of $\lambda = 1$ and $\widehat{X} = \widehat{X}_1 = \begin{bmatrix} 1 \\ a \end{bmatrix}$, we get

$$\Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 0$$

$$\Rightarrow a = 0 \qquad (1)$$

Putting the value of $\lambda = 2$ and $\widehat{X} = \widehat{X}_2 = \begin{bmatrix} 1 \\ b \end{bmatrix}$, we get

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 0$$

$$\Rightarrow -1 + 2b = 0$$

$$\Rightarrow b = \frac{1}{2} \qquad (2)$$

From Eqs. (1) and (2), we get

$$a + b = 0 + \frac{1}{2} = \frac{1}{2}$$

Ans. (b)

58. How many of the following matrices have an eigenvalue 1?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(a) One (b) Two
(c) Three (d) Four

(GATE 2008, 2 Marks)

Solution: Eigenvalues of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ can be calculated using characteristic equation,

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 0-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) - \lambda = 0$$

$$\lambda = 0, 1$$

Eigenvalues of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is given by

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = 0$$

$$\lambda = 0, 0$$

Eigenvalues of $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is given by

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 + 1 = 0$$

$$(1-\lambda)^2 = -1$$

$$1 - \lambda = i, -i$$

$$\lambda = (1-i),\ (1+i)$$

Eigenvalues of $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$ is given by

$$\begin{bmatrix} -1-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix}$$

$$(-1-\lambda)(-1-\lambda) = 0$$
$$(1+\lambda)^2 = 0$$
$$\lambda = -1, -1$$

So, only one matrix has an eigenvalue of 1 which is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Ans. (a)

59. A square matrix B is skew-symmetric if

(a) $B^T = -B$ (b) $B^T = B$
(c) $B^{-1} = B$ (d) $B^{-1} = B^T$

(GATE 2009, 1 Mark)

Solution: A square matrix B is defined as skew-symmetric if and only if $B^T = -B$, by definition.

Ans. (a)

60. For a matrix $[M] = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix, $[M]^T = [M]^{-1}$. The value of x is given by

(a) $-\frac{4}{5}$ (b) $-\frac{3}{5}$
(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

(GATE 2009, 1 Mark)

Solution: Given that $M^T = M^1$

So, $M^T M = 1$

$$\Rightarrow \begin{bmatrix} \frac{3}{5} & x \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \left(\frac{3}{5}\right)^2 + x^2 & \left(\frac{3}{5}\cdot\frac{4}{5}\right) + \frac{3}{5}x \\ \left(\frac{4}{5}\cdot\frac{3}{5}\right) + \frac{3}{5}\cdot x & \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comparing both sides of a_{12}, we get

$$a_{12} = \left(\frac{3}{5}\right)\left(\frac{4}{5}\right) + \left(\frac{3}{5}\right)x = 0$$

$$\Rightarrow \frac{3}{5}x = -\frac{3}{5}\cdot\frac{4}{5} \Rightarrow x = -\frac{4}{5}$$

Ans. (a)

61. The trace and determinant of a 2 × 2 matrix are known to be −2 and −35, respectively. Its eigenvalues are

(a) −30 and −5 (b) −37 and −1
(c) −7 and 5 (d) 17.5 and −2

(GATE 2009, 1 Mark)

Solution: We know that

$$\text{Trace}(A) = -2$$

Also, $\sum \lambda_1 = \text{Trace}(A)$

Therefore, $\lambda_1 + \lambda_2 = -2$ (1)

Also,

$\Pi\lambda_i = |A| = -35 \Rightarrow \lambda_1\lambda_2 = -35 \Rightarrow \lambda_2 = \dfrac{-35}{\lambda_1}$ (2)

Now substituting λ_2 from Eq. (2) in Eq. (1), we get

$$\lambda_1 + \frac{-35}{\lambda_1} = -2 \Rightarrow \lambda_1^2 + 2\lambda_1 - 35 = 0$$
$$\Rightarrow (\lambda_1 + 7)(\lambda_1 - 5) = 0$$
$$\lambda_1 = -7 \text{ or } 2$$

If $\lambda_1 = -7$, then $\lambda_2 = 5$.

Ans. (c)

62. The eigenvalues of the matrix $\begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ are

(a) 3, 3 + 5j, 6 − j
(b) −6 + 5j, 3 + j, 3 − j
(c) 3 + j, 3 − j, 5 + j
(d) 3, −1 + 3j, −1 − 3j

(GATE 2009, 2 Marks)

Solution: Sum of eigenvalues = Trace(A) = (−1) + (−1) + 3 = 1

So, $\Sigma\lambda_i = 1$.

Only option (d) (3, −1 + 3j, − 1 − 3j) gives $\Sigma\lambda_i = 1$.

Ans. (d)

63. The eigenvalues of a skew-symmetric matrix are

(a) always zero
(b) always pure imaginary

(c) either zero or pure imaginary
(d) always real

(GATE 2010, 1 Mark)

Solution: Eigenvalues of a skew-symmetric matrix are either zero or pure imaginary.

Ans. (c)

64. The inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ is

(a) $\frac{1}{12}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$ (b) $\frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

(c) $\frac{1}{14}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$ (d) $\frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

(GATE 2010, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$$

Therefore,

$$A^{-1} = \begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}^{-1} = \frac{1}{|A|}(\text{adj}(A))$$
$$= \frac{1}{[(3+2i)(3-2i)+i^2]}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$
$$= \frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

Ans. (b)

65. For the set of equations

$$x_1 + 2x_2 + x_3 + 4x_4 = 2$$
$$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$$

the following statement is true:

(a) Only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$ exists
(b) There is no solution
(c) A unique non-trivial solution exists
(d) Multiple non-trivial solutions exist

(GATE 2010, 2 Marks)

Solution: The given equations are

$$x_1 + 2x_2 + x_3 + 4x_4 = 2$$
$$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$$

The augmented matrix is given by

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 3 & 6 & 3 & 12 & 6 \end{array}\right]$$

Performing Gauss elimination on this, we get

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 3 & 6 & 3 & 12 & 6 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{rank}(A) = \text{rank}(A|B) = 1$$

Therefore, the system is consistent. As the system has $\text{rank}(A) = \text{rank}(A|B) = 1$ which is less than the number of variables, only infinite (multiple) non-trivial solutions exist.

Ans. (d)

66. One of the eigenvectors of the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ is

(a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(GATE 2010, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

Eigenvalues of A are calculated using the characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0 \Rightarrow \lambda^2 - 5\lambda + 4 = 0$
$\Rightarrow \lambda = 1, 4$

The eigenvalue is given by $[A - \lambda I]\widehat{X} = 0$

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 1$, we get

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad (1)$$

$$x_1 + 2x_2 = 0 \quad (2)$$

Solution is $x_2 = k$ and $x_1 = -2k$

$$\widehat{X}_1 = \begin{bmatrix} -2k \\ k \end{bmatrix} \Rightarrow x_1 : x_2 = -2 : 1$$

The option (a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in same ratio of $x_1 : x_2$.

Ans. (a)

67. An eigenvector of $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is

(a) $[-1 \quad 1 \quad 1]^T$ (b) $[1 \quad 2 \quad 1]^T$
(c) $[1 \quad -1 \quad 2]^T$ (d) $[2 \quad 1 \quad -1]^T$

(GATE 2010, 2 Marks)

Solution: Given, $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Since P is triangular, the eigenvalues are the diagonal elements themselves. Eigenvalues are therefore $\lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 3$.

Now, the eigenvalue problem is $[A - \lambda]\widehat{X} = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda_1 = 1$, we get the eigenvector corresponding to this eigenvalue

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives the equations

$$x_2 = 0$$
$$x_2 + 2x_3 = 0$$
$$2x_3 = 0$$

The solution is $x_2 = 0$, $x_3 = 0$, and $x_1 = k$.

So, one eigenvector is $\widehat{X} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$.

Hence, $x_1 : x_2 : x_3 = k : 0 : 0$.

We need to find the other eigenvectors corresponding to the other eigenvalues because none of the eigenvectors given in the options matches with this ratio.

Now, corresponding to $\lambda_2 = 2$, we get the following set of equations by substituting $\lambda = 2$ in the eigenvalue problem,

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives

$$-x_1 + x_2 = 0$$
$$2x_3 = 0$$
$$x_3 = 0$$

Hence, the solution is $x_3 = 0, x_1 = k, x_2 = k$.

Therefore, $\widehat{X}_2 = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix} \Rightarrow x_1 : x_2 : x_3 = 1 : 1 : 0$.

As none of the eigenvectors given in the options is of this ratio, we need the third eigenvector also.

By putting $\lambda = 3$ in the eigenvalue problem, we get

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$-2x_1 + x_2 = 0$$
$$-x_2 + 2x_3 = 0$$

Putting $x_1 = k$, we get $x_2 = 2k$ and $x_3 = x_2/2 = k$

Therefore, $\widehat{X}_3 = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} \Rightarrow x_1 : x_2 : x_3 = 1 : 2 : 1$.

Only the eigenvector given in option (b) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is in this ratio.

Ans. (b)

68. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$$

If the eigenvalues of A are 4 and 8, then

(a) $x = 4,\ y = 10$ (b) $x = 5,\ y = 8$
(c) $x = -3,\ y = 9$ (d) $x = -4,\ y = 10$

(GATE 2010, 2 Marks)

Solution: Sum of eigenvalues = Trace $(A) = 2 + y$
Product of eigenvalues $= |A| = 2y - 3x$

Therefore,

$$4 + 8 = 2 + y \quad (1)$$
$$y = 10$$
$$4 \times 8 = 2y - 3x \quad (2)$$

Substituting the value of y in Eq. (2), we get

$$20 - 3x = 32$$
$$\Rightarrow x = -12 / 4 = -3$$

Hence, $x = -4$ and $y = 10$.

Ans. (d)

69. Eigenvalues of a real symmetric matrix are always

(a) positive (b) negative
(c) real (d) complex

(GATE 2011, 1 Mark)

Solution: Eigenvalues of a symmetric matrix are always real.

Ans. (c)

70. Consider the following system of equations:

$$2x_1 + x_2 + x_3 = 0$$
$$x_2 - x_3 = 0$$
$$x_1 + x_2 = 0$$

This system has

(a) a unique solution
(b) no solution
(c) infinite number of solutions
(d) five solutions

(GATE 2011, 2 Marks)

Solution: The system can be represented as

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is given by

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

Now, rank of matrix A is given by

$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$
$$= (2)(0+1) + (1)(-1-1) = 0$$

Hence, rank $(A) \neq 3$. Now, considering a minor from the matrix, we get

$$\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 - 0 = 2 \neq 0$$

Thus, rank $(A) = 2$.

Therefore, the rank of augmented matrix is 2.

Now, rank (A) = rank $(A|B)$ = 2 < number of unknown variables = 3
So, the system has infinite number of solutions.

Ans. (c)

71. The system of equations

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \mu$$

has no solution for values of λ and μ given by

(a) $\lambda = 6$, $\mu = 20$ (b) $\lambda = 6$, $\mu \neq 20$
(c) $\lambda \neq 6$, $\mu = 20$ (d) $\lambda \neq 6$, $\mu \neq 20$

(GATE 2011, 2 Marks)

Solution: The augmented matrix for the system of equations is

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} [A \mid B]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 0 & 0 & \lambda - 6 & \mu - 20 \end{array}\right]$$

If $\lambda = 6$ and $\mu \neq 20$, then rank$(A|B) = 3$ and rank$(A) = 2$, which is the required condition.

Hence, the given system of equations has no solution for $\lambda = 6$ and $\mu \neq 2$.

Ans. (b)

72. Consider the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$$

Which one of the following options provides the correct values of the eigenvalues of the matrix?

(a) 1, 4, 3 (b) 3, 7, 3
(c) 7, 3, 2 (d) 1, 2, 3

(GATE 2011, 2 Marks)

Solution: As the given matrix is upper triangular, its eigenvalues are the diagonal elements themselves, which are 1, 4 and 3. Hence, the correct option is the first one.

Ans. (a)

73. Let A be the 2×2 matrix with elements $a_{11} = a_{12} = a_{21} = +1$ and $a_{22} = -1$. Then the eigenvalues of the matrix A^{19} are

(a) 1024 and -1024

(b) $1024\sqrt{2}$ and $-1024\sqrt{2}$

(c) $4\sqrt{2}$ and $-4\sqrt{2}$

(d) $512\sqrt{2}$ and $-512\sqrt{2}$

(GATE 2012, 1 Mark)

Solution: We have

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Eigenvalues of A are the roots of the characteristic polynomial given as follows:

$$|A-\lambda I| = 0$$
$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(-1-\lambda)-1 = 0$$
$$-(1-\lambda)(1+\lambda)-1 = 0$$
$$\lambda^2 - 2 = 0$$
$$\lambda = \pm\sqrt{2}$$

Eigenvalues of A are $\sqrt{2}$ and $-\sqrt{2}$, respectively.

So, eigenvalues of $A^{19} = \left(\sqrt{2}\right)^{19}, \left(-\sqrt{2}\right)^{19}$

$$= 2^{19/2}, -2^{19/2}$$
$$= 2^9 \cdot 2^{1/2},\ -2^9 \cdot 2^{1/2}$$
$$= 512\sqrt{2},\ -512\sqrt{2}.$$

Ans. (d)

74. The system of algebraic given below has
$$x+2y+z=4$$
$$2x+y+2z=5$$
$$x-y+z=1$$

(a) a unique solution of $x=1$, $y=1$ and $z=1$
(b) only two solutions of ($x=1$, $y=1$, $z=1$) and ($x=2$, $y=1$, $z=0$)
(c) infinite number of solutions
(d) no feasible solution

(GATE 2012, 2 Marks)

Solution: The given system is
$$x+2y+z=4$$
$$2x+y+2z=5$$
$$x-y+z=1$$

Use Gauss elimination method as follows:

The augmented matrix is given by

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 1 & 2 & 5 \\ 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & -3 & 0 & -3 \end{array}\right]$$
$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, rank (A) = rank $(A|B)$ = 2 and the system is consistent.

Now, rank $(A|B)$ is less than the number of unknown variables. Therefore, we have infinite number of solutions.

Ans. (c)

75. For the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, one of the normalized eigenvectors is given as

(a) $\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$

(c) $\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

(GATE 2012, 2 Marks)

Solution: The matrix is given by
$$A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$$

Characteristic equation is given by
$$|A-\lambda I| = 0$$
$$\Rightarrow \begin{vmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (5-\lambda)(3-\lambda) - 3 = 0$$
$$\Rightarrow \lambda^2 - 8\lambda + 12 = 0$$
$$\Rightarrow \lambda = 2, 6$$

Now, to find eigenvectors
$$[A-\lambda I]\widehat{X} = 0$$
$$\Rightarrow \begin{bmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 2$ in the above equation, we get
$$\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow 3x_1 + 3x_2 = 0 \quad (1)$$
$$x_1 + x_2 = 0 \quad (2)$$

Eqs. (1) and (2) are the same equations whose solution is given by
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix}$$

Hence, the eigenvector is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$.

Ans. (b)

76. The eigenvalues of matrix $\begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$ are

(a) -2.42 and 6.86
(b) 3.48 and 13.53
(c) 4.70 and 6.86
(d) 6.86 and 9.50

(GATE 2012, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$$

The characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 9-\lambda & 5 \\ 5 & 8-\lambda \end{vmatrix} = 0$$

$$(9-\lambda)(8-\lambda) - 25 = 0$$

$$\Rightarrow \lambda^2 - 17\lambda + 47 = 0$$

$$\Rightarrow \lambda = \frac{17 \pm \sqrt{289 - 188}}{2}$$

So, eigenvalues are $\lambda = 3.48$ and 13.53.

Ans. (b)

77. Given that

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

the value A^3 is

(a) $15A + 12I$ (b) $19A + 30I$
(c) $17A + 15I$ (d) $17A + 15I$

(GATE 2012, 2 Marks)

Solution: We have $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Characteristic equation of A is given by

$$\begin{vmatrix} -5-\lambda & -3 \\ 2 & 0-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-\lambda) + 6 = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 6 = 0$$

So, $A^2 + 5A + 6I = 0$ (by Cayley–Hamilton theorem)

$$\Rightarrow A^2 = -5A - 6I$$

Multiplying by A on both sides, we have

$$A^3 = -5A^2 - 6A$$

$$\Rightarrow A^3 = -5(-5A - 6I) - 6A = 19A + 30I$$

Ans. (b)

78. There are three matrixes $P(4 \times 2)$, $Q(2 \times 4)$ and $R(4 \times 1)$. The minimum number of multiplication required to compute the matrix PQR is

(GATE 2013, 1 Mark)

Solution: If we multiply QR first, then $Q_{2\times4} \times R_{4\times1}$ will have multiplication number 8.

Therefore, $P_{4\times2}\, QR_{2\times1}$ will have minimum number of multiplication $= (8 + 8) = 16$.

79. The dimension of the null space of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3

(GATE 2013, 1 Mark)

Solution: We know that

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

Order of matrix = 3

Now, $|A| = (-1)(-1-0) + (-1)(0+1) = 0$

Hence, rank $(A) = 2$

Therefore, dimension of null space of $A = 3 - 2 = 1$.

Ans. (b)

80. Which one of the following options does not equal $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y \\ 1 & z & z^2 \end{vmatrix}$?

(a) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$

(GATE 2013, 1 Mark)

Solution: We know that

(i) In a matrix if we add or subtract any constant to any row or column, then the determinant of the matrix remains same.
(ii) In a matrix if we apply any row or column transformation, then also the determinant of the matrix remains same.
(iii) In a matrix if we interchange two rows or columns, then the determinant of the resultant matrix is multiplied by -1.

So by concepts (i) and (ii), matrix in options (b), (c) and (d) gives the same result as our given matrix.
By concept (iii), matrix in option a gives a different result

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -\begin{vmatrix} 1 & x(x+1) & y+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$$

Hence, the answer is (a).

Ans. (a)

81. If the A-matrix of the state space model of a SISO linear time interval system is rank deficient, the transfer function of the system must have

(a) a pole with a positive real part
(b) a pole with a negative real part
(c) a pole with a positive imaginary part
(d) a pole at the origin

(GATE 2013, 1 Mark)

Solution: A pole at the origin.

Ans. (d)

82. Choose the correct set of functions, which are linearly dependent.

(a) $\sin x, \sin^2 x$ and $\cos^2 x$
(b) $\cos x, \sin x$ and $\tan x$
(c) $\cos 2x, \sin^2 x$ and $\cos^2 x$
(d) $\cos 2x, \sin x$ and $\cos x$

(GATE 2013, 1 Mark)

Solution: The correct set of linearly dependent functions is $\cos x, \sin x$ and $\tan x$.

Ans. (b)

83. The equation $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has

(a) no solution
(b) only one solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
(c) non-zero unique solution
(d) multiple solutions

(GATE 2013, 1 Mark)

Solution: The system of equations given is

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 2x_2 = 0$$

$$x_1 - x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

Hence, x_1 and x_2 are having infinite number of solutions or multiple solutions.

Ans. (d)

84. All the eigenvalues of a symmetric matrix are

(a) complex with non-zero positive imaginary part
(b) complex with non-zero negative imaginary part
(c) real
(d) pure imaginary

(GATE 2013, 1 Mark)

Solution: All the eigenvalues of symmetric matrix $[A^T = A]$ are purely real.

Ans. (c)

85. The minimum eigenvalue of the following matrix is

$$\begin{bmatrix} 3 & 5 & 2 \\ 5 & 12 & 7 \\ 2 & 7 & 5 \end{bmatrix}$$

(a) 0 (b) 1
(c) 2 (d) 3

(GATE 2013, 1 Mark)

Solution: Characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 5 & 2 \\ 5 & 12-\lambda & 7 \\ 2 & 7 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)[(12-\lambda)(5-\lambda) - 49] - 5[5(5-\lambda) - 14] + 2[35 - 2(12-\lambda)] = 0$$

$$\Rightarrow (3-\lambda)[60 - 17\lambda + \lambda^2 - 49] - 5(25 - 5\lambda - 14) + 2(35 - 242\lambda) = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 - 17\lambda + 11 - 5)(11 - 5\lambda) + (11 - 2\lambda) = 0$$

$$\Rightarrow 3\lambda^2 - 51\lambda + 33 - \lambda^3 + 17\lambda^2 - 11\lambda - 55 + 25\lambda + 22 + 4\lambda^2 = 0$$

$$-\lambda^3 + 24\lambda^2 - 37\lambda = 0$$

$$\lambda^3 - 24\lambda^2 + 37\lambda = 0$$

$$\lambda(\lambda^2 - 24\lambda + 37) = 0$$

$$\lambda_1 = 0$$

$$\lambda_2, \lambda_3 = \frac{24 \pm \sqrt{(24)^2 - 4(37)}}{2}$$

$$\lambda_2 = 22$$

$$\lambda_3 = 2$$

Hence, $\lambda_{\min} = 0$.

Ans. (a)

86. Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. It is given that determinant $(I_m + AB) =$ determinant$(I_n + BA)$, where I_K is the $k \times k$ identity matrix. Using the above property, the determinant of the matrix given below is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(a) 2 (b) 5
(c) 8 (d) 16

(GATE 2013, 2 Marks)

Solution: Take the determinant of given matrix

$$\begin{aligned} \text{Determinant} &= 2[2(4-1) - 1(2-1) + 1(1-2)] - 1[1(4-1) \\ &\quad - 1(2-1) + 1(1-2)] + 1[1(2-1) - 2(2-1) \\ &\quad + 1(1-1)] - 1[1(1-2) - 2(1-2) + 1(1-1)] \\ &= 2[6-1-1] - 1[3-1-1] + 1[1-2+0] \\ &\quad - 1[-1+2+0] = 2(4) - 1(1) + 1(-1) - 1(1) \\ &= 8 - 1 - 1 - 1 = 5 \end{aligned}$$

Hence, the answer is 5.

Ans. (b)

87. One pair of eigenvectors corresponding to the two eigenvalues of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 \\ -j \end{bmatrix}, \begin{bmatrix} j \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} j \\ 1 \end{bmatrix}$

(GATE 2013, 2 Marks)

Solution: We have

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues are calculated using characteristic equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$\therefore \ \lambda = \pm i$

To find eigenvector,

For $\lambda = +i$

$$\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore,

$$-ix_1 - x_2 = 0$$

$$x_1 - ix_2 = 0$$

Clearly, $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -j \end{bmatrix}$ and $\begin{bmatrix} j \\ 1 \end{bmatrix}$ satisfy

Now, for $\lambda = -i$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore,

$$ix_1 - x_2 = 0$$

$$x_1 + ix_2 = 0$$

Clearly, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} j \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ j \end{bmatrix}$ satisfy

Thus, the two eigenvalues of the matrix are $\begin{bmatrix} 1 \\ -j \end{bmatrix}, \begin{bmatrix} j \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ j \end{bmatrix}, \begin{bmatrix} j \\ 1 \end{bmatrix}$.

Ans. (d)

88. A matrix has eigenvalues -1 and -2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, respectively. The matrix is

(a) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

(GATE 2013, 2 Marks)

Solution: We know that

$$AX = \lambda X$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$a - b = -1 \quad (1)$$

$$c - d = 1 \quad (2)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$a - 2b = -2 \quad (3)$$

$$c - 2d = 4 \quad (4)$$

Subtracting Eq. (1) from Eq. (3), we get

$$b = 1$$

Substituting value of b in Eq. (1), we get

$$a - 1 = -1 \Rightarrow a = 0$$

Subtracting Eq. (2) from Eq. (4), we get

$$-d = 3 \Rightarrow d = -3$$

Substituting value of d in Eq. (2), we get

$$c - (-3) = 1 \Rightarrow c = 4$$

Therefore, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Ans. (d)

89. Consider the following system of equations:

$$3x + 2y = 1$$
$$4x + 7z = 1$$
$$x + y + z = 3$$
$$x - 2y + 7z = 0$$

The number of solutions for this system is ________ .

(GATE 2014, 1 Mark)

Solution: The given equations are

$$3x + 2y = 1$$
$$4x + 7z = 1$$
$$x + y + z = 3$$
$$x - 2y + 7z = 0$$

$$\text{Augmented matrix } \rho(A:B) = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 7 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & -2 & 7 & 0 \end{bmatrix}$$

Applying row operations, the resultant matrix will be

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & 0 & -15 & -21 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3 =$ number of variables. So equations have a unique solution.

Ans. unique

90. The value of dot product of the eigenvectors corresponding to any pair of different eigenvalues of a 4-by-4 symmetric positive definite matrix is ________.

(GATE 2014, 1 Mark)

Solution: The eigenvectors for symmetric positive matrix corresponding to different eigenvalues are orthogonal. Hence, their dot product is zero always.

Ans. 0

91. If the matrix A is such that

$$A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 \end{bmatrix}$$

then the determinant of A is equal to ________.

(GATE 2014, 1 Mark)

Solution: The determinant of the given matrix A is

$$\text{Determinant of } A = \begin{vmatrix} 2 & 18 & 10 \\ -4 & -36 & -20 \\ 7 & 63 & 35 \end{vmatrix}$$

$$= \{2((-36\times 35)-(-20\times 63)) + 4((18\times 35)-(10\times 63)) + 7((-20\times 18)-(-36\times 10))\}$$
$$= \{2((-1260)-(-1260)) + 4((630)-(630)) + 7((-360)-(-360))\}$$
$$= \{2(0)+4(0)+7(0)\}$$
$$= 0$$

Ans. 0

92. Which one of the following statements is TRUE about every $n\times n$ matrix with only real eigenvalues?

(a) If the trace of the matrix is positive and the determinant of the matrix is negative, at least one of its eigenvalues is negative.

(b) If the trace of the matrix is positive, all its eigenvalues are positive.

(c) If the determinant of the matrix is positive, all its eigenvalues are positive.

(d) If the product of the trace and determinant of the matrix is positive, all its eigenvalues are positive.

(GATE 2014, 1 Mark)

Solution: If the trace of the matrix is positive and the determinant of the matrix is negative, then at least one of its eigenvalues is negative.

Determinant of matrices = Product of eigenvalues

Ans. (a)

93. For matrices of same dimension M, N and scalar c, which one of these properties DOES NOT ALWAYS hold?

(a) $(M^T)^T = M$

(b) $(cM)^T = c(M)^T$

(c) $(M + N)^T = M^T + N^T$

(d) $MN = NM$

(GATE 2014, 1 Mark)

Solution: Matrix multiplication is not commutative in general.

Ans. (d)

94. A real (4×4) matrix A satisfies the equation $A^2 = I$, where I is the (4×4) identity matrix. The positive eigenvalue of A is ________.

(GATE 2014, 1 Mark)

Solution: $A^2 = I \Rightarrow A = A^{-1} \Rightarrow$ if λ is an eigenvalue of A, then $\dfrac{1}{\lambda}$ is also its eigenvalue. Since, we require positive eigenvalue, $\lambda = 1$ is the only possibility as no other positive number is self-inversed.

Ans. 1

95. The determinant of matrix A is 5 and the determinant of matrix B is 40. The determinant of matrix AB is ________.

(GATE 2014, 1 Mark)

Solution: $|AB| = |A| \cdot |B| = (5) \cdot (40) = 200$

Ans. 200

96. Given a system of equations:

$$x + 2y + 2z = b_1$$
$$5x + y + 3z = b_2$$

Which of the following is true regarding its solutions?

(a) The system has a unique solution for any given b_1 and b_2
(b) The system will have infinitely many solutions for any given b_1 and b_2
(c) Whether or not a solution exists depends on the given b_1 and b_2
(d) The system would have no solution for any values of b_1 and b_2

(GATE 2014, 1 Mark)

Solution: We are given a set of two equations

$$x + 2y + 2z = b_1$$
$$5x + y + 3z = b_2$$
$$[A / B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 5 & 1 & 3 & b_2 \end{array}\right]$$

Applying Row transformation,

$$R_2 \to R_2 - 5R_1$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -9 & -7 & b_2 - 5b_1 \end{array}\right]$$

Thus, $\text{rank}(A) = \text{rank}(A/B) <$ number of unknowns, for all values of b_1 and b_2.

Therefore, the equations have infinitely many solutions, for any given b_1 and b_2.

Ans. (b)

97. Which one of the following statements is true for all real symmetric matrices?

(a) All the eigenvalues are real.
(b) All the eigenvalues are positive.
(c) All the eigenvalues are distinct.
(d) Sum of all the eigenvalues is zero.

(GATE 2014, 1 Mark)

Solution: Eigenvalues of a real symmetric matrix are all real.

Ans. (a)

98. Two matrices A and B are given below:

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}; \qquad B = \begin{bmatrix} p^2 + q^2 & pr + qs \\ pr + qs & r^2 + s^2 \end{bmatrix}$$

If the rank of matrix A is N, then the rank of matrix B is

(a) $N/2$ (b) $N-1$ (c) N (d) $2N$

(GATE 2014, 1 Mark)

Solution: Rank of a matrix is unaltered by the elementary transformations, i.e. row/column operators. (Here B is obtained from A by applying row/column operations on A).

Since rank of A is N. Therefore, rank of B is also N.

Ans. (c)

99. Given that the determinant of the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is -12, the determinant of the matrix

$\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is

(a) -96 (b) -24 (c) 24 (d) 96

(GATE 2014, 1 Mark)

Solution: Taking factor 2 common from all the three rows, we have

$$\begin{vmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{vmatrix} = (2)^3 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 8 \times -12 = -96$$

Ans. (a)

100. The matrix form of the linear system $\dfrac{dx}{dt} = 3x - 5y$ and $\dfrac{dy}{dt} = 4x + 8y$ is

(a) $\dfrac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(b) $\dfrac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & 4 \\ 8 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(c) $\dfrac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & -5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(d) $\dfrac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 4 & 8 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(GATE 2014, 1 Mark)

Solution: As $\dfrac{dx}{dt} = 3x - 5y$ and $\dfrac{dy}{dt} = 4x + 8y$, the matrix form is $\dfrac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Ans. (a)

101. One of the eigenvectors of the matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ is

(a) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ 9 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(GATE 2014, 1 Mark)

Solution: For the matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$, the eigenvalues are found as follows:

$$\begin{vmatrix} -5 & 2 \\ -9 & 6 \end{vmatrix} - \lambda\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda - 12 = 0 \Rightarrow \lambda = 4, -3$$

For finding eigenvectors

$$\begin{bmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

For $\lambda = 4$, we have the characteristic equation as $-9x_1 + 2x_2 = 0$ and for $\lambda = -3$, we have $-2x_1 + 2x_2 = 0$

So, eigenvector corresponding to eigenvalue 4 is $\begin{bmatrix} 2/9 \\ 9/2 \end{bmatrix}$ and that corresponding to eigenvalue -3 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Ans. (d)

102. Consider a 3×3 real symmetric matrix S such that two of its eigenvalues are $a \neq 0$, $b \neq 0$ with respective eigenvectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. If $a \neq b$ then

$x_1y_1 + x_2y_2 + x_3y_3$ equals

(a) a (b) b (c) ab (d) 0

(GATE 2014, 1 Mark)

Solution: Since the matrix is symmetric with distinct eigenvalues, the eigenvectors of the matrix are orthogonal. So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0 \Rightarrow x_1y_1 + x_2y_2 + x_3y_3 = 0$$

Ans. (d)

103. Which one of the following equations is a correct identity for arbitrary 3 × 3 real matrices P, Q and R?

(a) $(Q + R) = PQ + RP$

(b) $(P - Q)^2 = P^2 - 2PQ + Q^2$

(c) $\det(P + Q) = \det P + \det Q$

(d) $(P + Q)^2 = P^2 + PQ + QP + Q^2$

(GATE 2014, 1 Mark)

Solution: The correct answer is option (d).

Ans. (d)

104. Given the matrices $J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and $K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, the product $K^T JK$ is ________.

(GATE 2014, 1 Mark)

Solution: We are given,

$$J = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix} \text{ and } K = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Now we have to calculate $K^T JK$, we get

$$\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 & 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 23$$

Ans. 23

105. The sum of eigenvalues of the matrix, $[M]$ is

where $[M] = \begin{bmatrix} 215 & 650 & 795 \\ 655 & 150 & 835 \\ 485 & 355 & 550 \end{bmatrix}$

(a) 915 (b) 1355 (c) 1640 (d) 2180

(GATE 2014, 1 Mark)

Solution: Sum of the eigenvalues = Trace of the matrix = 215 + 150 + 550 = 915

Ans. (a)

106. The determinant of matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is ________ .

(GATE 2014, 1 Mark)

Solution: The given matrix is $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$

The determinant is given by

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix}$$

$$= \left[1(0-1) - 2(6-0) + 3(3-0)\right]$$

$$= -3\left[0 - 1(4-3) + 3(0-9)\right] = 88$$

Ans. 88

107. There are 5 bags labeled 1 to 5. All the coins in a given bag have the same weight. Some bags have coins of weight 10 gm, others have coins of weight 11 gm. I pick 1, 2, 4, 8, 16 coins, respectively, from bags 1 to 5. Their total weight comes out to be 323 gm. Then the product of the labels of the bags having 11 gm coins is ________.

(GATE 2014, 2 Marks)

Solution: Let the weight of coins in the respective bags (1 through 5) be a, b, c, d and e – each of which can take one of two values namely 10 or 11 (gm).

Now, the given information on total weight can be expressed as the following equation:

$$a + 2b + 4c + 8d + 16e = 323$$
$$\Rightarrow a \text{ must be odd} \Rightarrow a = 11$$

The equation then becomes

$$11 + 2b + 4c + 8d + 16e = 323$$
$$\Rightarrow 2b + 4c + 8d + 16e = 312$$
$$\Rightarrow b + 2c + 4d + 8e = 156$$
$$\Rightarrow b \text{ must be odd} \Rightarrow b = 10$$

The equation then becomes

$$10 + 2c + 4d + 8e = 156$$
$$\Rightarrow 2c + 4d + 8e = 146$$
$$\Rightarrow c + 2d + 4e = 73$$
$$\Rightarrow c \text{ must be odd} \Rightarrow c = 11$$

The equation now becomes

$$11 + 2d + 4e = 73$$
$$\Rightarrow 2d + 4e = 62$$
$$\Rightarrow d + 2e = 31$$
$$\Rightarrow d = 11 \text{ and } e = 10$$

Therefore, bags labeled 1, 3 and 4 contain 11 gm coins $\Rightarrow$ Required product $= 1 \times 3 \times 4 = 12$.

Ans. 12

108. The product of the non-zero eigenvalues of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is ________.

(GATE 2014, 2 Marks)

Solution: Eigenvector is $AX = \lambda X$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Convert the given matrix in diagonal matrix. The diagonal entries are called eigenvalues.
Eigenvalues = 2, 3
Product of eigenvalues = $2 \times 3 = 6$

Ans. 6

109. Consider the matrix

$$J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is obtained by reversing the order of the columns of the identity matrix I_6. Let $P = I_6 + \alpha J_6$, where α is a non-negative real number. The value of α for which $\det(P) = 0$ is ________.

(GATE 2014, 2 Marks)

Solution: Let $|P| = 1 - \alpha^2$.

$$P = I_2 + \alpha\, J_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$$

$$P = I_4 + \alpha\, J_4 = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & 1 & 0 \\ \alpha & 0 & 0 & 1 \end{bmatrix}$$

Now, let

$$|P| = (1)\begin{vmatrix} 1 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - (\alpha)\begin{vmatrix} 0 & 1 & \alpha \\ 0 & \alpha & 1 \\ \alpha & 0 & 0 \end{vmatrix}$$

$$= (1-\alpha^2) - (\alpha)[\alpha(1-\alpha^2)] = (1-\alpha^2)^2$$

Similarly, if $P = I_6 + \alpha J_6$, then we get

$$|P| = \left(1-\alpha^2\right)^3$$

Therefore

$$|P| = 0 \Rightarrow \alpha = -1, 1$$

Since α is non-negative, so $\alpha = 1$.

Ans. 1

110. The system of linear equations

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 14 \end{pmatrix}$$

has

(a) a unique solution

(b) infinitely many solutions

(c) no solution

(d) exactly two solutions

(GATE 2014, 2 Marks)

Solution:

$$[A/B] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 3 & 0 & 1 & -4 \\ 1 & 2 & 5 & 14 \end{array}\right]$$

Using row transformation,

$$R_2 \to 2R_2 - 3R_1 \text{ and } R_3 \to 2R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 0 & -3 & -7 & -23 \\ 0 & 3 & 7 & 23 \end{array}\right] \xrightarrow{R_3 = R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 0 & -3 & -7 & -23 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since rank (A) = rank (A/B) < number of unkowns, equations have infinitely many solutions.

Ans. (a)

111. The maximum value of the determinant among all 2×2 real symmetric matrices with trace 14 is ______.

(GATE 2014, 2 Marks)

Solution: Let the 2×2 real symmetric matrix be $\begin{bmatrix} y & x \\ x & z \end{bmatrix}$.

Now det $= yz - x^2$ and trace is given as $y + z = 14$

$$\Rightarrow z = 14 - y \qquad (1)$$

Let $f = yz - x^2$ (det) $= -x^2 - y^2 + 14y$, using Eq. (1). Using maxima and minima of a function of two variables, f is maximum at $x = 0$ and $y = 7$. Therefore, maximum value of the determinant is 49.

Ans. 49

112. Which one of the following statements is NOT true for a square matrix A?

(a) If A is upper triangular, the eigenvalues of A are the diagonal elements of it

(b) If A is real symmetric, the eigenvalues of A are always real and positive

(c) If A is real, the eigenvalues of A and A^T are always the same

(d) If all the principal minors of A are positive, all the eigenvalues of A are also positive

(GATE 2014, 2 Marks)

Solution: Consider a real symmetric matrix

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Characteristic equation is

$$|A - \lambda I| = 0 \Rightarrow (1+\lambda)^2 - 1 = 0 \Rightarrow \lambda + 1 = \pm 1$$

Therefore

$$\lambda = 0, -2 \quad \text{(not positive)}$$

Thus, option (b) is not true. Options (a), (c) and (d) are true using properties of eigenvalues.

Ans. (b)

113. A system matrix is given as follows:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$$

The absolute value of the ratio of the maximum eigenvalue to the minimum eigenvalue is ______.

(GATE 2014, 2 Marks)

Solution: The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & -1 \\ -6 & -11-\lambda & 6 \\ -6 & -11 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

So $\lambda = -1, -2$, and -3 are the eigenvalues of A.

Now $\lambda_{max} = -1$ and $\lambda_{min} = -3$. Hence

$$\left|\frac{\lambda_{max}}{\lambda_{min}}\right| = \left|\frac{-1}{-3}\right| = \frac{1}{3}$$

Ans. (1/3)

114. The rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is ________ .

(GATE 2014, 2 Marks)

Solution: We have the matrix,

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$$

Using row transformation,

$$R_2 \to 3R_2 + R_1;\ R_3 \to 6R_3 - 14R_1$$

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ 0 & 42 & 28 & 58 \\ 0 & -84 & -56 & -116 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2$$

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = number of non-zero rows = 2.

Ans. 2

115. A scalar valued function is defined as $f(x) = x^T Ax + b^T x + c$, where A is a symmetric positive definite matrix with dimension $n \times n$; b and x are vectors of dimension $n \times 1$. The minimum value of $f(x)$ will occur when x equals

(a) $(A^T A)^{-1}\ b$ (b) $-(A^T A)^{-1}\ b$

(c) $-\left(\dfrac{A^{-1}b}{2}\right)$ (d) $\dfrac{A^{-1}b}{2}$

(GATE 2014, 2 Marks)

Solution: The minimum value of $f(x)$ will occur when $x = -\left(\dfrac{A^{-1}b}{2}\right)$.

Ans. (c)

116. For the matrix A satisfying the equation given below, the eigenvalues are

$$[A]\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) (1, $-j$, j) (b) (1, 1, 0)

(c) (1, 1, -1) (d) (1, 0, 0)

(GATE 2014, 2 Marks)

Solution:

$$[A]\begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\Rightarrow |A|\begin{vmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$\Rightarrow |A|\begin{vmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{vmatrix} \Rightarrow |A| = -1$$

Product of eigenvalues = -1

Thus, option (c) is correct.

Ans. (c)

117. The larger of the two eigenvalues of the matrix $\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$ is ________.

(GATE 2015, 1 Mark)

Solution: The given matrix is $\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$.

The characteristic equation is given by $|A - \lambda I| = 0$

Hence,

$$\begin{vmatrix} 4-\lambda & 5 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda - 6)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 6, -1$$

Therefore, larger eigenvalue is 6.

Ans. 6

118. In the given matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, one of the eigenvalues is 1. The eigenvectors corresponding to the eigenvalue 1 are

(a) $\{\alpha(4,2,1) \mid \alpha \neq 0, \alpha \in R\}$

(b) $\{\alpha(-4,2,1) \mid \alpha \neq 0, \alpha \in R\}$

(c) $\{\alpha(\sqrt{2},0,1) \mid \alpha \neq 0, \alpha \in R\}$

(d) $\{\alpha(-\sqrt{2},0,1) \mid \alpha \neq 0, \alpha \in R\}$

(GATE 2015, 1 Mark)

Solution: Let X be an eigenvector corresponding to eigenvalue $\lambda = 1$, then

$$AX = \lambda X \Rightarrow (A-\lambda)X = 0$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0$$

$$\Rightarrow -y + 2z = 0 \text{ and } x + 2y = 0$$

$$\Rightarrow y = 2z \text{ and } \frac{x}{-2} = y$$

Therefore,

$$\frac{x}{-2} = y = 2z \Rightarrow \frac{x}{-4} = \frac{y}{2} = \frac{z}{1} = \alpha\,(\text{say})$$

$$\Rightarrow X = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}\alpha;\ \alpha \neq 0$$

Thus, eigenvectors are $\{\alpha(-4,2,1)|\alpha \neq 0,\ \alpha \in R|\}$

Ans. (b)

119. The value of p such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigen-vector of the matrix $\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}$ is ________.

(GATE 2015, 1 Mark)

Solution: We are given,

$$\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}$$

We have,

$$AX = \lambda X \Rightarrow \begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12 \\ p+7 \\ 36 \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \\ 3\lambda \end{bmatrix} \Rightarrow \lambda = 12 \qquad (1)$$

$$2\lambda = p + 7 \qquad (2)$$

and $3\lambda = 36 \Rightarrow \lambda = 12$

Thus, Eq. (2) gives

$$p + 7 = 24 \Rightarrow p = 17$$

Ans. 17

120. Consider the system of linear equations:

$$x - 2y + 3z = -1$$
$$x - 3y + 4z = 1 \text{ and}$$
$$-2x + 4y - 3z = k$$

The value of k for which the system has infinitely many solutions is ________.

(GATE 2015, 1 Mark)

Solution: We are given,

$$x - 2y + 3z = -1$$
$$x - 3y + 4z = 1$$
$$-2x + 4y - 3z = k$$

$$\text{Augmented matrix (A/B)} = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & -3 & 4 & 1 \\ -2 & 4 & -6 & k \end{bmatrix}$$

Using row transformation

$$R_2 \to R_2 - R_1, R_3 \to R_3 + 2R_1$$

We get

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & k-2 \end{bmatrix}$$

The system will have infinitely many solutions if $p(A/B) = p(A) = r <$ number of variables

$$\Rightarrow k - 2 = 0$$
$$\Rightarrow k = 2$$

Ans. 2

121. Two sequences $[a, b, c]$ and $[A, B, C]$ are related as

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w_3^{-1} & w_3^{-2} \\ 1 & w_3^{-2} & w_3^{-4} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where $w_3 = e^{j\frac{2\pi}{3}}$

If another sequence $[p, q, r]$ is derived as

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w_3^{-1} & w_3^{2} \\ 1 & w_3^{2} & w_3^{4} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & w_3^{2} & 0 \\ 0 & 0 & w_3^{4} \end{bmatrix}\begin{bmatrix} A/3 \\ B/3 \\ C/3 \end{bmatrix}$$

then the relationship between the sequences $[p, q, r]$ and $[a, b, c]$ is

(a) $[p,\ q,\ r] = [b,\ a,\ c]$ (b) $[p,\ q,\ r] = [b,\ c,\ a]$
(c) $[p,\ q,\ r] = [c,\ a,\ b]$ (d) $[p,\ q,\ r] = [c,\ b,\ a]$

(GATE 2015, 1 Mark)

Solution: Consider,

$$\begin{bmatrix}1 & 1 & 1\\ 1 & w_3^1 & w_3^2\\ 1 & w_3^2 & w_3^1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & w_3^2 & 0\\ 0 & 0 & w_3^4\end{bmatrix} = \begin{bmatrix}1 & w_3^2 & w_3^4\\ 1 & w_3^3 & w_3^6\\ 1 & w_3^4 & w_3^5\end{bmatrix}$$

Since $w_3^4 = w_3^{(3+1)} = w_3^1; w_3^6 = w_3^3 = w_3^0 = 1; w_3^5 = w_3^2$

$$\frac{1}{3}\begin{bmatrix}1 & w_3^2 & w_3^1\\ 1 & 1 & 1\\ 1 & w_3^1 & w_3^2\end{bmatrix}\begin{bmatrix}A\\ B\\ C\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & w_3^2 & w_3^1\\ 1 & 1 & 1\\ 1 & w_3^1 & w_3^2\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & w_3^{-1} & w_3^{-2}\\ 1 & w_3^{-2} & w_3^{-1}\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}1+w_3^2+w_3^1 & 1+w_3+w_3^{-1} & 1+w_3^0+w_3^0\\ 1+1+1 & 1+w_3^{-1}+w_3^{-2} & 1+w_3^{-2}+w_3^{-1}\\ 1+w_3^1+w_3^2 & 1+w_3^0+w_3^0 & 1+w_3^{-1}+w_3^1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

Since, $w_3 = e^{\frac{j2\pi}{3}}$

$$w_3^2 + w_3^1 = w_3 + w_3^{-1} = w_3^{-1} + w_3^{-2} = -1$$

Thus,

$$\begin{bmatrix}p\\ q\\ r\end{bmatrix} = \frac{1}{3}\begin{bmatrix}0 & 0 & 3\\ 3 & 0 & 0\\ 0 & 3 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}c\\ a\\ b\end{bmatrix}$$

Ans. (c)

122. The value of x for which all the eigenvalues of the matrix given below are real is

$$\begin{bmatrix}10 & 5+j & 4\\ x & 20 & 2\\ 4 & 2 & -10\end{bmatrix}$$

(a) $5 + j$ (b) $5 - j$
(c) $1 - 5j$ (d) $1 + 5j$

(GATE 2015, 1 Mark)

Solution: The matrix is given by,

$$A = \begin{bmatrix}10 & 5+j & 4\\ x & 20 & 2\\ 4 & 2 & -10\end{bmatrix}$$

Given that all eigenvalues of A are real.

Thus, A is Hermitian.

$$A^{+} = A\text{, i.e. } (\bar{A})^T = A$$

$$\begin{bmatrix}10 & \bar{x} & 4\\ 5-j & 20 & 2\\ 4 & 2 & -10\end{bmatrix} = \begin{bmatrix}10 & 5+j & 4\\ x & 20 & 2\\ 4 & 2 & -10\end{bmatrix} \Rightarrow x = 5-j$$

Ans. (b)

123. For $A = \begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$, the determinant of $A^{\mathrm{T}}A^{-1}$ is

(a) $\sec^2 x$ (b) $\cos 4x$
(c) 1 (d) 0

(GATE 2015, 1 Mark)

Solution: We are given,

$$A = \begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$$

$$A^T = \begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

$$|A| = 1 + \tan^2 x = \sec^2 x$$

$$A^{-1} = \frac{1}{\sec^2 x}\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}\frac{1}{\sec^2 x}\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

$$= \frac{1}{\sec^2 x}\begin{bmatrix}1-\tan^2 x & -\tan x - \tan x\\ \tan x + \tan x & -\tan^2 x + 1\end{bmatrix}$$

$$= \frac{1}{\sec^2 x}\begin{bmatrix}1-\tan^2 x & -2\tan^2 x\\ 2\tan x & 1-\tan^2 x\end{bmatrix}$$

$$\left|A^T A^{-1}\right| = \frac{1}{\sec^2 x}[(1-\tan^2 x)^2 + 4\tan^2 x]$$

$$= \frac{1}{\sec^2 x}[1 + \tan^4 x - 2\tan^2 x + 4\tan^2 x]$$

$$= \frac{1}{\sec^2 x}(1 + \tan^4 x + 2\tan^2 x) = \frac{1}{\sec^2 x}(1+\tan^2 x)^2$$

$$= \frac{1}{\sec^2 x}(\sec^2 x)^2 = \sec^2 x$$

Ans. (a)

124. If the sum of the diagonal elements of a 2×2 matrix is -6, then the maximum possible value of determinant of the matrix is ______.

(GATE 2015, 1 Mark)

Solution: Sum of the diagonals elements is -6 for 2×2 matrix.

The possible eigenvalues are

$$\begin{array}{lll} -1,-5 & -5,-1 & -8,2 \\ -2,-3 & -4,-2 & -9,3 \;--- \\ -3,-1 & -3,-3 & -10,4 \end{array}$$

Maximum possible value of determinant is $(-3)\times(-3)=9$.

Ans. 9

125. The maximum value of 'a' such that the matrix

$$\begin{bmatrix} -3 & 0 & -2 \\ 1 & -1 & 0 \\ 0 & a & -2 \end{bmatrix}$$

has three linearly independent real eigenvectors is

(a) $\dfrac{2}{3\sqrt{3}}$ (b) $\dfrac{1}{3\sqrt{3}}$

(c) $\dfrac{1+2\sqrt{3}}{3\sqrt{3}}$ (d) $\dfrac{1+\sqrt{3}}{3\sqrt{3}}$

(GATE 2015, 1 Mark)

Solution: The characteristic equation of A is

$$|A-XI|=0$$

$$\Rightarrow f(x)=x^3+6x^2+11x+6+2a$$

$$=(x+1)(x+2)(x+3)+2a=0$$

$f(x)$ cannot have all 3 roots (if any) equal since if $f(x)=(x-k)^3$, then comparing coefficients, we get

$$6=-3k, 3k^2=11$$

No such k exists.

(a) Thus $f(x)=0$ has repeated (2) roots (say) α,α,β

or

(b) $f(x)=0$ has real roots (say) α,β,γ.

Now,

$$f'(x)=0 \Rightarrow x_1=\frac{-6-\sqrt{3}}{3}\approx -2.577a$$

$$x_2=\frac{-6+\sqrt{3}}{3}\approx -1.422$$

At x_1, $f(x)$ has relative maxima and at x_2, $f(x)$ has relative minima.

In all possible cases, we have

$$f(x_2)\le 0 \Rightarrow 2\left(a-\frac{\sqrt{3}}{9}\right)\le 0 \Rightarrow a\le \frac{1}{3\sqrt{3}}$$

Ans. (b)

126. We have a set of three linear equations in three unknowns. '$X\equiv Y$' means X and Y are equivalent statements and '$X \not\equiv Y$' means X and Y are not equivalent statements.

P: There is a unique solution.

Q: The equations are linearly independent.

R: All eigenvalues of the coefficient matrix are non-zero.

S: The determinant of the coefficient matrix is nonzero.

Which one of the following is TRUE?

(a) $P\equiv Q\equiv R\equiv S$

(b) $P\equiv R\not\equiv Q\equiv S$

(c) $P\equiv Q\not\equiv R\equiv S$

(d) $P\not\equiv Q\not\equiv R\not\equiv S$

(GATE 2015, 1 Mark)

Solution: The correct answer is option (a)

Ans. (a)

127. At least one eigenvalue of a singular matrix is

(a) Positive

(b) Zero

(c) Negative

(d) Imaginary

(GATE 2015, 1 Mark)

Solution: The correct answer is 'Zero'.

Ans. (b)

128. The lowest eigenvalue of the 2×2 matrix $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ is ________.

(GATE 2015, 1 Mark)

Solution: Let

$$A=\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Characteristic equation of A is

$$|A-\lambda I|=0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix}=0$$

$$\Rightarrow \lambda^2-7\lambda+10=0\Rightarrow \lambda=2,5$$

The lowest eigenvalue is 2.

Ans. 2

129. If any two columns of a determinant $P=\begin{bmatrix} 4 & 7 & 8 \\ 3 & 1 & 5 \\ 9 & 6 & 2 \end{bmatrix}$ are interchanged, which one of the following statements regarding the value of the determinant is CORRECT?

(a) Absolute value remains unchanged but sign will change.
(b) Both absolute value and sign will change.
(c) Absolute value will change but sign will not change.
(d) Both absolute value and sign will remain unchanged.

(GATE 2015, 1 Mark)

Solution: The correct answer is option (a).

Ans. (a)

130. For what value of p, the following set of equations will have no solution?

$$2x + 3y = 5$$
$$3x + py = 10$$

(GATE 2015, 1 Mark)

Solution: We are given the equations,

$$2x + 3y = 5$$
$$3x + py = 10$$
$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & p \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$
$$\text{AX} = \text{B}$$

Augmented matrix

$$[\text{A} / \text{B}] = \begin{bmatrix} 2 & 3 & 5 \\ 3 & p & 10 \end{bmatrix}$$

Using, row transformation, we get

$$R_2 \to 2R_2 - 3R_1$$
$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 2p-9 & 5 \end{bmatrix}$$

System will have no solution if $\rho(A/B) \neq \rho(A)$

$$\Rightarrow 2p - 9 = 0$$
$$\Rightarrow p = \frac{9}{2} = 4.5$$

Ans. 4.5

131. Let $A = [a_{ij}]$, $1 \le i, j \le n$ with $n \ge 3$ and $a_{ij} = i, j$. The rank of A is

(a) 0 (b) 1
(c) $n - 1$ (d) n

(GATE 2015, 1 Mark)

Solution: Given $A = \left[a_{ij}\right]$, $1 \le i, j \le n$, $n \ge 3$ and $a_{ij} = i \cdot j$

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 & --- \\ 2 & 4 & 6 & --- \\ 3 & 6 & 9 & --- \\ - & - & - & --- \end{bmatrix}$$

Applying row transformation,

$$R_2 - 2R_1, R_3 - 3R_1, \ldots$$

Thus, every row will be zero row, except first row in echelon form.

Thus, $\rho(A) = 1$.

Ans. (b)

132. Let A be an $n \times n$ matrix with rank r $(0 < r < n)$. Then $Ax = 0$ has p independent solutions, where p is

(a) r (b) n (c) $n - r$ (d) $n + r$

(GATE 2015, 1 Mark)

Solution: Given $AX = 0$

$$\rho(A_{n \times n}) = r\,(0 < r < n)$$

where ρ is the number of independent solution.
That is, nullity $= 0$
We know that,

$$\text{Rank} + \text{Nullity} = n$$
$$r + p = n \Rightarrow p = n - r$$

Ans. (c)

133. The following set of three vectors

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ 6 \\ x \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

is linearly dependent when x is equal to

(a) 0 (b) 1 (c) 2 (d) 3

(GATE 2015, 1 Mark)

Solution: For linearly depending vectors, $|A| = 0$.

$$\begin{vmatrix} 1 & x & 3 \\ 2 & 6 & 4 \\ 1 & x & 2 \end{vmatrix} = 0$$
$$\Rightarrow (12 - 4x) - x(4 - 4) + 3(2x - 6) = 0$$
$$\Rightarrow 2x = 6$$
$$\Rightarrow x = 3$$

Ans. (d)

134. For the matrix $\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$, if $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector, the corresponding eigenvalue be ________.

(GATE 2015, 1 Mark)

Solution: Let $A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let eigenvalue be λ. So,

$$Ax = \lambda x$$
$$\Rightarrow (A - \lambda I)(x) = 0$$

where $$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, $$\lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 4-\lambda+3 \\ 3+4-\lambda \end{pmatrix} = 0 \Rightarrow \lambda = 7$$

Ans. 7

135. Consider the following 2 × 2 matrix A where two elements are unknown and are marked by a and b. The eigenvalues of this matrix are −1 and 7. What are the values of a and b?

$$A = \begin{pmatrix} 1 & 4 \\ b & a \end{pmatrix}$$

(a) $a = 6,\ b = 4$ (b) $a = 4,\ b = 6$
(c) $a = 3,\ b = 5$ (d) $a = 5,\ b = 3$

(GATE 2015, 2 Marks)

Solution: Given that $\lambda_1 = -1$ and $\lambda_2 = 7$ are eigenvalues of A.
Using the properties of matrices,

$$\lambda_1 + \lambda_2 = tr(A) \qquad (1)$$
$$\lambda_1 \cdot \lambda_2 = |A| \qquad (2)$$

$$\Rightarrow 6 = 1 + a \Rightarrow a = 5$$

Also,

$$a - 4b = -7 \Rightarrow 5 - 4b = -7 \Rightarrow b = 3$$

Ans. (d)

136. Perform the following operations on the matrix $\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$

(i) Add the third row to the second row.
(ii) Subtract the third column from the first column.

The determinant of the resultant matrix is ______.

(GATE 2015, 2 Marks)

Solution:

Let $A = \begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$

Applying row transformation,

$$R_2 \to R_2 + R_3$$
$$\sim \begin{bmatrix} 3 & 4 & 45 \\ 20 & 11 & 300 \\ 13 & 2 & 195 \end{bmatrix}$$

Applying column transformation,

$$C_1 \to C_2 - C_3$$
$$\sim \begin{bmatrix} -42 & 4 & 45 \\ -280 & 11 & 300 \\ -182 & 2 & 195 \end{bmatrix} = B \quad \text{(resultant matrix)}$$

Now,

$$|B| = \begin{vmatrix} -42 & 4 & 3 \\ 280 & 11 & 300 \\ -182 & 2 & 195 \end{vmatrix} = (-14)(15)\begin{vmatrix} 3 & 4 & 3 \\ 20 & 11 & 20 \\ 13 & 2 & 13 \end{vmatrix} = 0$$

Ans. 0

137. If the following system has non-trivial solution

$$px + qy + rz = 0$$
$$qx + ry + pz = 0$$
$$rx + py + qz = 0$$

then which one of the following options is TRUE?

(a) $p - q + r = 0$ or $p = q = -r$
(b) $p + q - r = 0$ or $p = -q = r$
(c) $p + q + r = 0$ or $p = q = r$
(d) $p - q + r = 0$ or $p = -q = -r$

(GATE 2015, 2 Marks)

Solution: For non-trivial solution, $|A| = 0$.

That is $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

Applying column transformation,

$$C_1 \to C_1 + C_2 + C_3$$

$$|(p+q+r)|\begin{vmatrix} 1 & q & r \\ 1 & r & p \\ 1 & p & q \end{vmatrix} = 0$$

Applying row transformation,

$$[R_2 \to R_2 - R_1; R_3 \to R_3 - R_1]$$

$$p+q+r = 0 \text{ or } \begin{vmatrix} 1 & q & r \\ 0 & r-q & p-r \\ 0 & p-q & q-r \end{vmatrix} = 0$$

$$\Rightarrow (r-q)^2 - (p-q)(p-r) = 0$$
$$\Rightarrow p^2 + q^2 + r^2 - pq - qr - pr = 0$$
$$\Rightarrow (p-q)^2 + (q-r)^2 + (r-p)^2 = 0$$
$$\Rightarrow p-q = 0, q-r = 0, r-p = 0$$
$$\Rightarrow p = q = r$$

Ans. (c)

138. For a given matrix $P = \begin{bmatrix} 4+3i & -i \\ i & 4-3j \end{bmatrix}$, where $i = \sqrt{-1}$, the inverse of matrix P is

(a) $\dfrac{1}{24}\begin{bmatrix} 4-3i & i \\ -i & 4+3i \end{bmatrix}$

(b) $\dfrac{1}{25}\begin{bmatrix} i & 4-3i \\ 4+3i & -i \end{bmatrix}$

(c) $\dfrac{1}{24}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

(d) $\dfrac{1}{25}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$

(GATE 2015, 2 Marks)

Solution: We are given

$$P = \begin{bmatrix} 4+3i & -i \\ i & 4-3j \end{bmatrix},$$

$$|P| = (4+3i)(4-3i) - (i)(-i) = 16 + 9 - 1 = 24$$

$$\text{adj } P = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$$

Therefore,

$$P^{-1} = \frac{1}{24}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$$

Ans. (c)

139. The smallest and largest eigenvalues of the following matrix are:

$$\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$$

(a) 1.5 and 2.5

(b) 0.5 and 2.5

(c) 1.0 and 3.0

(d) 1.0 and 2.0

(GATE 2015, 2 Marks)

Solution: Let $A = \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 & 2 \\ 4 & -4-\lambda & 6 \\ 2 & -3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$
$$\Rightarrow (\lambda-1)(\lambda^2 - 3\lambda + 2) = 0$$
$$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-2) = 0$$
$$\Rightarrow \lambda = 1,\ 2$$

Ans. (d)

140. The two eigenvalues of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$ have a ratio of 3:1 for $p = 2$. What is another value of p for which the eigenvalues have the same ratio of 3:1?

(a) -2 (b) 1

(c) 7/3 (d) 14/3

(GATE 2015, 2 Marks)

Solution: Let $A = \begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$

Given that two eigenvalues of A are in 3:1.
Ratio for $p = 2$.
Characteristic equation $\lambda^2 - 4\lambda + 3 = 0$ (by substituting $p = 2$)

$$\Rightarrow \lambda = 1,3$$

If we take $p = \dfrac{14}{3}$ then $A = \begin{bmatrix} 2 & 1 \\ 1 & \dfrac{14}{3} \end{bmatrix}$

$$\Rightarrow \lambda^2 - \left(2 + \frac{14}{3}\right)\lambda + \left(\frac{28}{3} - 1\right) = 0$$

$$\Rightarrow \lambda^2 - \frac{20}{3}\lambda + \frac{25}{3} = 0 \Rightarrow 3\lambda^2 - 20\lambda + 25 = 0$$

$$\Rightarrow \lambda = 5, \frac{5}{3}$$

Eigenvalues are in the ratio 3:1.

Therefore, $p = \frac{14}{3}$

Ans. (d)

141. A system is represented in state-space as $\dot{X} = Ax + Bu$, where $A = \begin{bmatrix} 1 & 2 \\ \alpha & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The value of α for which the system is not controllable is ________ .

(GATE 2015, 2 Marks)

Solution: For a system to be uncontrollable, its controllability determinant should be equal to zero.

$$Q_c = |BAB| = 0$$

$$AB = \begin{bmatrix} 1 & 2 \\ \alpha & 6 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 3 \\ \alpha + 6 \end{bmatrix}$$

$$Q_c = |BAB| \rightarrow \begin{vmatrix} 1 & 3 \\ 1 & \alpha + 6 \end{vmatrix} = 0 \Rightarrow \alpha + 6 - 3 = 0$$

$$\Rightarrow \alpha = -3$$

Ans. -3

142. Two eigenvalues of a 3×3 real matrix P are $(2 + \sqrt{-1})$ and 3. The determinant of P is __________.

(GATE 2016, 1 Mark)

Solution: For 3×3 real matrix, the eigenvalues must be $(2 + \sqrt{-1}, 2 - \sqrt{-1}$ and $3) = (2+i, 2-i$ and $3)$. Now, determinant of the matrix P

$$\begin{aligned} &= \text{Product of eigenvalues} \\ &= (2+i)(2-i)(3) \\ &= (2^2 - i^2) \times 3 \\ &= (4+1) \times 3 \\ &= 15 \end{aligned}$$

Therefore, the determinant of the matrix P is 15.

Ans. 15

143. Suppose that the eigenvalues of matrix A are 1, 2, 4. The determinant of $(A^{-1})^T$ is __________.

(GATE 2016, 1 Mark)

Solution: It is given that for matrix A, the eigenvalues are 1, 2, 4. Therefore,

$$\det(A) = \text{Product of eigenvalues}$$

$$\text{trace}(A) = \text{Sum of eigenvalues}$$

Now, the eigenvalues of A^{-1} are 1, 1/2, 1/4. Therefore,

$$|A^{-1}| = 1 \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

$$|A^{-1}|^{T} = |A^{-1}| = \frac{1}{8} = 0.125$$

Ans. 0.125

144. Let $M^4 = I$, (where I denotes the identity matrix) and $M \neq I$, $M^2 \neq I$ and $M^3 \neq I$. Then, for any natural number k, M^{-1} equals:

(a) M^{4k+1} (b) M^{4k+2}
(c) M^{4k+3} (d) M^{4k}

(GATE 2016, 1 Mark)

Solution: It is given that

$$M^4 = I \qquad (1)$$

On multiplying both sides of Eq. (1) with M^{-1}, we get

$$M^{-1}(M^4) = M^{-1}$$

$$\Rightarrow M^{-1} = M^3 = M^{0+3} = M^{4k+3} \quad (\text{for } k=0)$$

Therefore,

$$M^{-1} = M^{4k+3}$$

Ans. (c)

145. The value of x for which the matrix $A = \begin{pmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{pmatrix}$ has zero as an eigenvalue is __________.

(GATE 2016, 1 Mark)

Solution: For the eigenvalue of a matrix is to be zero, then the determinant of the matrix is also zero:

$$|A| = 0 \Rightarrow \begin{vmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{vmatrix} = 0$$

$$\Rightarrow 3[(-63+7x)+52] - 2[(-81+9x)+78] + 4(-36+42) = 0$$

$$\Rightarrow 21x - 33 - 18x + 6 + 24 = 0$$

$$\Rightarrow 3x - 3 = 0$$

$$\Rightarrow x = 1$$

Ans. 1

146. Consider a 2×2 square matrix: $A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$, where x is unknown. If the eigenvalues of the matrix A are $(\sigma + j\omega)$ and $(\sigma - j\omega)$, then x is equal to

(a) $+j\omega$ (b) $-j\omega$
(c) $+\omega$ (d) $-\omega$

(GATE 2016, 1 Mark)

Solution: It is given that

$$A = \begin{vmatrix} \sigma & x \\ \omega & \sigma \end{vmatrix}$$

Now,

Determinant of A = Product of eigenvalues

$$\Rightarrow \sigma^2 - \omega x = (\sigma + j\omega)(\sigma - j\omega) = \sigma^2 + \omega^2$$

$$\Rightarrow -\omega x = \omega^2 \Rightarrow x = -\omega$$

Ans. (d)

147. Consider a 3×3 matrix with every element being equal to 1. Its only non-zero eigenvalue is ________.

(GATE 2016, 1 Mark)

Solution: Det $[A - \lambda I]$

$$= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$\lambda = 1 \times$ order of matrix

$= 1 \times 3 = 3$

Eigenvalues are 0, 0, 3.

Ans. 3

148. A 3×3 matrix P is such that, $P^3 = P$. Then the eigenvalues of P are

(a) 1, 1, -1
(b) 1, $0.5 + j0.866$, $0.5 - j0.866$
(c) 1, $-0.5 + j0.866$, $-0.5 - j0.866$
(d) 0, 1, -1

(GATE 2016, 1 Mark)

Solution: Using Cayley–Hamilton theorem, we get

$$\lambda^3 = \lambda$$

$$\lambda = 0, 1, -1$$

Ans. (d)

149. The solution to the system of equations

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2 \\ -30 \end{Bmatrix}$$

is

(a) 6, 2 (b) -6, 2 (c) -6, -2 (d) 6, -2

(GATE 2016, 1 Mark)

Solution: Given that

$$D = \begin{vmatrix} 2 & 5 \\ -4 & 3 \end{vmatrix} = 26$$

$$D_1 = \begin{vmatrix} 2 & 5 \\ -30 & 3 \end{vmatrix} = 156$$

$$D_2 = \begin{vmatrix} 2 & 5 \\ -4 & -30 \end{vmatrix} = -52$$

Thus,

$$x = \frac{D_1}{D} = 6$$

$$y = \frac{D_2}{D} = -2$$

Ans. (d)

150. The condition for which the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$$

are positive is

(a) $k > 1/2$ (b) $k > -2$ (c) $k > 0$ (d) $k < -1/2$

(GATE 2016, 1 Mark)

Solution: Given that

$$A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$$

Characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & k-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2k - 2\lambda - k\lambda + \lambda^2) - 1 = 0$$

$$\Rightarrow \lambda^2 - (k+2)\lambda + 2k - 1 = 0$$

$$\Rightarrow \lambda = \frac{(k+2) \pm \sqrt{(k+2)^2 - 4(2k-1)}}{2}$$

For λ to be positive, we have

$$(k+2) > \sqrt{(k+2)^2 - 4(2k-1)}$$

$$\Rightarrow (k+2)^2 > (k+2)^2 - 4(2k-1)$$

$$\Rightarrow -4(2k-1) < 0$$

$$\Rightarrow 2k - 1 > 0$$

$$\Rightarrow k > 1/2$$

Ans. (a)

151. A real square matrix A is called skew-symmetric if

(a) $A^T = A$ (b) $A^T = A^{-1}$
(c) $A^T = -A$ (d) $A^T = A + A^{-1}$

(GATE 2016, 1 Mark)

Solution: We know that a square matrix can be written as

$$[A] = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

where $\frac{1}{2}(A + A^T)$ is a symmetric and $\frac{1}{2}(A - A^T)$ is a skew symmetric matrix.
Now, if $A = -A^T$ or $A^T = -A$

$$[A] = \frac{1}{2}(A - A) + \frac{1}{2}[A - (-A)]$$

Symmetric part becomes zero, whereas skew symmetric part is left. Therefore, a square matrix is called a skew symmetric matrix if $A^T = -A$.

Ans. (c)

152. If the entries in each column of a square matrix M add up to 1, then an eigenvalue of M is

(a) 4 (b) 3 (c) 2 (d) 1

(GATE 2016, 1 Mark)

Solution: Eigenvalue increases by increasing the values of entries. If it is increased by 1 then one eigenvalue will be equal to 1.

Ans. (d)

153. Let A_1, A_2, A_3, and A_4 be four matrices of dimensions 10×5, 5×20, 20×10, and 10×5, respectively. The minimum number of scalar multiplications required to find the product $A_1A_2A_3A_4$ using the basic matrix multiplication method is ________.

(GATE 2016, 2 Marks)

Solution: Here, there are five possible cases:

A(B(CD))
A((BC)D)
((AB)C)D
(A(BC))D
(AB)(CD)

The scalar multiplications required are 1750, 1500, 3500, 2000, 3000, respectively. Therefore, the minimum number of scalar multiplications is 1500.

Ans. 1500

154. A sequence $x[n]$ is specified as $\begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$, for $n \geq 2$. The initial conditions are $x[0]=1$, $x[1]=1$, and $x[n]=0$ for $n<0$. The value of $x[12]$ is ________.

(GATE 2016, 2 Marks)

Solution: It is given that

$$\begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Substituting $n=2$, we get

$$\begin{bmatrix} x(2) \\ x(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Substituting $n=3$, we get

$$\begin{bmatrix} X(3) \\ X(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Therefore, $x(3)=3$ and $x(2)=2$. Hence, we can write the recursive relation as follows:

$$x(n)=x(n-1)+x(n-2)$$

That is,

$$x(2)=x(1)+x(0)=1+1=2$$
$$x(3)=x(2)+x(1)=2+1=3$$
$$x(4)=x(3)+x(2)=3+2=5$$
$$x(5)=x(4)+x(3)=5+3=8$$
$$x(6)=x(5)+x(4)=8+5=13$$
$$x(7)=x(6)+x(5)=13+8=21$$
$$x(8)=x(7)+x(6)=21+13=34$$
$$x(9)=x(8)+x(7)=34+21=55$$
$$x(10)=x(9)+x(8)=55+34=89$$

Similarly, we get

$$x(11)=89+55=144$$
$$x(12)=144+89=233$$

Ans. 233

155. The matrix $A = \begin{pmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{pmatrix}$ has $\det(A)=100$ and trace$(A)=14$. The value of $|a-b|$ is __________.

(GATE 2016, 2 Marks)

Solution: Since trace(A) is 14, we get

$$a+5+2+b=14$$

$$\Rightarrow a+b=7 \qquad (1)$$

and since $\det(A)=100$, we get

$$5\begin{vmatrix} a & 3 & 7 \\ 0 & 2 & 4 \\ 0 & 0 & b \end{vmatrix} = 100 = 5\times 2\times a\times b = 100$$

$$\Rightarrow 10ab = 100$$

$$\Rightarrow ab = 10 \qquad (2)$$

Solving Eqs. (1) and (2), we get $a=5$, $b=2$ or $a=2$, $b=5$. Therefore,

$$|a-b|=3$$

Ans. 3

156. Let the eigenvalues of a 2×2 matrix A be 1, -2 with eigenvectors x_1 and x_2, respectively. Then the eigenvalues and eigenvectors of the matrix $A^2 - 3A + 4I$ would, respectively, be

(a) 2, 14; x_1, x_2

(b) 2, 14; $x_1 + x_2$, $x_1 - x_2$

(c) 2, 0; x_1, x_2

(d) 2, 0; $x_1 + x_2$, $x_1 - x_2$

(GATE 2016, 2 Marks)

Solution: Eigenvalues of

$$A^2 - 3A + 4I$$

are $(1)^2 = 3(1) + 4$ and $(-2)^2 - 3(-2) + 4 = 2, 14$

Ans. (a)

157. Let A be a 4×3 real matrix with rank 2. Which one of the following statements is TRUE?

(a) Rank of A^TA is less than 2.

(b) Rank of A^TA is equal to 2.

(c) Rank of A^TA is greater than 2.

(d) Rank of A^TA can be any number between 1 and 3.

(GATE 2016, 2 Marks)

Solution: Rank (A^TA) = Rank (A)

Ans. (b)

158. Let $P = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Consider the set S of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $a^2 + b^2 = 1$ where $\begin{pmatrix} a \\ b \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}$. Then S is

(a) a circle of radius $\sqrt{10}$

(b) a circle of radius $\dfrac{1}{\sqrt{10}}$

(c) an ellipse with major axis along $\begin{pmatrix}1\\1\end{pmatrix}$

(d) an ellipse with minor axis along $\begin{pmatrix}1\\1\end{pmatrix}$

(GATE 2016, 2 Marks)

Solution: Given that

$$\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$

Then
$$3x + y = a$$
$$x + 3y = b$$
$$a^2 + b^2 = 1$$

Therefore,

$$(3x+y)^2 + (x+3y)^2 = 1$$
$$10x^2 + 10y^2 + 12xy = 1$$
$$\Rightarrow a = 10, b = 10 \text{ and } h = 6$$

This represents an ellipse. Then length of semi-axis is

$$(ab - h^2)r^4 - (a+b)r^2 + 1 = 0$$
$$\Rightarrow r = \frac{1}{2} \text{ or } \frac{1}{4}$$

Length of minor axis $= 2r = 2\left(\frac{1}{4}\right) = \frac{1}{2}$

Equation of minor axis is

$$\left(a - \frac{1}{r^2}\right)x + hy = 0$$
$$\Rightarrow (10-16)x + 6y = 0$$
$$\Rightarrow y - x = 0$$

Similarly major axis equation is

$$y + x = 0$$

So, $\begin{bmatrix}1\\1\end{bmatrix}$ lies on the minor axis.

Ans. (d)

159. The number of linearly independent eigenvectors of matrix $A = \begin{vmatrix}2 & 1 & 0\\0 & 2 & 0\\0 & 0 & 3\end{vmatrix}$ is ______.

(GATE 2016, 2 Marks)

Solution: As two eigenvalues of given matrix are identical, the eigenvectors resulting from these identical eigenvalues would be identical/linearly dependent. Therefore, number of linearly independent eigenvectors $= n - 1 = 2$.

Ans. 2

160. Consider the following linear system:

$$x + 2y - 3z = a$$
$$2x + 3y + 3z = b$$
$$5x + 9y - 6z = c$$

This system is consistent if, a, b and c satisfy the equation

(a) $7a - b - c = 0$ (b) $3a + b - c = 0$
(c) $3a - b + c = 0$ (d) $7a - b + c = 0$

(GATE 2016, 2 Marks)

Solution: Equations are given as

$$x + 2y - 3z = a$$
$$2x + 3y + 3z = b$$
$$5x + 9y - 6z = c$$

Matrix form of equation is

$$\begin{bmatrix}1 & 2 & -3\\2 & 3 & 3\\5 & 9 & 6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$$

Solving by row operation, we have

$$\begin{bmatrix}1 & 2 & -3\\2 & 3 & 3\\5 & 9 & -6\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}\xrightarrow{R_2 \to R_1 - R_2}\begin{bmatrix}1 & 2 & -3\\0 & +1 & -9\\5 & 9 & -6\end{bmatrix}\begin{bmatrix}a\\2a-b\\c\end{bmatrix}$$

$$\downarrow R_3 \to 5R_1 - R_3$$

$$\begin{bmatrix}1 & 2 & -3\\0 & 1 & -9\\0 & 1 & -9\end{bmatrix}\begin{bmatrix}a\\2a-b\\5a-c\end{bmatrix}$$

Now,

$$x - 9y = (2a - b)$$
$$x - 9y = (5a - c)$$
$$- \quad + \quad -$$
$$0 = (2a - b) - (5a - c)$$

Hence, $3a + b - c = 0$

Ans. (b)

161. Consider the 5×5 matrix

$$A = \begin{bmatrix}1 & 2 & 3 & 4 & 5\\5 & 1 & 2 & 3 & 4\\4 & 5 & 1 & 2 & 3\\3 & 4 & 5 & 1 & 2\\2 & 3 & 4 & 5 & 1\end{bmatrix}$$

It is given that A has only one real eigenvalue. Then the real eigenvalue of A is

(a) -2.5 (b) 0 (c) 15 (d) 25

(GATE 2017, 1 Mark)

Solution:

$$[A - \lambda I][X] = 0$$

$$\Rightarrow [A]\cdot[X] = \lambda[X] \quad (1)$$

Let

$$[X] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

From Eq. (1), we have

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

Therefore,

$$a + 2b + 3c + 4d + 5e = \lambda a \quad (2)$$

$$5a + b + 2c + 3d + 4e = \lambda b \quad (3)$$

$$4a + 5b + c + 2d + 3e = \lambda c \quad (4)$$

$$3a + 4b + 5c + d + 2e = \lambda d \quad (5)$$

$$2a + 3b + 4c + 5d + e = \lambda e \quad (6)$$

Adding Eqs. (2) and (6), we have

$$15\ (a + b + c + d + e) = \lambda\ (a + b + c + d + e)$$

$$\Rightarrow \lambda = 15$$

Ans. (c)

162. The rank of the matrix $M = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

(GATE 2017, 1 Mark)

Solution: The rank of matrix $M = \begin{bmatrix} 5 & 10 & 10 \\ 1 & 0 & 2 \\ 3 & 6 & 6 \end{bmatrix}$

$$R_1 \leftrightarrow R_2 = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 10 & 10 \\ 3 & 6 & 6 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 - 5R_1 \text{ and } R_3 - 3R_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 10 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 - \frac{1}{10}R_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\text{Rank }(M) = 2$$

Ans. (c)

163. Consider the following statements about the linear dependence of the real-valued functions $y_1 = 1$, $y_2 = x$ and $y_3 = x^2$, over the field of real numbers.

I. y_1, y_2 and y_3 are linearly independent on $-1 \le x \le 0$.

II. y_1, y_2 and y_3 are linearly dependent on $0 \le x \le 1$.

III. y_1, y_2 and y_3 are linearly independent on $0 \le x \le 1$.

IV. y_1, y_2 and y_3 are linearly dependent on $-1 \le x \le 0$.

Which one among the following is correct?

(a) Both I and II are true.

(b) Both I and III are true.

(c) Both II and IV are true.

(d) Both III and IV are true.

(GATE 2017, 1 Mark)

Solution: Given that

$$y_1 = 1, \qquad y_2 = x, \qquad y_3 = x^2$$

From Wronskian's matrix, if

$$|W| = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} = 0$$

functions are linearly dependent.
Here, $f(x) = 1$, $g(x) = x$, $h(x) = x^2$

$$|W| = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$$

$$\Rightarrow |W| \ne 0$$

So, functions are linear independent. Therefore, statements (I) and (III) are correct.

Ans. (b)

164. The rank of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \text{ is } \_\_\_\_\_\_\_\_.$$

(GATE 2017, 1 Mark)

Solution:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_4 \to R_4 + R_5$, we get

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_4$, we get

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_2 \to R_2 + R_3$, we get

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Therefore, rank of the matrix is 4.

Ans. (4)

165. The matrix $A = \begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$ has three distinct eigenvalues and one of its eigenvectors is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Which one of the following can be another eigenvector of A?

(a) $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(GATE 2017, 1 Mark)

Solution: This matrix A is symmetric and all its eigenvalues are distinct. Therefore, all the eigenvalues are orthogonal. We know,

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The corresponding orthogonal vector matches option (c), i.e. $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ as $X_1^T \cdot X_2 = 0$.

Ans. (c)

166. The product of eigenvalues of the matrix P is

$$P = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

(a) –6 (b) 2 (c) 6 (d) –2

(GATE 2017, 1 Mark)

Solution: The product of eigenvalues of P is equal to the determinant of P, i.e.

$$= 2\,(3 - 6) - 0 + 1 \times 8$$
$$= 2$$

Ans. (b)

167. The determinant of a 2×2 matrix is 50. If one eigenvalue of the matrix is 10, the other eigenvalue is ______.

(GATE 2017, 1 Mark)

Solution: The determinant of a matrix is equal to the product of eigenvalues. Thus, other eigenvalue will be 50/10 = 5.

Ans. 5

168. The matrix P is the inverse of a matrix Q. If I denotes the identity matrix, which one of the following options is correct?

(a) $PQ = I$ but $QP \neq I$
(b) $QP = I$ but $PQ \neq I$
(c) $PQ = I$ and $QP = I$
(d) $PQ - QP = I$

(GATE 2017, 1 Mark)

Solution: Given

$$P = Q^{-1} \quad (1)$$

Also

$$Q^{-1} \cdot Q = I \quad (2)$$

Multiplying Q on both sides of Eq. (1), we have

$$PQ = Q^{-1} \cdot Q \quad (3)$$

Substituting Eq. (2) in Eq. (3), we have

$$PQ = I$$

Again multiplying Eq. (1) by Q, we get

$$QP = QQ^{-1} \Rightarrow QP = I \qquad (\text{since } QQ^{-1} = I)$$

Therefore,

$$PQ = I \quad \text{and} \quad QP = I$$

Ans. (c)

169. Consider the following simultaneous equations (with c_1 and c_2 being constants):

$$3x_1 + 2x_2 = c_1$$
$$4x_1 + x_2 = c_2$$

The characteristic equation for these simultaneous equations is

(a) $\lambda^2 - 4\lambda - 5 = 0$
(b) $\lambda^2 - 4\lambda + 5 = 0$
(c) $\lambda^2 + 4\lambda - 5 = 0$
(d) $\lambda^2 + 4\lambda + 5 = 0$

(GATE 2017, 1 Mark)

Solution: From the given simultaneous equations, we have

$$[A] = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Now,

$$[A - \lambda I] = 0$$

$$\Rightarrow [A - \lambda I] = \begin{bmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{bmatrix}$$

$$\Rightarrow (3-\lambda)(1-\lambda) - 8 = 0$$

$$\Rightarrow 3 - \lambda^2 - 4\lambda - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

Ans. (a)

170. Let A be $n \times n$ real-valued square symmetric matrix of rank 2 with $\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2 = 50$. Consider the following statements.

I. One eigenvalue must be in [–5, 5].
II. The eigenvalue with the largest magnitude must be strictly greater than 5.

Which of the above statements about eigenvalues of A is/are necessarily CORRECT?

(a) Both I and II
(b) I only
(c) II only
(d) Neither I nor II

(GATE 2017, 2 Marks)

Solution: Given that

Rank of $A_{n \times n}$ symmetric matrix = 2

Let $\lambda_1, \lambda_2, 0, 0, \ldots, 0$ be the eigenvalues. So $(n - 2)$ eigenvalues are zero.

Given that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^2 = 50 = \text{Trace of } (AA^T)$$

$$= \lambda_1^2 + \lambda_2^2 + 0 + 0 + \cdots + 0$$

$$\Rightarrow \lambda_1^2 + \lambda_2^2 = 50$$

As eigenvalue lies in [–5, 5], statement I is true. Also as eigenvalues are $\lambda_1 = \pm 5$, $\lambda_2 = \pm 5$, statement II is false.

Ans. (b)

171. If the characteristic polynomial of a 3×3 matrix M over $\mathbb{R}$ (the set of real numbers) is $\lambda^3 - 4\lambda^2 + a\lambda + 30$, $a \in \mathbb{R}$ and one eigenvalue of M is 2, then the largest among the absolute value of the eigenvalues of M is ____________.

(GATE 2017, 2 Marks)

Solution: The characteristic equation of M is

$$\lambda^3 - 4\lambda^2 + a\lambda + 30 = 0$$

Substituting $\lambda = 2$ in the above equation as 2 is one of the root of the equation, we get

$$a = -11$$

Now, the characteristic equation is

$$\lambda^3 - 4\lambda^2 - 11\lambda + 30 = 0$$
$$\Rightarrow (\lambda - 2)(\lambda^2 - 2\lambda - 15) = 0$$
$$\Rightarrow \lambda = 2, -3, 5$$

Therefore, the maximum eigenvalue among the absolute values of the eigenvalues is 5.

Ans. (5.0)

172. The eigenvalues of the matrix given below are

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

(a) (0, −1, −3) (b) (0, −2, −3)
(c) (0, 2, 3) (d) (0, 1, 3)

(GATE 2017, 2 Marks)

Solution:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & -3 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(\lambda^3 + 4\lambda + 3) = 0$$
$$\Rightarrow \lambda = 0 \text{ or } \lambda = -1, -3$$

Ans. (a)

173. Consider the matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Which one of the following statements about P is incorrect?

(a) Determinant of P is equal to 1
(b) P is orthogonal
(c) Inverse of P is equal to its transpose
(d) All eigenvalues of P are real numbers

(GATE 2017, 2 Marks)

Solution:

$$|P| = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - 0\right) - 0 + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{2} + \frac{1}{2}$$
$$= 1$$

$$P \cdot P^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

P is an orthogonal matrix. Inverse of P is its transpose only. Thus, option (d) is incorrect.

Ans. (d)

174. Consider the matrix

$$A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$$

whose eigenvectors corresponding to eigenvalues λ_1 and λ_2 are

$$X_1 = \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} \lambda_2 - 80 \\ 70 \end{bmatrix}$$

respectively. The value of $X_1^T X_2$ is ______.

(GATE 2017, 2 Marks)

Solution: For the given case

$$X_1^T X_2 = [70\lambda_1 - 50]\begin{bmatrix} \lambda_1 - 80 \\ 70 \end{bmatrix}$$
$$= 70(\lambda_2 - 80 + (\lambda_1 - 50)70)$$
$$= 0$$

Ans. 0

175. Consider the matrix $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$. Which one of the following statements is TRUE for the eigenvalues and eigenvectors of this matrix?

(a) Eigenvalue 3 has a multiplicity of 2, and only one independent eigenvector exists.
(b) Eigenvalue 3 has a multiplicity of 2, and two independent eigenvectors exist.

(c) Eigenvalue 3 has a multiplicity of 2, and no independent eigenvector exists.
(d) Eigenvalues are 3 and –3, and two independent eigenvectors exist.

(GATE 2017, 2 Marks)

Solution:

$$[A] = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$$

For eigenvector, we have

$$[A - \lambda I]\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 5-3 & -1 \\ 4 & 1-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Now,

$$2x - y = 0$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$4x - 2y = 0$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For eigenvalue, we have

$$[A - \lambda I] = \begin{bmatrix} 5-\lambda & -1 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (5-\lambda)(1-\lambda) + 4 = 0$$

$$\Rightarrow 5 - 5\lambda - \lambda + \lambda^2 + 4 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda - 3)^2 = 0$$

$$\Rightarrow \lambda = 3$$

So, only one independent eigenvector exists.

Ans. (a)

176. If $A = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$, AB^T is equal to

(a) $\begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 40 \\ 42 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 43 & 27 \\ 34 & 50 \end{bmatrix}$ (d) $\begin{bmatrix} 38 & 32 \\ 28 & 56 \end{bmatrix}$

(GATE 2017, 2 Marks)

Solution:

$$\text{A} = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}; \ \text{B} = \begin{bmatrix} 3 & 7 \\ 8 & 4 \end{bmatrix}$$

Now,

$$AB^{\text{T}} = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 35+3 & 20+8 \\ 14+18 & 8+48 \end{bmatrix} = \begin{bmatrix} 38 & 28 \\ 32 & 56 \end{bmatrix}$$

Ans. (a)

CHAPTER 2

CALCULUS

LIMITS

Limits are used to define continuity, derivatives and integrals.

Consider a function $f(x)$. Now, if x approaches a value c, and if for any number $\alpha > 0$, we find a number $\beta > 0$ such that $|f(x) - l| < \alpha$ whenever $0 < |x - c| < \beta$, then l is called the limit of function $f(x)$.

Limits are denoted as

$$\lim_{x \to c} f(x) = l$$

For example, if we have a function $f(x) = \dfrac{x^2 - 4}{x - 2}$, its limit does not exist for $x = 2$ since $f(1) = \infty$. However, as x moves closer or approaches the value 2, $f(x)$ approaches the limit 4.

Left-Hand and Right-Hand Limits

If the values of a function $f(x)$ at $x = c$ can be made as close as desired to the number l_1 at points closed to and on the left of c, then l_1 is called left-hand limit.

It is denoted by

$$\lim_{x \to c^-} f(x) = l_1$$

If the values of a function $f(x)$ at $x = c$ can be made as close as desired to the number l_2 at points on the right of and close to c, then l_2 is called right-hand limit.

It is denoted by

$$\lim_{x \to c^+} f(x) = l_2$$

For example, let us calculate the left-hand and right-hand limits of the following function:

$$f(x) = \begin{cases} \dfrac{|x-1|}{x-1}, & x \neq 1 \\ 0\;\;\;\;\;\;\;\;, & x = 1 \end{cases} \quad \text{at } x = 1$$

L.H.L. of $f(x)$ at $x = 1$

$$= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$$

$$= \lim_{h \to 0} \frac{|1-h-1|}{1-h-1} = \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} (-1) = -1$$

R.H.L. of $f(x)$ at $x = 1$

$$= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0} \frac{|1+h-1|}{1+h-1} = \lim_{h\to 0} \frac{|h|}{h}$$

$$= \lim_{h\to 0} 1 = 1$$

PROPERTIES OF LIMITS

Suppose we have $\lim_{x\to c} f(x) = a, \lim_{x\to c} g(x) = b$ and if a and b exist, some of the important properties of limits are:

1. $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = a \pm b$
2. $\lim_{x\to c} (f(x)\cdot g(x)) = \lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x) = a\cdot b$
3. $\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{a}{b}$, where $b \neq 0$
4. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$, where k is a constant
5. $\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)| = |a|$
6. $\lim_{x\to c} (f(x))g(x) = a^b$
7. If $\lim_{x\to c} f(x) = \pm\infty$, then $\lim_{x\to c} \frac{1}{f(x)} = 0$

Some of the useful results of limits are given as follows:

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x\to 0} \frac{\tan x}{x} = 1$
3. $\lim_{x\to 0} \cos x = 1$
4. $\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
5. $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
6. $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$, if $(a > 0)$
7. $\lim_{x\to 0} \frac{x^n - a^n}{x-a} = na^{n-1}$
8. $\lim_{x\to\infty} \frac{\log x}{x^m} = 0$, if $(m > 0)$

L'Hospital's Rule

If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$ for all x, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

CONTINUITY AND DISCONTINUITY

A function $f(x)$ at any point $x = c$ is continuous if

$$\lim_{x\to c} f(x) = f(c)$$

and $$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

A function $f(x)$ is continuous for an open interval (a, b) if it is continuous at every point on the interval (a, b).

A function $f(x)$ is continuous on a closed interval $[a, b]$ if

1. f is continuous for (a, b)
2. $\lim_{x\to a^+} f(x) = f(a)$
3. $\lim_{x\to b^-} f(x) = f(b)$

If the conditions for continuity are not satisfied for a function $f(x)$ for a point or an interval, then the function is said to be discontinuous.

DIFFERENTIABILITY

Consider a real-valued function $f(x)$ defined on an open interval (a, b). The function is said to be differentiable for $x = c$, if

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ exists for every } c \in (a, b)$$

A function is always continuous at a point if the function is differentiable at the same point. However, the converse is not always true.

MEAN VALUE THEOREMS

Rolle's Theorem

Consider a real-valued function defined in the closed interval $[a, b]$, such that

1. It is continuous on the closed interval $[a, b]$.
2. It is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then, according to Rolle's theorem, there exists a real number $c \in (a,b)$ such that $f'(c) = 0$.

Lagrange's Mean Value Theorem

Consider a function $f(x)$ defined in the closed interval $[a, b]$, such that

1. It is continuous on the closed interval $[a, b]$.
2. It is differentiable on the closed interval (a, b).

Then, according to Lagrange's mean value theorem, there exists a real number $c \in (a,b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Cauchy's Mean Value Theorem

Consider two functions $f(x)$ and $g(x)$, such that

1. $f(x)$ and $g(x)$ both are continuous in $[a, b]$.
2. $f'(x)$ and $g'(x)$ both exist in (a, b).

Then there exists a point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Taylor's Theorem

If $f(x)$ is a continuous function such that $f'(x)$, $f''(x), \ldots, f^{n-1}(x)$ are all continuous in $[a, a + h]$ and $f^n(x)$ exists in $(a, a + h)$ where $h = b - a$, then according to Taylor's theorem,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a)$$

Maclaurin's Theorem

If the Taylor's series obtained in Section 2.5.4 is centered at 0, then the series we obtain is called the Maclaurin's series. According to Maclaurin's theorem,

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{h^n}{n!}f^n(0)$$

FUNDAMENTAL THEOREM OF CALCULUS

There are two parts of the fundamental theorem of calculus that are used widely. It links the concept of the derivative of the function with the concept of the integral.

According to the first part of the fundamental theorem of calculus or the first fundamental theorem of calculus, if f is continuous on the closed interval $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

According to the second part of the fundamental theorem of calculus or the second fundamental theorem of calculus, if f is a continuous real-valued function defined on a closed interval $[a, b]$ and F is a function defined for all x in $[a, b]$, by

$$F(x) = \int_a^x f(t)\,dt$$

then F is continuous on $[a, b]$, differentiable on the open interval (a, b) and $F'(x) = f(x)$ for all x in (a, b).

DIFFERENTIATION

Some of the important properties of differentiation are:

1. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
2. $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
3. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}(g(x)) + \frac{d}{dx}(f(x)) \cdot g(x)$
4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}f(x) \cdot g(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$

Some of the derivatives of commonly used functions are given as follows:

1. $\frac{d}{dx}x^n = nx^{n-1}$
2. $\frac{d}{dx}\ln x = \frac{1}{x}$
3. $\frac{d}{dx}\log_a x = \log_a e \cdot \left(\frac{1}{x}\right)$

4. $\dfrac{d}{dx} e^x = e^x$

5. $\dfrac{d}{dx} a^x = a^x \log_e a$

6. $\dfrac{d}{dx} \sin x = \cos x$

7. $\dfrac{d}{dx} \cos x = -\sin x$

8. $\dfrac{d}{dx} \tan x = \sec^2 x$

9. $\dfrac{d}{dx} \sec x = \sec x \tan x$

10. $\dfrac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

11. $\dfrac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

12. $\dfrac{d}{dx} \sin \text{h}\, x = \cos \text{h}\, x$

13. $\dfrac{d}{dx} \cos \text{h}\, x = \sin \text{h}\, x$

14. $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$

15. $\dfrac{d}{dx} \cos^{-1} x = \dfrac{-1}{\sqrt{1-x^2}}$

16. $\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1+x^2}$

17. $\dfrac{d}{dx} \operatorname{cosec}^{-1} x = -\dfrac{1}{x\sqrt{x^2-1}}$

18. $\dfrac{d}{dx} \sec^{-1} x = -\dfrac{1}{x\sqrt{x^2-1}}$

19. $\dfrac{d}{dx} \cot^{-1} x = \dfrac{1}{1+x^2}$

APPLICATIONS OF DERIVATIVES

Increasing and Decreasing Functions

Any function $f(x)$ is said to be increasing on an interval (a, b) if

$$x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2) \text{ for all values of } x_1, x_2 \in (a, b)$$

A function $f(x)$ is said to be strictly increasing on an interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all values of } x_1, x_2 \in (a, b)$$

A function $f(x)$ is said to be decreasing on an interval (a, b) if

$$x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2) \text{ for all values of } x_1, x_2 \in (a, b)$$

A function $f(x)$ is said to be strictly decreasing on an interval (a, b) if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all values of } x_1, x_2 \in (a, b)$$

A monotonic function is any function $f(x)$ which is either increasing or decreasing on an interval (a, b).

Some important conditions for increasing and decreasing functions are:

1. Consider $f(x)$ to be continuous on $[a, b]$ and differentiable on (a, b). Now,

(a) If $f(x)$ is strictly increasing on (a, b), then $f'(x) > 0$ for all $x \in (a, b)$.
(b) If $f(x)$ is strictly decreasing on (a, b), then $f'(x) < 0$ for all $x \in (a, b)$.

2. Consider $f(x)$ to be a differentiable real function defined on an interval (a, b). Now,

(a) If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on (a, b).
(b) If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).

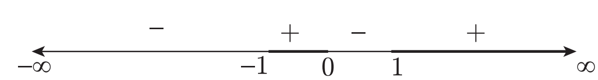
For example, let us find the intervals for which $f(x) = x^4 - 2x^2$ is increasing or decreasing.

$$f(x) = x^4 - 2x^2$$

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

For $f(x)$ to be increasing, $f'(x) > 0$

$$\therefore \quad 4x\,(x^2 - 1) > 0$$
$$\Rightarrow x\,(x^2 - 1) > 0$$
$$\Rightarrow x(x + 1)(x - 1) > 0$$

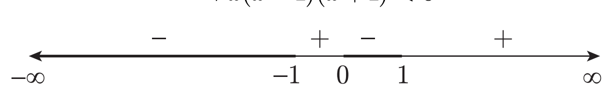


$\Rightarrow -1 < x < 0$ or $x > 1$

$\therefore x \in (-1, 0) \cup (1, \infty)$

For $f(x)$ to be decreasing, $f'(x) < 0$

$$\Rightarrow 4x(x^2 - 1) < 0$$
$$\Rightarrow x(x^2 - 1) < 0$$
$$\Rightarrow x(x - 1)(x + 1) < 0$$



$\Rightarrow x < -1$ or $0 < x < 1$

$\therefore x \in (-\infty, -1) \cup (0, 1)$

Maxima and Minima

Suppose $f(x)$ is a real-valued function defined at an internal (a, b). Then $f(x)$ is said to have maximum value, if there exists a point y in (a, b) such that

$$f(x) = f(y) \text{ for all } x \in (a, b)$$

Suppose $f(x)$ is a real-valued function defined at the interval (a, b). Then $f(x)$ is said to have minimum value, if there exists a point y in (a, b) such that

$$f(x) \geq f(y) \text{ for all } x \in (a, b)$$

Local maxima and local minima of any function can be calculated as:

Consider that $f(x)$ be defined in (a, b) and $y \in (a, b)$. Now,

1. If $f'(y) = 0$ and $f'(x)$ changes sign from positive to negative as 'x' increases through 'y', then $x = y$ is a point of local maximum value of $f(x)$.
2. If $f'(y) = 0$ and $f'(x)$ changes sign from negative to positive as 'x' increases through 'y', then $x = y$ is a point of local minimum value of $f(x)$.

For example,

1. $f(x) = x^3 - 6x^2 + 12x$

   Find all points of local maxima and local minima.

   $$y = f(x) = x^3 - 6x^2 + 12x$$

   $$\frac{dy}{dx} = 3x^2 - 12x + 12 = 3\,(x-2)^2$$

   For a local maximum or local minimum, we have

   $$\frac{dy}{dx} = 0 \Rightarrow 3(x-2)^2 = 0 \Rightarrow x = 2$$

   We observe that dy/dx does not change sign as increased through $x = 2$. Hence, $x = 2$ is neither a point of local maximum nor local minimum.

2. Find all points of local maxima and minima of the function,

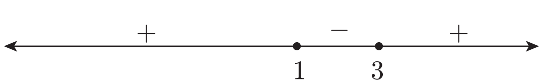
   $$f(x) = x^3 - 6x^2 + 9x$$

   Now, $f'(x) = 3x^2 - 12x + 9 = 3\,(x^2 - 4x + 3)$

   For a local maximum or local minimum, we have

   $$f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0 \Rightarrow x = 1, 3$$

   The change in signs of $f'(x)$ are shown in the following figure.



   $f'(x)$ changes sign from positive to negative as 'x' increases through 1.

   Therefore, $x = 1$ is a point of local maximum.

   Also, $f'(x)$ changes sign from negative to positive as 'x' increases through 3.

   Therefore, $x = 3$ is a point of local minimum.

Some important properties of maximum and minima are given as follows:

1. If $f(x)$ is continuous in its domain, then at least one maxima and minima lie between two equal values of x.
2. Maxima and minima occur alternately, i.e. no two maxima or minima can occur together.

For example, let us find all points of maxima and minima for the following points:

$$f(x) = 2x^3 - 21x^2 + 36x$$

Now, $f'(x) = 6x^2 - 42x + 36$

For local maxima or minima,

$$f'(x) = 0$$

$\therefore 6x^2 - 42x + 36 = 0$

$\Rightarrow (x - 1)(x - 6) = 0$

$\Rightarrow x = 1, 6$

Now, to test for maxima and minima,

$$f''(x) = 12x - 42$$

At $x = 1$,

$f''(1) = -30 < 0$, hence $x = 1$ is a point of local maximum.

At $x = 6$,

$f''(6) = 72 - 42$

$= 30 > 0$, hence $x = 6$ is a point of local minimum.

Maximum and minimum values in a closed interval $[a, b]$ can be calculated using the following steps:

1. Calculate $f'(x)$.
2. Put $f'(x) = 0$ and find value(s) of x. Let $c_1, c_2, \ldots, c_n$ be values of x.
3. Take the maximum and minimum values out of the values $f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)$. The maximum and minimum values obtained are the absolute maximum and absolute minimum values of the function, respectively.

For example, let us find the points of maxima and minima for $f(x) = 2x^3 - 24x + 107$ in the interval $[0, 3]$.

$$f'(x) = 6x^2 - 24$$

Now, $f'(x) = 0$

$\Rightarrow 6x^2 - 24 = 0 \Rightarrow x^2 = \dfrac{24}{6} = 4$

$\therefore x = \pm\, 2$

But, $x = -2 \notin [0, 3]$

$\therefore$ $x = 2$ is the only stationary point.

$f(0) = 107$

$f(2) = 2(2)^3 - 24(2) + 107$

$= 16 - 48 + 107 = 75$

$f(3) = 2(3)^3 - 24(3) + 107 = 89$

The maximum value of $f(x)$ is 107 at $x = 0$.

The minimum value of $f(x)$ is 75 at $x = 2$.

Therefore, the points of maxima and minima are 0 and 2, respectively.

PARTIAL DERIVATIVES

Partial differentiation is used to find the partial derivative of a function of more than one independent variable.

The partial derivatives of $f(x, y)$ with respect to x and y are defined by

$$\frac{\partial f}{\partial x} = \lim_{a \to 0} \frac{f(x + ay) - f(x, y)}{a}$$

$$\frac{\partial f}{\partial x} = \lim_{b \to 0} \frac{f(x, y + b) - f(x, y)}{b}$$

and the above limits exist.

$\partial f / \partial x$ is simply the ordinary derivative of f with respect to x keeping y constant, while $\partial f / \partial x$ is the ordinary derivative of f with respect to y keeping x constant.

Similarly, second-order partial derivatives can be calculated by

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ and is, respectively, denoted by $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$.

A homogenous function is an expression in which every term is of the same degree. Thus, a homogeneous function of x and y of degree n can be represented as

$$a_0 x^n + a_1 x^{n-1} y + a_1 x^{n-2} y^2 + \cdots + a_n y^n$$

Euler's theorem on homogeneous function $f(x, y)$ of degree 'n' is given by

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

If $u = f(x, y)$ where $x = g_1(t)$ and $y = g_2(t)$, then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

In the above equation, the term $\frac{\partial u}{\partial t}$ is called total differential coefficient of u with respect to t while $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are partial derivatives of u. Some of the important results from the above relation are given as follows:

1. If $u = f(x, y)$ and $y = f(x)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x}$$

2. If $u = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}$$

and
$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}$$

INTEGRATION

We have already discussed differentiation in the previous sections of this chapter. In this section, we discuss the other main operation of calculus, integration.

Given a function f of a real variable x and an interval $[a, b]$ of the real line, the definite integral $\int_a^b f(x)dx$ is defined as the area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$.

METHODS OF INTEGRATION

Integration, unlike differentiation, is not straightforward. Some of the integrals can be solved directly from the table, however, in most of the calculations we need to apply one or the other techniques of integration. In this section, we discuss those techniques in order to make the integration problems easier to solve.

Integration Using Table

Some of the common integration problems can be solved by directly referring the tables and computing the results. Table 1 shows the result of some of the common integrals we use.

Table 1 | Table of common integrals

Integration	Result
$\int \frac{1}{ax+b}dx$	$\frac{1}{a}\ln\lvert ax+b\rvert + C$ where C is a constant
$\int \frac{1}{(ax+b)^2}dx$	$-\frac{1}{a(ax+b)}+C$
$\int \frac{1}{(ax+b)^n}dx$	$-\frac{1}{a(n-1)(ax+b)^{n-1}}+C$
$\int \frac{1}{a^2+x^2}dx$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$
$\int \frac{f'(x)}{f(x)}dx$	$\ln\lvert f(x)\rvert + C$
$\int \sin^2 xdx$	$\frac{x}{2}-\frac{1}{2}\sin x\cos x + C$
$\int \sin^3 xdx$	$-\cos x+\frac{1}{3}\cos^3 x + C$
$\int \sin^n xdx$	$-\frac{1}{n}\sin^{n-1} x\cos x+\frac{n-1}{n}\int \sin^{n-2} xdx + C$
$\int \cos^2 xdx$	$\frac{x}{2}+\frac{1}{2}\sin x\cos x + C$
$\int \cos^3 xdx$	$\sin x-\frac{1}{3}\sin^3 x + C$
$\int \cos^n xdx$	$\frac{1}{n}\cos^{n-1} x\sin x+\frac{n-1}{n}\int \cos^{n-2} xdx + C$
$\int \cos^n xdx$	$\frac{\tan^{-1} x}{n-1}-\int \tan^{n-2} xdx + C$
$\int \frac{dx}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$
$\int \frac{dx}{\sqrt{x^2 \pm a^2}}$	$\ln\left\lvert x+\sqrt{x^2 \pm a^2}\right\rvert + C$
$\int x\sin nxdx$	$\frac{1}{n^2}(\sin nx - nx\cos nx)+C$
$\int x\cos nxdx$	$\frac{1}{n^2}(\cos nx + nx\sin nx)+C$

(*Continued*)

Table 1 | Continued

Integration	Result
$\int e^{ax}\sin bxdx$	$\frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}+C$
$\int e^{ax}\cos bxdx$	$\frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}+C$
$\int x^2\sin nxdx$	$\frac{1}{n^3}(-n^2x^2\operatorname{cosn} x + 2\cos nx + 2nx\sin nx)+C$
$\int x^2\cos nxdx$	$\frac{1}{n^3}(n^2x^2\operatorname{sinn} x - 2\sin nx + 2nx\cos nx)+C$

Integration Using Substitution

There are occasions when it is possible to perform integration using a substitution to solve a complex piece of integration. This has the effect of changing the variable, the integrand and even the limits of integration (for definite integrals).

To integrate a differential $f(x)dx$ which is not in the table, we first take a function $u = u(x)$ so that the given differential can be rewritten as a differential $g(u)du$ which does appear in the table. Then, if $\int g(u)du = G(u)+C$, we know that

$$\int f(x)\,dx = G(u(x)) + C.$$

Integration by Parts

Sometimes we can recognize the differential to be integrated as a product of a function which is easily differentiated and a differential which is easily integrated.

Integration by parts is a technique for performing integration (definite and indefinite) by expanding the differential of a product of function $d(uv)$ and expressing the original integral in terms of a known integral $\int vdu$.

Using the product rule for differentiation, we have

$$d(uv) = udv + vdu$$

Integrating both sides, we get

$$\int d(uv) = uv = \int udv + \int vdu$$

Rearranging the above equation, we get

$$\int udv = uv - \int vdu$$

Integration by Partial Fraction

If the integrand is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, the fraction needs to be expressed in *partial fractions* before integration takes place.

The point of the partial fractions expansion is that integration of a rational function can be reduced to the following formulae, once we have determined the roots of the polynomial in the denominator. The formula which come handy while working with partial fractions are given as follows:

$$\int \frac{1}{x-a}\,dx = \ln(x-a) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{x}{a^2+x^2}\,dx = \frac{1}{2}\ln\left(a^2+x^2\right) + C$$

Integration Using Trigonometric Substitution

Trigonometric substitution is used to simplify certain integrals containing radical expressions. Depending on the function we need to integrate, we substitute one of the following expressions to simplify the integration:

(a) For $\sqrt{a^2-x^2}$, use $x = a\sin\theta$.

(b) For $\sqrt{a^2+x^2}$, use $x = a\tan\theta$.

(c) For $\sqrt{x^2-a^2}$, use $x = a\sec\theta$.

DEFINITE INTEGRALS

If a function $f(x)$ is defined in the interval $[a, b]$, then the definite integral of the function is given by

$$\int_a^b f(x)\cdot dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is an integral of $f(x)$, a is called the lower limit and b is the upper limit of the integral.

Geometrically, a definite integral represents the area bounded by curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$.

Some of the important properties of definite integrals are given as follows:

1. The value of definite integrals remains the same with change of variables of integration provided the limits of integration remain the same.

$$\int_a^b f(x)\cdot dx = \int_a^b f(y)\cdot dy$$

2. $\displaystyle\int_a^b f(x)\cdot dx = -\int_b^a f(x)\cdot dx$

3. $\displaystyle\int_a^b f(x)\cdot dx = \int_a^c f(x)\cdot dx + \int_c^b f(x)\cdot dx \quad$ if $a < c < b$

4. $\displaystyle\int_0^{2a} f(x)\cdot dx = \int_0^a f(x)\cdot dx + \int_0^a f(2a-x)\cdot dx$

5. $\displaystyle\int_0^a f(x)\cdot dx = \int_0^a f(a-x)\cdot dx$

6. $\displaystyle\int_{-a}^a f(x)\cdot dx = 2\int_0^a f(x)\cdot dx$, if the function is even.

$\displaystyle\int_{-a}^a f(x)\cdot dx = 0$, if the function is odd.

7. $\displaystyle\int_0^{na} f(x)\cdot dx = n\int_0^a f(x)\cdot dx$ if $f(x) = f(x+a)$

IMPROPER INTEGRALS

An improper integral is a definite integral that has either or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration. Improper integrals cannot be computed using a normal Riemann integral.

Such an integral is often written symbolically like a standard definite integral, perhaps with infinity as a limit of integration.

$$\lim_{b\to\infty}\int_a^b f(x)dx,\ \lim_{a\to-\infty}\int_a^b f(x)dx$$

Figure 1 shows the graph of improper integral $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$.

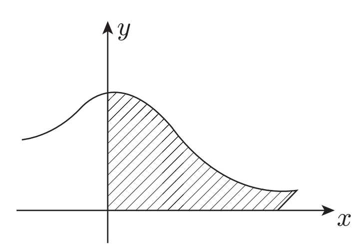


Figure 1 | An improper integral of the first kind.

The integral, $\int_1^\infty x^{-2}dx$ is an example of improper integral. This can be solved as follows:

$$\int_1^y x^{-2}dx = 1-\frac{1}{y}$$

$$\Rightarrow \int_1^\infty x^{-2}dx = \lim_{y\to\infty}\int_1^y x^{-2}dx = \lim_{y\to\infty}\left(1-\frac{1}{y}\right) = 1$$

Improper integrals of the generalized form $\int_a^b f(x)dx$ with one of the limits being infinite and the other being non-zero may also be expressed as finite integrals over transformed functions. Let

$$t = \frac{1}{x}$$

Differentiating both sides, we get

$$dt = -\frac{dx}{x^2}$$

$$dx = -x^2 dt = -\frac{dt}{t^2}$$

Substituting the value of dx in the generalized form, we get

$$\int_a^b f(x)dx = -\int_{1/a}^{1/b}\frac{1}{t^2}f\left(\frac{1}{t}\right)dt = \int_{1/b}^{1/a}\frac{1}{t^2}f\left(\frac{1}{t}\right)dt$$

Now, considering that $f(x)$ diverges as $(x-a)^\gamma$ for $\gamma \in [0,1]$, let

$$x = t^{1/(1-\gamma)} + a$$

Differentiating both sides, we get

$$dx = \frac{1}{(1-\gamma)}t^{[1/(1-\gamma)-1]}dt = \frac{1}{(1-\gamma)}t^{[1-(1-\gamma)]/(1-\gamma)}dt$$

$$= \frac{1}{(1-\gamma)}t^{\gamma/(1-\gamma)}dt$$

$$t = (x-a)^{1-\gamma}$$

$$\Rightarrow \int_a^b f(x)dx = \frac{1}{1-\gamma} = \int_0^{(b-a)^{1-\gamma}} t^{\gamma/(1-\gamma)} f\left(t^{1/(1-\gamma)}+a\right)dt$$

Now, considering that $f(x)$ diverges as $(x+b)^\gamma$ for $\gamma \in [0,1]$, let

$$x = b - t^{1/(1-\gamma)}$$

Differentiating both sides, we get

$$dx = -\frac{1}{(\gamma-1)}t^{[\gamma/(1-\gamma)]}dt$$

$$t = (b-x)^{1-\gamma}$$

$$\Rightarrow \int_a^b f(x)dx = \frac{1}{1-\gamma} = \int_0^{(b-a)^{1-\gamma}} t^{\gamma/(1-\gamma)} f\left(b-t^{1/(1-\gamma)}\right)dt$$

DOUBLE INTEGRATION

Suppose we have a function $f(x, y)$ defined in a closed area Q of xy plane. Now, we can divide Q into n sub-regions ΔQ_k of area ΔA_k, $k = 1, 2, \ldots, n$. Let (a_k, b_k) be any arbitrary point of ΔQ_k. Hence, the sum can be given as

$$\sum_{k=1}^{n} f(a_k, b_k)\,\Delta A_k$$

Now, let us consider

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(a_k, b_k)\,\Delta A_k$$

where limit is taken such that 'n' increases indefinitely and the largest linear dimension of each ΔQ_k approaches zero.

If the limit exists, then the double integral of $f(x, y)$ over region Q is denoted by

$$\iint_Q f(x, y)dA$$

Some of the important properties of double integrals are:

1. When x_1, x_2 are functions of y and y_1, y_2 are constants, then $f(x, y)$ is integrated with respect to x keeping y constant within the limits x_1, x_2 and the resulting expression is integrated with respect to y between the limits y_1, y_2.

$$\iint_Q f(x,y)dxdy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)dxdy$$

2. When y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x, y)$ is first integrated with respect to y, keeping x constant and between the limits y_1, y_2 and the resulting expression is integrated with respect to x within the limits x_1, x_2.

$$\iint_Q f(x,y)dxdy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dydx$$

3. When x_1, x_2, y_1 and y_2 are constants, then

$$\iint_Q f(x,y)dxdy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)dxdy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dydx$$

CHANGE OF ORDER OF INTEGRATION

As already discussed, if limits are constant

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)dxdy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dydx$$

Hence, in a double integral, the order of integration does not change the final result provided the limits are changed accordingly.

However, if the limits are variable, the change of order of integration changes the limits of integration.

TRIPLE INTEGRALS

Suppose we have a function $f(x, y, z)$ defined in a closed three-dimensional region Q. Now, we can divide Q into n sub-regions ΔQ_k of ΔV_k, $k = 1, 2, \ldots, n$. Let (a_k, b_k, c_k) be any point of ΔQ_k.

Hence, the sum can be given as

$$\sum_{k=1}^{n} f(a_k, b_k, c_k)\,\Delta V_k$$

Now, let us consider

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(a_k, b_k, c_k)\,\Delta V_k$$

where limit is taken such that n increases indefinitely and the largest linear dimensions of each ΔQ_k approaches zero.

If the limit exists, then the triple integral of $f(x, y, z)$ over region Q is denoted by

$$\iiint_Q f(x,y,z)dV$$

The limit of $f(x, y, z)$ is continuous in Q.

APPLICATIONS OF INTEGRALS

Integration is used in a wide variety of calculations ranging from the most fundamental to advance physical and mathematical calculations. In this section, we discuss the methods of integration to calculate the area of curves, length of the curves and the volume of revolution.

Area of Curve

Area bounded by the Cartesian curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is $\int_a^b y dx$.

As shown in Fig. 2, area bounded by the polar curve $r = f(\theta)$ and the radii vectors $\theta = \alpha, \beta$ is $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$.

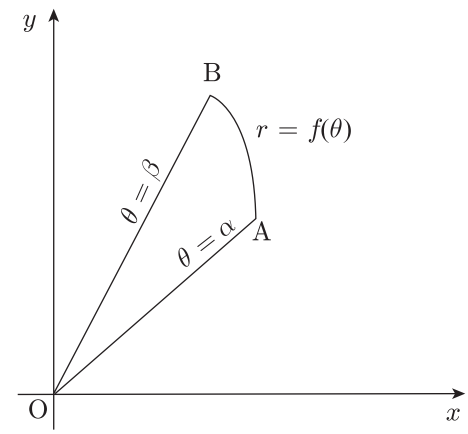


Figure 2 | Polar curve from A to B given by $r = f(\theta)$.

Length of Curve

Consider Fig. 3. The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $y = b$ is $\int_a^b \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}dx$.

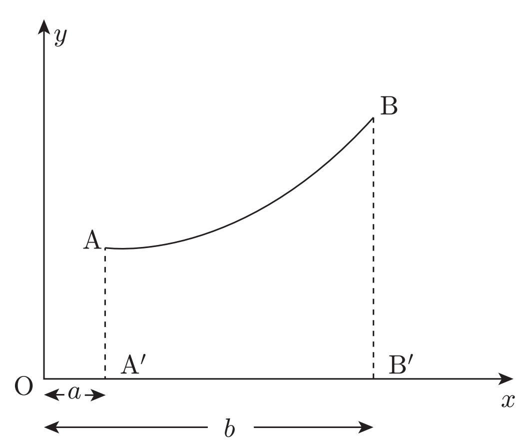


Figure 3 | Curve between A to B given by $y = f(x)$.

The length of the arc of the polar curve $r = f(\theta)$ between the points where $\theta = \alpha, \beta$ is $\int_{\alpha}^{\beta} \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta.$

The length of the arc of the curve $x = f(t), y = g(t)$ between the points where $t = a$ and $t = b$ is

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} dt.$$

Volumes of Revolution

The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, b is $\int_a^b \pi y^2 dx$.

Similarly, the volume of the solid generated by the revolution about the y-axis, of the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, b is $\int_a^b \pi x^2 dy$.

Consider Fig. 4. The volume of the solid generated by the revolution about any axis $\mathrm{A'B'}$ of the area bounded by the curve AB, the axis and the two perpendiculars on the axes $\mathrm{AA'}$ and $\mathrm{BB'}$ is $\int_{\mathrm{OA'}}^{\mathrm{OB'}} \pi (\mathrm{CD})^2 d(\mathrm{OD})$.

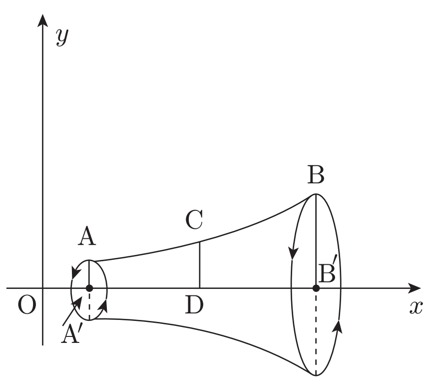


Figure 4 Volume of solid generation by revolution of the curve AB about any axis $\mathrm{A'B'}$.

Consider the polar curve shown in Fig. 5. The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha, \beta$ is given as follows:

(a) For $\theta = 0$, (about the line OX)

$$\int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \sin\theta d\theta$$

(b) For $\theta = \frac{\pi}{2}$, (about the line OY)

$$\int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \cos\theta d\theta$$

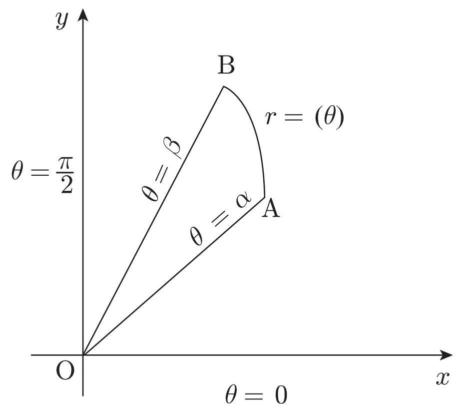


Figure 5 Polar curve from A to B given by $r = f(\theta)$ for $\theta = \alpha$ and $\theta = \beta$.

FOURIER SERIES

Fourier series is a way to represent a wave-like function as a combination of sine and cosine waves. It decomposes any periodic function into the sum of a set of simple oscillating functions (sines and cosines). The Fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx dx$$

The values of a_0, a_n and b_n are known as Euler's formulae.

Conditions for Fourier Expansion

Fourier expansion can be performed on any function $f(x)$ if it fulfills the Dirichlet conditions. The Dirichlet conditions are given as follows:

1. $f(x)$ should be periodic, single-valued and finite.
2. $f(x)$ should have a finite number of discontinuities in any one period.
3. $f(x)$ should have a finite number of maxima and minima.

Fourier Expansion of Discontinuous Function

While deriving the values of a_0, a_n, b_n, we assumed $f(x)$ to be continuous. However, a function may be expressed as a Fourier transform even if the function has a finite number of points of finite discontinuity.

Let us say that we have a function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ and $f(x)$ is defined by

$$\begin{aligned} f(x) &= \phi_1(x), \quad \alpha < x < c \\ &\ \phi_2(x), \quad c < x < \alpha + 2\pi \end{aligned}$$

where c is the point of discontinuity.

Now, we can define the Euler's formulae as follows:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\left[\int_\alpha^c \phi_1(x)dx + \int_c^{\alpha+2\pi} \phi_2(x)dx\right] \\ a_n &= \frac{1}{\pi}\left[\int_\alpha^c \phi_1(x)\cos nx dx + \int_c^{\alpha+2\pi} \phi_2(x)\cos nx dx\right] \\ b_n &= \frac{1}{\pi}\left[\int_\alpha^c \phi_1(x)\sin nx dx + \int_c^{\alpha+2\pi} \phi_2(x)\sin nx dx\right] \end{aligned}$$

At $x = c$, there is a finite jump in the graph of function. Both the limits, left-hand limit (i.e. $f(c - 0)$) and right-hand limit (i.e. $f(c + 0)$), exist and are different. At such a point, Fourier series gives the value of $f(x)$ as the arithmetic mean of these two limits. Hence, at $x = c$,

$$f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$$

Change of Interval

Till now, we have talked about functions having periods of 2π. However, often the period of the function required to be expanded is some other interval (say $2c$). Then, the Fourier expansion is given as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$\begin{aligned} a_0 &= \frac{1}{c}\int_\alpha^{\alpha+2c} f(x)dx \\ a_n &= \frac{1}{c}\int_\alpha^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}dx \\ b_n &= \frac{1}{c}\int_\alpha^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}dx \end{aligned}$$

Fourier Series Expansion of Even and Odd Functions

We already know that a function $f(x)$ is said to be even if $f(-x) = f(x)$ and $f(x)$ is said to be even if $f(-x) = -f(x)$.

Now, a periodic function $f(x)$ defined in $(-c, c)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$\begin{aligned} a_0 &= \frac{1}{c}\int_{-c}^{c} f(x)dx \\ a_n &= \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx \\ b_n &= \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx \end{aligned}$$

Case 1:

When $f(x)$ is an even function,

$$\begin{aligned} a_0 &= \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{2}{c}\int_0^c f(x)dx \\ a_n &= \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx \end{aligned}$$

However, since $f(x)\sin\frac{n\pi x}{c}$ is even, $b_n = \frac{1}{c}\int_{-c}^{c} f(x) \sin\frac{n\pi x}{c}dx = 0$.

Thus, if a periodic function $f(x)$ is even, its Fourier expansion contains only cosine terms, a_0 and a_n.

Case 2:

When $f(x)$ is an odd function,

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = 0$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = 0$$

However,

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx.$$

Thus, if a periodic function $f(x)$ is odd, its Fourier expansion contains only sine terms and b_n.

Half Range Series

Sometimes, it is required to obtain a Fourier expansion of a function $f(x)$ for the range $(0, c)$, which is half the period of the Fourier series. The Fourier expansion of such a function of half the period, therefore, consists of sine or cosine terms only.

If $f(x)$ is required to be expanded as a sine series in $0 < x < c$, then we extend the function reflecting it in the origin, so that $f(x) = -f(-x)$. The extended function is odd in $(-c, c)$ and the expansion can be given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

where $b_n = \dfrac{2}{c}\displaystyle\int_0^c f(x) \sin \frac{n\pi x}{c} dx$

If $f(x)$ is required to be expanded as a cosine series in $0 < x < c$, we extend the function reflecting it in the y-axis, so that $f(x) = f(-x)$. The extended function is even in $(-c, c)$ and the expansion can be given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$$

where

$$a_0 = \frac{2}{c}\int_0^c f(x) dx$$
$$a_n = \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi x}{c} dx$$

VECTORS

Vector is any quantity that has magnitude as well as direction. If we have two points A and B, then vector between A and B is denoted by $\overrightarrow{AB}$.

Position vector is a vector of any points, A, with respect to the origin, O. If A is given by the coordinates x, y and z.

$$\left|\overrightarrow{AP}\right| = \sqrt{x^2 + y^2 + z^2}$$

1. *Zero vector* is a vector whose initial and final points are same. Zero vectors are denoted by $\vec{0}$. They are also called null vectors.
2. *Unit vector* is a vector whose magnitude is unity or one. It is in the direction of given vector $\vec{A}$ and is denoted by $\hat{A}$.
3. *Equal vectors* are those which have the same magnitude and direction regardless of their initial points.

Addition of Vectors

According to triangle law of vector addition, as shown in Fig. 6,

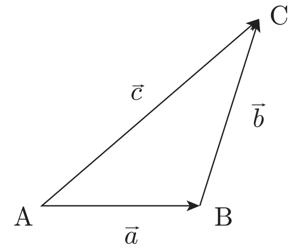


Figure 6 | Triangle law of vector addition.

$$\vec{c} = \vec{a} + \vec{b}$$

According to parallelogram law of vector addition, as shown in Fig. 7,

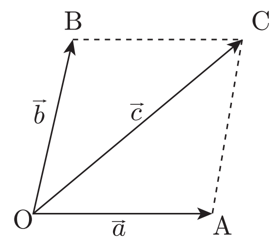


Figure 7 | Parallelogram law of vector addition.

$$\vec{c} = \vec{a} + \vec{b}$$

If we have two vectors represented by adjacent sides of parallelogram, then the sum of the two vectors in magnitude and direction is given by the diagonal of the parallelogram. This is known as parallelogram law of vector addition.

Some important properties of vector addition are:

1. If we have two vectors $\vec{a}$ and $\vec{b}$
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$
2. If we have three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$
$$\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$$

Multiplication of Vectors

1. **Multiplication of a vector with scalar:** Consider a vector $\vec{a}$ and a scalar quantity k. Then
$$|k\vec{a}| = |k||\vec{a}|$$
2. **Multiplication of a vector with another vector using dot product:** Dot product or scalar product of two vectors is given by
$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos \theta$$

where $\left|\vec{a}\right|$ = magnitude of vector $\vec{a}$, $\left|\vec{b}\right|$ = magnitude of vector $\vec{b}$ and θ = angle between $\vec{a}$ and $\vec{b}, 0 \le \theta \le \pi$.

The result of dot product of two vectors is a scalar. Dot product is zero when both the vectors are perpendicular to each other. Dot product is maximum when both the vectors are in the same direction and is minimum when both the vectors are in the opposite direction.

Multiplication of Vectors Using Cross Product

The cross or vector product of two vectors is given by

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n},$$

where $|\vec{a}|$ = magnitude of vector $\vec{a}$

$|\vec{b}|$ = magnitude of vector $\vec{b}$

θ = angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$

$\hat{n}$ = Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

The result of $\vec{a} \times \vec{b}$ is always a vector.

Cross product is zero if the vectors are in the same direction or in the opposite direction (i.e. $\theta = 0$ or $180°$). Cross product is maximum if the angle between the two vectors is $90°$, and it is minimum if the angle between the two vectors is $270°$.

Some important laws of dot product are:

1. $A \cdot B = B \cdot A$
2. $A \cdot B + C = A \cdot B + A \cdot C$
3. $k(A \cdot B) = (kA) \cdot B = A \cdot (kB)$
4. $i \cdot i = j \cdot j = k \cdot k = 1,\ i \cdot j = j \cdot k = k \cdot i = 0$

Some important laws of cross product are:

1. $A \times B = -B \times A$
2. $A \times (B + C) = A \times B + A \times C$
3. $m(A \times B) = (ma) \times B = A \times (mB)$
4. $i \times i = j \times j = k \times k = 0,\ i \times j = k,\ j \times k = i,\ k \times i = j$
5. If $A = A_1 i + A_2 j + A_3 k$ and $B = B_1 i + B_2 j + B_3 k$, then

$$A \times B = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Derivatives of Vector Functions

The derivative of vector $A(x)$ is defined as

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x}$$

if the above limits exists.

If $A(x) = A_1(x)i + A_2(x)j + A_3(x)k$, then

$$\frac{dA}{dx} = \frac{dA_1}{dx} i + \frac{dA_2}{dx} j + \frac{dA_3}{dx} k$$

If $A(x, y, z) = A_1(x, y, z)i + A_2(x, y, z)j + A_3(x, y, z)k$, then

$$dA = \frac{dA}{dx} dx + \frac{dA}{dy} dy + \frac{dA}{dz} dz$$

$$\frac{d}{dy}(A \cdot B) = A \frac{dB}{dy} + \frac{dA}{dy} B$$

$$\frac{d}{dz}(A \times B) = A \times \frac{dB}{dz} + \frac{dA}{dz} \times B$$

A unit vector perpendicular to two given vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{c} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Gradient of a Scalar Field

If we have a scalar function $a(x, y, z)$, then the gradient of this scalar function is a vector function which is defined by

$$\text{grad } \vec{a} = \nabla a = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

The gradient is basically defined as a vector of the same magnitude and direction as that of the maximum space rate of change of $\vec{a}$.

Divergence of a Vector

If we have a differentiable vector $\vec{A}(x, y, z)$, then divergence of vector $\vec{A}$ is given by

$$\text{div } \vec{A} = \nabla \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

where A_x, A_y and A_z are the components of vector $\vec{A}$.

Curl of a Vector

The curl of a continuously differentiable vector $\vec{A}$ is given by

$$\text{curl } \vec{A} = \nabla \vec{A} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (A_x i + A_y j + A_z k)$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) i + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) j + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) k$$

where A_x, A_y and A_z are the components of vector $\vec{A}$.

Thus, the curl of a vector $\vec{A}$ is defined as a vector function of space obtained by taking the vector product of $\vec{A}$.

Some important points of divergence and curl are:

1. $\nabla \cdot \nabla \vec{A} = \nabla^2 A = \dfrac{\partial^2 \vec{A}}{\partial x^2} + \dfrac{\partial^2 \vec{A}}{\partial y^2} + \dfrac{\partial^2 \vec{A}}{\partial z^2}$
2. $\nabla \times \nabla \vec{A} = 0$
3. $\nabla \cdot \nabla \times \vec{A} = 0$
4. $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) \times \nabla^2 \vec{A}$
5. $\nabla (\nabla \cdot \vec{A}) = \nabla \times (\nabla \times \vec{A}) + \nabla^2 \vec{A}$
6. $\nabla (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$
7. $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$
8. $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
9. $\nabla \times (\vec{A} \times \vec{B}) = (B \cdot \nabla)\, A - B\, (\nabla \cdot A) - (A \nabla)\, B + A\, (\nabla B)$

Directional Derivative

The directional derivative of a multivariate differentiate function $f(x, y, z)$ is the rate at which the function changes at a point (x_0, y_0, z_0) in the direction of a vector v.

The directional derivative of a scalar function, $f(x) = f(x_1, x_2, \ldots, x_n)$ along a vector $v = (v_1, \ldots, v_n)$ is a function defined by the limit

$$\nabla_v f(x) = \lim_{h \to 0} \frac{f(x + h\hat{v}) - f(x)}{h}$$

If the function f is differentiable at x, then the directional derivative exists alone any vector v,

$$\nabla_v f(x) = \nabla f(x) \cdot \hat{v}$$

where ∇ on the right-hand side of the equation is the gradient and $\hat{v}$ is the unit vector given by $\hat{v} = \dfrac{v}{|v|}$.

The directional derivative is also often written as follows:

$$\frac{d}{dv} = \hat{v} \cdot \nabla = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$$

Scalar Triple Product

The scalar triple product of three vectors is defined as the dot product of one of the vectors with the cross product of the other two vectors.

Thus, the scalar product of three vectors a, b and c is defined as

$$[a, b, c] = a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$$

The volume of a parallelepiped whose sides are given by the vectors a, b and c, as shown in Fig. 8, can be calculated by the absolute value of the scalar triple product.

$$V_{\text{parallelepiped}} = |a \cdot (b \times c)|$$

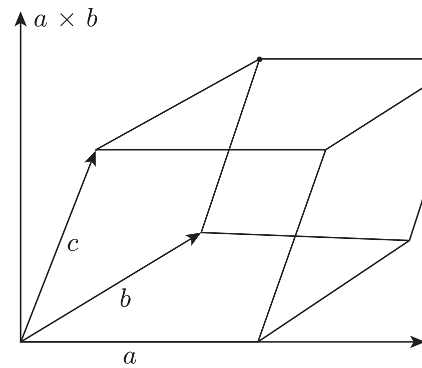


Figure 8 | Parallelepiped with sides given by the vectors a, b and c.

Vector Triple Product

The vector triple product of any three vectors is defined as the cross product of one vector with the cross product of the other two vectors.

Hence, if we have three vectors a, b and c, then the vector triple product is given by

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$$

LINE INTEGRALS

Let c be a curve from points $A(a_1, b_1)$ and $B(a_2, b_2)$ on the xy plane. Let $P(x, y)$ and $Q(x, y)$ be single-valued functions defined at all points of c. Now, c is divided into n parts and $(n - 1)$ points are chosen as $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1})$. Let us define $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$, $k = 1, 2, \ldots, n$ where $(a_1, b_1) \equiv (x_0, y_0)$ and $(a_2, b_2) \equiv (x_n, y_n)$.

Let us say that points (α_k, β_k) are chosen so that they lie on the curve between points (x_{k-1}, y_{k-1}) and (x_k, y_k).

Now, consider the sum,

$$\sum_{k=1}^{n} \{P(\alpha_k, \beta_k)\, \Delta x_k + Q(\alpha_k, \beta_k)\, \Delta y_k\}$$

The limit of the sum as $n \to \infty$ is taken in such a way that all the quantities Δx_k and Δy_k approach zero, and if such limit exists, it is called a line integral along the curve C and is denoted by

$$\int_c [P(x,y)dx + Q(x,y)dy]$$

The limit exists if P and Q are continuous at all points of C. To understand better, refer to Fig. 9.

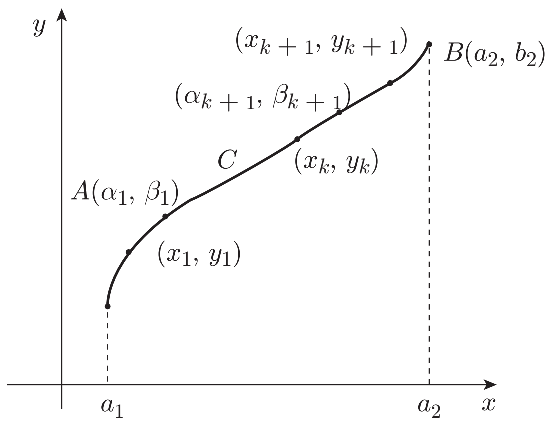


Figure 9 | Line integral.

In the same way, a line integral along a curve C in the three-dimensional space is given by

$$\int_c [A_1 dx + A_2 dy + A_3 dz]$$

where A_1, A_2 and A_3 are functions of x, y and z, respectively.

SURFACE INTEGRALS

Consider C to be a two-sided surface having a projection C' on the xy plane. The equation for C is $z = f(x, y)$, where f is a continuous single-valued function for all values of x and y.

Now, divide C' into n sub-regions of area ΔA_k where $k = 0, 1, 2, \ldots, n$ and join a vertical column on each of the corresponding sub-regions to intersect C in an area ΔC_p.

Suppose $g(x, y, z)$ is single valued and continuous for all values of C. Now, consider the sum

$$\sum_{k=1}^{n} g(\alpha_k, \beta_k, \gamma_k)\, \Delta c_p$$

where $(\alpha_k, \beta_k, \gamma_k)$ is any arbitrary point of ΔC_k. If the limit of this sum, $n \to \infty$ is in such a way that each $\Delta C_k \to 0$ exists, the resulting limit is called the surface integral of $g(x, y, z)$ over C.

The surface integral is denoted by

$$\iint_C g(x, y, z) \cdot ds$$

Figure 10 graphically represents surface integrals.

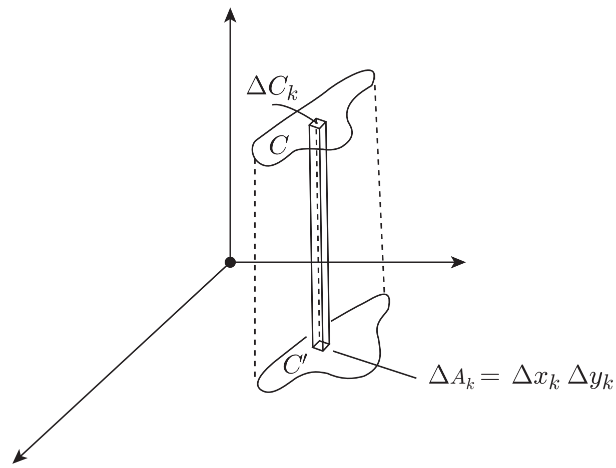


Figure 10 | Surface integrals.

STOKES' THEOREM

Let S be an open, two-sided surface bounded by a simple closed curve C. Also, if $\vec{V}$ is a single-valued, continuous function, then according to Stokes' theorem, "The line integral of the tangential component of a vector $\vec{V}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\vec{V}$ taken over any surface S having C as the boundary."

It is denoted by

$$\int_c A \cdot dr = \iint_s (\nabla \times A) \cdot n ds$$

GREEN'S THEOREM

Let S be a surface bounded by a simple closed curve C. Let $f_1(x, y)$ and $f_2(x, y)$ be continuous functions and $\dfrac{\partial f_1}{\partial x}$ and $\dfrac{\partial f_2}{\partial y}$ be continuous partial derivatives in S, then according to Green's theorem,

$$\iint_s \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} \right) dx dy = \oint_c (f_2 dx + f_1 dy)$$

GAUSS DIVERGENCE THEOREM

Let S be a closed surface bounding a region of volume V. Assuming the outward drawn normal to the surface as positive normal and considering α, β and γ as the angles which this normal makes with x, y and z axes, respectively.

Also, if A_1, A_2 and A_3 are continuous and have continuous partial derivatives in the region, then

$$\iiint_v \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_s (A_1 \cos \alpha + A_2 \cos \beta + A_3 \cos \gamma) dS$$

$$\Rightarrow \iiint_v \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_s (A_1 dydz + A_2 dzdx + A_3 dxdy)$$

In vector form, with $A = A_1 i + A_2 j + A_3 k$ and $\hat{n} = \cos \alpha i + \cos \beta j + \cos \gamma k$,

$$\iiint_v \nabla \cdot A dV = \iint_s A \cdot \hat{n} dS$$

Divergence theorem states that the surface integral of the normal components of a vector $\vec{A}$ taken over a closed surface is equal to the integral of the divergence of $\vec{A}$ taken over the volume enclosed by the surface.

SOLVED EXAMPLES

1. What is the value of $\lim\limits_{x \to 0} \dfrac{\sin\left[\frac{4}{3}x\right]}{x}$?

Solution: We have

$$\lim_{x \to 0} \frac{\sin\left[\frac{4}{3}x\right]}{x}$$

$$= \lim_{\frac{4}{3}x \to 0} \frac{4}{3} \frac{\sin\left[\frac{4}{3}x\right]}{\frac{4}{3}x}$$

$$= \frac{4}{3} \lim_{\frac{4}{3}x \to 0} \frac{\sin\left[\frac{4}{3}x\right]}{\frac{4}{3}x}$$

$$= \frac{4}{3} \times 1 = \frac{4}{3}$$

2. What is the value of $\lim\limits_{x \to 0} \dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$?

Solution: When $x \to 2$, $\dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8} = \dfrac{0}{0}$

Hence, we apply L'Hospital's rule,

$$\lim_{x \to 2} \frac{3x^2 - 12x + 11}{2x - 6} = \frac{3(2)^2 - 12(2) + 11}{2(2) - 6}$$

$$= \frac{12 - 24 + 11}{-2} = \frac{-1}{-2} = \frac{1}{2}$$

3. What is the value of $\lim\limits_{x \to \infty} \dfrac{x + \sin x}{x - x \cos x}$?

Solution: We have

$$\boxed{\lim_{x \to \infty} \frac{x + \sin x}{x - x \cos x}} = \lim_{x \to \infty} \frac{1 + \frac{\sin x}{x}}{1 + \frac{\cos x}{x}} = \frac{1 - \lim\limits_{x \to \infty} \frac{\sin x}{x}}{1 + \lim\limits_{x \to \infty} \frac{\cos x}{x}}$$

$$\boxed{= \frac{1 - 0}{1 + 0} = 1}$$

4. What is the value of $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{x}$?

Solution: We have

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x}$$

$$= \lim_{x \to 0} \frac{2 \sin^2 x}{x}$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \cdot \sin x = 2\left(\lim_{x \to 0} \frac{\sin x}{x}\right)\left(\lim_{x \to 0} \sin x\right)$$

$$= 2(1)(0) = 0$$

5. If a function is given by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

Find out whether or not $f(x)$ is continuous at $x = 0$.

Solution: We have

L.H.L. at $x = 0$

$$= \lim_{x \to 0} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \left[\frac{\sin(-h)}{-h} + \cos(-h)\right] = 1 + 1 = 2$$

R.H.L. at $x = 0$

$$= \lim_{x \to 0} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \left[\frac{\sin h}{h} + \cos h\right] = 1 + 1 = 2$$

Also, we know that $f(0) = 2$. Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$.

Hence, $f(x)$ is continuous at $x = 0$.

6. Discuss the continuity of the function $f(x)$ at $x = 1/2$, where

$$f(x) = \begin{cases} 1/2^{-x}, & 0 \le x < 1/2 \\ 1, & x = 1/2 \\ 3/2^{-x}, & 1/2 < x \le 1 \end{cases}$$

Solution: We have

LHL at $x = 1/2$

$$= \lim_{x\to 1/2^-} f(x) = \lim_{x\to 1/2} (1/2 - x)$$

$$= -1/2 - 1/2 = 0$$

RHL at $x = 1/2$

$$= \lim_{x\to 1/2^+} f(x) = \lim_{x\to 1/2} (3/2 - x)$$

$$= 3/2 - 1/2 = 1$$

Since, $\lim_{x\to \frac{1}{2}^-} f(x) \neq \lim_{x\to \frac{1}{2}^+} f(x)$

Hence, $f(x)$ not continuous at $x = \frac{1}{2}$.

7. Discuss the continuity of $f(x) = 2x - |x|$ at $x = 0$.

Solution: We have

$$f(x) = 2x - |x| = \begin{cases} 2x - x, & \text{if } x \ge 0 \\ 2x - (-x), & \text{if } x < 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} x, & \text{if } x \ge 0 \\ 3x, & \text{if } x < 0 \end{cases}$$

Now,

L.H.L. at $x = 0$

$$= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3x = 3 \times 0 = 0$$

R.H.L. at $x = 0$

$$= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$

and $f(0) = 0$

$$= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} x\, f(x) = f(0)$$

So, $f(x)$ is continuous at $x = 0$.

8. For what value of λ is the function $f(x)$ continuous at $x = 3$?

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ \lambda, & x = 3 \end{cases}$$

Solution: Since $f(x)$ is continuous at $x = 3$,

$$\lim_{x\to 3} f(x) = f(3)$$

$$\lim_{x\to 3} f(x) = \lambda$$

$$\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lambda$$

$$\lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)} = \lambda$$

$$\lim_{x\to 3} (x+3) = \lambda$$

$$3 + 3 = \lambda$$

$$\lambda = 6$$

9. Discuss the differentiability of the function

$$f(x) = \begin{cases} x - 1, & \text{if } x < 2 \\ 2x - 3, & \text{if } x \ge 2 \end{cases}$$

Solution: At $x = 2$

$$(\text{L.H.D. at } x = 2) = \lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x\to 2} \frac{(x-1) - (4-3)}{x - 2}$$

$$= \lim_{x\to 2} \frac{x-2}{x-2} = \lim_{x\to 2} 1 = 1$$

$$(\text{R.H.D. at } x = 2) = \lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x\to 2} \frac{(2x-3) - (4-3)}{x - 2}$$

$$= \lim_{x\to 2} \frac{2(x-2)}{x-2} = \lim_{x\to 2} 2 = 2$$

Therefore, L.H.D. $\neq$ R.H.D.

Hence, $f(x)$ is not differentiable at $x = 2$.

10. Discuss the differentiability of $f(x) = x|x|$ at $x = 0$.

Solution: We have

$$f(x) = x|x| = \begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$$

$$(\text{L.H.D. at } x = 0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x\to 0} \frac{-x^2 - 0}{x - 0} = \lim_{x\to 0} (-x) = 0$$

$$(\text{R.H.D. at } x = 0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x\to 0} \frac{x^2 - 0}{x - 0} = \lim_{x\to 0} x = 0$$

Therefore, (L.H.D. at $x = 0$) = (R.H.D. at $x = 0$)

Hence, $f(x)$ is differentiable at $x = 0$.

11. Discuss the applicability of Rolle's theorem for the function $f(x) = x^2 - 5x + 6$ on the interval [2, 3].

Solution: We know that

(i) $f(x)$ is continuous on [2, 3]

(ii) $f(x)$ is differentiable on [2, 3]

[$\because$ a polynomial function is differentiable everywhere]

(iii) $f(2) = (2)^2 - 5(2) + 6 = 0$

$f(3) = (3)^2 - 5(3) + 6 = 0$

Thus, $f(2) = f(3)$.

Hence, Rolle's theorem is applicable. Therefore, there exists a value $c \in (2,3)$ such that $f'(c) = 0$.

We have

$$f(x) = x^2 - 5x + 6 \quad \Rightarrow \quad f'(x) = 2x - 5$$
$$f'(x) = 2x - 5 = 0 \quad \Rightarrow \quad x = 2.5$$

Thus, $c = 2.5 \in (2,3)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

12. Discuss the applicability of Rolle's theorem for $f(x) = |x|$ on $[-1, 1]$.

Solution: We have

$$f(x) = \begin{cases} -x, & \text{when } -1 \le x < 0 \\ x, & \text{when } 0 \le x \le 6 \end{cases}$$

Function $f(x)$ is continuous and differentiable at all points $x < 0$ and $x > 0$, since it is a polynomial function.

However, we have to check for continuity and differentiability at $x = 0$.

Now, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0}(-x) = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} x = 0$

$f(0) = 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Thus, $f(x)$ is continuous at $x = 0$ and hence continuous on $[-1, 1]$.

Checking for differentiability,

$$(\text{L.H.D. at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$
$$= \lim_{x \to 0} \frac{-x - 0}{x - 0} = \lim_{x \to 0}(-1) = -1$$

$$(\text{R.H.D. at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$
$$= \lim_{x \to 0} \frac{x - 0}{x} = \lim_{x \to 0} 1 = 1$$

(L.H.D. at $x = 0$) $\neq$ (R.H.D. at $x = 0$)

Hence, $f(x)$ is not differentiable at $x = 0 \in (-1, 1)$.

Thus, Rolle's theorem is not applicable.

13. Verify Rolle's theorem for the function $f(x) = \sin x + \cos x - 1$ on $[0, \pi/2]$.

Solution: As sin x and cos x are continuous and differentiable everywhere, $f(x) = \sin x + \cos x - 1$ is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$.

Now, $f(0) = \sin 0 + \cos 0 - 1 = 0 + 1 - 1$
$= 0$

$f(\pi/2) = \sin \pi/2 + \cos \pi/2 - 1 = 1 + 0 - 1$
$= 0$

$\therefore f(0) = f(\pi/2)$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore, Rolle's theorem is applicable, i.e. there exists $c \in (0, \pi/2)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin x + \cos x - 1 \quad \Rightarrow f'(x) = \cos x - \sin x$$

Also, $f'(x) = 0$

$\therefore f'(x) = \cos x - \sin x = 0$

$\Rightarrow \sin x = \cos x$

$\Rightarrow \tan x = 1$

$\Rightarrow x = \pi/4$

Thus, $c = \pi/4 \in (0, \pi/2)$ such that $f'(c) = 0$.

14. Verify Lagrange's mean value theorem for $f(x) = x(x - 2)$ on [1, 3].

Solution: We have

$$f(x) = x(x - 2) = x^2 - 2x$$

We know that a polynomial function is continuous and differentiable everywhere. So, $f(x)$ is continuous on [1, 3] and differentiable on (1, 3).

Hence, $f(x)$ satisfies both the conditions of Langrage's mean value theorem on [1, 3], and hence there exists at least one real number $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

Now, $f(x) = x^2 - 2x$

$\Rightarrow f'(x) = 2x - 2$

$$f(3) = 9 - 6 = 3 \text{ and } f(1) = 1 - 2 = -1$$

$$f'(x) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow\ 2x - 2 = \frac{3 - (-1)}{3 - 1}$$

$$\Rightarrow\ 2x - 2 = 2 \Rightarrow x = 2$$

Thus, $c = 2 \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1}$.

Hence, Lagrange's mean value theorem is verified for $f(x)$ on $[1, 3]$.

15. Verify Lagrange's mean value theorem for $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$.

Solution: $\sin x$ and $\sin 2x$ are continuous and differentiable everywhere, therefore $f(x)$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

Thus, $f(x)$ satisfies both the conditions of Lagrange's mean value theorem.

There exists at least one $c \in (0, \pi)$ such that $f'(c) = \dfrac{f(\pi) - f(0)}{\pi - 0}$

$$f(x) = 2\sin x + \sin 2x$$

$$f'(x) = 2\cos x + 2\cos 2x$$

$f(0)$ and $f(\pi) = 2\sin \pi + \sin 2\pi = 0$

$$\therefore\ f'(x) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$\Rightarrow\quad 2\cos x + 2\cos 2x = \frac{0 - 0}{\pi - 0}$$

$$\Rightarrow\quad 2\cos x + 2\cos 2x = 0$$

$$\Rightarrow\quad \cos x + \cos 2x = 0$$

$$\Rightarrow\quad \cos 2x = -\cos x$$

$$\Rightarrow\quad \cos 2x = \cos(\pi - x)$$

$$\Rightarrow\quad 2x = \pi - x$$

$$\Rightarrow\quad 3x = \pi \quad \Rightarrow x = \pi/3$$

Thus, $c = \pi/3 \in (0, \pi)$ such that $f'(c) = \dfrac{f(\pi) - f(0)}{\pi - 0}$.

Hence, Lagrange's mean value theorem is verified.

16. Differentiate the function $f(x) = \tan^{-1}\left(\dfrac{\sqrt{1 + x^2} + 1}{x}\right)$.

Solution: Let $y = \tan^{-1}\left(\dfrac{\sqrt{1 + x^2} + 1}{x}\right)$, put $x = \tan\theta$, i.e. $\theta = \tan^{-1} x$.

Now, after substituting the value, we get

$$y = \tan^{-1}\left(\frac{\sqrt{1 + \tan^2\theta} + 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta + 1}{\tan\theta}\right) = \tan^{-1}\left[\frac{\frac{1}{\cos\theta} + 1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left(\frac{1 + \cos\theta}{\sin\theta}\right) = \tan^{-1}\left[\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]$$

$$= \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi}{2} - \frac{1}{2}\tan^{-1} x$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{2}\cdot\frac{1}{1 + x^2} = \frac{-1}{2(1 + x^2)}$$

17. Differentiate the function $f(x) = \dfrac{e^x + \sin x}{1 + \log x}$

Solution: We have

$$f(x) = \frac{e^x + \sin x}{1 + \log x}$$

$$f'(x) = \frac{(1 + \log x)\frac{d}{dx}(e^x + \sin x) - (e^x + \sin x)\frac{d}{dx}(1 + \log x)}{(1 + \log x)^2}$$

$$= \frac{(1 + \log x)(e^x + \cos x) - (e^x + \sin x)\left[0 + \frac{1}{x}\right]}{(1 + \log x)^2}$$

$$= \frac{(1 + \log x)(e^x + \cos x) - \frac{e^x + \sin x}{x}}{(1 + \log x)^2}$$

18. Differentiate the function $f(x) = x^{x^x}$.

Solution: Let $y = x^{x^x}$, then

$$y = ex^x \cdot \log x$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x\log x^x}\frac{d}{dx}(x^x \cdot \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^{x^x}\frac{d}{dx}(e^{x\log x} \cdot \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^{x^x}\left[\log x \cdot e^{x\log x}\frac{d}{dx}(x\log x) + e^{x\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$= x^{x^x}\left[\log x \cdot e^{x\log x}\frac{d}{dx}(x\log x) + e^{x\log x}\cdot\frac{1}{x}\right]$$

$$= x^{x^x}\left[\log x \cdot x^x\left(x\cdot\frac{1}{x}+\log x\right) + x^x\cdot\frac{1}{x}\right]$$

$$= x^{x^x}\left[x^x(1+\log x)\cdot\log x + \frac{x^x}{x}\right]$$

$$= x^{x^x}\cdot x^x\left[(1+\log x)\cdot\log x + \frac{1}{x}\right]$$

19. If $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$. Hence, show that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots - \infty = \frac{1}{4}(\pi-2)$.

Solution: Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x\cos nx dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad (n \ne 1)$$

$$= \frac{1}{2\pi}\left\{\frac{1-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right\} = 0, \quad \text{when } n \text{ is odd}$$

$$\text{and } -\frac{2}{\pi(n^2-1)}, \quad \text{when } n \text{ is even}$$

When $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\cos x dx = \frac{1}{2\pi}\int_0^{\pi}\sin 2x dx$$

$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x\sin nx dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos\overline{n-1}x - \cos\overline{n+1}x\right]dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin\overline{n-1}x}{n-1} - \frac{\sin\overline{n+1}x}{n+1}\right]_0^{\pi} = 0 \quad (n \ne 1)$$

When $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\sin x dx$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi}(1-\cos 2x)dx\right]$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

Hence,

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right] + \frac{1}{2}\sin x \qquad (1)$$

Putting $x = \frac{\pi}{2}$ in Eq. (1), we get 1

$$= \frac{1}{\pi} - \frac{2}{\pi}\left(-\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots\infty\right) + \frac{1}{2}$$

Hence, $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots\infty = \frac{1}{4}(\pi - 2)$.

20. Prove that the function $f(x) = x^3 - 6x^2 + 12x - 9$ is strictly increasing on R.

Solution: We have

$$f(x) = x^3 - 6x^2 + 12x - 9, \quad x \in R$$

Differentiating with respect to x, we get

$$f'(x) = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4)$$

$$= 3(x-2)^2 \ge 3 \quad (\because (x-2)^2 \ge 0 \text{ for all } x \in R)$$

$\Rightarrow f'(x) > 0$ for all $x \in R$

$\Rightarrow f(x)$ is strictly increasing function for all $x \in R$.

21. Find the intervals in which the function $f(x) = x^4 - \frac{x^3}{3}$ is decreasing.

Solution: We have

$$f(x) = x^4 - \frac{x^3}{3}$$

$$\Rightarrow \quad f'(x) = 4x^3 - x^2 = x^2(4x-1)$$

For $f(x)$ to be decreasing, we have $f'(x) < 0$

$$x^2(4x-1) < 0$$

$$\Rightarrow \quad (4x-1) < 0 \qquad (\because x^2 > 0)$$

$$\Rightarrow \quad 4x < 1 \quad \Rightarrow x < \frac{1}{4}$$

$$\Rightarrow \quad x \in \left(-\infty, \frac{1}{4}\right)$$

Hence, $f(x)$ is decreasing on $\left(-\infty, \frac{1}{4}\right)$.

22. Find the points of local maxima and the corresponding maximum values of the function,

$$f(x) = 2x^3 - 21x^2 + 36x$$

Solution: We have

$$f(x) = 2x^3 - 21x^2 + 36x$$
$$f'(x) = 6x^2 - 42x + 36$$

For local maximum, we have $f'(x) = 0$

$$\Rightarrow \quad 6x^2 - 42x + 36 = 0 \quad \Rightarrow (x-1)(x-6) = 0$$
$$\Rightarrow x = 1, 6$$

Thus, $x = 1$ and $x = 6$ are the possible points of local maxima or minima.

Now, $f''(x) = 12x - 42$

At $x = 1$, we have

$$f''(1) = 12 - 42 = -30 < 0$$

Hence, $x = 1$ is a point of local maximum.

The local maximum value is $f(1) = 2 - 21 + 36 - 20 = -3$

At $x= 6$, we have

$$f''(6) = 12(6) - 42 = 30 > 0$$

Hence, $x = 6$ is a point of local maximum.

23. Find the points of local maxima or minima for the function,

$$f(x) = x^3 + x$$

Solution: We have

$$f(x) = x^3 + x$$
$$f'(x) = 3x^3 + 1$$

For a maximum or minimum, we have

$$f'(x) = 0 \quad \Rightarrow \quad 3x^2 + 1 = 0 \quad \Rightarrow \quad x = \pm \frac{i}{\sqrt{3}}$$

This gives the imaginary values of x, hence $f'(x) \neq 0$ for any real value of x.

Hence, $f(x)$ does not have a maximum or minimum.

24. Find the absolute maximum and minimum values of $f(x) = \sin x + \frac{1}{2}\cos 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Solution: We have

$$f(x) = \sin x + \frac{1}{2}\cos 2x \quad \text{in } \left[0, \frac{\pi}{2}\right]$$

Differentiating with respect to x, we get

$$f'(x) = \cos x - \sin 2x$$

For absolute maximum and absolute minimum,

$$f'(x) = 0$$
$$\Rightarrow \cos x - 2\sin x \cos x = 0$$
$$\Rightarrow \cos x(1 - 2\sin x) = 0$$
$$\Rightarrow \cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{4}$$
$$x = \frac{\pi}{2} \text{ or } \frac{\pi}{6}$$

Now, $f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\cos\pi = 1 - \frac{1}{2} = \frac{1}{2}$$
$$f(0) = \sin 0 + \frac{1}{2}\cos 0 = 0 + \frac{1}{2} = \frac{1}{2}$$

The absolute maximum value $= 3/4$.

The absolute minimum value $= 1/2$.

25. Derive the Taylor's series expansion of $\frac{\sin x}{x - \pi}$ at $x = \pi$.

Solution: The Taylor's series expansion of $f(x)$ around $x = \pi$ is

$$f(x) = f(\pi) + \frac{x-\pi}{\lfloor 1} f'(x) + \frac{(x-\pi)^2}{\lfloor 2} f''(\pi) + \cdots$$

Now, $f(\pi) = \lim_{x\to\pi} \frac{\sin x}{x-\pi} = \frac{0}{0}$

Hence, we apply L'Hospital's rule,

$$\lim_{x\to\pi} \frac{\cos x}{1} = -1$$

Similarly by using L'Hospital's rule, we can show that

$$f'(\pi) = 0$$
$$f''(\pi) = -1/6$$

So, the expansion is $f(x) = -1 + (-1/6)(x-\pi)^2 + \cdots$

Thus, $f(x) = -1 - \frac{(x-\pi^2)}{\underline{|3}} + \cdots$

26. Expand $e^{\sin x}$ by Maclaurin's series up to the term containing x^4.

Solution: We have

$$f(x) = e^{\sin x}$$
$$f'(x) = e^{\sin x}\cos x \cdot f(x)\cdot \cos x$$
$$f''(x) = f'(x)\cos x - f(x)\sin x \quad f''(0) = 1$$
$$f'''(x) = f''(x)\cos x - 2f'(x)\sin x$$
$$-f(x)\cos x, f'''(0) = 0$$
$$f''''(x) = f'''(x)\cos x - 3f'(x)\sin x$$
$$-3f'(x)\cos x\cdot f(x)\sin x,$$
$$f''''(0) = 0$$

and so on.

Substituting the values of $f(0), f'(0)$, etc. in the Maclaurin's series, we get

$$e^{(\sin x)} = 1 + x\cdot 1 + \frac{x^2\cdot 1}{2!} + \frac{x^3\cdot 0}{3!} + \frac{x^4\cdot(-3)}{4!} + \cdots$$
$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

27. Expand $\log\tan[(\pi/4)+x]$ in ascending power of x till x^5.

Solution: Using Taylor's theorem, we know that

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h)$$
$$+ \frac{x^3}{3!}f'''(h) + \cdots \quad (1)$$

If $f(x+h) = \log\tan[x+(\pi/4)]$

then $f(x) = \log\tan x$ and $f(h) = \log\tan(h)$ (2)

Differentiating Eq. (2) successively with respect to h, we get

$$f'(h) = \frac{\sec^2 h}{\tanh} = 2\operatorname{cosec} 2h$$
$$f''(h) = -4\operatorname{cosec} 2h\cot 2h$$
$$f'''(h) = -8 - [\operatorname{cosec}2h\cot^2 2h + 5\operatorname{cosec}^3 2h]$$
$$f''''(h) = -16[\operatorname{cosec}2h\cot^3 2h + 5\operatorname{cosec}^3 2h\cot 2h]$$
$$f'''''(h) = 32\,[\operatorname{cosec}2\text{h}\cot^4 2h + 3\cot^2 2h\operatorname{cosec}^3 2h$$
$$+ 5\operatorname{cosec}^5 2h + 15\operatorname{cosec}^3 2h\cot^2 2h]$$

Now, substituting the value of $h = \pi/4$, we get

$$f(\pi/4) = 0$$
$$f'(\pi/4) = 2$$
$$f''(\pi/4) = 0$$
$$f'''(\pi/4) = 8$$
$$f''''(\pi/4) = 0$$
$$f'''''(\pi/4) = 160$$

Putting these values in Eq. (1), we have

$$f(x+h) = 0 + 2x - \frac{x^2}{2!}(0) + \frac{x^3}{3!}\times 8 + \frac{x^4}{4!}\times(0)$$
$$+ \frac{x^5}{5!}\times 160 = 2x + \frac{4}{3}x^3 + \frac{4}{3}x^5 + \cdots$$

28. Evaluate $\int \frac{2x}{\sqrt{1-x^2-x^4}}\,dx$

Solution: Let $x^2 = t$

Then, $d(x^2) = dt \Rightarrow 2x\cdot dx = dt \Rightarrow dx = \frac{dt}{2x}$

Therefore,

$$f(x) = \int \frac{2x}{\sqrt{1-x^2-x^4}}\,dx = \int \frac{dt}{\sqrt{1-t-t^2}}$$
$$= \int \frac{dt}{\sqrt{-(t^2+t-1)}}$$
$$= \int \frac{dt}{\sqrt{-\left(t^2+t+\frac{1}{4}-\frac{1}{4}-1\right)}}$$
$$= \int \frac{dt}{\sqrt{-\left[\left(t+\frac{1}{2}\right)^2-\frac{5}{4}\right]}}$$
$$= \int \frac{dt}{\sqrt{5/4-\left(t+\frac{1}{2}\right)^2}} = \int \frac{dt}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(t+\frac{1}{2}\right)^2}}$$
$$f(x) = \sin^{-1}\left[\frac{t+1/2}{\sqrt{5}/2}\right] + \mathrm{C}$$
$$= \sin^{-1}\left(\frac{2t+1}{\sqrt{5}}\right) + \mathrm{C}$$

Substituting $t = x^2$,

$$f(x) = \sin^{-1}\left(\frac{2x^2+1}{\sqrt{5}}\right) + \mathrm{C}$$

29. Find the Fourier series expansion of $f(x) = 2x - x^2$ in (0, 3) and hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots - \infty = \frac{\pi}{12}$.

Solution: The required series is of the form,

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $l = 3/2$.

Then

$$a_0 = \frac{1}{l}\int_0^{2l} (2x - x^2)dx = \frac{2}{3}\left|x^2 - \frac{x^3}{3}\right|_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{\sin 2n\pi x}{2n\pi/3} - (2-2x)\frac{-\cos 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}\left[(2-6)\cos 2n\pi - 2\right] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{-\cos 2n\pi x/3}{2n\pi/3} - (2-2x)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{\cos 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\left\{-\frac{6}{n^2\pi^2}\cos 2n\pi - \frac{27}{4n^3\pi^3}(\cos 2n\pi - 1)\right\}$$

$$= \frac{3}{n\pi}$$

Substituting the values of a_0, a_n, b_n in Eq. (1), we get

$$2x - x^2 = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

30. If $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, prove that $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}$. Hence, show that $\frac{1}{1\cdot 3} - \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots - \infty = \frac{1}{4}(\pi - 2)$

Solution: Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x\cos nx dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad (n \neq 1)$$

$$= \frac{1}{2\pi}\left\{\frac{1-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1} - 1}{n-1}\right\} = 0,$$

when n is odd

and $-\frac{2}{\pi(n^2 - 1)}$, when n is even.

When $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\cos x dx = \frac{1}{2\pi}\int_0^{\pi}\sin 2x dx$$

$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi}\sin x\sin nx dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos\overline{n-1}x - \cos\overline{n+1}x\right]dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin\overline{n-1}x}{n-1} - \frac{\sin\overline{n+1}x}{n+1}\right]_0^{\pi} = 0 \quad (n \neq 1)$$

When $n = 1$,

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x \sin x dx = \frac{1}{2\pi}\left[\int_0^{\pi}(1-\cos 2x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

Hence, $f(x) = \dfrac{1}{\pi} - \dfrac{2}{\pi}\left[\dfrac{\cos 2x}{2^2-1} + \dfrac{\cos 4x}{4^2-1} + \dfrac{\cos 6x}{6^2-1} + \cdots\right] + \dfrac{1}{2}\sin x \quad (1)$

Putting $x = \dfrac{\pi}{2}$ in Eq. (1), we get $1 = \dfrac{1}{\pi} - \dfrac{2}{\pi}\left(-\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} - \dfrac{1}{5\cdot 7} + \cdots\infty\right) + \dfrac{1}{2}$

Hence, $\dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \cdots\infty = \dfrac{1}{4}(\pi - 2)$

31. Evaluate $\int x\sin^{-1} dx$.

Solution: We have

$$f(x) = \int x\sin^{-1} x \cdot dx$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}\cdot dx$$

$$= \frac{x^2}{2}\cdot\sin^{-1} x + \frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}\cdot dx = \frac{x^2}{2}\cdot\sin^{-1} x + \frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}\cdot dx$$

$$= \frac{x^2}{2}\cdot\sin^{-1} x + \frac{1}{2}\left[\int\frac{1-x^2}{\sqrt{1-x^2}}\cdot dx - \int\frac{1}{\sqrt{1-x^2}}\cdot dx\right]$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\left[\int\sqrt{1-x^2}dx - \int\frac{1}{\sqrt{1-x^2}}dx\right]$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\left[\left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right) - \sin^{-1} x\right] + C$$

$$= \frac{1}{2}x^2\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} - \frac{1}{4}\sin^{-1} x + C$$

32. Evaluate $\displaystyle\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$.

Solution: Let

$$f(x) = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \quad (1)$$

Since, $\displaystyle\int_0^a f(x)\cdot dx = \int_0^a f(a-x)dx$

$$f(x) = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}\cdot dx \quad (2)$$

Adding Eqs. (1) and (2), we get

$$2f(x) = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}\cdot dx$$

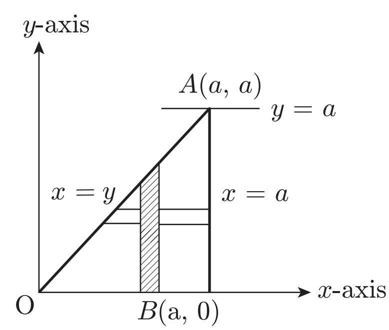
$$2f(x) = \int_0^a dx$$

$$2f(x) = a$$

$$f(x) = \frac{a}{2}$$

33. Change the order of integration in $\displaystyle\int_0^a\int_y^a \frac{x\cdot dxdy}{x^2+y^2}$ and evaluate the same.

Solution: From the limit of integration, it is clear that region of integration is bounded by $x = y$, $x = a$, $y = 0$ and $y = a$. Hence, region of integration can be formed as follows:



Hence, it is clear that region of integration is given by ΔOAB and divided into horizontal strips. For changing the order of integration, we divide the region into vertical strips.

Now, the new limits become 0 to x for y and 0 to a for x.

$$\Rightarrow \int_0^a\int_y^a \frac{x}{x^2+y^2}dx\,dy = \int_0^a\int_0^x \frac{x}{x^2+y^2}dy\,dx$$

$$= \int_0^a x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$= \int_0^a \frac{\pi}{4}dx = \frac{\pi}{4}[x]_0^a = \frac{\pi a}{4}$$

34. Find the area of the triangle whose vertices are $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$.

Solution: Let $\vec{a}, \vec{b}, \vec{c}$ be the positive vectors of points $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$, respectively.

Then, $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} - 3\hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + \hat{k}$.

We have

Area of triangle $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

Now, $\overrightarrow{AB} = \vec{b} - \vec{a} = \hat{i} - \hat{j} - 3\hat{k} - (3\hat{i} - \hat{j} + 2\hat{k}) = 2\hat{i} - 5\hat{k}$

$$\overrightarrow{AC} = \vec{c} - \vec{a} = 4\hat{i} - 3\hat{j} + \hat{k} - (3\hat{i} - \hat{j} + 2\hat{k}) = \hat{i} - 2\hat{j} - \hat{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= (0 - 10)\hat{i} - (2 + 5)\hat{j} + (4 - 0)\hat{k}$$

$$= -10\hat{i} - 7\hat{j} + 4\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-10)^2 + (-7)^2 + (4)^2} = \sqrt{165}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{165}$$

35. Prove that $\int_1^2\int_3^4 (xy + e^y)dy \cdot dx = \int_3^4\int_1^2 (xy + e^y)dx \cdot dy$

Solution: We have

$$\text{L.H.S.} = \int_1^2\int_3^4 (xy + e^y)dy \cdot dx = \int_1^2 \left[\frac{xy^2}{2} + e^y\right]_3^4 dx$$

$$= \int_1^2 \left(8x + e^4 - \frac{9}{2}x - e^3\right)dx$$

$$= \int_1^2 \left[\frac{7}{2}x + e^4 - e^3\right] \cdot dx = \left[\frac{7}{4}x^2 + (e^4 - e^3)x\right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3)$$

$$= \frac{21}{4} + e^4 - e^3$$

Now, $\text{R.H.S.} = \int_3^4\int_1^2 (xy + e^y)\,dx \cdot dy$

$$= \int_3^4 \left[\frac{x^2y}{2} + e^y \cdot x\right]_1^2 \cdot dy$$

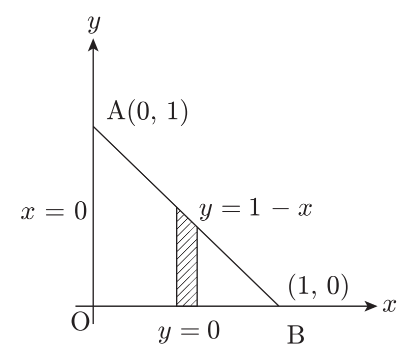
$$\int_3^4 \left[2y + 2e^y - \frac{y}{2} - e^y\right] \cdot dy = \int_3^4 \left(\frac{3}{2}y + e^y\right) \cdot dy$$

$$= \left(\frac{3}{4}y^2 + e^y\right)_3^4 = 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3$$

Therefore, L.H.S. = R.H.S. Hence proved.

36. Evaluate $\iint_R e^{2x+3y} \cdot dxdy$ over the triangle bounded by $x = 0, y = 0$ and $x + y = 1$.

Solution: The region R of integration is ΔAOB. Here, x varies from 0 to 1 and y varies from x-axis up to the line $x + y = 1$, i.e. from 0 to $1 - x$.



The region R can be expressed as follows:

$$0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1 - x$$

$$\text{Therefore, } \iint_R e^{2x+3y} \cdot dxdy = \int_0^1\int_0^{1-x} e^{2x+3y} \cdot dydx$$

$$= \int_0^1 \left[\frac{1}{3} \cdot e^{2x+3y}\right]_0^{1-x} \cdot dx = \frac{1}{3}\int_0^1 (e^{3-x} - e^{2x}) \cdot dx$$

$$= \frac{1}{3}\left[-e^{3-x} - \frac{1}{2}e^{2x}\right]_0^1 = -\frac{1}{3}\left[e^2 + \frac{1}{2}e^2 - e^3 - \frac{1}{2}\right]$$

$$= -\frac{1}{3}\left[-e^2(e - 1) + \frac{1}{2}(e^2 - 1)\right]$$

$$= -\frac{1}{6}(e - 1)\left[2e^2 - (e + 1)\right] = \frac{1}{6}(e - 1)(2e^2 - e - 1)$$

$$= \frac{1}{6}(e - 1)(e - 1)(2e + 1) = \frac{1}{6}(e - 1)^2(2e + 1)$$

37. Expand $f(x) = \begin{cases} \frac{1}{4} - x, & \text{if } 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \text{if } \frac{1}{2} < x < 1 \end{cases}$ as the Fourier series of sine terms.

Solution: Let $f(x)$ represents an odd function in $(-1, 1)$ so that $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$ where

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x dx$$

$$= 2\left[\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x dx + \int_{1/2}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x dx\right]$$

$$= 2\left|-\left(\frac{1}{4} - x\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right|_0^{1/2}$$

$$+ 2\left|-\left(x - \frac{3}{4}\right)\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_{1/2}^{1}$$

$$= 2\left[\frac{1}{4n\pi}\cos\frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{\sin n\pi/2}{n^2\pi^2}\right]$$

$$+ 2\left[-\frac{1}{4n\pi}\cos n\pi - \frac{1}{4n\pi}\cos\frac{n\pi}{2} - \frac{\sin n\pi/2}{n^2\pi^2}\right]$$

$$= \frac{1}{2n\pi}\left[1-(-1)^n\right] - \frac{4\sin n\pi/2}{n^2\pi^2}$$

Thus, $b_1 = \frac{1}{\pi} - \frac{4}{\pi^2}$, $b_2 = 0$, $b_3 = \frac{1}{3\pi} + \frac{4}{3^2\pi^2}$,

$b_4 = 0$, $b_5 = \frac{1}{5\pi} - \frac{4}{5^2\pi^2}$, $b_6 = 0$

Hence, $f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)$

$\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \cdots$

PRACTICE EXERCISE

1. What is the value of $\lim_{x\to 2}\frac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16}$?

(a) 3/16 (b) -2
(c) 1 (d) 0

2. If $f(x) = \begin{cases} 4x - 5, \text{ if } x \le 2 \\ x - \lambda, \text{ if } x > 2 \end{cases}$, then find λ for which $\lim_{x\to 2} f(x)$ exists.

(a) 1 (b) -1
(c) 5 (d) 2

3. Find the value of 'k' such that the following function is continuous at $x = 0$.

$$f(x) = \begin{cases} \dfrac{1-\cos 2x}{2x^2}, & x \ne 0 \\ h, & x = 0 \end{cases}$$

(a) 0 (b) 1 (c) 2 (d) 3

4. If $f(x) = \begin{cases} x^2 + 3x + a, & \text{for } x \le 1 \\ bx + 2, & \text{for } x > 1 \end{cases}$ is differentiable everywhere, find the values of a and b.

(a) $a = 4,\ b = 6$ (b) $a = 5,\ b = 3$
(c) $a = 3,\ b = 5$ (d) $a = 4,\ b = 2$

5. If Rolle's theorem holds for $f(x) = x^3 + bx^2 + ax$, $x \in [1,3]$ with $c = 2 + \frac{1}{\sqrt{3}}$, then what are the values of a and b?

(a) $a = 11,\ b = -6$ (b) $a = 8,\ b = -9$
(c) $a = -17,\ b = 1$ (d) $a = -1,\ b = -3$

6. Discuss the verification of Lagrange's mean value theorem for $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$.

(a) Not applicable
(b) Applicable for $c = \pm\pi/3$
(c) Applicable for $c = -\pi/3$
(d) Applicable for $c = \pi/3$

7. Find the absolute maximum and absolute minimum values of the function $f(x) = (1/2 - x)^2 + x^3 \in [-2, 2.5]$.

(a) Absolute maximum $= \frac{38}{3}$ and absolute minimum $= 0$
(b) Absolute maximum $= 112$ and absolute minimum $= \frac{10}{3}$
(c) Absolute maximum $= \frac{117}{2}$ and absolute minimum $= -8$
(d) Absolute maximum $= \frac{157}{8}$ and absolute minimum $= -\frac{7}{4}$

8. Differentiate $\tan^{-1}\left\{\frac{\sqrt{1-x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right\}$ with respect to $\cos^{-1} x^2$.

(a) 0 (b) $-1/2$
(c) 1/2 (d) $\cos x \cdot \sin x$

9. For what interval is $f(x)=\frac{x}{2}+\frac{2}{x}, x\neq 0$ increasing?

(a) $(-\infty,-2)\cup(2,\infty)$ (b) $(-\infty,-1)\cup(1,\infty)$
(c) $(-2, 2)$ (d) $(0,\infty)$

10. For what interval is $f(x)=x^4-2x^2$ decreasing?

(a) $(-1, 1)$ (b) $(-1,\infty)$
(c) $(-\infty,-1)\cup(0,1)$ (d) $(-\infty, 0)\cup(1,\infty)$

11. What is the value of $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\cdot dx$?

(a) $x+\frac{x}{2}+c$ (b) $\frac{\pi}{4}x-\frac{\pi}{2}+c$
(c) $\pi x-x^2+c$ (d) $\frac{\pi}{4}x-\frac{x^2}{2}+c$

12. Evaluate $\int \frac{\sin 2x}{a^2\sin^2 x+b^2\cos^2 x}\cdot dx$

(a) $\log\left|a^2\sin^2 x+b^2\cos^2 x\right|+c$

(b) $\frac{1}{(a^2-b^2)}\log\left|a^2\sin^2 x+b^2\cos^2\right|+c$

(c) $\cos 2x\cdot\log\left|\frac{1}{a^2\sin^2 x+b^2\cos^2 x}\right|+c$

(d) $\frac{1}{2}\log\left|a^2\sin^2 x+b^2\cos^2 x\right|+c$

13. What is the value of $\int_0^{2\pi} e^x\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cdot dx$?

(a) $e^{2\pi}+1$ (b) $\frac{-\sqrt{2}}{5}(e^{2\pi}+1)$
(c) $5/4$ (d) $-\frac{e^{2\pi}+1}{2\sqrt{2}}$

14. What is the value of $\int_1^2 \frac{\log x}{x^2}\cdot dx$?

(a) $\log e$ (b) $\frac{1}{2}\log 2$
(c) $\frac{1}{2}\log(e/2)$ (d) $\log 2e$

15. What is the value of $\int_0^1\int_0^{\sqrt{1+x^2}}\frac{dydx}{1+x^2+y^2}$?

(a) $\frac{\pi}{4}$ (b) $\log\pi/4$
(c) $\log\sqrt{2}$ (d) $\frac{\pi}{4}\log(\sqrt{2}+1)$

16. Find the area bounded by $y=2x-x^2$ and x-axis using integration.

(a) $4/3$ sq. units (b) $2/3$ sq. units
(c) 8 sq. units (d) $16\sqrt{2}$ sq. units

17. Evaluate $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

(a) $1/4$ (b) $-1/4$
(c) 0 (d) 1

18. Calculate by double integration, the volume generated by the revolution of the cardioid $r=a(1-\cos\theta)$ about its axis.

(a) $a^2/2$ (b) $4\pi a^2$
(c) $8\pi a^3/3$ (d) $8\pi a$

19. If the position vector $\vec{a}$ of the point $(5, n)$ is such that $|\vec{a}|=13$. Then what is the value of n?

(a) ± 12 (b) 0
(c) ± 1 (d) -4

20. Find the unit vector in the direction of $3\hat{i}-6\hat{j}+2\hat{k}$.

(a) $\frac{3}{2}\hat{i}-3\hat{j}+\hat{k}$ (b) $3/7\hat{i}-6/7\hat{j}+2/7\hat{k}$
(c) $3/5\hat{i}-6/5\hat{j}+2/5\hat{k}$ (d) $\hat{i}-2\hat{j}+2/3\hat{k}$

21. Find a vector of magnitude 9, which is perpendicular to both the vectors $\vec{a}=4\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$.

(a) $\frac{1}{3}(\hat{i}+\hat{j}+\hat{k})$ (b) $3(\hat{i}-\hat{j}+\hat{k})$
(c) $3\hat{i}-6\hat{j}-6\hat{k}$ (d) $-3\hat{i}+6\hat{j}+6\hat{k}$

22. If $\phi=x^2yz^3$ and $F=xz\hat{i}-y^2\hat{j}+2x^2y\hat{k}$, then what is the value of div(ϕF)?

(a) $3x^2yz^4-3x^2y^2z^3+6x^4y^2z^2$
(b) $x^2yz^4+x^2y^2z^3+2x^4y^2z^2$
(c) $xyz^3+xy^2z^3+4x^3yz$
(d) $-3x^2yz+3x^2y^2z^2+6x^4y^2z^2$

23. If $\phi=x^2yz^3$ and $F=xz\hat{i}-y^2\hat{j}+2x^2\hat{k}$, then what is the value of curl(ϕF)?

(a) $(2xy^4+3x^2y^3z)\hat{i}-(4x^3yz+8xy^2z^3)\hat{j}-(2xyz^3+x^3z^4)\hat{k}$

(b) $(2xyz+3yz)\hat{i}+(4xyz+3x^2y^2z^3)\hat{j}-(2xyz^3+x^3z^4)\hat{k}$

(c) $(4x^4yz^3+3x^2y^3z^2)\hat{i}+(4x^3yz^3-8x^3y^2z^3)\hat{j}$
$-(2xy^3z^3+x^3z^4)\hat{k}$

(d) $(x^4yz+3x^2y^3z^2)\hat{i}+(4x^3yz^3-2x^3y^2z^3)\hat{j}$
$-(2xy^3z^3+3x^3z^4)\hat{k}$

24. If a vector field is given by $F=\sin y\hat{i}+x(1+\cos y)\hat{j}$, then evaluate the line integral over a circular path given by $x^2+y^2=a^2, z=0$.

(a) $\frac{\pi}{2}a$ (b) 2π
(c) $2\pi^2a^2$ (d) πa^2

25. Evaluate $\int_c[(y-\sin x)dx+\cos x dy]$, where c is the plane triangle enclosed by the lines $y=0$, $x=\pi/2$ and $y=\frac{2}{\pi}x$.

(a) $-\frac{\pi}{4}-\frac{2}{\pi}$ (b) $\pi^2/2$
(c) $\frac{\pi}{2}+\frac{2}{\pi}$ (d) $\frac{2}{3}\pi^2-4\pi$

26. Evaluate $\int_c[(x+y)dx+(2x-z)dy+(y+z)dz]$, where c is the boundary of the triangle with vertices $(2,0,0)$, $(0,\ 3,\ 0)$ and $(0,\ 0,\ 6)$ and area of the triangle formed is $3\sqrt{14}$.

(a) $21\sqrt{14}$ (b) 21
(c) $\sqrt{14}$ (d) $\frac{7}{\sqrt{14}}$

27. Evaluate $\int\int_S A\cdot\hat{n}ds$, where S is a closed surface.

(a) V
(b) V^2
(c) V^3
(d) $3V$, where V is the volume enclosed by S.

28. Evaluate $\int_0^{\pi/2}\left[\int_0^{a\cos\theta} r\cdot\sqrt{a^2-r^2}\cdot dr\right]\cdot d\theta$

(a) $\frac{a^3}{18}(3\pi-4)$ (b) $\frac{a^3}{9}(3\pi-2)$
(c) $\frac{a^3}{18}(\pi-4)$ (d) $\frac{a^3}{9}(3\pi-4)$

29. Find the magnitude of the vector $\bar{a}=(\hat{i}+3\hat{j}-2\hat{k})\times(-\hat{i}+3\hat{k})$.

(a) 91 (b) 10
(c) $\sqrt{91}$ (d) $\sqrt{70}$

30. Find the area of a parallelogram which is represented by $\vec{A}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{B}=3\hat{i}-2\hat{j}+\hat{k}$.

(a) 8 square units (b) $\sqrt{3}$ square units
(c) $8\sqrt{3}$ square units (d) 16 square units

31. Evaluate $\int_0^1\frac{2x}{5x^2+1}dx$.

(a) $\frac{1}{5}\log 9$ (b) $\frac{1}{5}\log 6$
(c) $\frac{1}{10}\log 9$ (d) $\frac{1}{10}\log 6$

32. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}dzdydx$.

(a) $\pi/8$ (b) $\pi^2/8$
(c) $1/8$ (d) $\pi^2/16$

33. Find the intervals in which the function $f(x)=2x^3+9x^2+12x+24$ is increasing.

(a) $(-\infty,-2)\cup(-1,\infty)$ (b) $(-2,-1)$
(c) $(-\infty,-1)\cup(2,\infty)$ (d) $(-1,\infty)$

34. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, then what is the value of $\frac{dy}{dx}$?

(a) $\frac{1}{(x+1)^2}$ (b) $\frac{1}{(x-1)^2}$
(c) $-\frac{1}{(x-1)^2}$ (d) $-\frac{1}{(x+1)^2}$

35. Find the values of a and b for which the following function $f(x)$ is continuous at $x=1$.

$$f(x)=\begin{cases}3ax+b, & \text{if } x>1\\ 11, & \text{if } x=1\\ 5ax-2b, & \text{if } x<1\end{cases}$$

(a) $a=2,\ b=2$ (b) $a=3,\ b=2$
(c) $a=2,\ b=3$ (d) $a=3,\ b=3$

36. Evaluate $\int_0^{\pi}\sin^3\theta d\theta$.

(a) 4/3 (b) 2/3 (c) 8/5 (d) 6/5

37. What is the value of $\lim_{x\to 4}\frac{x^2-16}{\sqrt{x^2+9}-5}$?

(a) 0 (b) 5 (c) 10 (d) 16

38. Find the absolute maximum value and the absolute minimum value of

$$f(x) = (1/2 - x)^2 + x^3 \text{ in } [-2,\ 2]$$

(a) Absolute maximum $= \dfrac{17}{4}$ and absolute minimum value $= -9/4$

(b) Absolute maximum $= \dfrac{21}{2}$ and absolute minimum value $= -7/2$

(c) Absolute maximum $= \dfrac{23}{4}$ and absolute minimum value $= -7/4$

(d) Absolute maximum $= \dfrac{19}{2}$ and absolute minimum value $= -5/2$

39. Find the unit vector perpendicular to both the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $\hat{i} + 2\hat{j} - \hat{k}$.

(a) $\dfrac{1}{\sqrt{3}}(-\hat{i} + \hat{j} + \hat{k})$ (b) $(-\hat{i} + \hat{j} + \hat{k})$

(c) $\dfrac{1}{\sqrt{5}}(-2\hat{i} + \hat{j} + \hat{k})$ (d) $\dfrac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$

40. What is the work done in moving a particle in a force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along the straight line from (0, 0, 0) to (2, 1, 3)?

(a) 10 units (b) 12 units
(c) 14 units (d) 16 units

41. Find the Fourier series to represent $f(x) = x - x^2$ from $x = -\pi$ to $x = \pi$.

(a) $$x - x^2 = -\frac{\pi}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

(b) $$x - x^2 = -\pi + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

(c) $$x - x^2 = -\frac{\pi}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right] + 2\left[\frac{\sin x}{1^2} - \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} - \frac{\sin 4x}{4^2} + \cdots\right]$$

(d) $$x - x^2 = -\frac{\pi}{3} + 4\left[\frac{\cos x}{1} - \frac{\cos 2x}{2} + \frac{\cos 3x}{3} - \frac{\cos 4x}{4} + \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

42. Find the Fourier series for the function,

$$f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \pi/2 < t < \pi \end{cases}$$

(a) $f(t) = \dfrac{2}{\pi}\left(\sin t + \sin 2t + \dfrac{1}{3}\sin 3t + \cdots\right)$

(b) $f(t) = \dfrac{2}{\pi}\left(\sin t - \sin 2t + \dfrac{1}{3}\sin 3t - \cdots\right)$

(c) $f(t) = \dfrac{1}{\pi}\left(\sin t - \dfrac{1}{2}\sin 2t + \dfrac{1}{3}\sin 3t + \cdots\right)$

(d) $f(t) = \dfrac{1}{\pi}\left(1 + \sin t - \sin 2t + \dfrac{1}{3}\sin 3t + \cdots\right)$

43. Obtain the Fourier expansion of $x \sin x$ as a cosine series in $(0, \pi)$.

(a) $$x\sin x = 1 - \frac{1}{2}\cos x - \left\{\frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{3\cdot 5} + \frac{\cos 4x}{5\cdot 7} - \cdots\infty\right\}$$

(b) $$x\sin x = 1 - \cos x - 2\left\{\frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{3\cdot 5} + \frac{\cos 4x}{5\cdot 7} - \cdots\infty\right\}$$

(c) $$x\sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1\cdot 3} + \frac{\cos 3x}{3\cdot 5} + \frac{\cos 4x}{5\cdot 7} + \cdots\infty\right\}$$

(d) $$x\sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{3\cdot 5} + \frac{\cos 4x}{5\cdot 7} - \cdots\infty\right\}$$

44. Express $f(x) = x/2$ as a Fourier series in the interval $-\pi < x < \pi$.

(a) $x/2 = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots$

(b) $x/2 = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots$

(c) $x/2 = \sin x - \frac{1}{2^2}\sin 2x + \frac{1}{3^2}\sin 3x - \frac{1}{4^2}\sin 4x + \cdots$

(d) $x/2 = \sin x + \frac{1}{2^2}\sin 2x + \frac{1}{3^2}\sin 3x + \frac{1}{4^2}\sin 4x + \cdots$

45. Evaluate $\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x} \cdot dx$

(a) 0 (b) $-i$ (c) $\pi/2$ (d) i

ANSWERS

1. (a)	**6.** (d)	**11.** (d)	**16.** (a)	**21.** (d)	**26.** (b)	**31.** (b)	**36.** (a)	**41.** (a)
2. (b)	**7.** (d)	**12.** (b)	**17.** (c)	**22.** (a)	**27.** (d)	**32.** (b)	**37.** (c)	**42.** (b)
3. (b)	**8.** (b)	**13.** (b)	**18.** (c)	**23.** (c)	**28.** (a)	**33.** (a)	**38.** (c)	**43.** (d)
4. (c)	**9.** (a)	**14.** (c)	**19.** (a)	**24.** (d)	**29.** (c)	**34.** (d)	**39.** (a)	**44.** (a)
5. (a)	**10.** (c)	**15.** (d)	**20.** (b)	**25.** (a)	**30.** (c)	**35.** (b)	**40.** (d)	**45.** (d)

EXPLANATIONS AND HINTS

1. (a) We have

$$\lim_{x\to 2} \frac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16}$$

Now, if we put $x = 2$, then the above limit becomes $\frac{0}{0}$.

Hence,

$$\lim_{x\to 2} \frac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16} = \lim_{x\to 2} \frac{(x-2)(x^2-x-2)}{(x^2-4)^2}$$
$$= \lim_{x\to 2} \frac{(x-2)(x^2-x-2)}{(x+2)^2(x-2)^2}$$
$$= \lim_{x\to 2} \frac{(x^2-x-2)}{(x+2)^2(x-2)}$$
$$= \lim_{x\to 2} \frac{(x-2)(x+1)}{(x+2)^2(x-2)}$$
$$= \lim_{x\to 2} \frac{x+1}{(x+2)^2} = \frac{2+1}{(2+2)^2}$$
$$= \frac{3}{16}$$

2. (b) We have

$$f(x) = \begin{cases} 4x - 5, & \text{if } x \le 2 \\ x - \lambda, & \text{if } x > 2 \end{cases}$$

Now, $\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h)$

$$= \lim_{h\to 0} 4(2-h) - 5 = \lim_{h\to 0}(3-4h) = 3$$

and $\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h)$

$$= \lim_{h\to 0} (2+h) - \lambda = (2-\lambda)$$

Now, if limit exists, then

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$
$$\Rightarrow 3 = 2 - \lambda \Rightarrow \lambda = -1$$

3. (b) We know that

$$f(x) = \begin{cases} \frac{1-\cos 2x}{2x^2}, & x \ne 0 \\ k, & x = 0 \end{cases}$$

and $f(x)$ is continuous at $x = 0$.

Thus, $\lim_{x\to 0} f(x) = f(0)$

$$\Rightarrow \lim_{x\to 0} \frac{1-\cos 2x}{2x^2} = k$$
$$\Rightarrow \lim_{x\to 0} \frac{2\sin^2 x}{2x^2} = k$$
$$\Rightarrow \lim_{x\to 0} \left[\frac{\sin x}{x}\right]^2 = k$$
$$\Rightarrow k = 1$$

Hence, $f(x)$ is continuous at $x = 0$, for $k = 1$.

4. (c) We have

$$f(x) = \begin{cases} x^2 + 3x + a, & \text{for } x \le 1 \\ bx + 2, & \text{for } x > 1 \end{cases} \text{ is differentiable}$$

Now, $f(x)$ is differentiable and continuous at $x = 1$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x \to 1^-} (x^2 + 3x + a) = \lim_{x \to 1^+} (bx + 2) = 1 + 3 + a$$

$$\Rightarrow \quad 1 + 3 + a = b + 2 \text{ or } a - b = -2 \qquad (1)$$

Also since $f(x)$ is differentiable at $x = 1$, L.H.D. at $x = 1$ = R.H.D. at $x = 1$

Therefore, $\lim_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1}$

$$\Rightarrow \lim_{x \to 1} \frac{x^2 + 3x + a - (4 + a)}{x - 1} = \lim_{x \to 1} \frac{(bx + 2) - (4 + a)}{x - 1}$$

$$\Rightarrow \lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1} = \lim_{x \to 1} \frac{bx - 2 - a}{x - 1}$$

$$\Rightarrow \lim_{x \to 1} \frac{(x + 4)(x - 1)}{x - 1} = \lim_{x \to 1} \frac{bx - b}{x - 1} \qquad \text{[from Eq. (1)]}$$

$$\Rightarrow \lim_{x \to 1} (x + 4) = \lim_{x \to 1} b$$

$b = 5$

Putting value of b in Eq. (1), we get

$$a - 5 = -2 \Rightarrow a = 3$$

Hence, $a = 3$ and $b = 5$.

5. (a) We know that Rolle's theorem is applicable for $f(x)$ defined on $[1, 3]$ with $c = 2 + \dfrac{1}{\sqrt{3}}$.

Therefore, $f(1) = f(3)$ and $f'(c) = 0$.

$$\Rightarrow \quad 1 + b + a = 27 + 9b + 3a$$

and $3c^2 + 2bc + a = 0$

$$\Rightarrow \quad 2a + 8b + 26 = 0$$

and $3\left(2 + \dfrac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \dfrac{1}{\sqrt{3}}\right) + a = 0$

$$\Rightarrow a + 4b + 13 = 0 \qquad (1)$$

and $a + 4b + 13 + \dfrac{2}{\sqrt{3}}(b + 6) = 0 \qquad (2)$

Now, putting Eq. (1) in Eq. (2), we get

$$0 + \frac{2}{\sqrt{3}}(b + 6) = 0 \Rightarrow b = -6$$

Putting the value of b in Eq. (1), we get

$$a - 24 + 13 = 0 \Rightarrow a = 11$$

Hence, $a = 11$ and $b = -6$.

6. (d) Since $\sin x$ and $\sin 2x$ are continuous and differentiable everywhere, the function $f(x)$ is continuous on $[0, \pi]$ and differentiate on $(0, \pi)$. Hence, Lagrange's mean value theorem is applicable.

Thus, there exists at least one $c \in (0, \pi)$ such that $f'(c) = \dfrac{f(\pi) - f(0)}{\pi - 0}$.

Now, $f(x) = 2\sin x + \sin 2x$

$$f'(x) = 2\cos x + 2\cos 2x$$

$$f(0) = 0 \text{ and } f(\pi) = 2\sin\pi + \sin\pi = 0$$

Hence, $f'(x) = \dfrac{f(x) - f(0)}{\pi - 0} \Rightarrow 2\cos x + 2\cos 2x = \dfrac{0}{\pi}$

$$\Rightarrow \quad \cos x + \cos 2x = 0 \quad \Rightarrow \cos 2x = -\cos x$$

$$\Rightarrow \quad \cos 2x = \cos(\pi - x)$$

$$\Rightarrow \quad 2x = \pi - x \quad \Rightarrow x = \frac{\pi}{3}$$

Thus, $c = \dfrac{\pi}{3} \in (0, \pi)$ such that $f'(c) = \dfrac{f(x) - f(0)}{\pi - 0}$.

7. (d) We have

$$f(x) = \left[\frac{1}{2} - x\right]^2 + x^3, \; x \in [-2, 2.5]$$

$$f'(x) = -2\left(\frac{1}{2} - x\right) + 3x^2 = -1 + 2x + 3x^2$$

For maximum or minimum, we must have

$$f'(x) = 0$$

$$\Rightarrow \quad 3x^2 + 2x - 1 = 0$$

$$\Rightarrow \quad (3x - 1)(x + 1) = 0$$

$$\Rightarrow \quad x = 1/3, -1$$

Now, $f(-2) = \left[\dfrac{1}{2} + 2\right]^2 + (-2)^3 = \dfrac{25}{4} - 8 = \dfrac{-7}{4}$

$$f\left(\frac{1}{8}\right) = \left[\frac{1}{2} - \frac{1}{3}\right]^2 + \left[\frac{1}{3}\right]^3 = \frac{1}{36} - \frac{1}{27} = \frac{7}{108}$$

$$f(-1) = \left[\frac{1}{2} + 1\right]^2 + (-1)^3 = \frac{5}{4}$$

$$f(2.5) = \left[\frac{1}{2} - 2.5\right]^2 + (2.5)^3 = \frac{157}{8}$$

Hence, absolute maximum $= \dfrac{157}{8}$ and absolute minimum $= -\dfrac{7}{4}$.

8. (b) Let $u = \tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ and $v = \cos^{-1} x^2$.

Putting $x^2 = \cos\theta$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta/2}-\sqrt{2\sin^2\theta/2}}{\sqrt{2\cos^2\theta/2}+\sqrt{2\sin^2\theta/2}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\dfrac{\theta}{2}-\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}+\sin\dfrac{\theta}{2}}\right\}$$

Dividing numerator and denominator by $\cos\dfrac{\theta}{2}$, we get

$$u = \tan^{-1}\left\{\frac{1-\tan\theta/2}{1+\tan\theta/2}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$$\Rightarrow\ u = \frac{\pi}{4}-\frac{\theta}{2} = \frac{\pi}{4}-\frac{1}{2}\cos^{-1} x^2$$

Therefore, $\dfrac{du}{dx} = -\dfrac{1}{2}\times\dfrac{-2x}{\sqrt{1-x^4}} = \dfrac{x}{\sqrt{1-x^4}}$

$$\frac{dv}{dx} = \frac{-2x}{\sqrt{1-x^4}}$$

So,

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{x/\sqrt{1-x^4}}{-2x/\sqrt{1-x^4}} = \frac{-1}{2}$$

9. (a) We have

$$f(x) = \frac{x}{2}+\frac{2}{x}$$

$$\Rightarrow\quad f'(x) = \frac{1}{2}-\frac{2}{x^2} = \frac{x^2-4}{2x^2}$$

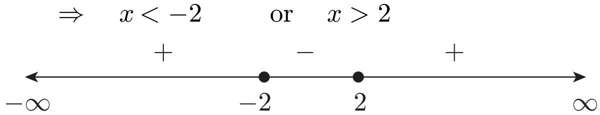
For $f(x)$ to be increasing,

$$f'(x) > 0$$

$$\Rightarrow\quad \frac{x^2-4}{2x^2} > 0$$

$$\Rightarrow\quad x^2-4>0 \Rightarrow (x-2)(x+2)>0$$

$$\Rightarrow\quad x<-2 \quad\text{or}\quad x>2$$



So, $f(x)$ is increasing on $(-\infty,-2)\cup(2,\infty)$.

10. (c) We have

$$f(x) = x^4-2x^2$$

$$\Rightarrow\quad f'(x) = 4x^3-4x = 4x(x^2-1)$$

For $f(x)$ to be decreasing,

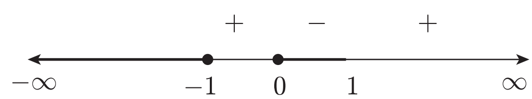
$$f'(x) < 0$$

$$\Rightarrow\quad 4x(x^2-1) < 0$$

$$\Rightarrow\quad x(x^2-1) < 0$$

$$\Rightarrow\quad x(x-1)(x+1) < 0$$

$$\Rightarrow\quad x<-1 \quad\text{or}\quad 0<x<1$$



Thus, $f(x)$ is decreasing on $(-\infty,-1)\cup(0,1)$.

11. (d) We have

$$\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\cdot dx$$

$$= \int \tan^{-1}\left\{\sqrt{\frac{1-\cos\left(\dfrac{\pi}{2}-x\right)}{1+\cos\left(\dfrac{\pi}{2}-x\right)}}\right\}\cdot dx$$

$$= \int \tan^{-1}\left\{\sqrt{\frac{2\sin^2\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)}{2\cos^2\left(\dfrac{-}{4}-\dfrac{x}{2}\right)}}\right\}\cdot dx$$

$$= \int \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\cdot dx = \int\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot dx$$

$$= \frac{\pi}{4}x-\frac{x^2}{2}+c$$

12. (b) We have

$$\int \frac{\sin 2x}{a^2\sin^2 x+b^2\cos^2 x}\cdot dx$$

Let $a^2\sin^2 x+b^2\cos^2 x = u$

and $(2a^2\sin x\cos x-2b^2\cos x\sin x)dx = du$

$$\Rightarrow\quad dx = \frac{du}{2a^2\sin x\cos-2b^2\cos x\sin x}$$

Therefore, $\displaystyle\int \frac{\sin 2x}{a^2\sin^2 x+b^2\cos^2 x}\cdot dx$

$$= \int \frac{\sin 2x}{u}\cdot\frac{du}{2a^2\sin x\cos x-2b^2\cos x\sin x}$$

$$= \int \frac{\sin 2x}{u} \cdot \frac{du}{2\sin x \cos x(a^2 - b^2)}$$

$$= \frac{1}{(a^2 - b^2)} \int \frac{1}{u} \cdot du$$

$$= \frac{1}{(a^2 - b^2)} \log |u| + \text{c}$$

Substituting the value of u, we get

$$\frac{1}{(a^2 - b^2)} \log | a^2 \sin^2 x + b^2 \cos^2 x | + \text{c}$$

13. (b) Let $A = \int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot dx$

On integrating by parts, we get

$$A = \left[\sin\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot e^x\right]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} e^x \cos\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot dx$$

$$= \left[\sin \frac{5\pi}{4} e^{2\pi} - \sin \frac{\pi}{4}\right] - \frac{1}{2}\left[\left\{e^x \cos\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\}_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx\right]$$

$$= \left[-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] - \frac{1}{2}\left[\left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}A\right]$$

$$= -\left(-\frac{e^{2\pi} + 1}{\sqrt{2}}\right) + \left(\frac{e^{2\pi} + 1}{2\sqrt{2}}\right) - \frac{1}{4}A$$

$$\frac{5A}{4} = -\left(-\frac{e^{2\pi} + 1}{\sqrt{2}}\right) + \left(\frac{e^{2\pi} + 1}{2\sqrt{2}}\right) = \frac{e^{2\pi} + 1}{2\sqrt{2}}(1 - 2)$$

$$= \frac{-e^{2\pi} + 1}{2\sqrt{2}}$$

$$A = -\frac{\sqrt{2}}{5}(2\pi + 1)$$

14. (c) We have

$$\int_1^2 \frac{\log x}{x^2} \cdot dx$$

On integrating by parts, we get

$$\int_1^2 \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{1/x^2} \cdot dx = \left[\log x \cdot \left(\frac{-1}{x}\right)\right]_1^2 - \int_1^2 \frac{1}{x}\left(\frac{-1}{x}\right) \cdot dx$$

$$= \left[-\frac{1}{x}\log x\right]_1^2 - \left[\frac{1}{x}\right]_1^2 = \left(\frac{-1}{2}\log 2\right) + (1 \cdot \log 1) - \left(\frac{1}{2} \cdot \frac{1}{1}\right)$$

$$= \frac{-1}{2}\log 2 + \frac{1}{2} = \frac{1}{2}(-\log 2 + 1)$$

$$= \frac{1}{2}(-\log 2 + \log e)$$

$$= \frac{1}{2}\log\left(\frac{e}{2}\right)$$

15. (d) We have

$$A = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{(1 + x^2) + y^2} \cdot dy\right] \cdot dx$$

$$= \int_0^1 \frac{1}{\sqrt{1 + x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1 + x^2}}\right]_0^{\sqrt{1+x^2}} \cdot dx$$

$$= \int_0^1 \frac{1}{\sqrt{1 + x^2}} [\tan^{-1} 1 - \tan^{-1} 0] \cdot dx = \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1 + x^2}}$$

$$= \frac{\pi}{4}\left[\log(x + \sqrt{1 + x^2})\right]_0^1 = \frac{\pi}{4}\left[\log(1 + \sqrt{2}) - \log 1\right]$$

$$= \frac{\pi}{4}\log(\sqrt{2} + 1)$$

16. (a) We have a parabola of equation,

$$y = 2x - x^2$$

Now, parabola is opening downward.

Now, parabola cuts x-axis, hence $y = 0$. Therefore,

$$0 = 2x - x^2$$
$$x(x - 2) = 0$$
$$\Rightarrow x = 0, 2$$

Thus, parabola cuts x-axis at (0, 0) and (2, 0).

Required area $= \int_0^2 y \cdot dx$

$$= \int_0^2 (2x - x^2)dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}\text{sq. units}$$

17. (c) We have

$$\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z} (x + y + z)dy\, dx\, dz$$

Firstly, integrating with respect to y keeping x and z constant, we have

$$A = \int_{-1}^{1}\int_0^z | xy + y^2/2 + yz |_{x-z}^{x+z} \; dx \cdot dz$$

$$= \int_{-1}^{1}\int_0^z \left[(x + z)(2z) + \frac{1}{2} \cdot 4xz\right] dx \cdot dz$$

$$= 2\int_{-1}^{1}\left|\frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z\right|_0^z dz$$

$$= 2\int_{-1}^{1}\left(\frac{z^3}{2} + z^3 + \frac{z^3}{2}\right)dz$$

$$= 4\left|\frac{z^4}{4}\right|_{-1}^{1} = 0$$

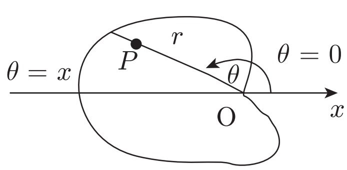
18. (c) The volume required is given by

$$V = \int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi\int_0^{\pi}\left|\frac{r^3}{3}\right|_0^{a(1-\cos\theta)} \sin\theta \cdot d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi}(1-\cos\theta)^3 \cdot \sin\theta \cdot d\theta$$

$$= \frac{2\pi a^3}{3}\left|\frac{(1-\cos\theta)^4}{4}\right|_0^{\pi} = \frac{8\pi a^3}{3}$$



19. (a) We have

$$\vec{a} = 5\hat{i} + n\hat{j}$$

Therefore, $|\vec{a}| = \sqrt{25+n^2}$

$$\Rightarrow \quad 13 = \sqrt{25+n^2} \Rightarrow 25 + n^2 = 169$$

$$\Rightarrow \quad n^2 = 144$$

$$\Rightarrow \quad n = \pm 12$$

20. (b) We have

$$\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Then, $|\vec{a}| = \sqrt{3^2 + (-6)^2 + 2^2} = 7$

So, unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}$$

21. (d) We have

$$\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$$
$$\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$$

Then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & +1 & -2 \end{vmatrix}$

$$= (2-3)\hat{i} - (-8+6)\hat{j} + (4-2)\hat{k}$$

$$= -\hat{i} + 2\hat{j} + 2\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$$

Now, we know that magnitude = 9.

Therefore, required vector

$$= 9\left[\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right] = \frac{9}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$$

$$= -3\hat{i} + 6\hat{j} + 6\hat{k}$$

22. (a) We have

$$\phi = x^2 y z^3$$

$$\vec{F} = xz\hat{i} - y^2\hat{j} + 2x^2 y\hat{k}$$

$$\text{div}(\phi F) = \nabla\cdot(\phi F) = \nabla\cdot(x^3 y z^4\hat{i} - x^2 y^2 z^3\hat{j} + 2x^4 y^2 z^3\hat{k})$$

$$= \frac{\partial}{\partial x}\left(x^3 y z^4\right) - \frac{\partial}{\partial y}(x^2 y^3 z^3) + \frac{\partial}{\partial z}(2x^4 y^2 z^3)$$

$$= 3x^2 y z^4 - 3x^2 y^2 z^3 + 6x^4 y^2 z^2$$

23. (c) We have

$$\phi = x^2 y z^3$$

$$F = xz\hat{i} - y^2\hat{j} + 2x^2 y\hat{k}$$

$$\text{curl }(\phi F) = \nabla\times(\phi F) = \nabla\times(x^3 y z^4\hat{i} - x^2 y^3 z^3\hat{j} + 2x^4 y^2 z^3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 y z^4 & -x^2 y^3 z^3 & 2x^4 y^2 z^3 \end{vmatrix}$$

$$= (4x^4 y z^3 + 3x^2 y^3 z^2)\hat{i} + (4x^3 y z^3 - 8x^3 y^2 z^3)\hat{j}$$
$$- (2xy^3 z^3 + x^3 z^4)\hat{k}$$

24. (d) Since particle moves in xy plane, $z = 0$.

Now, let $R = x\hat{i} + y\hat{j}$

then $dR = dx\hat{i} + dy\hat{j}$

Also, path given is $x^2 + y^2 = a^2$.

So, $x = a\cos t$ and $y = a\sin t$, where t varies from 0 to 2π.

Therefore,

$$\oint_C F\cdot dR = \oint_C (\sin y\,\hat{i} + x(1+\cos y)\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \oint_C [\sin y \cdot dx + x(1+\cos y)dy]$$

$$= \oint_C [[\sin y \cdot dx + x\cos y\, dy] + x\, dy]$$

$$= \oint_C [d(x\sin y) + x\, dy]$$

Now, substituting the values of x and y, we get

$$\int_0^{2\pi} [d(a\cos t \sin(a\sin t)) + a^2\cos^2 t\, dt]$$

$$= |\, a\cos t\sin(a\sin t)\,|_0^{2\pi} + \frac{a^2}{2}\left| t + \frac{\sin 2t}{2}\right|_0^{2\pi}$$

$[\because \cos^2 t = 1+\cos 2]$

$$= 0 + \frac{a^2}{2}[2\pi + 0 - (0+0)]$$

$$= \pi a^2$$

25. (a) We have

$$f_1 = y - \sin x \text{ and } f_2 = \cos x$$

Hence, by Green's theorem $\int_C [(y-\sin x)dx + \cos x dy]$

$$= \int\int_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\cdot dy$$

$$= \int_{x=0}^{\pi/2}\int_{y=0}^{2x/\pi} (-\sin x - 1)dy\,dx$$

$$= \int_{x=0}^{\pi/2} (-\sin x - 1)\,|\,y\,|_0^{2x/\pi}\ dx = \int_0^{\pi/2}\left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right)dx$$

$$= \frac{-2}{\pi}\int_0^{\pi/2}[x(\sin x + 1)]dx = \frac{-2}{\pi}\left\{|\,x(-\cos x + x)\,|_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos x + x)dx\right\}$$

$$= \frac{-2}{\pi}\left\{\frac{\pi^2}{4} - \left|-\sin x + \frac{x^2}{2}\right|_0^{\pi/2}\right\} = \frac{-\pi}{2} + \frac{2}{\pi}\left(-1 + \frac{\pi^2}{8}\right)$$

$$= -\frac{\pi}{4} - \frac{2}{\pi}$$

26. (b) We have

$$A = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$$

$$\text{curv } A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+y) & (2x-z) & (y+z) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z)\right]\hat{i} + \left[\frac{\partial}{\partial z}(x+y) - \frac{\partial}{\partial x}(y+z)\right]\hat{j} + \left[\frac{\partial}{\partial y}(2x-z) + \frac{\partial}{\partial z}(x+y)\right]\hat{k}$$

$$= 2\hat{i} + \hat{k}$$

Also, equation of plane through A, B, C is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \text{ or } 3x + 2y + z = 6.$$

Normal vector to this plane, $n = 3\hat{i} + 2\hat{j} + \hat{k}$

Therefore, $\hat{n} = \dfrac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{3^2 + 2^2 + 1^2}} = \dfrac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k})$

Hence, by Stokes' theorem,

$$\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$$

$$= \int_S (2\hat{i} + \hat{k})\cdot\left(\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}\right)ds$$

$$= (6+1)\frac{1}{\sqrt{14}}\int_s ds = \frac{7}{\sqrt{14}} \times 3\sqrt{14}$$

[where $\int_s ds$ = Area of triangle

$= 3\sqrt{14}$ sq. units] $= 21$

27. (d) By divergence theorem, we have

$$\int\int_s A\cdot\hat{n}\, ds = \int\iint_v \nabla\cdot A\, dv$$

$$\iint_v \int\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})\cdot dV$$

$$= \iint_V \int\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)dV = 3\iint_V\int dV$$

$= 3V$, where V is volume enclosed by S.

28. (a) Let $\displaystyle\int_0^{\pi/2}\left[\int_0^{a\cos\theta} r\cdot\sqrt{a^2 - r^2}\cdot dr\right]d\theta$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta - a^3)\cdot d\theta$$

$$= -\frac{a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right]$$

$$= \frac{a^3}{18}(3\pi - 4)$$

29. (c) We have

$$\bar{a} = (\hat{i} + 3\hat{j} - 2\hat{k}) \times (-\hat{i} + 0\hat{j} + 3\hat{k})$$

$$\bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix} = (9-0)\hat{i} - (3-2)\hat{j} + (0+3)\hat{k}$$

$$= 9\hat{i} - \hat{j} + 3\hat{k}$$

$$|\bar{a}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{81+1+9} = \sqrt{91}$$

30. (c) We know that the area of parallelogram whose adjacent sides are represented by $\vec{A}$ and $\vec{B}$ is given by $\vec{A} \times \vec{B}$.

Now,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= (2 \times 6)\hat{i} - (1-9)\hat{j} + (-2-6)\hat{k}$$

$$= 8\hat{i} + 8\hat{j} - 8\hat{k}$$

Thus, area of parallelogram

$$= \left|\vec{A} \times \vec{B}\right| = \sqrt{8^2 + 8^2 + (-8)^2}$$

$$= 8\sqrt{3} \text{ square units.}$$

31. (b) We have

$$\int_0^1 \frac{2x}{5x^2+1} dx$$

$$= \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1} \cdot dx = \frac{1}{5}\left[\log(5x^2+1)\right]_0^1$$

$$= \frac{1}{5}[\log 6 - \log 1]$$

$$= \frac{1}{5}\log 6$$

32. (b) We know,

$$A = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2)-z^2}} \cdot dz \cdot dy \cdot dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} \cdot dy \cdot dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} (\sin^{-1} - \sin^{-1} 0)\, dy \cdot dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} \cdot dy \cdot dx$$

$$= \int_0^1 \frac{\pi}{2}(y)_0^{\sqrt{1-x^2}} \cdot dx = \frac{\pi}{2}\int_0^1 (\sqrt{1-x^2} - 0) \cdot dx$$

$$= \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_0^1 = \frac{\pi}{4}[\sin^{-1} 1]$$

$$= \frac{\pi}{4} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

33. (a) We have

$$f(x) = 2x^3 + 9x^2 + 12x + 24$$

Differentiating with respect to x, we get

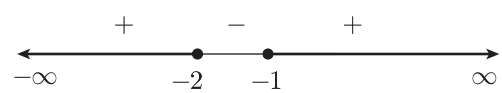
$$f'(x) = 2x^3 + 9x^2 + 12x + 24 = 6(x^2 + 3x + 2)$$

For $f(x)$ to be increasing function, we have $f'(x) > 0$

$$\Rightarrow \quad 6(x^2 + 3x + 2) > 0$$

$$\Rightarrow \quad (x^2 + 3x + 2) > 0$$

$$(x+1)(x+2) > 0$$



So, $f(x)$ is increasing on $(-\infty, -2) \cup (-1, \infty)$.

34. (d) We have

$$c\sqrt{1+y} + y\sqrt{1+x} = 0$$

$$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$$

$$\Rightarrow \quad x^2(1+y) = y^2(1+x) \quad \text{[Squaring both sides]}$$

$$\Rightarrow \quad x^2 - y^2 = y^2x - x^2y$$

$$\Rightarrow \quad (x+y)(x-y) = -xy(x-y)$$

$$\Rightarrow \quad x + y = -xy$$

$$\Rightarrow \quad x = -xy - y$$

$$\Rightarrow \quad y \cdot (1+x) = -x$$

$$\Rightarrow \quad y = \frac{-x}{1+x}$$

$$\frac{dy}{dx} = -\left\{\frac{(1+x) \cdot 1 - x(0+1)}{(1+x^2)}\right\}$$

$$= -\frac{1}{(x+1)^2}$$

35. (b) We have

L.H.L. at $x = 1$

$$= \lim_{x\to 1^-} f(x)$$

$$= \lim_{x\to 1^-} (5ax - 2b) = 5a - 2b$$

R.H.L. at $x = 1$

$$= \lim_{x\to 1^+} f(x)$$

$$= \lim_{x\to 1^+} (3ax + b) = 3a + b$$

Also, $f(1) = 11$.

Now, since $f(x)$ is continuous at $x = 1$,

$$= \lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$$

Thus, $5a - 2b = 11$ (1)

and $3a + b = 11$ (2)

Multiplying Eq. (2) by 2 and adding with Eq. (1), we get

$$11a = 33$$
$$a = 3$$
$$b = 11 - 9 = 2$$

Hence, the values of a and b are 3 and 2, respectively.

36. (a) We have

$$f(x) = \int_0^{\pi} \sin^3\theta \cdot d\theta$$

$$= \int_0^{\pi} (1 - \cos^2\theta)\sin\theta \cdot d\theta$$

Let $\cos\theta = t$

$$-\sin\theta \cdot d\theta = dt$$

At $\theta = 0$, $t = \cos 0 = 1$.

At $\theta = \pi$, $t = \cos\pi = -1$.

$$f(x) = -\int_{-1}^{+1} (1 - t^2)\,dt = \left| t - \frac{t^3}{3} \right|_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left[-1 + \frac{1}{3}\right]$$

$$= \frac{2}{3} + \frac{2}{3}$$

$$= \frac{4}{3}$$

37. (c) When $x \to 4$, the expression $\dfrac{x^2 - 16}{\sqrt{x^2 + 9} - 5} = \dfrac{0}{0}$.

Hence, rationalizing the denominator, we get

$$\lim_{x\to 4} \frac{x^2 - 16}{\sqrt{x^2 + 9} - 5} \times \frac{\sqrt{x^2 + 9} + 5}{\sqrt{x^2 + 9} + 5}$$

$$\Rightarrow \lim_{x\to 4} \frac{(x^2 - 16)}{(\sqrt{x^2 + 9})^2 - 5^2} = \lim_{x\to 4} \frac{(x^2 - 16)(\sqrt{x^2 + 9} + 5)}{(x^2 + 9) - 25}$$

$$= \lim_{x\to 4} \frac{(x^2 - 16)(\sqrt{x^2 + 9} + 5)}{(x^2 - 16)} = (\sqrt{16 + 9} + 5)$$

$$= 5 + 5$$

$$= 10$$

38. (c) We have

$$f(x) = \left[\frac{1}{2} - x\right]^2 + x^3, \quad \text{where} \quad x \in [-2, 2]$$

$$\Rightarrow f'(x) = -2(1/2 - x) + 3x^2 = -1 + 2x + 3x^2$$

For maximum or minimum,

$$f'(x) = 0$$
$$3x^2 + 2x - 1 = 0$$
$$(3x - 1)(x + 1) = 0 \Rightarrow x = 1/3, -1$$

Now, $f(-2) = \left(\dfrac{1}{2} + 2\right)^2 + (-2)^3 = \dfrac{25}{4} - 8 = \dfrac{-7}{4}$

$$f(1/3) = \left(\frac{1}{2} - \frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 = \frac{1}{36} + \frac{1}{27} = \frac{7}{108}$$

$$f(-1) = \left(\frac{1}{2} + 1\right)^2 + (-1)^3 = \frac{5}{4}$$

$$f(2) = \left(\frac{1}{2} - 2\right)^2 + (2)^3 = \frac{23}{4}$$

The absolute maximum value of $f(x) = 23/4$, and the absolute minimum value of $f(x) = -7/4$.

39. (a) We have two vectors, say

$$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k} \text{ and } \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= (2 - 6)\hat{i} - (-1 - 3)\hat{j} + (2 + 2)\hat{k}$$

$$= -4\hat{i} + 4\hat{j} + 4\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-4)^2 + (4)^2 + (4)^2} = 4\sqrt{3}$$

Hence, unit vector perpendicular to both the vectors is given by

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{(-4\hat{i} + 4\hat{j} + 4\hat{k})}{4\sqrt{3}} = \frac{1}{\sqrt{3}}(-\hat{i} + \hat{j} + \hat{k})$$

40. (d) We know that

$$\int_C F\cdot dr = \int_C [3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}]$$
$$(dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$= \int_C [3x^2dx + (2xz-y)dy + zdz]$$

The equations of the straight line from (0, 0, 0) to (2, 1, 3) are $x/2 = y/1 = z/3 = t$.

$\therefore x = 2t, y = t$ and $z = 3t$ are its parametric equations. The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$, respectively.

Hence, work done

$$= \int_C F\cdot dr = \int_0^1 \left[3(2t)^2 2dt + [(4t)(3t) - t]dt + (3t)3dt\right]$$

$$= \int_0^1 (36t^2 + 8t)\,dt = \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1 = 12 + 4$$
$$= 16$$

41. (a) We have

$$f(x) = x - x^2$$

Now,

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

Then,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)dx = \frac{1}{\pi}\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{-\pi}^{\pi} = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx dx$$

Integrating by parts, we get

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1-2x)\times\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2} \qquad \left[\because \cos n\pi = (-1)^n\right]$$

$\therefore a_1 = 4/1^2, a_2 = -4/2^2, a_3 = 4/3^2, a_4 = 4/4^2$, etc.

Finally,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi} = \frac{-2(-1)^2}{n}$$

$\therefore b_1 = 2/1, b_2 = -2/2, b_3 = 2/3, b_4 = -2/4$, etc.

Substituting the values of a and b in Eq. (1), we get

$$x - x^2 = -\frac{\pi}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

42. (b) Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \qquad (1)$$

Then,

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)dt + \int_{-\pi/2}^{\pi/2} (0)dt + \int_{\pi/2}^{\pi} (1)dt\right\}$$
$$= \frac{1}{\pi}\left\{|-x|_{-\pi}^{-\pi/2} + |x|_{\pi/2}^{\pi}\right\}$$
$$= \frac{1}{\pi}(\pi/2 - \pi + \pi - \pi/2) = 0$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)\cos nt dt + \int_{-\pi/2}^{\pi/2} (0)\cos nt dt + \int_{\pi/2}^{\pi} (1)\cos nt dt\right\}$$
$$= \frac{1}{\pi}\left\{\left|\frac{\sin nt}{n}\right|_{-\pi}^{-\pi/2} + \left|\frac{\sin nt}{n}\right|_{\pi/2}^{\pi}\right\}$$
$$= \frac{1}{n\pi}\left(\frac{\sin n\pi}{2} - \frac{\sin n\pi}{2}\right)$$
$$= 0$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1)\sin nt dt + \int_{-\pi/2}^{\pi/2} (0)\sin nt dt + \int_{\pi/2}^{\pi} (1)\sin nt dt\right\}$$
$$= \frac{1}{\pi}\left\{\left|\frac{\cos nt}{n}\right|_{-\pi}^{-\pi/2} + \left|\frac{\cos nt}{n}\right|_{\pi/2}^{\pi}\right\}$$
$$= \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$$\therefore b_1 = \frac{2}{\pi}, b_2 = -\frac{2}{\pi}, b_3 = \frac{2}{3\pi}, \text{ etc.}$$

Hence, substituting the values of a and b in Eq. (1),

we get $f(t) = \dfrac{2}{\pi}\left(\sin t - \sin 2t + \dfrac{1}{3}\sin 3t - \cdots\right)$.

43. (d) Let

$$x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x dx = \frac{2}{\pi}\left|x(-\cos x) - 1(-\sin x)\right|_0^{\pi}$$

$$= 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left(\sin \overline{n+1}x - \sin \overline{n-1}x\right)dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\left\{\frac{-\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\cdot\pi\left\{\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right\} (n \neq 1)$$

When $n = 1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x dx$$

$$= \frac{1}{\pi}\left|x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{2}\right)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left(-\frac{\pi \cos 2\pi}{2}\right) = -\frac{1}{2}$$

Hence,

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{3\cdot 5} + \frac{\cos 4x}{5\cdot 7} - \cdots \infty\right\}$$

44. (a) Since

$$f(-x) = -x/2 = -f(x)$$

$\therefore$ $f(x)$ is an odd function and hence

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx dx = \frac{2}{\pi}\int_0^{\pi} \frac{x}{2} \sin nx dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = -\frac{\cos n\pi}{n}$$

$\therefore b_1 = 1/1,\ b_2 = -1/2,\ b_3 = 1/3,\ b_4 = -1/4$, etc.

Hence, the series is given by

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots$$

45. (d) We have

$$\int_0^{\pi/2} \frac{\cos x + i \sin x}{\cos x - i \sin x} dx = \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}} dx$$

$$= \int_0^{\pi/2} e^{2ix} \cdot dx$$

$$= \left[\frac{e^{2ix}}{2i}\right]_0^{\pi/2} = \frac{1}{2i}\left[e^{i\pi} - e^0\right]$$

$$= \frac{1}{2i}[-1-1] \quad (\because e^{i\pi} = -1)$$

$$= \frac{-2}{2i} = \frac{-1}{-i} = i$$

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. If P, Q and R are three points having coordinates (3, −2, −1), (1, 3, 4), and (2, 1, −2) in XYZ space, then the distance from point P to plane OQR (O being the origin of the coordinate system) is given by

(a) 3 (b) 5
(c) 7 (d) 9

(GATE 2003, 1 Mark)

Solution: We are given that
$P(3, -2, -1)$, $Q(1, 3, 4)$, $R(2, 1, -2)$ and $O(0, 0, 0)$

Equation of plane OQR is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x - 0 & y - 0 & z - 0 \\ 1 & 3 & 4 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

$$\Rightarrow 2x - 2y + z = 0$$

Now, perpendicular distance of (x_1, y_1, z_1) from $ax + by + cz + d = 0$ is given by

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Therefore, perpendicular distance of (3, −2, −1) from plane $2x - 2y + z = 0$ is given by

$$\left|\frac{2\times 3 - 2\times(-2) + (-1)}{\sqrt{2^2 + (-2)^2 + 1^2}}\right| = 3$$

Ans. (a)

2. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, then dy/dx will be equal to

(a) $\sin\left(\frac{\theta}{2}\right)$ (b) $\cos\left(\frac{\theta}{2}\right)$

(c) $\tan\left(\frac{\theta}{2}\right)$ (d) $\cot\left(\frac{\theta}{2}\right)$

(GATE 2004, 1 Mark)

Solution: We are given that

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$$

Differentiating x and y with respect to θ, we get

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \quad (1)$$

$$\frac{dy}{d\theta} = a\sin\theta \quad (2)$$

Dividing Eq. (2) by Eq. (1), we get

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \frac{2a\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)}{a\times 2\cos^2\left(\frac{\theta}{2}\right)}$$

$$= \tan\frac{\theta}{2}$$

Ans. (c)

3. The volume of an object expressed in spherical coordinate is given by

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2 \sin\phi \, dr \, d\phi \, d\theta$$

The value of the integral is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

(GATE 2004, 2 Marks)

Solution: We have a triple integral,

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2 \sin\phi \, dr \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{r^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}[-\cos\phi]_0^{\pi/3} d\theta = \frac{1}{3}\times\frac{1}{2}\times\int_0^{2\pi} d\theta$$

$$= \frac{1}{3}\times\frac{1}{2}\times 2\pi = \frac{\pi}{3}$$

Ans. (a)

4. If $S = \int_1^{\infty} x^{-3} dx$, then S has the value

(a) −1/3 (b) 1/4
(c) 1/2 (d) 1

(GATE 2005, 1 Mark)

Solution: We have

$$S = \int_1^{\infty} x^{-3} dx$$

Now, integrating, we get

$$S = \left[\frac{x^{-2}}{-2}\right]_1^{\infty} = -\left[\frac{1}{2x^2}\right]_1^{\infty} = -\left[\frac{1}{\infty} - \frac{1}{2}\right] = \frac{1}{2}$$

Ans. (c)

5. Changing the order of the integration in the double integral $I = \int_0^8\int_{x/4}^2 f(x,y)dydx$ leads to $I = \int_r^s\int_p^q f(x,y)dxdy$. What is q?

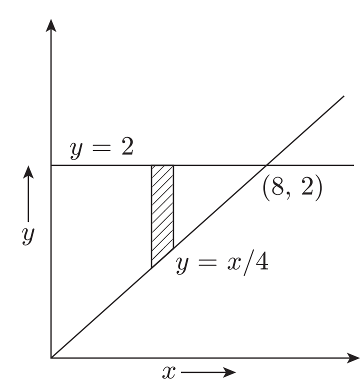
(a) $4y$ (b) $16y^2$
(c) x (d) 8

(GATE 2005, 1 Mark)

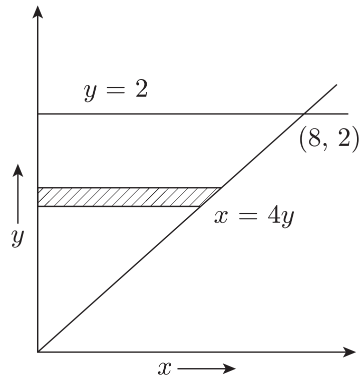
Solution: We have

$$I = \int_0^8\int_{x/4}^2 f(x\cdot y)dydx$$

Hence,



Now,



$$I = \int_0^2 \int_0^{4y} f(x \cdot y)dxdy$$

$\therefore \qquad q = 4y$

Ans. (a)

6. Stokes' theorem connects

(a) a line integral and a surface integral
(b) a surface integral and a volume integral
(c) a line integral and a volume integral
(d) gradient of a function and its surface integral

(GATE 2005, 1 Mark)

Solution: A line integral and a surface integral is related by Stokes' theorem.

Ans. (a)

7. A rail engine accelerates from its stationary position for 8 seconds and travels a distance of 280 m. According to the mean value theorem, the speedometer at a certain time during acceleration must read exactly

(a) 0 (b) 8 kmph
(c) 75 kmph (d) 126 kmph

(GATE 2005, 2 Marks)

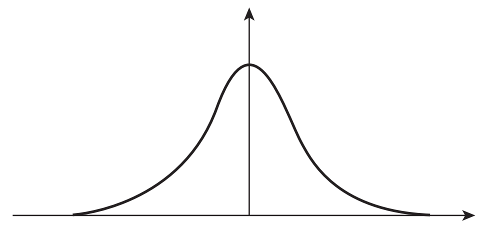
Solution: Since the position of rail engine $S(t)$ is a continuous and differentiable function, according to Lagrange's mean value theorem,

$$S'(t) = v(t) = \frac{S(8) - S(0)}{8 - 0} = \frac{(280 - 0)}{8}\,\text{m/sec}$$
$$= \frac{280}{8}\,\text{m/sec} = \frac{280}{8} \times \frac{3600}{1000}\,\text{kmph} = 126 \text{ kmph}$$

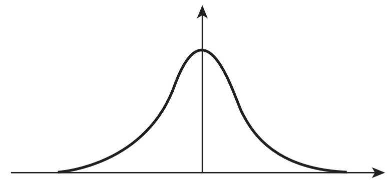
where $v(t)$ is the velocity of the rail engine.

Ans. (d)

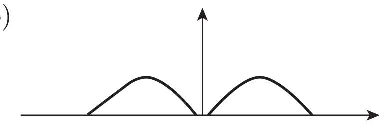
8. The derivative of the symmetric function drawn in the given figure will look like



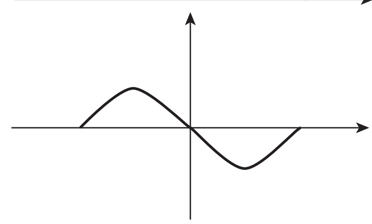
(a)



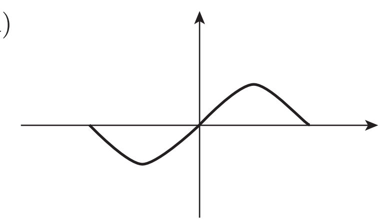
(b)



(c)

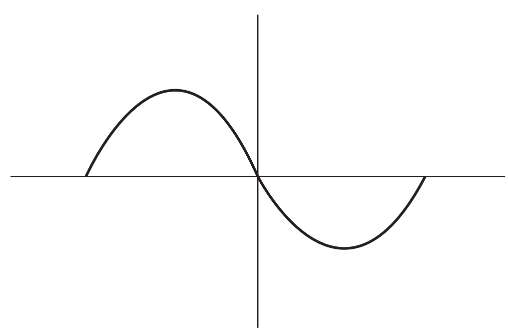


(d)



(GATE 2005, 2 Marks)

Solution: The given function has a negative slope in the positive half and a positive slope in the negative half. So, its differentiation curve is satisfied by option (c).



Ans. (c)

9. By a change of variable $x(u, v) = uv$, $y(u, v) = v/u$ in a double integral, the integrand $f(x, y)$ changes to $f(uv, v/u)\phi(u, v)$. Then $\phi(u, v)$ is

(a) $2v/u$ (b) $2uv$
(c) v^2 (d) 1

(GATE 2005, 2 Marks)

Solution:

$$\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u$$

and $\dfrac{\partial y}{\partial u} = -\dfrac{v}{u^2} \qquad \dfrac{\partial y}{\partial v} = \dfrac{1}{u}$

and $\phi(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -\dfrac{v}{u^2} & \dfrac{1}{u} \end{vmatrix} = \dfrac{v}{u} + \dfrac{v}{u} = \dfrac{2v}{u}$

Ans. (a)

10. For the scalar field $u = \dfrac{x^2}{2} + \dfrac{y^2}{3}$, magnitude of the gradient at the point (1, 3) is

(a) $\sqrt{\dfrac{13}{9}}$ (b) $\sqrt{\dfrac{9}{2}}$

(c) $\sqrt{5}$ (d) $\dfrac{9}{2}$

(GATE 2005, 2 Marks)

Solution: We have the scalar field,

$$u = \frac{x^2}{2} + \frac{y^2}{3}$$

Now, $\text{grad}\, u = i\dfrac{\partial u}{\partial x} + j\dfrac{\partial u}{\partial y} = xi + \dfrac{2}{3}yj$

At (1, 3),

$$\text{grad}\, u = (1)j + \left(\frac{2}{3}\cdot 3\right)j = i + 2j$$

$$|\text{grad}\ u| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Ans. (c)

11. The line integral $\int \vec{V}\cdot d\vec{r}$ of the vector $\vec{V}\cdot(\vec{r}) = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$ from the origin to the point $P(1, 1, 1)$ is

(a) 1
(b) 0
(c) −1
(d) cannot be determined without specifying the path

(GATE 2005, 2 Marks)

Solution: We are given that

$$\vec{V}\cdot(\vec{r}) = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$$

Hence, we can deduce that

$$f_x = 2xyz, f_y = x^2z, f_z = x^2y$$

On integrating, we get $f =$ potential function of $\vec{V} = x^2yz$.

Therefore, line integral of the vector function from point

$$A(0, 0, 0) \text{ to point } B(1, 1, 1) = f(B) - f(A)$$

$$= (x^2yz)_{1,1,1} - (x^2yz)_{(0,0,0)} = 1 - 0 = 1$$

Ans. (a)

12. Value of the integral $\oint_c (xydy - y^2dx)$, where c is the square cut from the quadrant by the lines $x = 1$ and $y = 1$ will be (use Green's theorem to change the line integral into double integral)

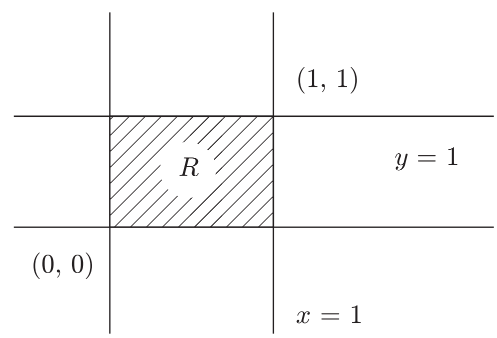
(a) $\dfrac{1}{2}$ (b) 1

(c) $\dfrac{3}{2}$ (d) $\dfrac{5}{3}$

(GATE 2005, 2 Marks)

Solution: We know that Green's theorem is given by

$$\oint_c \phi dx + \Psi dy = \iint_R \left(\frac{\partial \Psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy$$



Here $I = \oint_c (xydy - y^2dx) = \oint_c (-y^2)dx + (xy)dy$

Hence, we can deduce

$$\phi = -y^2,$$
$$\Psi = xy,$$
$$\frac{\partial \Psi}{\partial x} = y,$$
$$\frac{\partial \phi}{\partial y} = -2y$$

Substituting in Green's theorem, we get

$$I = \int_{y=0}^{1}\int_{x=0}^{1} [y - (-2y)]\, dxdy = \int_{y=0}^{1}\int_{x=0}^{1} 3ydxdy$$

$$= \int_{y=0}^{1} [3xy]_{x=0}^{1} dy = \int_{y=0}^{1} 3ydy$$

$$= \frac{3}{2}$$

Ans. (c)

13. $\nabla \times \nabla \times P$, where P is a vector, is equal to

(a) $P \times \nabla \times P - \nabla^2 P$ (b) $\nabla^2 P + \nabla(\nabla \times P)$
(c) $\nabla^2 P + \nabla \times P$ (d) $\nabla(\nabla \cdot P) - \nabla^2 P$

(GATE 2006, 1 Mark)

Solution: From the property of vector triple product, we have

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

Now putting, $A = \nabla$, $B = \nabla$ and $C = P$

We get $\nabla \times \nabla \times P = (\nabla \cdot P)\nabla - (\nabla \cdot \nabla)P$

$$= \nabla(\nabla \cdot P) - \nabla^2 P$$

Ans. (d)

14. $\iint (\nabla \times P) \cdot ds$, where P is a vector, is equal to

(a) $\oint P \cdot dl$ (b) $\oint \nabla \times \nabla \times P \cdot dl$
(c) $\oint \nabla \times P \cdot \mathrm{d}l$ (d) $\iiint \nabla \cdot P dv$

(GATE 2006, 1 Mark)

Solution: According to Stokes' theorem,

$$\iint (\nabla \times P) \cdot ds = \oint P \cdot dl$$

Ans. (a)

15. If $f(x) = \dfrac{2x^2 - 7x + 3}{5x^2 - 12x - 9}$, then $\lim\limits_{x \to 3} f(x)$ will be

(a) $-1/3$ (b) $5/18$
(c) 0 (d) $2/5$

(GATE 2006, 2 Marks)

Solution: We have to find

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \left(\frac{2x^2 - 7x + 3}{5x^2 - 12x - 9} \right)$$

Applying $x = 3$, we get the form of $\left(\dfrac{0}{0}\right)$. Hence, applying L'Hospital's rule, we get

$$\lim_{x \to 3} \left(\frac{4x - 7}{10x - 12} \right) = \frac{5}{18}$$

Ans. (b)

16. As x increased from $-\infty$ to ∞, the function $f(x) = \dfrac{e^x}{1 + e^x}$

(a) monotonically increases
(b) monotonically decreases
(c) increases to a maximum value and then decreases
(d) decreases to a minimum value and then increases

(GATE 2006, 2 Marks)

Solution: We have

$$f(x) = \frac{e^x}{1 + e^x}$$

Differentiating $f(x)$, we get

$$f'(x) = \frac{e^x(1 + e^x) - e^{2x}}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2}$$

Since e^x is positive for all values of x, $f'(x)$ is positive for all values of x and hence $f(x)$ monotonically increases.

Ans. (a)

17. Assuming $i = \sqrt{-1}$ and t as a real number, $\int_0^{\pi/3} e^{it} dt$ is

(a) $\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}$ (b) $\dfrac{\sqrt{3}}{2} - i\dfrac{1}{2}$
(c) $\dfrac{1}{2} + i\left(\dfrac{\sqrt{3}}{2}\right)$ (d) $\dfrac{1}{2} + i\left(1 - \dfrac{\sqrt{3}}{2}\right)$

(GATE 2006, 2 Marks)

Solution: We have

$$I = \int_0^{\pi/3} e^{it} dt$$

On integrating, we get

$$I = \left[\frac{e^{it}}{i}\right]_0^{\pi/3} = \left[\frac{\cos t + i \sin t}{i}\right]_0^{\pi/3} = \frac{1}{i}\left[\frac{1}{2} + i\frac{\sqrt{3}}{2} - 1\right]$$

$$= \left[-\frac{1}{2i} + \frac{\sqrt{3}}{2}\right] = \left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right]$$

Ans. (a)

18. The integral $\int_0^{\pi} \sin^3 \theta d\theta$ is given by

(a) 1/2 (b) 2/3
(c) 4/3 (d) 8/3

(GATE 2006, 2 Marks)

Solution: We have

$$I = \int_0^{\pi} \sin^3 \theta \cdot d\theta$$

$$= \int_0^{\pi} (1 - \cos^2 \theta) \sin \theta \, d\theta$$

Let $\cos \theta = t$, then $dt = -\sin\theta \, d\theta$

At $\theta = 0$, $t = \cos 0 = 1$

and at $\theta = \pi$, $t = \cos \pi = -1$

So,

$$I = -\int_{-1}^{1}(1-t^2)dt = \left|t - \frac{t^3}{3}\right|_{-1}^{1} = \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)$$

$$I = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

Ans. (c)

19. What is the area common to the circles $r = a$ and $r = 2a\cos\theta$?

(a) $0.524\ a^2$ (b) $0.614\ a^2$
(c) $1.047\ a^2$ (d) $1.228\ a^2$

(GATE 2006, 2 Marks)

Solution: The area common to the circles $r = a$ and $r = 2a\cos\theta$ is $1.228a^2$.

Ans. (d)

20. The expression $V = \int_0^H \pi R^2 (1 - h/H)^2 dh$ for the volume of a cone is equal to

(a) $\int_0^R \pi R^2 (1 - h/H)^2 dr$

(b) $\int_0^R \pi R^2 (1 - h/H)^2 dh$

(c) $\int_0^H 2\pi r H (1 - r/R) dh$

(d) $\int_0^R \pi r H \left(1 - \frac{r}{R}\right)^2 dr$

(GATE 2006, 2 Marks)

Solution: We consider options (a) and (d) only, because these contain variable r, as variable of integration. On integrating option (d), we get

$(1/3)\pi a^2 H,$ which is volume of cone.

Ans. (d)

21. The directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at point $P(2, 1, 3)$ in the direction of the vector $a = i - 2k$ is

(a) -2.785 (b) -2.145
(c) -1.789 (d) 1.000

(GATE 2006, 2 Marks)

Solution: We have

$$f = 2x^2 + 3y^2 + z^2,\ P(2,1,3),$$
$$a = i - 2k$$
$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$
$$= 4xi + 6yj + 2zk$$

at $P(2,1,3)$ $\quad \nabla f = 4\times 2\times i + 6\times 1\times j + 2\times 3\times k$

$$= 8i + 6j + 6k$$

The directional derivative of f in direction of vector $a = i - 2k$ is the component of grad f in the direction of vector a and is given by $\frac{a}{|a|}\cdot$ grad f

$$= \left[\frac{i - 2k}{\sqrt{1^2 + (-2)^2}}\right]\cdot(8i + 6j + 6k)$$
$$= \frac{1}{\sqrt{5}}[1.8 + 0.6 + (-2)6] = \frac{-4}{\sqrt{5}}$$
$$= -1.789$$

Ans. (c)

22. Equation of the line normal to function $f(x) = (x-8)^{2/3} + 1$ at $P(0, 5)$ is

(a) $y = 3x - 5$ (b) $y = 3x + 5$
(c) $3y = x + 15$ (d) $3y = x - 15$

(GATE 2006, 2 Marks)

Solution: Given that

$$f(x) = (x-8)^{2/3} + 1$$

Differentiating with respect to x, we get

$$f'(x) = \frac{2}{3}(x-8)^{-1/3}$$

Slope of tangent at point (0, 5)

$$m = \frac{2}{3}(0-8)^{-1/3} = -\frac{1}{3}$$

Slope of normal at point (0, 5)

$$m_1 = -\frac{1}{m} = 3$$

Equation of normal at point (0, 5)

$$y - 5 = 3(x - 0)$$
$$\Rightarrow \qquad y = 3x + 5$$

Ans. (b)

23. $\lim_{\theta\to\infty} \frac{\sin(\theta/2)}{\theta}$ is

(a) 0.5 (b) 1
(c) 2 (d) not defined

(GATE 2007, 1 Mark)

Solution: We have

$$\lim_{\theta\to\infty}\frac{\sin(\theta/2)}{\theta} = \lim_{\theta\to 0}\frac{\frac{1}{2}\times\sin\left(\frac{\theta}{2}\right)}{\theta\times\frac{1}{2}}$$
$$= \frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta/2}{\theta/2} = \frac{1}{2}$$
$$= 0.5$$

Ans. (a)

24. For the function e^{-x}, the linear approximation around $x = 2$ is

(a) $(3-x)e^{-2}$

(b) $1 - x$

(c) $[3+2\sqrt{2}-(1+\sqrt{2})x]e^{-2}$

(d) e^{-2}

(GATE 2007, 1 Mark)

Solution: The Taylor's series expansion of $f(x)$ allowed at $x = 2$ is

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \cdots$$

For linear approximation, we take only the first two terms and get

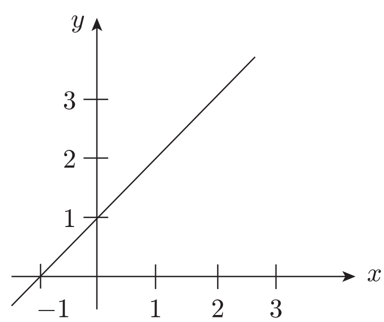
$$f(x) = f(2) + (x-2)f'(2)$$

Here, $f(x) = e^{-x}$ and $f'(x) = -e^{-x}$

$$\therefore \quad f(x) = e^{-2} + (x-2)(-e^{-2}) = (3-x)e^{-2}$$

Ans. (a)

25. The following plot shows a function y which varies linearly with x. The value of the integral $I = \int_1^2 y dx$ is



(a) 1.0 (b) 2.5

(c) 4.0 (d) 5.0

(GATE 2007, 1 Mark)

Solution: Equation of line with slope 1 and y-intercept of 1 is given by

$$y = x + 1$$

$$I = \int_1^2 y dx$$

$$= \int_1^2 (x+1)\,dx = \left(\frac{x^2}{2} + x\right)\Bigg|_1^2$$

$$= \left(\frac{4}{2} + 2\right) - \left(\frac{1}{2} + 1\right) = 2.5$$

Ans. (b)

26. What is the value of $\lim_{x\to 0} \dfrac{e^x - \left(1 + x + \dfrac{x^2}{2}\right)}{x^3}$?

(a) 0 (b) 1/6

(c) 1/3 (d) 1

(GATE 2007, 2 Marks)

Solution: We have

$$\lim_{x\to 0} \frac{e^x - \left(1 + x + \dfrac{x^2}{2}\right)}{x^3}$$

This is of the form $\left(\dfrac{0}{0}\right)$.

Applying L'Hospital's rule, we get

$$\lim_{x\to 0} \frac{e^x - 1 - x}{3x^2}, \text{ which is of the form } \left(\frac{0}{0}\right).$$

Applying L'Hospital's rule again, we get

$$\lim_{x\to 0} \frac{e^x - 1}{6x}, \text{ which is of the form } \left(\frac{0}{0}\right).$$

Applying L'Hospital's rule again, we get

$$\lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6}$$

Ans. (b)

27. Evaluate $\int_0^\infty \frac{\sin t}{t} dt$

(a) π (b) $\dfrac{\pi}{2}$

(c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{8}$

(GATE 2007, 2 Marks)

Solution: Let

$$I(\alpha) = \int_0^\infty e^{-\alpha x} \frac{\sin x}{x} dx \qquad (1)$$

$$\frac{\partial I}{\partial \alpha} = \int_0^\infty (-x)e^{-\alpha x}\left(\frac{\sin x}{x}\right)dx = -\int_0^\infty e^{-\alpha x} \sin x\, dx$$

Then integrating by parts, we get

$$\frac{\partial I}{\partial \alpha} = -\left[\frac{e^{-\alpha x}}{1+\alpha^2}(-\alpha \sin x - 1\cos x)\right]_0^\infty = -\frac{1}{1+\alpha^2}$$

$$\therefore \frac{\partial I}{\partial \alpha} = -\frac{1}{1+\alpha^2}$$

$$\partial I = -\frac{\partial\alpha}{1+\alpha^2}$$

On integrating, we get

$$I = -\tan^{-1}\alpha + C$$

$I(\infty) = -\dfrac{\pi}{2} + C$ but from Eq. (1), $I(\infty) = 0$

$$\therefore \quad -\frac{\pi}{2} + C = 0$$

$$\Rightarrow \quad C = \frac{\pi}{2}$$

$$\therefore \quad I(\alpha) = -\tan^{-1}\alpha + \frac{\pi}{2}$$

$$\Rightarrow \quad I(0) = \frac{\pi}{2} - \tan^{-1} 0$$

$$\therefore \quad I(0) = \frac{\pi}{2}$$

But from Eq. (1), we get

$$I(0) = \int_0^\infty \frac{\sin x}{x}\,dx$$

$$\therefore \quad \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Ans. (b)

28. The area of a triangle formed by the tips of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

(a) $\frac{1}{2}(\vec{a}-\vec{b})(\vec{a}-\vec{c})$

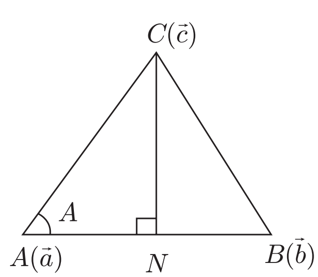
(b) $\frac{1}{2}|(\vec{a}-\vec{b})\times(\vec{a}-\vec{c})|$

(c) $\frac{1}{2}|\vec{a}\times\vec{b}\times\vec{c}|$

(d) $\frac{1}{2}(\vec{a}\times\vec{b})\cdot\vec{c}$

(GATE 2007, 2 Marks)

Solution:



In the figure, from C, draw $CN \perp AB$. From right-angled ΔCAN, we get

$$\sin A = \left|\frac{CN}{AC}\right| \Rightarrow |CN| = |AC|\sin A$$

$$\begin{aligned}\text{Area of } \Delta\, ABC &= \frac{1}{2}|AB|\times|CN| \\ &= \frac{1}{2}|AB|\cdot|AC|\sin A \\ &= \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|\end{aligned}$$

From the figure, $\overrightarrow{AB} = \vec{b}-\vec{a}$ and $\overrightarrow{AC} = \vec{c}-\vec{a}$.

$$\begin{aligned}\text{So, area of } \Delta ABC &= \frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})| \\ &= \frac{1}{2}|(\vec{a}-\vec{b})\times(\vec{a}-\vec{c})|\end{aligned}$$

Ans. (b)

29. Let x and y be two vectors in a three-dimensional space and $<x, y>$ denotes their dot product. Then the determinant of $\begin{bmatrix} <x,x> & <x,y> \\ <y,x> & <y,y> \end{bmatrix}$ is

(a) zero when x and y are linearly independent
(b) positive when x and y are linearly independent
(c) non-zero for all non-zero x and y
(d) zero only when either x or y is zero

(GATE 2007, 2 Marks)

Solution: Let $D = \begin{vmatrix} x\cdot x & x\cdot y \\ y\cdot x & y\cdot y \end{vmatrix}$

Let $x = x_1 i + x_2 j$ and $y = y_1 i + y_2 j$.

Then,

$$x\cdot x = x_1^2 + x_2^2$$
$$y\cdot y = y_1^2 + y_2^2$$
$$x\cdot y = x_1y_1 + x_2y_2$$

$$\begin{aligned}\therefore\ D &= \begin{vmatrix} {x_1}^2 + {x_2}^2 & x_1y_1 + x_2y_2 \\ x_1y_1 + x_2y_2 & {y_1}^2 + {y_2}^2 \end{vmatrix} \\ &= ({x_1}^2 + {x_2}^2)({y_1}^2 + {y_2}^2) - (x_1y_1 + x_2y_2)^2 \\ &= {x_2}^2{y_1}^2 + {x_1}^2{y_2}^2 - 2x_1y_1x_2y_2 \\ &= (x_2y_1 - x_1y_2)^2\end{aligned}$$

Now, $D = 0$. Thus,

$$x_2y_1 - x_1y_2 = 0$$

$$\Rightarrow \quad \frac{x_1}{x_2} = \frac{y_1}{y_2}$$

Hence, vectors $x_1 i + x_2 j$ and $y_1 i + y_2 j$ are linearly dependent.

$\therefore$ Linear dependence $\Rightarrow D = 0$

So, linear independence $\Rightarrow D \neq 0$

i.e. D is negative or positive.

However, since $D = (x_2 y_1 - x_1 y_2)^2$, it cannot be negative.

Hence, D is positive when x and y are linearly independent.

Ans. (b)

30. A velocity vector is given as $\vec{V} = 5xy\vec{i} + 2y^2\vec{j} + 3yz^2\vec{k}$. The divergence of this velocity vector at (1, 1, 1) is

(a) 9 (b) 10 (c) 14 (d) 15

(GATE 2007, 2 Marks)

Solution: We have

$$\vec{V} = 5xy\vec{i} + 2y^2\vec{j} + 3yz^2\vec{k} = v_1 i + v_2 j + v_3 k$$

$$\text{div}(\vec{V}) = \frac{dv_1}{dx} + \frac{dv_2}{dy} + \frac{dv_3}{dz} = 5y + 4y + 6yz$$

Now, at (1, 1, 1), we have

$$\text{div}(\vec{V}) = 5 \cdot 1 + 4 \cdot 1 + 6 \cdot 1 \cdot 1 = 15$$

Ans. (d)

31. The value of $\lim\limits_{x \to 8} \dfrac{x^{1/3} - 2}{(x-8)}$

(a) $\dfrac{1}{16}$ (b) $\dfrac{1}{12}$

(c) $\dfrac{1}{8}$ (d) $\dfrac{1}{4}$

(GATE 2008, 1 Mark)

Solution: Let us say

$$(x - 8) = h$$

$$\Rightarrow \quad x = 8 + h$$

$$\therefore \quad \lim_{h \to 0} \frac{(8+h)^{1/3} - 2}{h}$$

Putting the value of limit, we obtain the form $\left(\dfrac{0}{0}\right)$. Hence, apply L'Hospital's rule,

$$\lim_{h \to 0} \frac{\frac{1}{3}(8+h)^{\left(\frac{1}{3}-1\right)}}{1} = \frac{1}{3}(8)^{-\frac{2}{3}} = \frac{1}{12}$$

Ans. (b)

32. $\lim\limits_{x \to \infty} \dfrac{x - \sin x}{x + \cos x}$ equals

(a) 1 (b) -1 (c) ∞ (d) $-\infty$

(GATE 2008, 1 Mark)

Solution: We have

$$\lim_{x \to \infty} \frac{x - \sin x}{x + \cos x} = \lim_{x \to \infty} \frac{1 - \dfrac{\sin x}{x}}{1 + \dfrac{\cos x}{x}} = \frac{\lim\limits_{x \to \infty}\left(1 - \dfrac{\sin x}{x}\right)}{\lim\limits_{x \to \infty}\left(1 + \dfrac{\cos x}{x}\right)}$$

$$= \frac{1 - \lim\limits_{x \to \infty} \dfrac{\sin x}{x}}{1 + \lim\limits_{x \to \infty} \dfrac{\cos x}{x}}$$

$$= \frac{1-0}{1+0} = 1$$

Ans. (a)

33. In the Taylor's series expansion of e^x about $x = 2$, the coefficient of $(x - 2)^4$ is

(a) $1/4!$ (b) $2^4/4!$

(c) $e^2/4!$ (d) $e^4/4!$

(GATE 2008, 1 Mark)

Solution: $f(x)$ in the neighborhood of a is given by

$$f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$$

where $b_n = \dfrac{f^n(a)}{n!}$

$$f^4(x) = e^x \cdot f^4(2) = e^2$$

Therefore, coefficient of $(x-2)^4 = b_4 = \dfrac{f^4(2)}{4!} = \dfrac{e^2}{4!}$

Ans. (c)

34. Which of the following functions would have only odd powers of x in its Taylor's series expansion about the point $x = 0$?

(a) $\sin(x^3)$ (b) $\sin(x^2)$

(c) $\cos(x^3)$ (d) $\cos(x^2)$

(GATE 2008, 1 Mark)

Solution: We know that

$$\sin x = x - \frac{x^3}{\angle 3} + \frac{x^5}{\angle 5} - \frac{x^7}{\angle 7} + \cdots \qquad (1)$$

$$\cos x = 1 - \frac{x^2}{\angle 2} + \frac{x^4}{\angle 4} - \frac{x^6}{\angle 6} + \cdots \qquad (2)$$

From Eqs. (1) and (2), we can deduce that

$$\sin x^2 = x^2 - \frac{x^6}{\angle 3} + \frac{x^{10}}{\angle 5} - \frac{x^{14}}{\angle 7}$$

$$\cos x^2 = 1 - \frac{x^4}{\angle 2} + \frac{x^8}{\angle 4} - \frac{x^{12}}{\angle 6}$$

So, sin x^2 and cos x^2 have only even powers of x.

Similarly, $\sin x^3 = x^3 - \dfrac{x^9}{\angle 3} + \dfrac{x^{15}}{\angle 5} - \cdots$

$$\cos x^3 = 1 - \frac{x^6}{\angle 2} + \frac{x^{12}}{\angle 4} - \cdots$$

So, only $\sin(x^3)$ has all odd powers of x.

$\therefore$ The correct option is (a). Ans. (a)

35. The divergence of the vector field $(x - y)\hat{i} + (y - x)\hat{j} + (x + y + z)\hat{k}$ is

(a) 0 (b) 1 (c) 2 (d) 3

(GATE 2008, 1 Mark)

Solution: $\text{div}[(x - y)\hat{i} + (y - x)\hat{j} + (x + y + z)\hat{k}] =$

$$\frac{\partial}{\partial x}(x - y) + \frac{\partial}{\partial y}(y - x) + \frac{\partial}{\partial z}(x + y + z) = 1 + 1 + 1 = 3$$

Ans. (d)

36. In the Taylor's series expansion of exp(x) + sin(x) about the point $x = \pi$, the coefficient of $(x - \pi)^2$ is

(a) exp(π) (b) 0.5 exp(π)
(c) exp(π) + 1 (d) exp(π) − 1

(GATE 2008, 2 Marks)

Solution: We have

$$f(x) = e^x + \sin x$$

We wish to expand about $x = \pi$.

Taylor's series expansion about $x = a$ is given by

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \frac{(x - a)^3}{3!}f'''(a) + \cdots$$

Now, for $x = \pi$

$$f(x) = f(\pi) + (x - \pi)f'(\pi) + \frac{(x - \pi)^2}{2!}f''(\pi) + \cdots$$

The coefficient of $(x - \pi)^2$ is $\dfrac{f''(\pi)}{2!}$.

Here, $f(x) = e^x + \sin x$

$$f'(x) = e^x + \cos x$$

$$f''(x) = e^x - \sin x$$

$$f'''(\pi) = e^\pi - \sin \pi = e^\pi - 0 = e^\pi$$

The coefficient of $(x - \pi)^2$ is therefore $\dfrac{e^\pi}{2!} = 0.5\exp(\pi)$.

Ans. (b)

37. Let $f = y^x$. What is $\dfrac{\partial^2 f}{\partial x \partial y}$ at $x = 2$, $y = 1$?

(a) 0 (b) ln 2
(c) 1 (d) $\dfrac{1}{\ln 2}$

(GATE 2008, 2 Marks)

Solution: We have

$$f = y^x$$

Treating x as a constant, we get

$$\frac{\partial f}{\partial y} = xy^{x-1}$$

Now treating y as a constant, we get

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(y^{x-1}x) = y^{x-1} + x\,y^{x-1}\ln y$$

Substituting $x = 2$ and $y = 1$, we get

$$\frac{\partial^2 f}{\partial x \partial y} = 1^{(2-1)}(1 + 2 \cdot \ln 1) = 1$$

Ans. (c)

38. Which one of the following integrals is unbounded?

(a) $\displaystyle\int_0^{\pi/4} \tan x\,dx$ (b) $\displaystyle\int_0^{\infty} \frac{1}{x^2 + 1}dx$

(c) $\displaystyle\int_0^{\infty} xe^{-x}dx$ (d) $\displaystyle\int_0^{1} \frac{1}{1 - x}dx$

(GATE 2008, 2 Marks)

Solution: We consider all the four options one by one.

$$\int_0^{\frac{\pi}{4}} \tan x\,dx = \log\sqrt{2}$$

$$\int_0^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}$$

$$\int_0^{\infty} x\,e^{-x}\,dx$$

Integrating by parts, taking $u = x$ and $dv = e^{-x}$ dx, we get $du = dx$ and $v = -e^{-x}$.

So, $\displaystyle\int xe^{-x}dx = x(-e^{-x}) - \int -e^{-x}dx$

$$= -xe^{-x} - e^{-x}$$

$$= -e^{-x}(x + 1)$$

Now $\int_0^\infty xe^{-x}dx = [-e^{-x}(x+1)]_0^\infty = 1$

$$\int_0^1 \frac{1}{1-x}dx = \ln 0 - \ln 1 = -\infty - 0 = -\infty$$

Since only option (d) is unbounded, it is the required answer.

Ans. (d)

39. The length of the curve $y = \frac{2}{3}x^{3/2}$ between $x = 0$ and $x = 1$ is

(a) 0.27 (b) 0.67
(c) 1 (d) 1.22

(GATE 2008, 2 Marks)

Solution: We have

$$y = \frac{2}{3}x^{3/2} \quad (1)$$

Differentiating Eq. (1), we get

$$\frac{dy}{dx} = x^{1/2}$$

Length of the curve is given by

$$\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \int_0^1 \sqrt{1+x}dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$$

$$= 1.22$$

Ans. (d)

40. The value of the integral of the function $g(x,y) = 4x^3 + 10y^4$ along the straight line segment from point (0, 0) to point (1, 2) in the xy plane is

(a) 33 (b) 35 (c) 40 (d) 56

(GATE 2008, 2 Marks)

Solution: Equation of straight line from point (0, 0) to (1, 2) is given by

$$y - 0 = \frac{(2-0)}{(1-0)}(x-0)$$

or $\quad y = 2x$

We are given that

$$g(x,y) = 4x^3 + 10y^4$$

Substituting value of y, we get

$$g(x,y) = 4x^3 + 10(2x)^4 = 4x^3 + 160x^4$$

$$\int_0^1 (4x^3 + 160x^4)dx = \left(\frac{4x^4}{4} + \frac{160x^5}{5}\right)\Bigg|_0^1$$

$$= 1 + 32 = 33$$

Ans. (a)

41. The value of $\int_0^3\int_0^x (6-x-y)dxdy$ is

(a) 13.5 (b) 27.0
(c) 40.5 (d) 54.0

(GATE 2008, 2 Marks)

Solution: We have

$$\int_0^3\int_0^x (6-x-y)dxdy$$

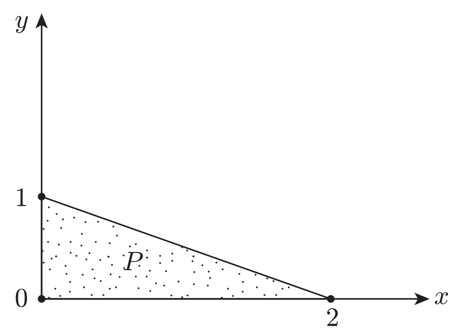
Now, calculating the double integral, we get

$$\int_0^3\int_0^x (6-x-y)dxdy = \int_0^3 \left[(6-x)y - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^3 \left[(6-x)x - \frac{x^2}{2}\right]dx = 13.5$$

Ans. (a)

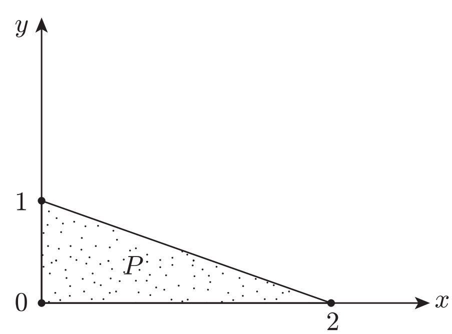
42. Consider the shaded triangular region P shown in the figure. What is $\iint_P xydxdy$?



(a) $\frac{1}{6}$ (b) $\frac{2}{9}$ (c) $\frac{7}{16}$ (d) 1

(GATE 2008, 2 Marks)

Solution:



The equation of the straight line with x-intercept = 2 and y-intercept = 1 is

$$\frac{x}{2} + \frac{y}{1} = 1$$

$\Rightarrow y = 1 - \dfrac{x}{2}$

$\Rightarrow x = 2 - 2y$

$$\int_0^1 \int_0^{(2-2y)} (xy\,dx)dy = \int_0^1 \left[\left(\frac{yx^2}{2} \right) \Bigg|_0^{2-2y} \right] dy$$

$$= \int_0^1 \frac{y}{2}(2-2y)^2 dy = \int_0^1 2y(1-y)^2 dy = \frac{1}{6}$$

Ans. (a)

43. The inner (dot) product of two non-zero vectors $\vec{P}$ and $\vec{Q}$ is zero. The angle (degrees) between the two vectors is

(a) 0 (b) 30 (c) 90 (d) 120

(GATE 2008, 2 Marks)

Solution: We know that

$$\vec{P} \cdot \vec{Q} = 0$$

$$\vec{P} \cdot \vec{Q} = |P|\,|Q| \cos\theta$$

If $\vec{P} \cdot \vec{Q} = 0$

$\Rightarrow \quad |P|\,|Q| \cos\theta = 0$

Since, P and Q are non-zero vectors

$\Rightarrow \cos\,\theta = 0$

$\Rightarrow \theta = 90^\circ$

Ans. (c)

44. The directional derivative of the scalar function $f(x, y, z) = x^2 + 2y^2 + z$ at point $P = (1,\ 1,\ 2)$ in the direction of the vector $\vec{a} = 3\hat{i} - 4\hat{j}$ is

(a) -4 (b) -2 (c) -1 (d) 1

(GATE 2008, 2 Marks)

Solution: We know that

$$\text{grad}\, f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = 2xi + 4yj + k$$

At point $P(1, 1, 2)$

$$\text{grad}\, f = 2i + 4j + k$$

Now directional derivative of f at $P(1, 1, 2)$ in the direction of vector $a = 3i - 4j$ is given by

$$\frac{a}{|a|} \text{grad}\ f = \left(\frac{3i - 4j}{\sqrt{25}} \right) \cdot (2i + 4j + k)$$

$$= \frac{1}{5}(3.2 - 4.4 + 0) = -2$$

Ans. (b)

45. Consider points P and Q in the xy plane, with $P = (1,\ 0)$ and $Q = (0,\ 1)$. The line integral $2\int_P^Q (xdx + ydy)$ along the semicircle with the line segment PQ as its diameter

(a) is -1

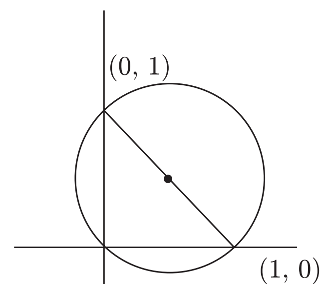
(b) is 0

(c) is 1

(d) depends on the direction (clockwise or anti-clockwise) of the semicircle.

(GATE 2008, 2 Marks)

Solution: Taking $f(x, y) = xy$, we can show that $xdx + ydy$ is exact. So, the value of the integral is independent of path.



$$= \int_p^0 (xdx + ydy)$$

$$= 2\int_1^0 xdx + 2\int_0^1 ydy$$

$$= 2\left[\frac{x^2}{2} \Bigg|_1^0 + \frac{y^2}{2} \Bigg|_{.0}^1 \right] = 0$$

$$\text{Integral} = f(Q) - f(P) = [xy]_{(0,1)} - [xy]_{(1,0)}$$

$$= 0 - 0 = 0$$

Ans. (b)

46. For a scalar function $f(x,\ y,\ z) = x^2 + 3y^2 + 2z^2$, the gradient at point $P(1,\ 2,\ -1)$ is

(a) $2i + 6j + 4k$ (b) $2i + 12j - 4k$

(c) $2i + 12j + 4k$ (d) $\sqrt{56}$

(GATE 2009, 1 Mark)

Solution: We have

$$f = x^2 + 3y^2 + 2z^2$$

$$\Delta f = \text{grad}\ f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(2x) + j(6y) + k(4z)$$

At $P(1,\ 2,\ -1)$,

$$\text{grad}\ f = i(2 \times 1) + j(6 \times 2) + k(4 \times -1)$$

$$= 2i + 12j - 4k$$

Ans. (b)

47. The divergence of the vector field $3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$ at a point (1, 1, 1) is equal to

(a) 7 (b) 4 (c) 3 (d) 0

(GATE 2009, 1 Mark)

Solution: Vector field is given as

$$\vec{f} = 3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k} = v_1 i + v_2 j + v_3 k$$

Divergence of vector field is given by

$$\begin{aligned}\text{Div}(f) = \nabla \cdot f &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \\ &= \frac{\partial}{\partial x}(3xz) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}\left(-yz^2\right) \\ &= 3z + 2x - 2zy\end{aligned}$$

$$\text{Div}(f)\big|_{(1,1,1)} = 3(1) + 2(1) - 2(1)(1) = 3$$

Ans. (c)

48. The distance between the origin and the point nearest to it on the surface $z^2 = 1 + xy$ is

(a) 1 (b) $\frac{\sqrt{3}}{2}$

(c) $\sqrt{3}$ (d) 2

(GATE 2009, 2 Marks)

Solution: Let the point be (x, y, z) on surface $z^2 = 1 + xy$.

Distance from origin $= l$

$$= \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

$$l = \sqrt{x^2 + y^2 + 1 + xy} \quad [\text{since } z^2 = 1 + xy \text{ is given}]$$

This distance is shortest when l is minimum. We need to find minima of $x^2 + y^2 + 1 + xy$.

Let $u = x^2 + y^2 + 1 + xy$

$$\frac{\partial u}{\partial x} = 2x + y$$

$$\frac{\partial u}{\partial y} = 2y + x$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \quad 2x + y = 0 \quad \text{and} \quad 2y + x = 0$$

Solving simultaneously, we get

$$x = 0 \text{ and } y = 0$$

is the only solution and so (0, 0) is the only stationary point.

Now, $$r = \frac{\partial^2 u}{\partial x^2} = 2$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2$$

Since $rt = 2 \times 2 = 4 > s^2 = 1$

We have case 1, i.e. either a maximum or minimum exists at (0, 0).

Now, since $r = 2 > 0$, so it is a minima at (0, 0).

Now at $x = 0$, $y = 0$, $z = \sqrt{1 + xy} = \sqrt{1 + 0} = 1$.

So, the point nearest to the origin on surface $z^2 = 1 + xy$ is (0, 0, 1).

The distance $l = \sqrt{0^2 + 0^2 + 1^2} = 1$.

Ans. (a)

49. The Taylor's series expansion of $\frac{\sin x}{x - \pi}$ at $x = \pi$ is given by

(a) $1 + \frac{(x-\pi)^2}{3!} + \cdots$ (b) $-1 - \frac{(x-\pi)^2}{3!} + \cdots$

(c) $1 - \frac{(x-\pi)^2}{3!} + \cdots$ (d) $-1 + \frac{(x-\pi)^2}{3!} + \cdots$

(GATE 2009, 2 Marks)

Solution: Taylor's series expansion of $f(x)$ around $x = \pi$ is

$$f(x) = f(\pi) + \frac{x-\pi}{1!}f'(\pi) + \frac{(x-\pi)^2}{2!}f''(\pi) + \cdots$$

Now, $f(\pi) = \lim\limits_{x\to\pi} \frac{\sin x}{x - \pi} = \lim\limits_{x\to\infty} \frac{\cos x}{1} = -1$

Similarly, by using L'Hospital's rule, we can show that

$$f'(\pi) = 0$$

and $$f''(\pi) = -1/6$$

So, the expansion is $f(x) = -1 + (-1/6)(x-\pi)^2 + \cdots$

$$\therefore \ f(x) = -1 - \frac{(x'-\pi)^2}{3!} + \cdots$$

Ans. (b)

50. $\int_0^{\pi/4} \frac{(1 - \tan x)}{(1 + \tan x)} dx$ evaluates to

(a) 0 (b) 1

(c) ln 2 (d) 1/2 ln 2

(GATE 2009, 2 Marks)

Solution: We have

$$\int_0^{\pi/4} \frac{(1-\tan x)}{(1+\tan x)}dx \qquad (1)$$

We know that

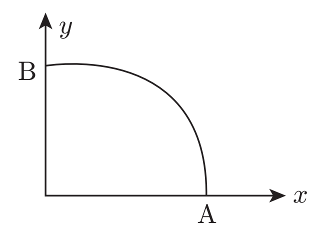
$$\tan\left(\frac{\pi}{4}-x\right) = \frac{1-\tan x}{1+\tan x}$$

Now, substituting the value of $\frac{1-\tan x}{1+\tan x}$ in Eq. (1), we get

$$\int_0^{\pi/4} \tan\left(\frac{\pi}{4}-x\right)dx = \ln\sqrt{2} = \frac{1}{2}\ln 2$$

Ans. (d)

51. A path AB in the form of one quarter of a circle of unit radius is shown in the figure. Integration of $(x+y)^2$ on path AB traversed in a counter-clockwise sense is



(a) $\frac{\pi}{2}-1$ (b) $\frac{\pi}{2}+1$

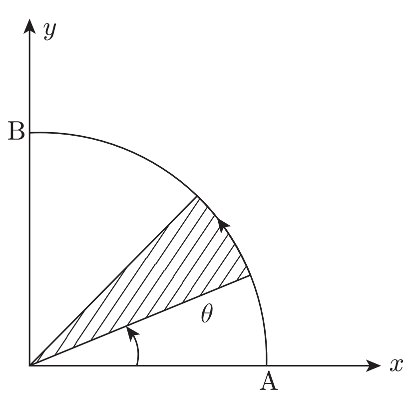
(c) $\frac{\pi}{2}$ (d) 1

(GATE 2009, 2 Marks)

Solution: Path AB: $x^2+y^2=1$

$$x = \cos\theta$$
$$y = \sin\theta$$

Along path AB, θ varies from $0°$ to $90°$ $[0 \text{ to } \pi/2]$



$$\int_{\text{Path AB}} (x+y)^2(rd\theta) = \int_0^{\pi/2} (\cos\theta+\sin\theta)^2 1\cdot d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta)d\theta$$

$$= \int_0^{\pi/2} (1+\sin 2\theta)d\theta$$

$$= \left(\theta + \frac{(-\cos 2\theta)}{2}\right)\Bigg|_0^{\pi/2}$$

$$= \frac{\pi}{2} - \frac{1}{2}\left[\cos 2\frac{\pi}{2} - \cos 0\right]$$

$$= \frac{\pi}{2} - \frac{1}{2}[-1-1] = \frac{\pi}{2}+1$$

Ans. (b)

52. The area enclosed between the curve $y^2 = 4x$ and $x^2 = 4y$ is

(a) $\frac{16}{3}$ (b) 8

(c) $\frac{32}{3}$ (d) 16

(GATE 2009, 2 Marks)

Solution: We are given two curves,

Curve 1: $y^2 = 4x$

Curve 2: $x^2 = 4y$

For intersection points of curves 1 and 2,

$$y^2 = 4x = 4\sqrt{4y} = 8\sqrt{y}$$

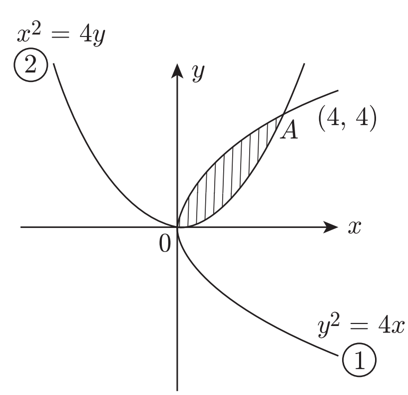
$$y^4 = \left(8\sqrt{y}\right)^2 \Rightarrow y(y^3-64) = 0$$

The solution is $y = 4$ and $y = 0$.

Therefore, $x = 4$, $x = 0$.

Hence, intersection points are $A(4, 4)$ and $O(0, 0)$.

The area enclosed between curves 1 and 2 is given by



$$\text{Area} = \int_{x_1}^{x_2} y_1 dx - \int_{x_1}^{x_2} y_2 dx$$

$$= -\int_0^4 \frac{x^2}{4}\,dx = 2\left.\frac{x^{3/2}}{3/2}\right|_0^4 - \left.\frac{x^3}{3\times 4}\right|_0^4$$

$$= \frac{4}{3}(4)^{3/2} - \frac{(4)^3}{3\times 4} = \frac{16}{3}$$

Alternately, the same answer could have been obtained by taking a double integral as follows:

$$\text{Required area} = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dx\,dy$$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx = \frac{16}{3}$$

Ans. (a)

53. $f(x, y)$ is a continuous function defined over $(x,y) \in [0,1]\times[0,1]$. Given the two constraints, $x > y^2$ and $y > x^2$, the volume under $f(x, y)$ is

(a) $\int_{y=0}^{y=1} \int_{x=y^2}^{x=\sqrt{y}} f(x,y)dx\,dy$

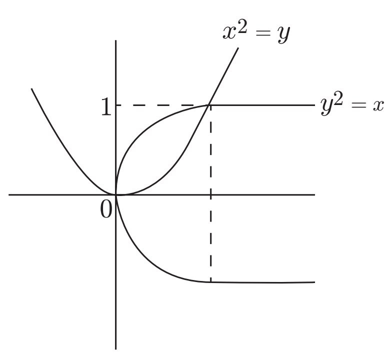
(b) $\int_{y=x^2}^{y=1} \int_{x=y^2}^{x=1} f(x,y)dx\,dy$

(c) $\int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x,y)dx\,dy$

(d) $\int_{y=0}^{y=\sqrt{x}} \int_{x=0}^{x=\sqrt{y}} f(x,y)dx\,dy$

(GATE 2009, 2 Marks)

Solution:



$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=\sqrt{y}} f(x,y)dx\,dy$$

Ans. (a)

54. For a scalar function $f(x,y,z) = x^2 + 3y^2 + 2z^2$, the directional derivative at point $P(1, 2, -1)$ in the direction of a vector $i - j + 2k$ is

(a) -18 (b) $-3\sqrt{6}$

(c) $3\sqrt{6}$ (d) 18

(GATE 2009, 2 Marks)

Solution: We have

$$f(x,y,z) = x^2 + 3y^2 + 2z^2$$

Also,

$$\Delta f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\therefore \quad \Delta f = i(2x) + j(6y) + k(4z)$$

At $P(1, 2, -1)$,

$$\Delta f = i(2\times 1) + j(6\times 2) + k(4\times -1) = 2i + 12j - 4k$$

The directional derivative in the direction of vector $a = i - j + 2k$ is given by

$$\frac{a}{|a|}\,\text{grad}\, f = \frac{i - j + 2k}{\sqrt{1^2 + (-1)^2 + 2^2}}(2i + 12j - 4k)$$

$$= \frac{1}{\sqrt{6}}(1\cdot 2 + (-1)\cdot 12 + 2(-4))$$

$$= -\frac{18}{\sqrt{6}} = -3\sqrt{6}$$

Ans. (b)

55. The $\lim\limits_{x\to 0} \dfrac{\sin\left[\frac{2}{3}x\right]}{x}$ is

(a) 2/3 (b) 1

(c) 3/2 (d) ∞

(GATE 2010, 1 Mark)

Solution: We have

$$\lim_{x\to 0}\frac{\sin\left[\frac{2}{3}x\right]}{x} = \lim_{\frac{2}{3}x\to 0}\frac{\frac{2}{3}\sin\left[\frac{2}{3}x\right]}{\frac{2}{3}x}$$

Let $\theta = \frac{2}{3}x$

$$\text{Required limit} = \frac{2}{3}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

$$= \frac{2}{3}\times 1 = \frac{2}{3}$$

Ans. (a)

56. What is the value of $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$?

(a) 0 (b) e^{-2}
(c) $e^{-1/2}$ (d) 1

(GATE 2010, 1 Mark)

Solution: We have

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n} = \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{n}\right]^2 = \left[\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}\right]^2$$

$$= (e^{-1})^2 = e^{-2}$$

Ans. (b)

57. The function $y = |2 - 3x|$

(a) is continuous $\forall x \in R$ and differentiable $\forall x \in R$
(b) is continuous $\forall x \in R$ and differentiable $\forall x \in R$, expect at $x = 3/2$
(c) is continuous $\forall x \in R$ and differentiable $\forall x \in R$, expect at $x = 2/3$
(d) is continuous $\forall x \in R$ except $x = 3$ and differentiable $\forall x \in R$

(GATE 2010, 1 Mark)

Solution: We have

$$y = |2-3x| = 2-3x, \quad y \ge 0$$
$$= 3x-2, \quad y < 0$$

Therefore, $y = 2-3x,\ x \le \frac{2}{3}$

$$= 3x-2,\ x > \frac{2}{3}$$

Since $(2 - 3x)$ and $(3x - 2)$ are polynomials, these are continuous at all points. We have to check at $x = \frac{2}{3}$.

Left limit at $x = \frac{2}{3}$ is $2-\left(3\times\frac{2}{3}\right) = 0$

Right limit at $x = \frac{2}{3}$ is $\left(3\times\frac{2}{3}\right) - 2 = 0$

$$f\left(\frac{2}{3}\right) = 2-3\times\frac{2}{3} = 0$$

Since, left limit = right limit = $f\left(\frac{2}{3}\right)$.

Therefore, function is continuous at $\frac{2}{3}$.

Hence, y is continuous $\forall x \in R$.

Now, since $(2 - 3x)$ and $(3x - 2)$ are polynomials, they are differentiable.

Now, at $x = \frac{2}{3}$ Left derivative $= -3$

Right derivative $= +3$

Left derivative $\neq$ Right derivative

$\therefore$ The function y is not differentiable at $x = \frac{2}{3}$.

So, we can say that y is differentiable $\forall x \in R$, except at $x = \frac{2}{3}$.

Ans. (c)

58. The value of the integral $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$ is

(a) $-\pi$ (b) $-\pi/2$
(c) $\pi/2$ (d) π

(GATE 2010, 1 Mark)

Solution: We have

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_{-\infty}^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(-\infty) = \frac{\pi}{2} - \left[\frac{-\pi}{2}\right]$$

$$= \pi$$

Ans. (d)

59. The value of the quantity P, where $P = \int_0^1 xe^x dx$, is equal to

(a) 0 (b) 1 (c) e (d) $1/e$

(GATE 2010, 1 Mark)

Solution: We have

$$P = \int_0^1 xe^x dx$$

Integrating by parts, we get

Let $u = x$, $v = e^x$

$$dv = e^x dx$$
$$du = dx$$
$$v = \int e^x dx = e^x$$

Now, $\int u dv = uv - \int v du$

$\therefore$ $\int xe^x dx = xe^x - \int e^x dx$

$$= xe^x - e^x + c$$

$$\int_0^1 xe^x dx = [xe^x - e^x]_0^1$$

$$= (1\cdot e^1 - e^1) - (0\cdot e^0 - e^0) = 0 - (-1)$$
$$= 1$$

Ans. (b)

60. The parabola arc $y=\sqrt{x}, 1 \le x \le 2$ is revolved around the x-axis. The volume of the solid of revolution is

(a) $\pi/4$ (b) $\pi/2$ (c) $3\pi/4$ (d) $3\pi/2$

(GATE 2010, 1 Mark)

Solution: The volume of a solid generated by revolution about the x-axis of the area bounded by curve $y=f(x)$, the x-axis and the ordinates $x=a$, $y=b$ is given by

$$\text{Volume } = \int_a^b \pi y^2 dx$$

Here, $a=1$, $b=2$ and $y=\sqrt{x} \Rightarrow y^2 = x$

$$\therefore \quad \text{Volume} = \int_1^2 \pi \cdot x \cdot dx$$

$$= \pi \cdot \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{\pi}{2}[x^2]_1^2 = \frac{\pi}{2}[2^2 - 1^2]$$

$$= \frac{3}{2}\pi$$

Ans. (d)

61. Divergence of the three-dimensional radial vector field $\vec{r}$ is

(a) 3 (b) $1/r$

(c) $\hat{i}+\hat{j}+\hat{k}$ (d) $3\left(\hat{i}+\hat{j}+\hat{k}\right)$

(GATE 2010, 1 Mark)

Solution: We have

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Now, divergence of vector is given by

$$\operatorname{div}\vec{r} = \nabla \cdot \vec{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1+1+1 = 3$$

Ans. (a)

62. Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$.

The optimal value of $f(x, y)$

(a) is a minimum equal to 10/3
(b) is a maximum equal to 10/3
(c) is a minimum equal to 8/3
(d) is a maximum equal to 8/3

(GATE 2010, 2 Marks)

Solution: We have

$$f(x,y) = 4x^2 + 6y^2 - 8x - 4y + 8$$

On differentiating, we get

$$\frac{\partial t}{\partial x} = 8x - 8$$

$$\frac{\partial f}{\partial y} = 12y - 4y$$

Putting, $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$, we get

$$8x - 8 = 0 \text{ and } 12y - 4y = 0$$

Given that $x = 1$ and $y = \dfrac{1}{3}$, $\left(1, \dfrac{1}{3}\right)$ is the only stationary point.

$$r = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1,\frac{1}{3}\right)} = 8$$

$$s = \left[\frac{\partial^2 f}{\partial x \partial y}\right]_{\left(1,\frac{1}{3}\right)} = 0$$

$$t = \left[\frac{\partial^2 f}{\partial y^2}\right]_{\left(1,\frac{1}{3}\right)} = 12$$

Since, $rt = 8 \times 12 = 48$

$$s^2 = 0$$

Since, $rt > s^2$, we have either a maxima or minima at $\left(1, \dfrac{1}{3}\right)$.

Also, since $r = \left[\dfrac{\partial f}{\partial x^2}\right]_{\left(1,\frac{1}{3}\right)} = 8 > 0$, the point $\left(1, \dfrac{1}{3}\right)$ is a point of minima.

The minimum value is

$$t\left(1\frac{1}{3}\right) = 4 \times 1^2 + 6 \times \frac{1}{3^2} - 8 \times 1 - 4 \times \frac{1}{3} + 8 = \frac{10}{3}$$

So, the optimal value of $f(x, y)$ is a minimum equal to $\dfrac{10}{3}$.

Ans. (a)

63. A parabolic cable is held between two supports at the same level. The horizontal span between the supports is L. The sag at the mid-span is h. The equation of the parabola is $y = 4h(x^2/L^2)$, where x is the horizontal coordinate and y is the vertical coordinate with the origin at the center of the cable. The expression for the total length for the cable is

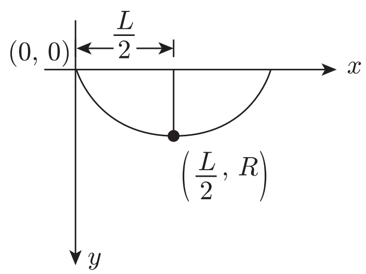
(a) $\int_0^L \sqrt{1+64\frac{h^2x^2}{L^4}}dx$ (b) $2\int_0^{L/2} \sqrt{1+64\frac{h^3x^2}{L^4}}dx$

(c) $\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}dx$ (d) $2\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}dx$

(GATE 2010, 2 Marks)

Solution: The length of curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$



Here, $$y = 4h\frac{x^2}{L^2} \quad (1)$$

$$\frac{dy}{dx} = 8h\frac{x}{L^2}$$

Since, $y = 0$ at $x = 0$

and $y = h$ at $x = \frac{L}{2}$

(As can be seen from Eq. (1), by substituting $x = 0$ and $x = L/2$)

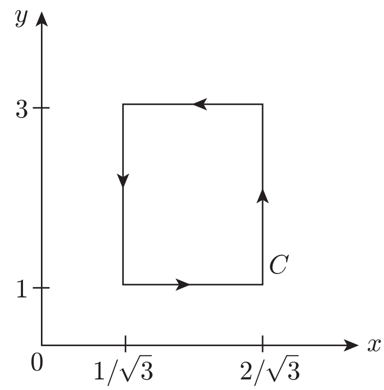
$$\therefore \frac{1}{2}(\text{Length of cable}) = \int_0^{L/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

$$= \int_0^{L/2} \sqrt{1+\left(\frac{8hx}{L^2}\right)^2}dx$$

$$\text{Lenght of cable} = 2\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^2}}dx$$

Ans. (d)

64. If $\vec{A} = xy\hat{a}_x + x^2\hat{a}_y$, $\oint_C \vec{A}\cdot d\vec{l}$ over the path shown in the figure is

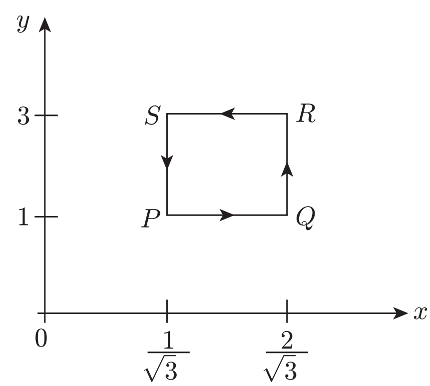


(a) 0 (b) $\frac{2}{\sqrt{3}}$ (c) 1 (d) $2\sqrt{3}$

(GATE 2010, 2 Marks)

Solution: We have

$$\vec{A} = xy\hat{a}_x + x^2\hat{a}_y$$
$$\vec{l} = x\hat{a}_x + y\hat{a}_y$$
$$d\vec{l} = dx\hat{a}_x + dy\hat{a}_y$$
$$\vec{A}\cdot d\vec{l} = xy\,dx + x^2dy$$



$P - Q$: $y = 1,\ dy = 0$

$$\int_P^Q \vec{A}\cdot d\vec{l} = \int_{1/\sqrt{3}}^{2/\sqrt{3}} x\,dx = \left.\frac{x^2}{2}\right|_{1/\sqrt{3}}^{2/\sqrt{3}} = \frac{1}{2}$$

$Q - R$: $x = \frac{2}{\sqrt{3}}, dx = 0$

$$\int_Q^R \vec{A}\cdot d\vec{l} = \int_1^3 \left(\frac{2}{\sqrt{3}}\right)^2 dy = \frac{4}{3}(3-1) = \frac{8}{3}$$

$R - S$: $y = 3,\ dy = 0$

$$\int_R^S \vec{A}\cdot d\vec{l} = \int_{2/\sqrt{3}}^{1/\sqrt{3}} 3x\,dx = \frac{3}{2}x^2\Big|_{2/\sqrt{3}}^{1/\sqrt{3}} = \frac{3}{2}\left(\frac{1}{3}-\frac{4}{3}\right) = \frac{-3}{2}$$

$S - P$: $x = \frac{1}{\sqrt{3}}, dx = 0$

$$\int_S^P \vec{A}\cdot d\vec{l} = \int_3^1 \left(\frac{1}{\sqrt{3}}\right)^2 dy = \frac{1}{3}(1-3) = \frac{-2}{3}$$

$$\text{So,} \oint_C \vec{A}\cdot d\vec{l} = \int_P^Q \vec{A}\cdot d\vec{l} + \int_Q^R \vec{A}\cdot d\vec{l} + \int_R^S \vec{A}\cdot d\vec{l} + \int_S^P \vec{A}\cdot d\vec{l}$$

$$= \frac{1}{2} + \frac{8}{3} - \frac{3}{2} - \frac{2}{3} = 1$$

Ans. (c)

65. Velocity vector of a flow field is given as $\vec{v} = 2xy\hat{i} - x^2 z\hat{j}$. The vortices vector at (1, 1, 1) is

(a) $4\hat{i} - \hat{j}$ (b) $4\hat{i} - \hat{k}$
(c) $\hat{i} - 4\hat{j}$ (d) $\hat{i} - 4\hat{k}$

(GATE 2010, 2 Marks)

Solution: Velocity vector is given as

$$\vec{v} = 2xy\hat{i} - x^2 z\hat{j}$$

Velocity vector = curl (vortices vector)

$$= \text{curl}(\vec{v})$$
$$= \nabla \times \vec{v}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy & -x^2 z & 0 \end{vmatrix}$$
$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-x^2 z)\right]\hat{i} - \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2xy)\right]\hat{j}$$
$$+ \left[\frac{\partial}{\partial x}(-x^2 z) - \frac{\partial}{\partial y}(2xy)\right]\hat{k}$$
$$= x^2\hat{i} + [-2xz - 2x]\hat{k}$$

At (1, 1, 1), we get vortices vector = $\hat{i} - 4\hat{k}$.

Ans. (d)

66. What is $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta}$ equal to?

(a) θ (b) $\sin\theta$
(c) 0 (d) 1

(GATE 2011, 1 Mark)

Solution: We have

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Ans. (d)

67. A series expansion for the function $\sin\theta$ is

(a) $1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \cdots$

(b) $\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \cdots$

(c) $1 + \theta + \dfrac{\theta^2}{2!} + \dfrac{\theta^3}{3!} + \cdots$

(d) $\theta + \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \cdots$

(GATE 2011, 1 Mark)

Solution: Series expansion for the function $\sin\theta$ is given by

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

Ans. (b)

68. If $f(x)$ is an even function and a is a positive real number, then $\int_{-a}^{a} f(x)dx$ equals

(a) 0 (b) a
(c) $2a$ (d) $2\int_0^a f(x)dx$

(GATE 2011, 1 Mark)

Solution: If $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

Ans. (d)

69. What should be the value of λ such that the function defined below is continuous at $x = \pi/2$?

$$f(x) = \begin{cases} \dfrac{\lambda\cos x}{\dfrac{\pi}{2} - x}, & \text{if } x \neq \pi/2 \\ 1, & \text{if } x = \pi/2 \end{cases}$$

(a) 0 (b) $2/\pi$
(c) 1 (d) $\pi/2$

(GATE 2011, 2 Marks)

Solution: If $f(x)$ is continuous at $x = \dfrac{\pi}{2}$,

$$\lim_{x \to \frac{\pi}{2}} \frac{\lambda\cos x}{\dfrac{\pi}{2} - x} = f\left(\frac{\pi}{2}\right) = 1 \qquad (1)$$

Since the limit is in the form $\frac{0}{0}$, we can use L'Hospital's rule on L.H.S. of Eq. (1) and get

$$\lim_{x\to\frac{\pi}{2}} \frac{-\lambda \sin x}{-1} = 1$$

$$\Rightarrow \quad \lambda \sin\frac{\pi}{2} = 1$$

$$\Rightarrow \quad \lambda = 1$$

Ans. (c)

70. The function $f(x) = 2x - x^2 + 3$ has

(a) a maxima at $x = 1$ and a minima at $x = 5$
(b) a maxima at $x = 1$ and a minima at $x = -5$
(c) only a maxima at $x = 1$
(d) only a minima at $x = 1$

(GATE 2011, 2 Marks)

Solution: We have

$$f(x) = 2x - x^2 + 3$$
$$f'(x) = 2 - 2x = 0$$

Now, $x = 1$ is the stationary point.

$$f''(x) = -2$$
$$f''(1) = -2 < 0$$

So, at $x = 1$ we have a relative maxima.

Ans. (c)

71. Given $i = \sqrt{-1}$, what will be the evaluation of the definite integral $\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}dx$?

(a) 0 (b) 2
(c) $-i$ (d) i

(GATE 2011, 2 Marks)

Solution: We have

$$\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos - i\sin x}dx$$

$$= \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}}dx = \int_0^{\pi/2} e^{2ix}dx$$

$$= \left[\frac{e^{2ix}}{2i}\right]_0^{\pi/2} = \frac{1}{2i}[e^{i\pi} - e^0]$$

$$= \frac{1}{2i}[-1-1] \quad (\because\ e^{i\pi} = -1)$$

$$= \frac{-2}{2i} = \frac{-1}{i} = i$$

Ans. (d)

72. What is the value of the definite integral $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}dx$?

(a) 0 (b) $a/2$
(c) a (d) $2a$

(GATE 2011, 2 Marks)

Solution: Let

$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}dx \qquad (1)$$

Since $\int_0^a f(x)dx = \int_0^q f(a-x)dx$, we have

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} \qquad (2)$$

Adding Eqs. (1) and (2), we get

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}}dx$$

$$\Rightarrow \quad 2I = \int_0^a dx$$

$$\Rightarrow \quad 2I = a$$

$$\Rightarrow \quad I = a/2$$

Ans. (b)

73. The two vectors [1, 1, 1] and [1, a, a^2], where $a = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$ are

(a) orthonormal (b) orthogonal
(c) parallel (d) collinear

(GATE 2011, 2 Marks)

Solution: The given vectors are [1, 1, 1] and [1, a, a^2].

Now, $a = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = \omega$

or $\quad a^2 = \omega^2$

Hence, the vectors are

$$x = [1, 1, 1] \text{ and } y = [1, \omega, \omega^2]$$

Now $x \cdot y = 1 \cdot 1 + 1 \cdot \omega + 1 \cdot \omega^2$

$$= 1 + \omega + \omega^2 = 0$$

So, x and y are orthogonal.

Ans. (b)

74. $\lim_{x\to 0}\left(\dfrac{1-\cos x}{x^2}\right)$ is

(a) 1/4 (b) 1/2
(c) 1 (d) 2

(GATE 2012, 1 Mark)

Solution: We have

$$\lim_{x\to 0}\frac{(1-\cos x)}{x^2}$$

Applying $x = 0$, we get

$$\frac{1-\cos 0}{0^2} = \frac{0}{0}$$

Hence, we use L'Hospital's rule

$$\lim_{x\to 0}\frac{(1-\cos x)}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \frac{0}{0}$$

Applying L'Hospital's rule again, we get

$$\lim_{x\to 0}\left[\frac{\cos x}{2}\right] = \frac{1}{2}$$

Ans. (b)

75. Consider the function $f(x) = |x|$ in the interval $-1 < x \le 1$. At the point $x = 0$, $f(x)$ is

(a) continuous and differentiable
(b) non-continuous and differentiable
(c) continuous and non-differentiable
(d) neither continuous nor differentiable

(GATE 2012, 1 Mark)

Solution: We have

$$f(x) = |x| = x \qquad x \ge 0$$
$$= -x \qquad x < 0$$

At $x = 0$,

L.H.L. $= 0$
R.H.L. $= -0 = 0$
and $f(0) = 0$

Therefore, L.H.L. = R.H.L. = $f(0)$.

Hence, $f(x)$ is continuous at $x = 0$.

Now, Left derivative (at $x = 0$) $= -1$
Right derivative (at $x = 0$) $= +1$
Left derivative $\neq$ Right derivative

Hence, $f(x)$ is not differentiable at $x = 0$.

So, $|x|$ is continuous and non-differentiable at $x = 0$.

Ans. (c)

76. The infinite series $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$ corresponds to

(a) $\sec x$ (b) e^x
(c) $\cos x$ (d) $1 + \sin^2 x$

(GATE 2012, 1 Mark)

Solution: By MacLaurin's series expansion, we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

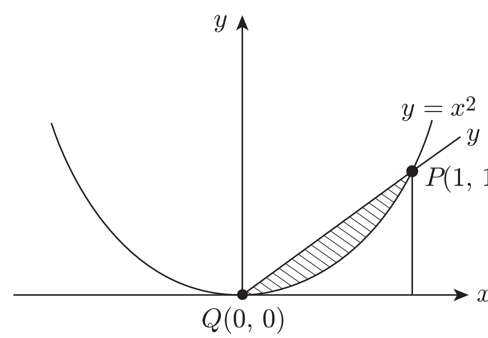
Ans. (b)

77. The area enclosed between the straight line $y = x$ and the parabola $y = x^2$ in the xy plane is

(a) 1/6 (b) 1/4
(c) 1/3 (d) 1/2

(GATE 2012, 1 Mark)

Solution: The area enclosed is shown in the shaded region:



The coordinates of point P and Q are obtained by solving

$$y = x$$

and $y = x^2$ simultaneously

i.e. $x = x^2$

$\Rightarrow$ $x(x-1) = 0$

$\Rightarrow$ $x = 0, x = 1$

Now, $x = 0 \Rightarrow y = 0$, which is $Q(0, 0)$.

And $x = 1 \Rightarrow y = 1^2 = 1$, which is $P(1, 1)$.

Hence, required area $= \displaystyle\int_0^1 x\,dx - \int_0^1 x^2\,dx$

$$\left[\frac{x^2}{x}\right]_2^1 - \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Ans. (a)

78. For the spherical surface $x^2 + y^2 + z^2 = 1$, the unit outward normal vector at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is given by

(a) $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$ (b) $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

(c) $\hat{k}$ (d) $\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

(GATE 2012, 1 Mark)

Solution: The spherical surface is given by

$$x^2 + y^2 + z^2 = 1$$

$$f(x, y, z) = x^2 + y^2 + z^2 = 0$$

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \qquad \text{at } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\text{grad } f = \frac{2}{\sqrt{2}}\hat{i} + \frac{2}{\sqrt{2}}\hat{j} + 2 \times 0 \times \hat{k}$$

$$= \sqrt{2}i + \sqrt{2}j + 0\hat{k}$$

$$|\text{grad } f| = \sqrt{2+2} = \sqrt{4} = 2$$

The unit outward normal vector at point P is

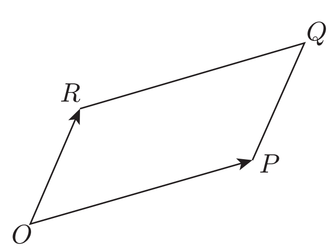
$$n = \frac{1}{|\text{grad } f|}(\text{grad } f) \text{ at } P$$

$$= \frac{1}{2}\left(\sqrt{2}i + \sqrt{2}j\right)$$

$$= \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$$

Ans. (a)

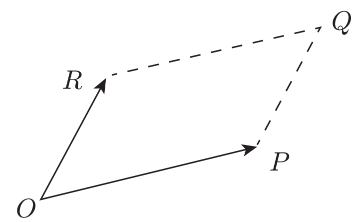
79. For the parallelogram $OPQR$ shown in the sketch, $\overrightarrow{OP} = a\hat{i} + b\hat{j}$ and $\overrightarrow{OR} = c\hat{i} + d\hat{j}$. The area of the parallelogram is



(a) $ad - bc$ (b) $ac + bd$
(c) $ad + bc$ (d) $ab - cd$

(GATE 2012, 2 Marks)

Solution:



The area of parallelogram $OPQR$, in the figure shown above, is the magnitude of the vector product

$$= |\overrightarrow{OP} \times \overrightarrow{OR}|$$

$$\overrightarrow{OP} = a\hat{i} + b\hat{j}$$

$$\overrightarrow{OR} = c\hat{i} + d\hat{j}$$

$$\overrightarrow{OP} \times \overrightarrow{OR} = \begin{vmatrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + (ad - bc)\hat{k}$$

$$\left|\overrightarrow{OP} \times \overrightarrow{OR}\right| = \sqrt{0^2 + 0^2 + (ad - bc)^2}$$

$$= ad - bc$$

Ans. (a)

80. The direction of vector A is radically outward from the origin, with $|A| = kr^n$ where $r^2 = x^2 + y^2 + z^2$ and k is a constant. The value of n for which $\nabla \cdot A = 0$ is

(a) -2 (b) 2
(c) 1 (d) 0

(GATE 2012, 2 Marks)

Solution: We have

$$\vec{A} = kr^n \hat{i}_r$$

Divergence of the vector is given by

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot kr^n)$$

$$= \frac{1}{r^2} \cdot \frac{\partial}{\partial r}(kr^{n+2})$$

$$= (n+2)k \cdot \frac{r^{n+1}}{r^2} = (n+2)kr^{n-1}$$

Now, since $\nabla \cdot \vec{A} = 0$,

$$n + 2 = 0$$

$$\Rightarrow \qquad n = -2$$

Ans. (a)

81. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$ is equal to

(a) 0 (b) ∞
(c) 1 (d) -1

(GATE 2013, 1 Mark)

Solution: We have

$$\lim_{x\to 0} \frac{\sin^2 x}{x}$$

Now,

$$\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 x$$
$$= 1\times 0 = 0$$

Ans. (a)

82. The maximum value of θ until which the approximation $\sin\theta = \theta$ holds to within 10% error is

(a) 10° (b) 18°
(c) 50° (d) 90°

(GATE 2013, 1 Mark)

Solution: We know that

$$10^\circ = \frac{10\pi}{180} = 0.1745 \text{ rad}$$
$$\sin 10^\circ = 0.1736$$

So, for $10^\circ \to \sin\theta \cong \theta$ holds within 10% error

$$18^\circ = \frac{18\times\pi}{180} = 0.3142$$
$$\sin 18^\circ = 0.3090$$

So, for $18^\circ \to \sin\theta \cong \theta$ holds within 10% error

$$50^\circ = \frac{50\times\pi}{180} = 0.8727$$
$$\sin 50^\circ = 0.766$$

So, for $50^\circ \to \sin\theta \cong \theta$ does not hold within 10% error

$$90^\circ = \frac{90\times\pi}{180} = 1.571$$
$$\sin 90^\circ = 1$$

So, for $90^\circ \to \sin\theta \cong \theta$ does not hold within 10% error.

So, the maximum value of θ for the approximation $\sin\theta \cong \theta$ holds to within 10% error is 18°.

Ans. (b)

83. Which one of the following functions is continuous at $x = 3$?

(a) $f(x) = \begin{cases} 2, & \text{if } x = 3 \\ x-1, & \text{if } x > 3 \\ \dfrac{x+3}{3}, & \text{if } x < 3 \end{cases}$

(b) $f(x) = \begin{cases} 4, & \text{if } x = 3 \\ 8-x, & \text{if } x \neq 3 \end{cases}$

(c) $f(x) = \begin{cases} x+3, & \text{if } x \leq 3 \\ x-4, & \text{if } x > 3 \end{cases}$

(d) $f(x) = \dfrac{1}{x^3-27}, \quad \text{if } x \neq 3$

(GATE 2013, 1 Mark)

Solution: Considering the first option,

$$\begin{cases} 2, & \text{if } x = 3 \\ x-1, & \text{if } x > 3 \\ \dfrac{x+3}{3}, & \text{if } x < 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x+3}{3} = 2$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x-1 = 2$$

So it is continuous at $x = 3$.

Hence, option (a) is correct.

Ans. (a)

84. Function f is known at the following points:

x	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0
$f(x)$	0	0.09	0.36	0.81	1.44	2.25	3.24	4.41	5.76	7.29	9.00

The value of $\int_0^3 f(x)dx$ computed using the trapezoidal rule is

(a) 8.983 (b) 9.003
(c) 9.017 (d) 9.045

(GATE 2013, 1 Mark)

Solution: According to Trapezoidal rule, we get

$$\int_0^3 f(x)dx = \frac{h}{2}\big[f(x_0) + f(x_{10}) + 2\big(f(x_1) + f(x_2) + \cdots + f(x_9)\big)\big]$$
$$= \frac{0.3}{2}\big[9.00 + 2(25.65)\big] = 9.045$$

Ans. (d)

85. The curl of the gradient of the scalar field defined by $V = 2x^2y + 3y^2z + 4z^2x$ is

(a) $4xy\,a_x + 6yz\,a_y + 8zx\,a_z$

(b) $4a_x + 6a_y + 8a_z$

(c) $(4xy + 4z^2)a_x + (2x^2 + 6yz)a_y + (3y^2 + 8zx)a_z$

(d) 0

(GATE 2013, 1 Mark)

Solution: The curl of the gradient of a scalar field is always zero.

$$\nabla \times \nabla V = 0$$

Ans. (d)

86. The divergence of the vector field $A = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ is

(a) 0 (b) 1/3
(c) 1 (d) 3

(GATE 2013, 1 Mark)

Solution: We know that

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1$$

$\therefore\ \nabla \cdot \vec{A} = 3$

Ans. (d)

87. For a vector E, which one of the following statements is not true?

(a) If $\nabla \cdot E = 0$, E is called solenoidal
(b) If $\nabla \times E = 0$, E is called conservative
(c) If $\nabla \times E = 0$, E is called irrotational
(d) If $\nabla \cdot E = 0$, E is called irrotational

(GATE 2013, 1 Mark)

Solution: We know that the cross product of irrotational vector is zero. Thus, for a vector to be irrotational, $\nabla \times E = 0$. Hence, option (d) is not true.

Ans. (d)

88. A vector field is given, $\vec{F} = y^2x\hat{a}_x - yz\hat{a}_y - x^2\hat{a}_z$, the line integral $\int \vec{F} \cdot \vec{dl}$ evaluated along a segment on the x-axis from $x = 1$ to $x = 2$ is

(a) −2.33 (b) 0
(c) 2.33 (d) 7

(GATE 2013, 1 Mark)

Solution: From $x = 1$ to $x = 2$ means $y = 0,\ z = 0$, $dy = 0$ and $dz = 0$.

Now,

$$\int \vec{F} \cdot \vec{dl} = \int (y^2x\hat{a}_x - x^2\hat{a}_z)(\hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz)$$
$$= \int y^2x dx - yz dy - x^2 dz$$

Putting $y = 0$, $z = 0$, $dy = 0$ and $dz = 0$, we get

$$\int \vec{F} \cdot \vec{dl} = 0$$

Ans. (b)

89. Consider a vector filed $\vec{A}(\vec{r})$. The closed loop line integral $\oint \vec{A} \cdot \vec{dl}$ can be expressed as

(a) $\oiint (\nabla \times \vec{A}) \cdot \vec{ds}$ over the closed surface bounded by the loop
(b) $\oiiint (\nabla \cdot \vec{A})\,dv$ over the closed volume bounded by the top
(c) $\iiint (\nabla \cdot \vec{A})\,dv$ over the open volume bounded by the loop
(d) $\iint (\nabla \times \vec{A}) \cdot \vec{ds}$ over the open surface bounded by the loop

(GATE 2013, 1 Mark)

Solution: We know that according to Stokes' theorem,

$$\oint_c \vec{A} \cdot \vec{dl} = \iint_s (\nabla \times \vec{A}) \cdot \vec{ds}$$

Ans. (d)

90. The value of $\int_0^{\pi/6} \cos^4 3a\theta\ \sin^3 6\theta\ d\theta$ is

(a) 0 (b) $\frac{1}{15}$

(c) 1 (d) $\frac{8}{3}$

(GATE 2013, 2 Marks)

Solution: We have

$$\int_0^{\frac{\pi}{6}} \cos^4 3\theta \sin^3 6\theta\ d\theta = 8\int_0^{\frac{\pi}{6}} \cos^6 3\theta \sin^3 3\theta \cos 3\theta\ d\theta$$

Let $\sin 3\theta = t$

$$\Rightarrow \quad 3\cos 3\theta \, d\theta = dt$$

$$\Rightarrow \quad I = \frac{8}{3}\int_0^1 (1-t^2)^3 t^3 \cdot dt$$

$$= \frac{8}{3}\int_0^1 (t^6 - 2t^2 + 3t^4)t^3 dt$$

$$= \frac{8}{3}\int_0^1 (t^3 - t^9 - 3t^5 + 3t^7)dt$$

$$= \frac{8}{3}\left[\frac{t^4}{4} - \frac{t^{10}}{10} - \frac{3t^6}{6} + \frac{3t^6}{8}\right]_0^1$$

$$= \frac{8}{3}\left[\frac{1}{4} - \frac{1}{10} - \frac{3}{6} + \frac{3}{8}\right]$$

$$= \frac{8}{3}\left[\frac{30-12-60+45}{120}\right]$$

$$= \frac{1}{15}$$

Ans. (b)

91. Suppose you break a stick of unit length at a point chosen uniformly at random. Then the expected length of the shorter stick is ________.

(GATE 2014, 1 Mark)

Solution: The length of shorter stick ranges between 0 to 0.5 meters, with each length equally possible. So, the average expected length is 0.25 m.

Ans. 0.25 m

92. Let the function

$$f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \tan\theta \\ \sin(\pi/6) & \cos(\pi/6) & \tan(\pi/6) \\ \sin(\pi/3) & \cos(\pi/3) & \tan(\pi/3) \end{vmatrix}$$

where $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $f'(\theta)$ denote the derivative of f with respect to θ. Which of the following statements is/are **TRUE**?

(I) There exists $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ such that $f'(\theta) = 0$

(II) There exists $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ such that $f'(\theta) \neq 0$

(a) I only
(b) II only
(c) Both I and II
(d) Neither I nor II

(GATE 2014, 1 Mark)

Solution: By applying mean value theorem, both I and II found to be true.

Ans. (c)

93. A non-zero polynomial $f(x)$ of degree 3 has roots at $x = 1$, $x = 2$ and $x = 3$. Which one of the following must be TRUE?

(a) $f(0)\cdot f(4) < 0$
(b) $f(0)\cdot f(4) > 0$
(c) $f(0) + f(4) > 0$
(d) $f(0) + f(4) < 0$

(GATE 2014, 1 Mark)

Solution: Given that $f(x)$ has degree = 3. For $f(x) = 0$, roots are $x = 1, 2, 3$ which lie between 0 and 4. So, $f(0)$ and $f(4)$ have opposite signs. Therefore, $f(0)\cdot f(4)$ should be less than zero.

Ans. (a)

94. Consider the equation $(123)_5 = (x8)_y$ with x and y as unknown. The number of possible solutions is ________.

(GATE 2014, 1 Mark)

Solution: Converting both sides of $(123)_5 = (x8)_y$ into decimal number system, we get

$$1\times 5^2 + 2\times 5^1 + 3\times 5^0 = x\times y^1 + 8\times y^0$$

$$25 + 10 + 3 = xy + 8$$

$$38 = xy + 8 \Rightarrow xy = 30$$

Factors of 30 are -1, 2, 3, 15, 30

If $x = 1$, then $y = 30$

If $x = 2$, then $y = 15$

If $x = 3$, then $y = 10$

So, we have three solutions to the given problem.

Ans. 3

95. If $\int_0^{2\pi} |\, x\sin x \,|\, dx = k\pi$, then the value of k is equal to ________.

(GATE 2014, 1 Mark)

Solution: We are given

$$\int_0^{2\pi} |\, x\sin x\,|\, dx = k\pi$$

Now

$$|\sin x| = -\sin x \text{ [from } \pi \text{ to } 2\pi]$$

So

$$\int_0^{2\pi} |x\sin x \; dx| = \int_0^{\pi} x\sin x \; dx + \int_{\pi}^{2\pi} -x\sin x \; dx$$

$$= x(-\cos x) - 1(-\sin x)| - (-x\cos x + \sin x)|$$

$$= 4\pi$$

Ans. 4

96. For $0 \le t < \infty$, the maximum value of the function $f(t) = e^{-t} - 2e^{-2t}$ occurs at

(a) $t = \log_e 4$ (b) $t = \log_e 2$
(c) $t = 0$ (d) $t = \log_e 8$

(GATE 2014, 1 Mark)

Solution: $f'(t) = -e^{-t} + 4e^{-2t} = 0$

$$\Rightarrow e^{-t}\left[4e^{-t} - 1\right] \Rightarrow e^{-t} = \frac{1}{4} \Rightarrow t = \log_e 4$$

and $f''(t) < 0$ at $t = \log_e 4$.

Ans. (a)

97. The value of

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

is

(a) ln 2 (b) 1.0 (c) e (d) ∞

(GATE 2014, 1 Mark)

Solution: Using the standard limit, we have

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

Ans. (c)

98. The maximum value of the function $f(x) = \ln(1 + x) - x$ (where $x > -1$) occurs at $x =$ ________.

(GATE 2014, 1 Mark)

Solution: $f'(x) = 0 \Rightarrow \dfrac{1}{1+x} - 1 = 0$

$$\Rightarrow \frac{-x}{1+x} = 0 \Rightarrow x = 0$$

and $f''(x) = \dfrac{-1}{(1+x)^2} < 0$ at $x = 0$.

Ans. 0

99. The series $\sum_{n=0}^{\infty}\dfrac{1}{n!}$ converges to

(a) 2 ln 2 (b) $\sqrt{2}$ (c) 2 (d) e

(GATE 2014, 1 Mark)

Solution: $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$

$$= e \text{ as } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots, \forall x \text{ in } R$$

Ans. (d)

100. The magnitude of the gradient for the function $f(x,y,z) = x^2 + 3y^2 + z^3$ at the point (1,1,1) is ________.

(GATE 2014, 1 Mark)

Solution:

$$(\nabla f)_{P(1,1,1)} = [\vec{i}(2x) + \vec{j}(6y) + \vec{k}(3z^2)]_{P(1,1,1)}$$
$$= (2\vec{i} + 6\vec{j} + 3\vec{k})$$
$$\left|(\nabla f)_P\right| = \sqrt{4+36+9} = 7$$

Ans. 7

101. The directional derivative of $f(x,y) = \dfrac{xy}{\sqrt{2}}(x+y)$ at (1, 1) in the direction of the unit vector at an angle of $\dfrac{\pi}{4}$ with y-axis is given by ________.

(GATE 2014, 1 Mark)

Solution:

$$f = \frac{1}{\sqrt{2}}\left(x^2y + xy^2\right)$$

$$\Rightarrow \nabla f = i\left[\frac{2xy + y^2}{\sqrt{2}}\right] + j\left[\frac{x^2 + 2xy}{\sqrt{2}}\right]$$

At (1, 1),

$$\nabla f = \frac{3}{\sqrt{2}}i + \frac{3}{\sqrt{2}}j$$

$e =$ unit vector in the direction making an angle of $\pi/4$ with y-axis. So

$$e = \left(\sin\frac{\pi}{4}\right)\vec{i} + \left(\cos\frac{\pi}{4}\right)\vec{j}$$

Thus, directional derivative is

$$e \cdot \nabla f = 2\left(\frac{3}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 3$$

Ans. 3

102. Let $f(x) = xe^{-x}$. The maximum value of the function in the interval $(0, \infty)$ is

(a) e^{-1} (b) e (c) $1 - e^{-1}$ (d) $1 + e^{-1}$

(GATE 2014, 1 Mark)

Solution: We are given,

$$f(x) = xe^{-x}$$

For maximum value of the function, we calculate

$$f'(x) = 0 \Rightarrow e^{-x}(1-x) = 0 \Rightarrow x = 1$$

Differentiating again, we get

$$f'(x) = 0 \Rightarrow e^{-x}(1-x) = 0 \Rightarrow x = 1$$

$$f''(x) = -e^{-x} - \left[e^{-x} - xe^{-x}\right] < 0 \text{ at } x = 1$$

Thus, maximum value is $f(1) = e^{-1}$.

Ans. (a)

103. Minimum of the real-valued function $f(x) = (x-1)^{2/3}$ occurs at x equal to

(a) $-\infty$ (b) 0 (c) 1 (d) ∞

(GATE 2014, 1 Mark)

Solution: We are given a real-valued function,

$$f(x) = (x-1)^{2/3}$$

Differentiating, we get

$$f'(x) = \frac{2}{3\cdot(x-1)^{1/3}} \left\} \begin{array}{l} \text{is negative, } \forall x < 1 \text{ or} \\ \qquad \forall x \text{ in } (1-h, 1) \\ \text{is positive, } \forall x > 1 \text{ or} \\ \qquad \forall x \text{ in } (1, 1+h) \end{array} \right.$$

Thus, h is positive and small.

Therefore, f has local minima at $x = 1$ and the minimum value is '0'.

Ans. (c)

104. A particle, starting from origin at $t = 0$ s, is traveling along x-axis with velocity

$$v = \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\text{m/s}$$

At $t = 3$ s, the difference between the distance covered by the particle and the magnitude of displacement from the origin is _________.

(GATE 2014, 1 Mark)

Solution: At $t = 3$ s, the distance covered by the particle is $1 + 1 + 1 = 3$ m and displacement from the origin is -1.

Therefore, difference between the distance covered by the particle and the magnitude of displacement from the origin is $3 - |-1| = 2$.

Ans. 2

105. Let $\nabla\cdot(fv) = x^2y + y^2z + z^2x$, where f and v are scalar and vector fields, respectively. If $v = yi + zj + xk$, then $v\cdot\nabla f$ is

(a) $x^2y + y^2z + z^2x$ (b) $2xy + 2yz + 2zx$
(c) $x + y + z$ (d) 0

(GATE 2014, 1 Mark)

Solution: We have,

$$\nabla\cdot(f\vec{v}) = f(\nabla\cdot\vec{v}) + \nabla f\cdot\vec{v} \qquad (1)$$

Now, $\nabla\cdot v = 0 + 0 + 0 = 0$

Thus, Eq. (1) becomes

$$x^2y + y^2z + z^2x = f(0) + \vec{v}\nabla f$$

Therefore, $\vec{v}\cdot\nabla f = x^2y + y^2z + z^2x$.

Ans. (a)

106. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{1-\cos x}$ is

(a) 0 (b) 1
(c) 3 (d) not defined

(GATE 2014, 1 Mark)

Solution: $\lim\limits_{x\to 0}\dfrac{x-\sin x}{1-\cos x}$ is a (0/0) form.

Applying L'Hospital rule, we have,

$\lim\limits_{x\to 0}\dfrac{1-\cos x}{\sin x} = \dfrac{0}{0} \Rightarrow \lim\limits_{x\to 0}\dfrac{\sin x}{\cos x} = \dfrac{0}{1} = 0$ (on applying the rule again)

Ans. (a)

107. Which one of the following describes the relationship among the three vectors: $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+3\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+4\hat{k}$?

(a) The vectors are mutually perpendicular
(b) The vectors are linearly dependent
(c) The vectors are linearly independent
(d) The vectors are unit vectors

(GATE 2014, 1 Mark)

Solution: The given vectors $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+3\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+4\hat{k}$ are linearly independent if the determinant in the below expression is non-zero.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & 6 & 1 \end{vmatrix} = 1(12-6) - 2(1-6) + 5(1-3)$$

$$= 6 + 10 - 10 = 6 \neq 0$$

So, the vectors are linearly independent.

Ans. (c)

108. $\underset{x\to 0}{\text{Lt}}\dfrac{e^{2x}-1}{\sin 4x}$ is equal to

(a) 0 (b) 0.5 (c) 1 (d) 2

(GATE 2014, 1 Mark)

Solution: The given expression $\underset{x\to 0}{\text{Lt}}\dfrac{e^{2x}-1}{\sin 4x}$ is of (0/0) form.

Applying L'Hospital rule, we have

$$\underset{x\to 0}{\text{Lt}}\frac{2e^{2x}}{4\cos 4x} = \frac{2}{4} = 0.5$$

Ans. (b)

109. Curl of vector $\overrightarrow{F} = x^2z^2\hat{i} - 2xy^2z\hat{j} + 2y^2z^3\hat{k}$ is

(a) $(4yz^3 - 2xy^2)\hat{i} + 2x^2z\hat{j} - 2y^2z\hat{k}$

(b) $(4yz^3 - 2xy^2)\hat{i} - 2x^2z\hat{j} - 2y^2z\hat{k}$

(c) $2xz^2\hat{i} - 4xyz\hat{j} + 6y^2z^2\hat{k}$

(d) $2xz^2\hat{i} + 4xyz\hat{j} + 6y^2z^2\hat{k}$

(GATE 2014, 1 Mark)

Solution: For $\overrightarrow{F} = x^2z^2i - 2xy^2zj + 2y^2z^3k$, the curl $\overrightarrow{F}$ is given by

$$\nabla \times \overrightarrow{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z^2 & -2xy^2z & 2y^2z^3 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2y^2z^3) - \frac{\partial}{\partial z}(-2xy^2z)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(2y^2z^3) - \frac{\partial}{\partial z}(x^2z^2)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(-2xy^2z) - \frac{\partial}{\partial y}(x^2z^2)\right]$$

$$= \hat{i}(4yz^3 - 2xy^2) + \hat{j}(2x^2z) + \hat{k}(-2y^2z)$$

Ans. (a)

110. The best approximation of the minimum value attained by $e^{-x}\sin(100x)$ for $x \geq 0$ is ________.

(GATE 2014, 1 Mark)

Solution: Differentiating $e^{-x}\sin(100x)$ w.r.t. x, we get

$$100e^{-x}\cos(100x) - e^{-x}\sin(100x)$$

Now,

$$\frac{d[e^{-x}\sin(100x)]}{dx} = 0$$

$$\Rightarrow 100e^{-x}\cos(100x) - e^{-x}\sin(100x) = 0$$

$$\Rightarrow 100\cos(100x) = \sin(100x) \Rightarrow \tan(100x) = 100$$

$$\Rightarrow 100x = \tan^{-1}(100) \approx 90$$

Taking second differential, we get

$$\frac{d^2[e^{-x}\sin(100x)]}{dx^2}$$

$$= 10000e^{-x}\sin(100x) + 100e^{-x}\cos(100x)$$

$$- e^{-x}\cos(100x) + e^{-x}\sin(100x)$$

$$\approx 8.19 \times 10^{-40} \times 10^4 = 8.19 \times 10^{-36}$$

Ans. 8.19×10^{-36}

111. If a function is continuous at a point,

(a) the limit of the function may not exist at the point
(b) the function must be derivable at the point
(c) the limit of the function at the point tends to infinity
(d) the limit must exist at the point and the value of limit should be same as the value of the function at that point

(GATE 2014, 1 Mark)

Solution: A function $f(x)$ is said to be continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists such that $\lim_{x \to a} f(x) = f(a)$.

Ans. (d)

112. Divergence of the vector field $x^2zi + xyj - yz^2k$ at (1, −1, 1) is

(a) 0 (b) 3 (c) 5 (d) 6

(GATE 2014, 1 Mark)

Solution: For $\overrightarrow{F} = x^2zi + xyj - yz^2k$, the divergence is given by

$$\nabla \cdot \overrightarrow{F} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(-yz^2)$$

$$= 2xz + x - 2yz$$

At (1, −1, 1), div $\overrightarrow{F}$ = 2 + 1 + 2 = 5

Ans. (c)

113. The value of the integral $\int_0^2 \frac{(x-1)^2\sin(x-1)}{(x-1)^2\cos(x-1)}$ is

(a) 3 (b) 0 (c) −1 (d) −2

(GATE 2014, 1 Mark)

Solution: Suppose

$$I = \int_0^2 \frac{(x-1)^2\sin(x-1)}{(x-1)^2\cos(x-1)}$$

We have

$$I = \int_0^2 \frac{(2-x-1)^2\sin(2-x-1)}{(2-x-1)^2\cos(2-x-1)}$$

$$= \int_0^2 \frac{(1-x)^2\sin(1-x)}{(1-x)^2\cos(1-x)} = -I$$

Therefore, $I + I = 0 \Rightarrow 2I = 0 \Rightarrow I = 0$

Ans. (b)

114. $\lim_{x \to \infty}\left(\frac{x + \sin x}{x}\right)$ equals to

(a) $-\infty$ (b) 0 (c) 1 (d) ∞

(GATE 2014, 1 Mark)

Solution: We have to find,

$$\lim_{x\to\infty}\left(\frac{x+\sin x}{x}\right)$$

$$\lim_{x\to\infty}\left(\frac{x+\sin x}{x}\right)=\lim_{x\to\infty}\left(1+\frac{\sin x}{x}\right)=1+0=1$$

Ans. (c)

115. With reference to the conventional Cartesian (x, y) coordinate system, the vertices of a triangle have the following coordinates: $(x_1, y_1) = (1, 0)$; $(x_2, y_2) = (2, 2)$; and $(x_3, y_3) = (4, 3)$. The area of the triangle is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $\frac{4}{5}$ (d) $\frac{5}{2}$

(GATE 2014, 1 Mark)

Solution:

$$\text{Area of the triangle} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 4 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left|1(2-3)-2(0-3)+4(0-2)\right|=\frac{1}{2}\left|-3\right|=\frac{3}{2}$$

Ans. (a)

116. Given

$x(t) = 3\sin(1000\,\pi\,t)$ and $y(t) = 5\cos\left(1000\,\pi t+\frac{\pi}{4}\right)$

The $x-y$ plot will be

(a) a circle

(b) a multi-loop closed curve

(c) a hyperbola

(d) an ellipse

(GATE 2014, 1 Mark)

Solution: The phase difference of the two signals,

$$\phi=\frac{\pi}{4}$$

The ellipse will occur in $x-y$ plot.

Ans. (d)

117. A vector is defined as

$$f = y\hat{i} + x\hat{j} + z\hat{k}$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in Cartesian (x, y, z) coordinate system.

The surface integral $\oiint f\cdot ds$ over the closed surface S of a cube with vertices having the following coordinates: (0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,0,1), (1,0,1), (1,1,1), (0,1,1) is ________ .

(GATE 2014, 1 Mark)

Solution: From Gauss divergence theorem, we have

$$\int_s \vec{f}\cdot\vec{ds}=\int_v \nabla\vec{f}\cdot dv=\int_v\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right)dxdydz$$

$$=\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(0+0+1)\,dxdydz=1$$

$$\left[\because \vec{f}=y\hat{i}+x\hat{j}+z\hat{k}\Rightarrow f_1=y, f_2=x, f_3=z\right]$$

118. Gradient of a scalar variable is always

(a) a vector (b) a scalar

(c) a dot product (d) zero

(GATE 2014, 1 Mark)

Solution: Gradient $\nabla=\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}$

If f is a scalar point function,

$$\nabla f=\text{grad } f=\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\text{ is a vector.}$$

Ans. (a)

119. If $f(x)$ is a real and continuous function of x, the Taylor series expansion of $f(x)$ about its minima will **NEVER** have a term containing

(a) first derivative (b) second derivative

(c) third derivative (d) any higher derivative

(GATE 2014, 1 Mark)

Solution: For a real-valued function, $y = f(x)$

Taylor series expansion about 'a'

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots$$

For minima at $x = a$, $f'(a) = 0$.

So, Taylor series expansion of $f(x)$ about 'a' will never contain first derivative term.

Ans. (a)

120. The function $f(x) = x\sin x$ satisfies the following equation $f''(x) + f(x) + t\cos x = 0$. The value of t is ________.

(GATE 2014, 2 Marks)

Solution: We are given

$$f(x) = x\sin x \quad (1)$$

We are also given that $f''(x) + f(x) + t\cos x = 0$

Differentiating Eq. (1) w.r.t. x, we get

$$f'(x) = \sin x + x\cos x \quad (2)$$

Differentiating Eq. (2) w.r.t. x, we get

$$f''(x) = \cos x + x(-\sin x) + \cos x = 2\cos x - x\sin x$$
$$2\cos x - x\sin x + x\sin x + t\cos x = 0$$
$$2\cos x + t\cos x = 0 \Rightarrow t = -2$$

Ans. −2

121. A function $f(x)$ is continuous in the interval [0,2]. It is known that $f(0) = f(2) = -1$ and $f(1) = 1$.

Which one of the following statements must be true?

(a) There exists a y in the interval (0,1) such that $f(y) = f(y + 1)$.
(b) For every y in the interval (0,1), $f(y) = f(2 - y)$.
(c) The maximum value of the function in the interval (0.2) is 1.
(d) There exists a y in the interval (0,1) such that $f(y) = f(2 - y)$.

(GATE 2014, 2 Marks)

Solution: Let $g(x) = f(x) - f(x+1)$ defined in [0, 1]

$$g(0) = f(0) - f(1) = -1 - 1 = -2$$
$$g(1) = f(1) - f(2) = 1 + 1 = +2$$

$g(0)$ is negative and $g(1)$ is positive. Therefore, there exists some x for which $g(x) = 0$.

$$\Rightarrow g(x) = 0 = f(x) - f(x+1) \Rightarrow f(x) = f(x+1)$$

Ans. (a)

122. The value of the integral given below is

$$\int_0^{\pi} x^2 \cos x \; dx$$

(a) -2π (b) π (c) $-\pi$ (d) 2π

(GATE 2014, 2 Marks)

Solution: We have to find the value of

$$\int_0^{\pi} x^2 \cos x \; dx$$

$$\Rightarrow \int_0^{\pi} x^2 \cos x \, dx = x^2 \sin x - 2x(-\cos x)$$
$$+ 2(-\sin x)[0 \text{ to } \pi] = -2\pi$$

Ans. (a)

123. The Taylor's series expansion of $3 \sin x + 2 \cos x$ is

(a) $2 + 3x - x^2 - \dfrac{x^3}{2} + \cdots$

(b) $2 - 3x + x^2 - \dfrac{x^3}{2} + \cdots$

(c) $2 + 3x + x^2 - \dfrac{x^3}{2} + \cdots$

(d) $2 - 3x - x^2 + \dfrac{x^3}{2} + \cdots$

(GATE 2014, 2 Marks)

Solution: We are given the expression,

$$3 \sin x + 2 \cos x$$

Using Taylor's series,

$$3\left(x - \frac{x^3}{3!} + \cdots\right) + 2\left(1 - \frac{x^2}{2!} + \cdots\right) = 2 + 3x - x^2 - \frac{x^3}{2} + \cdots$$

Ans. (a)

124. The volume under the surface $z(x, y) = x + y$ and above the triangle in the xy plane defined by $\{0 \le y \le x \text{ and } 0 \le x \le 12\}$ is ________.

(GATE 2014, 2 Marks)

Solution: We have to find the area under the surface $z(x, y) = x + y$ and above the triangle in the xy plane defined by $\{0 \le y \le x$ and $0 \le x \le 12\}$.

$$\text{Volume} = \iint_R Z(x, y)\, dy dx = \int_{x=0}^{12} \int_{y=0}^{x} (x + y)\, dy dx$$

$$= \int_{x=0}^{12} \left[xy + \frac{y^2}{2}\right]_0^x dx = \int_{x=0}^{12} \frac{3}{2} x^2 dx = \frac{3}{2}\left[\frac{x^3}{3}\right]_0^{12} = 864$$

Ans. 864

125. If $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ and $|\vec{r}| = r$, then $\text{div}(r^2 \nabla(\ln r)) =$ ________.

(GATE 2014, 2 Marks)

Solution: We are given $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ and $|\vec{r}| = r$.

Then,

$$\nabla(\ln r) = \frac{\vec{r}}{r^2} \Rightarrow \text{div}\left(r^2 \nabla(\ln r)\right) = \text{div}(\vec{r}) = 3$$

$$\left[\begin{aligned} \nabla(\ln r) &= \sum \hat{a}_x \frac{\partial}{\partial x}(\ln r) = \sum \hat{a}_x \left(\frac{1}{r}\right)\left(\frac{x}{r}\right) \\ &= \frac{1}{r^2} \sum \hat{a}_x x = \frac{\vec{r}}{r^2} \end{aligned}\right]$$

Ans. 3

126. The maximum value of $f(x) = 2x^3 - 9x^2 + 12x - 3$ in the interval $0 \le x \le 3$ is ________.

(GATE 2014, 2 Marks)

Solution: We are given

$$f(x) = 2x^3 - 9x^2 + 12x - 3$$

Then

$$f'(x) = 6x^2 - 18x + 12 = 0 \Rightarrow x = 1,\ 2 \in [0,\ 3]$$

Now, $f(0) = -3$; $f(3) = 6$ and $f(1) = 2$; $f(2) = 1$. Hence, $f(x)$ is maximum at $x = 3$ and the maximum value is 6.

Ans. 6

127. For a right-angled triangle, if the sum of the lengths of the hypotenuse and a side is kept constant, in order to have maximum area of the triangle, the angle between the hypotenuse and the side is

(a) 12° (b) 36° (c) 60° (d) 45°

(GATE 2014, 2 Marks)

Solution: Let x be the opposite side, y be the adjacent side and z be the hypotenuse of a right-angled triangle.
We are given that,

$$z + y = k \ \text{(constant)} \tag{1}$$

If angle between them is θ, then area

$$A = \frac{1}{2}xy = \frac{1}{2}(z\sin\theta)(z\cos\theta) = \frac{z^2}{4}\sin 2\theta$$

Now, according to Eq. (1), we get

$$z + z\sin\theta = k \Rightarrow z = \frac{k}{1+\sin\theta}$$

Therefore

$$A = \frac{k^2}{4}\left[\frac{\sin 2\theta}{(1+\sin\theta)^2}\right]$$

In order to have maximum area, $\dfrac{dA}{d\theta} = 0$.

$$\Rightarrow \frac{k^2}{4}\left[\frac{(1+\sin\theta)^2(2\cos 2\theta) - \sin 2\theta(\cos\theta)\cdot 2(1+\sin\theta)}{(1+\sin\theta)^4}\right] = 0$$

Ans. (c)

128. Let g: $[0,\ \infty) \to [0,\ \infty)$ be a function defined by $g(x) = x - [x]$, where $[x]$ represents the integer part of x. (That is, it is the largest integer which is less than or equal to x).

The value of the constant term in the Fourier series expansion of $g(x)$ is ________ .

(GATE 2014, 2 Marks)

Solution: Clearly, $g(x)$ is a periodic function with period '1'.
Consider, $g(x) = x - [x]$ for $0 < x < 1$

The constant term in the Fourier series expansion of

$$g(x) \text{ is } A_0 = \frac{a_0}{2} = \int_0^1 g(x)\,dx = \int_0^1 x\,dx - \int_0^1 [x]\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 - \int_0^1 (0)\,dx = \frac{1}{2}$$

129. To evaluate the double integral $\int_0^8\left(\int_{y/2}^{(y/2)+1}\left(\dfrac{2x-y}{2}\right)dx\right)dy$, we make the substitution $u = \left(\dfrac{2x-y}{2}\right)$ and $v = \dfrac{y}{2}$. The integral will reduce to

(a) $\int_0^4\left(\int_0^2 2u\,du\right)dv$

(b) $\int_0^4\left(\int_0^1 2u\,du\right)dv$

(c) $\int_0^4\left(\int_0^1 u\,du\right)dv$

(d) $\int_0^4\left(\int_0^2 u\,du\right)dv$

(GATE 2014, 2 Marks)

Solution: We have to substitute,

$$u = \frac{2x-y}{2} \tag{1}$$

$$v = \frac{y}{2} \tag{2}$$

$$x = \frac{y}{2} \Rightarrow u = 0;\ x = \frac{y}{2} + 1 \Rightarrow u = 1$$

$$y = 0 \Rightarrow v = 0;\ y = 8 \Rightarrow v = 4$$

From Eqs. (1) and (2), we get

$$x = u + v \tag{3}$$

$$y = 2v \tag{4}$$

Using Jacobian transformation,

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

Therefore

$$\int_0^8 \left(\int_{y/2}^{(y/2)+1} \left(\frac{2x-y}{2} \right) dx \right) dy = \int_{v=0}^{4} \left(\int_{u=0}^{1} (u)|J|\,du \right) dv$$

$$= \int_0^4 \left(\int_0^1 2u du \right) dv$$

Ans. (b)

130. The minimum value of the function $f(x) = x^3 - 3x^2 - 24x + 100$ in the interval $[-3, 3]$ is

(a) 20 (b) 28 (c) 16 (d) 32

(GATE 2014, 2 Marks)

Solution: We are given

$$f(x) = x^3 - 3x^2 - 24x + 100$$

Differentiating w.r.t. x, we get

$$f'(x) = 0 \Rightarrow x^2 - 2x - 8 = 0$$

$$\Rightarrow x = -2, 4 \in [-3, 3]$$

Now,

$$f(-3) = 118,\ f(3) = 28$$

and $$f(-2) = 128,\ f(4) = 44$$

Thus, $f(x)$ is minimum at $x = 3$ and the minimum value is $f(3) = 28$.

Ans. (b)

131. A differentiable non-constant even function $x(t)$ has a derivative $y(t)$, and their respective Fourier transforms are $X(\omega)$ and $Y(\omega)$. Which of the following statements is TRUE?

(a) $X(\omega)$ and $Y(\omega)$ are both real
(b) $X(\omega)$ is real and is $Y(\omega)$ imaginary
(c) $X(\omega)$ and $Y(\omega)$ are both imaginary
(d) $X(\omega)$ is imaginary and $Y(\omega)$ is real

(GATE 2014, 2 Marks)

Solution: We have

$$y(t) = \frac{d}{dt} x(t)$$

Applying Fourier transform,

$$Y(\omega) = j\omega X(\omega)$$

$\Rightarrow$ if $X(\omega)$ is real, $Y(\omega)$ is imaginary.

Ans. (b)

132. The integral $\oint_C (ydx - xdy)$ is evaluated along the circle $x^2 + y^2 = \frac{1}{4}$ traversed in counter clockwise direction. The integral is equal to

(a) 0 (b) $-\frac{\pi}{4}$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

(GATE 2014, 2 Marks)

Solution: The given integral is $\oint_C (ydx - xdy)$ where C is $x^2 + y^2 = 1/4$.

According to Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial x} \right) dxdy$$

where R is region included in C. So

$$\oint_C (ydx - xdy) = \iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial x} \right) dxdy$$

$$= \iint_R (-1-1) dxdy$$

$$= -2 \times \text{region } R = -2 \times \pi r^2$$

$$= -2 \times \pi \left(\frac{1}{2} \right)^2 = -\frac{\pi}{2}$$

Ans. (c)

133. The value of the integral $\int_0^2 \int_0^x e^{x+y}\, dy\, dx$ is

(a) $\frac{1}{2}(e-1)$ (b) $\frac{1}{2}(e^2-1)^2$

(c) $\frac{1}{2}(e^2-e)$ (d) $\frac{1}{2}\left(e - \frac{1}{e}\right)^2$

(GATE 2014, 2 Marks)

Solution: We have to find the value of

$$\int_0^2 \int_0^x e^{x+y}\, dy\, dx$$

$$\int_0^2 \int_0^x e^{x+y}\, dy\, dx = \int_0^2 \int_0^x e^x\, dx \int_0^x e^y\, dy$$

$$= \int_0^2 e^x \mid e^y \mid_0^x dx = \int_0^2 e^{2x} - e^x\ dx$$

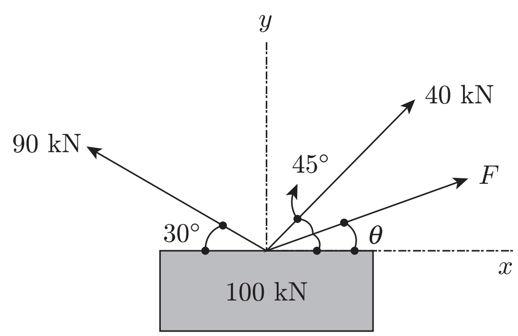
$$= \int_0^2 e^{2x}\ dx - \int_0^2 e^x\ dx$$

$$= \frac{(e^4 - 1)}{2} - (e^2 - 1)$$

$$= \frac{1}{2}(e^4 - 2e^2 + 1) = \frac{1}{2}(e^2 - 1)^2$$

Ans. (b)

134. A box of weight 100 kN shown in the figure is to be lifted without swinging. If all forces are coplanar, the magnitude and direction (θ) of the force (F) with respect to x-axis should be



(a) $F = 56.389$ kN and $\theta = 28.28°$

(b) $F = -56.389$ kN and $\theta = -28.28°$

(c) $F = 9.055$ kN and $\theta = 1.414°$

(d) $F = -9.055$ kN and $\theta = -1.414°$

(GATE 2014, 2 Marks)

Solution: For no swinging,

$$\sum F_H = 0$$

$$\Rightarrow 90\cos 30° = F\cos\theta + 40\cos 45°$$

$$\Rightarrow F\cos\theta = 49.658 \quad (1)$$

Also,

$$\sum F_V = 0$$

$$\Rightarrow 100 = 90\sin 30° + 40\sin 45° + F\sin\theta$$

$$\Rightarrow F\sin\theta = 26.715 \quad (2)$$

Solving Eqs. (1) and (2), we get

$$F = 56.389 \text{ kN and } \theta = 28.28°$$

Ans. (a)

135. A particle moves along a curve whose parametric equations are: $x = t^3 + 2t$, $y = -3e^{-2t}$ and $z = 2$ sin ($5t$), where x, y and z show variations of the distance covered by the particle (in cm) with time t (in s). The magnitude of the acceleration of the particle (in cm/s^2) at $t = 0$ is ________ .

(GATE 2014, 2 Marks)

Solution: The equations of the curve are given as

$$x = t^3 + 2t,\ y = -3e^{-2t} \text{ and } z = 2\sin(5t)$$

$$a_x = \frac{\partial^2 x}{\partial t^2} = 6t$$

$$a_y = \frac{\partial^2 y}{\partial t^2} = -12e^{-2t}$$

$$a_z = \frac{\partial^2 z}{\partial t^2} = -50\sin 5t$$

$$a = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = (6t)\hat{i} - \left(12e^{-2t}\right)\hat{j} - (50\sin 5t)\hat{k}$$

$$a(t = 0) = -12\hat{j}$$

Thus, $|a|_{t=0} = 12$ cm/s^2

Ans. 12 cm/s^2

136. The expression $\lim\limits_{\alpha \to 0} \dfrac{x^\alpha - 1}{\alpha}$ is equal to

(a) $\log x$ (b) 0
(c) $x \log x$ (d) ∞

(GATE 2014, 2 Marks)

Solution: We are given the expression,

$$\lim_{\alpha = 0} \frac{x^\alpha - 1}{\alpha}$$

Since substituting $\alpha = 0$, the expression takes the form of $\dfrac{0}{0}$.

Hence, by L'Hospital rule,

$$L = \frac{\frac{d}{d\alpha}\left(x^{\alpha - 1}\right)}{\frac{d}{d\alpha}(\alpha)} = \lim_{\alpha \to 0} \frac{x^\alpha \cdot \log x}{1} = \log x$$

Ans. (a)

137. With reference to a standard Cartesian (x, y) plane, the parabolic velocity distribution profile of fully developed laminar flow in x-direction between two parallel, stationary and identical plates that are separated by distance, h, is given by the expression

$$u = -\frac{h^2}{8\mu}\frac{dp}{dx}\left[1 - 4\left(\frac{y}{h}\right)^2\right]$$

In this equation, the $y = 0$ axis lies equidistant between the plates at a distance $h/2$ from the two plates, p is the pressure variable and μ is the dynamic viscosity term. The maximum and average velocities are, respectively

(a) $u_{\max} = -\dfrac{h^2}{8\mu}\dfrac{dp}{dx}$ and $u_{\text{average}} = \dfrac{2}{3}u_{\max}$

(b) $u_{\text{max}} = \dfrac{h^2}{8\mu}\dfrac{dp}{dx}$ and $u_{\text{average}} = \dfrac{2}{3}u_{\text{max}}$

(c) $u_{\text{max}} = -\dfrac{h^2}{8\mu}\dfrac{dp}{dx}$ and $u_{\text{average}} = \dfrac{3}{8}u_{\text{max}}$

(d) $u_{\text{max}} = \dfrac{h^2}{8\mu}\dfrac{dp}{dx}$ and $u_{\text{average}} = \dfrac{3}{8}u_{\text{max}}$

(GATE 2014, 2 Marks)

Solution: $u = \dfrac{-h^2}{8\mu}\left(\dfrac{dp}{dx}\right)\left[1-4\left(\dfrac{y}{h}\right)^2\right]$

Maximum velocity is at $y = 0$.

$$u_{\text{max}} = \frac{-h^2}{8\mu}\left(\frac{dp}{dx}\right)$$

$$u_{\text{average}} = \frac{Q}{A} = \frac{\int u\cdot dA}{A} = \frac{\int u_{\text{max}}\left(1-\frac{4y^2}{h^2}\right)\cdot dA}{A}$$

$$= \frac{2\int_0^{h/2} u_{\text{max}}\left(1-\frac{4y^2}{h^2}\right)dy\times 1}{h\times 1} = \frac{2u_{\text{max}}}{h}\left[y-\frac{4y^3}{3h^2}\Bigg|_0^{h/2}\right]$$

$$= \frac{2u_{\text{max}}}{h}\left[\frac{h}{2}-\frac{4h^3}{24h^2}\right] = \frac{2}{3}u_{\text{max}}$$

Ans. (a)

138. If $g(x) = 1 - x$ and $h(x) = \dfrac{x}{x-1}$, then $\dfrac{g[h(x)]}{h[g(x)]}$ is

(a) $\dfrac{h(x)}{g(x)}$ (b) $\dfrac{-1}{x}$

(c) $\dfrac{g(x)}{h(x)}$ (d) $\dfrac{x}{(1-x)^2}$

(GATE 2015, 1 Mark)

Solution: We have,

$$g(x) = 1 - x \tag{1}$$

$$h(x) = \frac{x}{x-1} \tag{2}$$

Replace x by $h(x)$ in Eq. (1), we get

$$g[h(x)] = 1 - h(x) = 1 - \frac{x}{x-1} = \frac{-1}{x-1}$$

Replacing x by $g(x)$ in Eq. (2), we get

$$h[g(x)] = \frac{g(x)}{g(x)-1} = \frac{1-x}{-x}$$

$$\Rightarrow \frac{g[h(x)]}{h[g(x)]} = \frac{x}{(x-1)(1-x)} = \frac{\frac{x}{x-1}}{1-x} = \frac{h(x)}{g(x)}$$

Ans. (a)

139. The value of $\lim\limits_{x\to\infty}(1+x^2)e^{-x}$ is

(a) 0 (b) $\dfrac{1}{2}$ (c) 1 (d) ∞

(GATE 2015, 1 Mark)

Solution: If we substitute the value of the limit in the given equation, we get

$$\lim_{x\to\infty}(1+x^2)e^{-x} = \lim_{x\to\infty}\frac{1+x^2}{e^x} \text{ is } \frac{\infty}{\infty}\text{ form.}$$

$$= \lim_{x\to\infty}\frac{2x}{e^x} \text{ (using L-Hospital's rule)}$$

Therefore,

$$\lim_{x\to\infty}(1+x^2)e^{-x} = \lim_{x\to\infty}\frac{2x}{e^x} = 0$$

Ans. (a)

140. A function $f(x) = 1 - x^2 + x^3$ is defined in the closed interval $[-1, 1]$. The value of x, in the open interval $(-1, 1)$ for which the mean value theorem is satisfied, is

(a) $-\dfrac{1}{2}$ (b) $-\dfrac{1}{3}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{1}{2}$

(GATE 2015, 1 Mark)

Solution: By Lagrange's mean value theorem,

$$f'(x) = \frac{f(1)-f(-1)}{1-(-1)} = \frac{2}{2} = 1$$

$$-2x + 3x^2 = 1$$

$$\Rightarrow 3x^2 - 2x - 1 = 0$$

Hence, $x = 1, -\dfrac{1}{3}$

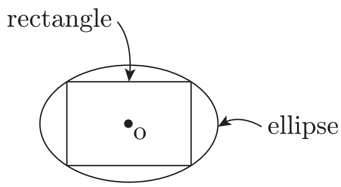
Thus, $x = -\dfrac{1}{3}$ only lies in $(-1, 1)$.

Ans. (b)

141. The maximum area (in square units) of a rectangle whose vertices lie on the ellipse $x^2 + 4y^2 = 1$ is ________.

(GATE 2015, 1 Mark)

Solution: Let $2x$, $2y$ be the length and breadth, respectively, of the rectangle inscribed in the ellipse $x^2 + 4y^2 = 1$.



Then, area of the rectangle $= (2x)(2y) = 4xy$

Consider,

$$f = (\text{Area})^2 = 16x^2y^2 = 4x^2(1-x^2) \left(\because y^2 = \frac{1-x^2}{4}\right)$$

$$f'(x) = 0 \Rightarrow x(1-2x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$$

Therefore,

$$y^2 = \frac{1}{8} \Rightarrow y = \frac{1}{\sqrt{8}}$$

$$f''(x) = 8 - 48x^2 < 0$$

when $x = \dfrac{1}{\sqrt{2}}$

$\Rightarrow f$ is maximum at $x = \dfrac{1}{\sqrt{2}}$

Thus, the maximum area is $4\left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{8}}\right)$, that is, 1.

Ans. 1

142. A vector $\vec{P}$ is given by $\vec{P} = x^3y\hat{a}_x - x^2y^2\hat{a}_y - x^2yz\hat{a}_2$. Which one of the following statements is TRUE?

(a) $\vec{P}$ is solenoidal, but not irrotational

(b) $\vec{P}$ is irrotational, but not solenoidal

(c) $\vec{P}$ is neither solenoidal nor irrotational

(d) $\vec{P}$ is both solenoidal and irrotational

(GATE 2015, 1 Mark)

Solution: We are given,

$$\vec{P} = x^3y\hat{a}_x - x^2y^2\hat{a}_y - x^2yz\hat{a}_2$$

$$\nabla P = 3x^2y - 2x^2y - x^2y = 0$$

It is solenoidal,

$$\nabla \times P = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^3y & -x^2y^2 & -x^2yz \end{vmatrix}$$

$$= \hat{a}_x(-x^2y) - \hat{a}_y(-2xyz) + \hat{a}_z(-2xy^2 - x^3) \neq 0$$

So P is solenoidal but not irrotational.

Ans. (a)

143. If C denotes the counterclockwise unit circle, the value of the contour integral $\dfrac{1}{2\pi j}\oint_c \text{Re}(z)\,dz$ is __________ .

(GATE 2015, 1 Mark)

Solution: We are given,

$$\frac{1}{2\pi j}\oint_C \text{Re}(z)dz \text{ where } C \text{ is } |z| = 1$$

Put

$$z = e^{j\theta} \Rightarrow d\theta = je^{j\theta}d\theta$$

$$\frac{1}{2\pi j}\int_0^{2\pi} \text{Re}(e^{j\theta})je^{j\theta}d\theta = \frac{1}{2\pi j}\int_0^{2\pi} \cos\theta j(\cos\theta + j\sin\theta)d\theta$$

$$= \frac{j}{2\pi j}\left[\int_0^{2\pi} \cos^2\theta d\theta - \int_0^{2\pi} \cos\theta\sin\theta d\theta\right]$$

$$= \frac{j}{2\pi j}(\pi - 0) = \frac{1}{2}$$

Ans. 1/2

144. Consider a function $\vec{f} = \dfrac{1}{r^2}\hat{r}$, where r is the distance from the origin and $\hat{r}$ is the unit vector in the radial direction. The divergence of this function over a sphere of radius R, which includes the origin, is

(a) 0 (b) 2π (c) 4π (d) $R\pi$

(GATE 2015, 1 Mark)

Solution: The given function is $\vec{f} = \dfrac{1}{r^2}a_{\hat{r}}$

$$\Rightarrow \nabla f = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta f_\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial \phi}$$

$$\Rightarrow \nabla f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 + \frac{1}{r^2}\right) + 0 + 0 = 0$$

Ans. (a)

145. If a continuous function $f(x)$ does not have a root in the interval $[a, b]$, then which one of the following statements is TRUE?

(a) $f(a)\cdot f(b) = 0$ (b) $f(a)\cdot f(b) < 0$
(c) $f(a)\cdot f(b) > 0$ (d) $f(a)/f(b) \leq 0$

(GATE 2015, 1 Mark)

Solution: We know that according to intermediate value theorem, if $f(a) \cdot f(b) < 0$ then $f(x)$ has at least one root in (a, b). $f(x)$ does not have roots in (a, b) means $f(a) \cdot f(b) > 0$.

Ans. (c)

146. Match the following:

P. Stokes' theorem
Q. Gauss's theorem
R. Divergence theorem
S. Cauchy's integral theorem

1. $\oiint D \cdot ds = Q$
2. $\oint f(z)dz = 0$
3. $\iiint (\nabla \cdot A)dv = \oiint A \cdot ds$
4. $\iint (\nabla \times A) \cdot ds = \oint A \cdot dl$

	P	Q	R	S
(a)	2	1	4	3
(b)	4	1	3	2
(c)	4	3	1	2
(d)	3	4	2	1

(GATE 2015, 1 Mark)

Solution: The correct answer is option (b).

Ans. (b)

147. The volume enclosed by the surface $f(x, y) = e^x$ over the triangle bounded by the lines $x = y$; $x = 0$; $y = 1$ in the xy plane is ______.

(GATE 2015, 1 Mark)

Solution: The triangle is bounded by $x = y$, $x = 0$, $y = 1$ in xy plane.

$$\text{Required volume} = \iint_{OAB} f(x,y)dxdy$$

$$= \int_{x=0}^{1}\int_{y=x}^{1} e^x dxdy$$

$$= \int_{x=0}^{1} e^x (y)_x^1 dx = \int_{x=0}^{1} e^x(1-x)dx = \int_{x=0}^{1}(e^x - xe^x)dx$$

$$= (e^x)_0^1 - (e^x(x-1))_0^1 = (e^1 - 1) - [0 - (-1)] = e - 2$$

Ans. $e - 2$

148. Curl of vector $V(x, y, x) = 2x^2 i + 3z^2 j + y^3 k$ at $x = y = z = 1$ is

(a) $-3i$ (b) $3i$ (c) $3i - 4j$ (d) $3i - 6k$

(GATE 2015, 1 Mark)

Solution: Curl of vector

$$V(x,y,z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 & 3z^2 & y^3 \end{vmatrix}$$

$$= i(3y^2 - 6z) + j(0-0) + k(0-0)$$

$$= (3y^2 - 6z)i\Big|_{x=y=z=1} = -3i$$

Ans. (a)

149. If the fluid velocity for a potential flow is given by $V(x, y) = u(x, y)i + v(x, y)j$ with usual notations, then the slope of potential line at (x, y) is

(a) $\dfrac{v}{u}$ (b) $-\dfrac{u}{v}$
(c) $\dfrac{v^2}{u^2}$ (d) $\dfrac{u}{v}$

(GATE 2015, 1 Mark)

Solution: Fluid velocity is given by $V(x,y) = u(x,y)i + v(x,y)j$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$$d\phi = udx + vdy$$

$$\frac{dy}{dx} = -\frac{u}{v}$$

Ans. (b)

150. A small ball of mass 1 kg moving with a velocity of 12 m/s undergoes a direct central impact with a stationary ball of mass 2 kg. The impact is perfectly elastic. The speed (in m/s) of 2 kg mass ball after the impact will be ______.

(GATE 2015, 1 Mark)

Solution: For elastic collision,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad (1)$$

[Conservation of momentum]

$$m_1 = 1 \text{ kg}, \ u_1 = 12 \text{ m/s}$$
$$m_2 = 2 \text{ kg}, \ u_2 = 0 \text{ m/s}$$

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \quad (2)$$

[Energy conservation]

For Eq. (1),

$$v_1 + 2v_2 = 12 \quad (3)$$

From Eq. (2),

$$\frac{1}{2}\times 1\times 144 + \frac{1}{2}\times 2\times 0 = \frac{1}{2}\times 1\times v_1^2 + \frac{1}{2}\times 2\times v_2^2$$

$$\Rightarrow 144 = v_1^2 + 2v_2^2 \quad (4)$$

From Eqs. (3) and (4), we get

$$144 = 144 + 4v_2^2 - 48v_2 + 2v_2^2$$
$$\Rightarrow 6v_2^2 - 48v_2 = 0$$
$$\Rightarrow 6v_2(v_2 - 8) = 0$$
$$\Rightarrow v_2 = 8 \text{ m/s}$$

Ans. 8 m/s

151. At $x = 0$, the function $f(x) = |x|$ has

(a) A minimum
(b) A maximum
(c) A point of inflexion
(d) Neither maximum nor minimum

(GATE 2015, 1 Mark)

Solution: For negative values of x, $f(x)$ will be positive.
For positive values of x, $f(x)$ will be positive.

Therefore, minimum value of $f(x)$ will occur at $x = 0$.

Ans. (a)

152. The surface integral $\iint_s \frac{1}{\pi}(9xi - 3yj)\cdot n\,dS$ over the sphere given $x^2 + y^2 + z^2 = 9$ is ________ .

(GATE 2015, 1 Mark)

Solution: By Gauss divergence theorem,

$$\int_S \vec{F}\cdot\bar{n}ds = \int_V \operatorname{div}\vec{F}\,dV$$

Here,

$$\bar{F} = 9x\hat{i} - 3y\hat{j}$$
$$\operatorname{div}\bar{F} = 9 - 3 = 6$$

Thus,

$$\iint_s \frac{1}{\pi}(9x\hat{i} - 3y\hat{j})\cdot\hat{n}dS = \frac{1}{\pi}\int_V 6dV = \frac{1}{\pi}\cdot 6V$$
$$= \frac{1}{\pi}6\left(\frac{4}{3}\pi r^3\right) = 8(3)^3 = 216$$

Ans. 216

153. Let ϕ be an arbitrary smooth real-valued scalar function and $\vec{V}$ be an arbitrary smooth vector valued function in a three-dimensional space. Which one of the following is an identity?

(a) $\text{Curl}(\phi\vec{V}) = \nabla(\phi\text{Div}\vec{V})$
(b) $\text{Div}\vec{V} = 0$
(c) $\text{Div Curl } \vec{V} = 0$
(d) $\text{Div } (\phi\vec{V}) = \phi\text{Div}\vec{V}$

(GATE 2015, 1 Mark)

Solution: Div Curl $\vec{V} = 0$ is an identity.

Ans. (c)

154. The value of $\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x + x\cos x}\right)$ is ________ .

(GATE 2015, 1 Mark)

Solution: Given that

$$\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x + x\cos x}\right) \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\cos x}{2\cos x + \cos x - x\sin x}\right) \quad \text{(L-Hospital Rule)}$$
$$= \frac{-1}{3}$$

Ans. $-(1/3)$

155. A swimmer can swim 10 km in 2 h when swimming along the flow of a river. While swimming against the flow, she takes 5 h for the same distance. Her speed in still water (in km/h) is ________ .

(GATE 2015, 1 Mark)

Solution: Let speed in still water be x and speed of river be y.

$$2 = \frac{10}{x + y} \Rightarrow x + y = 5$$
$$5 = \frac{10}{x - y} \Rightarrow x - y = 2$$

On solving, we get

$$2x = 7$$

$$x = 3.5$$

Thus, the speed of swimmer in still water = 3.5 km/h

Ans. 3.5 km/h

156. The value of $\lim_{x\to 0} \dfrac{1-\cos(x^2)}{2x^4}$ is

(a) 0 (b) $\dfrac{1}{2}$

(c) $\dfrac{1}{4}$ (d) Undefined

(GATE 2015, 1 Mark)

Solution: We are given

$$\lim_{x\to 0} \frac{1-\cos(x^2)}{2x^4}$$

Applying the value of limit, we get the form of (0/0).

Using L'Hospital rule,

$$\lim_{x\to 0} \frac{(\sin x^2)2x}{-8x^3} = \frac{0}{0}$$

Applying again,

$$= \lim_{x\to 0} \frac{(\cos x^2)(4x^2) + (\sin x^2)2}{24x^2}$$

$$= \lim_{x\to 0} \frac{(-\sin x^2)(2x)(4x^2) + (\cos x^2)(8x) + 2(\cos x^2)2x}{48x}$$

$$= \lim_{x\to 0} \frac{(-8\sin x^2)(x^3) + 8(\cos x^2)(x) + 4(\cos x^2)x}{48x}$$

$$= \lim_{x\to 0} \frac{\left\{-8\left[(\cos x^2)(2x)(x^3) + (\sin x^2)(3x^2)\right] + 8\left[(-\sin x^2)(2x) + \cos x^2\right] + 4\left[(-\sin x^2)(2x) + \cos x^2\right]\right\}}{48}$$

$$= \frac{12}{48} = \frac{1}{4}$$

Ans. (c)

157. While minimizing the function $f(x)$, necessary and sufficient conditions for a point x_0 to be a minima are:

(a) $f'(x_0) > 0$ and $f''(x_0) = 0$

(b) $f'(x_0) < 0$ and $f''(x_0) = 0$

(c) $f'(x_0) = 0$ and $f''(x_0) < 0$

(d) $f'(x_0) = 0$ and $f''(x_0) > 0$

(GATE 2015, 1 Mark)

Solution: The conditions are $f'(x_0) = 0$ and $f''(x_0) > 0$.

Ans. (d)

158. $\lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{2x}$ is equal to

(a) e^{-2} (b) e

(c) 1 (d) e^2

(GATE 2015, 1 Mark)

Solution: We have

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}$$

Rearranging, we get

$$= \left[\lim_{x\to\infty}\left(x+\frac{1}{x}\right)^{x}\right]^2 = e^2$$

Ans. (d)

159. The double integral $\int_0^a \int_0^y f(x,y)\, dxdy$ is equivalent to

(a) $\int_0^x \int_0^y f(x,y) dx\ dy$

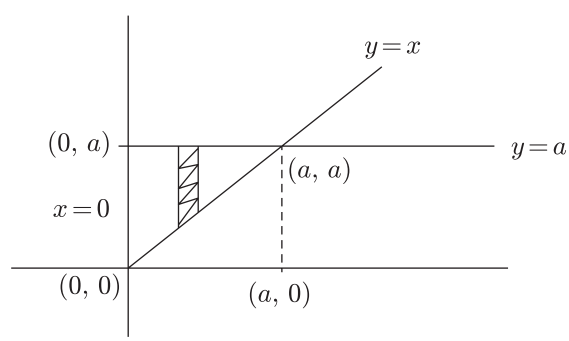
(b) $\int_0^a \int_x^y f(x,y) dx\ dy$

(c) $\int_0^a \int_x^a f(x,y) dy\ dx$

(d) $\int_0^a \int_0^a f(x,y) dx\ dy$

(GATE 2015, 1 Mark)

Solution: Given double integral $\int_0^x \int_0^y f(x,y) dxdy$ and $x = 0$ to $x = y$, $y = 0$ to $y = a$.



By applying change of order of integration,

$$\int_{x=0}^{a} \int_{y=x}^{a} f(x,y)\, dydx$$

Ans. (c)

160. The magnitude of the directional derivative of the function $f(x,y) = x^2 + 3y^2$ m a direction normal to the circle $x^2 + y^2 = 2$, at the point (1,1), is

(a) $4\sqrt{2}$ (b) $5\sqrt{2}$ (c) $7\sqrt{2}$ (d) $9\sqrt{2}$

(GATE 2015, 1 Mark)

Solution: Let $f(x,y) = x^2 + 3y^2$

Let $\phi = x^2 + y^2 - 2$ and P is (1, 1)

Normal to the surface, $\phi = \nabla\phi$, then

$$i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} = 2xi + 2yj$$

$$\nabla\phi\big|_{(1,1)} = 2i + 2j$$

Let $\vec{a} = \nabla\phi = 2i + 2j$, then

We need to calculate magnitude of directional derivatives of f along $\vec{a}$ at (1, 1)

Magnitude of directional derivations $= \nabla f \cdot \hat{a}$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} = 2xi + 6yj$$

$$\nabla f\big|_{(1,1)} = 2i + 6j$$

$$|\vec{a}| = \sqrt{4+4} = 2\sqrt{2}$$

$$\hat{a} = \frac{2i+2j}{2\sqrt{2}} = \frac{i+j}{\sqrt{2}}$$

Magnitude of directional derivative

$$= (2i+6j)\left(\frac{i+j}{\sqrt{2}}\right) = \frac{2+6}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

Ans. (a)

161. A scalar function in the xy-plane is given by $\phi\ (x, y) = x^2 + y^2$. If $\hat{i}$ and $\hat{j}$ are unit vectors in the x and y directions, the direction of maximum increase in the value of ϕ at (1, 1) is along:

(a) $-2\hat{i} + 2\hat{j}$ (b) $2\hat{i} + 2\hat{j}$

(c) $-2\hat{i} - 2\hat{j}$ (d) $2\hat{i} - 2\hat{j}$

(GATE 2015, 1 Mark)

Solution: Direction of maximum increase

$$= \nabla\phi \text{ at } (1, 1)$$

$$= (2x\hat{i} + 2y\hat{j}) \text{ at } (1, 1)$$

$$= (2\hat{i} + 2\hat{j})$$

Ans. (b)

162. $\displaystyle\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2}dx =$ ____________.

(GATE 2015, 2 Marks)

Solution: We have,

$$\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2}dx$$

Putting

$$1/x = t$$

$$\Rightarrow -\frac{1}{x^2}dx = dt$$

and

$$x = 2/\pi \Rightarrow t = \pi/2$$
$$x = 1/\pi \Rightarrow t = \pi$$

$$\int_{\pi/2}^{\pi} \cos t dt$$

$$\left(\text{since} \int_a^b f(x)dx = -\int_b^a f(x)dx\right)$$

$$= (\sin t)_{\pi/2}^{\pi} = \sin\pi - \sin(\pi/2) = -1$$

Ans. –1

163. $\displaystyle\sum_{x=1}^{99} \frac{1}{x(x+1)} =$ ____________.

(GATE 2015, 2 Marks)

Solution: We are given

$$\sum_{x=1}^{99} \frac{1}{x(x+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots + \frac{1}{99(100)}$$

$$= \frac{2-1}{1(2)} + \frac{3-2}{2(3)} + \frac{4-3}{3(4)} + \cdots + \frac{100-99}{99(100)}$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} + \cdots + \frac{1}{98} - \frac{1}{99} + \frac{1}{99} - \frac{1}{100}$$

$$= 1 - \frac{1}{100} = \frac{99}{100} = 0.99$$

Ans. 0.99

164. Let $f(x) = x^{-(1/3)}$ and A denote the area of the region bounded by $f(x)$ and the x-axis, when x varies from -1 to 1. Which of the following statements is/are TRUE?

(i) f is continuous in $[-1, 1]$
(ii) f is not bounded in $[-1, 1]$
(iii) A is non-zero and finite

(a) (ii) only
(b) (iii) only
(c) (ii) and (iii) only
(d) (i), (ii) and (iii)

(GATE 2015, 2 Marks)

Solution: Since $f(0) \to \infty$

Therefore, f is not bounced in $[-1,1]$ and hence f is not continuous in $[-1,1]$.

$$A = \int_{-1}^{0} f(x)dx = \int_{-1}^{0} x^{-1/3}dx + \int_{0}^{1} x^{-1/3}dx$$
$$= \frac{2}{3}(x^{2/3})_{-1}^{0} + \frac{2}{3}(x^{2/3})_{0}^{1} = 0$$

Thus, only statement (ii) is true.

Ans. (a)

165. If for non-zero x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 25$ where $a \neq b$ then $\int_{1}^{2} f(x)dx$ is

(a) $\frac{1}{a^2 - b^2}\left[a(\ln 2 - 25) + \frac{47b}{2}\right]$
(b) $\frac{1}{a^2 - b^2}\left[a(2\ln 2 - 25) - \frac{47b}{2}\right]$
(c) $\frac{1}{a^2 - b^2}\left[a(2\ln 2 - 25) + \frac{47b}{2}\right]$
(d) $\frac{1}{a^2 - b^2}\left[a(\ln 2 - 25) - \frac{47b}{2}\right]$

(GATE 2015, 2 Marks)

Solution: We are given,

$$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 25 \qquad (1)$$

$$\Rightarrow af\left(\frac{1}{x}\right) + bf(x) = x - 25 \qquad (2)$$

Solving Eqs. (1) and (2), we get

$$f(x) = \frac{1}{a^2 + b^2}\left[a\left(\frac{1}{x} - 25\right) - b(x - 25)\right]$$

Therefore,

$$\int_{1}^{2} f(x)dx = \frac{1}{a^2 - b^2}\left[a\{\ln x - 25x\}_1^2 - b\left\{\frac{x^2}{2} - 25x\right\}_1^2\right]$$
$$= \frac{1}{a^2 - b^2}\left[a\{\ln 2 - 25\} - b\left\{\frac{3}{2} - 25\right\}\right]$$
$$= \frac{1}{a^2 - b^2}\left[a\{\ln 2 - 25\} + \frac{47}{2}b\right]$$

Ans. (a)

166. Let $f(n) = n$ and $g(n) = n^{(1+\sin n)}$, where n is a positive integer. Which of the following statements is/are correct?

(i) $f(n) = O(g(n))$
(ii) $f(n) = \Omega(g(n))$

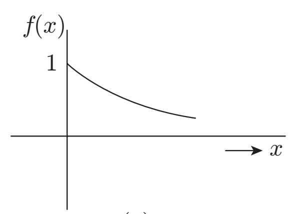
(a) only (i)
(b) only (ii)
(c) Both (i) and (ii)
(d) Neither (i) nor (ii)

(GATE 2015, 2 Marks)

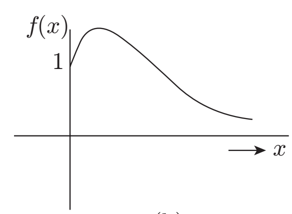
Solution: As $-1 \leq \sin x \leq 1$, neither of them is true.

Ans. (d)

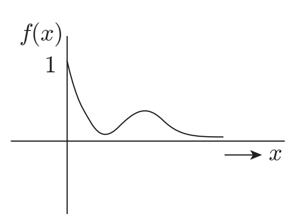
167. Which one of the following graphs describes the function $f(x) = e^{-x(x^2+x+1)}$?



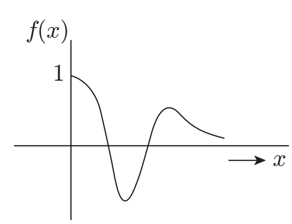
(a)



(b)



(c)



(d)

(GATE 2015, 2 Marks)

Solution: $f(1) = e^{-x}(x^2 + x + 1)$
$f(0) = 1$
$f(0.5) = 1.067$

For positive values of x, the function never goes negative.

Ans. (b)

168. The value of the integral $\int\limits_{-\infty}^{\infty} 12\cos(2\pi t)\dfrac{\sin(4\pi t)}{4\pi t}\,dt$ is ________.

(GATE 2015, 2 Marks)

Solution: The integral is given as,

$$\int\limits_{-\infty}^{\infty} 12\cos(2\pi t)\frac{\sin(4\pi t)}{4\pi t}\,dt$$

$$= \frac{12}{4\pi}\int\limits_{0}^{\infty}\frac{2\cos 2\pi t\sin 4\pi t}{t}\,dt$$

$$= \frac{3}{\pi}\left[\int\limits_{0}^{\infty}\frac{\sin 6\pi t\,dt}{t} + \int\limits_{0}^{\infty}\frac{\sin 2\pi t\,dt}{t}\right]$$

$[\because \sin A - \cos B = \sin(A+B) + \sin(A-B)]$

$$= \frac{3}{\pi}\left[\int\limits_{0}^{\infty} e^{0t}\,\frac{6\sin 6\pi t}{t}\,dt + \int\limits_{0}^{\infty} e^{0t}\,\frac{\sin 2\pi t}{t}\,dt\right]$$

$$= \frac{3}{\pi}\left[L\left\{\frac{\sin 6\pi t}{t}\right\} + L\left\{\frac{\sin 2\pi t}{t}\right\}\right] \text{with } s = 0$$

$$= \frac{3}{\pi}\left[\int\limits_{s}^{\infty}\frac{6\pi}{s^2+36\pi^2}\,ds + \int\limits_{s}^{\infty}\frac{2\pi}{s^2+4\pi^2}\,ds\right] \text{with } s = 0$$

$$= \frac{3}{\pi}\left[6\pi\cdot\frac{1}{6\pi}\tan^{-1}\left(\frac{s}{6\pi}\right) + 2\pi\cdot\frac{1}{\pi}\tan^{-1}\left(\frac{s}{2\pi}\right)\Bigg|_{0}^{\infty}\right] \text{with } s = 0$$

$$= \frac{3}{\pi}\left[\tan^{-1}\infty - \tan^{-1}\left(\frac{s}{6\pi}\right) + \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{2\pi}\right)\right]$$

$$= \frac{3}{\pi}\left[\frac{\pi}{2} - \tan^{-1}0 + \frac{\pi}{2} - \tan^{-1}0\right]$$

$$= \frac{3}{\pi}\left[\frac{\pi}{2} - 0 + \frac{\pi}{2} - 0\right] = \frac{3}{\pi}\times\pi = 3$$

Ans. 3

169. A vector field $D = 2\rho^2 a_\rho + z a_z$ exists inside a cylindrical region enclosed by the surfaces $\rho = 1$, $z = 0$ and $z = 5$. Let S be the surface bounding this cylindrical region. The surface integral of this field on $S\left(\oiint\limits_S D\cdot ds\right)$ is ________.

(GATE 2015, 2 Marks)

Solution: Vector field is given by

$$D = 2\rho^2 a\rho + z a_z$$

$$\oint\limits_S D\cdot ds = \int_V (\nabla\cdot D)\cdot dv$$

$$\nabla\cdot D = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot 2\rho^2) + 0 + 1$$

$$= \frac{1}{\rho}\cdot 2(3)\rho^2 + 1 = 6\rho + 1$$

$$\int_V (\nabla\cdot D)dv = \int\limits_{\rho=0}^{1}\int\limits_{\phi=0}^{2\pi}\int\limits_{z=0}^{5}(6\rho+1)\rho\,d\rho\,d\phi\,dz$$

$$= \left(\frac{6\rho^3}{3} + \frac{\rho^2}{2}\right)\Bigg|_0^1 (2\pi)(5) = \left(2 + \frac{1}{2}\right)10\pi$$

$$\int_V (\nabla\cdot D)dv = 78.53$$

Ans. 78.53

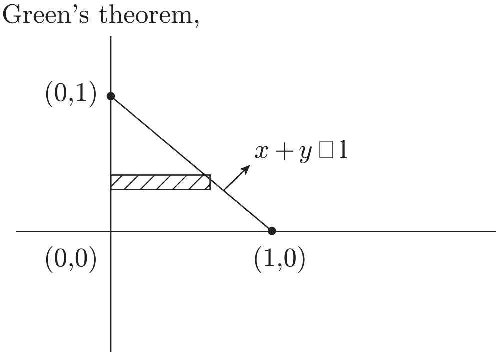
170. The value of $\int\limits_c [(3x - 8y^2)dx + (4y - 6xy)dy]$, (where C is the region bounded by $x = 0$, $y = 0$ and $x + y = 1$) is ________ .

(GATE 2015, 2 Marks)

Solution: The limits can be decided as

$$x = 0,\ x = 1 - y$$
$$y = 0,\ y = 1$$

By Green's theorem,



$$\int\limits_c (3x - 8y^2)dx + (4y - 6xy)dy$$

$$= \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$= \int\limits_{y=0}^{1}\int\limits_{x=0}^{1-y}[-6y - (-16y)]dxdy$$

$$= \int_{y=0}^{1}\left[\int_{x=0}^{1-y} 10y\,dx\right]dy = 10\int_{y=0}^{1} yx\Big|_0^{1-y}\,dy$$

$$= 10\int_{y=0}^{1} y[(1-y)-0]\,dy = 10\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Bigg|_0^1$$

$$= 10\left(\frac{1}{3}-\frac{1}{3}\right) = \frac{5}{3} = 1.66$$

Ans. 1.66

171. Consider a spatial curve in three-dimensional space given in parametric form by $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = \frac{2}{\pi}t$, $0 \le t \le \frac{\pi}{2}$. The length of the curve is ________.

(GATE 2015, 2 Marks)

Solution: The length of the curve

$$= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2 + \left(\frac{2}{\pi}\right)^2}\,dt$$

$$= \int_0^{\pi/2} \sqrt{(\sin^2 t) + (\cos^2 t) + \left(\frac{4}{\pi^2}\right)}\,dt$$

$$= \int_0^{\pi/2} \sqrt{1+\left(\frac{4}{\pi^2}\right)}\,dt = \int_0^{\pi/2} \sqrt{1+\left(\frac{4}{\pi^2}\right)}\cdot t\Big|_0^{\pi/2}$$

$$= \int_0^{\pi/2} \sqrt{1+\left(\frac{4}{\pi^2}\right)}\cdot\frac{\pi}{2}$$

$$= 1.8622$$

Ans. 1.8622

172. In a two-dimensional steady flow field, in a certain region of the x-y plane, the velocity component in the x-direction is given by $v_x = x^2$ and the density varies as $\rho = (1/x)$. Which of the following is a valid expression for the velocity component in the y-direction, v_y?

(a) $v_y = -x/y$ (b) $v_y = x/y$

(c) $v_y = -xy$ (d) $v_y = xy$

(GATE 2015, 2 Marks)

Solution:

$$\frac{\partial}{\partial x}(\rho y) + \frac{\partial}{\partial y}(\rho V) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}\left(\frac{1}{x}\cdot V\right)V = 0$$

$$\Rightarrow 1 + \frac{\partial}{\partial y}\left(\frac{V}{x}\right) = 0 \Rightarrow \frac{\partial}{\partial y}\left(\frac{V}{x}\right) = -1 \Rightarrow V = -xy$$

Ans. (c)

173. In a survey work, three independent angles X, Y and Z were observed with weights W_X, W_Y, W_Z, respectively. The weight of the sum of angles X, Y and Z is given by

(a) $\dfrac{1}{[(1/W_X)+(1/W_Y)+(1/W_Z)]}$

(b) $\left(\dfrac{1}{W_X}+\dfrac{1}{W_Y}+\dfrac{1}{W_Z}\right)$

(c) $W_X + W_Y + W_Z$

(d) $W_X^2 + W_Y^2 + W_Z^2$

(GATE 2015, 2 Marks)

Solution: The weight of the sum of angles X, Y and Z is given by

$$\frac{1}{[(1/W_X)+(1/W_Y)+(1/W_Z)]}$$

Ans. (a)

174. The acceleration–time relationship for a vehicle subjected to non-uniform acceleration is

$$\frac{dv}{dt} = (\alpha - \beta v_0)e^{-\beta t}$$

where v is the speed in m/s, t is the time in s, α and β are parameters, and v_0 is the initial speed in m/s. If the accelerating behavior of a vehicle, whose driver intends to overtake a slow moving vehicle ahead, is described as

$$\frac{dv}{dt} = (a - \beta v)$$

Considering $\alpha = 2$ m/s^2, $\beta = 0.05$ s^{-1} and $\frac{dv}{dt} = 1.3$ m/s^2 at $t = 3$ s, the distance (in m) travelled by the vehicle in 35 s is ____________.

(GATE 2015, 2 Marks)

Solution: We have,

$$\frac{dV}{dt} = (\alpha - \beta V_0)e^{-\beta t}$$

Integrating, we get

$$\int dV = \int (\alpha - \beta V_0)\times e^{-\beta t}\cdot dt = \frac{(\alpha - \beta V_0)e^{-\beta t}}{-\beta}$$

$t = 0,\ V = V_0$

$$\Rightarrow V_0 = \frac{(\alpha - \beta V_0)}{-\beta} + C$$

$$C = V_0 + \frac{\alpha - \beta V_0}{\beta} \Rightarrow C = \frac{\alpha}{\beta}$$

$$\Rightarrow V = \frac{\alpha - (\alpha\beta V_0)\times e^{-\beta t}}{\beta}$$

$$x = \frac{\alpha t_0}{\beta} + \frac{\alpha - \beta V_0}{\beta^2}(e^{-\beta t_0} - 1)$$

$$\left.\frac{dv}{dt}\right|_{t=3} = (\alpha - \beta V_0)e^{-3\beta} = 1.3$$

$$\Rightarrow \alpha - \beta V_0 = \frac{1.3}{e^{-3\beta}}$$

$$x = \frac{\alpha t_0}{\beta} + \frac{1.3}{\beta^2(e^{-3\beta})}(e^{-3\beta t_0} - 1)$$

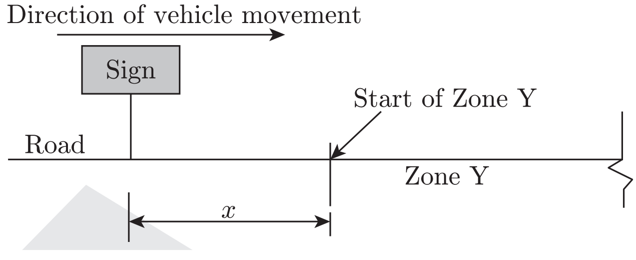
$$= \frac{2\times 35}{0.05} + \frac{1.3}{\beta^2(e^{-3\beta})}(e^{-\beta t_0} - 1)$$

$$= \frac{2\times 35}{0.05} + \frac{1.3(e^{-35\times 0.05} - 1)}{(0.05)^2(e^{-3\times 0.05})}$$

$$= 1400 - 499.17 = 900.83\,\text{m}$$

Ans. 900.83 m

175. A sign is required to be put up asking drivers to slow down to 30 km/h before entering zone Y (see figure). On this road, vehicles require 174 m to slow down to 30 km/h (the distance of 174 m includes the distance travelled during the perception – reaction time of drivers). The sign can be read by 6/6 vision drivers from a distance of 48 m. The sign is placed at a distance of x m from the start of zone Y so that even a 6/9 vision driver can slow down to 30 km/h before entering the zone. The minimum value of x is ________ m.



(GATE 2015, 2 Marks)

Solution: For a 6/6 vision person, driver can see from a distance of 48 m.
For a 6/9 vision person, driver can see from distance $= 48\times\frac{6}{9} = 32$ m
The vehicle requires 174 m to slow down to 30 km/h.
So, minimum distance $X = 174 - 32 = 142$ m

Ans. 142 m

176. The velocity components of a two-dimensional plane motion of a fluid are

$$u = \frac{y^3}{3} + 2x - x^2y \text{ and } v = xy^2 - 2y - \frac{x^3}{3}$$

The correct statement is

(a) Fluid is incompressible and flow is irrotational.
(b) Fluid is incompressible and flow is rotational.
(c) Fluid compressible and flow is irrotational.
(d) Fluid is compressible and flow is rotational.

(GATE 2015, 2 Marks)

Solution: For incompressible flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

For irrotational flow,

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \quad (2)$$

$$\frac{\partial}{\partial x}\left(\frac{y^3}{3} + 2x - x^2y\right) + \frac{\partial}{\partial y}\left(xy^2 - 2y - \frac{x^3}{3}\right) = 0$$
$$= 2 - 2xy + 2xy - 2 = 0$$

$$\frac{\partial}{\partial x}\left(xy^2 - 2y - \frac{x^3}{3}\right) - \frac{\partial}{\partial x}\left(\frac{y^3}{3} + 2x - x^2y\right)$$
$$= y^2 - x^2 - y^2 + x^2 = 0$$

Ans. (a)

177. A vector $u = -2y\hat{i} + 2x\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors in x and y directions, respectively. Evaluate the line integral

$$I = \oint_C u\cdot dr$$

where C is a closed loop formed by connecting points (1, 1), (3, 1), (3, 2) and (1, 2) in that order. The value of I is ________.

(GATE 2015, 2 Marks)

Solution: Given that

$$u = -2y\hat{i} + 2x\hat{j}$$
$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

So, $$\oint_c u \cdot dx = -\oint_c 2y \cdot dx + \oint_c 2x \cdot dx \quad (1)$$

Now, for $A \to B$, $y = 1$, $dy = 0$, x changes from 1 to 3.

$$(1) \Rightarrow \oint_c u \cdot dr = -\int_1^3 2 \times 1dy = -4$$

For $B \to C$, $x = 2x$, $dx = 0$, y changes from 1 to 2.

$$(1) \Rightarrow \oint_c u \cdot dr = \int_1^2 2 \times 3dy = 6$$

For $C \to D$, $y = 2$, $dy = 0$, x changes from 3 to 1.

$$(1) \Rightarrow \oint_c u \cdot dr = -\int_3^1 2 \times 2dx = 8$$

For $D \to A$, $x = 1$, $dx = 0$, y changes from 2 to 1.

$$(1) \Rightarrow \oint_c u \cdot dr = \int_2^1 2 \times 1dy = -2$$

For whole loop, $\oint_c u \cdot dr = -4 + 6 + 8 - 2 = 8$

Ans. 8

178. $\lim_{x\to 4} \dfrac{\sin(x-4)}{x-4} =$ ________.

(GATE 2016, 1 Mark)

Solution: We have

$$\lim_{x\to 4} \frac{\sin(x-4)}{x-4} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$\Rightarrow \lim_{x\to 4} \frac{\cos(x-4)}{1} \qquad \text{(using L-Hospital's rules)}$$

That is, $\cos 0 = 1$.

Ans. 1

179. Let $f(x)$ be a polynomial and $g(x) = f'(x)$ be its derivatives. If the degree of $(f(x)+f(-x))$ is 10, then the degree of $(g(x)-g(-x))$ is ________.

(GATE 2016, 1 Mark)

Solution:
Let $f(x)$ be a polynomial of degree n; therefore, $f'(x)$ is a polynomial of degree $(n-1)$.

Now, it is given that $f(x)+f(-x)$ is a polynomial of degree 10. Hence, $g(x)-g(-x)$ is a polynomial of degree 9.

If the degree of $(f(x)+f(x))=10$, then the largest even exponent of x in $f(x)$ is 10. No restriction on odd degree terms.

Now, the largest odd exponent of x in $g(x)$ is 9. Thus, the degree of $(g(x)-g(-x))$ is 9. The odd powers in f becomes even in derivative and $g(x)-g(-x)$ retains only odd powers.

Ans. 9

180. Given the following statements about a function f: $\mathbb{R}\to\mathbb{R}$, select the right option:

P: If $f(x)$ is continuous at $x=x_0$, then it is also differentiable at $x=x_0$.

Q: If $f(x)$ is continuous at $x=x_0$, then it may not be differentiable at $x=x_0$.

R: If $f(x)$ is differentiable at $x=x_0$, then it is also continuous at $x=x_0$.

(a) P is true, Q is false, R is false
(b) P is false, Q is true, R is true
(c) P is false, Q is true, R is false
(d) P is true, Q is false, R is true

(GATE 2016, 1 Mark)

Solution: If the function is differentiable, then it is continuous. However, vice versa is not true. If the function is continuous, then it may/may not be differentiable.

Ans. (b)

181. As x varies from -1 to $+3$, which one of the following describes the behaviour of the function $f(x)=x^3-3x^2+1$?

(a) $f(x)$ increases monotonically.
(b) $f(x)$ increases, then decreases and increases again.
(c) $f(x)$ decreases, then increases and decreases again.
(d) $f(x)$ increases and then decreases.

(GATE 2016, 1 Mark)

Solution: We have

$$f(x) = x^3 - 3x^2 + 1$$
$$\Rightarrow f'(x) = 3x^2 - 6x$$

For maxima or minima, we get

$$f'(x) = 0 \Rightarrow 3x^2 - 6x = 0$$

That is,

$$3x(x-2) = 0 \Rightarrow x = 0 \text{ or } x = 2$$

Now, $\quad f''(x) = 6x - 6$

- At $x=0$, $f''(0) = -6$ (i.e. maxima).
- At $x=2$, $f''(2) = 6$ (i.e. minima).

Therefore, $f(x)$ increases until $x=0$, then decreases until $x=2$, then increases once again.

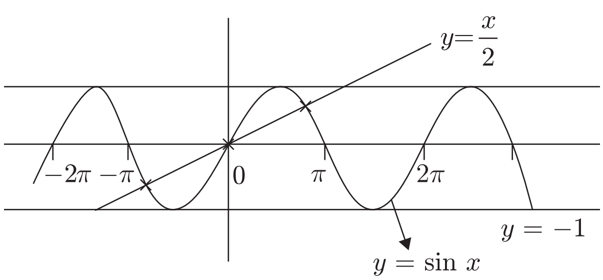
Ans. (b)

182. How many distinct values of x satisfy the equation $\sin(x)=x/2$, where x is in radians?

(a) 1 (b) 2
(c) 3 (d) 4 or more

(GATE 2016, 1 Mark)

Solution: Let $y=\sin x$ and $y=x/2$ be two curves. The solutions of $\sin x=x/2$ are the points intersected these two curves.



From the graph shown here, we see that the intersection is at three points.

Ans. (c)

183. Consider the time-varying vector $\vec{I} = \hat{x}\,15\cos(\omega t) + \hat{y}\,5\sin(\omega t)$ in Cartesian coordinates, where $\omega>0$ is a constant. When the vector magnitude $|\vec{I}|$ is at its minimum value, the angle θ that $\vec{I}$ makes with the x-axis (in degrees, such that $0\le\theta\le 180$) is __________.

(GATE 2016, 1 Mark)

Solution: We have

$$\vec{I} = \hat{x}\,15\cos(\omega t) + \hat{y}\,5\sin(\omega t)$$

Therefore,

$$\left|\vec{I}\right| = \sqrt{(15\cos\omega t)^2 + (5\sin\omega t)^2} = \sqrt{25 + 200\cos^2\omega t}$$

Here, $\left|\vec{I}\right|$ is minimum when $\cos(\omega t)=0$. Hence, $\vartheta=\omega t=90°$.

Ans. 90

184. The integral $\int_0^1 \frac{dx}{\sqrt{1-x}}$ is equal to __________.

(GATE 2016, 1 Mark)

Solution: It is given that

$$I = \int_0^1 \frac{dx}{\sqrt{1-x}}$$

Substituting $x=\sin^2\theta$, we get

$$dx = 2\sin\theta\cos\theta d\theta$$

Therefore,

$$I = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin 2\theta}}(2\sin\theta\cos\theta\,d\theta)$$
$$= \int_0^{\pi/2} \frac{1}{\cos\theta}(2\sin\theta\,\cos\theta\,\,d\theta)$$
$$= 2(-\cos\theta)_0^{\pi/2} = (-2)(0-1) = 2$$

Ans. 2

185. The maximum value attained by the function $f(x) = x(x-1)(x-2)$ in the interval $[1, 2]$ is ________.

(GATE 2016, 1 Mark)

Solution: Given that

$$f(x) = x(x-1)\,(x-2)$$
$$= x(x^2 - 3x + 2) = x^3 - 3x^2 + 2x$$

$$f'(x) = 3x^2 - 6x + 2$$

Therefore, there are no critical points in the given interval. Checking value at end points, that is, 1 and 2.

$$f(1) = 0$$
$$f(2) = 0$$

Hence, maximum value of $f(x)$ in the interval $[1, 2] = 0$.

Ans. 0

186. The value of the line integral

$$\int_C (2xy^2dx + 2x^2ydy + dz)$$

along a path joining the origin (0, 0, 0) and the point (1, 1, 1) is

(a) 0 (b) 2
(c) 4 (d) 6

(GATE 2016, 1 Mark)

Solution: $\vec{F} = xy^2\hat{i} + 2x^2y\hat{j} + \hat{k}$
As $\vec{\nabla} \times \vec{F} = 0$

$$F = \nabla\phi$$

Let $\phi_x = 2xy^2,\ \phi_y = 2xy^2,\ \phi_z = 1$
Therefore,

$$\phi = x^2y^2 + z + c$$

Hence,

$$\int_{(0,0,0)}^{(1,1,1)} \vec{F} \cdot di = \int_{(0,0,0)}^{(1,1,1)} d\phi = x^2y^2 + 2\Big|_{0,0,0}^{1,1,1} = 2$$

Ans. (b)

187. Let $f(x)$ be a real, periodic function satisfying $f(-x) = -f(x)$. The general form of its Fourier series representation would be

(a) $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx)$
(b) $f(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$
(c) $f(x) = a_0 + \sum_{k=1}^{\infty} a_{2k} \cos(kx)$
(d) $f(x) = \sum_{k=1}^{\infty} a_{2k+1} \sin(2k+1)x$

(GATE 2016, 1 Mark)

Solution: $f(-x) = -f(x)$ implies odd function. Odd functions have only sine terms in their Fourier expression. Therefore, correct option is (b).

Ans. (b)

188. The values of x for which the function

$$f(x) = \frac{x^2 - 3x - 4}{x^2 + 3x - 4}$$

is **NOT** continuous are

(a) 4 and –1 (b) 4 and 1
(c) –4 and 1 (d) –4 and –1

(GATE 2016, 1 Mark)

Solution: $f(x) = \dfrac{x^2 - 3x - 4}{x^2 + 3x - 4}$

The function will certainly be discontinuous where the denominator $x^2 + 3x - 4$ is zero. Therefore,

$$x^2 + 3x - 4 = 0$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm \sqrt{25}}{2} = \frac{-3 \pm 5}{2}$$

$$\Rightarrow x = 1, -4$$

Thus, the given function is not continuous at $x = -4$ and $x = 1$.

Ans. (c)

189. $\lim\limits_{x \to 0} \dfrac{\log_e(1+4x)}{e^{3x} - 1}$ is equal to

(a) 0 (b) $\dfrac{1}{12}$ (c) $\dfrac{4}{3}$ (d) 1

(GATE 2016, 1 Mark)

Solution: Substituting $x = 0$, we find the function takes a form $\dfrac{0}{0}$. Therefore, we can use L'Hospital's rule.

$$\lim_{x \to 0} \frac{\log_e(1+4x)}{e^{3x} - 1} = \lim_{x \to 0} \frac{4}{(1+4x)3e^{3x}}$$

$$= \frac{4}{(1+0)3e^0} = \frac{4}{3}$$

Ans. (c)

190. The optimum value of the function $f(x) = x^2 - 4x + 2$ is

(a) 2 (maximum) (b) 2 (minimum)
(c) –2 (maximum) (d) –2 (minimum)

(GATE 2016, 1 Mark)

Solution: Given $f(x) = x^2 - 4x + 2$

$$\frac{d}{dx} f(x) = 2x - 4$$

(a) For optimum value, $\dfrac{d}{dx} f(x) = 0$

$$2x - 4 = 0$$
$$x = +2$$

(b) $\dfrac{d^2}{dx^2} f(x) = +2 \rightarrow$ minima/minimum

At $x = +2$,

$$f(x) = (2)^2 - 4 \times 2 + 2 = -2$$

Ans. (d)

191. The Fourier series of the function,

$$f(x) = 0, \quad -\pi < x \le 0$$
$$= \pi - x, \quad 0 < x < \pi$$

in the interval $[-\pi, \pi]$ is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right]$$
$$+\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

The convergence of the above Fourier series at $x = 0$ gives

(a) $\sum_{n-1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (b) $\sum_{n-1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

(c) $\sum_{n-1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (d) $\sum_{n-1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$

(GATE 2016, 1 Mark)

Solution: The convergence will be $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Putting $x = 0$, we get

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Ans. (c)

192. What is the value of $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2 + y^2}$?

(a) 1 (b) −1
(c) 0 (d) Limit does not exist

(GATE 2016, 1 Mark)

Solution: $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2 + y^2}$

Dividing the above expression by xy, we get

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{1}{\left(\frac{x}{y}\right) + \left(\frac{y}{x}\right)}$$

Let $\frac{x}{y} = t$

then $\lim_{t \to \frac{0}{0}} \frac{1}{t + \frac{1}{t}}$

or $\lim_{t \to \frac{0}{0}} \frac{t}{t^2 + 1}$

Here $\frac{0}{0}$ is undefined. Hence, limit not exist.

Ans. (d)

193. The integral $\frac{1}{2\pi}\iint_D (x + y + 10)dx\,dy$, where D denotes the disc: $x^2+y^2 \le 4$, evaluates to __________.

(GATE 2016, 2 Marks)

Solution: It is given that

$$I = \frac{1}{2\pi}\iint (x+y+10)dx\,dy$$

and $x^2+y^2 \le 4$

Substituting $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rdrd\theta$, we get

$$I = \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{2}(r\cos\theta + r\sin\theta + 10)\,rdrd\theta$$
$$= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\left[\left(\frac{r^3}{3}\right)\cos\theta + \left(\frac{r^3}{3}\right)\sin\theta + 5r^2\right]_0^2 d\theta$$
$$= \frac{1}{2\pi}\left\{\int_{\theta=0}^{2\pi}\left[\left(\frac{8}{3}\right)\cos\theta + \left(\frac{8}{3}\right)\sin\theta + 20\right]d\theta\right\}$$
$$= \frac{1}{2\pi[0+0+20(2\pi)]} = 20$$

Ans. 20

194. In the following integral, the contour C encloses the points $2\pi j$ and $-2\pi j$: $-\frac{1}{2\pi}\oint_C \frac{\sin z}{(z-2\pi j)^3}dz$.

The value of the integral is __________.

(GATE 2016, 2 Marks)

Solution: We have

$$-\frac{1}{2\pi}\oint_C \frac{\sin z}{(z-2\pi j)^3}dz$$

$\oint[\sin z/(z-2\pi j)^3]$, $dz = 2\pi i$ (sum of residues)

Pole is at $z = 2\pi j$.

Therefore,

$$[\text{Residue}] = \left[\left(\frac{1}{2!}\right)\times\left(\frac{d^2}{dz^2}\right)\right](\sin z)\Big|_{z\to 2\pi j}$$
$$= \frac{1}{2}\left[\frac{d}{dz}(\cos z)\right]$$
$$= -\frac{1}{2}\sin z = -\left(\frac{1}{2}\right)\sin 2\pi i = +\left(\frac{1}{2}\right)\times i\sin h(2\pi)$$
$$= +\left(\frac{i}{2}\right)\left(\frac{e^{2\pi} - e^{-2\pi}}{2}\right) = +i(133.87)$$

Since $\sin i\theta = -i\sin h\theta$ and $\sin hi\theta = -i\sin\theta$, we get

$$-\frac{1}{2\pi}\oint_C \frac{\sin z}{(z-2\pi j)^3}dz$$
$$= -\frac{1}{2\pi}\times 2\pi i\times(+133.87)i$$
$$= -133.87$$

Ans. –133.87

195. The region specified by $\left\{(\rho,\varphi,z): 3\le\rho\le 5, \frac{\pi}{8}\le \varphi\le\frac{\pi}{4}, 3\le z\le 4.5\right\}$ in cylindrical coordinates has volume of ________.

(GATE 2016, 2 Marks)

Solution:
The region of the cylinder specified is

$$3\le\rho\le 5; \frac{\pi}{8}\le\varphi\le\frac{\pi}{4}; 3\le z\le 4.5$$

Now, we can write the differential volume of the cylinder as

$$dV = \rho d\rho\ \rho d\phi dz$$

Therefore, the volume is

$$V = \int_{\rho=3}^{5}\int_{\varphi=\pi/8}^{\pi/4}\int_{z=3}^{4.5}\rho d\rho d\varphi\, dz$$
$$= \left.\frac{\rho^2}{2}\right|_3^5 \times \varphi\Big|_{\pi/8}^{\pi/4}\times z\Big|_3^{4.5}$$
$$= \frac{1}{2}(25-9)\times\left(\frac{\pi}{4}-\frac{\pi}{8}\right)\times(4.5-3)$$
$$= 4.71\,\text{m}^3$$

Ans. 4.71 m^3

196. Suppose C is the closed curve defined as the circle $x^2+y^2=1$ with C oriented anticlockwise. The value of $\oint(xy^2dx + x^2ydy)$ over the curve C equals ________.

(GATE 2016, 2 Marks)

Solution: By Green's theorem, we have

$$\int xy^2dx + x^2y\ dy = \int\int\left[\frac{d}{dx}(x^2y) - \frac{d}{dy}(xy^2)\right]dx\ dy$$
$$= \int\int[2xy - 2xy]dx\ dy = 0$$

Ans. 0

197. If the vectors $e_1=(1, 0, 2)$, $e_2=(0, 1, 0)$ and $e_3=(-2, 0, 1)$ form an orthogonal basis of the three-dimensional real space $\mathbb{R}^3$, then the vector $u=(4, 3, -3)\in\mathbb{R}^3$ can be expressed as

(a) $u=-\frac{2}{5}e_1-3e_2-\frac{11}{5}e_3$ (b) $u=-\frac{2}{5}e_1-3e_2+\frac{11}{5}e_3$

(c) $u=-\frac{2}{5}e_1+3e_2+\frac{11}{5}e_3$ (d) $u=-\frac{2}{5}e_1+3e_2-\frac{11}{5}e_3$

(GATE 2016, 2 Marks)

Solution: It is given that

$$e_1 = (1,\ 0,\ 2) = i+2k$$
$$e_2 = (0,\ 1,\ 0) = j$$
$$e_3 = (-2,\ 0,\ 1) = -2i+k$$
$$u = (4,\ 3, -3) = 4i+3j-3k \quad (1)$$

Let us consider

$$u = (-25e_1 + 3e_2 - 115e_3)$$
$$= -25(i+2k) + 3(j) - 115(-2i+k)$$
$$= (-25i+225i) + 3j + k(-45-115)$$
$$= 4i+3j-3k \quad (2)$$

As we see, Eqs. (1) and (2) are same. Therefore, option (d) is correct.

Ans. (d)

198. A triangle in the xy-plane is bounded by the straight lines $2x=3y$, $y=0$ and $x=3$. The volume above the triangle and under the plane $x+y+z=6$ is ________.

(GATE 2016, 2 Marks)

Solution: It is given that

$$x\text{-limits}: 0 \text{ to } 3$$
$$y\text{-limits}: 0 \text{ to} \left(\frac{2}{3}\right)x$$

The volume is

$$V = \iint_R z\,dx\,dy$$
$$= \int_{x=0}^{3}\int_{y=0}^{(2/3)x}(6-x-y)\,dy\,dx$$
$$= \int_{\infty}^{3}\left[(6-x)\frac{2x}{3} - \frac{1}{2}\left(\frac{4}{9}x^2\right)\right]dx$$

$$= \int_{\infty}^{3}\left(4x - \frac{2}{3}x^2 - \frac{2}{9}x^2\right)dx$$

$$= \left[2x^2 - \frac{8}{9}\left(\frac{x^3}{3}\right)\right]_0^3 = 2(9) - \frac{8}{27}(3^3) = 10 \text{ cubic unit}$$

Ans. 10 cubic unit

199. Let $S = \sum_{n=0}^{\infty} n\alpha^n$ where $|\alpha| < 1$. The value of α in the range $0 < \alpha < 1$, such that $S = 2\alpha$ is ________.

(GATE 2016, 2 Marks)

Solution: The standard solution for

$$\sum_{n=0}^{\infty} n\alpha^n = \frac{\alpha}{(1-\alpha)^2}$$

Given that, $S = 2\alpha$, therefore,

$$2\alpha = \frac{\alpha}{1-\alpha^2}$$

$$2(1-\alpha^2) = 1 \Rightarrow \alpha = 0.29$$

Ans. 0.29

200. Given the following polynomial equation:

$$s^3 + 5.5s^2 + 8.5s + 3 = 0$$

the number of roots of the polynomial, which have real parts strictly less than -1, is ________.

(GATE 2016, 2 Marks)

Solution: For the given polynomial equation,
$s^3 + 5.5s^2 + 8.5 + 3 = 0$

Let $s = (r - 1)$

$\Rightarrow (r-1)^3 + (5.5)(r-1)^2 + 8.5(r-1) + 3 = 0$

$\Rightarrow r^3 + 2.5r^2 + 0.5r - 1 = 1$

r^3	1	0.5
r^2	2.5	-1
r_1	0.9	0
r^0	-1	

One sign change implies two roots of the polynomial lie to the left of $s = -1$.

Ans. 2

201. The value of the integral $2\int_{-\infty}^{\infty}\left(\frac{\sin 2\pi t}{\pi t}\right)dt$ is equal to

(a) 0 (b) 0.5

(c) 1 (d) 2

(GATE 2016, 2 Marks)

Solution: We know that $\frac{\sin 2\pi t}{\pi t}$ is an even function. Therefore,

$$2\int_{-\infty}^{\infty} \frac{\sin(2\pi t)}{\pi t}dt = 2 \times 2\int_{0}^{\infty} \frac{\sin(2\pi t)}{\pi t}dt$$

$$\int_{0}^{\infty} \frac{\sin(2\pi t)}{\pi t}dt = \frac{1}{2}$$

Therefore,

$$4\int_{-\infty}^{\infty} \frac{\sin(2\pi t)}{\pi t} = 4 \times \frac{1}{2} = 2$$

Ans. (d)

202. The line integral of the vector field $F = 5xz\hat{i} + (3x^2 + 2y)\hat{j} + x^2z\hat{k}$ along a path from (0, 0, 0) to (1, 1, 1) parametrised by (t, t^2, t) is ________.

(GATE 2016, 2 Marks)

Solution: Given that

$$E = 5xz\hat{i} + (3x^2 + 2y)\hat{j} + x^2z\hat{k}$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 5t^2dt + (3t^2 + 2t^2)2t\ dt + t^3dt$$

$$= 4.41$$

Ans. 4.41

203. Consider the function $(x) = 2x^3 - 3x^2$ in the domain $[-1, 2]$. The global minimum of (x) is ______.

(GATE 2016, 2 Marks)

Solution: Given that

$$f(x) = 2x^3 - 3x^2$$

in the domain $[-1, 2]$. At boundaries,

$$f(-1) = -5$$

$$f(2) = 4$$

For maxima or minima,

$$f'(x) = 0$$

$$\Rightarrow 6x^2 - 6x = 0$$

$$\Rightarrow x = 0, 1$$

Now,

$$f''(x) = 12x - 6$$

$$\Rightarrow f''(0) = -6 \text{ (negative)}$$

$$\Rightarrow f''(1) = 6 \text{ (positive)}$$

Thus, $f(x)$ maxima of 0 at $x = 0$ and minima −1 at $x = 1$. Thus, global minima in the given domain is −5, as already checked.

Ans. −5

204. The value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2} dx$$

evaluated using contour integration and the residue theorem is

(a) $-\pi \sin(1)/e$ (b) $-\pi \cos(1)/e$
(c) $\sin(1)/e$ (d) $\cos(1)/e$

(GATE 2016, 2 Marks)

Solution: Let

$$f(z) = \frac{e^{tz}}{z^2 + 2z + 2}$$

Poles of $f(z)$ are

$$z = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2}$$
$$= -1 \pm i$$

Now,

$$\text{Res } f(z) = [z - (i-1)] \times \frac{e^{t(t-1)}}{[z - (i-1) \times [z + (i+1)]]}$$

$$= \lim_{z \to t-1} \frac{e^{-1-t}}{z + (i+1)}$$

$$= \frac{e^{-t-1}}{2i}$$

Now,

$$\int \frac{e^{tz}}{z^2 + 2s + 2} dz = 2\pi i \left(\frac{e^{-t-1}}{2i} \right)$$

$$= (e^{-t-1})$$
$$= \pi e^{-t} e^{-1}$$

$$\int \frac{\cos z + i \sin z}{x^2 + 2z + 2} dz = \pi[\cos(1) - i \sin(1)]e^{-1}$$

Comparing the imaginary parts, one finds

$$\int \frac{\sin z}{z^2 + 2z + 2} = \frac{-\pi \sin(1)}{e}$$

Ans. (a)

205. A scalar potential φ has the following gradient:

$$\nabla \varphi = yz\hat{i} + xz\hat{j} + xy\hat{k}.$$

Consider the integral $\int_C \nabla \varphi \cdot d\vec{r}$ on the curve $\vec{r} = x\hat{i} + yj + z\hat{k}$. The curve C is parameterized as follows:

$$\begin{cases} x = t \\ y = t^2 \text{ and } 1 \le t \le 3. \\ z = 3t^2 \end{cases}$$

The value of the integral is ______.

(GATE 2016, 2 Marks)

Solution: Given that

$$\nabla \phi = yzi + xz\hat{j} + xy\hat{k}$$

Thus,

$$\int_c \nabla \phi \cdot d\vec{r} = \int (yz\hat{i} + xz\hat{j} + xy\hat{k}] \cdot [dx\hat{i} + dy\hat{j} + dz\hat{k}]$$

$$= \int (yzdx + xzdy + xy + dz)$$

$$= \int_1^3 [t^2(3t)^2 dt + t(3t^2)2t\, dt + t(t^2)6t\, dt]$$

$$= \int_1^3 [3t^4 + 6t^4 + 6t^4]dt = 15\left[\frac{t^5}{5}\right]_1^3$$

$$= \frac{15}{5} \times (3^5 - 1^5) = 726$$

Ans. 726

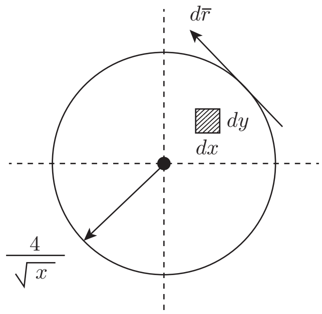
206. The value of the line integral $\oint_C \bar{F} \cdot \bar{r}' ds$, where C is a circle of radius $\frac{4}{\sqrt{\pi}}$ units is ______.

Here, $\bar{F}(x, y) = y\hat{i} + 2x\hat{j}$ and $\bar{r}'$ is the unit tangent vector on the curve C at an arc length s from a reference point on the curve. $\hat{i}$ and $\hat{j}$ are the basis vectors in the x-y Cartesian reference. In evaluating the line integral, the curve has to be traversed in the counter-clockwise direction.

(GATE 2016, 2 Marks)

Solution: $\overrightarrow{dr} = \vec{r}.ds$

$$\oint_c \vec{F}\cdot dr = \oint (y\hat{i} + 2x\hat{j})\cdot \overrightarrow{dr}$$



Applying Green's theorem, we get

$$\oint_c (y\hat{i} + 2x\hat{j}\cdot \overrightarrow{dr} = \int_A \left[\frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(y)\right] dxdy = \int_A dx\ dy$$

$$= \left\{\text{Area of the circle of radius } \frac{4}{\sqrt{x}} \text{ units}\right\}$$

$$= x\left(\frac{4}{\sqrt{x}}\right)^2 = 16$$

Ans. 16

207. $\lim_{x\to\infty} \sqrt{x^2 + x - 1 - x}$ **is**

(a) 0 (b) ∞ (c) 1/2 (d) $-\infty$

(GATE 2016, 2 Marks)

Solution: Let $x = 1/t$
then

$$\lim_{x\to\infty} \sqrt{x^2 + x - 1} - x = \lim_{t\to 0} \sqrt{\frac{1}{t^2} + \frac{1}{t} - 1} - \left(\frac{1}{t}\right)$$

$$= \lim_{t\to 0} \frac{\sqrt{1 + t - t^2} - 1}{t}$$

Since the function has $\frac{0}{0}$ form now, we can apply L'Hospital's rule,

$$\lim_{t\to 0} \frac{\sqrt{1 + t - t^2} - 1}{t} = \lim_{t\to 0} \left\{\left[\frac{1\times(1-2t)}{2\sqrt{1 + t - t^2}} - 0\right] / (1)\right\}$$

Applying limit now,

$$\lim_{x\to\infty} \sqrt{x^2 + x - 1} - x = \frac{1}{2}$$

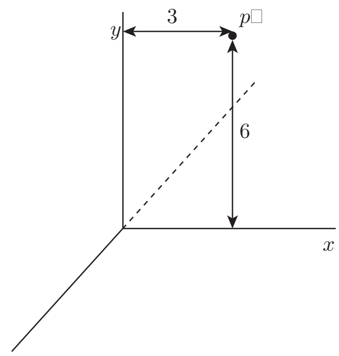
Ans. (c)

208. A point P(1, 3, –5) is translated by $2\hat{i} + 3\hat{j} - 4\hat{k}$ and then rotated counter clockwise by 90° about the z-axis. The new position of the point is

(a) (–6, 3, –9) (b) (–6, –3, –9)
(c) (6, 3, –9) (d) (6, 3, 9)

(GATE 2016, 2 Marks)

Solution:



As point $P(1, 3, -5)$ is translated by $2\hat{i} + 3\hat{j} - 4\hat{k}$, the new position P'' will be (1 + 2, 3 + 3, –5 – 4) ≡ (3, 6, –9).

Now OP' is rotated counter clockwise by 90° about the z axis. Let the new position P'' be (x, y, z).

Since P'' has been obtained by rotating P'' about z-axis, its z-coordinate will not change. Therefore,

$$P'' = (x, y, -9)$$

But vectors $3i + 6j$ and $x\hat{i} + y\hat{j}$ must be orthogonal,

$$(3\hat{i} + 6\hat{j})\cdot(x\hat{i} + y\hat{j}) - |A||B|\cos 90° = 0$$

$$\Rightarrow 3x + 6y = 0$$

Only option (a) satisfies the above equation.

Ans. (a)

209. The value of $\int_0^\infty \frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}dx$ is

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 1

(GATE 2016, 2 Marks)

Solution: $\int_0^\infty \frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}\cdot dx = [\tan^{-1} x]_0^\infty + \frac{\pi}{2}$

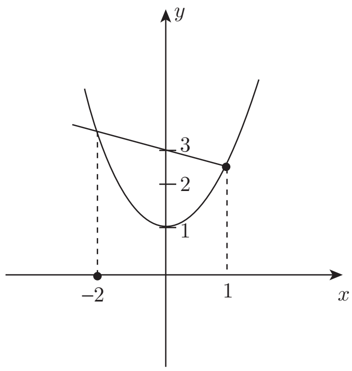
$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

Ans. (b)

210. The area of the region bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ is

(a) $\frac{59}{6}$ (b) $\frac{9}{2}$ (c) $\frac{10}{3}$ (d) $\frac{7}{6}$



(GATE 2016, 2 Marks)

Solution: Parabola, $y = x^2 + 1$ (1)

at $x = 0$, $y = 1$

Straight line,

$$x + y = 3 \Rightarrow y = 3 - x \quad (2)$$

at $x = 0$, $y = 3$

Equating Equations (1) and (2), we get

$$x^2 + 1 = 3 - x$$
$$x^2 + x - 2 = 0$$
$$(x + 2)\ (x - 1) = 0$$
$$x = -2,\ 1$$

Now area between the given curves,

$$A = \int_{-2}^{1} (3 - x)dx - \int_{-2}^{1} (x^2 + 1)dx$$

$$A = \int_{-2}^{1} (3 - x - x^2 - 1)dx$$

$$A = \left[3x - \frac{x^2}{2} - \frac{x^3}{3} - x\right]_{-2}^{1} = \frac{9}{2}$$

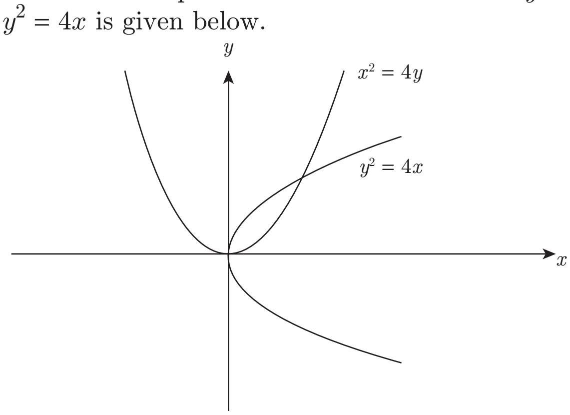
Ans. (b)

211. The angle of intersection of the curves $x^2 = 4y$ and $y^2 = 4x$ at point (0, 0) is

(a) 0° (b) 30° (c) 45° (d) 90°

(GATE 2016, 2 Marks)

Solution: Graphical view of curves $x^2 = 4y$ and $y^2 = 4x$ is given below.



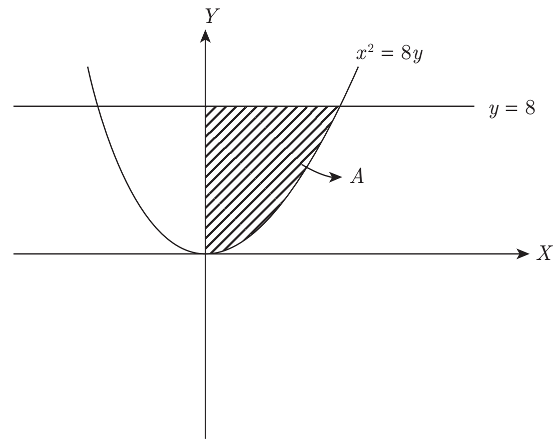
At point (0, 0), both curves intersect at 90° angle. Hence, intersection angle is 90°.

Ans. (d)

212. The area between the parabola $x^2 = 8y$ and the straight line $y = 8$ is ______.

(GATE 2016, 2 Marks)

Solution:



Curve, $x^2 = 8y$

$$\Rightarrow x = \sqrt{8y}$$

Therefore,

$$f(y) = \sqrt{8y}$$

Area under the curve ($y = 8$) is

$$A = \int_{0}^{8} f(y) \cdot dy$$

$$A = \int_{0}^{8} \sqrt{8y}\ dy$$

$$A = \sqrt{8}\left[\frac{2}{3}y^{3/2}\right]_{0}^{8}$$

$$A = \sqrt{8} \times \frac{2}{3} \times 8^{3/2}$$

$$A = 42.67 \text{ units}$$

Therefore, total area = $2A$

$$= 2 \times 42.67 = 85.33 \text{ units}$$

Ans. 85.33

213. The quadratic approximation of $f(x) = x^3 - 3x^5 - 5$ at the point $x = 0$ is

(a) $3x^2 - 6x - 5$ (b) $-3x^2 - 5$
(c) $-3x^2 + 6x - 5$ (d) $3x^2 - 5$

(GATE 2016, 2 Marks)

Solution: $f(x) = x^3 - 3x^2 - 5$, $f'(x) = 3x^2 - 6x$

$$f''(x) = 6x - 6$$

At $x = 0$, we have

$$f(0) = -5,\ f'(0) = 0,\ f''(0) = -6,\ f'''(0) \Rightarrow \text{NA}$$

Now approximation of $f(x)$ is given as (at $x = a$)

$$f(x) = f(a) + f'(a)\frac{x-a}{L_1} + f''(a)\frac{(x-a)^2}{L^2} + \cdots$$

At $x = 0$, we have

$$f(x) = -5 + 0 \times \frac{(x-0)}{L_1} + (-6)\frac{(x-0)^2}{2}$$
$$= -5 + 0 - 3x^2$$
$$= -3x^2 - 5$$

Ans. (b)

214. Let $c_1, \ldots, c_n$ be scalars, not all zero, such that $\sum_{i=1}^{n} c_i\alpha_i = 0$, where α_i are column vectors in R^n.

Consider the set of linear equations $Ax = b$, where $A = [a_1 \ldots a_n]$ and $b = \sum_{i=1}^{n}\alpha_i$. The set of equations has

(a) a unique solution at $x = J_n$, where J_n denotes an n-dimensional vector of all 1

(b) no solution

(c) infinitely many solutions

(d) finitely many solutions

(GATE 2017, 1 Mark)

Solution: For the system, $Ax = b$, the vectors a_1, $a_2, \ldots, a_n$ are linearly dependent.

Therefore, rank $(A) < n$.

Hence, the system has infinitely many solutions.

Ans. (c)

215. If $f(x) = R\sin\left(\frac{\pi x}{2}\right) + S$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x)dx = \frac{2R}{\pi}$, then the constants R and S are, respectively,

(a) $\frac{2}{\pi}$ and $\frac{16}{\pi}$ (b) $\frac{2}{\pi}$ and 0

(c) $\frac{4}{\pi}$ and 0 (d) $\frac{4}{\pi}$ and $\frac{16}{\pi}$

(GATE 2017, 1 Mark)

Solution:

$$f(x) = R\sin\left(\frac{\pi x}{2}\right) + S \tag{1}$$

$$\Rightarrow f'(x) = R\cos\left(\frac{\pi x}{2}\right)\frac{\pi}{2}$$

$$\Rightarrow f'\left(\frac{1}{2}\right) = R\cos\left(\frac{\pi}{4}\right)\frac{\pi}{2} \tag{2}$$

As given,

$$f'\left(\frac{1}{2}\right) = \sqrt{2} \tag{3}$$

From Eqs. (2) and (3), we get

$$\frac{R}{\sqrt{2}} \times \frac{\pi}{2} = \sqrt{2}$$

$$\Rightarrow \quad R = \frac{4}{\pi} \tag{4}$$

Now, integrating Eq. (1), we get

$$\int f(x)dx = \int R\sin\left(\frac{\pi x}{2}\right) + S \tag{5}$$

Substituting Eq. (4) in Eq. (5), we get

$$\int f(x)dx = \int \frac{4}{\pi}\sin\left(\frac{\pi x}{2}\right) + \int S\,dx$$

$$= \frac{4}{\pi} \times \frac{-\cos\frac{\pi x}{2}}{\frac{\pi}{2}} + Sx$$

$$= \frac{-8}{\pi^2}\cos\frac{\pi x}{2} + Sx$$

Taking limits from 0 to 1, we get

$$\int_0^1 f(x)dx = \frac{-8}{\pi^2}\left(\cos\frac{\pi}{2} - \cos(0)\right) + S(1-0) \tag{6}$$

As given

$$\int_0^1 f(x)dx = \frac{2R}{\pi}$$

Also

$$R = \frac{4}{\pi}$$

Hence, from Eq. (6), solving for S, we get

$$\frac{-8}{\pi^2} + S = \frac{2R}{\pi}$$

$$\Rightarrow S = 0$$

Ans. (c)

216. Consider a quadratic equation $x^2 - 13x + 36 = 0$ with coefficients in a base b. The solutions of this equation in the same base b are $x = 5$ and $x = 6$. Then $b =$ ____________.

(GATE 2017, 1 Mark)

Solution:

$$x^2 - 13x + 36 = 0$$

The solutions of this equation are 4 and 9.

Now, 13 in base 10 can be represented as

$$1 \times 10^1 + 3 \times 10^0$$

and 36 in base 10 as

$$3 \times 10^1 + 6 \times 10^0$$

Therefore, 13 in base b can be represented as

$$1 \times b^1 + 3 \times b^0 = b + 3$$

and 36 as

$$3 \times b^1 + 6 \times b^0 = 3b + 6$$

As the quadratic equation can be represented as

$$x^2 - (\text{Sum of roots})x + \text{Product of roots}$$

Now, the equation will be

$$x^2 - (b + 3)x + (3b + 6) = 0$$

As it is given, $x = 6$ is a solution of this equation in same base b. Therefore,

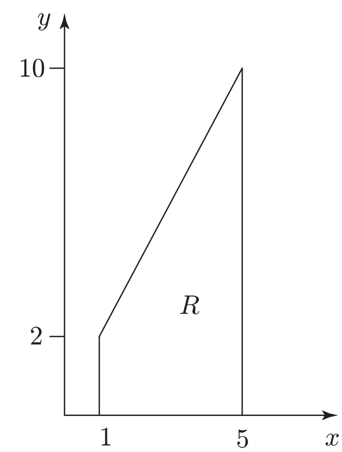
$$6^2 - (b+3)6 + (3b+6) = 0$$

$$\Rightarrow -3b + 24 = 0$$

$$\Rightarrow b = 8$$

Ans. (8.0)

217. Let $I = c\int\int_R xy^2 dxdy$, where R is the region shown in the figure and $c = 6 \cdot 10^{-4}$. The value of I equals _________. (Give the answer up to two decimal places.)



(GATE 2017, 1 Mark)

Solution: $I = c\iint xy^2 dxdy$

$$= c\int_{x=1}^{5}\int_{y=0}^{2x} xy^2 dydx$$

$$= c\int_1^5 \left.\frac{xy^3}{3}\right|_0^{2x} dx = c\int_1^5 \frac{8}{3}x^4 dx$$

$$= \frac{8c}{3}\left.\frac{x^5}{5}\right|_1^5 = \frac{8c}{3}(625 - 0.2)$$

$$= \frac{8}{3}(6 \times 10^{-4})(625 - 0.2) = 0.99968$$

Ans. (0.99)

218. Let x and y be integers satisfying the following equations.

$$2x^2 + y^2 = 34$$
$$x + 2y = 11$$

The value of $(x + y)$ is _______.

(GATE 2017, 1 Mark)

Solution:

$$2x^2 + y^2 = 34 \quad (1)$$
$$x + 2y = 11 \quad (2)$$

From Eq. (2), we have

$$x + 2y = 11 \Rightarrow x = 11 - 2y \quad (3)$$

Substituting the value of x in Eq. (1), we have

$$2(11 - 2y)^2 + y^2 = 34$$
$$\Rightarrow 2(121 + 4y^2 - 44y) + y^2 = 34$$
$$\Rightarrow 242 + 8y^2 - 88y + y^2 = 34$$

$$\Rightarrow 9y^2 - 88y + 208 = 0$$
$$\Rightarrow y = 5.77 \text{ or } 4$$

From Eq. (3), we have

$$x = 3 \text{ at } y = 4$$

Therefore,

$$x + y = 3 + 4 = 7$$

Ans. (7)

219. Let $y^2 - 2y + 1 = x$ and $\sqrt{x} + y = 5$. The value of $x + \sqrt{y}$ equals _______. (Give the answer up to three decimal places.)

(GATE 2017, 1 Mark)

Solution:

$$y^2 - 2y + 1 = x \quad (1)$$

and

$$\sqrt{x} + y = 5 \quad (2)$$

From Eq. (1),

$$y^2 - 2y + 1 = x$$
$$\Rightarrow (y-1)^2 = x$$
$$\Rightarrow y - 1 = \sqrt{x} \quad (3)$$

Put this value in Eq. (2),

$$y - 1 + y = 5$$
$$\Rightarrow y = 3$$

From Eq. (3),

$$3 - 1 = \sqrt{x}$$
$$\Rightarrow x = 4$$

Therefore,

$$x + \sqrt{y} = 4 + \sqrt{3} = 4 + 1.732 = 5.732$$

Ans. (5.732)

220. The value of $\lim_{x\to 0} \dfrac{x^3 - \sin x}{x}$ is

(a) 0 (b) 3 (c) 1 (d) –1

(GATE 2017, 1 Mark)

Solution:

$$\lim_{x\to 0}\frac{x^3 - \sin x}{x} = \lim_{x\to 0}\left(x^2 - \frac{\sin x}{x}\right)$$
$$= 0 - 1$$
$$= -1$$

Ans. (d)

221. The divergence of the vector $-y\hat{i} + x\hat{j}$ is _______.

(GATE 2017, 1 Mark)

Solution: Let $\vec{F} = -y\hat{i} + x\hat{j}$. Thus,

$$\text{div}(\vec{F}) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)$$
$$= 0$$

Ans. 0

222. Let x be a continuous variable defined over the interval $(-\infty, \infty)$ and $f(x) = e^{-x-e^{-x}}$. The integral $g(x) = \int f(x)dx$ is equal to

(a) $e^{e^{-x}}$ (b) $e^{-e^{-x}}$
(c) $e^{-e^{x}}$ (d) e^{-x}

(GATE 2017, 1 Mark)

Solution:

$$f(x) = e^{-x-e^{-x}} \quad (1)$$
$$g(x) = \int f(x)\, dx \quad (2)$$

Using Eq. (1) in Eq. (2), we get

$$g(x) = \int \frac{e^{-x}}{e^{e^{-x}}}\, dx \quad (3)$$

Let $e^{-x} = t$. So $-e^{-x}dx = dt$. Therefore, Eq. (3) can be written as

$$g(x) = \int -\frac{dt}{e^t} = \int -e^{-t}dt$$
$$\Rightarrow g(x) = e^{-t}$$
$$\Rightarrow g(x) = e^{-e^{-x}} \quad (\text{Since } e^{-x} = t)$$

Ans. (b)

223. $\lim_{x\to 0}\left(\dfrac{\tan x}{x^2 - x}\right)$ is equal to ___________.

(GATE 2017, 1 Mark)

Solution: $\lim_{x\to 0}\dfrac{\tan x}{x^2 - x}$

From L'Hospital's rule, we have

$$\operatorname*{lt}_{x\to 0}\frac{\sec^2 x}{2x^2 - 1} = -1$$

At $x = 0$, we have

$$\frac{\sec^2 0}{0 - 1} = -1$$

Ans. (–1)

224. The divergence of the vector field $V = x^2\hat{i} + 2y^3\hat{j} + z^4\hat{k}$ at $x = 1, y = 2, z = 3$ is ____________.

(GATE 2017, 1 Mark)

Solution:

$$V = x^2\hat{i} + 2y^3\hat{j} + z^4\hat{k}$$

$$\text{Divergence }(V) = \frac{\partial}{\partial x}(x^2)\hat{i} + \frac{\partial}{\partial y}(2y^3)\hat{j} + \frac{\partial}{\partial z}(z^4)\hat{k}$$

$$= 2x + 6y^2 + 4z^3$$

Therefore,

$$\left[\text{Divergence }(V)\right]_{(x=1,y=2,z=3)} = 2\times 1 + 6\times 2^2 + 4\times 3^3$$

$$= 134$$

Ans. (134)

225. The value of $\lim\limits_{x\to 1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$

(a) is 0
(b) is –1
(c) is 1
(d) does not exist

(GATE 2017, 2 Marks)

Solution:

$$\lim_{x\to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \frac{1-2+1}{1-3+2} = \frac{0}{0}$$

Therefore, applying L'Hospital's rule, we have

$$\lim_{x\to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \lim_{x\to 1} \frac{7x^6 - 10x^4}{3x^2 - 6x} = \frac{7-10}{3-6} = 1$$

Ans. (c)

226. Let u and v be two vectors in R^2 whose Euclidean norms satisfy $\|u\| = 2\,\|v\|$. What is the value of α such that $w = u + \alpha v$ bisects the angle between u and v?

(a) 2
(b) 1/2
(c) 1
(d) –1/2

(GATE 2017, 2 Marks)

Solution: To find α such that $w = u + \alpha v$ bisects the angle between u and v, we can choose two vectors of equal length, i.e.

$$\|u\| = \|v\|$$

Then $u + v$ bisects the angle between u and v.

Given that $\|u\| = 2\,\|v\|$, so u is twice as long as v.

So if the vector $2v$ has same length as u and hence $u + 2v$ will bisect angle between u and $2v$. Now, angle between u and v is same as angle between u and $2v$. Since 2 is only a scalar, hence the required vector is $u + 2v$. So

$$w = u + 2v$$
$$\Rightarrow \alpha = 2$$

Ans. (a)

227. If the ordinary generating function of a sequence $\{a_n\}_{n=0}^{\infty}$ is $\dfrac{1+z}{(1-z)^3}$, then $a_3 - a_0$ is equal to ________.

(GATE 2017, 2 Marks)

Solution:

$$\frac{1+z}{(1-z)^3} = (1+z)(1-z)^{-3}$$

$$= (1+z)(1+3z+6z^2+10z^3+\cdots)$$

$$= (1 + 4z + 9z^2 + 16z^3 + \cdots)$$

As from equation, $a_0 = 1$ and $a_3 = 16$. Therefore,

$$a_3 - a_0 = 16 - 1 = 15$$

Ans. (15.0)

228. Let $F(x) = e^{x+x^2}$ for real x. From among the following, choose the Taylor series approximation of $F(x)$ around $x = 0$, which includes all powers of x less than or equal to 3.

(a) $1 + x + x^2 + x^3$

(b) $1 + x + \dfrac{3}{2}x^2 + x^3$

(c) $1 + x + \dfrac{3}{2}x^2 + \dfrac{7}{6}x^3$

(d) $1 + x + 3x^2 + 7x^3$

(GATE 2017, 2 Marks)

Solution: Given,

$$F(x) = e^{x+x^2}$$

Taylor series is defined as

$$F(x) = F(a) + (x-a)F'(a) + \frac{(x-a)^2 F''(a)}{2!} + \frac{(x-a)^3 F'''(a)}{3!}$$

Taylor series around $x = 0$ is given by

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \frac{x^3}{3!}F'''(0)$$

Now,

$$F(x) = e^{x+x^2} \Rightarrow F(0) = e^0 = 1$$

$$F'(x) = e^{x+x^2} \cdot (2x+1) \Rightarrow F'(0) = 1$$

$$F''(x) = e^{x+x^2} \cdot 2 + (2n+1)e^{x+x^2} \cdot (2n+1)$$

$$\Rightarrow F''(x) = e^{x+x^2}(4x^2 + 4x + 3)$$

$$\Rightarrow F''(0) = 3$$

$$F'''(x) = e^{x+x^2}(8x+4) + (4x^2+4x+3)e^{x+x^2} \cdot$$

$$(2x+1)F'''(x) = e^{x+x^2}(4x^2+4x+3)$$

$$\Rightarrow F'''(x) = e^{x+x^2}[(8x+4) + (2x+1)(4x^2+4x+3)]$$

$$\Rightarrow F'''(0) = 7$$

Therefore, Taylor series around $x = 0$ is

$$F(x) = 1 + x + \frac{3x^2}{2} + \frac{7x^2}{6}$$

Ans. (c)

229. A three-dimensional region R of finite volume is described by

$$x^2 + y^2 \le z^3; \qquad 0 \le z \le 1$$

where x, y, z are real. The volume of R (up to two decimal places) is ____________.

(GATE 2017, 2 Marks)

Solution: Let $x^2 + y^2 = l^2$

Volume revolution is about z-axis.

Volume of region,

$$R = \int_0^1 \pi l^2 dz$$

Here,

$$l = \sqrt{x^2 + y^2}$$

$$l^2 = x^2 + y^2 = z^3$$

So, volume of region

$$R = \int_0^1 \pi z^3 dz = \left.\frac{\pi z^4}{4}\right|_0^1 = \frac{\pi}{4} = 0.7853$$

Ans. (0.78)

230. Let $I = \int_C (2z\,dx + 2y\,dy + 2x\,dz)$ where x, y, z are real, and let C be the straight line segment from point A: (0, 2, 1) to point B: (4, 1, –1). The value of I is ____________.

(GATE 2017, 2 Marks)

Solution: The straight line from A (0, 2, 1) to B (4, 1, –1) is

$$\frac{x-0}{4-0} = \frac{y-2}{1-2} = \frac{z-1}{-1-1}$$

$$\Rightarrow \frac{x}{4} = \frac{y-2}{-1} = \frac{z-1}{-2} = a\,(\text{say})$$

$$\Rightarrow x = 4a, y = 2-a, z = 1-2a$$

$$dx = 4da,\ dy = -da,\ dz = -2da$$

For $x = 0$,

$$a = 0$$

For $x = 4$,

$$a = 1$$

Therefore,

$$I = \int_C (2zdx + 2ydy + 2xdz)$$

$$= \int_{t=0}^{1} 2(1-2a)4da + 2(2-a)(-da) + 2(4a)(-2da)$$

$$= \int_{t=0}^{1} (-30a+4)da = \left[\frac{-30a^2}{2} + 4a\right]_0^1 = -11$$

Ans. (–11)

231. The values of the integrals

$$\int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} dy \right) dx$$

and

$$\int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} dx \right) dy \quad \text{are}$$

(a) same and equal to 0.5
(b) same and equal to –0.5
(c) 0.5 and –0.5, respectively
(d) –0.5 and 0.5, respectively

(GATE 2017, 2 Marks)

Solution:

$$I_1 = \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dydx = \int_0^1 \left[\int_0^1 \left(\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right) dy \right] dx$$

$$= \int_0^1 \left[2x \left(\frac{-1}{2(x+y)^2} \right)_0^1 + \left(\frac{1}{x+y} \right)_0^1 \right] dx$$

$$= \int_0^1 \frac{1}{(x+1)^2} dx = -\left[\frac{1}{x+1} \right]_0^1 = 0.5$$

and

$$I_2 = \int_0^1 \left(\int_0^1 \frac{x-y}{(x+y)^3} dx \right) dy = \int_0^1 \left[\int_0^1 \left(\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right) dx \right] dy$$

$$= \int_0^1 \left[\frac{-1}{x+y} + \frac{2y}{2(x+y)} \right]_0^1 dy \quad = \int_0^1 \frac{-1}{(1+y)^2} dy$$

$$= -\left[\frac{-1}{y+1} \right]_0^1 = -0.5$$

Therefore, option (c) is the correct answer.

Ans. (c)

232. The minimum value of the function $f(x) = \frac{1}{3}x(x^2 - 3)$ in the interval $-100 \le x \le 100$ occurs at $x =$ ________.

(GATE 2017, 2 Marks)

Solution:

$$f(x) = \frac{1}{3}x(x^2 - 3) \text{ in } [-100, 100]$$

$$= \frac{1}{3}x^3 - x \text{ in } [-100, 100]$$

$$f'(x) = \frac{1}{3}(3x^2) - 1 = x^2 - 1 = 0 \Rightarrow (x-1)(x+1) = 0$$

$x = -1, 1$ are the stationary points.

$$f''(x) = 2x$$

$$f''(-1) = -2 < 0 \qquad \text{(maxima)}$$

$$f''(1) = +2 > 0 \qquad \text{(minima)}$$

For $x = 1$,

$$f(1) = \frac{1}{3}(1)^3 - 1 = \frac{-2}{3}$$

Now,

$$f(-100) = -333433.33$$

and

$$f(100) = 333233.33$$

(Since $x = 100$ and -100 are end points of interval)
Therefore, minimum value occurs at $x = -100$.

Ans. (–100)

233. A function $f(x)$ is defined as

$$f(x) = \begin{cases} e^x & x < 1 \\ \ln x + ax^2 + bx, & x \ge 1 \end{cases}, \text{ where } x \in \mathbb{R}.$$

Which one of the following statements is TRUE?

(a) $f(x)$ is **NOT** differentiable at $x = 1$ for any values of a and b.
(b) $f(x)$ is differentiable at $x = 1$ for the unique values of a and b.
(c) $f(x)$ is differentiable at $x = 1$ for all the values of a and b such that $a + b = e$.
(d) $f(x)$ is differentiable at $x = 1$ for all values of a and b.

(GATE 2017, 2 Marks)

Solution:

$$f(x) = \begin{cases} e^x, & x < 1 \\ \ln x + ax^2 + bx, & x \ge 1 \end{cases}$$

$$\text{LHD} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = e$$

$$\text{RHD} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = 1 + 2a + b$$

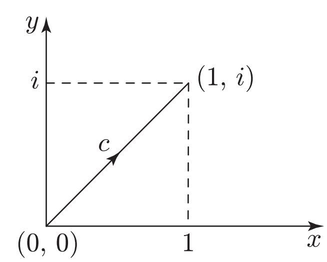
Hence,

$$\text{LHD} \ne \text{RHD}$$

Therefore, $f(x)$ cannot be differentiated at $x = 1$.

Ans. (a)

234. Consider the line integral $I = \int_C (x^2 + iy^2) dz$, where $z = x + iy$. The line c is shown in the figure below.



The value of I is

(a) $\frac{1}{2}i$ (b) $\frac{2}{3}i$ (c) $\frac{3}{4}i$ (d) $\frac{4}{5}i$

(GATE 2017, 2 Marks)

Solution: Clearly,

$$y = x$$

$$\Rightarrow dy = dx \qquad (1)$$

Therefore,

$$I = \int_C (x^2 + iy^2) dz = \int_C (x^2 + iy^2)(dx + idy)$$

$$= \int_C (x^2 + ix^2)(dx + idx)$$

$$= \int_0^1 x^2dx + ix^2dx + ix^2dx - x^2dx$$

$$= \frac{2i}{3}$$

Ans. (b)

235. Let $g(x) = \begin{cases} -x, & x \le 1 \\ x+1, & x \ge 1 \end{cases}$ and $f(x) = \begin{cases} 1-x, & x \le 0 \\ x^2, & x > 0 \end{cases}$.

Consider the composition of f and g, i.e. $(fog)(x) = f(g(x))$. The number of discontinuities in $(fog)(x)$ present in the interval $(-\infty, 0)$ is

(a) 0 (b) 1 (c) 2 (d) 4

(GATE 2017, 2 Marks)

Solution: $fog(x)$ is continuous at $x < 0$. Therefore, there are no discontinuities.

Ans. (a)

236. For the vector $\vec{V} = 2yz\hat{i} + 3xz\hat{j} + 4xy\hat{k}$, the value of $\nabla \cdot (\nabla \times \vec{V})$ is ______.

(GATE 2017, 2 Marks)

Solution: For any vector V

$$\nabla \cdot (\nabla \times \vec{V}) = 0$$

Ans. 0

237. The surface integral $\iint_s \vec{F} \cdot \hat{n} dS$ over the surface S of the sphere $x^2 + y^2 + z^2 = 9$, where $\vec{F} = (x + y)\hat{i} + (x + z)\hat{j} + (y + z)\hat{k}$ and $\hat{n}$ is the unit outward surface normal, yields ______

(GATE 2017, 2 Marks)

Solution: Given that

$$\vec{F} = (x + y)\hat{i} + (x + z)\hat{j} + (y + z)\hat{k}$$

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(x+z) + \frac{\partial}{\partial z}(y+z)$$

$$= 1 + 0 + 1$$

$$= 2$$

Using divergence theorem,

$$\iint_s \vec{F} \cdot \hat{n} dS = \int_V \text{div}\vec{F}\, dV$$

where V is the volume of sphere $x^2 + y^2 + z^2 = 9$. Thus,

$$\iint_s \vec{F} \cdot \hat{n} dS = \int_V 2dV$$

$$= 2 \times \frac{4\pi \times 27}{3}$$

$$= 226.19$$

Ans. 226.19

238. The tangent to the curve represented by $y = x \ln x$ is required to have 45° inclination with the x-axis. The coordinates of the tangent point would be

(a) (1.0) (b) (0.1) (c) (1.1) (d) $(\sqrt{2}, \sqrt{2})$

(GATE 2017, 2 Marks)

Solution:

$$\frac{dy}{dx} = \tan 45° \quad \text{(inclination of 45° with } x\text{-axis)} \quad (1)$$

Given,

$$y = x \ln x \quad (2)$$

$$\Rightarrow \frac{dy}{dx} = \ln x + 1 \quad (3)$$

From Eqs. (1) and (3), we get

$$\tan 45° = \ln x + 1$$

$$\Rightarrow 1 = \ln x + 1$$

$$\Rightarrow \ln x = 0$$

$$\Rightarrow x = 1$$

Substituting the value of x in Eq. (2), we have

$$y = 1 \times \ln 1 = 0$$

Therefore, the coordinates of the tangent point are (1, 0).

Ans. (a)

239. Consider the following definite integral:

$$I = \int_0^1 \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}} dx$$

The value of the integral is

(a) $\frac{\pi^3}{24}$ (b) $\frac{\pi^3}{12}$ (c) $\frac{\pi^3}{48}$ (d) $\frac{\pi^3}{64}$

(GATE 2017, 2 Marks)

Solution: $$I = \int_0^1 \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}} dx \quad (1)$$

Let

$$\sin^{-1}x = t$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$$

Therefore, from Eq. (1), we have

$$I = \int_0^{\pi/2} t^2 dt = \left[\frac{t^3}{3}\right]_0^{\pi/2}$$

$$= \frac{\pi^3}{24}$$

Ans. (a)

CHAPTER 3

DIFFERENTIAL EQUATIONS

INTRODUCTION

A differential equation is an equation which involves independent and dependent variables and their derivatives or differentials.

For example, $\frac{\partial x}{\partial y} + \frac{x}{y} = x^2$ and $\frac{\partial y}{\partial x} + xy = y^3$ are differential equations.

The order of a differential equation is the highest differential in the equation. The degree of a differential equation is the power of the highest differential in the equation.

For example, the equation

$$\left(\frac{\partial^2 y}{\partial x^2}\right) + \left(\frac{\partial y}{\partial x}\right) = xy + \left(\frac{\partial^3 y}{\partial x^3}\right)^2$$

has order = 3 and degree = 2.

Ordinary differential equation is a differential equation which involves only one independent variable.

For example, $\frac{dx}{dy} + xy = 1$ is an ordinary differential equation.

SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a differential equation is any relation between variables which is free of derivatives and which satisfies the differential equation.

A general solution is the solution in which the number of arbitrary constants and the order of the differential equation are same.

A particular solution is the solution which can be obtained by giving particular values to arbitrary constants of general solution.

Variable Separable

If all functions of x and dx can be arranged on one side and y and dy on the other side, then the variables are separable. The solution of this equation is found by integrating the functions of x and y.

$$\int f(x)dx = \int g(y)dy + C \qquad (1)$$

Homogeneous Equation

Homogeneous equations are of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \tag{2}$$

where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree in x and y. Homogeneous functions are those in which all the terms are of nth degree. To solve homogeneous equation,

1. Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dy}{dx}$. (3)
2. Separate v and x and then integrate.

$$\begin{aligned}&\frac{dy}{dx} = f(y/x)\\ \Rightarrow\ & y/x = v\\ \Rightarrow\ & \frac{dv}{f(v)-v} = \frac{dx}{x}\\ \Rightarrow\ & \int\frac{dv}{f(v)-v} = \log x + C\end{aligned} \tag{4}$$

Linear Equation of First Order

If a differential equation has its dependent variables and its derivatives occur in the first degree and are not multiplied together, then the equation is said to be linear. The standard equation of a linear equation of first order is given as

$$\frac{dy}{dx} + Py = Q \tag{5}$$

where P and Q are functions of x.

Integrating factor = (I.F.) = $e^{\int P\cdot dx}$

$$y\cdot e^{\int P\cdot dx} = \int Q\cdot e^{\int P\cdot dx}dx + C$$

$$\Rightarrow y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C \tag{6}$$

Exact Differential Equation

A differential equation of the form $M(x, y)\ dx + N(x, y)\ dy = 0$, if $df = M\ dx + N\ dy$ where $f(x, y)$ is a function, is called an exact differential equation.

Solution of differential equation is $f(x, y) = C$.

The necessary condition for the differential equation to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{7}$$

The exact differential equation can be solved using the following steps:

1. First, integrate M with respect to x keeping y constant.
2. Next, integrate the terms of N with respect to y which do not involve x.
3. Then, the sum of the two expressions is equated to an arbitrary constant to give the solution.

$$\int M\cdot dx + \int N\cdot dy = C \tag{8}$$

Integrating Factor

A differential equation which is not exact can be converted into an exact differential equation by multiplication of a suitable function. This function is called integrating factor.

Some of the important formulae are:

1. If the given equation of the form $M\,dx + N\,dy = 0$ is homogeneous and $Mx + Ny \neq 0$, then

$$\text{I.F.} = \frac{1}{Mx + Ny} \tag{9}$$

2. If the given equation is of the form $f(x,y)\,y\partial x + g(x,y)\,x\partial y = 0$ and $Mx - Ny \neq 0$, then

$$\text{I.F.} = \frac{1}{Mx - Ny} \tag{10}$$

3. If the given equation is of the form $Mdx + Ndy = 0$ and $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ where $f(x)$ is a function of x, then

$$\text{I.F.} = e^{\int f(x)\cdot\partial x} \tag{11}$$

4. If the given equation is of the form $Mdx + Ndy = 0$ and $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ where $f(y)$ in a function of y, then

$$\text{I.F.} = e^{\int f(y)\cdot\partial y} \tag{12}$$

Clairaut's Equation

An equation of the form $y = px + f(p)$, where $p = \frac{dy}{dx}$ is called a Clairaut's equation.

We know that

$$y = px + f(p) \tag{13}$$

Differentiating Eq. (13) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$
$$\Rightarrow [x + f'(p)]\frac{dp}{dx} = 0$$

Therefore,

$$[x + f'(p)] = 0 \quad \text{or} \quad \frac{dp}{dx} = 0$$

Now, if $\frac{dp}{dx} = 0$, then $p = c$. (14)

Thus, eliminating p from Eqs. (13) and (14), we get

$$y = cx + f(c) \tag{15}$$

as the general solution of Eq. (13).

Hence, the solution of Clairaut's equation is obtained on replacing p by c.

LINEAR DIFFERENTIAL EQUATION

Linear differential equations are the equations whose differential variables and their derivatives appear only in first degree and are not multiplied together.

The general form of a differential equation of nth order is denoted by

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} p_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + p_n y = X \tag{16}$$

where p_1, p_2, …, p_n and X are functions of x.

If operator $\frac{d}{dx}$ is denoted by D, then substituting it in above equation, we get

$$D^n y + p_1 D^{n-1} y + p_2 D^{n-2} y + \cdots + p_n y = X$$
$$\Rightarrow f(D)y = X \tag{17}$$

where $f(D) = D^n + p_1 D^{n-1} + \cdots + p_n = 0$.

As already mentioned before, the equation can be generalized as

$$f(D)y = X$$

The general solution of the above equation can be given as

$$y = \text{(Complementary Function)} + \text{(Particular Integral)}$$

The general form of the linear differential equation with constant coefficients is given as

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}}{\partial x^{n-2}} + \cdots + k_n y = X \tag{18}$$

where k_1, k_2, …, k_n are constants and X is a function of x only.

The equation

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}}{dx^{n-2}} + \cdots + k_n y = 0$$

where k_1, k_2, …, k_n are constants can also be denoted as

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)y = 0$$

and the equation $D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n = 0$ is called the auxiliary equation (A.E.).

Now, if m_1, m_2, …, m_n are the roots, then the complementary functions can be calculated using the following cases:

Case I: If the roots of A.E. are distinct and real, i.e. $m_1 \neq m_2 \neq m_3 \neq \ldots \neq m_n$, then

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots \tag{19}$$

Case II: If some roots of A.E. are real and some are equal, i.e. $m_1 = m_2 \neq m_3$, then

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 c^{m_3 x} + \cdots \tag{20}$$

Case III: If the roots of A.E. are complex, i.e. $\alpha + i\beta, \alpha - i\beta, m_3, \ldots, m_n$, then

$$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \tag{21}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

If the complex roots are equal, then

$$y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x] + \cdots + c_n e^{m_n x} \tag{22}$$

PARTICULAR INTEGRALS

Consider the equation given in the previous section

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)y = X$$

The particular integral of the equation is given by

$$\text{P.I.} = \frac{1}{D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n} X$$

The following cases arise for particular integrals:

1. When $X = e^{ax}$, then

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(a)} e^{ax}, \text{ if } f(a) \neq 0 \tag{23}$$

If $f(a) = 0$, then

$$\text{P.I.} = \frac{x}{f'(a)} e^{ax}, \text{ if } f'(a) \neq 0 \tag{24}$$

If $f'(a) = 0$, then

$$\text{P.I.} = \frac{x^2}{f''(a)} e^{ax}, \text{ if } f''(a) \neq 0 \tag{25}$$

2. When $X = \sin\ ax$, then

$$\text{P.I.} = \frac{1}{f(D^2)} \sin ax$$

$$= \frac{1}{f(-a^2)} \sin ax, \text{ if } f(-a^2) \neq 0 \tag{26}$$

3. When $X = \cos\ ax$, then

$$\text{P.I.} = \frac{1}{f(D^2)} \cos ax$$

$$= \frac{1}{f(-a^2)} \cos ax, \text{ if } f(-a^2) \neq 0 \tag{27}$$

4. When $X = x^m$, then

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m \tag{28}$$

Expansion of $[f(D)]^{-1}$ is to be carried up to the term D^m because $(m + 1)^{\text{th}}$ and higher derivatives of x^m are zero.

5. When $X = e^{ax} v(x)$, then

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} v(x)$$

$$\text{P.I.} = e^{ax} \frac{1}{f(D + a)} v(x) \tag{29}$$

6. When $X = x\ v(x)$, then

$$\text{P.I.} = \frac{1}{f(D)} x\, v(x)$$

$$= \left[x - \frac{f'(D)}{f(D)}\right] \cdot \frac{1}{f(D)} v(x) \tag{30}$$

Thus, the following steps should be used to calculate the complete solution:

(a) Find the auxiliary equation (A.E.) from the differential equation.
(b) Calculate the complementary function using the cases given before.
(c) Calculate particular integral (P.I.) using the cases explained before.
(d) To find the complete solution, use the following expression:

$$y = \text{C.F.} + \text{P.I.}$$

HOMOGENEOUS LINEAR EQUATION

Cauchy's homogeneous linear equation is given by

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + k_2 x^{n-2} \frac{\partial^{n-2}}{\partial x^{n-2}} + \cdots + k_n y = X \tag{31}$$

where k_1, k_2, …, k_n are constants and X is a function of x.

The above equation [Eq. (31)] can be reduced to linear differential equation with constant coefficients by the following substitution:

$$x = e^t \tag{32}$$

Taking natural log on both sides, we get

$$\log x = t \tag{33}$$

Equation (33) can be solved by methods mentioned in the section 3.2.

Legendre's homogeneous linear equation is given by

$$(ax + b)^n + k_1 (ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}}$$
$$+ k_2 (ax + b)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = X \tag{34}$$

where k_1, k_2, …, k_n are constants and X is a function of x.

Equation (34) can be reduced to Cauchy's form by substituting $ax + b = t$, and then it can be further reduced to linear differential equation with constant coefficients by the above-mentioned methods in Section 3.2.

BERNOULLI'S EQUATION

The equation $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x and can be reduced to Leibnitz's linear equation and is called the Bernoulli's equation.

To solve the Bernoulli's equation, divide both sides by y^n, so that

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \tag{35}$$

Putting $y^{1-n} = z$, we get

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

Therefore, Eq. (35) becomes

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

$$\Rightarrow \frac{dz}{dx} + P(1-n)z = Q(1-n)$$

which is Leibnitz's linear equation in z and can be solved using the methods mentioned under Integrating Factor.

EULER–CAUCHY EQUATIONS

There are two types of Euler–Cauchy equations namely homogeneous Euler–Cauchy equations and non-homogeneous Euler–Cauchy equations.

Homogeneous Euler–Cauchy Equation

An ordinary differential equation is called homogeneous Euler–Cauchy equation, if it is in the form

$$ax^2y'' + bxy' + cy = 0 \tag{36}$$

where a, b and c are constants. This equation has two linearly independent solutions, which depend on the following quadratic equation:

$$am^2 + (b - a)\,m + c = 0 \tag{37}$$

This quadratic equation is known as characteristic equation. Let us consider Eq. (36) for $x > 0$ as this equation is singular for $x = 0$.

1. If the roots of the characteristic equation are real and distinct, say m_1 and m_2, two linearly independent solutions of the differential equations are x^{m1} and x^{m2}.
2. If the roots of characteristic equation are real and equal, that is, $m_1 = m_2 = m$; the linearly independent solutions of the differential equation are x^m and $x^m \ln x$. The general solution to Eq. (36) for the first case is given by

$$y = c_1x^{m1} + c_2x^{m2} \tag{38}$$

The general solution to Eq. (36) for the second case is given by

$$y = (c_1 + c_2 \ln x)\, x^m \tag{39}$$

3. If the roots of characteristic equation are complex conjugate, that is, $m_1 = \alpha + j\beta$ and $m_2 = \alpha - j\beta$, Linearly independent solutions of differential Eq. (36) are $x^\alpha \cos(\beta \ln x)$ and $x^\alpha \sin(\beta \ln x)$.

The general solution is given by

$$y = x^\alpha\,[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)] \tag{40}$$

Non-homogeneous Euler–Cauchy Equation

The non-homogeneous Euler–Cauchy equation is of the form

$$ax^2y'' + bxy' + cy = \tilde{r}(x) \tag{41}$$

Here, a, b and c are constants. The method of variation of parameters to find solution to non-homogeneous Euler–Cauchy equation is given in Eq. (41). As a first step, divide Eq. (41) by ax^2 so as to make coefficient of y'' as unity.

$$y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = \frac{\tilde{r}(x)}{ax^2} \tag{42}$$

or

$$y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = r(x) \tag{43}$$

$$r(x) = \frac{\tilde{r}(x)}{ax^2}$$

The two linearly independent solutions y_1 and y_2 of the homogeneous part have been derived in the previous section. Particular solution to Eq. (41) is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x)r(x)}{W(y_1, y_2)}\,dx + y_2(x)\int \frac{y_1(x)r(x)}{W(y_1, y_2)}\,dx$$

General solution to non-homogeneous Euler–Cauchy equation is therefore given by the following equation:

$$y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x)$$

Please note that a homogeneous Euler–Cauchy equation can be transformed into linear constant coefficient homogeneous equation by changing the independent variable to $t = \ln x$ or $x = e^t$.

SOLVING DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

Laplace transform method is particularly useful for solving linear differential equations with constant coefficients and associated initial conditions. This method is also very simple and useful for finding solutions related to physical systems as the formulation of many physical problems leads to linear differential equations with initial conditions. Laplace transform method of solving linear differential equations produces particular solutions without there being any need for finding first the general solution and then evaluating the arbitrary constants. Laplace transform method can be used for solving linear constant coefficient differential equations by following the procedure outlined as under.

1. In the first step, take the Laplace transform of both sides of the given differential equation or set of differential equations and the given initial conditions. Laplace transforms are discussed in detail in the chapter on transform theory.
2. This leads to an algebraic equation or a system of algebraic equations in the Laplace transform of the required solution.
3. The required solution is then obtained by solving for this Laplace transform and then taking its inverse. One common method for solving the Laplace transform is the method of partial fractions. Inverse Laplace transforms are discussed in detail in the chapter on transform theory.

VARIATION OF PARAMETERS METHOD

The method of variation of parameters is a general method that can be used to solve non-homogeneous linear ordinary differential equations of the form

$$y'' + py' + qy = x \tag{44}$$

where p, q and x are functions of x.

The particular integral in this case is given by

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx \tag{45}$$

Here, y_1 and y_2 are the solutions of $y'' + py' + qy = 0$ and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1 and y_2.

The proof of the above equation for the particular integral is as follows.

Let the complementary function of $y'' + py' + qy = X_N$ also called general solution to the associated homogeneous equation $y'' + py + qy = 0$ be given by $y = C_1 y_1 + C_2 y_2$.

Replacing parameters C_1 and C_2 by unknown functions $u(x)$ and $v(x)$,

$$y = uy_1 + vy_2 \tag{46}$$

Differentiating the above equation with respect to x, we get

$$y' = uy_1' + vy_2' + u'y_1 + v'y_2 = uy_1' + vy_2' \tag{47}$$

This is with the assumption that $u'y_1 + v'y_2 = 0$

Differentiating Eq. (47) and substituting in Eq. (44) and noting that y_1 and y_2 satisfy Eq. (45), we get

$$u'y_1' + v'y_2' = X \tag{48}$$

Solving Eqs. (47) and (48), we get

$$u' = \frac{y_2 X}{W} \text{ and } v' = \frac{y_1 X}{W}$$

where $W = (y_1 y_2' - y_2 y_1')$

Integrating u' and v', we get

$$u = -\int \frac{y_2 X}{W}\, dx \text{ and } v = \int \frac{y_1 X}{W}\, dx \tag{49}$$

Substituting for (u) and (v) in Eq. (46), we get Eq. (45).

SEPARATION OF VARIABLES METHOD

The process used for finding the general solution of ordinary differential equations, in which we find the general solution and then determine the values of arbitrary constants from initial values, is not applicable to solving problems that involve partial differential equations. It is because of the reason that general solution of a partial differential equation contains arbitrary functions, which make it difficult to satisfy the given boundary conditions.

Separation of variables method can be used to solve most of the boundary value problems involving linear partial differential equations. Some common examples of such partial differential equations representing engineering problems include the one-dimensional heat flow equation given by

$$\frac{\partial u}{dt} = c^2 \frac{\partial^2 u}{dx^2}$$

wave equation representing vibrations of a stretched string given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

two-dimensional heat flow equation that becomes two-dimensional Laplace equation in steady state and given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

transmission line equations, two-dimensional wave equation and so on.

In the 'separation of variables' method, the solution is expressed as a product of two unknown functions. Each of these unknown functions depends only on one of the independent variables. By this, we are able to write the resulting equation so that one side depends only on one variable and the other side depends upon remaining variables. Each side in this case must be equal to a constant. The unknown functions can then be determined once the actual solution is given by the super position of solutions to these functions. The process of using the method of 'separation of variables' to solve partial differential equations is best illustrated with the help of following example.

Let us consider the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

Let us assume $Z = X(x) \cdot Y(y)$

Here, $X(x)$ is a function of (x) alone and $Y(y)$ is a function of (y) only. Substituting for (z) in the given partial differential equation.

$$X''Y - 4X'Y + XY' = 0$$

or $$Y(X'' - 4X') = -XY'$$

or $$\frac{X'' - 4X'}{X} = \frac{-Y'}{Y} \tag{50}$$

Equation (50) can be true only if each side is equal to same constant, say a.

$$\frac{X'' - 4X'}{X} = a \text{ gives } X'' - 4X' - aX = 0 \tag{51}$$

$$-\frac{Y'}{Y} = a \text{ gives } Y' + aY = 0 \tag{52}$$

In the next step, we shall find solutions to Eqs. (51) and (52). The characteristic equation representing Eq. (51) is given by

$$m^2 - 4m - a = 0$$

or $$m = 2 + \sqrt{4+a},\ 2 - \sqrt{4+a}$$

The solution of Eq. (51) can then be written as

$$X = C_1 e^{(2-\sqrt{4+a})x} + C_2 e^{(2-\sqrt{4+a})x}$$

The characteristic equation representing Eq. (52) is given by

$$m + a = 0,\ m = -a$$

Therefore, $$Y = C_3 e^{-ay} \tag{53}$$

Substituting the values of x and y in the expression for Z.

$$Z = \left[C_1 e^{(2+\sqrt{4+a})x} + C_2 e^{(2-\sqrt{4+a})x}\right] \cdot C_3 e^{-ay}$$

or $$Z = \left[k_1 e^{(2+\sqrt{4+a})x} + k_2^{(2-\sqrt{4+a})x}\right] \cdot e^{-ay} \tag{54}$$

Constants k_1 and k_2 are determined from given boundary conditions.

ONE-DIMENSIONAL DIFFUSION (HEAT FLOW) EQUATION

Heat flow through a homogeneous bar of uniform cross-section with its side walls impervious to heat and conduction of heat through a solid rod with no radiation are some important examples of one-dimensional heat flow. One-dimensional heat flow phenomenon is expressed by the following partial differential equation.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{55}$$

where u is temperature at a distance (x) at time (t), c^2 is $k/s\rho$, k is thermal conductivity, s is specific heat and ρ is volumetric density.

(c^2) is also known as diffusivity of the substance. Heat flow equation can be solved by using the method of separation of variables as follows:

Let us assume $u(x, t) = X(x) \cdot T(t)$

Here, $X(x)$ is a function of (x) only and $T(t)$ is a function of (t) only. Substituting for $u(x, t)$ in the partial differential equation

$$XT' = c^2 X''T$$

which gives $$\frac{X''}{X} = \frac{T'}{c^2 T} \tag{56}$$

If each side is equated to a constant, say (a), then

$$\frac{d^2 X}{dx^2} - aX = 0 \tag{57}$$

and $$\frac{dT}{dt} - ac^2 T = 0 \tag{58}$$

There can be three possible solutions for Eqs. (57) and (58) when 'a' is positive, 'a' is negative and 'a' is equal to zero.

When 'a' is positive and equal to say (p^2), then

$$X = C_1 e^{px} + C_2 e^{-px}$$

and $$T = C_3 e^{c^2 p^2 t}$$

when 'a' is negative and equal to say $(-p^2)$, then

$$X = C_4 \cos px + C_5 \sin px$$

and $$T = C_6 e^{-c^2 p^2 t}$$

when 'a' is zero, then

$$X = C_7 X + C_8$$

and $$T = C_9$$

The three possible solutions of $u(x, t)$ therefore are

$$u = [C_1 e^{px} + C_2 e^{-px}] C_3 e^{c^2 p^2 t} \tag{59}$$

$$u = \left[C_4 \cos px + C_5 \sin px\right] C_6 e^{-c^2 p^2 t} \tag{60}$$

$$u = (C_7 x + C_8) \cdot C_9 \tag{61}$$

Since (u) has to decrease with time (t); the solution represented by Eq. (60) is the only suitable solution of the one-dimensional heat flow equation.

Therefore, $$u(x,t) = [C_4 \cos px + C_5 \sin px] \cdot C_6 e^{-c^2 p^2 t} \tag{62}$$

SECOND ORDER ONE-DIMENSIONAL WAVE EQUATION

The second order one-dimensional wave equation gives the transverse vibrations of an elastic string that is fixed at the ends and is tightly stretched. Also, tension to which the string has been subjected is much larger than the mass of the string so that effects of gravity can be ignored. Also, there are no other external forces acting on the string. One-dimensional wave equation is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

where y is displacement along transverse direction, x is distance along length of string, t is time, $c^2 = T/m$, T is tension, the string is subjected to and m is the mass of string.

The wave equation can be solved as follows:

Let us assume $z = X(x) \cdot T(t)$

where $X(x)$ is a function of (x) alone and $T(t)$ is a function of (t) only.

This gives

$$\frac{\partial^2 y}{dt^2} = X \cdot T'' \text{ and } \frac{\partial^2 Y}{\partial x^2} = X'' \cdot T$$

Substituting for $\dfrac{\partial^2 y}{\partial t^2}$ and $\dfrac{\partial^2 y}{\partial x^2}$ in wave equation, we get

$$XT'' = c^2 X'' T$$

or $$\frac{X''}{X} = \frac{1}{c^2} \times \frac{T''}{T} = k \text{ (constant)}$$

This gives $X'' - kX = 0$ and $T'' - kc^2 T = 0$

These two equations can be solved for three cases for (i) 'k' positive (ii) 'k' negative and (iii) 'k' equal to zero.

When 'k' is positive and equal to say p^2, then

$$X = C_1 e^{px} + C_2 e^{-px}$$

and $$T = C_3 e^{cpt} + C_4 e^{-cpt}$$

when 'k' is negative and equal to say $-p^2$, then

$$X = C_5 \cos px + C_6 \sin px$$

and $$T = C_7 \cos cpt + C_8 \sin cpt$$

when 'k' is equal to zero, then

$$X = C_9 x + C_{10}$$

and $$T = C_{11} t + C_{12}$$

Now we have three sets of solutions. Since we are dealing with the problem of vibrations, (y) must therefore be periodic function of (x) and (t). Therefore, the right solution must involve trigonometric terms. Therefore, the right solution is given by

$$X = C_5 \cos px + C_6 \sin px$$

and $$T = C_7 \cos cpt + C_8 \sin cpt$$

Substituting the values of X and T, we get the following solution to the wave equation.

$$y = (C_1 \cos px + C_2 \sin px)\,(C_3 \cos cpt + C_4 \sin cpt)$$

TWO-DIMENSIONAL LAPLACE EQUATION

Two-dimensional heat flow such as the one through a metal plate is described by the following partial differential equation.

$$\frac{\partial u}{\partial t} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \tag{63}$$

where u is temperature, t is time, $c^2 = (k / \rho s)$ is diffusivity, k is thermal conductivity, ρ is density and s is specific heat.

Equation (63) describes the temperature distribution of the plane in the transient state. In the steady state, the temperature is independent of time, that is, $(\partial u / \partial t) = 0$. The above equation then reduces to

$$\frac{\partial u}{\partial t} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0$$

or

$$c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0 \tag{64}$$

Equation (64) is known as two-dimensional Laplace equation. It describes the two-dimensional heat flow phenomenon in the steady state. Solution to two-dimensional Laplace equation is derived as follows:

Let us assume that $u = X(x).\ Y(y)$

Substituting for (u) in Eq. (64), we get

$$X'' \cdot Y + X \cdot Y'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} \tag{65}$$

Since (x) and (y) are independent variables, Eq. (65) is valid only if both R.H.S and L.H.S of Eq. (65) are equal to a constant.

Therefore, $X'' - kX = 0$ and $Y'' + kY = 0$

Solving these equations for 'k' positive, 'k' negative and 'k' equal to zero, we get the following solutions:

(i) When 'k' is positive and equal to p^2 (say)

$$X = C_1 e^{px} + C_2 e^{-px}$$

and $$Y = C_3 \cos py + C_4 \sin py$$

This gives

$$u = (C_1 e^{px} + C_2 e^{-px})\,(C_3 \cos py + C_4 \sin py) \tag{66}$$

(ii) When 'k' is negative and equal to $-p^2$ (say),

$$X = C_5 \cos px + C_6 \sin px$$

$$Y = C_7 e^{py} + C_8 e^{-py}$$

This gives $u = (C_5 \cos px + C_6 \sin px)$

$$(C_7\, e^{py} + C_8\, e^{-py}) \tag{67}$$

(iii) When 'k' is zero,

$$X = C_9 x + C_{10}$$

and $$Y = C_{11} y + C_{12}$$

Therefore,

$$u = (C_9 x + C_{10})\,(C_{11} y + C_{12}) \tag{68}$$

Out of the three possible solutions given by Eqs. (66)–(68), the one that is consistent with given boundary conditions is considered.

SOLVED EXAMPLES

1. Determine the order and degree of

$$\frac{\left[1 + (dy/dx)^2\right]^{3/2}}{d^2y/dx^2} = K.$$

Solution: The given differential equation when written as a polynomial in derivatives becomes

$$K^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ and its power is 2.

The order is 2 and degree is 2.

2. Solve $dy/dx = (x + y + 1)^2$, if $y\,(0) = 0$.

Solution: Putting $x + y + 1 = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 1$

Thus, the given equation becomes $\frac{dt}{dx} - 1 = t^2$ or $\frac{dt}{dx} = 1 + t^2$

Integrating both sides, we get $\int \frac{dt}{1+t^2} = \int dx + c$

or $$\tan^{-1} t = x + c$$

$$\Rightarrow \quad \tan^{-1}(x + y + 1) = x + c$$

$$\Rightarrow \quad x + y + 1 = \tan(x + c)$$

when $x = 0,\ y = 0$

$$1 = \tan\,(c)$$

$$\Rightarrow \quad c = \pi/4$$

Thus, the solution is given by $x + y + 1 = \tan(x + \pi/4)$.

3. Solve the differential equation $(x^2 - y^2)\ dx + 2xy\ dy = 0$, given that $y = 1$ when $x = 1$.

Solution: We have

$$(x^2 - y^2)\,dx + 2xy\,dy = 0$$

$$(x^2 - y^2)\,dx = -2xy\,dy$$

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy} = \frac{y^2 - x^2}{2xy} \tag{1}$$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (1), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x \cdot vx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow \quad x \cdot \frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} = -\left[\frac{v^2 + 1}{2v}\right]$$

$$\Rightarrow \quad \frac{2v}{v^2 + 1} \cdot dv = -\frac{dx}{x}, \quad x \neq 0$$

$$\Rightarrow \quad \int \frac{2v}{v^2 + 1} \cdot dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log(v^2 + 1) = -\log|x| + c$$

$$\Rightarrow \quad \log(v^2 + 1) + \log|x| = \log c$$

$$\Rightarrow \quad (v^2 + 1)|x| = c$$

Now, putting $v = y/x$

$$(y^2/x^2 + 1)|x| = c$$

$$\Rightarrow \quad (x^2 + y^2) = c|x| \qquad (2)$$

Substituting $x = 1$ and $y = 1$, we get

$$c = 2$$

Putting value of $c = 2$ in Eq. (2), we get

$$x^2 + y^2 = 2x \quad \text{or} \quad x^2 + y^2 = 2(-x)$$

$x = 1$ and $y = 1$ do not satisfy $x^2 + y^2 = 2(-x)$.
Hence, $x^2 + y^2 = 2x$ is the required solution.

4. Solve the differential equation $\frac{dy}{dx} - \frac{y}{x} = 2x^2$, $x > 0$.

Solution: We know

$$\frac{dy}{dx} + \left(\frac{-1}{x}\right)y = 2x^2$$

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -\frac{1}{x} \text{ and } Q = 2x^2$$

Now,

$$\text{I.F.} = e^{\int P \cdot dx} = e^{\int -1/x \cdot dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying both sides with I.F., we get

$$\frac{1}{x} \cdot \frac{dy}{dx} - \frac{1}{x^2} \cdot y = 2x$$

Integrating both sides w.r.t. x, we get

$$y \cdot \left(\frac{1}{x}\right) = \int 2x \cdot dx + C$$

$$\Rightarrow \quad y \cdot \frac{1}{x} = x^2 + C$$

$\Rightarrow$ $y = x^3 + Cx$, $x > 0$ is the required solution.

5. Solve the differential equation:

$$(2x - y + 1)dx + (2y - x - 1)dy = 0$$

Solution: We know

$$(2x - y + 1)dx + (2y - x - 1)dy = 0$$

$$M = (2x - y + 1), N = (2y - x - 1)$$

$$\frac{\partial M}{\partial y} = -1$$

$$\frac{\partial N}{\partial x} = -1$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given equation is exact. Hence, the general solution is given by

$$\int_{y=\text{const}} M \cdot dx + \int_{\substack{\text{Terms of } N \text{ not} \\ \text{containing } x}} N\,dy = C$$

$$\Rightarrow \quad \int (2x - y + 1)\,dx + \int (2y - 1)\,dy = C$$

$$\Rightarrow \quad x^2 - xy + x + y^2 - y = C$$

$\Rightarrow$ $x^2 + y^2 + x - y - xy = C$, which is the general solution.

6. Solve $y'' + y' + (1/4)y = 0$.

Solution: We have $y'' + y' + (1/4)y = 0$
The characteristic equation is

$$m^2 + m + 1/4 = 0$$

$$\Rightarrow \quad [m + 1/2]^2 = 0$$

Hence, the roots of the equation are $-1/2$ and $-1/2$.

The roots are equal and real roots.

The general solution is given by

$$y = (c_1 + c_2x)e^{(-1/2)x} \qquad (1)$$

Differentiating Eq. (1), we get

$$y' = c_2 e^{(-1/2)x} - (1/2)(c_1 + c_2 x)e^{(-1/2)x} \quad (2)$$

Given $y(0) = 2$ and putting value in Eq. (1), we get

$$2 = [c_1 + c_2(0)]e^{(-1/2)\times 0}$$

$$\Rightarrow \quad c_1 = 2$$

Given $y'(0) = 4$ and putting value in Eq. (2), we get

$$4 = c_2 e^{(-1/2)\times 0} - (1/2)\big(2 + c_2(0)\big)e^{(-1/2)\times 0}$$

$$\Rightarrow \quad 4 = c_2 - 1$$

$$\Rightarrow \quad c_2 = 5$$

Thus, the particular solution is $y = (2 + 5x)e^{(-1/2)x}$.

7. Solve the following differential equation:

$$y'' - 5y' + 6y = 0$$

Solution: We have

$$y'' - 5y' + 6y = 0$$

The characteristic equation is given by

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow \quad m = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{6}{2}, \frac{4}{2} = 3, 2$$

Hence, the roots are distinct and real.

The roots are real and distinct. Hence, the general solution is given by $y(x) = c_1 e^{2x} + c_2 e^{3x}$.

8. Solve the differential equation: $16y'' - 8y' + 5y = 0$.

Solution: We have

$$16y'' - 8y' + 5y = 0$$

The characteristic equation is given by

$$16m^2 - 8m + 5 = 0$$

$$\Rightarrow m = \frac{8 \pm \sqrt{64 - 320}}{32} = \frac{8 \pm 16i}{32} = \frac{1 \pm 2i}{4} = \frac{1}{4} \pm \frac{1}{2}i$$

The roots are complex and distinct. Therefore, the general solution is given by

$$y = e^{x/4}\left[A\cos\frac{x}{2} + B\sin\frac{x}{2}\right]$$

9. Solve the differential equation:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}.$$

Solution: We have

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$$

$$\Rightarrow \quad (D^2 + 6D + 9)y = 5e^{3x}$$

Auxiliary equation is $D^2 + 6D + 9 = 0$

$$\Rightarrow \quad (D + 3)^2 = 0 \Rightarrow D = -3, -3$$

Complementary function $= (c_1 + c_2 x)e^{-3x}$

$$\text{Particular integral} = \frac{1}{D^2 + 6D + 9}\cdot 5e^{3x}$$

$$= \frac{5e^{3x}}{(3)^2 + 6(3) + 9} = \frac{5e^{3x}}{36}$$

The complete solution is given by $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow \quad y = (c_1 + c_2 x)e^{-3x} + \frac{5e^{3x}}{36}$$

10. Solve $(D^3 - 1)y = x^5 + 3x^4 - 2x^3$.

Solution: We have

$$(D^3 - 1)y = x^5 + 3x^4 - 2x^3$$

The characteristic equation is

$$m^3 - 1 = 0 \Rightarrow m = 1, \frac{-1 \pm \sqrt{3}i}{2}$$

The complementary function is given by

$$\text{C.F.} = Ae^x + e^{-x/2}\cdot\left[B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right]$$

The particular integral is given by

$$\text{P.I.} = \frac{1}{D^3 - 1}(x^5 + 3x^4 - 2x^3)$$

$$= \frac{-1}{(1 - D^3)}(x^5 + 3x^4 - 2x^3)$$

$$= -(1 - D^3)^{-1}(x^5 + 3x^4 - 2x^3)$$

$$= -(1 + D^3 + D^6 + D^9)(x^5 + 3x^4 - 2x^3)$$

$$= -(x^5 + 3x^4 - 2x^3 + 60x^2 + 72x - 12)$$

The complete solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$$y = Ae^x + e^{-x/2}\left(B\cos\frac{\sqrt{3}}{2} + C\sin\frac{\sqrt{3}}{2}x\right)$$

$$-(x^5 + 3x^4 - 2x^3 + 60x^2 + 72x - 12)$$

11. Find solution to the homogeneous Euler–Cauchy equation

$$x^2y'' - xy' - 3y = 0 \text{ for } x > 0$$

Solution: The characteristic equation is given by

$$m^2 - 2m - 3 = 0$$

This gives $m = -1, 3$

The general solution is therefore given by

$$y = \frac{C_1}{x} + C_2x^3$$

12. Find solution to the following Euler–Cauchy ordinary differential equation for $x > 0$

$$x^2y'' - 3xy' + 5y = 0$$

Solution: The characteristic equation is given by

$$m^2 - 4m + 5 = 0$$

Solving the characteristic equation, we get

$$m = (2 + i), (2 - i)$$

The general solution to the equation for $x > 0$ is given by

$$y = x^2 [C_1 \cos (\ln x) + C_2 \sin (\ln x)]$$

13. Consider the following non-homogeneous Euler–Cauchy equation:

$$x^2y'' - xy' - 3y = \frac{\ln x}{x};\ x > 0$$

Find solution to this equation.

Solution: The generalised solution is given by

$$y = C_1y_1 (x) + C_2y_2 (x) + y_p (x)$$

The characteristic equation is given by

$$m^2 - 2m - 3 = 0$$

which gives $m = -1, 3$

This gives $y_1(x) = \dfrac{1}{x}$ and $y_2(x) = x^3$

$$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$$

where $$u(x) = -\int \frac{y_2(x)r(x)}{W(y_1, y_2)} dx$$

and $$v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)} dx$$

Now $$W(y_1, y_2) = 4x$$

and $$r(x) = \frac{\ln x}{x^3}$$

Therefore, $u(x) = -\displaystyle\int \frac{\ln x}{4x} dx = -\frac{(\ln x)^2}{8}$

and $$v(x) = \int \frac{\ln x}{4x^5} dx = -\frac{\ln x}{16x^4} - \frac{1}{64x^4}$$

This leads to

$$y_p(x) = -\frac{(\ln x)^2}{8x} - \frac{\ln x}{16x} - \frac{1}{64x}$$

Therefore, the general solution to the Euler–Cauchy equation is given by

$$y(x) = \frac{A}{x} + Bx^3 - \frac{(\ln x)^2}{8x} - \frac{\ln x}{16x}$$

14. Solve the following differential equation using Laplace transform method.

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

It is given that

$$y(0) = \frac{dy}{dt}(0) = 0 \text{ and } \frac{d^2y}{dt^2}(0) = 6$$

Solution: Taking Laplace transform of both sides of differential equation, we get

$$[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] + 2[s^2Y(s) - sy(0)$$
$$- y'(0)] - [sY(s) - y(0)] - 2Y(s) = 0$$

Substituting the given initial conditions, we get

$$(s^3 + 2s^2 - s - 2)\ Y(s) = 6$$

$$\text{or } Y(s) = \frac{6}{(s^3 + 2s^2 - s - 2)} = \frac{6}{(s-1)(s+2)(s+1)}$$

Using partial fractions, we get

$$\frac{6}{(s^3 + 2s^2 - s - 2)} = \frac{6}{6(s-1)} + \frac{6}{-2(s+1)} + \frac{6}{3(s+2)}$$
$$= \frac{1}{(s-1)} - \frac{3}{(s+1)} + \frac{2}{(s+2)}$$

Taking inverse Laplace transform, we get

$$y(t) = L^{-1}\frac{1}{(s-1)} - 3L^{-1}\frac{1}{(s+1)} + 2L^{-1}\frac{1}{(s+2)}$$

$$\Rightarrow y(t) = e^t - 3e^{-t} + 2e^{-2t}$$

15. Solve the following differential equation using Laplace transform method given that $y(0) = 1$ and $y'(0) = 0$.

$$y''(t) - 3y'(t) + 2y(t) = 4$$

Solution: Taking Laplace transform on both sides of given differential equation, we get the following:

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - g(0)] + 2Y(s) = \frac{4}{s}$$

Substituting initial conditions, we get

$$[s^2Y(s) - s] - 3sY(s) + 3 + 2Y(s) = \frac{4}{s}$$

$$Y(s)[s^2 - 3s + 2] - s + 3 = \frac{4}{s}$$

or $$Y(s)[s^2 - 3s + 2] = \frac{4}{s} + s - 3$$

$$= \frac{4 + s^2 - 3s}{s} = \frac{s^2 - 3s + 4}{s}$$

or $$Y(s)(s-2)(s-1) = \frac{s^2 - 3s + 4}{s}$$

or $$Y(s) = \frac{s^2 - 3s + 4}{s(s-2)(s-4)}$$

Using partial fraction expansion,

$$Y(s) = \frac{2}{s} - \frac{2}{(s-1)} + \frac{1}{(s-1)(s-2)}$$

Taking inverse Laplace transform, we get the desired solution as

$$y(t) = 2 - 2e^t + e^{2t}$$

16. Use Laplace transform method to solve the differential equation $y'' + 4y' + 5y = 10e^t$ given that $y(0) = 1,\ y'(0) = 2$.

Solution: Taking Laplace transform on both sides of the given differential equation,

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 5Y(s) = \frac{10}{(s-1)}$$

or $$(s^2 + 4s + 5)Y(s) - s - 6 = \frac{10}{s-1}$$

or $$(s^2 + 4s + 5)Y(s) = \frac{10}{s-1} + s + 6$$

Solving for $Y(s)$, we get

$$Y(s) = \frac{s+6}{(s^2 + 4s + 5)} + \frac{10}{(s-1)(s^2 + 4s + 5)}$$

Using method of partial fraction expansion, we get

$$Y(s) = \frac{1}{(s+2)^2 + 1} + \frac{1}{(s-1)}$$

Using inverse Laplace transform, we get the desired solution.

$$y(t) = e^{-2t}\sin t + e^t$$

17. Solve the following differential equation using the method of variation of parameters.

$$\frac{d^2y}{dx^2} + 4y = \tan 2x$$

Solution: The characteristics function can be determined after finding roots of characteristic equation given by $m^2 + 4 = 0$, which gives $m = \pm 2i$. Therefore, complementary function,

$$y = C_1 \cos 2x + C_2 \sin 2x$$

Now $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = \tan 2x$

$$\text{Therefore, } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Therefore,

$$\text{Particular integral } = y_1 \int \frac{y_2 x}{W}dx + y_2 \int \frac{y_1 x}{W}dx$$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2}dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2}dx$$

$$= \frac{-1}{2}\cos 2x \int (\sec 2x - \cos 2x)dx + \frac{1}{2}\sin 2x \int \sin 2x\, dx$$

$$= -\frac{1}{4}\cos 2x[\log(\sec 2x + \tan 2x) - \sin 2x]$$

$$-\frac{1}{4}\sin 2x \cos 2x = -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Therefore, complete solution is given by

$$y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

18. Consider the following non-homogeneous ordinary differential equation.

$$y'' - 2y' - 3y = xe^{-x}$$

Determine general solution using method of variation of parameters.

Solution: The homogeneous part of the equation is given by

$$y'' - 2y' - 3y = 0$$

The roots of characteristic equation are -1 and 3.

Therefore, two linearly independent solutions of homogeneous part are

$$y_1(x) = e^{-x} \text{ and } y_2(x) = e^{3x}$$

Now $$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$$

where $$u(x) = -\int \frac{y_2(x)r(x)}{W}\,dx$$

$$u(x) = -\int \frac{y_2(x)r(x)}{W}\,dx$$

Here, $r(x) = xe^{-x}$

W is Wronskian given by

$$\begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} = 3e^{2x} + e^{2x} = 4e^{2x}$$

Therefore, $$u(x) = -\int \frac{x}{4}\,dx = -\frac{x^2}{8}$$

$$v(x) = \int \frac{xe^{-4x}}{4}\,dx = -\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}$$

Therefore,

$$y_p(x) = -\frac{x^2}{8}e^{-x} + e^{3x}\left(-\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}\right)$$

Therefore, general solution is given by:

$$y(x) = C_1e^{-x} + C_2e^{3x} - \frac{x^2}{8}e^{-x} + e^{3x}\left(-\frac{x}{16}e^{-4x} - \frac{1}{64}e^{-4x}\right)$$

19. Solve the following partial differential equation using the method of separation of variables. It is given that $u(x, 0) = 4e^{-3x}$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + u$$

Solution: Let us assume that $u(x, t) = X(x)\ T(t)$ Substituting for $u(x, t)$ in the given partial differential equation, we get

$$X'T = XT' + XT$$

or $$(X' - X)T = XT'$$

or $$\frac{X' - X}{X} = \frac{T'}{T} = a \ \text{(say)}$$

Therefore, $X' - aX - X = 0$

or $$\frac{X'}{X} = 1 + a;\ \ X' - (1+a)X = 0$$

and $$\frac{T'}{T} = a;\ \ T' - aT = 0$$

Solution to $X' - (1+a)X = 0$ is given by

$$X = C_1e^{(1+a)x}$$

Solution to $T' - aT = 0$ is given by

$$T = C_2e^{at}$$

Now $$u(x,\ T) = XT = C_1C_2e^{at}\cdot e^{(1+a)x}$$

$$u(x,\ 0) = C_1C_2e^{(1+a)x} = 4e^{-3x}$$

This gives $C_1C_2 = 4$ and $1 + a = -3$ or $a = -4$

Therefore, $u = 4e^{-4t}\cdot e^{-3x}$

$$= 4e^{-(3x\,+\,4t)}$$

20. Solve the following partial differential equation using the method of separation of variables.

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$$

It is given that $u(0,\ y) = 8e^{-3y}$.

Solution: Let us assume $u(x,\ y) = X(x)\cdot Y(y)$ Substituting for $4(x,\ y)$ in the given partial differential equation.

$$X'Y = 4XY'$$

$$\frac{X'}{4X} = \frac{Y'}{Y} = k \ \text{(say)}$$

Therefore, $X' - 4kX = 0$ and $Y' - kY = 0$

Solving for the above two equations, we get

$$X = C_1e^{4kx}$$

and $$Y = C_2e^{ky}$$

Substituting for X and Y in the equation for $u(x,\ y)$, we get

$$u(x,\ y) = C_1C_2e^{4kx}\cdot e^{ky}$$

It is given that $u(0,\ y) = 8e^{-3y}$

This gives $C_1C_2e^{ky} = 8e^{-3y}$

That is, $C_1C_2 = 8$ and $k = -3$

Substituting these values in the expression for $u(x,\ y)$, we get

$$u(x,\ y) = 8e^{-12x}\cdot e^{-3y} = 8e^{(-12x\,-\,3y)} = 8e^{-3(4x\,+\,y)}$$

Therefore, $$u(x,\ y) = 8e^{-3(4x\,+\,y)}$$

21. Solve the following partial differential equation using the method of separation of variables.

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

It is given that $u(x,\ 0) = 4e^{-x}$.

Solution: Let us assume that

$$u(x,\ y) = X(x)\cdot Y(y)$$

Substituting for $u(x, y)$ in the given partial differential equation,

$$3X'Y + 2XY' = 0$$

or $$3X'Y = -2XY'$$

or $$\frac{3X'}{2X} = \frac{Y'}{Y} = a \text{ (say)}$$

Therefore, the two equations are

$$3X' - 2aX = 0$$

and $$Y' - aY = 0$$

Solutions to these equations are given by

$$X = C_1 e^{\frac{2}{3}ax}$$

and $$Y = C_2 e^{ay}$$

This leads to $u(x, y) = C_1 C_2 e^{\frac{2}{3}ax} \cdot e^{ay}$

It is given that $u(x, 0) = 4e^{-x}$

or $$C_1 C_2 e^{\frac{2}{3}ax} = 4e^{-x}$$

which gives $C_1 C_2 = 4$

and $$\frac{2a}{3} = -1$$

or $$a = -\frac{3}{2}$$

Substituting the values of $(C_1 \cdot C_2)$ and (a), we get

$$u(x, y) = 4e^{-x} \cdot e^{\frac{3}{2}y} = 4e^{-\left(x + \frac{3}{2}y\right)}$$

$$u(x, y) = 4e^{-\frac{1}{2}(2x+3y)}$$

22. Solve the following one-dimensional heat conduction equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are as follows:

(i) $u(x, 0) = 2 \sin n\pi x$

(ii) $u(0, t) = 0$

(iii) $u(1, t) = 0$

where $t > 0$ and $0 < x < 1$.

Solution: We can write the solution for the given partial differential equation as follows:

$$u(x, t) = (C_1 \cos px + C_2 \sin px) \cdot e^{-p^2 t}$$

$$u(0, t) = C_1 e^{-p^2 t}$$

Since it is given that $u(0, t) = 0$,

Therefore, $$C_1 e^{-p^2 t} = 0; \quad C_1 = 0$$

The expression for $u(x, t)$ reduces to

$$u(x, t) = C_2 \sin px e^{-p^2 t}$$

Now $u(1, t) = C_2 \sin p \cdot e^{-p^2 t} = 0$ gives $\sin p = 0$

This gives $p = n\pi$

The equation for $u(x, t)$ reduces to

$$u(x, t) = C_2 e^{-(n\pi)^2 t} \cdot \sin n\pi x = b_n e^{-(n\pi)^2 t} \cdot \sin n\pi x$$

where $b_n = C_2$

The general solution to the given partial differential equation is expressed by

$$u(x, t) = \sum b_n e^{-n^2\pi^2 t} \cdot \sin n\pi x$$

$$u(x, 0) = \sin n\pi x = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \cdot \sin n\pi x$$

Comparing both sides, we get $b_n = 1$

Therefore, the desired solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x$$

23. The two ends of a solid rod, that is 20-cm long, have temperatures of 30°C and 80°C until it reaches steady state condition. If the temperature of the ends were changed to 40°C and 60°C, respectively, determine the temperature distribution in the rod at a given time (t).

Solution: The generalised form of one-dimensional heat conduction equation is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where the terms have their usual meaning.

In the steady state condition, temperature is independent of time (t) and depends only on distance (x).

The heat equation then reduces to

$$\frac{\partial^2 u}{\partial x^2} = 0$$

Its solution is given by $u = a + bx$

Constants (a) and (b) can be determined from the known values of temperature (u) at $x = 0$ and $x = 20$.

For $x = 0,\ a = 30$

For $x = 20,\ 30 + 20b = 80$, which gives $b = \dfrac{5}{2}$

Therefore, initial conditions (expressed by steady state) are given by

$$u(x,0) = 30 + \frac{5}{2}x$$

The boundary conditions are

$$u(0,\ t) = 40 \text{ and } u(20,\ t) = 60$$

The steady state temperature is given by

$$u(x,0) = 40 + \frac{60-40}{20}x = 40 + x$$

Temperature in the intermediate period can be expressed by

$$u(x,\ t) = u_s(t) + u_t(x,\ t)$$

where $u_s(t)$ is the steady state temperature distribution as given by the equation $u(x,\ 0) = 40 + x$ and $u_t(x,\ t)$ is the transient temperature distribution, which decreases to zero as the time increases. Since $u(x, t)$ satisfies the one-dimensional heat equation, then

$$u(x,t) = (40+x) + \sum_{n=1}^{\infty}(a_n \cos px + b_n \sin px)e^{-p^2 t}$$

$$u(0,t) = 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2 t}$$

This gives $a_n = 0$

The equation for $u(x,\ t)$ reduces to

$$u(x,t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin pxe^{p^{-2}t}$$

It is also given that,

$$u(20,t) = 60 = 40 + 20 + \sum_{n=1}^{\infty} b_n \sin 20pe^{-p^2 t}$$

or
$$\sum_{n=1}^{\infty} b_n \sin 20pe^{-p^2 t} = 0$$

This gives $\sin 20p = 0$

or
$$p = \frac{n\pi}{20}$$

The equation for $u(x,\ t)$ reduces to

$$u(x,t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20}e^{-n\pi t/20}$$

Now $\quad u(x,0) = 30 + \dfrac{5}{2}x = 40 + x + \displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20}$

or
$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20} = \frac{3x}{2} - 10$$

where

$$b_n = \frac{2}{20}\int_0^{20}\left(\frac{3x}{2} - 10\right)\sin\frac{n\pi x}{20}\,dx = \frac{-20}{n\pi}(1 + 2\cos n\pi)$$

The desired solution is therefore given by

$$u(x,t) = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(1+2\cos n\pi)}{n}\sin\frac{n\pi x}{20}e^{-(n\pi/20)^2 t}$$

24. A string fixed at points $x = 0$ and $x = l$ is tightly stretched. It is released from its initial position given by $y = y_0 \sin^3\left(\dfrac{\pi x}{l}\right)$. Prove that the displacement $y(x,\ t)$ is given by

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}$$

Solution: The equation describing transverse vibrations of the string is given by the following wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are given by

$$y(0,\ t) = 0 \text{ and } y(l,\ t) = 0$$

The initial conditions are given by

$$y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

and
$$\left[\frac{\partial y}{\partial t}\right]_{t=0} = 0$$

The vibration of the string is periodic and therefore its solution is given by

$$y(x,\ t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$$

It is given that $y(0,\ t) = y(1,\ t) = 0$

Therefore, $\quad C_1\ (C_3 \cos cpt + C_4 \sin cpt) = 0$

If this is true for all ‘t’, $C_1 = 0$

This reduces expression for $y(x,\ t)$ to

$$y(x,\ t) = C_2 \sin px\ (C_3 \cos cpt + C_4 \sin cpt)$$

Also, $y(l,\ t) = 0$

Therefore, $C_2 \sin pl\ (C_3 \cos cpt + C_4 \sin cpt) = 0$

If it is true for all 't', then

$pl = n\pi;\ p = \dfrac{n\pi}{l}$ where 'n' is an integer

Therefore,

$$y(x,t) = C_2 \sin\frac{n\pi x}{l}\left[C_3 \cos\frac{cn\pi t}{l} + C_4 \sin\frac{cn\pi t}{l}\right]$$

$$\frac{\partial Y}{\partial t} = C_2 \sin\frac{n\pi x}{l} \times \frac{cn\pi}{l}\left[-C_3 \sin\frac{cn\pi t}{l} + C_4 \cos\frac{cn\pi t}{l}\right]$$

$$\left.\frac{\partial T}{\partial t}\right|_{t=0} = \left(C_2 \sin\frac{n\pi x}{l}\right) \times \frac{cn\pi}{l} \times C_4 = 0$$

This gives $C_4 = 0$

Therefore, $y(x,t) = C_2 C_3 \sin\dfrac{n\pi x}{l} \cos\dfrac{n\pi ct}{l}$

$$= b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

Adding all such solutions, the general solution of the wave equation is given by

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}$$

It is given that

$$y(x,0) = y_0 \sin^3\frac{\pi x}{l}$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = y_0 \sin^3\frac{\pi x}{l}$

or $\quad y_0\left[\dfrac{3\sin \pi x/l - \sin 3\pi x/l}{4}\right]$

$$= b_1 \sin\frac{\pi x}{l} + b_2 \frac{\sin 2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \cdots$$

Comparing both sides, we get

$b_1 = 3\ y_0/4,\ b_2 = 0,\ b_3 = -y_0/4,\ b_4 = b_5 = \ldots = 0$

The expression for $y(x,\ t)$ can therefore be written as follows:

$$y(x,t) = \frac{3y_0}{4} \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l} - \frac{y_0}{4} \sin\frac{3\pi x}{l} \cos\frac{3\pi ct}{l}$$

and hence the proof.

PRACTICE EXERCISES

1. The order and degree of the differential equation $y + \dfrac{dy}{dx} = \dfrac{1}{4}\displaystyle\int y \cdot dx$ are

(a) order = 2 and degree = 1
(b) order = 1 and degree = 2
(c) order = 1 and degree = 1
(d) order = 2 and degree = 2

2. Find the solution to $(x+1)\dfrac{dy}{dx} = 2xy$.

(a) $\log x = \log(y+1) + C$
(b) $\log y = 1/x + 1$
(c) $\log y = 2[x - \log|x+1|] + C$
(d) $y/x = C$

3. Find the solution to $x\dfrac{dy}{dx} = y - x\tan[y/x]$.

(a) $|\cos x/y| = |xC|$ (b) $|\sin y/x| = |C/x|$
(c) $|\sin y/x| = |yC|$ (d) $|\sec x/y| = |C/y|$

4. Find the solution to $\dfrac{dy}{dx} + y\sec x = \tan x$.

(a) $y(\sec x + \tan x) = \sec x + \tan x - x + C$
(b) $y\tan x = \sec x + x + C$
(c) $y(\operatorname{cosec} x + \cot x) = \operatorname{cosec} x + \cot x - x + C$
(d) $y \log\sec x = \tan x + C$

5. Find the solution to $\cos^2 x\dfrac{dy}{dx} + y = \tan x$.

(a) $y\ e^{\tan x} = \cos x\ \tan x + C$
(b) $y\ e^{\tan x} = e^{\tan x}\tan x + C$
(c) $y\ e^{\tan x} = (\tan x - 1) + C$
(d) $y\ e^{\tan x} = e^{\tan x}(\tan x - 1) + C$

6. Find the solution to $(x^2 - 2xy - y^2)dx - (x-y)^2 dy = 0\cdot$

(a) $\dfrac{x^3}{3} - x^2 y - y^2 x - x^2 = C$
(b) $3x^2 - y^2 x - x^2 - y^3/3 = C$
(c) $x^3/3 - x^2 y - y^2/2 = C$
(d) $\dfrac{x^3}{3} - x^2 y - y^2 x - \dfrac{y^3}{3} = C$

7. Find the solution to $(y^2 e^{xy^2} + 4x^3)dx + (2xy\, e^{xy^2} - 3y^2)dy = 0$.

(a) $e^{xy^2} + x^4/y^3 = C$ (b) $e^{xy^2} + x^4 - y^3 = C$

(c) $e^{xy^2} + xy^2 = C$ (d) $e^{xy^2} + x^3 + y^4 = C$

8. Find the solution to $\dfrac{dy}{dx} = \cos\left(y - \dfrac{xdy}{dx}\right)$

(a) $x = C^2 x + \cos c$ (b) $y = C(x-1) - \cos Cx$

(c) $y = cx + \cos^{-1} C$ (d) $y = \cos^{-1} x$

9. Find the solution to $9y'' + 6y' + y = 0$ for $y(0) = 4$ and $y'(0) = -1/3$.

(a) $y = (4+x)e^{-x/3}$ (b) $y = (4-x)e^{-x/3}$

(c) $y = (8-2x)e^{x/3}$ (d) $y = (1-x)e^{-x/3}$

10. Find the solution to $2y'' - 4y' + 8y = 0$.

(a) $y = e^x(A\sin\sqrt{3}x - B\cos\sqrt{3}x)$

(b) $y = e^x(A\cos\sqrt{3}x + B\sin\sqrt{3}x)$

(c) $y = e^{-x}(A\sin\sqrt{3}x + B\sin\sqrt{3}x)$

(d) $y = e^x(A\cos\sqrt{3}x - B\sin\sqrt{3}x)$

11. Find the solution to $y'' - 16y = 0$.

(a) $y = (c_1 + c_2)e^{4x}$ (b) $y = (c_1 + c_2)e^{-4x}$

(c) $y = c_1 e^{4x} - c_2 e^{-4x}$ (d) $y = c_1 e^{4x} + c_2 e^{-4x}$

12. Find the solution to $y'' - y = 0$.

(a) $y = c_1 e^x - c_2 e^x$ (b) $y = c_1(e^x + e^{-x})$

(c) $y = c_1 e^x + c_2 e^{-x}$ (d) $y = c_1 e^x - c_2 e^{-x}$

13. Find the solution to $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-4x}$.

(a) $y = (Ax+B)e^{-4x} + e^{-4x}$

(b) $y = (Ax+B)e^{-4x} + \dfrac{e^{-4x}}{9}$

(c) $y = (Ax+B)e^{-2x} + \dfrac{e^{-2x}}{9}$

(d) $y = (Ax+B)e^{4x} + e^{4x}$

14. Find the solution to $\left(\dfrac{d^2y}{dx^2} + 4\right)y = \cos 2x$.

(a) $y = A\cos 2x + B\sin 2x + \dfrac{x}{4}\sin 2x$

(b) $y = A\cos 2x + B\sin 2x$

(c) $y = \dfrac{x}{4}\sin 2x$

(d) $y = A\cos 2x + B\sin 2x - \dfrac{x}{4}\sin 2x$

15. Find the solution to $(D^2 - 5D + 6)y = e^x \cos 2x$.

(a) $y = c_1 e^{2x} + c_2 e^{3x} + \dfrac{e^x}{20}(3\sin 2x + \cos 2x)$

(b) $y = c_1 e^{2x} + c_2 e^{3x} - \dfrac{e^x}{20}$

(c) $y = c_1 e^{2x} + c_2 e^{3x} - \dfrac{e^x}{20}(3\cos 2x - \sin 2x)$

(d) $y = c_1 e^{2x} + c_2 e^{3x} - \dfrac{e^x}{20}(3\sin 2x + \cos 2x)$

16. Find the solution to the following homogeneous Euler–Cauchy equation

$$x^2 y'' - 5xy' + 9y = 0 \text{ for } x > 0$$

(a) $y = (C_1 + C_2 \ln x)x^3$

(b) $y = (C_1 \ln x + C_2)x^3$

(c) $y = (C_1 - C_2 \ln x)x^3$

(d) $y = (C_2 - C_1 \ln x)x^3$

17. A certain homogeneous Euler–Cauchy equation is represented by the following characteristic equation:

$$m^2 - 2m - 8 = 0$$

Find solution to the equation for $x > 0$.

(a) $m = 2, -4$

(b) $m = 2, 4$

(c) $m = -2, 4$

(d) $m = -2, -4$

18. A certain homogeneous Euler–Cauchy equation is represented by the characteristic equation that has roots equal to $(1 + 3i)$ and $(1 - 3i)$. Find solution to the equation for $x > 0$.

(a) $y = x\{C_1 \cos(\ln 3x) - C_2 \sin(\ln 3x)\}$

(b) $y = x\{C_1 \cos(\ln 3x) + C_2 \sin(\ln 3x)\}$

(c) $y = x\{C_1 \sin(\ln 3x) + C_2 \cos(\ln 3x)\}$

(d) None of these

19. Find the solution to the equation of Solved Example 16 for $x < 0$.

(a) $y = -\dfrac{C_1}{x} - C_2x^3$

(b) $y = \dfrac{C_1}{x} + C_2x^3$

(c) $y = -C_1x^3 - \dfrac{C_2}{x}$

(d) None of these

20. Find general solution to the Euler–Cauchy equation given by

$$x^2y'' - xy' + y = 0$$

(a) $y(x) = (C_1 + C_2 \ln x)x$

(b) $y(x) = (C_2 + C_1 \ln x)x$

(c) $y(x) = (C_1 - C_2 \ln x)x$

(d) $y(x) = (-C_1 - C_2 \ln x)x$

21. Find general solution to the following differential equation:

$$x^2y'' + 7xy' + 16y = 0$$

(a) $y(x) = C_2x^4 + C_1x^4 \ln x$

(b) $y(x) = C_1x^4 - C_2x^4 \ln x$

(c) $y(x) = C_1x^2 + C_2x^4 \ln x$

(d) $y(x) = C_1x^4 + C_2x^4 \ln x$

22. An Euler–Cauchy equation is represented by a characteristic equation $m^2 - 3m - 10 = 0$. Find the general solution for $x > 0$ and also for $x < 0$.

(a) $y = C_1x^5 + C_2x^{-2}$ for $x > 0$

$y = C_1x^5 - \dfrac{C_2}{x^2}$ for $x < 0$

(b) $y = C_1x^5 + \dfrac{C_2}{x^2}$ for $x > 0$

$y = -C_1x^5 + \dfrac{C_2}{x^2}$ for $x < 0$

(c) $y = C_1x^5 + \dfrac{C_2}{x^2}$ for $x > 0$

$y = C_1x^5 - \dfrac{C_2}{x^2}$ for $x < 0$

(d) None of these

23. Find the solution to the following differential equation given that $y(0) = 1$ and $y'(0) = 0$.

$$y''(t) + y(t) = 1$$

(a) 1 (b) 2

(c) -1 (d) None of these

24. Find the solution to the following differential equation given that $y(0) = 2$ and $y'(0) = -1$.

$$y'' - 3y' + 2y = 2e^{-t}$$

(a) $y(t) = \dfrac{e^{-t}}{3} + 4e^t + \dfrac{7}{3}e^{2t}$

(b) $y(t) = 3e^{-t} + 4e^t - \dfrac{7}{3}e^{2t}$

(c) $y(t) = \dfrac{e^{-t}}{3} + 4e^t - \dfrac{7}{3}e^{2t}$

(d) $y(t) = \dfrac{e^{-t}}{3} - 4e^t - \dfrac{7}{3}e^{2t}$

25. Find the solution to the following differential equation using Laplace transform method.

$$(D^2 + n^2)x = a \sin (nt + \alpha)$$

It is given that $x = 0$ at $t = 0$ and $Dx = 0$ at $t = 0$

(a) $\dfrac{a}{2n^2}[\sin nt \cos \alpha - nt \cos(nt + \alpha)]$

(b) $\dfrac{a}{2n^2}[\sin nt \cos \alpha + nt \cos(nt + \alpha)]$

(c) $\dfrac{a}{n^2}[\sin nt \cos \alpha - nt \cos(nt + \alpha)]$

(d) None of these

26. Find the solution to the following ordinary differential equation (ODE) using Laplace transform method.

$y'' - 5y' + 6y = 0$ given that $y(0) = 2$ and $y'(0) = 2$

(a) $y(t) = 4e^{2t} + 2e^{3t}$

(b) $y(t) = 4e^{3t} + 2e^{2t}$

(c) $y(t) = 4e^{2t} - 2e^{3t}$

(d) None of these

27. Find the solution to the following differential equation using Laplace transform method. It is given that $f(0^+) = f'(0^+) = 0$.

$$f''(t)+3f'(t)+2f(t)=e^{-t}$$

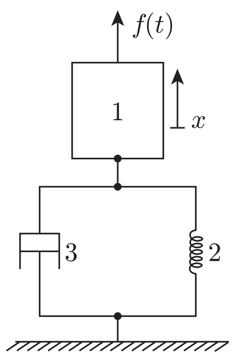
(a) $f(t)=e^{-2t}-e^{-t}+te^{-t}$

(b) $f(t)=e^{-2t}+e^{-t}-te^{-t}$

(c) $f(t)=e^{-2t}+e^{-t}+te^{-t}$

(d) $f(t)=e^{2t}+e^{t}+te^{-t}$

28. Refer to the mass spring damper system shown in the following figure. The system is at position $x = 0$ at $t = 0$. At $t = 0$, the initial velocity is 1 m/s. A force $f(t) = e^{-3t}$ is applied at this time. Find $x(t)$, that is, the trajectory of this mass spring damper system for $t \geq 0$ using method of Laplace transforms.



(a) $x(t)=1.5e^{-t}+2e^{-2t}+0.5e^{-3t}$

(b) $x(t)=1.5e^{-t}-2e^{-2t}+0.5e^{-3t}$

(c) $x(t)=-1.5e^{-t}-2e^{-2t}-0.5e^{-3t}$

(d) None of these

29. Find the solution to the following differential equation using Laplace transform method.

$$y''+y'-2y=4$$

It is given that $y(0) = 2,\ y'(0)=1$.

(a) $y(t)=-2-3e^{t}+e^{-2t}$

(b) $y(t)=2+3e^{t}+e^{-2t}$

(c) $y(t)=-2-3e^{-t}+e^{2t}$

(d) $y(t)=-2+3e^{t}+e^{-2t}$

30. Find the solution to the following equation by the method of variation of parameters.

$$y''-6y'+9y=e^{3x}/x^2.$$

(a) $y=(C_1+C_2x)e^{3x}-e^{3x}(1+\log x)$

(b) $y=(C_1-C_2x)e^{-3x}-e^{3x}(1+\log x)$

(c) $y=(C_1+C_2x)e^{-3x}-e^{-3x}(1+\log x)$

(d) None of these

31. Find general solution of the following non-homogeneous equation by using the variation of parameters method.

$$y''-5y'+6y=2e^{t}$$

(a) $y(t)=C_1e^{-3t}+C_2e^{-2t}+e^{-t}$

(b) $y(t)=C_1e^{3t}+C_2e^{2t}+e^{t}$

(c) $y(t)=C_1e^{3t}-C_2e^{2t}+e^{t}$

(d) None of these

32. For the following ordinary differential equation, use the method of variation of parameters to determine general solution.

$$y''-3y'-18y=t$$

(a) $y(t)=C_1e^{-6t}+C_2e^{-3t}+\dfrac{1}{18}t+\dfrac{1}{108}$

(b) $y(t)=C_1e^{6t}+C_2e^{-3t}-\dfrac{1}{18}t+\dfrac{1}{108}$

(c) $y(t)=-C_1e^{6t}-C_2e^{-3t}-\dfrac{1}{18}t+\dfrac{1}{108}$

(d) None of these

33. Find general solution for the following non-homogeneous ordinary differential equation using the method of variation of parameters.

$$y''+y=\tan x$$

(a) $y(x)=C_1\cos x+C_2\sin x-\cos x\ln|\sec x+\tan x|$

(b) $y(x)=C_1\cos x-C_2\sin x+\cos x\ln|\sec x+\tan x|$

(c) $y(x)=C_1\sin x+C_2\cos x-\cos x\ln|\sec x+\tan x|$

(d) $y(x)=-C_1\cos x-C_2\sin x+\cos x\ln|\sec x+\tan x|$

34. The steady state temperature of a rod of length l whose ends are kept at 40°C and 70°C is

(a) $\left[40+\dfrac{30x}{l}\right]$°C

(b) 55°C

(c) $\left[30+\dfrac{40x}{l}\right]$°C

(d) $\dfrac{30}{l}$°C

35. In the partial differential equation describing one-dimensional heat flow and given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

α^2 represents

(a) specific heat of the substance

(b) ratio of thermal conductivity to specific heat of substance

(c) diffusivity of the substance

(d) None of these

36. The partial differential equation describing transverse vibrations of string fastened on the two ends is given by

(a) $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$

(b) $\dfrac{\partial y}{\partial t} = c^2 \dfrac{\partial y}{\partial x}$

(c) $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial y}{\partial x}$

(d) $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$

37. When a vibrating string fastened to two points 'l' distance apart has an initial velocity (u_0), its initial conditions are given by

(a) $y(0, t) = y(l, t) = Y_{\max}$

(b) $y(0, t) = 0$, $y(l, t) = 0$ and $\left.\dfrac{\partial y}{\partial t}\right|_{t=0} = 0$

(c) $y(0, t) = \infty$, $y(l, t) = \infty$

(d) $y(0, t) = 0$, $y(l, t) = 0$, $\left.\dfrac{\partial y}{\partial t}\right|_{t=0} = \infty$

38. A tightly stretched string of length 'l' fastened at the two ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $\left(v_0 \sin^2 \dfrac{\pi x}{l}\right)$. Its initial conditions are

(a) $y(0, t) = y(l, t) = 0$

(b) $y(0, t) = y(l, t) = 0$ and $\left.\dfrac{\partial y}{\partial t}\right|_{t=0} = 0$

(c) $y(0, t) = y(l, t) = 0$ and $\left.\dfrac{\partial y}{\partial t}\right|_{t=0} = v_0 \sin^2 \dfrac{\pi x}{l}$

(d) None of these

39. In a two-dimensional heat flow in $x-y$ plane, the temperature along the normal to $x-y$ plane is

(a) zero

(b) given by two-dimensional Laplace equation

(c) given by three-dimensional Laplace equation

(d) none of these

40. The steady state behaviour of a two-dimensional heat flow is $x-y$ plane is given by

(a) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$

(b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = c^2$

(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(d) None of these

41. Two-dimensional steady state heat flow equation in polar coordinates is given by

(a) $r^2 \dfrac{\partial^2 u}{\partial r^2} + r \dfrac{\partial u}{\partial r} + \dfrac{\partial^2 u}{\partial \theta^2} = 0$

(b) $\dfrac{\partial^2 u}{\partial r^2} + r \dfrac{\partial u}{\partial r} + \dfrac{\partial^2 u}{\partial \theta^2} = 0$

(c) $r^2 \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{\partial^2 u}{\partial \theta^2} = 0$

(d) None of these

42. The three-dimensional heat flow is described in the transient state by the following partial differential equation.

(a) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right]$

(b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

(c) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

(d) $\dfrac{\partial u}{\partial t} = c^2 \left[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right]$

ANSWERS

1. (a)	**8.** (c)	**15.** (d)	**22.** (b)	**29.** (d)	**36.** (a)
2. (c)	**9.** (a)	**16.** (a)	**23.** (a)	**30.** (a)	**37.** (b)
3. (b)	**10.** (b)	**17.** (c)	**24.** (c)	**31.** (b)	**38.** (c)
4. (a)	**11.** (d)	**18.** (b)	**25.** (a)	**32.** (b)	**39.** (a)
5. (d)	**12.** (c)	**19.** (a)	**26.** (c)	**33.** (a)	**40.** (c)
6. (d)	**13.** (b)	**20.** (a)	**27.** (a)	**34.** (a)	**41.** (a)
7. (b)	**14.** (a)	**21.** (d)	**28.** (b)	**35.** (c)	**42.** (d)

EXPLANATIONS AND HINTS

1. (a) We have

$$y + \frac{dy}{dx} = \frac{1}{4}\int y \cdot dx$$

$$\Rightarrow \frac{dy}{dx} + \frac{d^2y}{dx^2} = \frac{1}{4}y \qquad \text{[on differentiating w.r.t. } x\text{]}$$

Differential equation is of order 2 and degree 1.

2. (c) We have

$$(x+1)\frac{dy}{dx} = 2xy$$

$$\Rightarrow (x+1)dy = 2xy\ dx$$

$$\Rightarrow \frac{dy}{y} = \frac{2x}{x+1}dx, \text{ if } x \neq -1$$

$$\Rightarrow \int \frac{dy}{y} = 2\int \frac{x}{x+1} \cdot dx$$

$$\Rightarrow \int \frac{1}{y} \cdot dy = 2\int \frac{x+1-1}{x+1} \cdot dx$$

$$\Rightarrow \int \frac{1}{y} \cdot dy = 2\int \left[1 - \frac{1}{x+1}\right] \cdot dx$$

$$\log y = 2[x - \log|x+1|] + C$$

3. (b) We have

$$x \cdot \frac{dy}{dx} = y - x\tan(y/x)$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \qquad \text{(i)}$$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = v - \tan v$$

$$\Rightarrow \quad x \cdot \frac{dv}{dx} = -\tan v \Rightarrow \cot v \cdot dv = \frac{-dx}{x}, \text{ if } x \neq 0$$

$$\int \cot v\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|\sin v| = -\log|x| + \log C$$

$$\Rightarrow \quad |\sin v| = \left|\frac{C}{x}\right|$$

$\Rightarrow |\sin(y/x)| = |C/x|$ is the required solution.

4. (a) We have

$$\frac{dy}{dx} + (\sec x)y = \tan x$$

This is of the form, $\frac{dy}{dx} + Py = Q$, where

$$P = \sec x,\ Q = \tan x$$

$$\text{I.F} = e^{\int P \cdot dx} = e^{\int \sec x} = e^{\log(\sec x + \tan x)}$$
$$= (\sec x + \tan x)$$

Multiplying both sides with I.F., we get

$$(\sec x + \tan x)\frac{dy}{dx} + y\sec x(\sec x + \tan x)$$
$$= \tan x(\sec x + \tan x)$$

Integrating both sides w.r.t. x, we get

$$y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int (\tan x \sec x + \tan^2 x)dx + C$$

$$= \int (\tan x \sec x + \sec^2 x - 1)dx + C$$

$\Rightarrow$ $y(\sec x + \tan x) = \sec x + \tan x - x + C$, which is the required solution.

5. (d) We have

$$\cos^2 x \cdot \frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \quad \frac{dy}{dx} + (\sec^2 x)y = \tan x \sec^2 x.$$

This is of the form, $\frac{dy}{dx} + Py = Q$, where

$$P = \sec^2 x, \quad Q = \tan x \sec^2 x$$

$$\text{I.F.} = e^{\int \sec^2 x \cdot dx} = e^{\tan x}$$

Multiplying both sides with I.F., we get

$$ye^{\tan x} = \int e^{\tan x} \cdot \tan x \sec^2 x \, dx + C$$

$\Rightarrow$ $ye^{\tan x} = \int t \cdot e^t \cdot dt + C$, where $t = \tan x$.

Now, integrating by parts

$$\Rightarrow \quad ye^{\tan x} = te^t - \int e^t \cdot dt + C$$

$$= te^t - e^t + C$$

$\Rightarrow$ $y^{e \tan x} = e^{\tan x}(\tan x - 1) + C$, which is the required solution.

6. (d) We have

$$(x^2 - 2xy - y^2)\,dx - (x+y)^2\,dy = 0$$

$$M = x^2 - 2xy - y^2, \quad N = -(x+y)^2 = -x^2y^2 - 2xy$$

$$\frac{\partial M}{\partial y} = -2x - 2y, \quad \frac{\partial N}{\partial x} = -2x - 2y$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, equation is exact. Now, general solution is given by

$$\underset{y = \text{const}}{\int M \cdot dx} + \underset{\substack{\text{Terms of } N \text{ not}\\ \text{containing } x}}{\int N \cdot dy} = C$$

$$\Rightarrow \quad \int (x^2 - 2xy - y^2)dx + \int -(y^2)dy = C$$

$$\Rightarrow \quad \frac{x^3}{3} - x^2y - y^2x - \frac{y^3}{3} = C$$

7. (b) We know

$$(y^2 e^{xy^2} + 4x^3)dx + (2x\,y e^{xy^2} - 3y^2)dy = 0$$

$$M = y^2 e^{xy^2} + 4x^3, \quad N = 2xy\,e^{xy^2} - 3y^2$$

Now,

$$\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 e^{xy^2}(2xy)$$

$$\frac{\partial N}{\partial x} = 2ye^{xy^2} + 2xy(e^{xy^2})(y^2)$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, equation is exact.

Now, general solution is given by

$$\underset{y = \text{const}}{\int M \cdot dx} = \underset{\substack{\text{Terms of } N \text{ not}\\ \text{containing } x}}{\int N \cdot dy} = C$$

$$\Rightarrow \quad \int (y^2 e^{xy^2} + 4x^3)\,dx + \int -3y^2 dy = C$$

$$\Rightarrow \quad y^2\left(\frac{e^{xy^2}}{y^2}\right) + \frac{4x^4}{4} - \frac{3y^3}{3} = C$$

$$\Rightarrow \quad e^{xy^2} + x^4 - y^3 = C$$

8. (c) We have

$$\frac{dy}{dx} = \cos\left(y - x\frac{dy}{dx}\right)$$

Putting $\frac{dy}{dx} = p$, we get

$$p = \cos(y - xp)$$

$$\Rightarrow \quad \cos^{-1} p = y - xp$$

$\Rightarrow y = px + \cos^{-1} p$, which is of the form $y = px + f(p)$

The general solution is given by

$$y = Cx + \cos^{-1}(C)$$

9. (a) We have

$$9y'' + 6y' + y = 0$$

The characteristic equation is given by

$$9m^2 + 6m + 1 = 0$$

$$\Rightarrow \quad m = \frac{-6 \pm \sqrt{36 - 36}}{2 \times 9} = \frac{-6}{18} \pm 0$$

$$= \frac{-1}{3}, \frac{-1}{3}$$

The roots are real and equal. The general solution is given by

$$y = (c_1 + c_2 x)e^{mx} \Rightarrow y = (c_1 + c_2 x)e^{-x/3} \quad (1)$$

Differentiating Eq. (1) w.r.t. x, we get

$$y' = \left[\frac{-1}{3}\right](c_1 + c_2 x)e^{-x/3} + c_2 e^{-x/3} \qquad (2)$$

Using $y\,(0) = 4$ in Eq. (1), we get

$$4 = (c_1 + c_2 + (0))\ e^0$$

$$\Rightarrow c_1 = 4$$

Using $y'\,(0) = -1/3$ in Eq. (2), we get

$$\frac{-1}{3} = \left(\frac{-1}{3}\right)\left(4 + c_2(0)\right)e^0 + c_2$$

$$\Rightarrow \quad \frac{-1}{3} = \left(\frac{-1}{3}\right)(4) + c_2$$

$$\Rightarrow \quad c_2 = \frac{4}{3} - \frac{1}{3} = 1$$

The particular solution is $y = (4 + x)\ e^{-x/3}$.

10. (b) We have

$$2y'' - 4y' + 8y = 0$$

The characteristic equation is given by

$$2m^2 - 4m + 8 = 0$$

$$\Rightarrow \quad m^2 - 2m + 4 = 0$$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{-12}}{2}$$

$$\Rightarrow \quad m = 2 \times \frac{(1 \pm \sqrt{3}i)}{2} \Rightarrow m = 1 \pm \sqrt{3}i$$

The roots are imaginary and distinct.

Hence, the solution is

$$y = e^x(A\cos\sqrt{3}x + B\sin\sqrt{3}x).$$

11. (d) We have

$$y'' - 16y = 0$$

The characteristic equation is given by

$$m^2 - 16 = 0 \Rightarrow m = \pm 4$$

The roots are real and distinct.

Hence, the general solution is given by $y = c_1 e^{4x} + c_2 e^{-4x}$.

12. (c) We have

$$y'' - y = 0$$

The characteristic equation is given by

$$m^2 - 1 = 0$$

$$(m + 1)(m - 1) = 0$$

$$m = -1,\ +1$$

The roots are real and distance. Hence, the general solution is given by

$$y = c_1 e^x + c_2 e^{-x}$$

13. (b) We have

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-4x}$$

The given equation can be written as

$$(D^2 + 2D + 1)y = e^{-4x}$$

The characteristic equation is given by

$$m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$$

The complementary function is given by

$$\text{C.F.} = (Ax + B)e^{-4x}$$

The particular integral is given by

$$\text{P.I.} = \frac{1}{D^2 + 2D + 1}e^{-4x} = \frac{1}{16 - 8 + 1}e^{-4x} = \frac{e^{-4x}}{9}$$

The complete solution is given by $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow \quad y = (Ax + B)e^{-4x} + \frac{e^{-4x}}{9}$$

14. (a) We have

$$\left(\frac{d^2y}{dx^2} + 4\right)y = \cos 2x$$

The differential equation can be written as

$$(D^2 + 4)y = \cos 2x$$

Auxiliary equation is $D^2 + 4 = 0$

$$D = \pm 2i$$

Complementary function is given by

$$\text{C. F.} = A\cos 2x + B\sin 2x$$

Particular integral is given by

$$\text{P.I.} = \frac{1}{(D^2 + 4)}\cos 2x$$

$$= \frac{x}{2D}\cos 2x$$

$$= \frac{x}{2}\left(\frac{1}{2}\sin 2x\right) = \frac{x}{4}\sin 2x$$

Complete solution, $y = \text{C.F.} + \text{P.I.}$

$$y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$$

15. (d) We have $(D^2 - 5D + 6)y = e^x \cos 2x$

Auxiliary equation is $(D^2 - 5D + 6) = 0$

$\Rightarrow \quad D = 2, 3$

Complementary function is given by

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$$

Particular integral is given by

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 5D + 6} e^x \cos 2x \\
&= e^x \cdot \frac{1}{(D+1)^2 - 5(D+1) + 6} \cdot \cos 2x \\
&= e^x \cdot \frac{1}{D^2 - 3D + 2} \cdot \cos 2x = e^x \cdot \frac{1}{-4 - 3D + 2} \\
&= -e^x \frac{1}{3D + 2} \cdot \cos 2x = -e^x \frac{3D - 2}{9D^2 - 4} \cos 2x \\
&= -e^x \frac{3D - 2}{9(-4) - 4} \cos 2x = \frac{e^x}{40}(3D - 2)\cos 2x \\
&= \frac{e^x}{40}(-60 \sin 2x - 2\cos 2x) \\
&= -\frac{e^x}{20}(3\sin 2x + \cos 2x)
\end{aligned}$$

The complete solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad y = c_1 e^{2x} + c_2 e^{3x} - \frac{e^x}{20}(3\sin 2x + \cos 2x)$$

16. (a) The characteristic equation representing the above equation is given by $m^2 - 6m + 9 = 0$, which gives $m = 3$, The general solution is therefore given by

$$y = (C_1 + C_2 \ln x)\, x^3$$

17. (c) $m = -2, 4$

The general solution is therefore given by

$$\frac{C_1}{x^2} + C_2 x^4$$

18. (b) $m_1 = (1 + 3i)$ and $m_2 = (1 - 3i)$

General solution is therefore given by

$$y = x[C_1 \cos(\ln 3x) + C_2 \sin(\ln 3x)]$$

19. (a) The solution for $x < 0$ can be determined from the solution for $x > 0$ by replacing x by $-x$.

Therefore, the solution is given by

$$y = -\frac{C_1}{x} - C_2 x^3$$

20. (a) The characteristic equation is given by $m^2 - 2m + 1 = 0$, which gives $m = 1, 1$

Therefore, general solution is given by

$$y(x) = (C_1 + C_2 \ln x)x$$

21. (d) The characteristic equation is given by

$$m^2 - 8m + 16 = 0$$

This gives $m = 4, 4$

Therefore, the general solution is given by

$$y(x) = C_1 x^4 + C_2 x^4 \ln x$$

22. (b) Roots of characteristic equation are $m = -2$, $+5$, therefore general solution for $x > 0$ is given by

$$y = C_1 x^5 + C_2 x^{-2} = C_1 x^5 + \frac{C_2}{x^2}$$

General solution for $x < 0$ is given by

$$y = -C_1 x^5 + C_2 x^{-2} = C_1 x^5 + \frac{C_2}{x^2}$$

Solution for $x < 0$ is found from solution for $x > 0$ by replacing (x) by $(-x)$.

23. (a) Taking Laplace transform on both sides, we get

$$[s^2 Y(s) - sy(0) - y'(0)] + Y(s) = 1/s$$

Substituting initial conditions, we get

$$s^2 Y(s) - s + Y(s) = \frac{1}{s}$$

or $$Y(s) = [s^2 + 1] = s + \frac{1}{s} = \frac{s^2 + 1}{s}$$

or $$Y(s) = \frac{1}{s}\, 1/\text{s}$$

Taking inverse Laplace transform, we get

$$y(t) = L^{-1}\left(\frac{1}{s}\right) = 1$$

24. (c) Taking Laplace transform on both sides of the given differential equation and then substituting the given initial conditions, we get

$$[s^2 Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0) + 2Y(s) = \frac{2}{(s+1)}$$

or $$(s^2 Y(s) - 2s + 1) - 3sY(s) + 6 + 2Y(s) = \frac{2}{(s+1)}$$

or $$Y(s)[s^2 - 3s + 2] - 2s + 7 = \frac{2}{(s+1)}$$

or
$$Y(s)[(s-2)(s-1)] = \frac{2}{s+1} + 2s - 7$$
$$= \frac{2 + 2s^2 + 2s - 7s - 7}{s+1}$$
$$= \frac{2s^2 - 5s - 5}{(s+1)}$$

or
$$Y(s) = \frac{2s^2 - 5s - 5}{(s+1)(s-1)(s-2)}$$

Using partial fraction expansion, we get
$$Y(s) = \frac{1/3}{(s+1)} + \frac{4}{(s-1)} + \frac{-7/3}{(s-2)}$$

Taking inverse Laplace transform, we get
$$y(t) = \frac{1}{3}e^{-t} + 4e^{t} - \frac{7}{3}e^{2t}$$

25. (a) Taking Laplace transform on both sides,
$$[s^2X(s) - sx(0) - x'(0)] + n^2X(s) = aL\sin(nt+\alpha)$$

Now $\sin(nt+\alpha) = (\sin nt \cos\alpha + \cos nt \sin\alpha)$

Therefore, Laplace transform of a sin $(nt + \alpha)$ is given by
$$a\cos\alpha \times \frac{n}{s^2+n^2} + a\sin\alpha \times \frac{s}{s^2+n^2}$$

Therefore, $(s^2+n^2)X(s) = \dfrac{na\cos\alpha}{s^2+n^2} + \dfrac{as\sin\alpha}{s^2+n^2}$

or
$$X(s) = an\cos\alpha \times \frac{1}{(s^2+n^2)^2} + a\sin\alpha \times \frac{s}{(s^2+n^2)^2}$$

Taking inverse Laplace transform, we get
$$x = an\cos\alpha \times \frac{1}{2n^2}(\sin nt - nt\cos nt) + a\sin\alpha \times \frac{t}{2n}\sin n$$
$$= a[\sin nt\cos\alpha - nt\cos(nt-\alpha)]/2n^2$$
$$= \frac{a}{2n^2}[\sin nt\cos\alpha - nt\cos(nt+\alpha)]$$

26. (c) Taking Laplace transform on both sides of the given differential equation, we get
$$[s^2Y(s) - sy(0) - y'(0)] - 5[sY(s) - y(0) + 6Y(s) = 0$$

Substituting the initial conditions, we get
$$(s^2Y(s) - 2s - 2) - 5(sY(s) - 2) + 6Y(s) = 0$$
$$Y(s)\ (s^2 - 5s + 6) - 2s + 8 = 0$$
$$\Rightarrow Y(s) = \frac{2s-8}{(s^2-5s+6)} = \frac{2s-8}{(s-2)(s-3)}$$

Using partial fraction expansion, we get
$$Y(s) = \frac{4}{s-2} - \frac{2}{s-3}$$

Taking inverse Laplace transform, we get
$$y(t) = 4e^{2t} - 2e^{3t}$$

27. (a) Taking Laplace transform on both sides, we get
$$s^2F(s) + 3sF(s) + 2F(s) = \frac{1}{s+1}$$

or
$$(s^2+3s+2)F(s) = \frac{1}{s+1}$$

or
$$F(s)(s+1)(s+2) = \frac{1}{s+1}$$

or
$$F(s) = \frac{1}{(s+1)^2(s+2)}$$

Using method of partial fraction expansion, we get
$$F(s) = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform, we get
$$f(t) = e^{-2t} - e^{-t} + te^{-t}$$

28. (b) The above system is represented by the following differential equation:
$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{-3t}$$

The initial conditions are
$$x(0) = 0,\ x'(0) = 1$$

Taking Laplace transform on both sides of differential equation:
$$[s^2X(s) - 1] + 3sX(s) + 2X(s) = \frac{1}{s+3}$$

or
$$X(s)[s^2+3s+2] = 1 + \frac{1}{s+3} = \frac{s+4}{s+3}$$
$$X(s) = \frac{s+4}{(s+3)(s+2)(s+1)}$$

Using partial fraction expansion, we get
$$X(s) = \frac{1.5}{s+1} - \frac{2}{s+2} + \frac{0.5}{s+3}$$

Taking inverse Laplace transform, we get
$$x(t) = 1.5e^{-t} - 2e^{-2t} + 0.5e^{-3t}$$

29. (d) Taking Laplace transform on both sides of differential equation.

$$[s^2Y(s) - sy(0) - y'(0)] + sY(s) - y(0) - 2Y(s) = \frac{4}{s}$$

or $$s^2Y(s) - 2s - 1 + sY(s) - 2 - 2Y(s) = \frac{4}{s}$$

or $$Y(s)[s^2 + s - 2] = \frac{4}{s} + 2 + 1 + 2s = \frac{4}{s} + 2s + 3$$

$$= \frac{2s^2 + 3s + 4}{s}$$

or $$Y(s) = \frac{2s^2 + 3s + 4}{s(s^2 + s - 2)} = \frac{2s^2 + 3s + 4}{s(s+2)(s-1)}$$

Using partial fraction expansion, we get

$$Y(s) = \frac{-2}{s} + \frac{3}{s-1} + \frac{1}{s+2}$$

Taking inverse Laplace transform, we get

$$y(t) = -2 + 3e^t + e^{-2t}$$

30. (a) Given equation can be written as

$$(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$$

The characteristic equation has two equal roots equal to 3.

Therefore, complementary function $= (C_1 + C_2x)e^{3x}$

Now $y_1 = e^{3x}$ and $y_2 = xe^{3x}$

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x} + 3xe^{6x} - 3e^{6x} = e^{6x}$$

Therefore,

$$\text{Particular integral } = y_1\int \frac{y_2X}{W}dx + y_2\int \frac{y_1X}{W}dx$$

$$= -e^{3x}\int \frac{xe^{3x}}{e^{6x}} \times \frac{e^{3x}}{x^2}dx + xe^{3x}\int \frac{e^{3x}}{e^{6x}} \times \frac{e^{3x}}{x^2}dx$$

$$= -e^{3x}\int \frac{dx}{x} + xe^{3x}\int \frac{dx}{x^2} = -e^{3x}(\log x + 1)$$

Complete solution is therefore given by

$$y = (C_1 + C_2x)e^{3x} - e^{3x}(\log x + 1)$$

31. (b) As a first step, we find fundamental solutions to the homogeneous equation. The characteristics equation is given by

$m^2 - 5m + 6 = 0$, which gives $m = 3, 2$

Therefore, $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$

In the next step, we complete the Wronskian.

$$W = \begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix} = e^{3t} \times 2e^{3t} - 3e^{3t} \times e^{2t} = -e^{5t}$$

Now, $$u_1' - \frac{y_2f}{w} = -e^{2t}(2e^t)(-e^{=5t}) = 2e^{-2t}$$

$$u_2' = e^{3t}(2e^t)(-e^{-5t}) = 2e^{-t}$$

Integrating u_1' and u_2', we get

$$u_1 = -e^{-2t} \text{ and } u_2 = 2e^{-t}$$

The particular solution is given by

$$y_p = (-e^{-2t})\ (e^{3t}) + (2e^{-t})\ (e^{2t}) = e^t$$

General solution is therefore given by

$$y(t) = C_1e^{3t} + C_2e^{2t} + e^t$$

32. (b) The corresponding homogeneous equation is

$$y'' - 3y' - 18y = 0$$

The characteristic equation is given by

$m^2 - 3m - 18 = 0$, which gives $m = -3, 6$

Two linearly independent solutions to the ordinary differential equation are given by

$$y_1(t) = e^{6t},\ y_2(t) = e^{-3t}$$

The Wronskian $W(t)$ is given by

$$W(t) = y_1y_2' - y_1'y_2 = e^{6t}(-3e^{-3t}) - (6e^{6t})e^{-3t} = -9e^{3t}$$

The particular solution is given by

$$y_p = -e^{6t}\int \frac{e^{-3t}.t}{-9e^{3t}}dt + e^{-3t}\int \frac{3^{6t}.t}{-9e^{3t}}dt$$

Now

$$\int \frac{te^{-3t}}{-9e^{3t}}dt = -\frac{1}{9}\int te^{-6t}dt = \frac{1}{54}te^{-6t} + \frac{1}{324}e^{-6t}$$

and $$\int \frac{te^{6t}}{-9e^{3t}}dt = -\frac{1}{9}\int te^{3t}dt = -\frac{1}{27}te^{3t} + \frac{1}{81}e^{3t}$$

Substituting for integral, we get

$$y_p = e^{-6t}\left[\frac{1}{54}te^{-6t} + \frac{1}{324}e^{-6t}\right] + e^{-3t}\left[-\frac{1}{27}te^{3t} + \frac{1}{81}e^{3t}\right]$$

$$= -\frac{1}{18}t + \frac{1}{108}$$

The general solution to the differential equation is given by

$$y(t) = C_1e^{6t} + C_2e^{-3t} - \frac{1}{18}t + \frac{1}{108}$$

33. (a) The homogeneous part is given by

$$y'' + y = 0$$

The roots of characteristic equation are $\pm\, i$.

The linearly independent solutions of the homogeneous part are given by

$$y_1(x) = \cos x \text{ and } y_2(x) = \sin x$$

$$y_p(x) = y_1(x)u(x) + y_2(x)v(x)$$

Wronskian,

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u(x) = -\int \frac{y_2(x)r(x)}{W}\,dx$$

and $\quad v(x) = \int \dfrac{y_1(x)r(x)}{W}\,dx$

Here, $\quad r(x) = \tan x$

$$u(x) = -\int \sin x \tan x\; dx = -\ln|\sec x + \tan x| + \sin x$$

$$v(x) = \int \sin x\; dx = -\cos x$$

$$y_p(x) = -\cos x \ln|\sec x + \tan x| + \sin x \cos x - \sin x \cos x$$

$$\Rightarrow y_p(x) = -\cos x \ln|\sec x + \tan x|$$

Therefore, general solution is given by

$$y(x) = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|$$

34. (a) The steady state temperature is given by

$$T_1 + \frac{(T_2 - T_1)x}{l}$$

35. (c)

36. (a) This is the standard second order one-dimensional wave equation.

37. (b) The initial transverse velocity of any point on the string is zero. Also, the displacement of points that are fastened is also zero.

38. (c)

39. (a) In the case of two-dimensional heat flow in x–y plane, the following equation describes temperature distribution in x–y plane in the transient state.

$$\frac{\partial u}{\partial t} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

40. (c) Temperature distribution in the steady state is given by two-dimensional Laplace equation.

41. (a) This is a standard form of two-dimensional Laplace equation in polar coordinates.

42. (d) This is a standard equation for three-dimensional heat flow.

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. The solution of the differential equation $\dfrac{dy}{dx} + y^2 = 0$ is

(a) $y = \dfrac{1}{x+c}$

(b) $y = \dfrac{-x^3}{3} + c$

(c) ce^x

(d) unsolvable as equation is non-linear

(GATE 2003, 2 Marks)

Solution: Given differential equation

$$\frac{dy}{dx} + y^2 = 0$$

$$\Rightarrow \frac{dy}{y^2} = dx$$

On integrating both sides, we get $-\int \dfrac{dy}{y^2} = \int dx$

$$\frac{1}{y} = x + c$$

$$\Rightarrow y = \frac{1}{x+c}$$

Ans. (a)

2. Biotransformation of an organic compound having concentration (x) can be modeled using an ordinary differential equation $\dfrac{dx}{dt} + kx^2 = 0$, where k is the reaction rate constant. If $x = a$ at $t = 0$, the solution of the equation is

(a) $x = ae^{-kt}$ (b) $\dfrac{1}{x} = \dfrac{1}{a} + kt$

(c) $x = a(1 - e^{-kt})$ (d) $x = a + kt$

(GATE 2004, 2 Marks)

Solution: We have,

$$\frac{dx}{dt} = -kx^2$$

This is in variable separable form,

$$\Rightarrow \frac{dx}{x^2} = -kdt$$

Integrating both sides,

$$\int \frac{dx}{x^2} = -\int kdt$$

$$-\frac{1}{x} = -kt + C$$

$$\Rightarrow \frac{1}{x} = kt + C'$$

At $t = 0$ and $x = a$

$$-\frac{1}{a} = -k(0) + C'$$

$$\Rightarrow C' = \frac{1}{a}$$

$$\frac{1}{x} = kt + \frac{1}{a}$$

Ans. (b)

3. The following differential equation has

$$3\left(\frac{d^2y}{dt^2}\right) + 4\left(\frac{dy}{dt}\right)^3 + y^2 + 2 = x.$$

(a) Degree = 2, order = 1
(b) Degree = 1, order = 2
(c) Degree = 4, order = 3
(d) Degree = 2, order = 3

(GATE 2005, 1 Mark)

Solution: The highest derivative term of the equation is 2, hence order = 2.
The power of highest derivative term is 1, hence degree = 1.

Ans. (b)

4. The solution of the first-order differential equation $x'(t) = -3x(t)$, $x(0) = x_0$ is

(a) $x(t) = x_0 = x_0 e^{-3t}$ (b) $x(t) = x_0 e^{-3}$
(c) $x(t) = x_0 e^{-1/3}$ (d) $x(t) = x_0 e^{-1}$

(GATE 2005, 1 Mark)

Solution: Given

$$x(t) = -3x(t)$$

$$\Rightarrow \frac{dx}{dt} = -3x$$

$$\Rightarrow \frac{dx}{x} = -3dt$$

Integrating both sides, we get

$$\Rightarrow \int \frac{dx}{x} = \int -3dt$$

$$\Rightarrow \quad \ln x = -3t + C$$

$$\Rightarrow \quad x = e^{-3t+C} = e^C \times e^{-3t} \quad \text{(i)}$$

Putting $e^C = C_1$ in Eq. (i), we get

$$x = C_1 \times e^{-3t}$$

Now putting initial condition, $x(0) = x_0$

$$x_0 = C_1 e^0 = C_1$$

$$\Rightarrow C_1 = x_0$$

Therefore, solution is $x(t) = x_0 e^{-3t}$.

Ans. (a)

5. The solution of the following differential equation is given by

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

(a) $y = e^{2x} + e^{-3x}$ (b) $y = e^{2x} + e^{3x}$
(c) $y = e^{-2x} + e^{3x}$ (d) $y = e^{-2x} + e^{-3x}$

(GATE 2005, 1 Mark)

Solution: We have

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

The equation can be written as

$$(D^2 - 5D + 6)y = 0$$

Auxiliary equation is given by

$$D^2 - 5D + 6 = 0$$

$$\Rightarrow (D-2)(D-3) = 0$$

$$\Rightarrow D = 2, 3$$

Therefore, $y = e^{2x} + e^{3x}$.

Ans. (b)

6. The solution of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 0$; $y(0) = 1$; $\frac{dy}{dx}\left(\frac{\pi}{4}\right) = 0$ in the range $0 < x < \frac{\pi}{4}$ is given by

(a) $e^{-x}\left(\cos 4x + \frac{1}{4}\sin 4x\right)$

(b) $e^{x}\left(\cos 4x - \frac{1}{4}\sin 4x\right)$

(c) $e^{-4x}\left(\cos 4x - \frac{1}{4}\sin x\right)$

(d) $e^{-4x}\left(\cos 4x - \frac{1}{4}\sin 4x\right)$

(GATE 2005, 2 Marks)

Solution: We have

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 0$$
$$y(0) = 1$$
$$\frac{dy}{dx}\left(\frac{\pi}{4}\right) = 0$$

This is a linear differential equation which can be written as

$$D^2 + 2D + 17 = 0$$
$$D = -1 \pm 4i$$

$$\therefore \quad y = C_1 e^{(-1+4i)x} + C_2 e^{(-1-4i)x}$$
$$= e^{-x} C_1 e^{4xi} + C_2 e^{-4xi}$$

$$= e^{-x}[C_1(\cos 4x + i\sin 4x)]$$
$$+C_2\left[\cos(-4x) + i\sin(-4x)\right]$$
$$= e^{-x}\left[(C_1 + C_2)\cos 4x + (C_1 - C_2)i\sin 4x\right]$$

Let $C_1 + C_2 = C_3$ and $(C_1 - C_2)i = C_4$

$$y = e^{-x}(C_3\cos 4x + C_4\sin 4x) \qquad \text{(i)}$$

Also, we know that $y(0) = 1$. Applying this value in Eq. (i), we get

$$1 = e^0(C_3\cos 0 + C_4\sin 0)$$

$$\Rightarrow \quad C_3 = 1$$

$$\frac{dy}{dx} = e^{-x}(-4C_3\sin 4x + 4C_4\cos 4x)$$
$$-e^{-x}[C_3\cos 4x + C_4\sin 4x]$$
$$= e^{-x}[(-4C_3 - C_4)\sin 4x + (4C_4 - C_3)\cos 4x]$$

Now, at $x = \frac{\pi}{4}, \frac{dy}{dx} = 0$. Thus,

$$-(-4C_4 - C_3)e^{-\pi/4} = 0$$

$$\Rightarrow 4C_4 = C_3$$
$$\Rightarrow C_4 = \frac{C_3}{4} = \frac{1}{4} \qquad [\because C_3 = 1]$$

Hence, $C_3 = 1$ and $C_4 = \frac{1}{4}$. Applying these values in Eq. (1), we get

$$y = e^{-x}\left(\cos 4x + \frac{1}{4}\sin 4x\right)$$

Ans. (a)

7. Transformation to the linear form by substituting $v = y^{1-n}$ to the equation $\frac{dy}{dt} + p(t)y = q(t)y^n; n > 0$ will be

(a) $\frac{dv}{dt} + (1-n)pv = (1-n)q$

(b) $\frac{dv}{dt} + (1-n)pv = (1+n)q$

(c) $\frac{dv}{dt} + (1+n)pv = (1-n)q$

(d) $\frac{dv}{dt} + (1+n)pv = (1+n)q$

(GATE 2005, 2 Marks)

Solution: The given equation is

$$\frac{dy}{dt} + p(t)y = q(t)y^n; n > 0 \qquad \text{(i)}$$

Substituting $v = y^{1-n}$ in Eq. (i), we get

$$\frac{dv}{dt} = (1-n)y^{-n}\frac{dy}{dt}$$
$$\Rightarrow \frac{dy}{dt} = \frac{1}{(1-n)y^{-n}}\frac{dv}{dt}$$

Substituting dy/dt in the given differential equation, we get

$$\frac{1}{(1-n)y^{-n}}\frac{dv}{dt} + p(t)y = q(t)y^n$$

$$\Rightarrow \frac{dv}{dt} + p(t)(1-n)y^{1-n} = q(t)(1-n)$$

Now since $y^{1-n} = v$, we get

$$\frac{dv}{dt} + (1-n)pv = (1-n)q \qquad \text{(ii)}$$

Equation (ii) is linear with v as dependent variable and t as independent variable.

Ans. (a)

Common Data for Questions 7 and 8: The complete solution of the ordinary differential equation $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$ is $y = c_1e^{-x} + c_2e^{-3x}$.

8. Then p and q are

(a) $p = 3, q = 3$ (b) $p = 3, q = 4$
(c) $p = 4, q = 3$ (d) $p = 4, q = 4$

(GATE 2005, 2 Marks)

Solution: The given equation is

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

The equation can be written as

$$(D^2 + pD + q)y = 0$$

$$\therefore \quad D^2 + pD + q = 0$$

Its solution is $y = C_1e^{-x} + C_2e^{-3x}$.

So the roots of $D^2 + pD + q = 0$ are $\alpha = -1$ and $\beta = -3$.

Sum of roots $= -p = -(-1-3) \Rightarrow p = 4$ and

product of roots $= q = (-1)(-3) \Rightarrow q = 3$.

Ans. (c)

9. Which of the following is a solution of the differential equation $\dfrac{d^2y}{dx^2} + p\dfrac{dy}{dx} + (q+1) = 0$?

(a) e^{-3x} (b) xe^{-x}
(c) xe^{-2x} (d) x^2e^{-2x}

(GATE 2005, 2 Marks)

Solution: The given equation is

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + (q+1) = 0$$

This equation can be written as

$$\left[D^2 + pD + (q+1)\right]y = 0$$

Substituting $p = 4$ and $q = 3$, we get

$$(D^2 + 4D + 4)y = 0$$

$$\Rightarrow D^2 + 4D + 4 = 0$$

$$(D+2)^2 = 0$$

$$\therefore \quad D = -2, -2 \Rightarrow y = (c_1x + c_2)e^{-2x}$$

Out of the options given, $y = xe^{-2x}$ is the only answer in the required form if $c_1 = 1$ and $c_2 = 0$.

Ans. (c)

10. The solution of the differential equation $\dfrac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

(a) $(1+x)e^{+x^2}$ (b) $(1+x)e^{-x^2}$
(c) $(1-x)e^{+x^2}$ (d) $(1-x)e^{-x^2}$

(GATE 2006, 1 Mark)

Solution: The given equation is

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

This is a first-order linear differential equation.

Integrating factor is given by

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

Solution is obtained using

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$$

$$ye^{x^2} = \int e^{-x^2}e^{x^2}\,dx + c$$

$$ye^{x^2} = x + c$$

At $\quad x = 0, y = 1$ (given)

$$\Rightarrow \quad 1e^0 = 0 + c$$

$$\Rightarrow \quad c = 1$$

So, the solution is $ye^{x^2} = x + 1$

$$\Rightarrow \quad y = e^{-x^2}(x+1)$$

Ans. (b)

11. A spherical naphthalene ball exposed to the atmosphere loses volume at a rate proportional to its instantaneous surface area due to evaporation. If the initial diameter of the ball is 2 cm and the diameter reduces to 1 cm after 3 months, the ball completely evaporates in

(a) 6 months (b) 9 months
(c) 12 months (d) infinite time

(GATE 2006, 2 Marks)

Solution: We have

$$\frac{dV}{dt} = -kA \qquad \text{(i)}$$

where $V = \dfrac{4}{3}\pi r^3$ and $A = 4\pi r^2$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

Substituting above values in Eq. (i), we get

$$4\pi^2\frac{dr}{dt} = -k(4\pi r^2)$$

$$\Rightarrow \frac{dr}{dt} = -k$$

$$\Rightarrow dr = -kdt$$

Integrating on both sides, we get

$$r = -kt + C \tag{2}$$

Substituting $t = 0$ and $r = 1$ in Eq. (2), we get

$$1 = k \times 0 + C$$

$$\Rightarrow \quad C = 1$$

$$\therefore \quad r = -kt + 1 \tag{3}$$

Now at $t = 3$ months, $r = 0.5\,\text{cm}$

$$\therefore \quad 0.5 = -k \times 3 + 1$$

$$\Rightarrow \quad k = \frac{0.5}{3}$$

Now substituting this value of k in Eq. (3) we get

$$r = -\frac{0.5}{3}t + 1$$

Putting $r = 0$ (ball completely evaporates) in the above equation and solving for t gives

$$0 = -\frac{0.5}{3}t + 1$$

$$\Rightarrow \quad t = 6\,\text{months}$$

Ans. (a)

12. For $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 3e^{2x}$, the particular integral is

(a) $\dfrac{1}{15}e^{2x}$ (b) $\dfrac{1}{5}e^{2x}$

(c) $3e^{2x}$ (d) $C_1e^{-x} + C_2e^{-3x}$

(GATE 2006, 2 Marks)

Solution: We have

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3e^{2x}$$

The given equation can be written as

$$(D^2 + 4D + 3)y = 3e^{2x}$$

Particular integral is given by

$$\text{P.I.} = \frac{1}{D^2 + 4D + 3}3e^{2x}$$

Now, since $\dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax}$

$$\text{P.I.} = 3\frac{e^{2x}}{(2)^2 + 4(2) + 3} = \frac{3e^{2x}}{15} = \frac{e^{2x}}{5}$$

Ans. (b)

13. The solution for the differential equation $\dfrac{dy}{dx} = x^2y$ with the condition that $y = 1$ at $x = 0$ is

(a) $y = e^{\frac{1}{2x}}$ (b) $\ln(y) = \dfrac{x^3}{3} + 4$

(c) $\ln(y) = \dfrac{x^2}{2}$ (d) $y = e^{\frac{x^3}{3}}$

(GATE 2007, 1 Mark)

Solution: We have

$$\frac{dy}{dx} = x^2y$$

Hence, by variable separable method, we have

$$\frac{dy}{y} = x^2dx$$

$$\int\frac{dy}{y} = \int x^2dx$$

$$\Rightarrow \quad \ln y = \frac{x^3}{3} + C_1$$

$$\Rightarrow \quad y = e^{\left(\frac{x^3}{3}+C_1\right)} = e^{C_1}\cdot e^{\frac{x^3}{3}}$$

$$y = C \times e^{\frac{x^3}{3}}$$

Also, we know that $y = 1$ at $x = 0$. Therefore,

$$1 = C \times e^{\frac{0}{3}}$$

$$\Rightarrow \quad C = 1$$

Therefore, $y = e^{\frac{x^3}{3}}$ is the solution. Ans. (d)

14. The degree of the differential equation $\dfrac{d^2x}{dt^2} + 2x^3 = 0$ is

(a) 0 (b) 1
(c) 2 (d) 3

(GATE 2007, 1 Mark)

Solution: The degree of a differential equation is the power of its highest order derivative after the differential equation is made free of radicals and fractions, if any, in derivative power.

Hence, here the degree is 1, which is power of $\dfrac{d^2x}{dt^2}$.

Ans. (b)

15. A body originally at 60°C cool down to 40°C in 15 minutes when kept in air at a temperature of 25°C. What will be the temperature of the body at the end of 30 minutes?

(a) 35.2°C (b) 31.5°C
(c) 28.7°C (d) 15°C

(GATE 2007, 2 Marks)

Solution: We know that Newton's law of cooling is given by

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

Calculating using variable separable form, we get

$$\frac{d\theta}{\theta - \theta_0} = -k\,dt$$

$$\int \frac{d\theta}{\theta - \theta_0} = \int -k\,dt$$

$$\Rightarrow \qquad \ln\left(\theta - \theta_0\right) = -kt + C_1$$

$$\Rightarrow \qquad \theta - \theta_0 = C \cdot e^{-kt} \quad (\text{where } C = e^{c_1})$$

$$\theta = \theta_0 + C \cdot e^{-kt}$$

We are given, $\theta_0 = 25°\text{C}$

Now at $t = 0$ and $\theta = 60°\text{C}$,

$$60 = 25 + C \cdot e^0$$

$$\Rightarrow \qquad C = 35$$

$$\therefore \qquad \theta = 25 + 35e^{-kt}$$

At $t = 15$ minutes and $\theta = 40°\text{C}$,

$$40 = 25 + 35e^{(-k\times 15)}$$

$$\Rightarrow e^{-15k} = \frac{3}{7} \qquad \text{(i)}$$

Now at $t = 30$ minutes,

$$\theta = 25 + 35e^{-30k} = 25 + 35(e^{-15k})^2$$

Now substituting $e^{-15k} = \frac{3}{7}$ from Eq. (i), we get

$$\theta = 25 + 35 \times \left(\frac{3}{7}\right)^2 = 31.428°\text{C} \approx 31.5°\text{C}$$

Ans. (b)

16. The solution of $\frac{dy}{dx} = y^2$ with initial value $y(0) = 1$ bounded in the interval is

(a) $-\infty \le x \le \infty$ (b) $-\infty \le x \le 1$
(c) $x < 1, x > 1$ (d) $-2 \le x \le 2$

(GATE 2007, 2 Marks)

Solution: Given that

$$\frac{dy}{dx} = y^2$$

$$\Rightarrow \qquad \int \frac{dy}{y^2} = \int dx$$

$$\Rightarrow \qquad -\frac{1}{y} = x + C$$

$$\therefore \qquad y = -\frac{1}{x + C}$$

Also, when $x = 0$, $y = 1$

$$\Rightarrow \qquad 1 = -\frac{1}{0 + C} \Rightarrow C = -1$$

$$\therefore \qquad y = -\frac{1}{x - 1}$$

y is bounded when $x - 1 \neq 0$

i.e. $x \neq 1$

$$\Rightarrow x < 1 \text{ or } x > 1$$

Ans. (c)

17. Which of the following is a solution to the differential equation $\frac{dx(t)}{dt} + 3x(t) = 0$?

(a) $x(t) = 3e^{-t}$ (b) $x(t) = 2e^{-3t}$
(c) $x(t) = -\frac{3}{2}t^2$ (d) $x(t) = 3t^2$

(GATE 2008, 1 Mark)

Solution: We have

$$\frac{dx(t)}{dt} + 3x(t) = 0$$

$$\Rightarrow \qquad \frac{dx}{dt} = -3x$$

$$\Rightarrow \qquad \frac{dx}{x} = -3dt$$

$$\Rightarrow \qquad \int \frac{dx}{x} = -3dt$$

$$\ln x = -3t + c$$

$$\Rightarrow \qquad x = e^{-3t} + e^c$$

$$\Rightarrow \qquad x = c_1 e^{-3t} \quad (c_1 = e^c)$$

$$\Rightarrow \qquad x = c_1 e^{-3t}$$

Hence, from the given options, $x(t) = 2e^{-3t}$ is the correct answer.

Ans. (b)

18. The general solution of $\dfrac{d^2y}{dx^2}+y=0$ is

(a) $y=P\cos x+Q\sin x$ (b) $y=P\cos x$

(c) $y=P\sin x$ (d) $y=P\sin^2 x$

(GATE 2008, 1 Mark)

Solution: We have

$$\frac{d^2y}{dx^2}+y=0$$

This equation can be written as

$$D^2+1=0$$

$$\Rightarrow \quad D=\pm i=0\pm 1i$$

Therefore, the general solution is given by

$$\begin{aligned} y &= e^{0x}\left[C_1\cos(1\times x)+C_2\sin(1\times x)\right] \\ &= C_1\cos x+C_2\sin x \\ &= P\cos x+Q\sin x \end{aligned}$$

where P and Q are arbitrary constants.

Ans. (a)

19. Given that $\ddot{x}+3x=0$ and $x(0)=1$, $\dot{x}(0)=0$, what is $x(1)$?

(a) -0.99 (b) -0.16
(c) 0.16 (d) 0.99

(GATE 2008, 1 Mark)

Solution: We have

$$\ddot{x}+3x=0$$

Auxiliary equation is given by

$$D^2+3=0$$

$$\Rightarrow D=\pm\sqrt{3}i$$

Therefore, the solution is given by

$$x=A\cos\sqrt{3}t+B\sin\sqrt{3}t \qquad (1)$$

We know that at $t=0$, $x=1$

$$\Rightarrow \quad A=1$$

Now $\quad \dot{x}=\sqrt{3}(B\cos\sqrt{3}t-A\sin\sqrt{3}t)$

At $\quad t=0,\ \dot{x}=0$

$$\Rightarrow \quad B=0$$

Applying values of A and B in Eq. (1), we get

So $x=\cos\sqrt{3}t$

$$x(1)=\cos\sqrt{3}=0.99$$

Ans. (d)

20. The solution of $\dfrac{dy}{dx}=-\dfrac{x}{y}$ at $x=1$ and $y=\sqrt{3}$ is

(a) $x-y^2=-2$ (b) $x+y^2=4$

(c) $x^2-y^2=-2$ (d) $x^2+y^2=4$

(GATE 2008, 2 Marks)

Solution: We have

$$\frac{dy}{dx}=\frac{-x}{y}$$

$$\Rightarrow \quad y\,dy=-x\,dx$$

$$\Rightarrow \quad \int y\,dy=\int -x\,dx$$

$$\frac{y^2}{2}=-\frac{x^2}{2}+C$$

Also, we know that at $x=1, y=\sqrt{3}$. Therefore,

$$\frac{(\sqrt{3})^2}{2}=\frac{-t^2}{2}+C$$

$$\Rightarrow \quad C=2$$

$\therefore$ The solution is $\dfrac{y^2}{2}=\dfrac{-x^2}{2}+2$

$$\Rightarrow \quad x^2+y^2=4$$

Ans. (d)

21. It is given that $y''+2y'+y=0$, $y(0)=0$, $y(1)=0$. What is $y(0.5)$?

(a) 0 (b) 0.37
(c) 0.62 (d) 1.13

(GATE 2008, 2 Marks)

Solution: We have

$$y''+2y'+y=0$$

This equation can be written as

$$(D^2+2D+1)y=0$$

Also, auxiliary equation is given as

$$\Rightarrow \quad D^2+2D+1=0$$

$$\Rightarrow \quad (D+1)^2=0$$

$$\Rightarrow \quad D=-1,-1$$

$$\therefore \quad y=\left(C_1+C_2x\right)e^{-x}$$

$$y(0)=0\Rightarrow 0=\left(C_1+C_2(0)\right)e^{-0}$$

$$\Rightarrow \quad C_1=0$$

$$y(1)=0\Rightarrow 0=\left(C_1+C_2\right)e^{-1}$$

$$\Rightarrow \quad C_1+C_2=0$$

$$\Rightarrow \quad C_2=0$$

$$\therefore \quad y=\left(0+0x\right)e^{-x}=0 \text{ is the solution}$$

$$\therefore \quad y(0.5)=0$$

Ans. (a)

22. The order of the differential equation $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$ is

(a) 1 (b) 2
(c) 3 (d) 4

(GATE 2009, 1 Mark)

Solution: The highest derivative of differential is 2. Hence, the order of the equation is 2.

Ans. (b)

23. The solution of the differential equation $3y\frac{dy}{dx} + 2x = 0$ represents a family of

(a) ellipses (b) circle
(c) parabola (d) hyperbola

(GATE 2009, 2 Marks)

Solution: We know that

$$3y\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{3y}$$

$$\Rightarrow \quad 3y\,dy = -2x\,dx$$

$$\Rightarrow \quad \int 3y\,dy = \int -2x\,dx$$

$$\Rightarrow \quad \frac{3}{2}y^2 = -2\times\frac{x^2}{2} + C$$

$$\Rightarrow \quad 3y^2 + 2x^2 = C$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{1}{2}\right)} + \frac{y^2}{\left(\frac{1}{3}\right)} = C$$

$$\Rightarrow \quad \frac{x^2}{\left(\frac{1}{2}C\right)} + \frac{y^2}{\left(\frac{1}{3}C\right)} = 1$$

which is the equation of a family of ellipses.

Ans. (a)

24. Match List I with List II and select the correct answer using the codes given below the lists:

List I	**List II**
A. $\frac{dy}{dx} = \frac{y}{x}$	1. Circles
B. $\frac{dy}{dx} = -\frac{y}{x}$	2. Straight lines
C. $\frac{dy}{dx} = \frac{x}{y}$	3. Hyperbola
D. $\frac{dy}{dx} = -\frac{x}{y}$	

Codes:

	A	**B**	**C**	**D**
(a)	2	3	3	1
(b)	1	3	2	1
(c)	2	1	3	3
(d)	3	2	1	2

(GATE 2009, 2 Marks)

Solution:

A. $\frac{dy}{dx} = \frac{y}{x}$

$$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \int\frac{dy}{y} = \int\frac{dx}{x}$$

$$\log y = \log x + \log c = \log cx$$

$y = cx$, which is the equation of a straight line.

B.
$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{y} = \frac{-dx}{x} \Rightarrow \int\frac{dy}{y} = -\int\frac{dx}{x}$$

$$\log y = -\log x + \log c$$

$$\log y + \log x = \log c$$

$$\log yx = \log c$$

$$yx = c$$

$y = c/x$, which is the equation of a hyperbola.

C.
$$\frac{dy}{dx} = \frac{x}{y}, ydy = xdx \Rightarrow \int ydy = \int xdx$$

$$\frac{y^2}{2} - \frac{x^2}{2} = \frac{c^2}{2} \rightarrow \text{const}$$

$$y^2 - x^2 = c^2$$

$\frac{y^2}{c^2} - \frac{x^2}{c^2} = 1$, which is the equation of a hyperbola.

D.
$$\frac{dy}{dx} = \frac{-x}{y} \Rightarrow \int ydy = -\int xdx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{c^2}{2}$$

$$\frac{y^2}{2} + \frac{x^2}{2} = \frac{c^2}{2}$$

$x^2 + y^2 = c^2$, which is the equation of a circle.

Ans. (a)

25. The solution of $x\frac{dy}{dx} + y = x^4$ with the condition $y(1) = \frac{6}{5}$ is

(a) $y = \frac{x^4}{5} + \frac{1}{x}$ (b) $y = \frac{4x^4}{5} + \frac{4}{5x}$

(c) $y = \frac{x^4}{5} + 1$ (d) $y = \frac{x^5}{5} + 1$

(GATE 2009, 2 Marks)

Solution: The given differential equation is

$$x\frac{dy}{dx} + y = x^4$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{y}{x}\right) = x^3 \quad (1)$$

The standard form of Leibnitz linear equation is given by

$$\frac{dy}{dx} + Py = Q \quad (2)$$

where P and Q are functions of x and the solution is given by

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$$

From Eq. (1), $P = \frac{1}{x}$ and $Q = x^3$. Therefore,

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Hence, the solution is given by

$$yx = \int x^3 \cdot x\,dx + c$$

$$yx = \frac{x^5}{5} + c$$

Also, we are given that $y(1) = \frac{6}{5}$

Hence, $x = 1$, $y = \frac{6}{5}$

$$\Rightarrow \quad \frac{6}{5}(1) = \frac{1}{5} + c$$

$$\Rightarrow \quad c = \frac{6}{5} - \frac{1}{5} = 1$$

Therefore, $yx = \frac{x^5}{5} + 1$

$$\Rightarrow \quad y = \frac{x^4}{5} + \frac{1}{x}$$

Ans. (a)

26. The Blasius equation, $\frac{d^3f}{d\eta^3} + \frac{f\,d^2f}{2d\eta^2} = 0$, is a

(a) second-order non-linear ordinary differential equation
(b) third-order non-linear ordinary differential equation
(c) third-order linear ordinary differential equation
(d) mixed order non-linear ordinary differential equation

(GATE 2010, 1 Mark)

Solution: $\frac{d^3f}{dn^3} + \frac{fd^2f}{2dn^2} = 0$ is third order $\left(\frac{d^3f}{dn^3}\right)$ and it is non-linear, because the product $f \times \frac{d^2f}{dn^2}$ is not allowed in linear differential equation.

Ans. (b)

27. A function $n(x)$ satisfies the differential equation $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$, where L is a constant. The boundary conditions are $n(0) = k$ and $n(\infty) = 0$. The solution to this equation is

(a) $n(x) = k\exp(x/L)$ (b) $n(x) = k\exp(-x/\sqrt{L})$
(c) $n(x) = k^2\exp(-x/L)$ (d) $n(x) = k\exp(-x/L)$

(GATE 2010, 1 Mark)

Solution: We know that

$$\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$$

The equation can be written as

$$\left(D^2 - \frac{1}{L^2}\right)n(x) = 0$$

$$\Rightarrow D^2 - \frac{1}{L^2} = 0$$

$$\Rightarrow D^2 = \frac{1}{L^2} \Rightarrow D = \pm\frac{1}{L}$$

Therefore, the solution is

$$n(x) = c_1e^{-1/Lx} + c_2e^{1/Lx} \quad (1)$$

Now, at $x = 0$, we have

$$n(0) = c_1 + c_2 = K$$

and at $x = \infty$, we have

$$n(\infty) = c_1e^{-\infty} + c_2e^{\infty} = 0$$

$$\Rightarrow \quad c_2e^{\infty} = 0$$

$$\Rightarrow \quad c_2 = 0$$

Hence, $c_1 = K$.

Now substituting values of c_1 and c_2, we get the required solution.

Thus, the solution is $n(x) = ke^{-x/L}$.

Ans. (d)

28. The order and degree of the differential equation $\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$ are, respectively,

(a) 3 and 2 (b) 2 and 3
(c) 3 and 3 (d) 3 and 1

(GATE 2010, 1 Mark)

Solution: The given equation is

$$\frac{d^3y}{dx^3} + 4\sqrt{\left(\frac{dy}{dx}\right)^3 + y^2} = 0$$

Removing radicals, we get

$$\left(\frac{d^3y}{dx^3}\right)^2 = 16\left[\left(\frac{dy}{dx}\right)^3 + y^2\right]$$

$\therefore$ The order is 3 because the highest differential is $\dfrac{d^3y}{dx^3}$.

The degree is 2 because the power of highest differential is 2.

Ans. (a)

29. The solution to the ordinary differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$ is

(a) $y = c_1e^{3x} + c_2e^{-2x}$ (b) $y = c_1e^{3x} + c_2e^{2x}$

(c) $y = c_1e^{-3x} + c_2e^{2x}$ (d) $y = c_1e^{-3x} + c_2e^{-2x}$

(GATE 2010, 2 Marks)

Solution: We have

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

The equation can be written as

$$(D^2 + D - 6)y = 0$$

The auxiliary equation is given by

$$D^2 + D - 6 = 0$$
$$\Rightarrow (D+3)(D-2) = 0$$
$$\Rightarrow D = -3, 2$$

Therefore, the solution is given by $y = c_1e^{-3x} + c_2e^{2x}$.

Ans. (c)

30. For the differential equation $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 8x = 0$ with initial conditions $x(0) = 1$ and $\left.\dfrac{dx}{dt}\right|_{t=0} = 0$, the solution is

(a) $x(t) = 2e^{-6t} - e^{-2t}$ (b) $x(t) = 2e^{-2t} - e^{-4t}$

(c) $x(t) = 2e^{-6t} + e^{-4t}$ (d) $x(t) = 2e^{-2t} + e^{-4t}$

(GATE 2010, 2 Marks)

Solution: We are given that

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8 = 0$$

$$x(0) = 1 \text{ and } \left.\frac{dx}{dt}\right|_{t=0} = 0$$

The equation can be written as

$$D^2 + 6D + 8 = 0$$
$$\Rightarrow (D+4)(D+2) = 0$$
$$\Rightarrow D = -2, -4$$

$\therefore$ The solution is given by

$$x = c_1e^{-2t} + c_2e^{-4t} \quad (1)$$

As $x(0) = 1$, we have

$$c_1 + c_2 = 1 \quad (2)$$

Now differentiating Eq. (1) w.r.t. t, we get

$$\frac{dx}{dt} = -2c_1e^{-2t} - 4c_2e^{-4t}$$

As $\left[\dfrac{dx}{dt}\right]_{t=0} = 0$, we have

$$2c_1 + 4c_2 = 0 \quad (3)$$

Multiplying Eq. (2) with 2 and subtracting it from Eq. (3), we get

$$2c_2 = -2 \Rightarrow c_2 = -1$$

Substituting value of c_2 in Eq. (2), we get

$$c_1 - 1 = 1 \Rightarrow c_1 = 2$$

Substituting values of c_1 and c_2 in Eq. (2), we get the required solution

$$x(t) = 2e^{-2t} - e^{-4t}$$

Ans. (b)

31. With K as a constant, the possible solution for the first-order differential equation $\dfrac{dy}{dx} = e^{-3x}$ is

(a) $-\dfrac{1}{3}e^{-3x} + K$ (b) $-\dfrac{1}{3}e^{3x} + K$

(c) $-3e^{-3x} + K$ (d) $-3e^{-x} + K$

(GATE 2011, 1 Mark)

Solution: We have

$$\frac{dy}{dx} = e^{-3x}$$

The equation can be solved using the following method:

$$\int dy = \int e^{-3x}$$

$$y = \frac{e^{-3x}}{-3} + K$$

Thus, the solution is given by $y = -\dfrac{1}{3}e^{-3x} + K$.

Ans. (a)

32. The solution of the differential equation $\dfrac{dy}{dx} = ky,\ y(0) = c$ is

(a) $x = ce^{-ky}$ (b) $x = ke^{cy}$

(c) $y = ce^{kx}$ (d) $y = ce^{-kx}$

(GATE 2011, 1 Mark)

Solution: We have

$$\frac{dy}{dx} = ky$$

$$\Rightarrow \quad \frac{dy}{y} = k\,dx$$

Integrating on both sides, we get

$$\ln y = kx + A$$

Putting $x = 0$, we get

$$\ln y(0) = A$$

$$\therefore \quad A = \ln c \quad [\because y(0) = c]$$

Hence,

$$\ln y = kx + \ln c$$

$$\Rightarrow \quad \ln y - \ln c = kx$$

$$\Rightarrow \quad \ln\left(\frac{y}{c}\right) = kx$$

$$\Rightarrow \quad \frac{y}{c} = e^{kx} \Rightarrow y = ce^{kx}$$

Ans. (c)

33. Consider the differential equation $\frac{dy}{dx} = (1+y^2)x$. The general solution with constant is

(a) $y = \tan\frac{x^2}{2} + \tan c$ (b) $y = \tan^2\left(\frac{x}{2} + c\right)$

(c) $y = \tan^2\left(\frac{x}{2}\right) + c$ (d) $y = \tan\left(\frac{x^2}{2} + c\right)$

(GATE 2011, 2 Marks)

Solution: The given differential equation is

$$\frac{dy}{dx} = (1+y^2)x$$

The equation can be solved using the following method:

$$\int \frac{dy}{1+y^2} = \int x\,dx$$

$$\Rightarrow \tan^{-1} y = \frac{x^2}{2} + c$$

$$\Rightarrow y = \tan\left(\frac{x^2}{2} + c\right)$$

Thus, the solution is given by $y = \tan\left(\frac{x^2}{2} + c\right)$.

Ans. (d)

34. The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x$, with the condition that $y = 1$ at $x = 1$, is

(a) $y = \frac{2}{3x^2} + \frac{x}{3}$ (b) $y = \frac{x}{2} + \frac{1}{2x}$

(c) $y = \frac{2}{3} + \frac{x}{3}$ (d) $y = \frac{2}{3x} + \frac{x^2}{3}$

(GATE 2011, 2 Marks)

Solution: We have

$$\frac{dy}{dx} + \frac{y}{x} = x$$

This is a linear differential equation $\frac{dy}{dx} + Py = Q$ with $P = \frac{1}{x}$ and $Q = x$.

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Solution is given by

$$y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$$

$$\Rightarrow yx = \int (x)(x)\,dx + C$$

$$\Rightarrow yx = \int x^2\,dx + C$$

$$\Rightarrow yx = \frac{x^3}{3} + C$$

$$\Rightarrow \quad y = \frac{x^2}{3} + \frac{C}{x}$$

Now, we are also given that $y(1) = 1$.

$$\Rightarrow \quad \frac{1^2}{3} + \frac{C}{1} = 1 \Rightarrow C = \frac{2}{3}$$

Hence, the solution is $y = \frac{x^2}{3} + \frac{2}{3x}$.

Ans. (d)

35. With initial condition $x(1) = 0.5$, the solution of the differential equation, $t\frac{dx}{dt} + x = t$ is

(a) $x = t - \frac{1}{2}$ (b) $x = t^2 - \frac{1}{2}$

(c) $x = \frac{t^2}{2}$ (d) $x = \frac{t}{2}$

(GATE 2012, 1 Mark)

Solution: The given differential equation is

$$t\frac{dx}{dt} + x = t$$

$$\Rightarrow \frac{dx}{dt} + \frac{x}{t} = 1$$

Also, we are given the initial condition $x(1) = \frac{1}{2}$.

The equation is of the form,

$$\frac{dx}{dt} + Px = Q$$

where $P = \dfrac{1}{t}$ and $Q = 1$

$$\text{Integrating factor} = e^{\int P dt} = e^{\int \frac{1}{t} dt} = e^{\log_e t} = t$$

The solution is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dt + C$$
$$x \cdot t = \int 1 \cdot t \cdot dt + C$$
$$xt = \frac{t^2}{2} + C$$
$$x = \frac{t}{2} + \frac{C}{t}$$

Now putting $x(1) = \dfrac{1}{2}$, we get

$$\Rightarrow \quad \frac{1}{2} = \frac{C}{1} + \frac{1}{2}$$
$$\Rightarrow \quad C = 0$$

So, $x = \dfrac{t}{2}$ is the required solution.

Ans. (d)

36. The solution of the ordinary differential equation $\dfrac{dy}{dx} + 2y = 0$ for the boundary condition $y = 5$ at $x = 1$ is

(a) $y = e^{-2x}$ (b) $y = 2e^{-2x}$
(c) $y = 10.95e^{-2x}$ (d) $y = 36.95e^{-2x}$

(GATE 2012, 2 Marks)

Solution: We are given that

$$\frac{dy}{dx} + 2y = 0$$

Also, $$y(1) = 5$$

$$\frac{dy}{dx} = -2y$$
$$\int \frac{dy}{y} = \int -2\, dx$$
$$\Rightarrow \quad \ln y = -2x + c$$
$$\Rightarrow \quad y = e^{-2x} \cdot e^{c} = c_1 e^{-2x} \qquad (1)$$
$$\therefore \quad y(1) = c_1 e^{-2} = 5 \Rightarrow c_1 = \frac{5}{e^{-2}}$$

Putting value of c_1 in Eq. (1), we get

$$y = \frac{5}{e^{-2}} e^{-2x} = 5e^2 e^{-2x} = 36.95e^{-2x}$$

Ans. (d)

37. Consider the differential equation $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 4y = 0$ with the boundary conditions of $y(0) = 0$ and $y(1) = 1$. The complete solution of the differential equation is

(a) x^2 (b) $\sin\left(\dfrac{\pi x}{2}\right)$
(c) $e^x \sin\left(\dfrac{\pi x}{2}\right)$ (d) $e^{-x} \sin\left(\dfrac{\pi x}{2}\right)$

(GATE 2012, 2 Marks)

Solution: We are given that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

Also, we have $y(0) = 0$ and $y(1) = 1$.

Option (a) satisfies the initial condition as well as equation as shown below:

If $y = x^2$

$$\Rightarrow \quad y(0) = 0 \text{ and } y(1) = 1^2 = 1$$

Also,

$$\frac{dy}{dx} = 2x$$
$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2$$
$$\Rightarrow x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2 \times 2 + r \times 2x - 4 \times x^2$$
$$= 2x^2 + 2x^2 - 4x^2 = 0$$

So, $y = x^2$ is the solution to this equation with the given boundary conditions.

Ans. (a)

38. The partial differential equation $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \dfrac{\partial^2 u}{\partial x^2}$ is a

(a) linear equation of order 2
(b) non-linear equation of order 1
(c) linear equation of order 1
(d) non-linear equation of order 2

(GATE 2013, 1 Mark)

Solution: A differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions of x, i.e. $f(x)$ is said to be linear equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

The given equation does not comply with the definition of linear equation; hence, it is a non-linear equation of order 2.

Ans. (d)

39. The solution to the differential equation $\dfrac{d^2u}{dx^2} - k\dfrac{du}{dx} = 0$, where k is constant, subjected to the boundary condition $u(0) = 0$ and $u(L) = U$, is

(a) $u = U\dfrac{x}{L}$ (b) $u = U\left(\dfrac{1-e^{kx}}{1-e^{kL}}\right)$

(c) $u = U\left(\dfrac{1-e^{-kx}}{1-e^{-kL}}\right)$ (d) $u = U\left(\dfrac{1+e^{kx}}{1+e^{kL}}\right)$

(GATE 2013, 2 Marks)

Solution: Considering option (a), i.e. $u = U\dfrac{X}{L}$.

Now, $u(0) = U\left(\dfrac{0}{L}\right) = 0$ and $u(L) = U\left(\dfrac{L}{L}\right) = U$

Hence, option (a) fulfills the boundary conditions.

Differentiating $u = U\left(\dfrac{x}{L}\right)$ w.r.t. x, we get

$$\frac{du}{dx} = \frac{U}{L} \qquad (1)$$

Differentiating Eq. (1) w.r.t. x, we get

$$\frac{d^2u}{dx^2} = 0 \qquad (2)$$

Now from Eqs. (1) and (2), we have

$$\frac{d^2u}{dx^2} - k\frac{du}{dx} \neq 0$$

Hence, option (a) is not the correct option.

Considering option (b), i.e. $u = U\left(\dfrac{1-e^{kx}}{1-e^{kL}}\right)$

Now, $$u(0) = U\left(\frac{1-e^{k(0)}}{1-e^{kL}}\right) = 0 \text{ and}$$

$$u(L) = U\left(\frac{1-e^{kL}}{1-e^{kL}}\right) = U$$

Hence, option (a) fulfills the boundary conditions.

Differentiating $u = U\left(\dfrac{1-e^{kx}}{1-e^{kL}}\right)$ w.r.t. x, we get

$$\frac{du}{dx} = \frac{-Uk}{(1-e^{kL})}\cdot e^{kx} \qquad (3)$$

Differentiation Eq. (3) w.r.t. x, we get

$$\frac{d^2u}{dx^2} = \frac{-Uk^2}{(1-e^{kL})}\cdot e^{kx} \qquad (4)$$

From Eqs. (3) and (4), we have

$$\frac{d^2u}{dx^2} - k\frac{du}{dx} = 0$$

Hence, $u = U\left(\dfrac{1-e^{kx}}{1-e^{kL}}\right)$ is the correct solution.

Ans. (b)

40. A system described by a linear constant coefficient and ordinary, first-order differential equation has an exact solution given by $y(t)$ for $t > 0$, when the forcing function is $x(t)$ and the initial condition is $y(0)$. If one wishes to modify the system so that the solution becomes $-2y(t)$ for $t > 0$, we need to

(a) change the initial condition to $-y(0)$ and the forcing function to $2x(t)$
(b) change the initial condition to $2y(0)$ and the forcing function to $-x(t)$
(c) change the initial condition to $j\sqrt{2}y(0)$ and the forcing function to $j\sqrt{2}x(t)$
(d) change the initial condition to $-2y(0)$ and the forcing function to $-2x(t)$

(GATE 2013, 2 Marks)

Solution: We have

$$\frac{dy(t)}{dt} + ky(t) = x(t)$$

Taking Laplace transformation of both sides, we have $sY(s) - y(0) + kY(s) = X(s)$

$$Y(s)[s+k] = X(s) + y(0)$$

$$\Rightarrow \qquad y(s) = \frac{X(s)}{s+k} + \frac{y(0)}{s+k}$$

Taking inverse Laplace transform, we have

$$y(t) = e^{-kt}\,x(t) + y(0)\,e^{-kt}$$

So if we want $-2y(t)$ as a solution, both $x(t)$ and $y(0)$ have to be multiplied by -2 and hence $x(t)$ and $y(0)$ be changed to $-2x(t)$ and $-2y(0)$, respectively.

Ans. (d)

41. The maximum value of the solution $y(t)$ of the differential equation $y(t) + \ddot{y}(t) = 0$ with initial condition $\dot{y}(0) = 1$ and $y(0) = 1$, for $t \geq 0$, is

(a) 1 (b) 2

(c) π (d) $\sqrt{2}$

(GATE 2013, 2 Marks)

Solution: We have

$$y(t) + \ddot{y}(t) = 0$$

This can be rewritten as

$$1 + D^2 = 0$$

$$\therefore \quad D = \pm i$$

Now, the solution is given by

$$y = C_1 e^{ix} + C_2 e^{-ix}$$
$$= A\cos x + B\sin x$$

We know that $y(0) = 1$

$$\Rightarrow 1 = A \times 1 + B \times 0$$
$$\Rightarrow A = 1$$

Differentiating y w.r.t. x, we get

$$\dot{y} = -A\sin x + b\cos x$$

It is given that $\dot{y}(0) = 1$

$$\therefore \quad 1 = -A \times 0 + B \times 1 \Rightarrow B = 1$$

Hence, $\quad y = \cos x + \sin x$

For maxima,

$$y' = -\sin x + \cos x = 0$$
$$\Rightarrow \quad \sin x = \cos x$$
$$\therefore \quad x = 45^\circ$$
$$y'' = -\cos x - \sin x$$

Therefore, $y'' < 0$ for $x = 45^\circ$ and hence maxima

$$y(\max) = \cos 45^\circ + \sin 45^\circ = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Ans. (d)

42. If the characteristic equation of the differential equation

$$\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$$

has two equal roots, then the values of α are

(a) ± 1 (b) 0, 0
(c) $\pm i$ (d) $\pm 1/2$

(GATE 2014, 1 Mark)

Solution: For equal roots, discriminant $b^2 - 4ac = 0$. Thus,

$$4\alpha^2 - 4 = 0 \Rightarrow \alpha = \pm 1$$

Ans. (a)

43. Which one of the following is a linear non-homogeneous differential equation, where x and y are the independent and dependent variables, respectively?

(a) $\dfrac{dy}{dx} + xy = e^{-x}$ (b) $\dfrac{dy}{dx} + xy = 0$

(c) $\dfrac{dy}{dx} + xy = e^{-y}$ (d) $\dfrac{dy}{dx} + e^{-y} = 0$

(GATE 2014, 1 Mark)

Solution: Option (a), $\dfrac{dy}{dx} + xy = e^{-x}$ is a first-order linear equation (non-homogeneous). Option (b), $\dfrac{dy}{dx} + xy = 0$ is a first-order linear equation (homogeneous). Options (c) and (d) are non-linear equations.

Ans. (a)

44. If $z = xy\ln(xy)$, then

(a) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$ (b) $y\dfrac{\partial z}{\partial x} = x\dfrac{\partial z}{\partial y}$

(c) $x\dfrac{\partial z}{\partial x} = y\dfrac{\partial z}{\partial y}$ (d) $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = 0$

(GATE 2014, 1 Mark)

Solution: We are given

$$z = xy\ln(xy)$$

$$\frac{\partial z}{\partial x} = y\left[x \times \frac{1}{xy} \times y + \ln xy\right] = y[1 + \ln xy]$$

and $\dfrac{\partial z}{\partial y} = x[1 + \ln xy] \Rightarrow x\dfrac{\partial z}{\partial x} = y\dfrac{\partial z}{\partial y}$

Ans. (c)

45. If a and b are constants, the most general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0 \text{ is}$$

(a) ae^{-t} (b) $ae^{-t} + bte^{-t}$
(c) $ae^{t} + bte^{-t}$ (d) ae^{-2t}

(GATE 2014, 1 Mark)

Solution: The auxiliary equation is given by

$$-m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$$

Thus, general solution is $x = (a + bt)e^{-t}$

Ans. (b)

46. The solution for the differential equation

$\frac{d^2x}{dt^2} = -9x$, with initial conditions $x(0) = 1$ and $\left.\frac{dx}{dt}\right|_{t=0} = 1$, is

(a) $t^2 + t + 1$

(b) $\sin 3t + \frac{1}{3}\cos 3t + \frac{2}{3}$

(c) $\frac{1}{3}\sin 3t + \cos 3t$

(d) $\cos 3t + t$

(GATE 2014, 1 Mark)

Solution: The differential equation is given by

$$\frac{d^2x}{dt^2} = -9x$$

Auxiliary equation of the given differential equation is given by

$$m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

Thus, solution is given by

$$x = a\cos 3t + b\sin 3t \qquad \text{(i)}$$

Differentiating Eq. (i), we get

$$\frac{dx}{dt} = -3a\sin 3t + 3b\cos 3t \qquad \text{(ii)}$$

Using $x(0) = 1$ and $\left.\frac{dx}{dt}\right|_{t=0} = 1$, Eqs. (i) and (ii) give

$$1 = a \text{ and } 1 = 3b \Rightarrow b = \frac{1}{3}$$

Therefore, $x = \cos 3t + \frac{1}{3}\sin 3t$

Ans. (c)

47. Consider the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$. Which of the following is a solution to this differential equation for $x > 0$?

(a) e^x (b) x^2 (c) $1/x$ (d) $\ln x$

(GATE 2014, 1 Mark)

Solution: The differential equation is given as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

which is Cauchy–Euler equation

$$\Rightarrow \left(\theta^2 - 1\right)\cdot y = 0 \text{ where } \theta = \frac{d}{dz} \text{ and } z = \log x \Rightarrow x = e^z$$

The auxiliary equation can be given as

$$m^2 - 1 = 0 \Rightarrow m = -1, 1$$

Therefore, solution is $y = C_1e^{-z} + C_2e^{z} = \frac{C_1}{x} + C_2x$

Thus, $1/x$ is a solution.

Ans. (c)

48. The solution of the initial value problem $\frac{dy}{dx} = -2xy$, $y(0) = 2$ is

(a) $1 + e^{-x^2}$ (b) $2e^{-x^2}$ (c) $1 + e^{x^2}$ (d) $2e^{x^2}$

(GATE 2014, 1 Mark)

Solution: Given that $\frac{dy}{dx} + 2xy = 0$

Let $p(x) = 2x$ and $q(x) = 0$, then integration factor is $e^{\int p dx} = e^{\int 2x dx} = e^{x^2}$. Therefore, the solution to the differential equation is

$$y(e^{x^2}) = \int e^{x^2} q(x)\, dx = 0 + c$$

or $$y = ce^{-x^2}$$

Given that $y(0) = 2$, so $2 = c$, hence the solution is $y = 2e^{-x^2}$.

Ans. (b)

49. The integrating factor for the differential equation $\frac{dP}{dt} + k_2 P = k_1 L_0 e^{-k_1 t}$ is

(a) $e^{-k_1 t}$ (b) $e^{-k_2 t}$ (c) $e^{k_1 t}$ (d) $e^{k_2 t}$

(GATE 2014, 1 Mark)

Solution: The given differential equation is given by

$$\frac{dP}{dt} + k_2 P = k_1 L_0 e^{-k_0 t}$$

The standard form of linear differential equations is

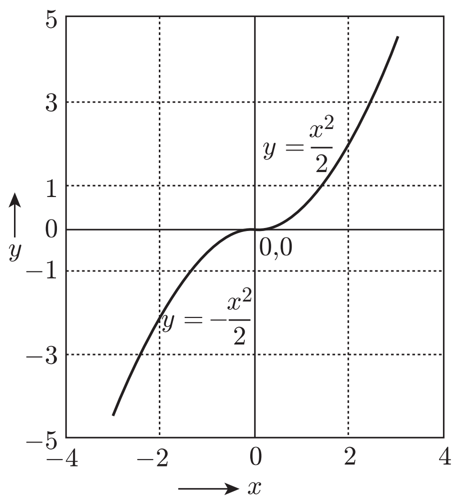
$$\frac{dy}{dx} + py = Q;\ \text{I.F.} = e^{\int P dx}$$

$$\Rightarrow \frac{dP}{dt} + k_2 P = \left(k_1 L_0 e^{-k_0 t}\right)$$

Integrating factor, I.F. $= e^{\int k_2 dt} = e^{k_2 t}$

Ans. (d)

50. The figure shows the plot of y as a function of x.



The function shown is the solution of the differential equation (assuming all initial conditions to be zero) is

(a) $\frac{d^2y}{dx^2} = 1$ (b) $\frac{dy}{dx} = x$

(c) $\frac{dy}{dx} = -x$ (d) $\frac{dy}{dx} = |x|$

(GATE 2014, 1 Mark)

Solution: By back tracking, from option (d), we get

$$\frac{dy}{dx} = |x| = \begin{cases} x \text{ for } x \geq 0 \\ -x \text{ for } x < 0 \end{cases}$$

Integrating we get

$$\int \frac{dy}{dx} = \int x dx \text{ for } x \geq 0 = \int -x dx \text{ for } x < 0$$

$$\Rightarrow y = \frac{x^2}{2} \text{ for } x \geq 0 = \frac{-x^2}{2} \text{ for } x < 0$$

Ans. (d)

51. With initial values $y(0) = y'(0) = 1$, the solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

at $x = 1$ is ________.

(GATE 2014, 2 Marks)

Solution: We can write the auxiliary equation as $m^2 + 4m + 4 = 0 \Rightarrow m - 2, -2$

Thus, solution is

$$y = (a + bx)e^{-2x} \quad \text{(i)}$$

$$y = (a + bx)(-2e^{-2x}) + e^{-2x}(b) \quad \text{(ii)}$$

Using $y(0) = 1$ and $y'(0) = 1$, Eqs. (i) and (ii) give $a = 1$ and $b = 3$. Therefore

$$y = (1 + 3x)e^{-2x}$$

At $x = 1,\ y = 4e^{-2} = 0.541$

Ans. 0.541

52. If $y = f(x)$ is the solution of $\frac{d^2y}{dx^2} = 0$ with boundary conditions $y = 5$ at $x = 0$, $\frac{dy}{dx} = 2$ at $x = 10$, $f(15)$ is ________.

(GATE 2014, 2 Marks)

Solution: Given that $\frac{d^2y}{dx^2} = 0$, $y = 5$ at $x = 0$, $\frac{dy}{dx}$ $= 2$ at $x = 10$

Here, $m^2 = 0 \Rightarrow m = 0, 0$, so we have $y_p = 0$ and $y_c = (a + bx)e^0 = a + bx$.

Now, given that $y = 5$ at $x = 0$. Substituting, we get $a = 5$.

Also given that $dy/dx = 2$ at $x = 10$. Substituting, we get $b = 2$.

Therefore, the general solution is $y = 5 + 2x$. At $x = 15$, $y = 5 + 30 = 35$.

Ans. 35

53. The general solution of the differential equation $\frac{dy}{dx} = \cos(x + y)$, with c as a constant, is

(a) $y + \sin(x + y) = x + c$

(b) $\tan\left(\frac{x+y}{2}\right) = y + c$

(c) $\cos\left(\frac{x+y}{2}\right) = x + c$

(d) $\tan\left(\frac{x+y}{2}\right) = x + c$

(GATE 2014, 2 Marks)

Solution: Given that $\frac{dy}{dx} = \cos(x + y)$.

Let $t = x + y$, then $\frac{dt}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$

So, we have $\frac{dt}{dx} - 1 = \cos t \Rightarrow \frac{1}{1 + \cos t} dt = dx$ (variable separable)

On integrating, we get

$$\int \frac{1}{1+\cos t}\,dt = \int dx \Rightarrow \int \frac{1}{2\cos^2(t/2)}\,dt$$
$$= x + c$$

$$\frac{1}{2}\int \sec^2\left(\frac{t}{2}\right)dt = x + c \Rightarrow \tan\left(\frac{t}{2}\right) = x + c$$
$$\Rightarrow \tan\left(\frac{x+y}{2}\right) = x + c$$

Ans. (d)

54. Consider two solutions $x(t) = x_1(t)$ and $x(t) = x_2(t)$ of the differential equation $\frac{d^2x(t)}{dt^2} + x(t) = 0$, $t > 0$ such that $x_1(0) = 1$, $\left.\frac{dx_1}{dt}\right|_{t=0} = 0$, $x_2(0) = 0$, $\left.\frac{dx_2}{dt}\right|_{t=0} = 1$

The Wronskian function, $W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ \frac{dx_1(t)}{dt} & \frac{dx_2(t)}{dt} \end{vmatrix}$ at $t = \pi/2$ is

(a) 1 (b) -1 (c) 0 (d) $\pi/2$

(GATE 2014, 2 Marks)

Solution: For the differential equation, $\frac{d^2x(t)}{dt^2} + x(t) = 0$, $m^2 + 1 = 0 \Rightarrow m = 0 \pm i$. Now, the complementary and particular solutions are $x_c = a\cos t + b\sin t$ and $x_p = 0$.

Now, for the general solution, $x = a\cos t + b\sin t$, let $x_1(t) = \cos t$ and $x_2(t) = \sin t$.
But it is given that

$$x_1(0) = 1,\ \left.\frac{dx_1}{dt}\right|_{t=0} = 0,\ x_2(0) = 0,\ \left.\frac{dx_2}{dt}\right|_{t=0} = 1$$

The Wronskian function,

$$W(t) = \left.\begin{vmatrix} x_1(t) & x_2(t) \\ \frac{dx_1(t)}{dt} & \frac{dx_2(t)}{dt} \end{vmatrix}\right|_{t=\pi/2} = \left.\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}\right|_{t=\pi/2}$$
$$= \cos^2 t + \sin^2 t = 1$$

Ans. (a)

55. Consider the following differential equation

$$\frac{dy}{dx} = x + \ln(y);\ y = 2 \text{ at } x = 0$$

The solution of this equation at $x = 0.4$ using Euler method with a step size of $h = 0.2$ is ________ .

(GATE 2014, 2 Marks)

Solution: We are given

$$\frac{dy}{dx} = x + \ln y$$

$$\frac{dy}{dx} = f(x,y) \Rightarrow f(x,y) = x + \ln y$$

given $x_0 = 0$ and $y_0 = 2$.

We have,

$$y_{n+1} = y_n + hf(x_n, y_n),\ n = 0, 1, 2, 3, \ldots$$

For $x = 0$,

$$y_1 = y_0 + hf(x_0, y_0)$$

For $h = 0.2$,

$$y_1 = y(x_1) = y(x_0 + h) = y(0 + 0.2) = y(0.2)$$

Similarly

$$y(0.2) = y_1 = 2 + 0.2f(0,2) = 2 + 0.2(0 + \ln 2)$$
$$= 2 + 0.2(0.69315) = 2.13863$$
$$y_2 = y(x_2) = y(x_1 + h)$$
$$= y(0.2 + 0.2) = y(0.4)$$

Thus,

$$y(0.4) = y_2 = y_1 + hf(x_1, y_1)$$
$$= 2.13863 + 0.2f(0.2, 2.13863)$$
$$= 2.13863 + 0.2[0.2 + h(2.13863)]$$
$$= 2.13863 + 0.2(0.2 + 0.76016) = 2.33066$$

56. The integrating factor for the differential equation $\frac{dy}{dx} - \frac{y}{1+x} = (1+x)$ is

(a) $\frac{1}{1+x}$ (b) $(1 + x)$

(c) $x(1 + x)$ (d) $\frac{x}{1+x}$

(GATE 2014, 2 Marks)

Solution: Given differential equation

$$\frac{dy}{dx} - \frac{y}{1+x} = 1 + x$$

$$\frac{dy}{dx} + Py = Q \Rightarrow P = \frac{-1}{1+x}$$

$$\text{I.F.} = e^{\int p\,dx} = e^{\int \frac{-1}{1+x}dx} = e^{\int -\log(1+x)} = \frac{1}{(1+x)}$$

Ans. (a)

57. The differential equation $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + x^3y = e^x$ is a

(a) non-linear differential equation of first degree

(b) linear differential equation of first degree

(c) linear differential equation of second degree

(d) non-linear differential equation of second degree

(GATE 2014, 2 Marks)

Solution: We are given that

$$\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+x^3y=e^x$$

This is a linear differential equation with order = 2 and degree = 1.

Ans. (b)

58. The solution of the differential equation $\frac{d^2y}{dt^2}+\frac{2dy}{dt}+y=0$ with $y(0)=y'(0)=1$ is

(a) $(2-t)e^t$ (b) $(1+2t)e^{-t}$

(c) $(2+t)e^{-t}$ (d) $(1-2t)e^t$

(GATE 2015, 1 Mark)

Solution: Differential equation is

$$\frac{d^2y}{dt^2}+\frac{2dy}{dt}+y=0$$

Thus,

$$D^2+2D+1=0\Rightarrow (D+1)^2=0\Rightarrow D=-1,-1$$

Thus, solution is $y(t)=(c_1+c_2t)e^{-t}\to$ C.F.

$$\Rightarrow y'(t)=c_2e^{-t}+(c_1+c_2t)(-e^{-t})$$

$y(0)=1; y'(0)=1$ gives $c_1=1$ and

$$c_2+c_1(-1)=1\Rightarrow c_2=2$$

Therefore, $y(t)=(1+2t)e^{-t}$

Ans. (b)

59. The general solution of the differential equation $\frac{dy}{dx}=\frac{1+\cos 2y}{1-\cos 2x}$ is

(a) $\tan y-\cot x=c$ (c is a constant)

(b) $\tan x-\cot y=c$ (c is a constant)

(c) $\tan y+\cot x=c$ (c is a constant)

(d) $\tan x+\cot y=c$ (c is a constant)

(GATE 2015, 1 Mark)

Solution: We are given that

$$\frac{dy}{dx}=\frac{1+\cos 2y}{1-\cos 2x}$$

$$\Rightarrow \frac{dy}{1-\cos 2y}=\frac{dx}{1+\cos 2x}$$

$$\Rightarrow \frac{dy}{2\sin^2 y}=\frac{dx}{2\cos^2 x}\ \text{(variable-separable)}$$

$$\Rightarrow \int \operatorname{cosec}^2 y dy=\int \sec^2 x dx$$

$$\Rightarrow -\cot y=\tan x+k$$

$$\Rightarrow -\tan x-\cot y=k$$

$$\Rightarrow \tan x+\cot y=c\ (\text{where } c=-k)$$

Ans. (d)

60. Consider the differential equation $\frac{dx}{dt}=10-0.2x$ with initial condition $x(0)=1$. The response $x(t)$ for $t>0$ is

(a) $2-e^{-0.2t}$

(b) $2-e^{0.2t}$

(c) $50-49e^{-0.2t}$

(d) $50-49e^{0.2t}$

(GATE 2015, 1 Mark)

Solution: Given differential equation is,

$$\frac{dx}{dt}=10-0.2x \quad x(0)=1$$

$$\Rightarrow \frac{dx}{dt}+(0.2)x=10$$

The auxiliary equation is given as

$$m+0.2=0\Rightarrow m=-0.2$$

Complementary solution,

$$x_c=Ce^{(-0.2)t}$$

$$x_p=\frac{1}{D+(0.2)}10e^{0t}=\frac{10e^{0t}}{0.2}=50e^{0t}=50$$

$$x=x_c+x_p=Ce^{(-0.2)t}+50$$

Given,

$$x(0)=1\Rightarrow C+50=1\Rightarrow C=49$$

$$x=50-49e^{(-0.2)t}$$

Ans. (c)

61. The contour on the $x-y$ plane, where the partial derivative of x^2+y^2 with respect to y is equal to the partial derivative of $6y+4x$ with respect to x, is

(a) $y=2$ (b) $x=2$

(c) $x+y=4$ (d) $x-y=0$

(GATE 2015, 1 Mark)

Solution: The partial derivative of $x^2 + y^2$ with respect to y is $0 + 2y \Rightarrow 2y$.

The partial derivative of $6y + 4x$ with respect to x is $0 + 4 = 4$.
Given that both are equal.

$$\Rightarrow 2y = 4 \Rightarrow y = 2$$

Ans. (a)

62. Consider the differential equation

$$\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = 0.$$

Given $x(0) = 20$ and $x(1) = 10/3$, where $e = 2.718$, the value of $x(2)$ is ________.

(GATE 2015, 1 Mark)

Solution: Given

$$\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = 0$$

$$x(0) = 20$$

$$x(1) = \frac{10}{e}$$

Auxiliary equation is given by

$$m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$$

Complementary solution, $x_e = c_1e^{-t} + c_2e^{-2t}$

Particular solution, $x_p = 0$
Finally,

$$x = x_c + x_p = c_1e^{-t} + c_2e^{-2t}$$

$$x(0) = 20 \Rightarrow 20 = c_1 + c_2 \qquad (1)$$

$$x(1) = \frac{10}{e} \Rightarrow \frac{10}{e} = c_1e^{-1} + c_2e^{-2}$$

$$\Rightarrow 10 = c_1 + c_2e^{-1} \qquad (2)$$

From Eq. (1), $c_1 = 20 - c_2$

Substituting in Eq. (2), we get

$$10 = (20 - c_2) + c_2e^{-1}$$

$$\Rightarrow 10 = c_2(e^{-1} - 1) + 20$$

$$\Rightarrow c_2 = \frac{10 - 20}{e^{-1} - 1} = -\frac{10}{e^{-1} - 1} = \frac{10}{1 - e^{-1}} = \frac{10e}{e - 1}$$

$$\Rightarrow c_1 = 20 - \frac{10e}{e - 1} = \frac{20e - 20 - 10e}{e - 1} = \frac{10e - 20}{e - 1}$$

$$x(t) = \frac{10e - 20}{e - 1}e^{-t} + \frac{10e}{e - 1}e^{-2t}$$

$$x(2) = \left(\frac{10e - 20}{e - 1}\right)e^{-2} + \left(\frac{10e}{e - 1}\right)e^{-4} = 0.8556$$

Ans. 0.8556

63. The initial velocity of an object is 40 m/s. The acceleration a of the object is given by the following expression:

$$a = -0.1v$$

where v is the instantaneous velocity of the object. The velocity of the object after 3 s will be ________ .

(GATE 2015, 1 Mark)

Solution: We are given the following expression:

$$a = -0.1v$$

$$\frac{dv}{dt} = -0.1v$$

$$\ln v = -0.1t + \ln k$$

$$v = ke^{-0.1t}$$

at $t = 0$; $v = 40$

Therefore, $k = 40$

$$v = 40e^{-0.1t}$$

At $t = 3$ s

$$V = 40e^{-0.1\times 3} = 29.6327 \text{ m/s}$$

Ans. 29.6327 m/s

64. Find the solution of $\frac{d^2y}{dx^2} = y$ which passes through the origin and the point $\left(\ln 2, \frac{3}{4}\right)$.

(a) $y = \frac{1}{2}e^x - e^{-x}$

(b) $y = \frac{1}{2}(e^x + e^{-x})$

(c) $y = \frac{1}{2}(e^x - e^{-x})$

(d) $y = \frac{1}{2}e^x + e^{-x}$

(GATE 2015, 1 Mark)

Solution: We are given that,

$$\frac{d^2y}{dx^2} = y$$

$$\Rightarrow (D^2 - 1)y = 0$$

$$D^2 - 1 = 0$$

$$\Rightarrow D = \pm 1$$

$$y = c_1e^x + c_2e^{-x}$$

Passes through (0,0) and $\left(\ln 2, \frac{3}{4}\right)$

At point (0, 0)

$$\Rightarrow 0 = c_1 + c_2 \quad (1)$$

At point (ln 2, 3/4)

$$\Rightarrow \frac{3}{4} = c_1 e^{\ln 2} + c_2 e^{-\ln 2} = c_1 \cdot 2 + \frac{c_2}{2}$$

$$\Rightarrow 2c_1 + \frac{1}{2}c_2 = \frac{3}{4} \quad (2)$$

Ans. (c)

65. Consider a linear ordinary differential equation: $\frac{dy}{dx} + p(x)y = r(x)$. Functions $p(x)$ and $r(x)$ are defined and have a continuous first derivative. The integrating factor of this equation is non-zero. Multiplying this equation by its integrating factor converts this into a:

(a) Homogeneous differential equation

(b) Non-linear differential equation

(c) Second-order differential equation

(d) Exact differential equation

(GATE 2015, 1 Mark)

Solution: Linear differential equation

$$y' + p(x)y = r(x)$$

Multiplying above equation by integrating factor $e^{\int p(x)dx}$ makes the equation exact.

Ans. (d)

66. A solution of the ordinary differential equation $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is such that $y(0) = 2$ and $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$. The value of $\frac{dy}{dt}(0)$ is ______ .

(GATE 2015, 2 Marks)

Solution: We are given

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

The auxiliary equation can be written as

$$m^2 + 5m + 6 = 0$$

$$\Rightarrow (m+2)(m+3) = 0 \Rightarrow m = -2, -3$$

Thus, the solution is given by

$$y(t) = C_1 e^{-3t} + C_2 e^{-2t}$$

$$y(0) = C_1 + C_2 = 2 \quad (1)$$

$$y(1) = -\left(\frac{1-3e}{e^3}\right) = -e^{-3} + 3e^{-2} = C_1 e^{-3} + C_2 e^{-2} \quad (2)$$

Solving Eqs. (1) and (2) for C_1 and C_2, we get

$$C_1 = -1 \text{ and } C_2 = -3$$

Thus,

$$y(t) = -e^{-3t} + 3e^{-2t}$$

$$\frac{dy(t)}{dt} = 3e^{-3t} - 6e^{-2t},\ \frac{dy(0)}{dt} = 3 - 6 = -3$$

Ans. −3

67. A differential equation $\frac{di}{dt} - 0.2i = 0$ is applicable over $-10 < t < 10$. If $i(4) = 10$, then $i(-5)$ is ________.

(GATE 2015, 2 Marks)

Solution: We are given,

$$\frac{di}{dt} - 0.2i = 0$$

$$\Rightarrow (D - 0.2)i(t) = 0$$

$$D = 0.2$$

$$i(t) = k \cdot e^{0.2t},\ -10 < t < 10$$

$$t = u;$$

$$10 = K \cdot e^{0.8} \Rightarrow K = 4.493$$

Therefore,

$$i(-5) = 4.493 \times e^{-1} \Rightarrow i(-5) = 1.65$$

Ans. 1.65

68. Consider the following differential equation $\frac{dy}{dt} = -5y$; initial condition: $y = 2$ at $t = 0$. The value of y at $t = 3$ is

(a) $-5e^{-10}$ (b) $2e^{-10}$
(c) $2e^{-15}$ (d) $-15e^{2}$

(GATE 2015, 2 Marks)

Solution: The given differential equation is

$$\frac{dy}{dt} = -5y$$

$$\Rightarrow \frac{dy}{y} = -5dt \quad \text{(Variables separable form)}$$

Integrating,

$$\ln y = -5t + c \quad (1)$$

when $y = 2$ at $t = 0$ (initial conditional, Eq. (1) gives)

$$c = \ln 2$$

$$\therefore \ln y = -5t + \ln 2 \Rightarrow \ln\left(\frac{y}{2}\right) = -5t \Rightarrow y = 2e^{-5t}$$

At $t = 3,\ y = 2e^{-15}$

Ans. (c)

69. Consider the following differential equation:

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

Which of the following is the solution of the above equation (c is an arbitrary constant)?

(a) $\frac{x}{y}\cos\frac{y}{x} = c$ (b) $\frac{x}{y}\sin\frac{y}{x} = c$

(c) $xy\cos\frac{y}{x} = c$ (d) $xy\sin\frac{y}{x} = c$

(GATE 2015, 2 Marks)

Solution: Given differential equation is

$$x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$$

$$\Rightarrow x(ydx + xdy)\cos\frac{y}{x} + \left(-\sin\frac{y}{x}\right)y(xdy - ydx) = 0$$

$$\Rightarrow (ydx + xdy)\cos\left(\frac{y}{x}\right) + \left(-\sin\frac{y}{x}\right)\frac{y(xdy - ydx)}{x} = 0$$

$$\Rightarrow (ydx + xdy)\cos\left(\frac{y}{x}\right) + (xy)\left(-\sin\frac{y}{x}\right)\left(\frac{xdy - ydx}{x^2}\right) = 0$$

By observing, the above equation is $d\left[(xy)\cos\frac{y}{x}\right] = 0$

By integrating, we get

$$xy\cos\left(\frac{y}{x}\right) = c$$

Ans. (c)

70. Consider the following second-order linear differential equation

$$\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$$

The boundary conditions are at $x = 0$, $y = 5$ and at $x = 2$, $y = 21$. The value of y at $x = 1$ is ________ .

(GATE 2015, 2 Marks)

Solution: Given

$$\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$$

$$y(0) = 5,\ y(2) = 21$$

Auxiliary equation is given by $m^2 = 0 \Rightarrow m = 0, 0.$

$$y_c = (c_1 + c_2x)e^{0x} = c_1 + c_2x$$

$$y_p = \frac{1}{D^2}(-12x^2 + 24x - 20)$$

$$= -12\frac{x^4}{12} + 24\cdot\frac{x^3}{6} - 20\cdot\frac{x^2}{2} = -x^4 + 4x^3 - 10x^2$$

$$y = c_1 + c_2x - 10x^2 + 4x^3 - x^4$$

For $x = 0$, $y = 5$

$$c_1 = 5$$

For $x = 2$, $y = 21$

$$21 = 5 + c_2(2) - 10(4) + 4(8) - 16 \Rightarrow c_2 = \frac{40}{2} = 20$$

Thus, value of y for $x = 1$ is given by

$$y(1) = 5 + 20(1) - 10 + 4 - 1 = 18$$

Ans. 18

71. Consider the systems, each consisting of m linear equations in n variables.

I. If $m<n$, then all such systems have a solution.

II. If $m>n$, then none of these systems has a solution.

III. If $m=n$, then there exists a system which has a solution.

Which one of the following is **CORRECT**?

(a) I, II and III are true.
(b) Only II and III are true.
(c) Only III is true.
(d) None of them is true.

(GATE 2016, 1 Mark)

Solution:

(i) **Case I:** Let us consider a system with two linear equations (m) of three variables (n):

$$x - y + z = 1$$

$$-x + y - z = 2$$

This system (i.e. when $m<n$) has no solution (inconsistent). Therefore, statement I is incorrect.

(ii) **Case II:** Let us consider a system with three linear equations (m) of two variables (n):

$$x + y = 2$$

$$x - y = 0$$

$$3x + y = 4$$

This system (i.e. when $m>n$) has a unique solution. Therefore, statement II is also incorrect.

(iii) **Case III:** Let us consider a system with two equations (m) of two variables (n):

$$x + y = 2$$

$$x - y = 0$$

This system (i.e. when $m=n$) has a solution: $x=1$ and $y=1$. Therefore, statement III is the correct one.

Ans. (c)

72. A function $y(t)$, such that $y(0) = 1$ and $y(1) = 3e^{-1}$, is a solution of the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0. \text{ Then } y(2) \text{ is}$$

(a) $5e^{-1}$ (b) $5e^{-2}$
(c) $7e^{-1}$ (d) $7e^{-2}$

(GATE 2016, 1 Mark)

Solution: For the given differential equation, the characteristic equation is

$$s^2 + 2s + 1 \Rightarrow s = -1, -1$$
$$y(t) = [C_1 + C_2 t]e^{-1\cdot t}$$

Given that $y(0) = 1$. Therefore,

$$y(0) = [C_1 + C_2(0)]e^0 = C_1 = 1$$

Also $y(1) = 3e^{-1}$

$$y(1) = [1 + C_2]e^{-1} = 3e^{-1}$$
$$\Rightarrow 1 + C_2 = 3 \Rightarrow C_2 = 2$$

Thus,

$$y(t) = (1 + 2t)e^{-t}$$
$$y(2) = 5e^{-2}$$

Ans. (b)

73. The type of partial differential equation $\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + 3\frac{\partial^2 p}{\partial x \partial y} + 2\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y} = 0$ is

(a) elliptic (b) parabolic
(c) hyperbolic (d) none of these

(GATE 2016, 1 Mark)

Solution: Comparing the given equation with the following equation

$$A\frac{\partial^2 p}{\partial x^2} + 2B\frac{\partial^2 p}{\partial x \partial y} + C\frac{\partial^2 p}{\partial y^2} + D\frac{\partial p}{\partial x} + E\frac{\partial p}{\partial y} = 0,$$

we have $A = 1$, $B = \frac{3}{2}$, $C = 1$

Here $B^2 - AC > 0$, thus equation is for hyperbola

Ans. (c)

74. The solution of the partial differential equation $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ is of the form

(a) $C\cos(kt)\left[C_1 e^{(\sqrt{(k/\alpha)x}} + C_2 e^{-(\sqrt{k/\alpha})x}\right]$

(b) $Ce^{kt}\left[C_1 e^{(\sqrt{k/\alpha})x} + C_2 e^{-(\sqrt{k/\alpha})x}\right]$

(c) $Ce^{kt}\left[C_1\cos(\sqrt{k/\alpha})x + C_2\sin(-\sqrt{k/\alpha})x\right]$

(d) $C\sin(kt)\left[C_1\cos(\sqrt{k/\alpha})x + C_2\sin(-\sqrt{k/\alpha})x\right]$

(GATE 2016, 1 Mark)

Solution: Roots of the given equation are m_1 and $m_2 = \pm\sqrt{\frac{k}{\alpha}}$

$$\text{Hence, solution} = Ce^{kt}\left(C_1 e^{\sqrt{\frac{k}{\alpha}}x} + C_2 e^{-\sqrt{\frac{k}{\alpha}}x}\right)$$

Ans. (b)

75. The ordinary differential equation $\frac{dx}{dt} = -3x + 2$, with $x(0)=1$ is to be solved using the forward Euler method. The largest time step that can be used to solve the equation without making the numerical solution unstable is ________.

(GATE 2016, 2 Marks)

Solution: We have

$$\frac{dx}{dt} = -3x + 2, \quad \text{with} \quad x(0) = 1$$

If $|1-3h|<1$, the solution is stable.

$$-1<1-3h<1$$
$$-2<-3h<0$$
$$0<h<2/3$$

Therefore,

$$h_{\max} = -\frac{2}{3} = 0.67$$

Ans. 0.67

76. The particular solution of the initial value problem given below is

$$\frac{d^2y}{dx^2} + \frac{12dy}{dx} + 36y = 0 \text{ with } y(0) = 3 \text{ and } \left.\frac{dy}{dx}\right|_{x=0} = -36$$

(a) $(3-18x)e^{-6x}$ (b) $(3+25x)e^{-6x}$
(c) $(3+20x)e^{-6x}$ (d) $(3-12x)e^{-6x}$

(GATE 2016, 2 Marks)

Solution: It is given that

$$\frac{d^2y}{dx^2} + \frac{12dy}{dx} + 36y = 0$$

Let $d/dx = D$. Therefore,

$$(D^2+12D+36)y=0$$

The auxiliary equation is

$$m^2+12m+36=0$$
$$(m+6)^2=0 \Rightarrow m=-6 \text{ and } -6$$

The complementary solution is

$$y=(c_1+c_2x)e^{-6x}$$

It is given that

$$y(0)=3=(c_1+0)e^{-0}$$
$$\Rightarrow c_1=3$$

Now,

$$\frac{dy}{dx} = (c_1+c_2x)(-6\times e^{-6x}) + e^{-6x}\times c_2$$

$$\left.\frac{dy}{dx}\right|_{x=0} = -36 = c_1(-6)+c_2 \Rightarrow c_2 = -36+18 = -18$$

Therefore,

$$y=(c_1+c_2x)e^{-6x}=(3-18x)e^{-6x}$$

Ans. (a)

77. Let $y(x)$ be the solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ with initial conditions $y(0)=0$ and $\left.\frac{dy}{dx}\right|_{x=0} = 1$. Then the value of $y(1)$ is ________.

(GATE 2016, 2 Marks)

Solution: The solution of the given differential equation is of the form

$$y(t) = (C_1 + C_2x)e^{2x}$$

Given that $y(0) = 0$, so,

$$0 = C_1$$

Therefore, $y(t) = C_2x\, e^{2x}$

$$y'(t) = C_2e^{2x} + 2C_2xe^{2x}$$

Given that $y'(0) = 1$, so

$$1 = C_2$$

Therefore,

$$y = xe^{2x}$$

Thus,

$$y(1) = e^2 = 7.38$$

Ans. 7.38

78. If $y = f(x)$ satisfies boundary value problem $y'' + 9y = 0$, $y(0) = 0$, $y(\pi/2) = \sqrt{2}$, then $y(\pi/4)$ is ______.

(GATE 2016, 2 Marks)

Solution: Given that

$$y'' + 9y = 0$$
$$\Rightarrow (D^2+9)y = 0$$
$$\Rightarrow D = \pm 3i$$

Complementary function can be written as

$$y = A\cos(3x) + B\sin(3x)$$

For the given constraints

$$y(0) = 0$$
$$\Rightarrow A\cos 0 + B\sin 0 = 0$$
$$\Rightarrow A = 0$$

Also,

$$y = \left(\frac{\pi}{2}\right) = \sqrt{2}$$

Therefore,

$$B\sin\left(3\frac{\pi}{2}\right) = \sqrt{2}$$
$$\Rightarrow B = -\sqrt{2}$$

Thus, the solution is

$$y = -\sqrt{2}\sin(3x)$$

Hence,

$$y\left(\frac{\pi}{4}\right) = -\sqrt{2}\sin\left(\frac{3\pi}{4}\right)$$
$$= -1$$

Ans. –1

79. The respective expressions for complimentary function and particular integral part of the solution of the differential equation $\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} = 108x^2$ are

(a) $[c_1 + c_2x + c_3\sin\sqrt{3}x + c_4\cos\sqrt{3}x]$ and $[3x^4 - 12x^2 + c]$

(b) $[c_2x + c_3\sin\sqrt{3}x + c_4\cos\sqrt{3}x]$ and $[5x^4 - 12x^2 + c]$

(c) $[c_2 + c_3\sin\sqrt{3}x + c_4\cos\sqrt{3}x]$ and $[3x^4 - 12x^2 + c]$

(d) $[c_1 + c_2x + c_3 \sin\sqrt{3}x + c_4 \cos\sqrt{3}x]$ and $[5x^4 - 12x^2 + c]$

(GATE 2016, 2 Marks)

Solution: Given that

$$\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} = 108x^2$$

Now for C.F., we have

$$D^4y + 3D^2y = 0$$
$$D^4 + 3D^2 = 0$$
$$D^2(D^2 + 3) = 0$$
$$\Rightarrow D = 0, 0, \pm\sqrt{3}i$$

Now,

$$\text{C.F.} = c_1 + c_2x + c_3 \sin\sqrt{3}x + c_4 \cos\sqrt{3}x$$

For P.I., we have

$$\text{P.I.} = \frac{108x^2}{(D^4 + D^2)} = \frac{180x^2}{D^2(D^2 + 1)}$$

$$\text{P.I.} = 3x^4 - 12x^2 + c$$

Ans. (a)

80. The general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 5y = 0$$

in terms of arbitrary constants K_1 and K_2 is

(a) $K_1e^{(-1+\sqrt{6})x} + K_2e^{(-1-\sqrt{6})x}$

(b) $K_1e^{(-1+\sqrt{8})x} + K_2e^{(-1-\sqrt{8})x}$

(c) $K_1e^{(-2+\sqrt{6})x} + K_2e^{(-2-\sqrt{6})x}$

(d) $K_1e^{(-2+\sqrt{8})x} + K_2e^{(-2-\sqrt{8})x}$

(GATE 2017, 1 Mark)

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 5y = 0$$

$$\Rightarrow (D^2 + 2D - 5)y = 0$$
$$\Rightarrow D^2 + 2D - 5 = 0$$
$$\Rightarrow D = \frac{-2 \pm \sqrt{4+20}}{2} = -1 \pm \frac{1}{2}\sqrt{24}$$
$$\Rightarrow D = -1 \pm \sqrt{6}$$

Therefore,

$$m_1 = -1 + \sqrt{6},\ \ m_2 = -1 - \sqrt{6}$$

We know

$$y(t) = K_1e^{m_1x} + K_2e^{m_2x}$$
$$\Rightarrow y(t) = K_1e^{(-1+\sqrt{6})x} + K_2e^{(-1-\sqrt{6})x}$$

Ans. (a)

81. Consider a function $f(x, y, z)$ given by $f(x, y, z) = (x^2 + y^2 - 2z^2)(y^2 + z^2)$.

The partial derivative of this function with respect to x at the point $x = 2$, $y = 1$ and $z = 3$ is _______.

(GATE 2017, 1 Mark)

Solution:

$$\frac{\partial f}{\partial z} = (x^2 + y^2 - 2z^2)(0) + (y^2 + z^2)(2x)$$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{x=2,y=1,z=3} = (1^2 + 3^2)(2 \times 2) = 40$$

Ans. (40)

82. Consider the following partial differential equation for $u(x, y)$ with the constant $c > 1$:

$$\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial x} = 0$$

Solution of this equation is

(a) $u(x, y) = f(x + cy)$ (b) $u(x, y) = f(x - cy)$
(c) $u(x, y) = f(cx + y)$ (d) $u(x, y) = f(cx - y)$

(GATE 2017, 1 Mark)

Solution: Given that

$$\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} = -c\frac{\partial u}{\partial x}$$

This is true for

$$u(x, y) = f(x - cy)$$

Ans. (b)

83. The differential equation

$$\frac{d^2y}{dx^2} + 16y = 0$$

for $y(x)$ with two boundary conditions

$$\left.\frac{dy}{dx}\right|_{x=0} = 1$$
$$\left.\frac{dy}{dx}\right|_{x=\pi/2} = -1$$

has
(a) no solution
(b) exactly two solutions
(c) exactly one solution
(d) infinitely many solutions

(GATE 2017, 1 Mark)

Solution: Given that

$$\frac{d^2y}{dx^2} + 16y = 0$$
$$D^2 + 16 = 0$$
$$D = 0 \pm 4i$$

Thus,

$$y_c = c_1 \cos 4x + c_2 \sin 4x$$
$$y_p = 0$$

Therefore,

$$y'(x) = -4c_1 \sin 4x + 4c_2 \cos 4x$$
$$y'(0) = 0 + 4c_2$$
$$1 = 4c_2$$
$$c_2 = \frac{1}{4}$$

Similarly,

$$y'\left(\frac{\pi}{2}\right) = 0 + 4c_2$$
$$c_2 = -\frac{1}{4}$$

$c_2 = 1/4$ and $c_2 = -1/4$ is not possible. Hence, there is no solution.

Ans. (a)

84. Consider the following partial differential equation:

$$3\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x \partial y} + 3\frac{\partial^2\phi}{\partial y^2} + 4\phi = 0$$

For this equation to be classified as parabolic, the value of B^2 must be ______________.

(GATE 2017, 1 Mark)

Solution: Given that

$$3\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x \partial y} + 3\frac{\partial^2\phi}{\partial y^2} + 4\phi = 0$$

Now, comparing the above equation to the following one:

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x \cdot \partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\phi = 0$$

We have

$$A = 3, C = 3$$

Therefore, for the equation to be parabolic, we have

$$B^2 - 4AC = 0$$
$$\Rightarrow B^2 - 4 \cdot 3 \cdot 3 = 0$$
$$\Rightarrow B^2 = 36$$

Ans. (36)

85. Let $w = f(x, y)$, where x and y are functions of t. Then, according to the chain rule, $\frac{dw}{dt}$ is equal to

(a) $\frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\frac{dt}{dt}$ (b) $\frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$

(c) $\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$ (d) $\frac{dw}{dx}\frac{\partial x}{\partial t} + \frac{dw}{dy}\frac{\partial t}{\partial t}$

(GATE 2017, 1 Mark)

Solution: Given

$$w = f(x, y)$$

According to the chain rule, we have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Ans. (c)

86. Which one of the following is the general solution of the first-order differential equation

$$\frac{dy}{dx} = (x + y - 1)^2$$

where x, y are real?

(a) $y = 1 + x + \tan^{-1}(x + c)$, where c is a constant.
(b) $y = 1 + x + \tan(x + c)$, where c is a constant.
(c) $y = 1 - x + \tan^{-1}(x + c)$, where c is a constant.
(d) $y = 1 - x + \tan(x + c)$, where c is a constant.

(GATE 2017, 2 Marks)

Solution: $$\frac{dy}{dx} = (x + y - 1)^2 \quad (1)$$

Let $$x + y = P \quad (2)$$

Differentiating Eq. (2), we get

$$1 + \frac{dy}{dx} = \frac{dp}{dx} \quad (3)$$

Ans. (d)

87. Consider the differential equation $(t^2 - 81)\frac{dy}{dt} + 5t\,y = \sin(t)$ with $y(1) = 2\pi$. There exists a unique solution for this differential equation when t belongs to the interval

(a) $(-2, 2)$ (b) $(-10, 10)$
(c) $(-10, 2)$ (d) $(0, 10)$

(GATE 2017, 2 Marks)

Solution: Rearranging the given differential equation, we have

$$\frac{dy}{dt} + \frac{5t}{t^2 - 81} y = \frac{\sin t}{t^2 - 81}$$

Let

$$P = \frac{5t}{t^2 - 81}; \quad Q = \frac{\sin t}{t^2 - 81}$$

Therefore,

$$\text{I.F.} = e^{\int P dt} = e^{\int \frac{5t}{t^2 - 81} dt}$$
$$= (t^2 - 81)^{5/2}$$

Now,

$$y(t^2 - 81)^{5/2} = \int \frac{\sin t}{t^2 - 81}(t^2 - 81)^{5/2} dt$$

The solution exists only if $t \neq \pm 9$. Options (b), (c) and (d) have either -9 or 9 or both. Therefore, option (a) is the correct answer.

Ans. (a)

88. Consider the differential equation

$$3y''(x) + 27y(x) = 0$$

with initial conditions

$$y(0) = 0$$
$$y'(0) = 2000$$

The value of y at $x = 1$ is ______.

(GATE 2017, 2 Marks)

Solution: Given that

$$3y''(x) + 27y(x) = 0$$
$$y(0) = 0$$
$$y'(0) = 2000$$

Auxiliary equation is

$$3D^2 + 27 = 0$$
$$D = 0 \pm 3i$$

Thus,

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$
$$y_p = 0$$

Using the boundary conditions,

$$c_1 = 0$$
$$c_2 = \frac{2000}{3}$$

Thus,

$$y = \frac{2000}{3} \sin 3x$$
$$y(1) = 94.08$$

Ans. 94.08

89. The solution of the equation $\frac{dQ}{dt} + Q = 1$ with $Q = 0$ at $t = 0$ is

(a) $Q(t) = e^{-t} - 1$ (b) $Q(t) = 1 + e^{-t}$
(c) $Q(t) = 1 - e^{t}$ (d) $Q(t) = 1 - e^{-t}$

(GATE 2017, 2 Marks)

Solution: $$\frac{dQ}{dt} + Q = 1$$

Now,

$$\text{I.F.} = e^{\int p dt} = e^{\int s dt} = e^t$$
$$Q(t) \cdot e^t = \int e^t \cdot dt = e^t + c$$

Given that at $t = 0, Q = 0$. From the above equation, we have

$$0 = 1 + c \Rightarrow c = -1$$

Therefore,

$$Q(t) = 1 - e^{-t}$$

Ans. (d)

90. Consider the equation $\frac{du}{dt} = 3t^2 + 1$ with $u = 0$ at $t = 0$. This is numerically solved by using the forward Euler method with a step size, $\Delta t = 2$. The absolute error in the solution at the end of the first time step is ____________.

(GATE 2017, 2 Marks)

Solution: $$\frac{du}{dt} = 3t^2 + 1$$

Let $f(u, t) = 3t^2 + 1$. Also $u_0 = 0$, $t_0 = 0$, $\Delta t = 2$.

From forward Euler method, we have

$$u_1 = u_0 + hf(u_0 t_0); \qquad t_1 = t_0 + h$$
$$= u_0 + h(3t_0^2 + 1); \qquad t_1 = 0 + 2$$
$$= 0 + 2\,(3 \cdot 0^2 + 1)$$
$$\Rightarrow u_1 = 2$$

$$\frac{du}{dt} = 3t^2 + 1 \Rightarrow du = (3t^2 + 1)dt$$

Integrating both sides, we get

$$\int du = \int_0^2 (3t^2 + 1)dt$$
$$\Rightarrow u = [t^3 + t]_0^2$$
$$\Rightarrow u = 10$$

Therefore,

Absolute error = Exact error − Approx. error = 10 − 2 = 8

Ans. (8)

91. Consider the following second-order differential equation:

$$y'' - 4y' + 3y = 2t - 3t^2$$

The particular solution of the differential equation is

(a) $-2 - 2t - t^2$ (b) $-2t - t^2$
(c) $2t - 3t^2$ (d) $-2 - 2t - 3t^2$

Solution: $2t - 3t^2 = y'' - 4y - 3y$

$$f(D) = D^2 - 4D + 3$$

$$\text{P.I.} = \frac{I}{f(D)}(2t - 3t^2)$$

$$= \frac{1}{D^2 - 4D + 3}(2t - 3t^2)$$

$$= \left[\frac{1}{1-D} - \frac{1}{3\left(1 - \frac{D}{3}\right)}\right]\left[t - \frac{3t^2}{2}\right]$$

$$= (1 + D + D^2 + \cdots)\left(t - \frac{3t^2}{2}\right) - \frac{1}{3}\left(t - \frac{D}{3} + \frac{D^2}{9} + \cdots\right)\left(t - \frac{32t^2}{2}\right)$$

$$= \left(t - \frac{3t^2}{2} + 1 - 3t - 3\right) - \frac{1}{3}\left(1 - \frac{3t^2}{2} + \frac{1}{3} - 8 - \frac{1}{3}\right)$$

$$= -2 - 2t - t^2$$

Ans. (a)

CHAPTER 4

COMPLEX VARIABLES

INTRODUCTION

Many engineering problems may be treated and solved by method involving complex numbers and complex functions.

A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{(-1)}$, is called a complex number.

x is called the real part of $x + iy$ and is written as $R(x + iy)$, whereas y is called the imaginary part and is written as $I(x + iy)$.

COMPLEX FUNCTIONS

If for each set of complex numbers which a variable z may assume, there corresponds one or more values of a variable w, then w is called a complex function of z or $w = f(z)$. If $z = x + iy$, then $w = f(z) = u(x, y) + iv(x, y)$, where u and v are real functions of x and y.

If to each value of z there corresponds only one value of w, then the function is called single-valued function.

Evidently, if w assumes more than one value of z, then the function is called multi-valued function.

For example, $w = 2z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function.

Exponential Function of Complex Variables

The exponential function of x when x is real is given by

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots + \infty \tag{1}$$

Similarly, the exponential function of a complex variable $z = x + iy$ is given by

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots + \infty \tag{2}$$

Now, we put $x = 0$ in Eq. (2), and we get $z = iy$. Therefore,

$$\begin{aligned} e^{iy} &= 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots + \frac{(iy)^n}{n!} + \cdots + \infty \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + i\left(4 - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right) \end{aligned}$$

$$\Rightarrow e^{iy} = \cos y + i\sin y \tag{3}$$

Hence, $e^{z} = e^{x} \cdot e^{iy} = e^{x}\left(\cos y + i\sin y\right)$ (4)

Also, $x + iy = r\left(\cos\theta + i\sin\theta\right) = re^{i\theta}$

Thus, exponential form of $z = re^{i\theta}$. (5)

Circular Function of Complex Variables

Considering Eq. (3), we have

$$e^{iy} = \cos y + i\sin y$$

Hence, $$e^{-iy} = \cos y - i\sin y \tag{6}$$

From Eqs. (3) and (6), we have

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

and $$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

Hence, the circular functions of the complex variable z can be defined by the equations:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{7}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{8}$$

$$\tan z = \frac{\sin z}{\cos z} \tag{9}$$

with cosec z, sec z and cot z as respective reciprocals of Eqs. (7), (8) and (9).

Now, $\cos z + i\sin z = \dfrac{e^{iz} + e^{-iz}}{2} + i\dfrac{e^{iz} - e^{-iz}}{2i} = e^{iz}$, where $z = x + iy$.

Also, $e^{iy} = \cos y + i\sin y$, where y is real.

Hence, $e^{i\theta} = \cos\theta + i\sin\theta$, where θ is real or complex.

This is called Euler's theorem.

Now, according to De Moivre's theorem for complex number,

$$\left(\cos\theta + i\sin\theta\right)^{n} = \left(e^{i\theta}\right)^{n} = \cos n\theta + i\sin n\theta,$$

whether θ is real or complex.

Hyperbolic Functions of Complex Variables

Hyperbolic functions can be given as

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \tag{10}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \tag{11}$$

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \tag{12}$$

$$\coth x = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} \tag{13}$$

$$\operatorname{sech} x = \frac{2}{e^{x} + e^{-x}} \tag{14}$$

$$\operatorname{cosech} x = \frac{2}{e^{x} - e^{-x}} \tag{15}$$

Hyperbolic and circular functions are related as follows:

$$\sin ix = i\ \sinh x$$
$$\cos ix = \cosh x$$
$$\tan ix = i\ \tanh x$$

If $\sin \mathrm{h}u = z$, then u is called the hyperbolic sine inverse of z and is represented as $u = \sin^{-1} z$.

The inverse hyperbolic functions like other inverse functions are many-valued functions.

Logarithmic Function of Complex Variables

If $z = x + iy$ and $w = u + iy$ are related such that $e^{w} = z$, then w is said to be a logarithm of x to the base e and is written as

$$w = \log e^{z} \tag{16}$$

Also, $$e^{w+2in\pi} = e^{w} \cdot e^{2in\pi} = z \quad [\because e^{2in\pi} = 1]$$

$$\Rightarrow \log z = w + 2\sin\pi \tag{17}$$

A complex logarithmic function is an inverse of the complex exponential function. The logarithm of a complex number has an infinite number of values and is a multi-valued function.

The principal value of a non-zero complex number $z = x + iy$ is the logarithm whose imaginary part lies in the interval $(-\pi, \pi]$. The general value of the logarithm of z is written as log z so as to distinguish it from its principal value which is written as log z. This principal value is obtained by taking $n = 0$ in log z.

From Eqs. (16) and (17), we have

$$\log\left(x + iy\right) = 2in\pi + \log\left(x + iy\right)$$

If $y = 0$, then $\log x = 2in\pi + \log x$.

This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real, whereas all other values are imaginary.

Now, we calculate real and imaginary parts of $\log(x+iy)$. Consider,

$$\log(x+iy) = 2in\pi + \log(x+iy)$$

Put $x = r\cos\theta$, $y = r\sin\theta$. Hence, $r = \sqrt{x^2+y^2}$

and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$\begin{aligned}\log(x+iy) &= 2in\pi + \log\left[r(\cos\theta + i\sin\theta)\right] \\ &= 2in\pi + \log(re^{i\theta}) \\ &= 2in\pi + \log r + i\theta \\ &= \log\sqrt{x^2+y^2} + i\left[2n\pi + \tan^{-1}\left(\frac{y}{x}\right)\right]\end{aligned}$$

The real and imaginary parts of $(\alpha+i\beta)^{x+iy}$ can be given as

$$(\alpha+i\beta)^{x+iy} = e^{(x+iy)\log(\alpha+i\beta)} = e^{(x+iy)[2in\pi+\log(\alpha+i\beta)]}$$

Put $\alpha = r\cos\theta$, $\beta = r\sin\theta$. Hence, $r = \sqrt{\alpha^2+\beta^2}$ and $\theta = \tan^{-1}\left(\dfrac{\beta}{\alpha}\right)$.

$$\begin{aligned}(\alpha+i\beta)^{x+iy} &= e^{(x+iy)\left[2in\pi+\log re^{i\theta}\right]} = e^{(x+iy)\left[\log r+i(2n\pi+\theta)\right]} \\ &= e^{A+iB} = e^{A}(\cos B + i\sin B)\end{aligned}$$

Hence, the required real part $= e^{A}\cos B$ and imaginary part $= e^{A}\sin B$, where $A = x\log r - y(2n\pi+\theta)$ and $B = y\log r + x(2n\pi+\theta)$.

LIMIT AND CONTINUITY OF COMPLEX FUNCTIONS

Definitions of limit and continuity for a function of a complex variable are analogous to those for a real variable. Hence, $f(z)$ is said to have the limit l as z approaches z_0, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|f(z) - l\right| < \varepsilon \text{ for } \left|z - z_0\right| < \delta$$

$f(z)$ is said to be continuous at z_0, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|f(z) - f(z_0)\right| < \varepsilon \text{ whenever } \left|z - z_0\right| < \delta$$

In other words, $f(z)$ is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$ exists.

DERIVATIVE OF COMPLEX VARIABLES

If $f(z)$ is a single-valued complex function in the z-plane, the derivative of $f(z)$ is denoted by

$$f'(z) = \lim_{\Delta z\to 0} f\frac{(z+\Delta z) - f(z)}{\Delta z} \tag{18}$$

provided the limit exists.

If the limit Eq. (18) exists for $z = z_0$, then $f(z)$ is called analytic at z_0. If the limit exists for all z in a region R, then $f(z)$ is called analytic in R. In order to be analytic, $f(z)$ must be single-valued and continuous. However, the converse is not necessarily true.

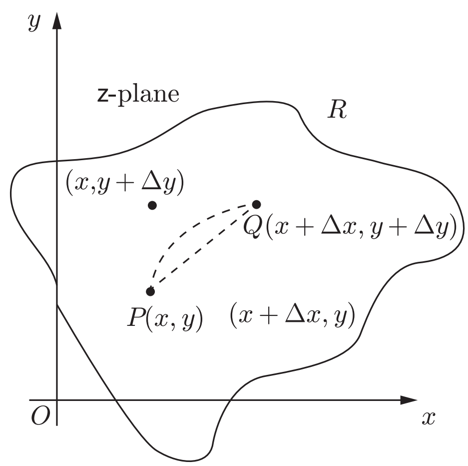


Figure 1 | A random region R.

Consider the region R in Fig. 1. Let us say that $P(z)$ is fixed and $Q(z+\Delta z)$ is an adjacent point. The point Q may approach P along any path (straight or curved) in the given region, i.e. Δz may tend to zero in any manner and $\dfrac{dw}{dz}$ may not exist. Hence, it is necessary for $\dfrac{dw}{dz}$ to exist. For this, we have the Cauchy–Riemann theorem.

Cauchy–Riemann Equations

A necessary condition for $w = u(x,y) + iv(x,y)$ to be analytic in a region R is that u and v satisfy the following equations:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial y} \tag{19}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{20}$$

Equations (19) and (20) are called Cauchy–Riemann equations.

If the partial derivatives in Eqs. (19) and (20) are continuous in R, the equations are sufficient conditions that $f(z)$ be analytic in R.

The derivative of $f(z)$ is then given by

$$f'(z) = \lim_{\Delta y \to 0}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x \qquad (21)$$

or $f'(z) = \lim\limits_{\Delta x \to 0}\left(\dfrac{\partial u}{i\partial y} + i\dfrac{\partial v}{i\partial y}\right)$

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = v_y - iu_y \qquad (22)$$

The real and imaginary parts of an analytic function are called conjugate functions.

INTEGRATION OF COMPLEX VARIABLES

Suppose $f(x)$ is a single-valued and continuous complex variable in a region R, then the integral of $f(x)$ along an arbitrary path c in R from point $z_1 = x_1 + iy_1$ to $z_2 = x_2 + iy_2$ is defined as

$$\int_c f(z)dz = \int_{(x_1,y_1)}^{(x_2,y_2)} (u+iv)(dx+idy)$$

$$= \int_{(x_1,y_1)}^{(x_2,y_2)} udx - vdy + i\int_{(x_1,y_1)}^{(x_2,y_2)} vdx + udy \qquad (23)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of the line integrals of real functions.

The rules for complex integration are similar to those for real integral.

An important result is

$$\left|\int_c f(x)dz\right| \le \int_c |f(x)||dz| \leqq M\int_c ds = ML \qquad (24)$$

where M is an upper bound of $|f(z)|$ on c, i.e. $|f(z)| \le M$, and L is the length of the path c.

CAUCHY'S THEOREM

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve c, then according to Cauchy's theorem,

$$\int_c f(z)\cdot dz = 0 \qquad (25)$$

Now, $f(z) = u(x,y) + iv(x,y)$ and $dz = dx + idy$, we get

$$\int_c f(z)\cdot dz = \int_c (udx - vdy) + i\int_c (vdx + udy) \qquad (26)$$

As already stated, $f'(z)$ is continuous. Therefore, $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ are also continuous in region R enclosed by the curve c. Hence, the Green's theorem can be applied to Eq. (26),

$$\int_c f(z)\cdot dz = -\iint_R \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]dxdy + i\iint_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]dxdy \qquad (27)$$

Now, $f(z)$ being analytic, u and v necessarily satisfy the Cauchy–Riemann equations. Thus, the integrands of the two double integrals in Eq. (27) vanish identically.

Hence, $$\iint_c f(z)\cdot dz = 0. \qquad (28)$$

Some of the important results that can be concluded are:

1. The line integral of a function $f(z)$, which is analytic in the region R, is independent of the path joining any two points of R.
2. **Extension of Cauchy's theorem:** If $f(z)$ is analytic in the region R between the two simple closed curves c and c_1, then
$$\int_c f(z)\cdot dz = \int_{c_1} f(z)\cdot dz \qquad (29)$$
3. If $c_1, c_2, c_3, \ldots$ be any number of closed curves within c, then
$$\int_c f(z)\cdot dz = \int_{c_1} f(z)\cdot dz + \int_{c_2} f(z)\cdot dz + \int_{c_3} f(z)\cdot dz + \cdots \qquad (30)$$

CAUCHY'S INTEGRAL FORMULA

If $f(x)$ is analytic within and on a simple closed curve c and α is any point interior to c, then

$$f(\alpha) = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{z-\alpha} \qquad (31)$$

Consider $\dfrac{f(z)}{z-\alpha}$ in Eq. (31), which is analytic at all points within c except at $z = \alpha$. Now, with α as center and r as radius, draw a small circle c_1 lying entirely within c.

Now, since $\frac{f(z)}{z-\alpha}$ is analytic in the region enclosed by c and c_1, we apply Cauchy's theorem and get

$$\int_c \frac{f(z)}{z-\alpha}dz = \int_{c_1} \frac{f(z)}{z-\alpha}dz \tag{32}$$

For any point on c_1, $z-\alpha = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$

$$\int_c \frac{f(z)}{z-\alpha}dz = \int_{c_1} \frac{f(\alpha + re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta}d\theta$$

$$= i\int_{c_1} f(\alpha + re^{i\theta})d\theta \tag{33}$$

As $r \to 0$, the integral in Eq. (33) approaches

$$i\int_{c_1} f(\alpha)d\theta = if(\alpha)\int_0^{2\pi} d\theta = 2\pi i f(\alpha)$$

Thus, $\int_c \frac{f(z)}{z-\alpha}dz = 2\pi i(\alpha)$

$$\text{i.e. } f(\alpha) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-\alpha}dz \tag{34}$$

Eq. (34) is the desired Cauchy's integral formula.

Now, differentiating both sides of Eq. (34) w.r.t. α, we get

$$f'(\alpha) = \frac{1}{2\pi i}\int_c \frac{\partial}{\partial\alpha}\left[\frac{f(z)}{z-\alpha}\right]dz = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-\alpha)^2}\cdot dz \tag{35}$$

Similarly, $f''(\alpha) = \frac{2!}{2\pi i}\int_c \frac{f(z)}{(z-\alpha)^3}\cdot dz$ (36)

Generally, we can write that

$$f^n(\alpha) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-\alpha)^{n+1}}dz \tag{37}$$

Hence, the above results from Eqs. (34)–(37) conclude that if a function $f(z)$ is analytic on the simple closed curve c then the values of the function and all its derivatives can be found at any point of c.

TAYLOR'S SERIES OF COMPLEX VARIABLES

If $f(z)$ is analytic inside a circle C with centre at α, then for z inside C,

$$f(z) = f(\alpha) + f'(\alpha)(z-\alpha) + \frac{f''(\alpha)}{2!}(z-\alpha)^2 + \cdots + \frac{f^n(\alpha)}{n!}(z-\alpha)^n \tag{38}$$

LAURENT'S SERIES OF COMPLEX VARIABLES

If $f(z)$ is analytic in the ring-shaped region R bounded by the two concentric circles C and C_1 of radii r and r_1 (such that $r > r_1$) and with center at α, then for all z in R

$$f(z) = \sum_{-\infty}^{\infty} \alpha(z-\alpha)^n = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots + a_{-1}(z-\alpha)^{-1} + a_{-2}(z-\alpha)^{-2} \tag{39}$$

where $a_n = \frac{1}{2\pi i}\oint \frac{f(t)}{(t-\alpha)^{n+1}}dt.$

ZEROS AND POLES OF AN ANALYTIC FUNCTION

A zero of an analytic function $f(z)$ is the value of z for which $f(z) = 0$.

If $f(z)$ is analytic in the neighborhood of a point $z = \alpha$, then by Taylor's theorem,

$$f(z) = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots + a_n(z-\alpha)^n$$

where $a_n = f^n\frac{(\alpha)}{n!}$.

If $a_0 = a_1 = a_2 = \cdots = a_{n-1} = 0$, then $f(z)$ has a zero of order n at $z = \alpha$. When $n = 1$, the zero is said to be simple.

A singular point of a function $f(z)$ is a value of z at which $f(z)$ fails to be analytic. If $f(z)$ is analytic everywhere in some region except at an interior point $z = \alpha$, then $z = \alpha$ is called an isolated singularity of $f(z)$. In such a case, $f(z)$ can be expanded in a Laurent's series around $z = \alpha$ giving

$$f(z) = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots + a_{-1}(z-\alpha)^{-1} + a_{-2}(z-\alpha)^{-2} + \cdots \tag{40}$$

where $a_n = \frac{1}{2\pi i}\int \frac{f(t)}{(t-\alpha)^{n+1}}dt.$

If all the negative powers of $(z - \alpha)$ in Eq. (40) are zero, then $f(x) = \sum_{n=0}^{\alpha} a_n(z-\alpha)^n$. In this case, the singularity can be removed by defining $f(z)$ at $z = \alpha$ in such a way that it becomes analytic at $z = \alpha$, and this singularity is called a removable singularity. Thus, if $\lim_{z\to\alpha} f(z)$ exists finitely, then $z = \alpha$ is a removable singularity.

If the number of negative powers of $(z - \alpha)$ in Eq. (40) is infinite, then $(z - \alpha)$ is called an essential singularity. In this case, $\lim_{z\to\alpha} f(z)$ does not exist.

If $f(z) = \dfrac{\phi(z)}{(z-\alpha)^2}$, where $\phi(z)$ is analytic everywhere in a region including $z = \infty$ and if n is a positive integer, then $f(z)$ has isolated singularity at $z = \alpha$, which is called a pole of order n. If $n = 1$, the pole is called a simple pole; if $n = 2$, the pole is called a double point and so on.

RESIDUES

The coefficient of $(z-\alpha)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at the point.

It can be found from the formula

$$a_{-1} = \lim_{z\to\alpha} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{z-1}}\left[(z-\alpha)^n f(z)\right] \qquad (41)$$

where n is the order of the pole.

For simple poles, the calculation of the residue reduces to

$$a_{-1} = \lim_{z\to\alpha} (z-\alpha) f(z) \qquad (42)$$

The residue of $f(z)$ at $z = \alpha$ can also be found by

$$\text{Res}\, f(\alpha) = \frac{1}{2\pi i} \oint_c f(z) \qquad (43)$$

Residue Theorem

If $f(z)$ is analytic in a region R except for a pole of order n at $z = \alpha$ and let C be a simple closed curve in R containing $z = \alpha$, then

$\oint_c f(z) dz = 2\pi i \times$ (sum of the residue at the singular points within C) (44)

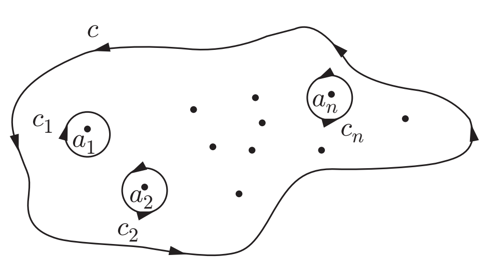
Calculation of Residues

1. If $f(z)$ has a simple pole at $z = \alpha$, then
$$\text{Res}\, f(\alpha) = \lim_{z\to\alpha} \left[(z-\alpha) f(z)\right]$$
2. If $f(z) = \phi(z)/\psi(z)$, where
$\psi(z) = (z-\alpha) f(z), f(\alpha) \neq 0$, then
$$\text{Res}\, f(\alpha) = \frac{\phi(\alpha)}{\psi'(\alpha)}$$
3. If $f(z)$ has a pole of order n at $z = \infty$, then
$$\text{Res}\, f(\alpha) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-\alpha)^n f(x)\right]\right\}_{z=\alpha}$$

SOLVED EXAMPLES

1. Prove the residue theorem.

Solution: Consider the following diagram:



Let us surround each of the singular points $a_1, a_2, \ldots, a_n$ by a small circle such that it encloses no other singular point. These circles $c_1, c_2, \ldots, c_n$ together with c form a multiple-connected region in which $f(z)$ is analytic.

Applying Cauchy's theorem, we have

$$\oint_c f(x)\cdot dz = \oint_{c_1} f(z)\cdot dz + \oint_{c_2} f(z)\cdot dz + \cdots + \oint_{c_n} f(z)\cdot dz$$
$$= 2\pi i[\text{Res}\, f(a_1) + \text{Res}\, f(a_2) + \cdots + \text{Res}\, f(a_n)]$$

which is the desired result.

2. Separate $\tan^{-1}(x+iy)$ into real and imaginary parts.

Solution: Let $\alpha + i\beta = \tan^{-1}(x+iy)$ (1)

Then $\alpha - i\beta = \tan^{-1}(x-iy)$ (2)

Adding Eqs. (1) and (2), we get

$$2\alpha = \tan^{-1}(x+iy) + \tan^{-1}(x-iy)$$
$$= \tan^{-1}\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}$$

Therefore, $\alpha = \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}$

Subtracting Eq. (2) from Eq. (1), we get

$$2i\beta = \tan^{-1}(x+iy) - \tan^{-1}(x-iy)$$
$$= \frac{\tan^{-1}(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$
$$= \tan^{-1} i\frac{2y}{1+x^2+y^2}$$
$$= i\tan^{-1}\frac{2y}{1+x^2+y^2} \quad [\because \tan^{-1} iz = i\tanh^{-1} z]$$
$$\beta = \frac{1}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2}$$

3. Show that $f(z) = z^3$ is analytic.

Solution: Let $z = x + iy$

$$\Rightarrow z^2 = (x+iy)(x+iy) = x^2 - y^2 + ixy$$
$$\Rightarrow z^3 = (x^2 - y^2 + ixy)(x+iy)$$
$$= (x^3 - 3xy^2) + (3x^2y - y^3)i$$

Now, $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$
$$\frac{\partial u}{\partial y} = -6xy$$
$$\frac{\partial v}{\partial x} = 6xy$$
$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

So, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

So, Cauchy–Riemann equations are satisfied and also the partial derivatives are continuous at all points. Hence, z^3 is analytic for every z.

4. If $w = \log z$, find dw/dz and determine if w is non-analytic.

Solution: We have

$$w = u + iv = \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1} y/x$$

Hence, $u = \frac{1}{2}\log(x^2+y^2)$ and $v = \tan^{-1} y/x$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$$
$$\Rightarrow \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$
$$\Rightarrow \frac{\partial v}{\partial x} = -\frac{y}{x^2+y^2}$$
$$\Rightarrow \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

Since the Cauchy–Riemann equations are satisfied and the partial derivatives are continuous except at (0, 0), w is analytic everywhere except at $z = 0$.

$$\therefore \frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$
$$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}(z \neq 0)$$

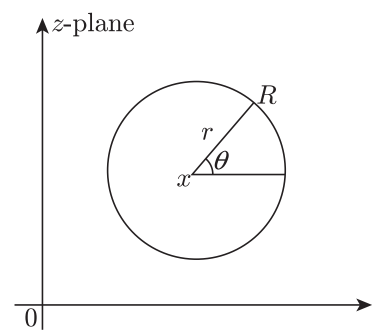
5. Prove that

(a) $\int_c \frac{dz}{z-\alpha} = 2\pi i$

(b) $\int_c (z-\alpha)^n dz = 0 \quad [n, \text{any integer} \neq -1]$

where c is circle $|z-\alpha| = r$.

Solution: The parametric equations of c are $z = \alpha = re^{i\theta}$, where θ varies from 0 to 2π as z describes c once in the positive (anti-clockwise) sense.



(a) $$\int_c \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{1}{re^{i\theta}} \cdot ire^{i\theta} \cdot d\theta \; [\because dz = ire^{i\theta}d\theta]$$
$$= i\int_0^{2\pi} d\theta = 2\pi i$$

(b) $$\int_c (z-\alpha)^n dz = \int_0^{2\pi} r^n e^{ni\theta} \cdot ire^{i\theta} d\theta$$
$$= ir^{n+1}\int_0^{2\pi} e^{(n+1)} d\theta$$
$$= \frac{r^{n+1}}{n+1}\left| e^{(n+1)\theta i} \right|_0^{2\pi}, \qquad \text{provided } n \neq -1$$
$$= \frac{r^{n+1}}{n+1}\left[e^{2(n+1)\pi i} - 1 \right] = 0 \qquad [\because e^{2(n+1)\pi i} = 1]$$

6. Evaluate $\int_c \frac{z^2-z+1}{z-1}\,dz$, where c is the circle and $|z|=1$.

Solution: Here $f(z)=z^2-z+1$ and $\alpha=1$

Since $f(z)$ is analytic within and on circle c: $|z|=1$ and $\alpha=1$ lies on c.

Therefore, by Cauchy's integral formula, we have

$$\frac{1}{2\pi i}\int_c \frac{f(z)}{z-\alpha} = f(\alpha)=1 \Rightarrow \int_c \frac{z^2-z+1}{z-1}dz = 2\pi i$$

7. Evaluate $\oint_c \frac{\sin^2 z}{(x-x/6)^3}dz$, where c is the circle $|z|=1$.

Solution: $f(z)=\sin^2 z$ is analytic inside the circle c.

We have $|z|=1$ and the point $\alpha=\pi/6$ (= 0.5 approx.) lies within c.

Therefore, by Cauchy's integral formula, we have

$$f''(\alpha)=\frac{2!}{2\pi i}\oint_c \frac{f(z)}{(z-\alpha)^3}dz$$

$$\Rightarrow \oint_c \frac{\sin^2 z}{(z-\pi/6)^3}dz = \pi i\left[\frac{d^2}{dz^2}(\sin^2 z)\right]_{z=\pi/6}$$

$$= \pi i(2\cos 2z)_{z=\pi/6}$$

$$= 2\pi i\cos\pi 3 = \pi i$$

8. Find the Taylor's expansion of $f(z)=\frac{2z^3+1}{z^2+z}$ about the point $z=i$.

Solution: We have

$$f(z)=\frac{2z^3+1}{z(z+1)} = 2z-2+\frac{2z+1}{z(z+1)}$$

$$=(2i-2)+2(z-i)+\frac{1}{z}+\frac{1}{z+1} \qquad (1)$$

Putting $z - i = t$, we get

$$\frac{1}{z}=\frac{1}{(t+i)}=\frac{1}{i}\left[1+\frac{t}{i}\right]^{-1} \quad \text{(Expanding by binomial theorem)}$$

$$=\frac{1}{i}\left[1-\frac{t}{i}+\frac{t^2}{i^2}-\frac{t^3}{i^3}+\frac{t^4}{i^4}-\cdots\infty\right]$$

$$=\frac{1}{i}+\frac{t}{i}+\frac{t^2}{i^3}-\frac{t^3}{i^4}+\frac{t^4}{i^5}-\cdots\infty$$

$$=-i+(z-i)+\sum_{n=2}^{\infty}(-1)^n\frac{(z-i)^n}{i^{n+1}} \qquad (2)$$

Similarly, $\frac{1}{z+1}=\frac{1}{t+i+1}=\frac{1}{1+i}\left[1+\frac{t}{1+i}\right]^{-1}$

$$=\frac{1}{1+i}\left[1-\frac{t}{1+i}+\frac{t^2}{(1+i)^2}-\frac{t^3}{(1+i)^3}+\frac{t^4}{(1+i)^4}-\cdots\infty\right]$$

$$=\frac{1-i}{2}-\frac{t}{2i}+\left[\frac{t^2}{(1+i)^3}-\frac{t^3}{(1+i)^4}+\frac{t^4}{(1+i)^5}-\cdots\infty\right]$$

$$=\frac{1}{2}-\frac{i}{2}-\frac{z-i}{2i}+\sum_{n=2}^{\infty}(-1)^n\frac{(z-i)^n}{(1+i)^{n+1}} \qquad (3)$$

Substituting values from Eqs. (2) and (3) in Eq. (1), we get

$$f(z)=\left[2i-2-i+\frac{1}{2}-\frac{i}{2}\right]+\left(2+1-\frac{1}{2i}\right)$$

$$+\sum_{n=2}^{\infty}(-1)^n\left\{\frac{1}{i^{n+1}}+\frac{1}{(1+i)^{n+1}}\right\}\times(z-i)^n$$

$$=\left(\frac{i}{2}-\frac{3}{2}\right)+\left(3+\frac{i}{2}\right)(z-i)$$

$$+\sum_{n=2}^{\infty}(-1)^n\left\{\frac{1}{i^{n+1}}+\frac{1}{(1+i)^{n+1}}\right\}(z-i)^n$$

9. Discuss the singularity of the function $\frac{e^{2z}}{(z-1)^4}$.

Solution: We have

$$f(x)=\frac{e^{2z}}{(z-1)^4}$$

Putting $t = z - 1$, we get

$$f(x)=\frac{e^{2(t+1)}}{t^4}=\frac{e^2}{t^4}\cdot e^{2t}$$

$$=\frac{e^2}{t^4}\left\{1+\frac{2t}{1!}+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\frac{(2t)^5}{5!}+\cdots\right\}$$

$$=e^2\left\{\frac{1}{t^4}+\frac{2}{t^3}+\frac{2}{t^2}+\frac{4}{3t}+\frac{2}{3}+\frac{4t}{15}+\cdots\right\}$$

Since there are four finite number of terms containing negative powers of $(z-1)$, $z=1$ is a pole of fourth order.

10. Find the sum of the residues of $f(z)=\frac{\sin z}{z\cos z}$ at its poles inside the circle $|z|=2$.

Solution: Now, $f(z)$ has simple poles at $z=0, \pm\pi/2, \pm 3\pi/2, \ldots$

Only the poles $z = 0$ and $z = \pm\pi/2$ lies inside $|z| = 2$.

Hence, $\text{Res}\, f(0) = \lim_{z\to 0}[z \cdot f(z)] = \lim_{z\to 0}\left[\frac{\sin z}{\cos z}\right] = 0$

$$\text{Res}\, f(\pi/2) = \lim_{z\to\pi/2}\left[\left(z - \frac{\pi}{2}\right) f(z)\right] = \lim_{z\to\pi/2}\left[\frac{(z-\pi/2)\sin z}{z\cos z}\right]$$

$$= \lim_{z\to\pi/2}\frac{(z-\pi/2)\cos z + \sin z}{\cos z - z\sin z}$$

$$= \frac{1}{-\pi/2} = -\frac{2}{\pi}$$

$$\text{Res}\, f(-\pi/2) = \lim_{z\to-\pi/2}\left[\frac{(z+\pi/2)\sin z}{z\cos z}\right]$$

$$= \lim_{z\to-\pi/2}\frac{(z+\pi/2)\cos z + \sin z}{\cos z - z\sin z}$$

$$= \frac{-1}{-\pi/2} = \frac{2}{\pi}$$

Hence, the sum of all the residues $= 0 - \frac{2}{\pi} + \frac{2}{\pi} = 0$

PRACTICE EXERCISE

1. If $w = \log z$, then what is the value of dw/dz?

(a) $1/z$ (b) z
(c) z^2 (d) $1/z^2$

2. What is the value of $\int_0^{2+i} (\bar{z})^2 dz$ along the line $y = x/2$?

(a) $5i$ (b) $\frac{5}{3}(2-i)$

(c) $3(i)$ (d) $\frac{5}{3}(i-2)$

3. What is the value of $\oint_c \frac{e^{2z}}{(z+i)^4}dz$, if c is the circle $|z| = 3$?

(a) ie^{-2} (b) πie^2

(c) $i\pi/3e^4$ (d) $\frac{8\pi i}{3}e^{-2}$

4. Evaluate $\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$, where c is the circle $|z| = 3$.

(a) $i(\sin\pi + \cos\pi)$ (b) πi
(c) $4\pi i$ (d) $(2+4\pi)i$

5. Find the Taylor's expression of $f(z) = \frac{1}{(z+1)^2}$ about the point $z = -i$.

(a) $\sum_{n=2}^{\infty}(-1)^n \frac{(z-i)^n}{(1+i)^{n+1}}$

(b) $\frac{i}{2}\left[\sum_{n=0}^{\infty}(-1)^n \frac{(n+1)(z+i)^n}{(1-i)^n}\right]$

(c) $\frac{i}{2}\sum_{n=2}^{\infty}(-1)^n \frac{(z-i)^n}{(1+i)^{n+1}}$

(d) $\frac{i}{2}\left[1 + \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)(z+i)^n}{(1+i)^n}\right]$

6. Find the Laurent's expansion of $f(z) = \frac{7z-2}{z(z+1)(z-2)}$ in the region $1 < z+1 < 3$.

(a) $1 - \frac{z+1}{3} + \frac{(z+1)^2}{3^2} - \frac{(z+1)^3}{3^3} + \cdots\infty$

(b) $\frac{-2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\infty$

$-\frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \frac{(z+1)^3}{3^3} + \cdots\infty\right]$

(c) $\frac{-2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\infty$

(d) $\frac{-2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \frac{(z+1)^3}{3^3} + \cdots\infty\right]$

7. What singularity does $f(x) = \frac{z - \sin z}{z^2}$ have?

(a) Removable singularity at $z = 0$
(b) Essential singularity at $z = 0$
(c) Isolated singularity at $z = 0$
(d) None of these

8. What singularity does $f(x) = \dfrac{e^{1/z}}{z^2}$ have?

(a) Removable singularity at $z = 0$

(b) Essential singularity at $z = 0$

(c) Isolated singularity at $z = 0$

(d) None of these

9. What is the residue at $z = \pi/2$ for $f(z) = \tan z$?

(a) 0 (b) 2

(c) 1 (d) −1

10. Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its pole of order 4.

(a) 27/16 (b) −8

(c) 8 (d) 101/16

ANSWERS

1. (a)	**3.** (d)	**5.** (d)	**7.** (a)	**9.** (d)
2. (b)	**4.** (c)	**6.** (b)	**8.** (b)	**10.** (d)

EXPLANATIONS AND HINTS

1. (a) We have

$$w = \log z$$

Let $w = u + iv = \log(x + iy)$

$$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$$

Thus, $u = \dfrac{1}{2}\log(x^2 + y^2)$, $v = \tan^{-1} y/x$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

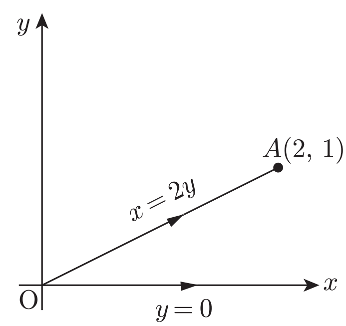
$$\frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right)$$

$$= \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{(x + iy)} = \frac{1}{z}(z \neq 0)$$

2. (b) Along the line OA, we have



$$x = 2y,\ z = (2 + i)y,\ \bar{z} = (2 - i)y$$

and $dz = (2 + i)dy$

$$\therefore I = \int_0^{2+i} (\bar{z})^2 dz = \int_0^1 (2 - i)^2 y^2 (2 + i)dy$$

$$= 5(2 - i)\left|\frac{y^3}{3}\right|_0^1 = \frac{5}{3}(2 - i)$$

3. (d) Suppose $f(z) = e^{2z}$ is analytic within the circle c, i.e. $|z| = 3$. Also $z = -1$ lies inside c.

Therefore, by Cauchy's integral formula, we have

$$f'''(\alpha) = \frac{3!}{2\pi i}\int_c \frac{f(z)dz}{(z-\alpha)^4}$$

$$\Rightarrow \int_c \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{6}\left|\frac{d^3(e^{2x})}{dz^3}\right|_{z=-1}$$

$$= \frac{\pi i}{3}\left[8e^{2z}\right]_{z=-1}$$

$$= \frac{8\pi i}{3}e^{-2}$$

4. (c) We know that

$f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within the circle $|z| = 3$ and $z = 1$, $z = 2$ lie inside this circle. Therefore,

$$\oint_c \frac{f(z)dz}{(z-1)(z-2)} = \oint_c (\sin \pi z^2 + \cos \pi z^2)$$

$$= \left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$= \oint_c \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}\,dz - \oint_c \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1}\,dz$$

By Cauchy's integral formula, we have

$$= 2\pi i\left[\sin \pi(2)^2 + \cos \pi(2)^2\right] - 2\pi i\left[\sin \pi(1)^2 + \cos \pi(1)^2\right]$$

$$= 2\pi i(0+1) - 2\pi i(0-1)$$

$$= 4\pi i$$

5. (d) We have

$$f(z) = \frac{1}{(z+1)^2}$$

Now, we need to find the Taylor's expansion of $f(z)$ about $z = -i$, i.e. in powers of $z + i$.

Putting $z + i = t$, we get

$$f(z) = \frac{1}{(t-i+1)^2} = (1-i)^{-2}\left[1 + t/(1-i)\right]^{-2}$$

$$= \frac{i}{2}\left[1 - \frac{2t}{1-i} + \frac{3t^2}{(1-i)^2} - \frac{4t^3}{(1-i)^3} + \cdots\right]$$

[Expanding by Binomial theorem]

$$= \frac{i}{2}\left[1 + \sum_{n=1}^{\infty}(-1)^n\frac{(n+1)(z+i)^n}{(1-i)^n}\right]$$

6. (b) We have

$$f(z) = \frac{7z-2}{z(z+1)(z-2)}$$

Putting $z + 1 = u$, we get

$$f(z) = \frac{7(u-1)-2}{u(u-1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)}$$

$$= -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

[Using partial fractions]

$$= -\frac{3}{u} + \frac{1}{u(1-1/u)} - \frac{2}{3(1-u/3)}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1 - \frac{u}{3}\right)^{-1}$$

Since $1 < u < 3$ or $\frac{1}{u} < 1$ or $\frac{u}{3} < 1$, we expand by Binominal theorem and get

$$f(z) = -\frac{3}{u} + \frac{1}{u}\left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \cdots\infty\right) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \cdots\infty\right)$$

$$= -\frac{2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \cdots\infty - \frac{2}{3}\left[1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \cdots\infty\right]$$

Hence,

$$f(z) = -\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\infty - \frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \frac{(z+1)^3}{3^3} + \cdots\infty\right]$$

which is valid in the region $1 < z + 1 < 3$.

7. (a) Here $z = 0$ is a singularity.

Also, $$\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right)\right\}$$

$$= \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \cdots$$

Since these are no negative powers of z in the expansion, $z = 0$ is a removable singularity.

8. (b) We have

$$f(z) = \frac{e^{1/z}}{z^2} = \frac{1}{z^2}\left\{1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots + \infty\right\}$$

$$= z^{-2} + z^{-3} + \frac{z^{-4}}{2} + \cdots + \infty$$

Since there are infinite number of terms in the negative powers of z, $f(z)$ has an essential singularity at $z = 0$.

9. (d) We have

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

At $z = \pi/2$,

$$\text{Res } f(\pi/2) = \lim_{z\to\pi/2} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \lim_{z\to\pi/2}\left(\frac{\sin z}{-\sin z}\right) = -1$$

10. (d) The prices of $f(z)$ are given by $(z-1)^4(z-2)(z-3) = 0$.

Therefore, $x = 1$ is a pole of order $= 4$, whereas $z = 2$ and $z = 3$ are simple poles.

Hence, we find

$$\text{Res } f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4\cdot\frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1}$$

$$= \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$$= \frac{1}{6}\frac{d^3}{dz^3}\left[z+5-\frac{8}{z-2}+\frac{27}{z-3}\right]$$

$$= \frac{1}{6}\left[-8\cdot\frac{(-1)^3 3!}{(z-2)^4}+\frac{27(-1)^3 3!}{(z-2)^4}\right]_{z=1}$$

$$= -\left[-8+\frac{27}{16}\right] = \frac{101}{16}$$

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. Consider likely applicability of Cauchy's integral theorem to evaluate the following integral counter clockwise around the unit circle c.

$$I = \oint_c \sec z dz,$$

z being a complex variable. The value of I will be

(a) $I = 0$: singularities set $= \phi$

(b) $I = 0$: singularities set $= \left\{\pm\frac{2n+1}{2}\pi, n = 0, 1, 2, \ldots\right\}$

(c) $I = \pi/2$: singularities set $= \{\pm n\pi = 0, 1, 2, \ldots\}$

(d) None of above

(GATE 2005, 2 Marks)

Solution: We have

$$\int \sec z dz = \int \frac{1}{\cos z}dz$$

The poles are at

$$z_0 = (n+1/2)\pi = \ldots,-3\pi/2,-\pi/2,\ \pi/2,3\pi/2,\ldots$$

None of these poles lie inside the unit circle $|z| = 1$.

Hence, sum of the residues at poles $= 0$.

Therefore, singularities set $= \phi$

$$I = 2\pi i\,[\text{sum of residue of } f(z) \text{ at the poles}]$$
$$= 2\pi i \times 0 = 0$$

Ans. (a)

2. Using Cauchy's integral theorem, the value of the integral (integration being taken in counterclockwise direction) $\oint_c \frac{z^3-6}{3z-i}dz$ is

(a) $\frac{2\pi}{81} - 4\pi i$ (b) $\frac{\pi}{8} - 6\pi i$

(c) $\frac{4\pi}{81} - 6\pi i$ (d) 1

(GATE 2006, 2 Marks)

Solution: Cauchy's integral theorem is given by

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz$$

i.e.

$$\oint_c \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$$

Now,

$$\oint_c \frac{z^3-6}{3z-1}dz = \frac{1}{3}\oint_c \frac{z^3-6}{\left(z-\frac{i}{3}\right)}$$

Applying Cauchy's integral theorem and using $f(z) = z^3 - 6$, we get

$$= \frac{1}{3}\left[2\pi i\, f\left(\frac{i}{3}\right)\right] = \frac{1}{3}\left\{2\pi i\left[\left(\frac{i}{3}\right)^3 - 6\right]\right\}$$

$$= \frac{1}{3}\left[2\pi i\left(\frac{i^3}{27}-6\right)\right] = \frac{2\pi}{81}i^4 - 4\pi i = \frac{2\pi}{81} - 4\pi i$$

Ans. (a)

3. The value of the contour integral $\oint\limits_{|z-i|=2} \frac{1}{z^2+4}dz$ in positive sense is

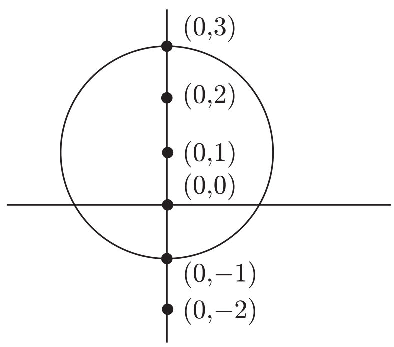
(a) $i\pi/2$ (b) $-\pi/2$
(c) $-i\pi/2$ (d) $\pi/2$

(GATE 2006, 2 Marks)

Solution: We have

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$

Pole (0, 2) lies inside the circle $|z-i| = 2$, whereas pole (0, −2) lies outside it, as can be seen from the following figure:



$$\begin{aligned}\int_C f(z)dz &= 2\pi i \text{ [Residue at these poles which are inside } C]\\ &= 2\pi i \operatorname{Res} f(2i)\\ &= 2\pi i \frac{1}{(2i+2i)} = \frac{\pi}{2}\end{aligned}$$

Ans. (d)

4. The value of $\oint\limits_C \frac{dz}{(1+z^2)}$, where C is the contour $|z - i/2| = 1$, is

(a) $2\pi i$ (b) π
(c) $\tan^{-1} z$ (d) $\pi i \tan^{-1} z$

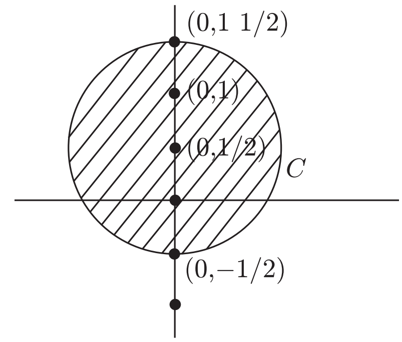
(GATE 2007, 2 Marks)

Solution: We know that

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+1)}$$

Hence, we have poles at i and $-i$, i.e. at (0, 1) and (0, −1).

From the figure of $|z-i/2| = 1$, we see that pole (0, 1), i.e. i is inside C, whereas pole (0, −1), i.e. $-i$ is outside C.



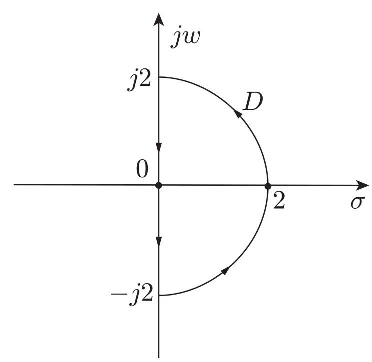
$|z - i/2| = 1$

So,

$$I = 2\pi i \operatorname{Res} f(i) = 2\pi i \cdot \frac{1}{(i-i)(i+i)} = \pi$$

Ans. (b)

5. If the semicircle D of radius 2 is as shown in the figure, then the value of the integral $\oint\limits_D \frac{1}{(s^2-1)}ds$ is



(a) $j\pi$ (b) $-j\pi$
(c) $-\pi$ (d) π

(GATE 2007, 2 Marks)

Solution: We have

$$\begin{aligned}I &= \oint \frac{1}{(s^2-1)}ds = \oint \frac{1}{(s+1)(s-1)}ds\\ &= 2\pi j \times (\text{Sum of residues})\end{aligned}$$

Pole $s = -1$ is not inside the contour D, but $s = 1$ is inside D.

Residue at pole $s = 1$ is

$$z = \lim_{s\to 1} \frac{(s-1)}{(s-1)(s+1)} = \frac{1}{2}$$

$$\Rightarrow \qquad \oint \frac{1}{(s^2-1)}ds = 2\pi j \times \frac{1}{2} = j\pi$$

Ans. (a)

6. The integral $\oint f(z)dz$ evaluated around the unit circle on the complex plane for $f(z) = \dfrac{\cos z}{z}$ is

(a) $2\pi i$ (b) $4\pi i$
(c) $-2\pi i$ (d) 0

(GATE 2008, 2 Marks)

Solution: We have

$$f(z) = \frac{\cos z}{z}$$

Now, $f(z)$ has simple pole at $z = 0$ and it is inside the unit circle on the complex plane. Therefore,

$$\text{Residue of } f(z) \text{ at } z = 0$$

$$\operatorname*{lt}_{z\to 0} f(z)\cdot z = \operatorname*{lt}_{z\to 0} \cos z = 1$$

$$\int_C f(z)dz = 2\pi i \text{ (Residue at } z = 0)$$
$$= 2\pi\, i \cdot 1 = 2\pi i$$

Ans. (a)

7. The residue of the function $f(z) = \dfrac{1}{(z+2)^2(z-2)^2}$ at $z = 2$ is

(a) $-\dfrac{1}{32}$ (b) $-\dfrac{1}{16}$

(c) $\dfrac{1}{16}$ (d) $\dfrac{1}{32}$

(GATE 2008, 2 Marks)

Solution: Since $\operatorname*{lt}_{z\to 2}\left[(z-2)^2 f(z)\right]$ is finite and non-zero, $f(z)$ has a pole of order two at $z = 2$. The residue at $z = a$ is given for a pole of order n as

$$\operatorname{Res} f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=2}$$

Here, $n = 2$ (pole of order 2) and $a = 2$.

$$\therefore \quad \operatorname{Res} f(2) = \frac{1}{1!}\left\{\frac{d}{dz}\left[(z-2)^2 f(z)\right]\right\}_{z=2}$$

$$= \left\{\frac{d}{dz}\left[(z-2)^2 \frac{1}{(z+2)^2(z-2)^2}\right]\right\}_{z=2}$$

$$= \left\{\frac{d}{dz}\left[\frac{1}{(z+2)^2}\right]\right\}_{z=2} = \left[-2(z+2)^{-3}\right]_{z=2}$$

$$= \frac{-2}{(2+2)^3} = -\frac{1}{32}$$

Ans. (a)

8. The analytic function $f(z) = \dfrac{z-1}{z^2+1}$ has singularities at

(a) 1 and -1 (b) 1 and i
(c) 1 and $-i$ (d) i and $-i$

(GATE 2009, 1 Mark)

Solution: We have

$$f(z) = \frac{z-1}{z^2+1} = \frac{z-1}{z^2+i^2} = \frac{z-1}{(z-i)(z+i)}$$

Therefore, the singularities are at $z = i$ and $-i$.

Ans. (d)

9. If $f(z) = c_0 + c_1 z^{-1}$, then $\oint\limits_{\text{unit circle}} \dfrac{1+f(z)}{z}dz$ is given by

(a) $2\pi c_1$ (b) $2\pi(1+c_0)$
(c) $2\pi j c_1$ (d) $2\pi j(1+c_0)$

(GATE 2009, 1 Mark)

Solution: We have

$$f(z) = c_0 + c_1 z^{-1}$$

$\oint\limits_{\text{unit circle}} \dfrac{1+f(z)}{z}dz$ has one pole at origin, which is inside the unit circle.

So, $\oint \dfrac{[1+f(z)]}{z}dz = 2\pi j$ [Residue $f(z)$ at $z = 0$]

$$= 2\pi j\left[1 + f(0)\right]$$

Since $\quad f(z) = c_0 + c_1 z^{-1} \Rightarrow f(0) = c_0$

$$\therefore \quad \oint\limits_{\text{unit circle}} \frac{1+f(z)}{z}dz = 2\pi j(1+c_0)$$

Ans. (d)

10. An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x,y) + i\,v(x,y)$, where $i = \sqrt{-1}$. If $u = xy$, the expression for v should be

(a) $\dfrac{(x+y)^2}{2} + k$ (b) $\dfrac{x^2+y^2}{2} + k$

(c) $\dfrac{y^2+x^2}{2} + k$ (d) $\dfrac{(x-y)^2}{2} + k$

(GATE 2009, 2 Marks)

Solution: We are given that $f(z) = u + iv$ is analytic.

Therefore, it must satisfy the Cauchy–Riemann equations. Now,

$$u_x = v_y$$

and $$v_x = -u_y \quad (1)$$

Here, since

$$u = xy \text{ (given)} \quad (2)$$

$$\Rightarrow \quad u_x = y \text{ and } u_y = x$$

Now substituting u_x and u_y in Eqs. (1) and (2), we get

$$v_y = y \quad (3)$$

and $$v_x = -x \quad (4)$$

Integrating Eqs. (3) and (4), we can now get v as follows:

$$v_y = y$$

$$\Rightarrow \quad \frac{\partial v}{\partial y} = y$$

$$\Rightarrow \quad \int \partial v = \int y \ \partial y$$

$$\Rightarrow \quad v = \frac{y^2}{2} + f(x) \quad (5)$$

From Eq. (5), we have $v_x = f'(x)$ (6)

Since from Eq. (4), we have $v_x = -x$

Substituting this in Eq. (6), we get

$$f'(x) = -x$$

$$\Rightarrow \quad \frac{df}{dx} = -x$$

$$\Rightarrow \quad \int df = \int -x \ dx$$

$$\Rightarrow \quad f = \frac{-x^2}{2} + k$$

Now substituting in Eq. (5), we get

$$v = \frac{y^2}{2} - \frac{x^2}{2} + k$$

$$v = \frac{y^2 - x^2}{2} + k$$

Ans. (c)

11. The value of the integral $\int_C \frac{\cos(2\pi z)}{(2z-1)(z-3)} dz$ (where C is a closed curve given by $|z| = 1$) is

(a) $-\pi i$ (b) $\frac{\pi i}{5}$

(c) $\frac{2\pi i}{5}$ (d) πi

(GATE 2009, 2 Marks)

Solution: Here,

$$I = \int_C \frac{\cos(2\pi z)}{(2z-1)(z-3)} dz = \frac{1}{2} \int_C \frac{\left[\frac{\cos(2\pi z)}{(z-3)}\right]}{\left[z - \frac{1}{2}\right]}$$

Since $z = 1/2$ is a point within $|z| = 1$ (the closed curve C), we can use Cauchy's integral theorem. Hence,

$$I = \frac{1}{2} f\left(\frac{1}{2}\right)$$

where $$f(z) = \frac{\cos(2\pi z)}{(z-3)}$$

[Notice that $f(z)$ is analytic on all points inside $|z| = 1$]. Therefore,

$$I = \frac{1}{2} \frac{\cos\left(2\pi \times \frac{1}{2}\right)}{\left(\frac{1}{2} - 3\right)} = \frac{2\pi i}{5}$$

Ans. (c)

12. The modulus of the complex number $\left(\frac{3+4i}{1-2i}\right)$ is

(a) 5 (b) $\sqrt{5}$

(c) $1/\sqrt{5}$ (d) 1/5

(GATE 2010, 1 Mark)

Solution: We have

$$Z = \frac{3+4i}{1-2i}$$

$$Z = \frac{(3+4i)(1+2i)}{(1-2i)(1+2i)} = \frac{-5+10i}{5} = -1 + 2i$$

$$|Z| = \sqrt{(-1)^2 (2)^2} = \sqrt{5}$$

Ans. (b)

13. The residue of a complex function $x(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles are

(a) $\frac{1}{2}, \frac{1}{2}$ and 1 (b) $\frac{1}{2}, \frac{1}{2}$ and -1

(c) $\frac{1}{2}$, 1 and $-\frac{3}{2}$ (d) $\frac{1}{2}$, -1 and $\frac{3}{2}$

(GATE 2010, 2 Marks)

Solution: We have

$$x(z) = \frac{1-2z}{z(z-1)(z-2)}$$

Poles are $z = 0$, $z = 1$ and $z = 2$.

Residue at $z = 0$

$$\text{Residue} = \text{Value of } \frac{1-2z}{(z-1)(z-2)} \text{ at } z = 0$$

$$= \frac{1-2\times 0}{(0-1)(0-2)} = \frac{1}{2}$$

Residue at $z = 1$

$$\text{Residue} = \text{Value of } \frac{1-2z}{z(z-2)} \text{ at } z = 1$$

$$= \frac{1-2\times 1}{(1)(1-2)} = 1$$

Residue at $z = 2$

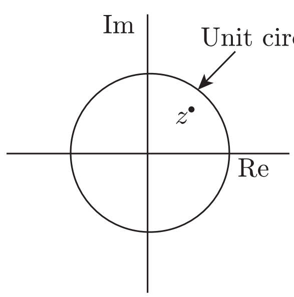
$$\text{Residue} = \text{Value of } \frac{1-2z}{z(z-1)} \text{ at } z = 2$$

$$= \frac{1-2\times 2}{2(2-1)} = -\frac{3}{2}$$

Therefore, the residue of $x(z)$ at its poles are $\frac{1}{2}$, 1 and $-\frac{3}{2}$.

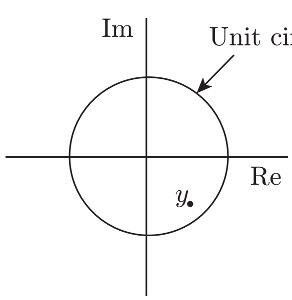
Ans. (c)

14. A point z has been plotted in the complex plane, as shown in the figure below:

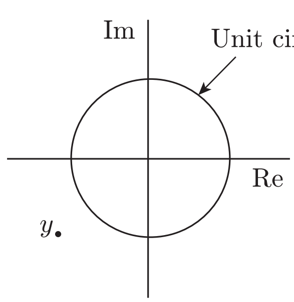


The plot of the complex number $y = \frac{1}{z}$ is?

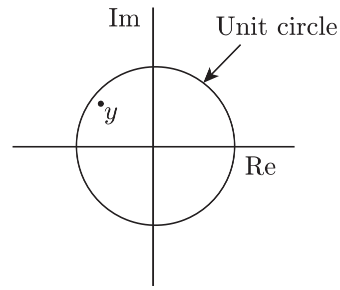
(a)



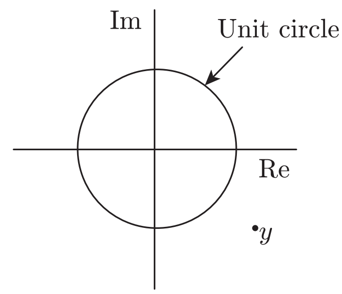
(b)



(c)

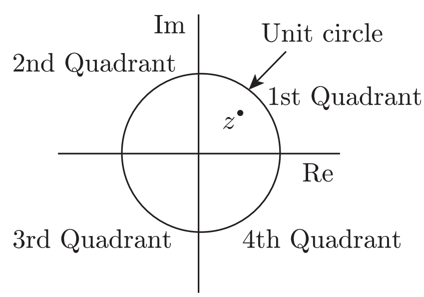


(d)



(GATE 2011, 1 Mark)

Solution: Let $z = a + bi$



Since z is shown inside the unit circle in the 1st quadrant, a and b both are +ve and $0 < \sqrt{a^2 + b^2} < 1$.

Now, $$\frac{1}{z} = \frac{1}{a+bi}$$

$$\frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

Since a, $b > 0$, we have

$$\frac{a}{\sqrt{a^2+b^2}} > 0$$

$$\frac{-b}{\sqrt{a^2+b^2}} < 0$$

So $\frac{1}{z}$ is in 4th quadrant.

$$\left|\frac{1}{z}\right| = \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{-b}{a^2+b^2}\right)^2} = \sqrt{\frac{1}{a^2+b^2}}$$

$$= \frac{1}{\sqrt{a^2+b^2}}$$

Since $$0 < \sqrt{a^2 + b^2} < 1$$

$$\frac{1}{\sqrt{a^2 + b^2}} > 1$$

So $\frac{1}{z}$ is outside the unit circle in the fourth quadrant.

Ans. (d)

15. For any analytic function, $f(x+iy) = u(x,y) + iv(x,y)$, u is given by $u = 3x^2 - 3y^2$. The expression for v considering K to be a constant is

(a) $3y^2 - 3x^2 + K$ (b) $6x - 6y + K$

(c) $6y - 6x + K$ (d) $6xy + K$

(GATE 2011, 2 Marks)

Solution: We have

$$f = u + iv \quad \text{and} \quad u = 3x^2 - 3y^2$$

To calculate v, we use the Cauchy–Riemann conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \qquad (2)$$

From Eq. (1), we have

$$6x = \frac{\partial v}{\partial y}$$

$$\Rightarrow \quad \int \partial v = \int 6x \partial y$$

$$v = 6\frac{x^2}{2} + f(x)$$

i.e. $$v = 3x^2 + f(x) \qquad (3)$$

Now applying equation Eq. (2), we get

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$$\Rightarrow \quad -6y = -\left[6x + \frac{df}{dx}\right]$$

$$\Rightarrow \quad 6x + \frac{df}{dx} = 6y$$

$$\frac{df}{dx} = 6y - 6x$$

By integrating, we get

$$f(x) = 6yx - 3x^2 + K$$

Substituting in Eq. (3), we get

$$v = 3x^2 + 6yx - 3x^2 + K$$

$$\Rightarrow \quad v = 6xy + K$$

Ans. (d)

16. If $x = \sqrt{-1}$, then the value of x^x is

(a) $e^{-\pi/2}$ (b) $e^{\pi/2}$

(c) x (d) 1

(GATE 2012, 1 Mark)

Solution: We have

$$x = i$$

Then in polar coordinates, we have

$$x = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{\pi}{2}i}$$

Now, $$x^x = i^i = (e^{\pi i/2})^i = e^{i^2\pi/2} = e^{-\pi/2}$$

Ans. (a)

17. Given $f(z) = \frac{1}{z+1} - \frac{2}{z+3}$. If C is a counterclockwise path in the z-plane such that $|z+1| = 1$, the value of $\frac{1}{2\pi j}\oint_C f(z)dz$ is

(a) -2 (b) -1

(c) 1 (d) 2

(GATE 2012, 1 Mark)

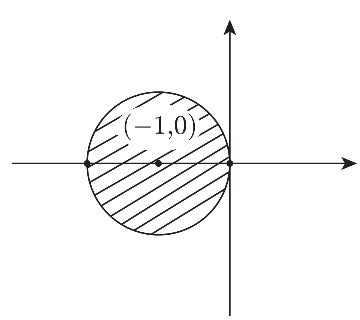
Solution: We are given that

$$f(z) = \frac{1}{z+1} - \frac{2}{z+3} = \frac{(z+3) - 2(z+1)}{(z+1)(z+3)}$$

$$= \frac{-z+1}{(z+1)(z+3)}$$

Hence, we have poles at -1 and -3, i.e. at $(-1, 0)$ and $(-3, 0)$.

From the figure of $|z+1| = 1$, we see that $(-1, 0)$ is inside the circle and $(-3, 0)$ is outside the circle.



Residue theorem states that

$$\frac{1}{2\pi j}\oint_C f(z)dz = \text{Residue of those poles which are}$$
inside C.

So the required integral $\frac{1}{2\pi j}\oint_C f(z)dz$ is given by the residue of function at pole (−1, 0) (which is inside the circle).

This residue is $\frac{-(-1)+1}{(-1+3)} = \frac{2}{2} = 1.$

Ans. (c)

18. Square roots of $-i$, where $i = \sqrt{-1}$, are

(a) $i, -i$

(b) $\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

(c) $\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

(d) $\cos\left(\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right), \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

(GATE 2013, 1 Mark)

Solution: We know that

$$-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right), \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$$

$$\Rightarrow (-i)^{1/2} = \left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]^{1/2}$$

$$= \cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$

Ans. (b)

19. The complex function tanh(s) is analytic over a region of the imaginary axis of the complex s-plane if the following is TRUE everywhere in the region for all integers n

(a) $\text{Re}(s) = 0$ (b) $\text{Im}(s) \neq n\pi$

(c) $\text{Im}(s) \neq \frac{n\pi}{3}$ (d) $\text{Im}(s) \neq \frac{(2n+1)\pi}{2}$

(GATE 2013, 1 Mark)

Solution: We know that

$$\tanh s = \frac{e^s - e^{-s}}{e^s + e^{-s}}$$

It is analytic if $e^s + e^{-s} \neq 0$. Therefore,

$$e^s \neq -e^{-s}$$

$$e^{2s} \neq -1$$

$$s \neq \frac{i(2n+1)}{2}\pi$$

$$\therefore \quad \text{Im}(s) \neq \frac{(2n+1)}{2}\pi$$

Ans. (d)

20. $\oint \frac{z^2-4}{z^2+4}dz$ evaluated anticlockwise around the circle $|z-1| = 2$, where $i = \sqrt{-1}$, is

(a) -4π (b) 0

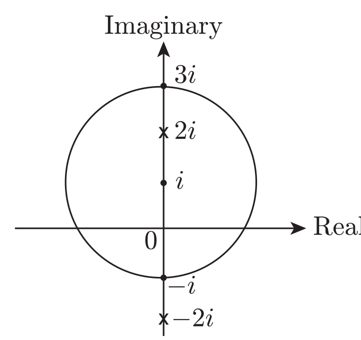
(c) $2 + \pi$ (d) $2 + 2i$

(GATE 2013, 2 Marks)

Solution: We have

$$\frac{z^2-4}{z^2+4} = \frac{z^2-4}{(z+2i)(z-2i)}$$

Hence, we have poles at $2i$ and $-2i$, i.e. $(0, 2i)$ and $(0, -2i)$



From the figure of $|z-i| = 2$, we see that the pole is inside C while the pole, $-2i$, is outside C. Therefore,

$$\oint \frac{z^2-4}{z^2+4}dz = 2\pi i \times \text{Res } F(z)$$

$$= 2\pi i \cdot \left.\frac{(z-2i)(z^2-4)}{(z+2i)(z-2i)}\right|_{z=2i}$$

$$= 2\pi i\frac{\left[(2i)^2-4\right]}{(2i+2i)} = -4\pi$$

Ans. (a)

21. C is a closed path in the z-plane given by $|Z| = 3$. The value of the integral $\oint_C \left(\frac{x^2 - x + 4}{z + 2j}\right) dz$ is

(a) $-\pi(1 + j2)$ (b) $4\pi(3 - j2)$
(c) $-4\pi(3 + j2)$ (d) $4\pi(1 - j2)$

(GATE 2014, 1 Mark)

Solution: $Z = -2j$ is a singularity lies inside $C : |Z| = 3$

Therefore, by Cauchy's integral formula,

$$\oint_C \frac{Z^2 - Z + 4j}{Z + 2j} dz = 2\pi j\left[Z^2 - Z + 4j\right]$$
$$= 2\pi j[-4 + 2j + 4j]$$
$$= -4\pi[3 + 2j]$$

Ans. (c)

22. All the values of the multi-valued complex function 1^i, where $i = \sqrt{-1}$, are

(a) purely imaginary
(b) real and non-negative
(c) on the unit circle
(d) equal in real and imaginary parts

(GATE 2014, 1 Mark)

Solution: We know that

$1 = \cos(2k\pi) + i\sin(2k\pi)$, where k is an integer

$= e^{i(2k\pi)}$

Therefore, $1^i = e^{-(2k\pi)}$

Hence, all values are real and non-negative.

Ans. (b)

23. The argument of the complex number $\frac{1+i}{1-i}$, where $i = \sqrt{-1}$ is

(a) $-\pi$ (b) $-\frac{\pi}{2}$ (c) $\frac{\pi}{2}$ (d) π

(GATE 2014, 1 Mark)

Solution: We have $z = \frac{1+i}{1-i}$. Multiplying the numerator and denominator with $1 + i$, we get

$$z = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1-1+2i}{1+1} = \frac{2i}{2} = i$$

This implies that $\arg(z) = \arg(i) = \tan^{-1}(y / x) = \tan^{-1}(\infty) = \pi/2$.

Ans. (c)

24. $z = \frac{2-3i}{-5+i}$ can be expressed as

(a) $-0.5 - 0.5i$ (b) $-0.5 + 0.5i$
(c) $0.5 - 0.5i$ (d) $0.5 + 0.5i$

(GATE 2014, 1 Mark)

Solution: We have,

$$z = \frac{2-3i}{-5+i} = \left(\frac{2-3i}{-5+i}\right) \times \left(\frac{-5-i}{-5-i}\right)$$
$$= \frac{-10-2i+15i+3i^2}{5^2-i^2} = \frac{-10+13i-3}{25+1}$$
$$= \frac{-13i-13}{26} = \frac{13(i-1)}{26} = \frac{(i-1)}{2} = 0.5i - 0.5$$

Ans. (b)

25. If $f^*(x)$ is the complex conjugate of $f(x) = \cos(x) + i\sin(x)$, then for real a and b, $\int_a^b f^*(x) f(x) dx$ is always

(a) positive (b) negative
(c) real (d) imaginary

(GATE 2014, 1 Mark)

Solution: We have

$$f(x) = \cos(x) + i\sin(x)$$

Then

$$f*(x) = \cos(x) - i\sin(x)$$

$$\int_a^b f^*(x) \cdot f(x) dx$$
$$= \int_a^b (\cos x - i\sin x)(\cos x + i\sin x) dx$$
$$= \int_a^b e^{-ix} \cdot e^{ix} dx = \int_a^b 1 \cdot dx = b - a \in R \; (\because a, b \in R)$$

$\Rightarrow$ Real for real a and b.

Ans. (c)

26. The real part of an analytic function $f(z)$ where $z = x + jy$ is given by $e^{-y}\cos(x)$. The imaginary part of $f(z)$ is

(a) $e^y \cos(x)$ (b) $e^{-y} \sin(x)$
(c) $-e^y \sin(x)$ (d) $-e^{-y} \sin(x)$

(GATE 2014, 2 Marks)

Solution: We have the real part $u = e^{-y}\cos x$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

[Using Cauchy-Riemann equations]

$$= e^{-y}\cos x dx - e^{-y}\sin x dy = d\left[e^{-y}\sin x\right]$$

On integrating, we get

$$v = e^{-y}\sin x$$

Ans. (b)

27. The line integral of function $F = yzi$, in the counterclockwise direction, along the circle $x^2 + y^2 = 1$ at $z = 1$ is

(a) -2π (b) $-\pi$ (c) π (D) 2π

(GATE 2014, 2 Marks)

Solution: We have to calculate the line integral =

$$\int_C \vec{F}\cdot d\vec{r} = \int_C yzdx$$

[C is circle $x^2 + y^2 = 1$ at $z = 1 \Rightarrow x = \cos\theta, y = \sin\theta$ and $\theta = 0$ to 2π]

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (\sin\theta)(1)(-\sin\theta d\theta)$$

$$= \int_0^{2\pi}\left(\frac{\cos 2\theta - 1}{2}\right)d\theta = \frac{1}{2}\left[\frac{\sin 2\theta}{2} - \theta\right]_0^{2\pi}$$

$$= -\pi$$

Ans. (b)

28. Integration of the complex function $f(z) = \dfrac{z^2}{z^2-1}$, in the counterclockwise direction, around $|z-1| = 1$, is

(a) $-\pi i$ (b) 0 (c) πi (d) $2\pi i$

(GATE 2014, 2 Marks)

Solution: $z = -1, 1$ are the simple poles of $f(z)$ and $z = 1$ lies inside $C : |z-1| = 1$. Therefore,

$$\oint_C f(z)dz = 2\pi i \times \left[\operatorname*{Res}_{z=1} f(z)\right] = 2\pi i \times \left[\lim_{z\to 1}(z-1)\cdot f(z)\right]$$

$$= 2\pi i \times \left[\lim_{z\to 1}\frac{z^2}{z+1}\right] = \pi i$$

Ans. (c)

29. An analytic function of a complex variable $z = x + iy$ is expressed as $f(z)=u(x,y)+ iv(x,y)$, where $i = \sqrt{-1}$. If $u(x,y) = 2xy$, then $v(x,y)$ must be

(a) $x^2 + y^2 +$ constant

(b) $x^2 - y^2 +$ constant

(c) $-x^2 + y^2 +$ constant

(d) $-x^2 - y^2 +$ constant

(GATE 2014, 2 Marks)

Solution: Given that $f(z)$ is analytic, so $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. Given that $u = 2xy$. So, $\dfrac{\partial u}{\partial y} = 2x$.

Checking each option for $-\dfrac{\partial v}{\partial x} = 2x$, we find that $v = -x^2 + y^2 +$ constant.

Ans. (c)

30. An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x,y)+ iv(x,y)$, where $i = \sqrt{-1}$. If $u(x,y) = x^2 - y^2$, then $v(x,y)$ in terms of x, y and a general constant c would be

(a) $xy + c$ (b) $\dfrac{x^2+y^2}{2} + c$

(c) $-2xy + c$ (d) $\dfrac{(x-y)^2}{2} + c$

(GATE 2014, 2 Marks)

Solution: The function $f(z) = w(x,y) + iv(x,y)$ is analytic if $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. Given that $u = x^2 - y^2$. So, $\dfrac{\partial u}{\partial y} = 2y$. Checking each option for $-\dfrac{\partial v}{\partial x} = 2y$, we find that $v = -2xy + c$.

Ans. (c)

31. If z is a complex variable, the value of $\displaystyle\int_5^{3i}\frac{dz}{z}$ is

(a) $-0.511 - 1.57i$ (b) $-0.511 + 1.57i$

(c) $0.511 - 1.57i$ (d) $0.511 + 1.57i$

(GATE 2014, 2 Marks)

Solution:

$$\int_5^{3i}\frac{dz}{z} = |\ln(z)|_5^{3i} = \ln(3i) - \ln 5$$

Now, $\ln(3i) - \ln 5$ can be expressed as $[\ln 3 + i(\pi/2)] - [\ln 5 + i(0)]$

The first part is of the form ln $r + i\theta$, where $r = \sqrt{x^2 + y^2}$ and θ is the argument of z. Therefore,

$$\ln z = \ln 3 - \ln 5 + i(1.57) = -0.511 + i(1.57)$$

Ans. (b)

32. The polar plot of the transfer $G(s) = \dfrac{10(s+1)}{(s+10)}$ for $0 \le \omega < \infty$ will be in the

(a) first quadrant
(b) second quadrant
(c) third quadrant
(d) fourth quadrant

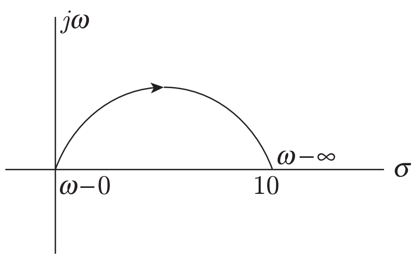
(GATE 2015, 1 Mark)

Solution: $G(s) = \dfrac{10(s+1)}{s+10}$

Put $s = j\omega$

$$G(j\omega) = \frac{10(j\omega+1)}{(j\omega+10)}$$

$$\omega = 0, \quad M = 1 < 0$$
$$\omega = \infty, \quad M = 10 < 0$$



So, zero is nearer to imaginary axis. Hence, plot will move in clockwise direction. It is the first quadrant.

Ans. (a)

33. Let $Z = x + iy$ be a complex variable, consider continuous integration is performed along the unit circle in anticlockwise direction. Which one of the following statements is NOT TRUE?

(a) The residue of $\dfrac{z}{z^2-1}$ at $z = 1$ is $\dfrac{1}{2}$

(b) $\oint_C z^2 dz = 0$

(c) $\dfrac{1}{2\pi i}\oint_C \dfrac{1}{z} dz = 1$

(d) $\bar{z}$ (complex conjugate of z) is an analytical function

(GATE 2015, 1 Mark)

Solution: $f(z) = \bar{z} = x - iy$

$$u = x,\ v = -y$$
$$\Rightarrow u_x = 1 \text{ and } v_x = 0$$
$$u_y = 0 \text{ and } v_y = -1$$

$\Rightarrow u_x \ne v_y$ that is, Cauchy-Riemann equations are not satisfied.

Thus, $\bar{z}$ is not analytic.

(a) $z = 1$ is a simple pole.

Therefore, residue $\left(\dfrac{z}{z^2-1}\right)$ at $z = 1$ is

$$\lim_{z\to 1}(z-1)\cdot\frac{z}{z^2-1} = \lim_{z\to 1}\frac{z}{z+1} = \frac{1}{2}$$

(b) Since z^2 is analytic everywhere

Therefore, using Cauchy's integral theorem,

$$\oint_C z^2 dz = 0$$

Ans. (d)

34. Let $f(z) = \dfrac{az+b}{cz+d}$. If $f(z_1) = f(z_2)$ for all $z_1 \ne z_2$, $a = 2$, $b = 4$ and $c = 5$, then d should be equal to __________ .

(GATE 2015, 1 Mark)

Solution: We are given

$$f(z) = \frac{az+b}{cz+d} \text{ if } f(z_1) = f(z_2), \text{for } z_1 \ne z_2$$

Also,

$a = 2,\ b = 4,\ c = 5$

$$f(z) = \frac{2z+4}{5z+d}$$

$$f(z_1) = f(z_2) \Rightarrow \frac{2z_1+4}{5z_1+d} = \frac{2z_2+4}{5_2+d}$$

$$\Rightarrow 10z_1z_2 + 20z_2 + 2dz_1 + 4d$$
$$= 10z_1z_2 + 20z_1 + 2dz_2 + 4d$$
$$\Rightarrow 20(z_2 - z_1) = 2d(z_2 - z_1) \Rightarrow d = 10$$

Ans. 10

35. If C is a circle of radius r with centre z_0, in the complex z-plane and if n is a non-zero integer, then $\oint_C \dfrac{dz}{(z-z_0)^{n+1}}$ equals

(a) $2\pi nj$ (b) 0 (c) $\dfrac{nj}{2\pi}$ (d) $2\pi n$

(GATE 2015, 1 Mark)

Solution: By Cauchy's integral formula,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i f^n(z_0)}{n!}$$

$$\oint_C \frac{dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} \times 0 = 0 \quad \left(\begin{array}{l}\text{Because } f(z) = 1, \\ f^n(z) = 0 \text{ at any } z_0\end{array}\right)$$

Ans. (b)

36. Given $f(z) = g(z) + h(z)$, where f, g, h are complex valued functions of a complex variable z. Which one of the following statements is TRUE?

(a) If $f(z)$ is differentiable at z_0, then $g(z)$ and $h(z)$ are also differentiable at z_0.

(b) If $g(z)$ and $h(z)$ are differentiable at z_0, then $f(z)$ is also differentiable at z_0.

(c) If $f(z)$ is continuous at z_0, then it is differentiable at z_0.

(d) If $f(z)$ is differentiable at z_0, then so are its real and imaginary parts.

(GATE 2015, 1 Mark)

Solution: Given $f(z) = g(z) + h(z)$

$f(z)$, $g(z)$ and $h(z)$ are complex variable functions.

Option (c) is not correct, since every continuous function need not be differentiable.

Let $g(z) = xh(z) = iy$

$$\Rightarrow g(z) = x + i(0) \quad h(z) = 0 + iy$$

Ans. (b)

37. Given two complex numbers $z_1 = 5 + (5\sqrt{3})i$ and $z_2 = \dfrac{z}{\sqrt{3}} + 2i$, the argument of $\dfrac{z_1}{z_2}$ in degrees is

(a) 0 (b) 30 (c) 60 (d) 90

(GATE 2015, 1 Mark)

Solution: We are given,

$z_1 = 5 + (5\sqrt{3})i$ and $z_2 = \dfrac{z}{\sqrt{3}} + 2i,$

$$\arg z_1 = \tan^{-1}\left(\frac{5\sqrt{3}}{5}\right) = \tan^{-1}(\sqrt{3}) = 60$$

$$\arg z_2 = \tan^{-1}\left(\frac{2}{\frac{2}{\sqrt{3}}}\right) = \tan^{-1}(\sqrt{3}) = 60°$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = 60° - 60° = 0$$

Ans. (a)

38. Given $i = \sqrt{-1}$, the value of the definite integral,

$$I = \int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}\,dx \text{ is}$$

(a) 1 (b) -1

(c) i (d) $-i$

(GATE 2015, 1 Mark)

Solution: We have to find the integral,

$$I = \int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x}\,dx$$

$$= \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}}\,dx = \int_0^{\pi/2} e^{2ix}\,dx = \left(\frac{e^{2ix}}{2}\right)_0^{\pi/2}$$

$$= \frac{1}{2}\left(e^{\frac{2ix}{2}} - e^0\right) = \frac{1}{2}(e^{ix} - e^0)$$

$$= \frac{1}{2}(\cos\pi + i\sin\pi - 1) = \frac{1}{2}(-1 + 0 - 1) = -1$$

Ans. (b)

39. The value of $\oint \dfrac{1}{z^2}\,dz$, where the contour is the unit circle traversed clockwise, is

(a) $-2\pi i$ (b) 0 (c) $-2\pi i$ (d) $4\pi i$

(GATE 2015, 1 Mark)

Solution: Given $\oint \dfrac{1}{z^2}\,dz$ where c is the unit circle.

Let $f(z) = \dfrac{1}{z^2}$, $f(z)$ is not analytic at $z = 0$.

By Laurent series, $f(z)$ can be expressed as

$$f(z) = \underbrace{(\text{zero tens})}_{\text{analytic part}} + \underbrace{\left(0\cdot\frac{1}{z} + 1\cdot\frac{1}{z^2}\right)}_{\text{principal part}}$$

Principal part has only two terms
Therefore, $z = 0$ is pole of order 2.

$$[\text{Res } f(z)]_{z=0} = \text{coefficient of } \frac{1}{z} = b_1 = 0$$

Therefore, by Cauchy's residue theorem.

$$\oint \frac{1}{z^2}\,dz = 2\pi i \times (\text{sum of residues}) = 2\pi i \times 0 = 0$$

Ans. (b)

40. A complex-valued function, $f(z)$, given below is analytic in domain D:

$$f(z) = u(x, y) + iv(x, y), \qquad z = x + iy$$

Which of the following is NOT correct?

(a) $\dfrac{df}{dz} = \dfrac{\partial v}{\partial y} + i\dfrac{\partial u}{\partial y}$ (b) $\dfrac{df}{dz} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

(c) $\dfrac{df}{dz} = \dfrac{\partial v}{\partial y} - i\dfrac{\partial u}{\partial y}$ (d) $\dfrac{df}{dz} = \dfrac{\partial v}{\partial y} + i\dfrac{\partial v}{\partial x}$

(GATE 2015, 1 Mark)

Solution:

$$f(z) = u(x,y) + iv(x,y)$$

$$f'(z) = \frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \quad (1)$$

For analytic function,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

So Eq. (1) can be written as

$$\frac{df}{dz} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Thus, option (a) cannot be deduced.

Ans. (a)

41. Consider the following complex function:

$$f(x) = \frac{9}{(x-1)(x+2)^2}$$

Which of the following is one of the residues of the above function?

(a) -1 (b) 9/16

(c) 2 (d) 9

(GATE 2015, 2 Marks)

Solution: We have $f(x) = \dfrac{9}{(x-1)(x+2)^2}$

$$\Rightarrow f(3) = \frac{9}{(z-1)(z+2)^2}$$

$z = 1$ is a simple pole and $z = -2$ is a pole of order 2. Hence,

$$[\text{Res } f(z)]_{z=1} = \lim_{z\to 1}(z-1)\frac{9}{(z-1)(z+2)^2} = \frac{9}{9} = 1$$

$$[\text{Res } f(z)]_{z=-2} = \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}\left[(z+2)^2 \cdot \frac{9}{(z-1)(z+2)^2}\right]$$

$$= \lim_{z\to 2}\frac{-9}{(z-1)^2} = \frac{-9}{9} = -1$$

Ans. (a)

42. For complex variable z, the value of the contour integral $\dfrac{1}{2\pi i}\displaystyle\int_c \frac{e^{-2z}}{z(z-3)}dz$ along the clockwise contour $c : |z| = 2$ (up to two decimal places) is __________.

(GATE 2015, 2 Marks)

Solution: We have to find the value of the contour integral,

$$\frac{1}{2\pi i}\int_c \frac{e^{-2z}}{z(z-3)}$$

$$c : |z| = 2 - 2 < z < 2$$

Poles are $z = 0.3$

$z = 3$ is outside the contour C.

$$R(0) = \frac{e^{-2\times 0}}{(0-3)} = -\frac{1}{3} = -0.33$$

Thus, the value of integral is -0.33.

Ans. -0.33

43. Consider the complex valued function $f(z)=2z^3+b|z|^3$, where z is a complex variable. The value of b for which the function $f(z)$ is analytic is __________.

(GATE 2016, 1 Mark)

Solution: Here, $|z|^3$ is differentiable at the origin, but it is not analytic. However, $2z^3$ is analytic everywhere. Therefore, b=0 for $f(z)$ to be analytic.

Ans. 0

44. For $f(z)=(\sin(z)/z^2)$, the residue of the pole at z=0 is __________.

(GATE 2016, 1 Mark)

Solution:

$$f(z)=\frac{\sin z}{z^2} = \left(\frac{1}{z^2}\right)\left[z-\left(\frac{z^3}{3!}\right)+\left(\frac{z^5}{5!}\right)-\left(\frac{z^7}{7!}\right)+\cdots\right]$$

$$= \left(\frac{1}{z}\right)-\left(\frac{z}{3!}\right)+\left(\frac{z^3}{5!}\right)-\cdots$$

Reside at z=0 is the coefficient of $(1/z)$ in the expansion of $f(z)$, that is residue R=1.

Ans. 1

45. The value of the integral

$$\oint_C \frac{2z+5}{\left(z-\frac{1}{2}\right)(z^2-4z+5)}dz$$

over the contour $|z| = 1$, taken in the anti-clockwise direction, would be

(a) $\frac{24\pi i}{13}$ (b) $\frac{48\pi i}{13}$

(c) $\frac{24}{13}$ (d) $\frac{12}{13}$

(GATE 2016, 1 Mark)

Solution: For the given integral

$$\int \frac{2z+5}{\left(z-\frac{1}{2}\right)(z^2-4z+5)}$$

Poles: $z = \frac{1}{2},\ 2 \pm i$

Of these, only $z = \left(\frac{1}{2}\right)$ lies inside $|z| = 1$.

Using residue theorem,

$$\int_C = 2\pi i(R_{1/2})$$

$$\text{Residue at}\ \frac{1}{2} = \lim_{z\to 1/2}\left[\frac{\left(z-\frac{1}{2}\right)(2z+5)dz}{\left(z-\frac{1}{2}\right)(z^2-4z+5)}\right] = \frac{24}{13}$$

Ans. (b)

46. Consider the function $f(z) = z + z^*$ where z is a complex variable and z^* denotes its complex conjugate. Which one of the following is TRUE?

(a) $f(z)$ is both continuous and analytic.

(b) $f(z)$ is continuous but not analytic.

(c) $f(z)$ is not continuous but is analytic.

(d) $f(z)$ is neither continuous nor analytic.

(GATE 2016, 1 Mark)

Solution: Let $z = A + iB$. Therefore,

$$z^* = A - iB$$

Therefore, $f(z) = 2A$ (continuous)

$$u = 2A \qquad v = 0$$

$$\frac{du}{dA} = 2 \qquad \frac{du}{dB} = 0$$

$$\frac{dv}{dA} = 0 \qquad \frac{dv}{dB} = 0$$

Therefore, it is not analytic.

Ans. (b)

47. $f(z) = u(x, y) + iv(x, y)$ is an analytic function of complex variable $z = x + iy$ where $i = \sqrt{-1}$. If $u(x, y) = 2\,xy$, then $v(x, y)$ may be expressed as

(a) $-x^2 + y^2$ + constant (b) $x^2 - y^2$ + constant

(c) $x^2 + y^2$ + constant (d) $-(x^2 + y^2)$ + constant

(GATE 2016, 1 Mark)

Solution: For the given values,

$$\frac{\partial u}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = 2x$$

For analytic function,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2x$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = 2y$$

Thus,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -2xdx + 2ydy$$

$$\Rightarrow v = -x^2 + y^2 + \text{constant}$$

Ans. (a)

48. A function f of the complex variable $z = x + iy$ is given as $f(x, y) = u(x, y) + iv(x, y)$, where $u(x, y) = 2kxy$ and $v(x, y) = x^2 - y^2$. The value of k, for which the function is analytic, is ______.

(GATE 2016, 1 Mark)

Solution: Given that

$$z = x + iy$$

$$f(z) = f(x, y) = u(x, y) + iv(x, y) = 2kxy + i(x^2 - y^2)$$

For $f(z)$ to be analytic, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$ must be satisfied.

$$\frac{\partial u}{\partial x} = 2ky \ \text{ and } \ \frac{\partial v}{\partial y} = -2y$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow k = -1$$

Also,

$$\frac{\partial v}{\partial x} = 2x \text{ and } \frac{\partial u}{\partial y} = 2kx$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \Rightarrow k = -1$$

Ans. –1

49. The values of the integral $\frac{1}{2\pi j}\oint_c \left(\frac{e^z}{z-2}\right) dz$ along a closed contour c in anti-clockwise direction for (i) the point z_0=2 inside the contour c, and (ii) the point z_0=2 outside the contour c, respectively, are

(a) (i) 2.72, (ii) 0 (b) (i) 7.39, (ii) 0
(c) (i) 0, (ii) 2.72 (d) (i) 0, (ii) 7.39

(GATE 2016, 2 Marks)

Solution: It is given that

$$f(z) = \frac{1}{2\pi j}\oint_c \left(\frac{e^z}{z-2}\right) dz$$

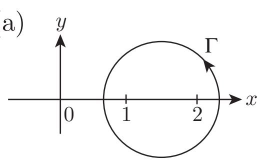
The singular point is at z=2. If z=2 lies inside the contour, then the value of $f(z)$ is the residue of $f(z)$ at z=2. Therefore,

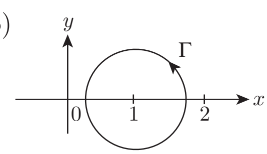
$$f(z) = \lim_{z\to 2}\left[\frac{e^z(z-2)}{z-2}\right] = e^2 = 7.39$$

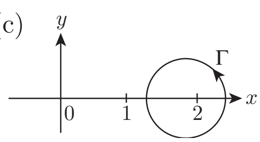
If z=2 is lies outside of the contour it means the given function is analytic, then the value of $f(z)$=0.

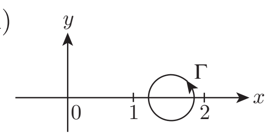
Ans. (b)

50. The value of $\oint_\Gamma \frac{3z-5}{(z-1)(z-2)} dz$ along a closed path Γ is equal to $(4\pi i)$, where $z = x + iy$ and $i = \sqrt{-1}$. The correct path Γ is

(a)


(b)


(c)


(d)


(GATE 2016, 2 Marks)

Solution: $\oint \frac{3z-5}{(z-1)(z-2)} dz = 4\pi i$

$$\text{LHS} = \oint\left[\left(\frac{2}{z-1}\right) + \left(\frac{1}{z_1-2}\right)\right] dz$$

$$= \oint \frac{2}{z-1} dz + \oint \frac{1}{z-2} dz$$

If the path is that shown in option (a), we have

$$\oint \frac{2}{z-1} dz + \oint \frac{1}{z-2} dz = 2\pi(i)\times 2 + 2\pi i(1)$$

If the path is that shown is option (b), we have

$$\oint \frac{2}{z-1} dz + \oint \frac{1}{z-2} dz = 2\pi i \times 2 + 0$$

If the path is that shown in option (c), we have

$$\oint \frac{2}{z-1} dz + \oint \frac{1}{z-c} dz = 0 + 2\pi i(1)$$

If the path is that shown in option (d), we have

$$\oint \frac{2}{z-1} dz + \oint \frac{1}{z-2} dz = 0 + 0$$

Since the path shown in option (b) produces the given value, option (b) is correct.

Ans. (b)

51. The residues of a function

$$f(z) = \frac{1}{(z-4)(z+1)^3} \quad \text{are}$$

(a) $\frac{-1}{27}$ and $\frac{-1}{125}$ (b) $\frac{1}{125}$ and $\frac{-1}{125}$

(c) $\frac{-1}{27}$ and $\frac{1}{5}$ (d) $\frac{1}{125}$ and $\frac{-1}{5}$

(GATE 2017, 1 Mark)

Solution:

$$f(z) = \frac{1}{(z-4)(z+1)^3}$$

Residue of $f(z)$ at $z = a$ is

$$f(z)\ (z-a)\big|_{z=a}$$

So, residue of $f(z) = \frac{1}{(z-4)(z+1)^3}$ at $z = 4$ is

$$\frac{1}{(z-4)(z+1)^3}(z-4)\bigg|_{z=4} = \left(\frac{1}{z+1}\right)^3\bigg|_{z=4} = \frac{1}{125}$$

Residue of $f(z)$ with multiple pole at $z = a$ with order n is

$$\frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}\right)[(z-a)^n f(z)]_{z=a}$$

Residue of $\frac{1}{(z-4)(z+1)^3}$ with poles at $z = -1$ is

$$\frac{1}{2!}\left[\frac{d^2}{dz^2}\left(\frac{1}{(z-4)(z+1)^3}\times(z+1)^3\right)\right]_{z=-1}$$

$$= \frac{1}{2}\left(\frac{d^2}{dz^2}\left(\frac{1}{z-4}\right)\right)\Bigg|_{z=-1}$$

$$= \frac{1}{2}\left(\frac{d}{dz}\left(\frac{-1}{(z-4)^2}\right)\right)\Bigg|_{z=-1}$$

$$= \frac{1}{2}\left[\frac{(z-4)^2\cdot 0-(-1)2(z-4)}{(z-4)^4}\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{2}{(z-4)^3}\right]_{z=-1} = -\frac{1}{125}$$

Ans. (b)

52. For a complex number z, $\lim_{z\to i}\frac{z^2+1}{z^3+2z-i(z^2+2)}$ is

(a) $-2i$ (b) $-i$ (c) i (d) $2i$

(GATE 2017, 1 Mark)

Solution: $\lim_{z\to i}\frac{z^2+1}{z^3+2z-i(z^2+2)}$ (% form)

On differentiating, we get

$$\lim_{z\to i}\frac{2z}{3z^2+2-i(2z)} = \frac{2i}{3i^2+2-i(2i)} = 2i$$

Ans. (d)

53. An integral I over a counter-clockwise circle C is given by

$$I = \int_C \frac{z^2-1}{z^2+1}e^z dz$$

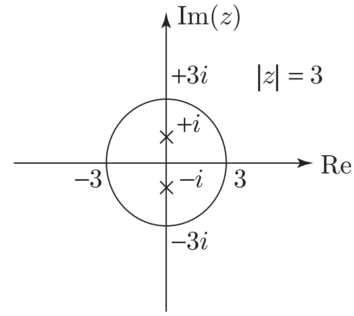
If C is defined as $|z| = 3$, then the value of I is

(a) $-\pi i\sin(1)$ (b) $-2\pi i\sin(1)$

(c) $-3\pi i\sin(1)$ (d) $-4\pi i\sin(1)$

(GATE 2017, 2 Marks)

Solution:



$$I = \oint\frac{z^2-1}{z^2+1}e^2 dz$$

$$= 2\pi i(\text{Res}(z=i)+\text{Res}(z=-i))$$

Residue of $f(z)$ at $z = a$,

$$\lim_{z\to a}(z-a)f(z)$$

Residue of $\frac{z^2-1}{z^2+1}e^z$ at $z = i$,

$$\lim_{z\to i}(z-i)\frac{(z^2-1)}{(z-i)(z+i)}e^z = \frac{-2}{i+i}e^i = \frac{-2}{2i}e^i$$

Residue of $\frac{z^2-1}{z^2+1}e^z$ at $z = -i$,

$$\lim_{z\to -i}(z+i)\frac{z^2-1}{(z+i)(z-i)}e^z = \frac{-2}{-2i}e^{-i}$$

Therefore,

$$I = \oint\frac{z^2-1}{z^2+1}e^z dz = 2\pi i(\text{Res}(z=i)+\text{Res}(z=-i))$$

$$\Rightarrow I = 2\pi i\left(\frac{-2}{2i}e^i+\frac{2}{2i}e^{-i}\right) = -4\pi i\left(\frac{e^i}{2i}-\frac{e^{-i}}{2i}\right)$$

$$\Rightarrow I = -4\pi i\left(\frac{e^i-e^{-i}}{2i}\right) = -4\pi i\sin(1)$$

Ans. (d)

54. If $f(z) = (x^2+ay^2)+ibxy$ is a complex analytic function of $z = x+iy$, where $i = \sqrt{-1}$, then

(a) $a = -1$, $b = -1$ (b) $a = -1$, $b = 2$
(c) $a = 1$, $b = 2$ (d) $a = 2$, $b = 2$

(GATE 2017, 2 Marks)

Solution: Given that the following function is analytic:

$$f(z) = (x^2+ay^2)+ibxy$$

Thus,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

For values,

$$u = x^2 + ay^2$$
$$v = bxy$$

one finds

$$2x = bx$$
$$2ay = by$$

Thus,

$$b = 2$$
$$a = -1$$

Ans. (b)

55. The solution of the equation $\frac{dQ}{dt} + Q = 1$ with $Q = 0$ at $t = 0$ is

(a) $Q(t) = e^{-t} - 1$ (b) $Q(t) = 1 + e^{-t}$
(c) $Q(t) = 1 - e^{t}$ (d) $Q(t) = 1 - e^{-t}$

(GATE 2017, 2 Marks)

Solution: $\frac{dQ}{dt} + Q = 1$

Now,

$$\text{I.F.} = e^{\int p dt} = e^{\int s dt} = e^t$$
$$Q(t) \cdot e^t = \int e^t \cdot dt = e^t + c$$

Given that at $t = 0$, $Q = 0$. From the above equation, we have

$$0 = 1 + c \Rightarrow c = -1$$

Therefore,

$$Q(t) = 1 - e^{-t}$$

Ans. (d)

CHAPTER 5

PROBABILITY AND STATISTICS

FUNDAMENTALS OF PROBABILITY

Probability is used to measure the degree of certainty or uncertainty of the occurrence of events.

Hence, if y is the total number of outcomes and x is the favorable number of outcomes then probability of occurrence of an event A is denoted by $P(A)$ and given by

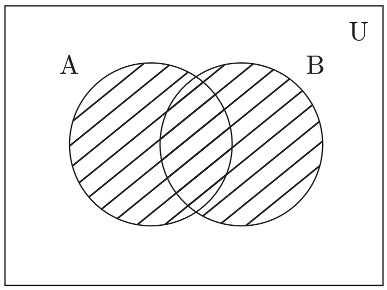
$$P(A) = \frac{x}{y}$$

A process to observe and measure the probability of an event is called an experiment. The result obtained through that experiment is the outcome of that experiment. An experiment which when repeated under identical conditions do not produce the same outcome every time is called a random experiment. The set of all the possible outcomes of a random experiment is called sample space.

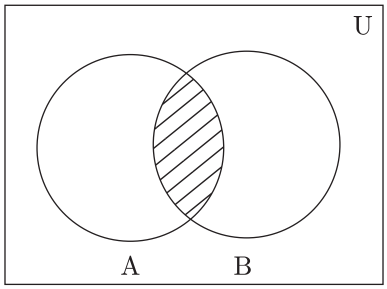
A set of the sample space associated with a random experiment is called an event.

Suppose we have two events A and B, then some of the operations performed on these events are as follows:

1. **Union of sets:** $A \cup B$

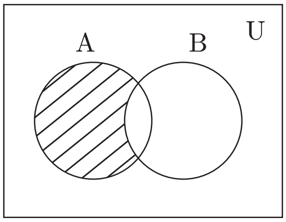


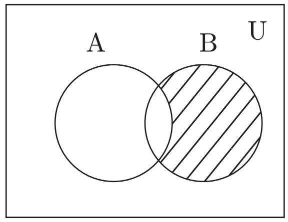
2. **Intersection of sets:** $A \cap B$



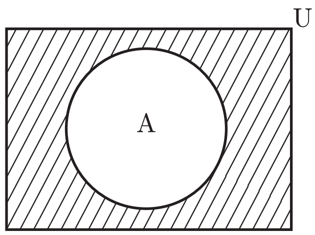
3. Difference of sets:

$A - B$ $\quad$ $B - A$

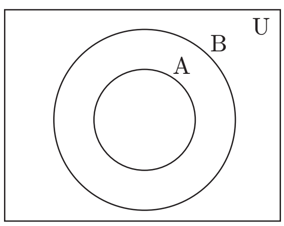




4. Complement of a set: A^c



5. A is subset of B: $A \subset B$



Types of Events

1. Each outcome of a random experiment is called an elementary event.
2. An event associated with a random experiment that always occurs whenever the experiment is performed is called a certain event.
3. An event associated with a random experiment that never occurs whenever the experiment is performed is called an impossible event.
4. If the occurrence of any one of two or more events, associated with a random experiment, presents the occurrence of all others, then the events are called mutually exclusive events.
5. If the union of two or more events associated with a random experiment includes all possible outcomes, then the events are called exhaustive events.
6. If the occurrence or non-occurrence of one event does not affect the probability of the occurrence or non-occurrence of the other, then the events are independent.
7. Two events are equally likely events if the probability of their occurrence is same.
8. An event which has a probability of occurrence equal to $1 - P$, where P is the probability of occurrence of an event A, is called the complementary event of A.

Approaches to Probability

These are two basic approaches of quantifying probability of an event.

1. **Classical approach:** Probability of an event E is calculated by the ratio of number of ways an event can occur to the number of ways a sample space can occur. This approach assumes that occurrence of all outcomes is equally probable or likely

$$P(E) = \frac{n(E)}{n(S)}$$

   where $n(E)$ is the number of favorable outcomes and $n(S)$ is the number of total outcomes or sample space.

2. **Frequency approach:** Probability of an event E is defined as the relative frequency of occurrence of E. This approach is used when all outcomes are not equally probable or likely

$$P(E) = \lim_{N \to \infty} \frac{n(E)}{N}$$

   where N is the number of times an experiment is performed and $n(E)$ is the number of times an event occurs.

Axioms of Probability

1. The numerical value of probability lies between 0 and 1.
   Hence, for any event A of S, $0 \leq P(A) \leq 1$.
2. The sum of probabilities of all sample events is unity. Hence, $P(S) = 1$.
3. Probability of an event made of two or more sample events is the sum of their probabilities.

Conditional Probability

Let A and B be two events of a random experiment. The probability of occurrence of A if B has already occurred and $P(B) \neq 0$ is known as conditional probability. This is denoted by $P(A/B)$. Also, conditional probability can be defined as the probability of occurrence of B if A has already occurred and $P(A) \neq 0$. This is denoted by$P(B/A)$.

Geometric Probability

Due to the nature of the problem or the solution or both, random events that take place in continuous sample space may invoke geometric imagery. Some popular problems such as Buffon's needle, Birds on a wire, Bertrand's paradox etc. arise in a geometrical domain. Hence. geometric probabilities can be considered as non-negative quantities with maximum value of 1 being assigned to subregions of a given domain subject to certain rules. If P is an expression of this assignment defined on a domain S, then

$$0 < P(A) \le 1, A \subset S \text{ and } P(S) = 1$$

The subsets of S for which P is defined are the random events that form a particular sample spaces. P is defined by the ratio of the areas so that if $\sigma(A)$ is defined as the area of set A, then

$$P(A) = \frac{\sigma(A)}{\sigma(s)}$$

Rules of Probability

Some of the important rules of probability are given as follows:

1. **Inclusion–Exclusion principle of probability:**
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
If A and B are mutually exclusive events, $P(A \cap B) = 0$ and then formula reduces to
$$P(A \cup B) = P(A) + P(B)$$
2. **Complementary probability:**
$$P(A) = 1 - P(A^{c})$$
where $P(A^{c})$ is the complementary probability of A.
3. $P(A \cap B) = P(A) * P(B/A) = P(B) * P(A/B)$
where $P(A/B)$ represents the conditional probability of A given B and $P(B/A)$ represents the conditional probability of B given A.
If $B_1, B_2, \ldots, B_n$ are pairwise disjoint events of positive probability, then
$$P(A)$$
$$= P\left(\frac{A}{B_1}\right)P(B_1) + P\left(\frac{A}{B_2}\right)P(B_2) + \cdots + P\left(\frac{A}{B_n}\right)P(B_n)$$
4. **Conditional probability rule:**
$$P(A \cap B) = P(B) * P(A/B)$$
$$\Rightarrow P(A / B) = \frac{P(A \cap B)}{P(B)}$$
$$\text{Or } P(B / A) = \frac{P(A \cap B)}{P(A)}$$
5. **Bayes' theorem:** Suppose we have an event A corresponding to a number of exhaustive events $B_1, B_2, \ldots, B_n$.
If $P(B_i)$ and $P(A/B_i)$ are given, then
$$P(B_i/A) = \frac{P(B_i)P(A/B_i)}{\sum P(B_i)P(A/B_i)}$$
6. **Rule of total probability:** Consider an event E which occurs via two different values A and B. Also, let A and B be mutually exhaustive and collectively exhaustive events.
Now, the probability of E is given as
$$P(E) = P(A \cap E) + P(B \cap E)$$
$$= P(A) * P(E/A) + P(B) * P(E/B)$$
This is called the rule of total probability.

STATISTICS

Statistics deals with the methods for collection, classification and analysis of numerical data for drawing valid conclusions and making reasonable decisions. The scope of statistics now includes collection of numerical data pertaining to almost every field, calculation of percentages, exports, imports, births—deaths, etc. Hence, it is useful in business, economics, sociology, biology, psychology, education, physics, chemistry and other related fields.

A value which is used to represent a given data is called central value and various methods of finding it are called measures of central tendency. Some measures of central tendency are arithmetic mean, median and mode. However, the central values are inadequate to give us a complete idea of the distribution as they do not tell us the extent to which the observations vary from the central value. Hence, to make better interpretation from the data, we should also have an idea how the observations are scattered around a central value.

The dispersion is the measure of variations in the values of variable. It measures the degree of scatterdness of the observations in a distribution around the central value. Some of the commonly used measures of dispersion are range, mean deviation, standard deviation and quartile deviation.

Arithmetic Mean

Arithmetic Mean for Raw data

Suppose we have values $x_1, x_2, \ldots, x_n$ and n are the total number of values, then arithmetic mean is given by

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where $\bar{x}$ = arithmetic mean.

Arithmetic Mean for Grouped Data (Frequency Distribution)

Suppose f_i is the frequency of x_i, then the arithmetic mean from frequency distribution can be calculated as

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i$$

where $N = \sum_{i=1}^{n} f_i$

Median

Arithmetic mean is the central value of the distribution in the sense that positive and negative deviations from the arithmetic mean balance each other.

Median is defined as the central value of a set of observations. It divides the whole series of observation into two parts in the sense that the numbers of values less than the median is equal to the number of values greater than the median.

Median for Raw Data

Suppose we have n numbers of ungrouped/raw values and let values be $x_1, x_2, ..., x_n$. To calculate median, arrange all the values in ascending or descending order.

Now, if n is odd, then median $= \left[\dfrac{(n+1)}{2}\right]^{\text{th}}$ value

If n is even, then median

$$= \frac{\left[\left(\dfrac{n}{2}\right)^{\text{th}} \text{value} + \left(\dfrac{n}{2}+1\right)^{\text{th}} \text{value}\right]}{2}$$

Median for Grouped Data

To calculate median of grouped values, identify the class containing the middle observation.

$$\text{Median} = L + \left[\frac{\dfrac{(N+1)}{2} - (F+1)}{f_m}\right] \times h$$

where L = lower limit of median class

N = total number of data items $= \sum f$

F = cumulative frequency of class immediately preceding the median class

f_m = frequency of median class

h = width of class

Mode

Mode of raw values of data is the value with the highest frequency or simply the value which occurs the most number of times.

Mode of Raw Data

Mode of raw data is calculated by simply checking which value is repeated the most number of times.

Mode of Grouped Data

Mode of grouped values of data is calculated by first identifying the modal class, i.e. the class which has the target frequency. The mode can then be calculated using the following formula:

$$\text{Mode} = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h$$

where

L = lower limit of modal class

f_m = frequency of modal class

f_1 = frequency of class preceding modal class

f_2 = frequency of class following modal class

h = width of model class

Relation Between Mean, Median and Mode

Empirical mode = 3 Median − 2 Mean

When an approximate value of mode is required, the given empirical formula for mode may be used.

There are three types of frequency distribution:

1. **Symmetric distribution:** It has lower half equal to upper half of distribution. For symmetric distribution,
$$\text{Mean} = \text{Median} = \text{Mode}$$
2. **Positively skewed distribution:** It has a longer tail on the right than on the left.
$$\text{Mode} \leq \text{Median} \leq \text{Mean}$$
3. **Negatively skewed distribution:** It has a long tail on the left than on the right.
$$\text{Mean} \leq \text{Median} \leq \text{Mode}$$

Figure 1 shows the three types of frequency distribution.

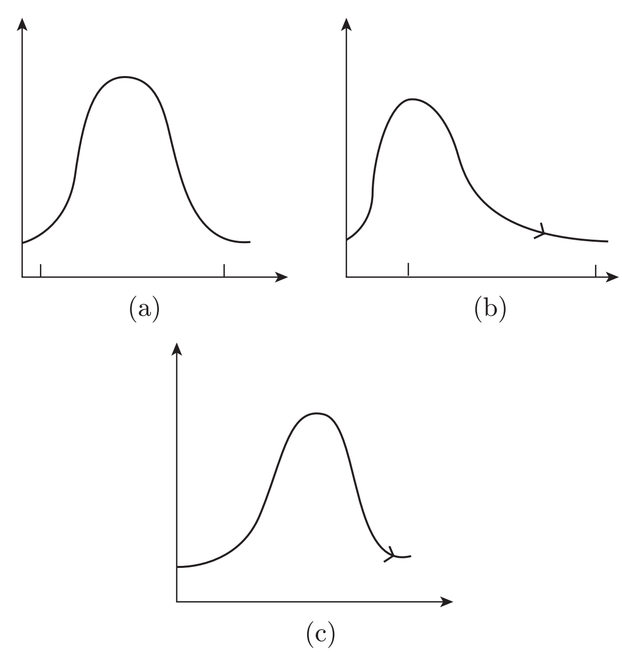


Figure 1 (a) Symmetrical frequency distribution, (b) positively skewed frequency distribution and (c) negatively skewed frequency distribution.

Geometric Mean

Geometric mean (G.M.) is a type of mean or average which indicates the central tendency or typical value of a set of numbers by using the product of their values.

Geometric Mean of Raw Data

Geometric mean of n numbers $x_1, x_2, ..., x_n$ is given by

$$\left(\underset{i=1}{\overset{n}{\pi}}\right)^{1/n} = \sqrt[n]{x_1 \cdot x_2 \cdot x_3 \cdot \cdots \cdot x_n}$$

Geometric Mean of Grouped Data

Geometric mean for a frequency distribution is given by

$$\log \text{G.M.} = \frac{1}{N}\sum_{i=1}^{n} f_i \log(x_i)$$

where $N = \sum_{i=1}^{n} f_i$.

Harmonic Mean

Harmonic mean (H.M.) is the special case of the power mean. As it tends strongly towards the least element of the list, it may (compared to the arithmetic mean) mitigate the influence of large outliers and increase the influence of small values.

Harmonic Mean of Raw Data

Harmonic mean of n numbers $x_1, x_2, x_3, ..., x_n$ is calculated as

$$\text{H.M.} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

Harmonic Mean of Grouped Data

Harmonic mean for a frequency distribution is calculated as

$$\text{H.M.} = \frac{N}{\sum_{i=1}^{n} (f_i / x_i)}$$

where $N = \sum_{i=1}^{n} f_i$.

Range

Range is the difference between two extreme observations of the distribution. Hence, range is calculated by subtracting the largest value and the smallest value.

Mean Deviation

The mean deviation (M.D.) is the mean of absolute of the differences in values from the mean, median or mode.

Mean Deviation of Raw Data

Suppose we have a given set of n values $x_1, x_2, x_3, ..., x_n$, then the mean deviation is given by

$$\text{M.D.} = \frac{1}{n}\sum_{i=1}^{n} \left|x_i - \bar{X}\right|$$

where $\bar{x}$ = mean.

The following steps should be followed to calculate mean deviation of raw data:

1. Compute the central value or average 'A' about which mean deviation is to be calculated.
2. Take mod of the deviations of the observations about the central value 'A', i.e. $\left|x_i - \bar{x}\right|$.
3. Obtain the total of these deviations, i.e. $\sum_{i=1}^{n} \left|x_2 - \bar{X}\right|$.
4. Divide the total obtained in step 3 by the number of observations.

Mean Deviation of Discrete Frequency Distribution

For a frequency deviation, the mean deviation is given by

$$\text{M.D.} = \frac{1}{N}\sum_{i=1}^{n} f_i \left|x_i - \bar{x}\right|$$

where $N = \sum_{i=1}^{n} f_i$.

The following steps should be followed to calculate mean deviation of discrete frequency deviation:

1. Calculate the central value or average 'A' of the given frequency distribution about which mean deviation is to be calculated.
2. Take mod of the deviations of the observations from the central value, i.e. $\left|x_i - \bar{x}\right|$.
3. Multiply these deviations by respective frequencies and obtain the total $\sum f_i \left|x_i - \bar{x}\right|$.
4. Divide the total obtained by the number of observations, i.e. $N = \sum_{i=1}^{n} f_i$ to obtain the mean deviation.

Mean Deviation of Grouped Frequency Distribution

For calculating the mean deviation of grouped frequency distribution, the procedure is same as for a discrete frequency distribution. However, the only difference is that we have to obtain the mid-points of the various classes and take the deviations of these mid-points from the given central value.

Standard Deviation

The variance of X is the arithmetic mean of the squares of all deviation of X from arithmetic mean of the observations. It is denoted by σ^2. Standard deviation (or root men square deviation) is the positive square root of the variation of X. It is denoted by σ.

Standard Deviation of Raw Data

If we have n values $x_1, x_2, ..., x_n$ of X, then

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\Rightarrow \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

The following steps should be followed to calculate standard deviation for discrete data:

1. Calculate mean $(\bar{X})$ for given observations.
2. Take deviations of observations from the mean, i.e. $(x_i - \bar{X})$.
3. Square the deviations obtained in the previous step and find
$$\sum_{i=1}^{n}(x_i - \bar{X})^2$$
4. Divide the sum by n to obtain the value of variance, i.e.
$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2$$
5. Take out the square root of variance to obtain standard deviation,
$$\sigma^2 = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2}$$

Standard Deviation of Discrete Frequency Distribution

If we have a discrete frequency distribution of X, then

$$\sigma^2 = \frac{1}{N}\left[\sum_{i=1}^{n}(x_i - \bar{X})^2\right]$$

$$\sigma = \sqrt{\frac{1}{N}\left[\sum_{i=1}^{n}(x_i - \bar{X})^2\right]}$$

where $N = \sum_{i=1}^{n} f$.

The following steps should be followed to calculate variance if discrete distribution of data is given:

1. Obtain the given frequency distribution.
2. Calculate mean $(\bar{X})$ for given frequency distribution.
3. Take deviations of observations from the mean, i.e. $(x_i - \bar{X})$.
4. Square the deviations obtained in the previous step and multiply the squared deviations by respective frequencies, i.e. $f_i(x_i - \bar{X})$.
5. Calculate the total, i.e. $\sum_{i=1}^{n} f_i(x_i - \bar{X})^2$.
6. Divide the sum $\sum_{i=1}^{n} f_i(x_i - \bar{X})^2$ by N, where $N = \sum f_i$, to obtain the variance, σ^2.
7. Take out the square root of the variance to obtain standard deviation, $\sigma = \sqrt{\frac{1}{n}\left[\sum_{i=1}^{n} f_i(x_i - \bar{X})^2\right]}$.

Standard Deviation of Grouped Frequency Distribution

For calculating standard deviation of a grouped frequency distribution, the procedure is same as for a discrete frequency distribution. However, the only difference is that we have to obtain the mid-point of the various classes and take the deviations of these mid-points from the given central point.

Coefficient of Variation

To compare two or more series which are measured in different units, we cannot use measures of dispersion. Thus, we require those measures which are independent of units.

Coefficient of variation (C.V.) is a measure of variability which is independent of units and hence can be used to compare two data sets with different units.

$$\text{C.V.} = \frac{\sigma}{\bar{X}} \times 100$$

where σ represents standard deviation and $\bar{X}$ represents mean.

PROBABILITY DISTRIBUTIONS

Random Variable

Any variable whose value is subject to variations due to randomness is called a random variable. A random variable does not have a fixed single value. Conversely, it can take a different set of values from a sample space in which each value has an associated probability.

Suppose S is sample space associated with a given random experiment. Then, random variable is a real-valued function X which assigns to each event $n \in S$ a unique real number $X(n)$.

Random variable can be discrete and continuous.

Discrete random variable is a variable that can take a value from a continuous range of values.

Continuous random variable is a variable that can take a value from a continuous range of values.

If a random variable X takes $x_1, x_2, \ldots, x_n$ with respective probabilities $P_1, P_2, \ldots, P_n$, then

$$\begin{array}{rccccc} X: & x_1 & x_2 & x_3 & \ldots & x_n \\ P(X): & P_1 & P_2 & P_3 & \ldots & P_n \end{array}$$

is known as the probability distribution of X.

Properties of Discrete Distribution

1. $\sum P(x) = 1$
2. $E(x) = \sum xP(x)$
3. $V(x) = E(x^2) - [E(x)]^2 = \sum x^2 P(x) - [\sum xP(x)]^2$

where $E(x)$ denotes the expected value or average value of a random variable x and $V(x)$ denotes the variable of a random variable x.

Properties of Continuous Distribution

1. Cumulative distribution function is given by
$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$
2. $E(x) = \int_{-\infty}^{\infty} xf(x)\,dx$
3. $V(x) = E(x^2) - [E(x)]^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \left[\int_{-\infty}^{\infty} xf(x)dx\right]^2$
4. $P(a < x < b) = P(a \le x \le b) = P(a < x \le b)$
$$= P(a \le x \le b) = \int_a^b f(x)\,dx$$

Types of Discrete Distribution

General Discrete Distribution

Suppose a discrete variable X is the outcome of some random experiment and the probability of X taking the value x_i is P_i, then

$$P(X = x_i) = P_i \text{ for } i = 1, 2, \ldots$$

where, $P(x_i) \ge 0$ for all values of i and $\sum P(x_i) = 1$.

The set of values x_i with their probabilities P_i of a discrete variable X is called a discrete probability distribution. For example, the discrete probability distribution of X, the number which is selected by picking a card from a well-shuffled deck is given by the following table:

$X = x_i$:	Ace	2	3	4	5	6	7
P_i :	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$
$X = x_i$:	8	9	10	Jack	Queen	King	
P_i :	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	

The distribution function $F(x)$ of discrete variable X is defined by

$$F(x) = P(X \le x) = \sum_{i=1}^{n} P_i$$

where x is any integer.

The mean value ($\bar{x}$) of the probability distribution of a discrete variable X is known as its expectation and is denoted by $E(x)$. If $f(x)$ is the probability density function of X, then

$$E(x) = \sum_{i=1}^{n} x_i f(x_i)$$

Variable of a distribution is given by

$$\sigma^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 f(x_i)$$

Binomial Distribution

Binomial distribution is concerned with trails of a respective nature whose outcome can be classified as either a success or a failure.

Suppose we have to perform n independent trails, each of which results in a success with probability P and in a failure with probability X which is equal to $1 - P$. If X represents the number of successes that occur in the n trails, then X is said to be binomial random variable with parameters (n, p).

The binomial distribution occurs when the experiment performed satisfies the following four assumptions of Bernoulli's trails.

1. They are finite in number.
2. There are exactly two outcomes: success or failure.
3. The probability of success or failure remains same in each trail.
4. They are independent of each other.

The probability of obtaining x successes from n trails is given by the binomial distribution formula,

$$P(X) = {}^nC_x P^x (1-p)^{n-x}$$

where P is the probability of success in any trail and $(1 - p)$ is the probability of failure.

Poisson Distribution

Poisson distribution is a distribution related to the probabilities of events which are extremely rare but which have a large number of independent opportunities for occurrence.

A random variable X, taking on one of the values 0, 1, 2, …, n, is said to be a Poisson random variable with parameters m if for some $m > 0$,

$$P(x) = \frac{e^{-m} m^x}{x!}$$

For Poisson distribution,

$$\text{Mean} = E(x) = m$$

$$\text{Variance} = V(x) = m$$

Therefore, the expected value and the variable of a Poisson random variable are both equal to its parameters m.

Hypergeometric Distribution

If the probability changes from trail to trail, one of the assumptions of binomial distribution gets violated; hence, binomial distribution cannot be used. In such cases, hypergeometric distribution is used; say a bag contains m white and n black balls. If y balls are drawn one at a time (with replacement), then the probability that x of them will be white is

$$P(x) = \frac{{}^mC_x^n C_{y-x}}{{}^{m+n}C_x}, \quad x = 0,\ 1,\ \ldots,\ y, \quad y \le m,\ n$$

This distribution is known as hypergeometric distribution.

For hypergeometric distribution,

$$\sum_{i=1}^{n} p(x) = 1, \text{ since } \sum_{i=1}^{n} {}^mC_x^n C_{y-x} = {}^{m+n}C_y$$

Geometric Distribution

Consider repeated trails of a Bernoulli experiment E with probability of success, P, and probability of failure, $q = 1 - p$. Let x denote the number of times E must be repeated until finally obtaining a success. The distribution is

$$p(x) = q^x p, \quad x = 0,\ 1,\ 2,\ \ldots,\ q = 1 - p$$

Also, $$\sum_{x=0}^{\infty} P(x) = P \sum_{x=0}^{\infty} q^x = p \frac{1}{1-q} = 1$$

The mean of geometric distribution $= q/p$.

The variance of geometric distribution $= q/p^2$.

Types of Continuous Distribution

General Continuous Distribution

When a random variable X takes all possible values in an interval, then the distribution is called continuous distribution of X.

A continuous distribution of X can be defined by a probability density function $f(x)$ which is given by

$$p(-\infty \le x \le \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$$

The expectation for general continuous distribution is given by

$$E(x) = \int_{-\infty}^{\infty} xf(x)dx$$

The variance for general continuous distribution is given by

$$V(x) = \sigma^2 = \int_{-\infty}^{\infty} \left(x - \bar{X}\right)^2 f(x)dx$$

Uniform Distribution

If density of a random variable X over the interval $-\infty < a < b < \infty$ is given by

$$f(x) = \frac{1}{b-a}, \quad a < x < b$$

Then the distribution is called uniform distribution.

The mean of uniform distribution is given by

$$E(x) = \int_a^b x \cdot F(x)dx$$

$$= \frac{1}{b-a}\left|\frac{x^2}{2}\right|_a^b$$

$$= \frac{a+b}{2}$$

In uniform distribution, x takes the values with the same probability.

The variance of uniform distribution is given by

$$V(x) = \sigma^2 = \frac{1}{12}(b-a)^2$$

Exponential Distribution

If density of a random variable x for $\lambda > 0$ is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then the distribution is called exponential distribution with parameter λ.

The cumulative distribution function $F(a)$ of an exponential random variable is given by

$$F(a) = P(x \leq a) = \int_0^a \lambda e^{-\lambda x} dx = (-e^{-\lambda x})^a$$
$$= 1 - e^{-\lambda a},\ a \geq 0$$

Mean for exponential distribution is given by

$$E(x) = 1/\lambda$$

Variance of exponential distribution is given by

$$V(x) = \frac{1}{\lambda^2}$$

Normal Distribution

A random variable X is a normal random variable with parameters μ and σ^2, if the probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty$$

where μ is mean for normal distribution and σ is standard deviation for normal distribution.

CORRELATION AND REGRESSION ANALYSES

Correlation

Until now we have discussed the analysis of observations on a single variable. In this topic, we discuss the cases where the changes in one variable are related to the changes in the other variable. Such simultaneous variation where the changes in one variable are associated with changes in the other is called *correlation*.

The correlation coefficient is a measure of linear association between two variables. The values of the correlation coefficient are always between -1 and 1. If an increase (or decrease) in the values of one variable corresponds to an increase (or decrease) in the other, the correlation is positive. Evidently, if an increase (or decrease) in the values of one variable corresponds to a decrease (or increase) in the other, the correlation is negative. A correlation coefficient of $+1$ indicates that the correlation is perfectly positive and a correlation coefficient of -1 indicates that the correlation is perfectly negative.

Coefficient of Correlation

Coefficient of correlation is defined as the numerical measure of correlation and can be calculated by the following relation:

$$r = \frac{\sum XY}{n\sigma_x\sigma_y}$$

where X is deviation from the mean $(x - \bar{x})$, Y is deviation from the mean $(y - \bar{y})$, σ_x is standard deviation of x-series, σ_y is standard deviation of y-series, and n is number of values of the two variables.

Coefficient of correlation for grouped data can be calculated using the following relation:

$$r = \frac{n\left(\sum fd_x d_y\right) - \left(\sum fd_x\right)\left(\sum fd_y\right)}{\sqrt{\left[\left\{n\sum fd_x^2 - \left(\sum fd_x^2\right)\right\} \times \left\{n\sum fd_y^2 - \left(\sum fd_y^2\right)\right\}\right]}}$$

where d_x is deviation of the central values from the assumed mean of x-series, d_y is deviation of the central values from the assumed mean of y-series, f is the frequency corresponding to the pair (x, y) and n is total number of frequencies $(= \sum f)$.

Lines of Regression

Sometimes, the dots of the scatter diagram tend to cluster along a well-defined direction which suggests a linear relationship between the variables x and y as shown in Fig. 2. Such a line giving the best fit for the given distribution of dots is known as line of regression.

The line giving the best possible mean values of y for each specified value of x is called the line of regression of y on x and the line giving the best possible mean values of x for each specified value of y is called the line of regression of x on y.

The regression coefficient of y on x is $r\dfrac{\sigma_y}{\sigma_x}$.

The regression coefficient of x on y is $r\dfrac{\sigma_x}{\sigma_y}$.

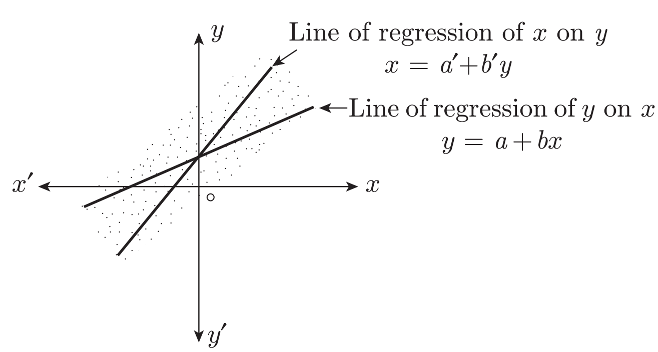


Figure 2 | Line of regression.

HYPOTHESIS TESTING

In statistics and probability theory, hypothesis testing is used for determining the probability that a given hypothesis is true. Let us illustrate the concept of hypothesis testing with the help of an example. Let us say that there are two schools of thoughts as regards how the presence of caffeine in coffee affects the alertness of people. The test is conducted on students of a class by dividing them into two groups; one that drank coffee with caffeine and the other that drank coffee without caffeine; their alertness (in terms of time spent sleeping in class) was tested and it was found that the group that drank coffee with caffeine slept less. One therefore tends to conclude that caffeine in coffee improves alertness. But the satistician may argue otherwise. He would say that difference in the average alertness of two groups was due to chance. There could be reasons other than the effect of caffeine, such as the students of caffeine-taker group having had a better sleep the previous night or being more interested in the class. He would argue that if the students were divided differently in two groups, the results would have been different and the conclusion drawn from the test might have been proven wrong. The satistician would therefore not conclude unless he performs a hypothesis test.

Hypothesis test allows us to make a rational decision between the hypothesis of real effects and chance explanations. Although the chance explanations cannot be completely eliminated, these may be unlikely if difference between the two groups is very large. A hypothesis would specify the required quantum of difference to conclude that the effects are real. The purpose of hypothesis testing is to eliminate false scientific conclusions as much as possible. The process of hypothesis testing consists of the following four major steps:

1. The first step is null hypothesis and alternative hypothesis. A null hypothesis (H_0) is a statistical hypothesis that is tested for possible rejection under the assumption that it is true. Rejection of null hypothesis would mean that the observations are not due to chance. The alternative hypothesis (H_a) is contrary to the null hypothesis. Alternative hypothesis believes that the observations are the result of a real effect with some amount of chance element included. The two hypotheses are stated in such a way that they are mutually exclusive. If one were true, other would be false and vice versa.
2. The second step is to identify a test statistic that can be used for assessing the truth of the null hypothesis. The analysis plan is to use sample data to either accept or reject the null hypothesis. The following elements need to be specified to complete the task of either accepting or rejecting null hypothesis. The first element is to choose a significance level. The significance level also denoted as α is the probability of rejecting the null hypothesis when it is true. A significance level of 0.02 indicates a 2% risk concluding that a difference exists when there is no actual difference. Researchers often choose significance level of 0.01, 0.05 or 0.1. Any value between 0 and 1 though can be used. The second and the important element is the test statistic and a sampling distribution. Computed from sample data, the test statistic might be a mean score, a proportion, difference between means, difference between proportions, etc. From the given sample distribution and test statistic, probabilities associated in the test statistic are computed. If the test statistic probability turns out to be less than the chosen significance level, the null hypothesis is rejected.
3. The third step is to analyze the sample data. It involves computing the P-value, which is the probability that a test statistic is at least as significant as the one observed, and would be obtained with the assumption that the null hypothesis was true. A smaller P-value signifies a stronger belief against the null hypothesis.
4. The P-value is next compared with the chosen (or an acceptable) significance level. If $P \leq \alpha$, then the observed effect is statistically significant. That is, null hypothesis is rejected and alternative hypothesis is valid.

HYPOTHESIS TESTING PROCEDURES FOR SOME COMMON STATISTICAL PROBLEMS

Hypothesis testing procedures for some common statistical problems are briefly described in the following paragraphs. The testing procedures discussed include hypothesis test procedures involving proportions, means, difference between proportions and difference between means.

Hypothesis Testing of a Proportion

The hypothesis testing of a proportion described as follows assumes that the following conditions are satisfied:

(a) The sampling method is a simple random sampling which implies that random sampling has a population of (N) objects; the sample consists of (n) objects and all possible samples of (n) objects are equally likely to occur.
(b) Each sample point can have two possible outcomes, namely, a success or a failure.
(c) The sample includes at least 10 successes and 10 failures.

(d) The population size is at least 20 times as big as the sample size.

State null and alternate hypothesis in such a way that they are mutually exclusive, that is, if one is true, then the other is false and vice versa. Choose an acceptable significance level. Use one-sample z-test to determine whether hypothesized population proportion differs significantly from the observed sample proportion.

The sample data are used for determining the test statistic and P-value in the following steps:

1. Compute the standard deviation (σ) of the sampling distribution from

$$\sigma = \sqrt{\frac{P \times (1-P)}{n}}$$

Here P is the hypothesized value of population proportion in the null hypothesis and n is the sample size.

2. The test statistic is a z-score defined by the following equation:

$$z = \frac{p - P}{\sigma}$$

Here p is the sample proportion and σ is the standard deviation of the sampling distribution.

3. The P-value is computed next. It is probability of observing a sample statistic as extreme as the test statistic. Since test statistic is a z-score, normal distribution calculator is used for assessing probability associated in the z-score.
4. The P-value is compared to the significance level; and if the P-value is less than the significance level, null hypothesis is rejected.

Hypothesis Testing of a Mean

Hypothesis testing of a mean described in the following paragraphs assumes that the following conditions are satisfied. (a) The sampling method is simple random sampling and (b) the sampling distribution is normal or nearly normal. Sampling distribution is nearly normal when population distribution is normal; population distribution is symmetric, unimodal, without outliers and the sample size is 15 or less; the population distribution is moderately skewed, unimodal, without outliers and the sample size is between 16 and 40; the sample size is greater than 40, without outliers.

The under mentioned steps are followed for hypothesis testing.

1. State null and alternative hypothesis. Three sets of null and alternate hypotheses are

   (a) H_0: $\mu = M$ and H_a: $\mu \neq M$ (two-tailed test)
   (b) H_0: $\mu \geq M$ and H_a: $\mu < M$ (one-tailed test)
   (c) H_0: $\mu \leq M$ and H_a: $\mu > M$ (one-tailed test).

   Here, μ is the population mean and M is the specified value of mean; the hypotheses describe how population mean is related to specified mean.
2. Choose an acceptable significance level. Use one-sample t-test to determine whether hypothesized mean differs significantly from the observed sample mean. The test involves finding the standard error (SE), degrees of freedom (DF), test statistic and the P-value associated with test statistic.
3. Standard error (SE) is given by

$$\text{SE} = s \times \sqrt{\left(\frac{1}{n}\right) \times \left(\frac{N-n}{N-1}\right)}$$

where s is the standard deviation of the sample, N is the population size and n is the sample size. If $N >> n$, the SE can be approximated by

$$\text{SE} = \frac{s}{\sqrt{n}}$$

4. The degrees of freedom (DF) is given by $(n - 1)$
5. The test statistic is a t-score (t) defined by

$$t = \frac{\bar{x} - \mu}{\text{SE}}$$

where $\bar{x}$ is sample mean and μ is hypothesized population mean in the null hypothesis.

6. P-value is computed next. Since test statistic is a t-score, t-distribution calculator is used for assessing the probability associated with t-score, given degrees of freedom is computed above.
7. Again, if P-value is less than the significance level, null hypothesis is rejected.

Hypothesis Testing of Difference Between Proportions

In the case of hypothesis testing of "difference between proportions", two-proportion z-test is the appropriate test procedure provided that the following conditions are satisfied:

(a) Sampling method is simple random sampling.
(b) Samples are independent, that is, occurrence of one does not affect the probability of occurrence of the other.
(c) Each sample includes at least 10 successes and 10 failures.
(d) Each population is at least 20 times as big as its sample.

Same set of steps are followed as described in the previous two cases. The steps for this testing are outlined as follows:

1. State the null and alternate hypotheses. The three possible sets of hypotheses can be stated as follows:

   (a) H_0: $P_1 - P_2 = 0$, H_a: $P_1 - P_2 \neq 0$ (two-tailed test)
   (b) H_0: $P_1 - P_2 \geq 0$, H_a: $P_1 - P_2 < 0$ (one-tailed test)
   (c) $P_1 - P_2 \leq 0$, H_a: $P_1 - P_2 > 0$ (one-tailed test)

   Here P_1 and P_2 are two population proportions, and different sets of hypotheses make a statement about the difference between the population proportions.
2. Choose an acceptable significance level.
3. Use "two-proportion z-test" to determine whether the hypothesized difference between proportions differs significantly from observed sample difference.
4. Using sample data to determine the test statistic and the P-value by computing the following.
5. When the null hypothesis states that there is no difference between population proportions, the null and alternative hypotheses for a two-tailed test are often stated as H_0: $P_1 - P_2$ and H_a: $P_1 \neq P_2$.
6. Since null hypothesis states that $P_1 = P_2$, a pooled sample proportion (p) is used for computing SE of sampling distribution as follows:

$$p = \frac{p_1 \times n_1 + p_2 \times n_2}{n_1 + n_2}$$

   where p_1 is the sample proportion from population-1, p_2 is the sample proportion from population-2, n_1 is the size of sample-1 and n_2 is the size of sample-2.
7. The standard error of the sampling distribution difference between two proportions is computed as follows:

$$\text{SE} = \sqrt{\left[p \times (1-p) \times \left\{\left(\frac{1}{n_1}\right) + \left(\frac{1}{n_2}\right)\right\}\right]}$$

8. The test statistic is a z-score test defined by

$$z = \frac{p_1 - p_2}{\text{SE}}$$

9. Since the test statistic is a z-score, normal distribution calculator is used for assessing the probability associated with z-score. P-value is compared with the significance level, and if it is less than that the null hypothesis is rejected.

Hypothesis Testing of Difference Between Means

In this case, "two-sample t-test" is the appropriate procedure provided that the following conditions are met with: (a) Simple random sampling is used, (b) samples are independent, (c) each population is at least 20 times larger than its respective sample and (d) sampling distribution is approximately normal. Different steps involved in hypothesis testing are outlined as follows.

1. There can be three different possible null and alternative hypotheses. Each makes a statement about the difference between two population means, μ_1 and μ_2. These are:

   (a) $\mu_1 - \mu_2 = d(H_0)$ and $\mu_1 - \mu_2 \neq d(H_a)$ (two-tailed test)
   (b) $\mu_1 - \mu_2 \geq d(H_0)$ and $\mu_1 - \mu_2 < d(H_a)$ (one-tailed test)
   (c) $\mu_1 - \mu_2 \leq d(H_0)$ and $\mathrm{i}_1 - \mu_2) > d(H_a)$ (one-tailed test).

2. Choose an acceptable significance level.
3. Use two-sample t-test as the test method.
4. Using sample data, compute standard error and degrees of freedom , test statistic and P-value associated with the test statistic.
5. Standard error is computed as follows:

$$\text{SE} = \sqrt{\left\{\left(\frac{s_1^2}{n_1}\right) + \left(\frac{s^2}{n_2}\right)\right\}}$$

   Here s_1 is the standard deviation of sample-1 and s_2 is the standard deviation of sample-2 and n_1 and n_2 are the size of the two samples.
6. Degrees of freedom is computed as follows

$$\text{DF} = \frac{\left[(s_1^2/n_1) + (s_2^2/n_2)\right]}{\left[\left\{(s_1^2/n_1)^2/(n_1-1)\right\} + \left\{(s_2^2/n_2)^2/(n_2-1)\right\}\right]}$$

   If DF does not turn out to be an integer, it is rounded off to the nearest whole number. It may also be approximated by the smallest of the two numbers given by $(n_1 - 1)$ and $(n_2 - 1)$.
7. The test statistic is a t-score test defined by

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - d}{\text{SE}}$$

   Here $\bar{x}_1$ is the mean of sample-1, $\bar{x}_2$ is the mean of sample-2 and d is the hypothesized difference between two population means and SE is the standard error.

8. t-Difference calculator is used for assessing the probability associated with the test statistic. The P-value is compared with the chosen significance level. If it is less than the significance level, null hypothesis is rejected.

BAYES' THEOREM

Bayes' theorem, also known as Bayes' rule, is a useful tool of calculating conditional probabilities. Bayes' theorem describes the formulation to compute the conditional probability of each of a set of possible causes for a given observed outcome from knowledge of the probability of each cause and the conditional probability of the outcome of each cause.

According to Bayes' theorem, if an event A corresponds to a number of exhaustive events B_1, B_2, B_3 …, B_n, and if $P(B_i)$ and $P(A/B_i)$ are given, then

$$P(B_i/A) = \frac{P(B_i)P(A/B_i)}{\Sigma P(B_i)P(A/B_i)}$$

Probabilities $P\ (B_i)$; $i = 1, 2, 3, \ldots, n$ are called *priori* probabilities because these exist before we get any information. Probabilities $P(A/B_i)$ are called *posteriori* probabilities because these are found after experiment results are known. Bayes' theorem can be proved as follows.

By multiplication law of probability,

$$P(AB_i) = P(A)P\left(\frac{B_i}{A}\right) = P(B_i)P\left(\frac{A}{B_i}\right)$$

This leads to

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i)P(A/B_i)}{P(A)}$$

From addition law of probability,

$$\begin{aligned} P(A) &= P(AB_1) + P(AB_2) + P(AB_3) \\ &\quad + \cdots + P(AB_n) \\ &= \sum P(AB_i) = \sum P(B_i)P\left(\frac{A}{B_i}\right) \end{aligned}$$

Substituting $p(A)$, we get

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i)P(A/B_i)}{\sum P(B_i)P(A/B_i)}$$

Bayes' theorem may also be written as

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i \cap A)}{P(B_1 \cap A) + P(B_2 \cap A) + \cdots P(B_n \cap A)}$$

because

$$P(B_i \cap A) = P(B_i)P\left(\frac{A}{B_i}\right)$$

Bayes' theorem should be considered for application when the following conditions exist:

1. The sample space is divided into a set of mutually exclusive events (B_1, B_2, …, B_n).
2. Within the sample space there exists an event, while $P(A) > 0$.
3. The analytical goal is to compute a conditional probability of the form $P(B_i/A)$.
4. At least one of the two following sets of probabilities is known, namely, $P(B_i \cap A)$ for each B_i and $P(B_i)$ and $P(A/B_i)$ for each B_i.

SOLVED EXAMPLES

1. A box contains 5 white and 10 black balls. Eight of them are placed in another box. What is the probability that the latter box contains 2 white and 6 black balls?

   Solution: The number of balls is 15. The number of ways in which 8 balls can be drawn out of 15 is ${}^{15}C_8$.
   The number of ways of drawing 2 white balls = ${}^{5}C_2$
   The number of ways of drawing 6 black balls = ${}^{10}C_6$
   Total number of ways in which 2 white and 6 red balls can be drawn is ${}^{5}C_2 \times {}^{10}C_6$.

   $$\text{Thus, the required probability} = \frac{{}^{5}C_2 \times {}^{10}C_6}{{}^{15}C_8} = \frac{140}{429}$$

2. Four cards are drawn at random from a pack of 52 playing cards. What is the probability of getting all the four cards of the same suit?

   Solution: Four cards can be drawn from a deck of 52 cards in ${}^{52}C_4$ ways; there are four suits in a deck, each of 13 cards.

Thus, total number of ways of getting all four cards of same suit is

$$^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4\left({}^{13}C_4\right)$$

Hence, required probability $= \dfrac{4\left({}^{13}C_4\right)}{{}^{52}C_4} = \dfrac{198}{20825}$

3. The letters of word SOCIETY are placed at random in a row. What is the probability that the three vowels come together?

Solution: The letters in the word SOCIETY can be arranged in 7! ways. The three vowels can be put together in 3! ways. And considering these three vowels as one letter, we have 5 letters which can be arranged in 5! ways.

Thus, favorable number of outcomes = 5! × 3!

Required probability $= \dfrac{5! \times 3!}{7!} = \dfrac{1}{7}$

4. In a race, the odds in favor of the four cars C_1, C_2, C_3, C_4 are 1:4, 1:5, 1:6, 1:7, respectively. Find the probability that one of them wins the race assuming that a dead heat is not possible.

Solution: The events are mutually exclusive because it is not possible for all the cars to cover the same distance at the same time. If P_1, P_2, P_3, P_4 are the probabilities of winning for the cars C_1, C_2, C_3, C_4, respectively, then

$$P_1 = \frac{1}{1+4} = \frac{1}{5}$$

$$P_2 = \frac{1}{1+5} = \frac{1}{6}$$

$$P_3 = \frac{1}{1+6} = \frac{1}{7}$$

$$P_4 = \frac{1}{1+7} = \frac{1}{8}$$

Hence, the chance that one of them wins = $P_1 + P_2 + P_3 + P_4$

$$= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840}$$

5. Given $P(A) = \dfrac{1}{4}$, $P(B) = \dfrac{1}{3}$ and $P(A \cup B) = \dfrac{1}{2}$, then what is the value of $P\left(\dfrac{A}{B}\right), P\left(\dfrac{B}{A}\right), P(A \cap B')$ and $P\left(\dfrac{A}{B'}\right)$?

Solution: We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{3} - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = \frac{1}{12}$$

Thus, $P\left(\dfrac{A}{B}\right) = P\dfrac{(A \cap B)}{P(B)} = \dfrac{1/12}{1/3} = \dfrac{1}{4}$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/2}{1/3} = \frac{1}{4}$$

$$P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$

$$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{1/6}{1 - P(B)} = \frac{1/6}{1 - \frac{1}{3}} = \frac{1}{4}$$

6. We have three bags. The first bag contains 1 white, 2 red and 3 black balls, the second bag contains 2 white, 3 red and 1 black bag, and the third bag contains 3 white, 1 red and 2 black balls. Two balls are chosen from a bag taken at random, which are found to be white and red. Find the probability that the balls drawn came from the second bag.

Solution: Let B_1, B_2, B_3 be the events of choosing first, second and third bag, respectively, and A be the event when two balls are white and red.
Now,

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

$$P(A/B_1) = \left({}^1C_1 \times {}^2C_1\right) \Big/ {}^6C_2 = \frac{2}{15}$$

Similarly, $P(A/B_2) = \left({}^2C_1 \times {}^3C_1\right) \Big/ {}^6C_2 = \dfrac{2}{5}$

and

$$P(A/B_3) = \left({}^3C_1 \times {}^1C_1\right) \Big/ {}^6C_2 = \frac{1}{5}$$

Now applying Bayes' theorem,

$P(B_2/A)$

$$= \frac{P(B_2)P(A/B_2)}{P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + P(B_3)P(A/B_3)}$$

$$= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{3} \times \frac{2}{15} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{1}{5}} = \frac{6}{11}$$

7. Calculate the arithmetic mean ($\overline{x}$) when $x = 1, 2, 5, 4, 6, 7, 5, 1, 8$.

Solution: Arithmetic mean for raw data is given by

$$\overline{x} = \frac{\sum x}{n} = \frac{1+2+5+4+6+7+5+1+1+8}{10}$$

$$= \frac{40}{10} = 4$$

8. Calculate the arithmetic mean of the following grouped frequency distribution.

x	5	10	15	20	25	30	35	40	45	50
f	2	10	8	5	6	9	2	4	3	1

Solution: We construct the table as follows:

x	f	fx
5	2	10
10	10	100
15	8	120
20	5	100
25	6	150
30	9	270
35	2	70
40	4	160
45	3	135
50	1	50
	$\sum f = 50$	$\sum fx = 1165$

The arithmetic mean of the given grouped frequency distribution $= \dfrac{1165}{50} = 23.3$

9. The weight of 8 students in a class is 57, 62, 48, 45, 55, 63, 56, 58. What is the median of the weight of the 8 students?

Solution: First, we arrange the weight in ascending order

45, 48, 55, 56, 57, 58, 62, 63

As the number of entries is even, we have two middle values, i.e. 4th and 5th values.

$$\text{Median} = \frac{56+57}{2} = 56.5$$

10. Consider the following table giving marks obtained by students in an exam.

Mark Range	Number of Students	Cumulative Frequency
0–20	2	2
20–40	3	5
40–60	10	15
60–80	15	30
80–100	20	50
	$\sum f = 50$	

Solution: Here $\dfrac{N+1}{2} = 25.5$

The class 60–80 is the median class because cumulative frequency is 30 > 25.5

$$\text{Median} = \frac{60 + [25.5 - (15+1)]}{15} \times 20$$

$$= 69.67$$

11. Find the mode of the data set: 50, 20, 25, 35, 50, 60, 60, 65, 50, 20.

Solution: In the above data set, 20 and 60 occurs twice whereas 50 occurs thrice. Hence, the mode of data set is 50.

12. Calculate the modal height from the data of 352 school students given in the following frequency distribution:

Height	Number of Students
3.0–3.5	12
3.5–4.0	37
4.0–4.5	79
4.5–5.0	152
5.0–5.5	65
5.5–6.0	7
	Total = 352

Solution: As 152 is the largest frequency, the modal class is 4.5–5.0.

Thus, $L = 4.5,\ f_0 = 152,\ f_1 = 79,\ f_2 = 65,\ h = 0.5$.

We know,

$$\text{Mode} = L + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h \tag{1}$$

stituting values in Eq. (1), we get

$$\text{Mode} = 4.5 + \frac{152-79}{2(152)-79-65} \times 0.5 = 4.5 + \frac{73}{320}$$

$$= 4.73$$

13. Calculate the mean deviation about mean from the following data:

x_i	3	9	17	23	27
f_i	8	10	12	9	5

Solution: For calculating mean deviation, we construct the following table:

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
3	8	24	12	96
9	10	90	6	60
17	12	204	2	24
23	9	207	8	72
27	5	135	12	60
	$\sum f_i = 44$	$\sum f_i x_i = 660$		$\sum f_i \lvert x_i - 15\rvert = 312$

$$\text{Mean} = \bar{X} = \frac{1}{N}\left(\sum f_i x_i\right) = \frac{660}{4} = 15$$

$$\text{Mean deviation} = \frac{1}{\sum f_i}\sum f_i \lvert x_i - \bar{x}\rvert = \frac{312}{44} = 7.09$$

14. The following table gives the distribution of income of 100 families in a city. Calculate the standard deviation.

Income (Rs):	0–1000	1000–2000	2000–3000	3000–4000	4000–5000	5000–6000
No. of families:	18	26	30	12	10	4

Solution: We calculate standard deviation by constructing the following table:

Income	Mid-Values, x_i	Number of Families, f_i	$u_i = \dfrac{x_i - 2500}{1000}$	$f_i u_i$	u_i^2	$f_i u_i^2$
0–1000	500	18	−2	−36	4	72
1000–2000	1500	26	−1	−26	1	26
2000–3000	2500	30	0	0	0	0
3000–4000	3500	12	1	12	1	12
4000–5000	4500	10	2	20	4	40
5000–6000	5500	4	3	12	9	36
		$\sum f_i = 100$		$\sum f_i u_i = -18$		$\sum f_i u_i^2 = 186$

Here, $N = 100,\ \sum f_i u_i = -18,\ \sum f_i u_i^2 = 186,\ h = 1000$

$$\text{Variance }(X) = h^2\left[\frac{1}{N}\left(\sum f_i u_i^2\right) - \left(\frac{1}{N}\sum f_i u_i\right)^2\right] = (1000)^2\left[\frac{186}{100} - \left(\frac{-18}{100}\right)^2\right] = 1827600$$

$$\text{Standard deviation} = \sqrt{\text{Var}(X)} = \sqrt{1827600} = 1351.88$$

15. The following values are calculated in respect of heights and weights of students in a class

	Height	**Weight**
Mean	162.6 cm	52.36 kg
Variance	127.69 cm^2	23.1361 kg^2

Find whether the weight shows greater variation or the height.

Solution: Let σ_1 and σ_2 denote standard deviations of height and weight, respectively. Similarly, $\bar{x}_1$ and $\bar{x}_2$ be the mean height and weight, respectively. Then, we have

$$\bar{x}_1 = 162.6$$

$$\bar{x}_2 = 52.36$$

$$\sigma_1^2 = 127.69 \Rightarrow \sigma_1 = 11.3$$

$$\sigma_2^2 = 23.1361 \Rightarrow \sigma_2 = 4.81$$

Now, coefficient of variation $= \frac{\sigma}{x} \times 100$

Then, coefficient of variation in heights $= \frac{11.3}{162.6} \times 100 = 6.95$

And coefficient of variation in weights $= \frac{4.81}{52.36} \times 100 = 18$

Hence, it is visible that weights show more variability then heights.

16. The probability density function of a random variable x is

x	0	1	2	3	4	5	6
$P(x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $p(x < 4), p(x \geq 5), p(3 < x \leq 6)$.

Solution: If X is a random variable, then

$\sum_{i=0}^{6} p(x_i) = 1$, i.e.

$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$\Rightarrow 49k = 1$

$\Rightarrow k = \frac{1}{49}$

$$P(x < 4) = k + 3k + 5k + 7k = 16k = 16/49$$

$$P(x \geq 5) = 11k + 13k = 24k = 24/49$$

$$P(3 < x \leq 6) = 9k + 11k + 13k = 33k = 33/49$$

17. X is a continuous random variable with probability density function given by

$$f(x) = kx \quad (0 \leq x < 2)$$
$$= 2k \ (2 \leq x < 4)$$
$$= -kx + 6k \ (4 \leq x < 6)$$

Find k and the mean value of X.

Solution: As the probability is unity,

$$\int_0^6 f(x) \cdot dx = 1$$

$$\Rightarrow \int_0^2 kx\, dx + \int_2^4 2k\, dx + \int_4^6 (-kx + 6k) dx = 1$$

$$\Rightarrow k\left|\frac{x^2}{2}\right|_0^2 + 2k|x|_2^4 + \left|-\frac{kx^2}{2} + 6kx\right|_4^6 = 1$$

$$\Rightarrow 2k + 4k + (-10k + 12k) = 1 \quad \Rightarrow \quad k = \frac{1}{8}$$

$$\text{Mean of } X = \int_0^6 x f(x)\, dx$$

$$= \int_0^2 kx^2\, dx + \int_2^4 2kx\, dx + \int_4^6 x(-kx + 6k) dx$$

$$= k\left|\frac{x^3}{3}\right|_0^2 + 2k\left|\frac{x^2}{2}\right|_2^4 + \left[-k\left|\frac{x^3}{3}\right|_4^6 + 6k\left|\frac{x^2}{2}\right|_4^6\right]$$

$$k(8/3) + k(12) - k(152/3) + 3k(20) = \frac{1}{8}(24) = 3$$

18. The probability that a dress manufactured by a factory will be defective is 1/10. If such dresses are manufactured, find the probability that

(a) exactly two will be defective

(b) at least two will be defective

Solution: The probability of a defective dress is $1/10 = 0.1$

Hence, the probability of non-defective dress is $1 - 0.1 = 0.9$

(a) The probability that exactly two will be defective

$$= {}^{12}C_2 (0.1)^2 (0.9)^{10} = 0.2301$$

(b) The probability that at least two will be defective

= 1− [probability that either none or one is non-defective]

$$= 1 - \left[{}^{12}C_0 (0.9)^{12} + {}^{12}C_1 (0.1)(0.9)^{11}\right] = 0.3412$$

19. If the probability of occurrence of disease from a certain injection is 0.01, determine the chance that out of 2000 individuals more than two will get the disease.

Solution: This follows a Poisson distribution as the probability of occurrence is very small.

$$\text{Mean} = m = 2000\ (0.001) = 2$$

Probability that more than 2 will get a disease

= 1− [probability that nobody gets the disease + probability that one gets the disease + probability that two gets the disease]

$$= 1 - \left[e^{-m} + \frac{me^{-m}}{1!} + \frac{m^2 e^{-m}}{2!}\right]$$

Now, we have $m = 2$

$$\text{Required probability} = 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}\right]$$

$$= 1 - \frac{5}{e^2} = 0.32$$

20. A random variable X has a uniform distribution over $(-3, 3)$, find k for which

$$P(X > k) = \frac{1}{3}$$

Also, evaluate $P(X < 2)$ and $P\left[|X-2|<2\right]$.

Solution: Density of $X =$

$$f(x) = \frac{1}{b-a} = \frac{1}{3-(-3)} = \frac{1}{6}$$

$$P(X > k) = 1 - P(X \le k) = 1 - \int_{-3}^{k} f(x)\,dx$$

$$1 - \frac{1}{6}\int_{-3}^{k} dx = 1 - \frac{1}{6}(k+3) = \frac{1}{3}$$

Hence, $k = 1$.

$$P(X<2) = \int_{-3}^{2} f(x)\,dx = \frac{1}{6}\int_{-3}^{2} dx = \frac{1}{6}(2+3) = \frac{5}{6}$$

$$\begin{aligned} P(|X-2|<2) &= P(2-2 < X < 2+2) \\ &= P(0 < x < 4) = \int_{0}^{3} f(x)\,dx \\ &= \frac{1}{6}\int_{0}^{3} dx = \frac{1}{6}(3-0) \\ &= \frac{1}{2} \end{aligned}$$

21. A manufacturer of metal pistons finds that on an average 12% of his pistons are rejected because they are either oversize or undersize. What is the probability that a batch of 10 pistons will contain no more than 2 rejects?

Solution: Let $X =$ number of rejected pistons
In this case, "success" means rejection.

Here, $n = 10,\ p = 0.12,\ q = 0.88$.

No rejects is the case when $x = 0$.

$$P(X) = C_x^n p^x q^{n-x} = C_0^{10}(0.12)^0(0.88)^{10} = 0.2785$$

One reject is the case when $x = 1$.

$$P(X) = C_1^{10}(0.12)^1(0.88)^9 = 0.37977$$

Two rejects is the case when $x = 2$.

$$P(X) = C_2^{10}(0.12)^2(0.88)^8 = 0.23304$$

So the probability of getting no more than 2 rejects is

Probability $= P(X \le 2)$

$= 0.2785 + 0.37977 + 0.23304 = 0.89131$

22. Find $V(X)$ for the following probability distribution:

X	8	12	16	20	24
$P(X)$	1/8	1/6	3/8	1/4	1/12

Solution: We know that

$$E(X) = 8\times\frac{1}{8} + 12\times\frac{1}{6} + 16\times\frac{3}{8} + 20\times\frac{1}{4} + 24\times\frac{1}{12} = 16$$

Also,

$$\begin{aligned} V(X) &= \sum\left[\{X-E(X)\}^2\cdot P(X)\right] \\ &= (8-16)^2\times\frac{1}{8} + (12-16)^2\times\frac{1}{6} + (16-16)^2\times\frac{3}{8} \\ &\quad + (20-16)^2\times\frac{1}{4} + (24-16)^2\times\frac{1}{12} \\ &= 20 \end{aligned}$$

23. Consider the following joint distribution of two random variables X and Y:

	$X = 1$	$X = 0$
$Y = 2$	0.4	0.1
$Y = 4$	0.2	0.3

Find the conditional distribution of X given Y.

Solution: We need to find the conditional probabilities for all possible combinations of values that X and Y can take. Thus,

$$P(X=1|Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{0.4}{0.4+0.1} = 0.8$$

$$P(X=1|Y=4) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{0.2}{0.2+0.3} = 0.4$$

$$P(X=0|Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{0.1}{0.4+0.1} = 0.2$$

$$P(X=0|Y=4) = \frac{P(X=0, Y=4)}{P(Y=4)} = \frac{0.3}{0.2+0.3} = 0.6$$

24. Ramesh is getting married tomorrow at an outdoor location. In recent years, it has rained only five days each year. Unfortunately, rain has been predicted for tomorrow by the weatherman. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it does not rain, he incorrectly forecasts rain 10% of the time. What is the probability that it would rain on Ramesh's wedding?

Solution: The sample space here is defined by two mutually exclusive events, it rains or it does not rain. Additionally, a third event occurs when the weatherman predicts rain. Let us assume the following:

Event A_1: It rains on Ramesh's wedding.

Event A_2: It does not rain on Ramesh's wedding.

Event B: The weatherman predicts rain.

The objective is to compute $P(A_1/B)$.

From Bayes' theorem,

$$P\left(\frac{A_1}{B}\right) = \frac{P(A_1)P(B / A_1)}{P(A_1)P(B/A_1) + P(A_2)P(B/A_2)} \quad (1)$$

Now

$$P(A_1) = \frac{5}{365} = 0.0136985 \text{ (it rains 5 days in a year)}$$

$$P(A_2) = \frac{360}{365} = 0.9863014 \text{ (it does not rain for 360 days in a year)}$$

$$P\left(\frac{B}{A_1}\right) = 0.9 \text{ (when it rains, weatherman predicts rain 90\% of time)}$$

$$P\left(\frac{B}{A_2}\right) = 0.1 \text{ (when it does not rain, weatherman predicts rain 10\% of time)}$$

Substituting these values in Eq. (1), we get

$$P\left(\frac{A_1}{B}\right) = \frac{0.0137 \times 0.9}{(0.0137 \times 0.9) + (0.986 \times 0.1)} = 0.111$$

25. There are three bags. The first bag contains one white, two red and three green balls. The second bag contains two white, three red and one green ball. The third bag contains three white, one red and two green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red ball. What is the probability that the two balls were drawn from the second bag?

Solution: Let B_1, B_2 and B_3 pertain to first, second and third bags chosen, respectively, and A represents the event that two balls are white and red.
The objective is to determine $P(B_2/A)$, that is, the chosen balls (represented by A) are from the second bag (represented by B_2). From Bayes' theorem,

$$P\left(\frac{B_2}{A}\right) = \frac{P(B_2)P(A/B_2)}{P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + P(B_3)P(A/B_3)} \quad (1)$$

Now $$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

$$P\left(\frac{A}{B_1}\right) = \frac{{}^1C_1 \times {}^2C_1}{{}^6C_2} = \frac{2}{15}$$

This is the probability that a white and a red ball are drawn from the first bag.

Similarly,

$$P\left(\frac{A}{B_2}\right) = \frac{{}^2C_1 \times {}^3C_1}{{}^6C_2} = \frac{2}{5}$$

and

$$P\left(\frac{A}{B_3}\right) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{1}{5}$$

Substituting the above computed values in Eq. (1), we get

$$P\left(\frac{B_2}{A}\right) = \frac{(1/3) \times (2/5)}{[(1/3) \times (2/15)] + [(1/3) \times (2/5)] + [(1/3) \times (1/5)]} = \frac{6}{11}$$

26. In a certain country, 51% of the people are male. One adult is randomly selected for a survey involving credit card usage. It is also learned that the selected survey subject was smoking a cigar. Also, 10% of males smoke cigars whereas only 2% of females smoke cigars. What is the probability that selected subject is a male?

Solution: Let:
$P(M)$ be the probability of randomly selecting an adult and getting a male;

$P(C/M)$ be the probability of getting someone who smokes cigar, given that the person is a male,

Then,

$P(C/\bar{M})$ is the probability of getting someone who smokes cigars given that the person is a female; and

$P(\bar{M})$ is the probability of randomly selecting an adult and getting a female.

The objective is to determine $P(M/C)$. From Bayes' theorem,

$$P\left(\frac{M}{C}\right) = \frac{P(M) \cdot P(C/M)}{P(M) \cdot P(C/M) + P(\bar{M}) \cdot P(C/\bar{M})} \quad (1)$$

Now $P(M) = 0.51$, $P(\bar{M}) = 0.49$, $P(C/M) = 0.1$ and $P(C/\bar{M}) = 0.02$.

Substituting these values in Eq. (1), the probability of selected person being a male, given that the person is a cigar smoker

$$= \frac{0.51 \times 0.1}{(0.51 \times 0.1) + (0.49 \times 0.02)} = \frac{0.051}{0.051 + 0.0098}$$

$$= \frac{0.051}{0.0608} \cong 0.84$$

PRACTICE EXERCISE

1. Out of all the words that can be formed from the letters of KING, what is the probability that a word chosen at random will start with K.

(a) 1/4 (b) 1/2
(c) 3/4 (d) 2/5

2. Two dice are thrown and the sum of the numbers which come up on the dice is noted. Consider the following events:

A = the sum is even
B = the sum is multiple of 3
C = the sum is less than 4
D = the sum is greater than 11

Which pairs of these events are mutually exclusive?

(a) A and B (b) B and C
(c) C and D (d) A and D

3. Three coins are tossed at once. What is the probability of getting exactly 2 tails?

(a) 1/8 (b) 3/8
(c) 5/8 (d) 1/4

4. Two dice are thrown simultaneously. What is the probability of getting a total of at least 10?

(a) 1/6 (b) 2/3
(c) 5/36 (d) 3/7

5. Four cards are drawn at random from a well-shuffled deck of 52 cards. What is the probability of getting all face cards?

(a) $\dfrac{141}{54145}$ (b) $\dfrac{325}{62522}$

(c) $\dfrac{82}{43131}$ (d) $\dfrac{99}{54145}$

6. Chances of solving a question by A, B and C are 1/2, 1/3 and 1/4, respectively. What is the probability that the question will be solved?

(a) 1/4 (b) 2/3
(c) 1/2 (d) 3/4

7. If A and B are two events such that $P(A) = 0.3$, $P(B) = 0.6$ and $P(B/A) = 0.5$, then what is the value of $P(A \cup B)$.

(a) 0.50 (b) 0.75
(c) 0.67 (d) 0.33

8. A bag contains 19 tickets, numbered from 1 to 19. A ticket is drawn and then another ticket is drawn without replacement. What is the probability that both the tickets will show even numbers?

(a) 2/19 (b) 11/19
(c) 4/19 (d) 10/19

9. What is the range of observations: 25, 5, 50, 30, 40, 30?

(a) 30 (b) 15
(c) 17.5 (d) 35

10. What is the arithmetic mean from the following table?

x_i	f_i
5	2
10	1
15	5
20	7
25	5

(a) 18 (b) 15
(c) 22 (d) 20

11. What is the arithmetic mean of the following set of observations: 28, 32, 15, 17, 11, 48, 43, 5, 7, 14?

(a) 22 (b) 28
(c) 19 (d) 25.5

12. What is the value of median of the following set of observations: 10, 15, 12, 8, 22?

(a) 10 (b) 15
(c) 12 (d) 11

13. What is the value of mode of the following set of observations: 5, 8, 8, 16, 24, 24, 30, 16, 24?

(a) 24 (b) 16
(c) 8 (d) None of these

14. Calculate the median of the following grouped data:

Marks	Number of Students
0–10	2
10–20	12
20–30	22
30–40	8
40–50	6

(a) 25 (b) 24.545
(c) 28.75 (d) 22.225

15. What is the geometric mean of 4, 9, 9 and 2?

(a) 4.092 (b) 7.655
(c) 6.245 (d) 5.045

16. Find the geometric mean of the following grouped data:

x	f
60–80	22
80–100	38
100–120	45
120–140	35
140–160	20

(a) 101.98 (b) 106.23
(c) 105.77 (d) 102.25

17. The marks obtained by some students of a class are given in the following table. Calculate the harmonic mean.

Marks (x)	20	21	22	23	24	25
Number of students (f)	4	2	7	1	3	1

(a) 17.71 (b) 19.27
(c) 20.88 (d) 21.91

18. Find the mean deviation from the mean of the following data: 6, 7, 10, 12, 13, 4, 8, 20.

(a) 3.75 (b) 4.25
(c) 5.65 (d) 3.05

19. Find the mean deviation of the following data:

Class	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Frequency	2	3	8	14	8	3	2

(a) 12 (b) 16
(c) 10 (d) 8

20. Find the standard deviation of the following data: 65, 68, 58, 44, 48, 45, 60, 62, 60, 50.

(a) 7.11 (b) 6.67
(c) 66.2 (d) 8.13

21. Calculate the variance from the data given below:

x_i	3.5	4.5	5.5	6.5	7.5	8.5	9.5
f_i	3	7	22	60	85	32	8

(a) 2.651 (b) 1.321
(c) 3.225 (d) 1.766

22. Calculate standard deviation for the following distribution:

Interval	20–30	30–40	40–50	50–60	60–70	70–80	80–90
Frequency	3	6	13	15	14	5	4

(a) 17.77 (b) 22.29
(c) 14.94 (d) 55.33

23. Find the mode of the following data:

Interval	5–10	10–15	15–20	20–25	25–30
Frequency	3	5	7	2	4

(a) 16.43 (b) 15
(c) 17.77 (d) 15.57

24. A random variable x has the following probability distribution:

x_i	−2	−1	0	1	2	3
p_i	0.1	k	0.2	$2k$	0.3	k

What is the value of k?

(a) 0.4 (b) 0.3
(c) 0.2 (d) 0.1

25. What is the variance of the number of heads in a simultaneous toss of three coins?

(a) 3/2 (b) 1/2
(c) 3/5 (d) 3/4

26. In an office, 70% members are male and 30% are female. A member is selected at random. Let $x = 0$ if the member is female and $x = 1$ if the member is male. What is the value of $E(X)$?

(a) 0.1 (b) 0.49
(c) 0.3 (d) 0.7

27. A company sells an A.C. which fails at a rate 1 out of 1000. If 500 A.C.s are purchased from this company, what is the probability of two of them failing within first year?

(a) 0.1172 (b) 0.0117
(c) 0.0758 (d) 0.7582

28. A dice is thrown 6 times. If “getting an odd number” is a “success”, what is the probability of at least 5 successes?

(a) $\dfrac{1}{64}$ (b) $\dfrac{7}{64}$

(c) $\dfrac{3}{32}$ (d) $\dfrac{63}{64}$

29. If X is uniformly distributed over (0, 10), calculate the probability $P[3 < X < 8]$.

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$
(c) $\frac{4}{5}$ (d) $\frac{3}{10}$

30. Suppose that the length of a call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at the booth, what is the probability that you have to wait more than 10 minutes?

(a) 0.6991 (b) 0.7102
(c) 0.2333 (d) 0.3679

31. In the case of hypothesis testing, one of the following statements is always true:

(a) The P-value is a probability.
(b) The P-value is always less than the chosen significance level.
(c) The P-value is always greater than the chosen significance level.
(d) The P-value is nothing but a test statistic.

32. In the case of hypothesis testing, null hypothesis is rejected when

(a) P-value associated with the test statistic is equal to the significance level
(b) P-value associated with the test statistic is greater than the significance level
(c) P-value associated with the test statistic is smaller than the significance level
(d) None of these

33. In the case of hypothesis testing of proportions, an appropriate test method is

(a) one sample z-test
(b) one sample t-test
(c) two-proportion z-test
(d) two-sample t-test

34. In the case of hypothesis testing of difference between proportions, an appropriate test method would be

(a) one-sample z-test
(b) two proportion z-test
(c) one sample t-test
(d) two sample t-test

35. While doing hypothesis testing of means, the null hypothesis was stated as H_0: $\mu = M$ and alternate hypothesis was stated as H_a: $\mu \neq M$; μ is the population mean and M is the specified value of mean. It is an example of

(a) one-tail test
(b) two-tail test
(c) either of one-tail or two-tail test
(d) None of these

36. In hypothesis testing of means, appropriate test method is

(a) one-sample z-test
(b) one-sample t-test
(c) two-sample t-test
(d) two-sample z-test

37. While carrying out hypothesis testing, the P-value of the test statistic was computed to be equal to 0.03. The significance level is so chosen as to have a 4% risk of making a wrong conclusion. In this case,

(a) null hypothesis can be rejected.
(b) null hypothesis cannot be rejected.
(c) given data has no relevance to rejection or acceptance of null hypothesis.
(d) None of these.

38. In the case of hypothesis testing of 'difference between means', the P-value associated with the test statistic, which is a t-score test, can be computed using

(a) normal distribution calculator
(b) t-distribution calculator
(c) either normal or t-distribution calculator can be used
(d) None of these

39. Normal distribution calculator is used for computing the P-value associated with

(a) hypothesis testing of means
(b) hypothesis testing of difference between means
(c) hypothesis testing of difference between proportions
(d) None of these

40. One of the following is a correct set of null and alternate hypothesis with reference to hypothesis testing

(a) H_0: $\mu \leq M$, H_a: $\mu > M$
(b) H_0: $\mu < M$, H_a: $\mu > M$
(c) H_0: $\mu = M$, H_a: $\mu < M$
(d) H_0: $\mu = M$, H_a: $\mu > M$

Here, μ is the population mean and M is the specified mean.

41. Mark the correct expression of Bayes' theorem.

(a) $$P\left(\frac{A_i}{B}\right) = \frac{P(A_i \cap B)}{P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)}$$

(b) $P\left(\frac{A_i}{B}\right) = \frac{P(B_i)P(A/B_i)}{\sum P(B_i)P(A/B_i)}$

(c) $P\left(\frac{A_i}{B}\right)$

$$= \frac{P(A_i \cap B)}{P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)}$$

(d) None of these

42. There are two bags, each containing white and black balls. One bag contains three times as many white balls as it contains black balls. The other bag contains three times as many black balls as it contains white balls. One of these bags is randomly chosen and from this bag, we select five balls at random; replacing each ball that has been selected, the result is that we find four white balls and one black ball. What is the probability that the chosen bag was the one with mainly white balls?

(a) 0.964 (b) 0.847
(c) 0.758 (d) 0.469

ANSWERS

1. (a)	**8.** (c)	**15.** (d)	**22.** (c)	**29.** (a)	**36.** (b)
2. (c)	**9.** (d)	**16.** (b)	**23.** (a)	**30.** (d)	**37.** (a)
3. (b)	**10.** (a)	**17.** (d)	**24.** (d)	**31.** (a)	**38.** (b)
4. (a)	**11.** (a)	**18.** (a)	**25.** (d)	**32.** (c)	**39.** (c)
5. (d)	**12.** (c)	**19.** (c)	**26.** (d)	**33.** (a)	**40.** (a)
6. (d)	**13.** (a)	**20.** (d)	**27.** (c)	**34.** (b)	**41.** (a)
7. (b)	**14.** (b)	**21.** (b)	**28.** (b)	**35.** (b)	**42.** (a)

EXPLANATIONS AND HINTS

1. (a) We know that

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ = words starting with K = 3!

$n(S)$ = total number of words = 4!

Hence, $P(E) = \frac{n(E)}{n(S)} = \frac{3!}{4!} = \frac{1}{4}$

2. (c) We have

$$S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

Now, A = the sum is even

$$= \begin{Bmatrix} (1,1), (2,2), (1,3), (1,5), (2,4), (2,6), (3,1), (3,3), (3,5), \\ (4,2), (4,4), (4,6), (5,1), (5,3), (5,5), (6,2), (6,4), (6,6) \end{Bmatrix}$$

B = the sum is a multiple of 3

$$= \begin{Bmatrix} (1,2), (2,1), (1,5), (5,1), (2,4), (4,2), \\ (3,3), (3,6), (6,3), (4,5), (5,4), (6,6) \end{Bmatrix}$$

C = the sum is less than 4

$$= \{(1,1), (1,2), (2,1)\}$$

D = the sum is greater than 11

$$= \{(6,6)\}$$

We observe that

$$A \cap B = \{(1,5), (2,4), (3,3), (4,2), (5,1), (6,6)\} \neq \phi$$

Thus, A and B are not mutually exclusive events.

$$B \cap C = \{(1,2), (2,1)\} \neq \phi$$

Thus, B and C are not mutually exclusive events.

$$C \cap D \neq \phi$$

Also,

$$A \cap D = \{(6,6)\} \neq \phi$$

Hence, C and D are mutually exclusive events.

3. (b) Sample space (S) associated with the random experiment of tossing three coins is given by

$$S = \begin{Bmatrix} \text{HHH, HHT, HTH, HTT,} \\ \text{THH, THT, TTH, TTT} \end{Bmatrix}$$

There are 8 elements in S. Hence, total outcomes = 8.

Now, E be events that we get exactly 2 tails,

$$E = \{\text{HTT, THT, TTH}\}$$

Favorable number of outcomes = 3

Hence, required probability $= \dfrac{3}{8}$

4. (a) We have

$$S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

Total outcomes = 36

Let A be the event of getting a total of at least 10.

Then, $A = \{(6,4), (4,6), (5,5), (6,5), (5,6), (6,6)\}$

$\therefore$ Favorable outcome = 6

Hence, required probability $= \dfrac{6}{36} = \dfrac{1}{6}$

5. (d) Four cards can be drawn from a deck of 52 cards in ${}^{52}C_4$ ways.

Hence, total outcomes $= {}^{52}C_4$

There are 12 face cards. Hence, 4 can be selected in ${}^{12}C_4$ ways.

Hence, favorable outcome $= {}^{12}C_4$

$$\therefore \text{ Required probability} = \frac{{}^{12}C_4}{{}^{52}C_4} = \frac{\dfrac{12!}{4!\,8!}}{\dfrac{52!}{4!\,8!}}$$

$$= \frac{12 \times 11 \times 10 \times 9}{52 \times 51 \times 50 \times 49} = \frac{11880}{6497400} = \frac{99}{54145}$$

6. (d) The probability that A can solve the question is $\dfrac{1}{2}$.

The probability that A cannot solve the question is $1 - \dfrac{1}{2} = \dfrac{1}{2}$.

Similarly, the probabilities that B and C cannot solve the question are

$$1 - \frac{1}{3} = \frac{2}{3} \quad \text{and} \quad 1 - \frac{1}{4} = \frac{3}{4}$$

Probabilities that A, B and C cannot solve the question $= \left(\dfrac{1}{2}\right)\left(\dfrac{2}{3}\right)\left(\dfrac{3}{4}\right) = \dfrac{1}{4}$.

Hence, the probability that at least one student will solve the question $= 1 - \dfrac{1}{4} = \dfrac{3}{4}$.

7. (b) We have

$$P(A \cap B) = P(A)\,P\left(\frac{B}{A}\right) = 0.3 \times 0.5 = 0.15$$

$$\therefore\ P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} \Rightarrow P\left(\frac{A}{B}\right) = \frac{0.15}{0.6} = \frac{1}{4}$$

Now,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cup B) = 0.3 + 0.6 - 0.15 = 0.75$$

8. (c) Let A be the event of drawing an even-numbered ticket in first draw and B be the event of drawing an even-numbered ticket in the second draw.

Then, required probability $= P(A \cap B) = P(A) \cdot P\left(\dfrac{B}{A}\right)$

19 tickets are numbered 1 to 19, and 9 are even numbered. Therefore,

$$P(A) = \frac{9}{19}$$

As the ticket drawn in the first draw is not replaced, the second ticket drawn is from remaining 18 tickets, out of which 8 are even numbered. Therefore,

$$P\left(\frac{B}{A}\right) = \frac{8}{18} = \frac{4}{9}$$

Hence, required probability $= P(A \cap B)$

$$= P(A) \cdot P\left(\frac{B}{A}\right)$$

$$= \frac{9}{14} \times \frac{4}{9} = \frac{4}{19}$$

9. (d) The range of a distribution is given by the difference between the maximum value and the minimum value.

Maximum observation = 50

Minimum observation = 15

Range = 35

10. (a) Let us reconstruct the following table:

x_i	f_i	$x_i f_i$
5	2	10
10	1	10
15	5	75
20	7	140
25	5	125
	$\sum f_i = 20$	$\sum f_i x_i = 360$

Arithmetic mean $(\bar{x}) = \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{360}{20} = 18$

11. (a) Number of observations = 10

Sum of all observations = 28 + 32 + 15 + 17 + 11 + 48 + 43 + 5 + 7 + 14 = 220

$$\text{Mean} = \frac{\sum x_i}{n} = \frac{220}{10} = 22$$

12. (c) We first arrange all the elements in ascending order:

8, 10, 12, 15, 22

As we have odd numbers of observations, median

$$= \left(\frac{n+1}{2}\right)^{\text{th}} \text{ term}$$

$$= \left(\frac{5+1}{2}\right)^{\text{th}} = 3^{\text{rd}} \text{ term}$$

Hence, median = 12

13. (a) Mode is the value which is repeated most number of times in the set of observations.

Hence, the mode is 24 because it occurs most number of times, i.e. 3.

14. (b) Let us construct the table as follows:

Marks (x_i)	f_i	Cumulative Frequency
0–10	2	2
10–20	12	14
20–30	22	36
30–40	8	44
40–50	6	50

Here, $\dfrac{n}{2} = \dfrac{50}{2} = 25$. Hence, 20–30 is the median class.

$$\text{Median} = 20 + \left[\frac{25-15}{22}\right] \times 10$$

$$= 20 + \frac{100}{22} = 24.545$$

15. (d) Geometric mean $= (x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{\frac{1}{2}}$

$$= (4 \times 9 \times 9 \times 2)^{\frac{1}{4}} = \sqrt[4]{4 \times 9 \times 9 \times 2}$$

$$= \sqrt[4]{648} = 5.04$$

16. (b) Let us construct the table as follows:

x	f	Mid-value	log x	$f \cdot \log x$
60–80	22	70	1.845	40.59
80–100	38	90	1.954	74.25
100–120	45	110	2,041	91.085
120–140	35	130	2,114	73,99
140–160	20	150	2,176	43.52
	$\sum f = 160$			$\sum f \cdot \log x = 324.2$

We have $N = 160$

Geometric mean = Antilog $\left[\dfrac{\sum f \log x}{N}\right]$

$$= \text{Antilog}\left[\frac{324.2}{160}\right] = \text{Antilog}[2.02625] = 106.23$$

17. (d) Let us construct the table as follows:

x	f	$\dfrac{1}{x}$	$f\left(\dfrac{1}{x}\right)$
20	4	0.0500	0.2000
21	2	0.0476	0.0952
22	7	0.0454	0.3178
23	1	0.0435	0.0435
24	3	0.0417	0.1251
25	1	0.0400	0.0400
	$\sum f = 18$		$\sum f\left(\dfrac{1}{x}\right) = 0.8216$

Now, we know that

$$\text{Harmonic mean} = \frac{N}{\sum f\left(\frac{1}{x}\right)} = \frac{18}{0.1968} = 21.91$$

18. (a) Let $\bar{x}$ be the mean of the given data. Then,

$$\bar{x} = \frac{6+7+10+12+13+4+8+20}{8} = \frac{80}{8} = 10$$

Let us construct the table as follows:

x_i	$\lvert x_i - \bar{x} \rvert$
6	4
7	3
10	0

(*Continued*)

Continued

x_i	$\lvert x_i - \bar{x} \rvert$
12	2
13	3
4	6
8	2
20	10
	$\sum\lvert x_i - \bar{x}\rvert = 30$

We have $\sum\lvert x_i - \bar{x}\rvert = 30$ and $n = 8$

$$\text{Mean deviation} = \frac{1}{n}\sum\lvert x_i - \bar{x}\rvert = \frac{30}{8} = 3.75$$

19. (c) We construct the table as follows:

Classes	Mid-values	f_i	$f_i x_i$	$\lvert x_i - \bar{x}\rvert = \lvert x_i - 45\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
10–20	15	2	30	30	60
20–30	25	3	75	20	60
30–40	35	8	280	10	80
40–50	45	14	630	0	0
50–60	55	8	440	10	80
60–70	65	3	195	20	60
70–80	75	2	150	30	60
		$\sum f_i = 40$	$\sum f_i x_i = 1800$		$\sum f_i \lvert x_i - \bar{x}\rvert = 400$

Now, from the table, we have $N = 40$ and $\sum f_i \lvert x_i - \bar{x}\rvert = 400$

$$\text{Mean deviation} = \frac{1}{N}\sum f_i\lvert x_i - \bar{X}\rvert = \frac{400}{40} = 10$$

20. (d) Suppose $\bar{x}$ is the mean of the given set of observations. Then

$$\bar{x} = \frac{65+68+58+44+48+45+60+62+60+50}{10}$$

$$\Rightarrow \bar{x} = \frac{560}{10} = 56$$

Let us construct the table as follows:

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
65	9	81
68	2	4

(*Continued*)

Continued

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
68	12	144
44	−12	144
48	−8	64
45	−11	121
60	4	16
62	6	36
60	4	16
50	−6	36
		$\sum(x_i - \bar{x})^2 = 662$

Now, we have $n = 10$ and

$$\sum(x_i - \bar{x})^2 = 662$$

Standard deviation,

$$\sigma = \sqrt{\frac{1}{n}\sum\left(x_i - \bar{x}\right)^2} = \sqrt{\frac{662}{10}} = \sqrt{66.2} = 8.13$$

21. (b) Let assumed mean = 6.5

We construct the table as follows:

x_i	f_i	$d_i = x_i - 6.5$	d_i^2	$f_i d_i$	$f_i d_i^2$
3.5	3	−3	9	−9	27
4.5	7	−2	4	−14	28
5.5	22	−1	1	−22	22
6.5	60	0	0	0	0
7.5	85	1	1	85	85
8.5	32	2	4	64	128
9.5					

Hence, from the table, we have

$$N = \sum f_i = 217,\ \sum f_i d_i = 128 \text{ and } \sum f_i d_{ii}^2 = 362$$

$$\text{Var}(X) = \left[\frac{1}{N}\sum f_i d_i^2\right] - \left[\frac{1}{N}\sum f_i d_i^2\right]^2$$

$$= \frac{362}{217} - \left(\frac{128}{217}\right)^2 = 1.668 - 0.347 = 1.321$$

22. (c) Let us construct the table as follows:

Interval	Frequency, f_i	Mid-values, x_i	$u_i = \dfrac{x_i - 55}{h}$	$f_i u_i$	u_i^2	$f_i u_i^2$
20–30	3	25	−3	−9	9	27
30–40	6	35	−2	−12	4	24
40–50	13	45	−1	−13	1	13
50–60	15	55	0	0	0	0
60–70	14	65	1	14	1	14
70–80	5	75	2	10	4	20
80–90	4	85	3	12	9	36
	$N = \sum f_i = 60$			$\sum f_i u_i = 2$		$\sum f_i u_i^2 = 134$

From the table, we have

$$N = 60, \sum f_i u_i = 2 \quad \text{and} \quad \sum f_i u_i^2 = 134$$

$$\text{Var}(x) = h^2\left[\frac{1}{N}\sum f_i u_i^2 - \left(\frac{1}{N}\sum f_i u_i^2\right)\right] = 100\left[\frac{134}{60} - \left(\frac{2}{60}\right)^2\right] = 222.9$$

Standard deviation $= \sqrt{222.9} = 14.94$

23. (a) Frequency of class interval 15–20 is maximum. Hence, it is the modal class.

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

Now, we have

$$l = 15, f_1 = 7, f_0 = 5, f_2 = 2 \text{ and } h = 5$$

$$\text{Mode} = 15 + \left(\frac{7-5}{2\times 7 - 5 - 2}\right)\times 5$$

$$= 15 + \left[\frac{2}{7}\right]\times 5$$

$$= 15 + \frac{10}{7}$$

$$= 15 + 1.43$$

$$= 16.43$$

24. (d) The sum of probabilities in a frequency distribution is always unity.

Therefore, $\quad 0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$$\Rightarrow 0.6 + 4k = 1$$

$$\Rightarrow 4k = 0.4$$

$$\Rightarrow k = 0.1$$

25. (d) Suppose X is the number of heads in a simultaneous toss of three coins.

Hence, X can take value 0, 1, 2, 3.

Now, $\quad P(x = 0) = P(\text{TTT}) = \dfrac{1}{8}$

$$P(x = 1) = P(\text{HTT}, \text{THT}, \text{TTH}) = \frac{3}{8}$$

$$P(x = 2) = P(\text{HHT}, \text{HTH}, \text{THH}) = \frac{3}{8}$$

and $\quad P(x = 3) = P(\text{HHH}) = \dfrac{1}{8}$

Hence, the probability distribution of X is given by

$$\begin{array}{rcccc} X: & 0 & 1 & 2 & 3 \\ p(X): & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

Now, let us construct the table as follows:

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	1/8	0	0
1	3/8	3/8	3/8
2	3/8	6/8	12/8
3	1/8	3/8	9/8
		$\sum p_i x_i = \dfrac{3}{8}$	$\sum p_i x_i^2 = 3$

From the table, we have

$$\sum p_i x_i = \frac{3}{2} \quad \text{and} \quad \sum p_i x_i^2 = 3$$

Now, $\text{Var}(x) = \sum p_i x_i^2 - \left(\sum p_i x_i\right)^2 = 3 - \left(\dfrac{3}{2}\right)^2 = \dfrac{3}{4}$

26. (d) The probability distribution of X is

X	0	1
$P(x)$	$\frac{3}{10}$	$\frac{7}{10}$

$$\therefore E(x) = \frac{3}{10} \times 0 + \frac{7}{10} \times 1 = \frac{7}{10} = 0.7$$

27. (c) For Poisson distribution,

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\lambda = np = 500 \times \frac{1}{1000} = \frac{1}{2}$$

$$P(x = 2) = \frac{e^{-1/2} (1/2)^2}{2!} = 0.0758$$

28. (b) Let p denote the probability of getting an odd number in a single throw of dice. Then,

$$p = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow q = 1 - p = \frac{1}{2}$$

Let X denotes the number of successes in 6 trials. Then, X is a binomial variable with parameter $n = 6$, $p = 1/2$.

The probability of r successes is given by

$$P(x = r) = {}^6\mathrm{C}_r \left(\frac{1}{2}\right)^{6-r} \left(\frac{1}{2}\right)^r, \qquad r = 0, 1, 2, \ldots, 6$$

$$\Rightarrow P(x = r) = {}^6\mathrm{C}_r \left(\frac{1}{2}\right)^6, \qquad r = 0, 1, 2, \ldots, 6$$

Probability of at least 5 successes $= P(X \geq 5)$

$$= P(x = 5) + P(x = 6)$$

$$= {}^6\mathrm{C}_5 \left(\frac{1}{2}\right)^6 + {}^6\mathrm{C}_6 \left(\frac{1}{2}\right)^6$$

$$= \frac{(6+1)}{64} = \frac{7}{64}$$

29. (a) For uniform distribution,

$$f(x) = \frac{1}{10 - 0} = \frac{1}{10}$$

$$P[3 < X < 8] = \int_3^8 \frac{1}{10} dx = \left[\frac{x}{10}\right]_3^8 = \frac{8}{10} - \frac{3}{10}$$

$$= \frac{5}{10} = \frac{1}{2}$$

30. (d) Let X denotes the length of the call made by the person in the booth. Therefore,

$$P[X > 10] = 1 - P[X < 10]$$

$$= 1 - \left(1 - e^{-\lambda \times 10}\right)$$

$$= e^{-10\lambda}$$

Now, $\lambda = \frac{1}{10}$

$$P[X > 10] = e^{-1} = 0.3679$$

31. (a) The P-value is the probability associated with the test statistic. It can either be smaller or greater than the significance level. It is not a test statistic.

32. (c) The solution is obvious if we examine the definitions of P-value and the significance level. The significance level is the probability of rejecting the null hypothesis when it is true. A significance level of 0.05 indicates a 5% risk, concluding that a difference exists when there is no actual difference. P-value is the probability associated with the test statistic and is a function of the observed sample results.

33. (a) The correct answer is one sample z-test.

34. (b) The correct answer is two proportion z-test.

35. (b) If $\mu \neq M$, there are two possibilities–either $\mu < M$ or $\mu > M$.

36. (b) The correct answer is one sample t-test.

37. (a) Null hypothesis is rejected when P-value is less than significance level. Significance level in the present case is 0.04.

38. (b) The correct answer is t-distribution calculator.

39. (c) Normal distribution calculator is used for computing the P-value associated with hypothesis testing of difference between proportions.

40. (a) Null and alternate hypotheses need to be mutually exclusive, that is, if one were true, then the other would be false and vice versa.

41. (a) The correct expression of Bayes' theorem is

$$P\left(\frac{A_i}{B}\right) = \frac{P(A_i \cap B)}{P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)}$$

42. (a) Let A_1 represent the bag with mostly white balls and A_2 represent the bag with mostly black balls. Let B represent the event four white balls and one black ball chosen from five selections. The objective is to find $P(A_1/B)$.

From Bayes' theorem,

$$P\left(\frac{A_1}{B}\right) = \frac{P(A_1)\cdot P(B/A_1)}{P(A_1)\cdot P(B/A_1) + P(A_2)\cdot P(B/A_2)}$$

Now $P(A_1) = P(A_2) = \frac{1}{2}$

$$P\left(\frac{B}{A_1}\right) = {}^5C_1\left(\frac{3}{4}\right)^4 \times \left(\frac{1}{4}\right)^1 = \frac{405}{1024}$$

$$P\left(\frac{B}{A_2}\right) = {}^5C_1\left(\frac{1}{4}\right)^4 \times \left(\frac{3}{4}\right)^1 = \frac{15}{1024}$$

Substituting these values, we get

$$P\left(\frac{A_1}{B}\right) = \frac{(405/1024)\times(1/2)}{[1/2][(405/1024)+(15/1024)]}$$

$$= \frac{405}{420} = 0.964$$

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. Let $P(E)$ denotes the probability of the event E. Given $P(A) = 1$, $P(B) = 1/2$, the values of $P(A/B)$ and $P(B/A)$, respectively, are

(a) 1/4, 1/2 (b) 1/2, 1/4

(c) 1/2, 1 (d) 1, 1/2

(GATE 2003, 1 Mark)

Solution: We are given,

$$P(A) = 1 \text{ and } P(B) = 1/2$$

Both events are independent.

So,

$P(A \cap B) = P(A) \cdot P(B) = 1 - 1/2 = 1/2$

$$P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$P(B \mid A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

Ans. (d)

2. A box contains 10 screws, 3 of which are defective. Two screws are drawn at random with replacement. The probability that none of the two screws is defective will be

(a) 100% (b) 50%

(c) 49% (d) None of these

(GATE 2003, 1 Mark)

Solution: Here, $n = 2$

$$x = 0 \text{ (no defective)}$$

$$P = P(\text{defective}) = 3/10$$

So,

$$P(x=0) = {}^2C_0\left(\frac{3}{10}\right)^0\left(1-\frac{3}{10}\right)^2 = 0.49 = 49\%$$

Ans. (c)

3. A box contains 5 black and 5 red balls. Two balls are randomly picked one after another from the box, without replacement. The probability for both balls being red is

(a) $\frac{1}{90}$ (b) $\frac{1}{2}$

(c) $\frac{19}{90}$ (d) $\frac{2}{9}$

(GATE 2003, 2 Marks)

Solution: Probability of drawing two red balls

$= P(\text{first is red})$
$\quad \times P(\text{second is red given that first is red})$

$$= \frac{5}{10}\times\frac{4}{9} = \frac{2}{9}$$

Ans. (d)

4. A program consists of two modules executed sequentially. Let $f_1(t)$ and $f_2(t)$, respectively, denote the probability density functions of time taken to execute the two modules. The probability density function of the overall time taken to execute the program is given by

(a) $f_1(t) + f_2(t)$ (b) $\int_0^t f_1(x)f_2(x)dx$

(c) $\int_0^1 f_1(x)f_2(t-x)dx$ (d) $\max\{f_1(t), f_2(t)\}$

(GATE 2003, 2 Marks)

Solution: Let the time taken for first and second modules be represented by x and y and total time $= t$.
$\therefore\ t = x + y$ is a random variable.

Now the joint density function,

$$g(t) = \int_0^t f(x,y)\,dx = \int_0^t f(x,t-x)\,dx$$

$$= \int_0^t f_1(x)f_2(t-x)\,dx$$

which is also called as convolution of f_1 and f_2, abbreviated as $f_1 * f_2$.

Therefore, the correct answer is option (c).

Ans. (c)

5. If a fair coin is tossed four times, then what is the probability that two heads and two tails will result?

(a) 3/8 (b) 1/2
(c) 5/8 (d) 3/4

(GATE 2004, 1 Mark)

Solution: Coin is tossed four times. Hence, the total outcomes $= 2^4 = 16$.

Favorable outcomes (the condition getting 2 heads and 2 tails) = {HHTT, HTHT, HTTH, TTHH, THTH, THHT} = 6

Hence, the required probability $= \dfrac{6}{16} = \dfrac{3}{8}$

Ans. (a)

6. A hydraulic structure has four gates which operate independently. The probability of failure of each gate is 0.2. Given that gate 1 has failed, the probability that both gates 2 and 3 will fail is

(a) 0.240 (b) 0.200
(c) 0.040 (d) 0.008

(GATE 2004, 2 Marks)

Solution: As all three gates are independent probability (gate 2 and gate 3 fail | gate 1 failed)

$$= P(\text{gate 2 and gate 3 fail})$$
$$= P(\text{gate 2}) \times P(\text{gate 3})$$
$$= 0.2 \times 0.2 = 0.04$$

Ans. (c)

7. An examination paper has 150 multiple choice questions of one mark each, with each question having four choices. Each incorrect answer fetches –0.25 marks. Suppose 1000 students choose all their answers randomly with uniform probability. The total sum of the expected marks obtained by all these students is

(a) 0 (b) 2550
(c) 7525 (d) 9375

(GATE 2004, 2 Marks)

Solution: Let the marks obtained per question be a random variable X.

Its probability distribution table is given as follows:

X	1	–0.25
$P(X)$	1/4	3/4

Expected marks per question

$$= E(x)$$
$$= \sum X\, P(X)$$
$$= 1 \times \frac{1}{4} + (-0.25) \times \frac{3}{4}$$
$$= \frac{1}{4} - \frac{3}{16}$$
$$= \frac{1}{16} \text{ marks}$$

Total marks expected for 150 questions $= \dfrac{1}{16} \times 150 = \dfrac{17}{8}$ marks per student.

Total expected marks of 1000 students $= \dfrac{17}{8} \times 1000 = 9375$ marks.

Ans. (d)

8. Two n bit binary strings, S_1 and S_2 are chosen randomly with uniform probability. The probability that the hamming distance between these strings (the number of bit positions where the two strings differ) is equal to d is

(a) ${}^n\mathrm{C}_d/2^2$ (b) ${}^n\mathrm{C}_d/2^d$
(c) $d/2^n$ (d) $1/2^d$

(GATE 2004, 2 Marks)

Solution: If hamming distance between two n bit strings is d, we are asking d out of n trials to be a success (success here means that the bits are different). So, this is a binomial distribution with n trials and d successes and the probability of success is

$$P = 2/4 = 1/2$$

(Because out of the four possibilities {(0, 0), (0, 1), (1, 0), (1, 1)}, only two, (0, 1) and (1, 0), are success.)

So,

$$P(X = d) = {}^n\mathrm{C}_d (1/2)^d (1/2)^{n-d} = \frac{{}^n\mathrm{C}_d}{2^n}$$

Ans. (a)

9. From a pack of regular playing cards, two cards are drawn at random. What is the probability that both cards will be Kings, if first card is not replaced?

(a) $\frac{1}{5}$ (b) $\frac{1}{52}$

(c) $\frac{1}{169}$ (d) $\frac{1}{221}$

(GATE 2004, 2 Marks)

Solution: Problems can be solved by hypergeometric distribution as follows:

$$P(X=2)=\frac{{}^4C_2\times{}^{48}C_0}{{}^{52}C_2}=\frac{1}{221}$$

Ans. (d)

10. A point is randomly selected with uniform probability in the $x-y$ plane within the rectangle with corners at (0, 0), (1, 0), (1, 2) and (0, 2). If p is the length of the position vector of the point, the expected value of p^2 is

(a) 2/3 (b) 1

(c) 4/3 (d) 5/3

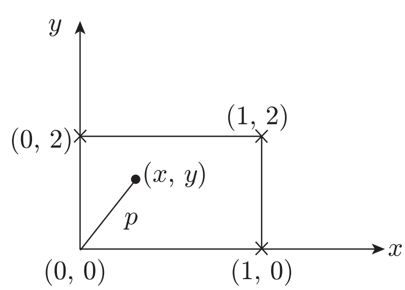
(GATE 2004, 2 Marks)

Solution: Length of position vector of point,

$$p=\sqrt{x^2+y^2}$$

$$p^2=x^2+y^2$$

$$E(p^2)=E(x^2+y^2)=E(x^2)+E(y^2)$$



Now x and y are uniformly distributed $0\le x\le 1$ and $0\le y\le 2$.

Probability density function of $x=\frac{1}{1-0}=1$

Probability density function of $y=\frac{1}{2-0}=1/2$

$$E(x^2)=\int_0^1 x^2p(x)dx=\int_0^1 x^2\cdot 1\cdot dx=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}$$

$$E(y^2)=\int_0^2 y^2p(y)dy=\int_0^2 y^2\cdot 1/2\cdot dy=\left[\frac{y^3}{6}\right]_0^2$$

$$=\frac{8}{6}=\frac{4}{3}$$

$$\therefore E(p^2)=E(x^2)+E(y^2)$$

$$=\frac{1}{3}+\frac{4}{3}=\frac{5}{3}$$

Ans. (d)

11. Let $f(x)$ be the continuous probability density function of a random variable X. The probability that $a<X\le b$ is

(a) $f(b-a)$ (b) $f(b)-f(a)$

(c) $\int_a^b f(x)dx$ (d) $\int_a^b xf(x)dx$

(GATE 2005, 1 Mark)

Solution: If $f(x)$ is the continuous probability density function of a random variable X, then

$$P(a<x\le b)=P(a\le x\le b)=\int_a^b f(x)dx$$

In general, we say that X is a uniform random variable on the interval (a, b) if its probability density function is given by

$$f(x)=\begin{cases}\dfrac{1}{\beta-\alpha} & \text{if } \alpha<x<\beta\\ 0 & \text{otherwise}\end{cases}$$

As $f(x)$ is a constant, all values of x between α and β are equally likely (uniform).

Ans. (c)

12. If P and Q are two random events, then the following is TRUE:

(a) Independence of P and Q implies that probability $(P\cap Q)=0$

(b) Probability $(P\cup Q)\ge$ Probability (P) + Probability (Q)

(c) If P and Q are mutually exclusive, then they must be independent

(d) Probability $(P\cap Q)\le$ Probability (P)

(GATE 2005, 1 Mark)

Solution: Option (a) is false because P and Q are independent

$$p(P\cap Q)=p(P)*p(Q)$$

which need not be zero.

Option (b) is false because

$$p(P\cup Q)=p(P)+p(Q)-p(P\cap Q)$$

$$\therefore p(P\cup Q)\le p(P)+p(Q)$$

Option (c) is false because independence and mutually exclusion are unrelated properties.

Now, as $P \cap Q \subseteq P$

$$\Rightarrow n(P \cap Q) \le n(P)$$
$$\Rightarrow pr(P \cap Q) \le pr(P)$$

Hence, option (d) is true.

Ans. (d)

13. A fair dice is rolled twice. The probability that an odd number will follow an even number is

(a) 1/2 (b) 1/6
(c) 1/3 (d) 1/4

(GATE 2005, 1 Mark)

Solution: Probability of getting an odd number $P_o = 3/6 = 1/2$

Probability of getting an even number $P_e = 3/6 = 1/2$

As both events are independent of each other,

$$P_{(o/e)} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Ans. (d)

14. Which one of the following statements is NOT true?

(a) The measure of skewness is dependent upon the amount of dispersion
(b) In a symmetric distribution, the values of mean, mode and median are the same
(c) In a positively skewed distribution: mean > median > mode
(d) In a negatively skewed distribution: mode > mean > median

(GATE 2005, 1 Mark)

Solution: Options (a), (b), (c) are true but option (d) is not true because in a negatively skewed distribution, mode > median > mean.

Ans. (d)

15. A lot has 10% defective items. Ten items are chosen randomly from this lot. The probability that exactly two of the chosen items are defective is

(a) 0.0036 (b) 0.1937
(c) 0.2234 (d) 0.3874

(GATE 2005, 1 Mark)

Solution: This problem can be done using binomial distribution because population is infinite.
Probability of defective item, $p = 0.1$

Probability of non-defective item, $q = 1 - p = 1 - 0.1 = 0.9$

Probability that exactly two of the chosen items are defective

$$= {}^{10}C_2 (p)^2 (q)^8 = {}^{10}C_2 (0.1)^2 (0.9)^8 = 0.1937$$

Ans. (b)

16. A single die is thrown twice. What is the probability that the sum is neither 8 nor 9?

(a) $\frac{1}{9}$ (b) $\frac{5}{36}$
(c) 1/4 (d) 3/4

(GATE 2005, 2 Marks)

Solution: Sample space $(S) = (6)^2 = 36$
Total ways in which sum is either 8 or 9 = (2, 6), (3, 5), (3, 6), (4, 4), (4, 5), (5, 3), (5, 4), (6, 2), (6, 3)

Hence, favorable outcomes = 27. Therefore,

Probability of sum neither being 8 nor 9 = 27/36 = 3/4

Ans. (d)

17. A fair coin is tossed three times in succession. If the first toss produces a head, then the probability of getting exactly two heads in three tosses is

(a) 1/8 (b) 1/2
(c) 3/8 (d) 3/4

(GATE 2005, 2 Marks)

Solution: We know that

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$\therefore$ P(2 heads in 3 tosses | first toss is head)

$$= \frac{P(\text{2 heads in 3 tosses and first toss is head})}{P(\text{first toss is head})}$$

P(first toss is head) = 1/2

P(2 head in 3 tosses and first toss is head) = $P(\text{HHT}) + P(\text{HTH}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

$$\therefore \qquad \text{Required probability} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Ans. (b)

18. The value of the integral $I = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{x^2}{8}\right) dx$ is

(a) 1 (b) π
(c) 2 (d) 2π

(GATE 2005, 2 Marks)

Solution: We have

$$I = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{\left(-x^2/8\right)}dx$$

$$\text{Comparing with } \frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} e^{\frac{(x-\mu)^2}{2\sigma^2}}\, dx = 1$$

We can provide μ and σ any value.

Here, putting $\mu = 0$

$$2\int_0^\infty \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} = 1$$

Given,

$$-\frac{x^2}{8} = -\frac{x^2}{2\sigma^2}$$

$\Rightarrow \sigma = 2,$

Now putting $\sigma = 2$ in the above equation, we get

$$\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{8}} = 1$$

Ans. (a)

19. A box contains 20 defective items and 80 non-defective items. If two items are selected at random without replacement, what will be the probability that both items are defective?

(a) $\frac{1}{5}$ (b) $\frac{1}{25}$
(c) $\frac{20}{99}$ (d) $\frac{19}{495}$

(GATE 2006, 1 Mark)

Solution: The problem can be solved by hyper-geometric distribution,

$$P(X=2) = \frac{{}^{20}C_2 \times {}^{80}C_2}{{}^{100}C_2} = \frac{19}{495}$$

Ans. (d)

20. A probability density function is of the form $p(x) = Ke^{-\alpha|x|}$, $x \in (-\infty, \infty)$. The value of K is

(a) 0.5 (b) 1
(c) 0.5α (d) α

(GATE 2006, 1 Mark)

Solution:

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} K\,e^{-\alpha|x|}\,dx = 1$$

$$\int_{-\infty}^{0} K\,e^{\alpha x}\,dx + \int_0^\infty K\,e^{-\alpha x}\,dx = 1$$

$$\Rightarrow \frac{K}{\alpha}\left(e^{\alpha x}\right)_{-\infty}^{0} + \frac{K}{-\alpha}\left(e^{-\alpha x}\right)_0^\infty = 1$$

$$\Rightarrow \frac{K}{\alpha} + \frac{K}{\alpha} = 1$$

$$2K = \alpha$$

$$\Rightarrow K = 0.5\alpha$$

Ans. (c)

21. Two fair dice are rolled and the sum r of the numbers turned up is considered:

(a) $P(r > 6) = (1/6)$
(b) $P(r/3$ is an integer$) = (5/6)$
(c) $P(r = 8 \mid r/4$ is an integer$) = (5/9)$
(d) $P(r = 6 \mid r/5$ is an integer$) = (1/18)$

(GATE 2006, 2 Marks)

Solution: If two fair dice are rolled, the probability distribution of r where r is the sum of the numbers on each dice is given by

γ	2	3	4	5	6	7	8	9	10	11	12
$P(\gamma)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The above table has been obtained by taking all different ways of obtaining a particular sum. For example, a sum of 5 can be obtained by (1, 4), (2, 3), (3, 2) and (4, 1).

$$P(x=5) = 4/36$$

Now let us consider option (a),

$$P(r > 6) = P(r > 7)$$
$$= \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{21}{36} = \frac{7}{12}$$

Therefore, option (a) is wrong.

Consider option (b),

$P(r/3$ is an integer$) = P(r=3) + P(r=6) + P(r{=}9) + P(r{=}12) = \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{12}{36} = \frac{1}{3}$

Therefore, option (b) is wrong.

Consider option (c),

$$P(r = 8 \mid r/4 \text{ is an integer}) = \frac{1}{36}$$

Now, $P(r/4 \text{ is an integer}) = P(r = 4) + P(r = 8) + P(r = 12) = \frac{3}{36} + \frac{5}{36} + \frac{1}{36} = \frac{9}{36} = \frac{1}{4}$

$P(r = 8 \text{ and } r/4 \text{ is an integer}) = P(r = 8) = 5/36$

$$P(r = 8 \mid r/4 \text{ is an integer}) = \frac{\frac{5}{36}}{\frac{1}{4}} = \frac{20}{36} = \frac{5}{9}$$

Hence, option (c) is correct.

Ans. (c)

22. Three companies X, Y and Z supply computers to a university. The percentage of computers supplied by them and the probability of those being defective are tabulated below:

Company	% of Computer	Probability of Being Defective
X	60%	0.01
Y	30%	0.02
Z	10%	0.03

Given that a computer is defective, the probability that it was supplied by Y is

(a) 0.1 (b) 0.2
(c) 0.3 (d) 0.4

(GATE 2006, 2 Marks)

Solution:
$S \rightarrow$ supply by Y, $d \rightarrow$ defective

Probability that the computer was supplied by Y, if the product is defective

$$P(S/d) = \frac{P(S \cap d)}{P(d)}$$

$$P(S \cap d) = 0.3 \times 0.02 = 0.006$$

$$P(d) = 0.6 \times 0.1 + 0.3 \times 0.02 + 0.1 \times 0.03 = 0.015$$

$$P(S/d) = \frac{0.006}{0.015} = 0.4$$

Ans. (d)

23. For each element in a set of size $2n$, an unbiased coin is tossed. The $2n$ coins tossed are independent. An element is chosen if the corresponding coin toss were head. The probability that exactly n elements are chosen is

(a) $\binom{2n}{n} \Big/ 4^n$ (b) $\binom{2n}{n} \Big/ 2^n$

(c) $1 \Big/ \binom{2n}{n}$ (d) $\frac{1}{2}$

(GATE 2006, 2 Marks)

Solution: The probability that exactly n elements are chosen is equal to the probability of getting n heads out of $2n$ tosses, that is

$$= {}^{2n}\mathrm{C}_n (1/2)^n (1/2)^{2n-n} \text{ (Binomial formula)}$$

$$= {}^{2n}\mathrm{C}_n (1/2)^n (1/2)^n$$

$$= \frac{{}^{2n}\mathrm{C}_n}{2^{2n}} = \frac{{}^{2n}\mathrm{C}_n}{\left(2^2\right)^n} = \frac{{}^{2n}\mathrm{C}_n}{4^n}$$

Ans. (a)

24. There are 25 calculators in a box. Two of them are defective. Suppose five calculators are randomly picked for inspection (i.e., each has the same chance of being selected). What is the probability that only one of the defective calculators will be included in the inspection?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{1}{5}$

(GATE 2006, 2 Marks)

Solution: As population is finite, hypergeometric distribution is applicable:

$$P(1 \text{ defective in 5 calculators}) = \frac{{}^2C_1 \times {}^{23}C_4}{{}^{25}C_5} = \frac{1}{3}$$

Ans. (b)

25. Consider the continuous random variable with probability density function,

$$f(t) = 1 + t \text{ for } -1 \leq t \leq 0$$
$$= 1 - t \text{ for } 0 \leq t \leq 1$$

The standard deviation of the random variable is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{6}}$

(c) $\frac{1}{3}$ (d) $\frac{1}{6}$

(GATE 2006, 2 Marks)

Solution: Mean is given by

$$\mu_t = E(t)\int_{-\infty}^{\infty} t \cdot f(t)dt = \int_{-1}^{1} t \cdot f(t)\,dt$$

$$= \int_{-1}^{0} t(1+t)dt + \int_{0}^{1} t(1-t)\,dt$$

$$= \left[\frac{t^2}{2} + \frac{t^3}{3}\right]_{-1}^{0} + \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_{0}^{1}$$

$$= -\left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] - \left[\frac{1}{6} + \frac{1}{6}\right] = 0$$

$$\text{Variance} = E(t^2) - [E(t)]^2$$

$$= \int_{-\infty}^{\infty} t^2 f(t)dt - [E(t)]^2 = \int_{-\infty}^{\infty} t^2 f(t)dt - (0)^2$$

$$= \int_{-1}^{0} (t^2 + t^3)dt + \int_{0}^{1} t^2(1-t)dt$$

$$= \left[\frac{t^3}{3} + \frac{t^4}{4}\right]_{-1}^{0} + \left[\frac{t^3}{3} - \frac{t^4}{4}\right]_{0}^{1}$$

$$= -\frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$$\text{Standard deviation} = \sqrt{\text{Variance}} = \frac{1}{\sqrt{6}}$$

Ans. (b)

26. A class of first year B.Tech. students is composed of four batches A, B, C and D, each consisting of 30 students. It is found that the sessional marks of students in Engineering Drawing in batch C have a mean of 6.6 and standard deviation of 2.3. The mean and standard deviation of the marks for the entire class are 5.5 and 4.2, respectively. It is decided by the course instructor to normalize the marks of the students of all batches to have the same mean and standard deviation as that of the entire class. Due to this, the marks of a student in batch C are changed from 8.5 to

(a) 6.0 (b) 7.0
(c) 8.0 (d) 9.0

(GATE 2006, 2 Marks)

Solution: Let the mean and standard deviation of the students of batch C be μ_c and σ_c, respectively, and the mean and standard deviation of the entire class of first year students be μ and σ, respectively. Now given, $\mu_c = 6.6$, $\sigma_c = 2.3$, $\mu = 5.5$ and $\sigma = 4.2$

In order to normalize batch C to entire class, the normalized score (Z scores) must be equated.
As

$$Z = \frac{x-\mu}{\sigma} = \frac{x-5.5}{4.2}$$

$$Z_c = \frac{x_c - \mu_c}{\sigma_c} = \frac{8.5-6.6}{2.3}$$

Equating these two and solving, we get

$$\frac{8.5-6.6}{2.3} = \frac{x-5.5}{4.2}$$

$$\Rightarrow \qquad x = 8.969 \simeq 9.0$$

Ans. (d)

27. Suppose we uniformly and randomly select a permutation from the 20! permutations of 1, 2, 3, …, 20. What is the probability that 2 appears at an earlier position than any other even number in the selected permutation?

(a) $\frac{1}{2}$ (b) $\frac{1}{10}$
(c) $\frac{9!}{20!}$ (d) None of these

(GATE 2007, 2 Marks)

Solution: Number of permutations with '2' in the first place = 19!

Number of permutations with '2' in the second place = $10 \times 18!$

Number of permutations with '2' in the third place $= 10 \times 9 \times 17!$

and so on until '2' is in the eleventh place. After that, it is not possible to satisfy the given condition, because there are only 10 odd numbers available to fill before '2'. So, the desired number of permutations which satisfies the given condition is

$$19! + 10 \times 18! + 10 \times 9 \times 17! + 10 \times 9 \times 8 \times 16! + \cdots + 10! \times 9!$$

Now, the probability of this happening is given by

$$\frac{19! + 10 \times 18! + 10 \times 9 \times 17! + \cdots + 10! \times 9!}{20!}$$

which is clearly not options (a), (b) or (c).

Therefore, correct option is (d), that is, none of these.

Ans. (d)

28. An examination consists of two papers, Paper 1 and Paper 2. The probability of failing in Paper 1 is 0.3 and that in Paper 2 is 0.2. Given that a student has failed in Paper 2, the probability of failing in Paper 1 is 0.6. The probability of a student failing in both the papers is

(a) 0.5 (b) 0.18
(c) 0.12 (d) 0.06

(GATE 2007, 2 Marks)

Solution: Let A denote the event of failing in Paper 1 and B denote the event of failing in Paper 2. Then we are given that

$P(A) = 0.3$

$P(B) = 0.2$

$P(A/B) = 0.6$

Probability of failing in both,

$P(A \cap B) = P(B) * P(A/B) = 0.2 * 0.6 = 0.12$

Ans. (c)

29. A loaded dice has the following probability distribution of occurrences

Dice value	1	2	3	4	5	6
Probability	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

If three identical dice as the above are thrown, the probability of occurrence of values 1, 5 and 6 on the three dice is

(a) same as that of occurrence of 3, 4, 5

(b) same as that of occurrence of 1, 2, 5

(c) 1/128

(d) 5/8

(GATE 2007, 2 Marks)

Solution:

Dice value	1	2	3	4	5	6
Probability	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$

As the dice are independent,

$$P(1,\ 5,\ 6) = \frac{1}{4} \times \frac{1}{8} \times \frac{1}{4} = \frac{1}{128}$$
$$P(3,\ 4,\ 5) = \frac{1}{8} \times \frac{1}{8} \times \frac{1}{8} = \frac{1}{512}$$
$$P(1,\ 2,\ 5) = \frac{1}{4} \times \frac{1}{8} \times \frac{1}{8} = \frac{1}{256}$$

Therefore, option (c), $P(1, 5 \text{ and } 6) = \dfrac{1}{128}$ is correct.

Ans. (c)

30. If the standard deviation of the spot speed of vehicles in a highway is 8.8 kmph and the mean speed of the vehicles is 33 kmph, the coefficient of variation in speed is

(a) 0.1517 (b) 0.1867

(c) 0.2666 (d) 0.3646

(GATE 2007, 2 Marks)

Solution:

$$\text{C.V.} = \frac{\sigma}{\mu} = \frac{8.8}{33} = 0.2666$$

Ans. (c)

31. Let X and Y be two independent random variables. Which one of the relations between expectation (E), variance (Var) and covariance (Cov) given below is FALSE?

(a) $E(XY) = E(X)E(Y)$

(b) $\text{Cov}(X, Y) = 0$

(c) $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

(d) $E(X^2Y^2) = (E(X))^2(E(Y))^2$

(GATE 2007, 2 Marks)

Solution:

Option (a) is true.

Option (b) is true.

Option (c) is true.

Option (d) is false.

Because $E(X^2Y^2) = (E(X))^2(E(Y))^2$

However, as X is not independent of X,

$$E(X^2) \neq [E(X)]^2$$

$$\therefore E(X^2Y^2) = E(X^2)E(Y^2) \neq [E(X)]^2\left[E(Y^2)\right]^2$$

Ans. (d)

32. A coin is tossed 4 times. What is the probability of getting heads exactly 3 times?

(a) 1/4 (b) 3/8

(c) 1/2 (d) 3/4

(GATE 2008, 1 Mark)

Solution: We know that

$$P = P(H) = 0.5$$

Probability of getting head exactly 3 times is

$$P(X = 3) = ^4C_3(0.5)^1 = \frac{1}{4}$$

Ans. (a)

33. A person on a trip has a choice between private car and public transport. The probability of using a private car is 0.45. While using the public transport, further choices available are bus and metro, out of which the probability of commuting by a bus is 0.55. In such a situation, the probability (rounded up to two decimals) of using a car, bus and metro, respectively, would be

(a) 0.45, 0.30 and 0.25

(b) 0.45, 0.25 and 0.30

(c) 0.45, 0.55 and 0.00

(d) 0.45, 0.35 and 0.20

(GATE 2008, 2 Marks)

Solution: We are given that $P(\text{Car}) = 0.45$

Now, probability of choosing a public transport = 0.55

Furthermore, probability of commuting by a bus = 0.55

Hence, $P(\text{Bus}) = 0.55 \times 0.55 = 0.30$

Now, probability of commuting by a metro = 0.45

$$P(\text{Metro}) = 0.55 \times 0.45 = 0.25$$

Ans. (a)

34. Aishwarya studies either computer science or mathematics everyday. If she studies computer science on a day, then the probability that she studies mathematics the next day is 0.6. If she studies mathematics on a day, then the probability that she studies computer science the next day is 0.4. Given that Aishwarya studies computer science on Monday, what is the probability that she studies computer science on Wednesday?

(a) 0.24 (b) 0.36

(c) 0.4 (d) 0.6

(GATE 2008, 2 Marks)

Solution: Let C denote computer science study and M denote maths study.

Now by rule of total probability, we total up the desired branches and get the answer as shown below:

P(C on Monday and C on Wednesday) = P(C on Monday, C on Tuesday and C on Wednesday) + P(C on Monday, M on Tuesday and C on Wednesday)

$$\begin{aligned} &= 1 \times 0.6 \times 0.4 + 1 \times 0.4 \times 0.4 \\ &= 0.24 + 0.16 \\ &= 0.40 \end{aligned}$$

Ans. (c)

35. If probability density function of a random variable X is

$$\begin{aligned} f(X) &= X^2 \text{ for } -1 \le X \le 1 \\ &= 0 \text{ for any other value of } x \end{aligned}$$

then the percentage probability $P\left(-\frac{1}{3} < x^2 < \frac{1}{3}\right)$ is

(a) 0.247 (b) 2.47

(c) 24.7 (d) 247

(GATE 2008, 2 Marks)

Solution: Given

$$\begin{aligned} f(x) &= x^2 \text{ for } -1 < x < 1 \\ &= 0 \text{ elsewhere} \end{aligned}$$

$$P\left(-\frac{1}{3} < x < \frac{1}{3}\right) = \int_{-\frac{1}{3}}^{\frac{1}{3}} f(x)dx = \int_{-\frac{1}{3}}^{\frac{1}{3}} x^2 dx = \left[\frac{x^3}{3}\right]_{-\frac{1}{3}}^{\frac{1}{3}}$$
$$= \frac{2}{81}$$

The probability expressed in percentage,

$$P = \frac{2}{81} \times 100 = 2.469\% = 2.47\%$$

Ans. (b)

36. Let X be a random variable following normal distribution with mean +1 and variance 4. Let Y be another normal variable with mean −1 and variance unknown. If $P(X \le -1) = P(Y \ge 2)$, the standard deviation of Y is

(a) 3 (b) 2

(c) $\sqrt{2}$ (d) 1

(GATE 2008, 2 Marks)

Solution: Given, $\mu_x = 1$, $\sigma_x^2 = 4 \Rightarrow \sigma_y = 2$

Also given, $\mu_y = -1$ and σ_y is unknown

Given, $P(X \le -1) = P(Y \ge 2)$

Converting into standard normal variates.

$$P\left(z \le \frac{-1-\mu_x}{\sigma_x}\right) = P\left(z \ge \frac{2-\mu_y}{\sigma_y}\right)$$

$$P\left(z \le \frac{-1-1}{2}\right) = P\left(z \ge \frac{2-(-1)}{\sigma_y}\right)$$

$$P(z \le -1) = P\left(z \ge \frac{3}{\sigma_y}\right) \quad \text{(i)}$$

Now, we know that in standard normal distribution,

$$P(z \le -1) = P(z \ge 1) \quad \text{(ii)}$$

Comparing Eqs. (i) and (ii), we can say that

$$\frac{3}{\sigma_y} = 1 \Rightarrow \sigma_y = 3$$

Ans. (a)

37. A fair coin is tossed 10 times. What is the probability that only the first two tosses will yield heads?

(a) $\left(\frac{1}{2}\right)^2$ (b) ${}^{10}C_2\left(\frac{1}{2}\right)^2$

(c) $\left(\frac{1}{2}\right)^{10}$ (d) ${}^{10}C_2\left(\frac{1}{2}\right)^{10}$

(GATE 2009, 1 Mark)

Solution: Probably of only the first two tosses being heads = P(H, H, T, T, T, …, T)

As each toss is independent, the required probability $= P(H)\times P(H)\times[P(T)]^8$

$$=\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^{10}$$

Ans. (c)

38. If two fair coins are flipped and at least one of the outcomes is known to be a head, then what is the probability that both outcomes are heads?

(a) 1/3 (b) 1/4

(c) 1/2 (d) 2/3

(GATE 2009, 1 Mark)

Solution: Let A be the event of head in one coin and B be the event of head in second coin.
The required probability is

$$P(A\cap B\mid A\cup B)=\frac{P[(A\cap B)\cap(A\cup B)]}{P(A\cup B)}$$

$$=\frac{P(A\cap B)}{P(A\cup B)}$$

$$P(A\cap B)=P(\text{both coin heads})$$

$$=P(\text{H, H})=\frac{1}{4}$$

P(HH, HT, TH) = 3/4

So, required probability $=\dfrac{\frac{1}{4}}{\frac{3}{4}}=\dfrac{1}{3}$

Ans. (a)

39. If three coins are tossed simultaneously, the probability of getting at least one head is

(a) 1/8 (b) 3/8

(c) 1/2 (d) 7/8

(GATE 2009, 1 Mark)

Solution: Here, total outcomes = 2^3 = 8.

The probability of getting zero heads when three coins are tossed = 1/8 (when all three are tails).

Hence, the probability of getting at least one head when three coins are tossed = 1 − 1/8 = 7/8.

Ans. (d)

40. An unbalanced dice (with 6 faces, numbered from 1 to 6) is thrown. The probability that the face value is odd is 90% of the probability that the face value is even. The probability of getting any even-numbered face is the same. If the probability that the face is even given that it is greater than 3 is 0.75, which one of the following options is closest to the probability that the face value exceeds 3?

(a) 0.453 (b) 0.468

(c) 0.485 (d) 0.492

(GATE 2009, 2 Marks)

Solution: It is given that

$$P(\text{odd}) = 0.9P(\text{even})$$
$$\sum p(x)=1$$

Now,

$$P(\text{odd}) + P(\text{even}) = 1$$
$$0.9P(\text{even}) + P(\text{even}) = 1$$
$$P(\text{even}) = 1/1.9 = 0.5263$$

Now, it is given that P(any even face) is same

$$\Rightarrow P(2)=P(4)=P(6)$$

Now,

$P(\text{even}) = P(2)$ or $P(4)$ or $P(6) = P(2) + P(4) + P(6)$

$\therefore\quad P(2) = P(4) = P(6) = 1/3P(\text{even})$

$= 1/3(0.5263) = 0.1754$

It is given that

$P(\text{even} \mid \text{face} > 3) = 0.75$

$$\Rightarrow\frac{P(\text{even}\cap\text{face}>3)}{P(\text{face}>3)}=0.75$$

$$\Rightarrow\frac{P(\text{face}=4,6)}{P(\text{face}>3)}=0.75$$

$$\Rightarrow P(\text{face}>3)=\frac{P(\text{face}=4,6)}{0.75}$$

$$=\frac{P(4)+P(6)}{0.75}$$

$$=\frac{0.1754+0.1754}{0.75}=0.4677\simeq0.468$$

Ans. (b)

41. The standard deviation of a uniformly distributed random variable between 0 and 1 is

(a) $\dfrac{1}{\sqrt{12}}$ (b) $\dfrac{1}{\sqrt{3}}$

(c) $\dfrac{5}{\sqrt{12}}$ (d) $\dfrac{7}{\sqrt{12}}$

(GATE 2009, 2 Marks)

Solution: $\sigma = \sqrt{\dfrac{(\beta-\alpha)^2}{12}} = \sqrt{\dfrac{(1-0)^2}{12}} = \dfrac{1}{\sqrt{12}}$

Ans. (a)

42. The standard normal probability function can be approximated as

$$F_{(X_N)} = \frac{1}{1+\exp(-1.7255\ x_n|X_n|^{0.12})}$$

where X_N = standard normal deviate. If mean and standard deviation of annual precipitation are 102 cm and 27 cm, respectively, the probability that the annual precipitation will be between 90 cm and 102 cm is

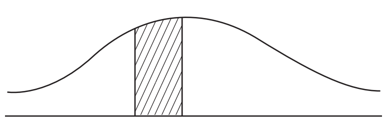
(a) 66.7% (b) 50.0%
(c) 33.3% (d) 16.7%

(GATE 2009, 2 Marks)

Solution: Here $\mu = 102$ cm and $\sigma = 27$ cm.

$$P(90 \le x \le 102) = P\left(\frac{90-102}{27} \le x \le \frac{102-102}{27}\right)$$
$$= P(-0.44 \le x \le 0)$$

This area is shown as follows:



The shaded area in the above figure is given by

$$F(0) - F(-0.44) = \frac{1}{1+\exp(0)}$$
$$= \frac{1}{1+\exp[-1.7255(-0.44)(0.44)0.12)]}$$
$$= 0.5 - 0.3345 = 1.1655 \simeq 16.55\%$$

Hence, the closest answer is 16.7%.

Ans. (d)

43. A box contains 2 washers, 3 nuts and 4 bolts. Items are drawn from the box at random one at a time without replacement. The probability of drawing 2 washers first followed by 3 nuts and subsequently the 4 bolts is

(a) 2/315 (b) 1/630
(c) 1/1260 (d) 1/2520

(GATE 2010, 2 Marks)

Solution: Box contains 2 washers, 3 nuts and 4 bolts.

Probability of drawing 2 washers first = $\left(\dfrac{2}{9}\times\dfrac{1}{8}\right)$

Probability of drawing 3 nuts after drawing 2 washers = $\left(\dfrac{3}{7}\times\dfrac{2}{6}\times\dfrac{1}{5}\right)$

Probability of drawing 4 bolts after drawing 4 bolts = $\left(\dfrac{4}{4}\times\dfrac{3}{3}\times\dfrac{2}{2}\times\dfrac{1}{1}\right)$

P(2 washers, then 3 nuts, then 4 bolts)

$$= \left(\frac{2}{9}\times\frac{1}{8}\right)\times\left(\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}\right)\times\left(\frac{4}{4}\times\frac{3}{3}\times\frac{2}{2}\times\frac{1}{1}\right) = \frac{1}{1260}$$

Ans. (c)

44. A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is

(a) 1/3 (b) 3/7
(c) 1/2 (d) 4/7

(GATE 2010, 2 Marks)

Solution: We have to calculate,

$$P(\text{II is red} \mid \text{I is white}) = \frac{P(\text{II is red and I is white})}{P(\text{I is white})}$$
$$= \frac{P(\text{I is white and II is red})}{P(\text{I is white})}$$

Probability of first removed ball being white = $\dfrac{4}{7}$

Probability of second removed ball being red = $\dfrac{3}{6}$

$$P(\text{II is red} \mid \text{I is white}) = \frac{\frac{4}{7}\times\frac{3}{6}}{\frac{4}{7}} = \frac{3}{6} = \frac{1}{2}$$

Ans. (c)

45. Consider a company that assembles computers. The probability of a faulty assembly of any computer is p. The company therefore subjects each computer to a testing process. This testing process gives the correct result for any computer with a probability of q. What is the probability of a computer being declared faulty?

(a) $pq + (1-p)(1-q)$ (b) $(1-q)p$
(c) $(1-p)q$ (d) pq

(GATE 2010, 2 Marks)

Solution: The probability of a faulty assembly of any computer $= p$

The probability of a computer being declared faulty from a faulty assembly $= pq$

The probability of a non-faulty assembly of any computer $= 1 - q$

The probability of a computer being declared faulty from a non-faulty assembly $= (1 - p)(1 - q)$

Required probability $= pq + (1 - p)(1 - q)$

Ans. (a)

46. A fair coin is tossed independently four times. The probability of the event "the number of times heads show up is more than the number of times tails show up" is

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$

(c) $\frac{1}{4}$ (d) $\frac{5}{16}$

(GATE 2010, 2 Marks)

Solution: Coin is tossed 4 times. Hence, the total outcomes $= 2^4 = 16$.

Favorable outcomes (the number of times heads show up is more than the number of times tails show up) = {HHHH, HHTH, HHHT, HTHH, THHH} = 5

Hence, the required probability $= \frac{5}{16}$.

Ans. (d)

47. There are two containers, with one containing 4 red and 3 green balls and the other containing 3 blue and 4 green balls. One ball is drawn at random from each container. The probability that one of the ball is red and the other is blue will be

(a) $\frac{1}{7}$ (b) $\frac{9}{49}$

(c) $\frac{12}{49}$ (d) $\frac{3}{7}$

(GATE 2011, 1 Mark)

Solution: Probability that one of the ball is red $= \frac{4}{7}$

Probability that one of the ball is blue $= \frac{3}{7}$

Probability that one of the ball is red and the other is blue $= \frac{4}{7} \times \frac{3}{7} = \frac{12}{49}$

Ans. (c)

48. A fair dice is tossed two times. The probability that the second toss results in a value that is higher than the first toss is

(a) 2/36 (b) 2/6

(c) 5/12 (d) 1/2

(GATE 2011, 1 Mark)

Solution: In the first toss, results can be 1, 2, 3, 4, 5.

For 1, the second toss results can be 2, 3, 4, 5, 6.

For 2, the second toss results can be 3, 4, 5, 6.

For 3, the second toss results can be 4, 5, 6.

For 4, the second toss results can be 5, 6.

For 5, the second toss results can be 6.

The required probability

$$= \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \frac{4}{6} + \frac{1}{6} \cdot \frac{3}{6} + \frac{1}{6} \cdot \frac{2}{6} + \frac{1}{6} \cdot \frac{1}{6} = \frac{5}{12}$$

Ans. (c)

49. If the difference between the expectation of the square of a random variable $(E[x])^2$ and the square of the expectation of the random variable $(E[x])^2$ is denoted by R, then

(a) $R = 0$ (b) $R < 0$

(c) $R \geq 0$ (d) $R > 0$

(GATE 2011, 1 Mark)

Solution: $V(x) = E(x^2) - [E(x)]^2 = R$

where $V(x)$ is the variance of x.

As variance is σ_x^2 and hence never negative, $R \geq 0$.

Ans. (c)

50. A deck of five cards (each carrying a distinct number from 1 to 5) is shuffled thoroughly. Two cards are then removed one at a time from the deck. What is the probability that the two cards are selected with the number on the first card being one higher than the number on the second card?

(a) $\frac{1}{5}$ (b) $\frac{4}{25}$

(c) 1.4 (d) $\frac{2}{5}$

(GATE 2011, 2 Marks)

Solution: The five cards are $\{1, 2, 3, 4, 5\}$

Sample space $= 5 \times 4$

Favorable outcome $= P\{(2,1),(3,2),(4,3),(5,4)\}$

$$= \frac{4}{5 \times 4} = \frac{1}{5}$$

Ans. (a)

51. Consider the finite sequence of random values $X = [x_1, x_2, \ldots, x_n]$. Let μ_x be the mean and σ_x be the standard deviation of X. Let another finite sequence Y of equal length be derived from this as $y_i = a * x_i + b$ where a and b are positive constant. Let μ_y be the mean and σ_y be the standard deviation of this sequence. Which one of the following statements is incorrect?

(a) Index position of mode of X in X is the same as the index position of mode of Y in Y.
(b) Index position of median of X in X is the same as the index position of median of Y in Y.
(c) $\mu_y = a\mu_x + b$
(d) $\sigma_y = a\sigma_x + b$

(GATE 2011, 2 Marks)

Solution: Standard deviation is affected by scale but not by shift of origin.

So, $y_i = ax_i + b$

$\Rightarrow$ $\sigma_y = a\sigma_x$

and not $\sigma_y = a\sigma_x + b$

Ans. (d)

52. An unbiased coin is tossed five times. The outcome of each toss is either a head or a tail. The probability of getting at least one head is

(a) $\frac{1}{32}$ (b) $\frac{13}{32}$
(c) $\frac{16}{32}$ (d) $\frac{31}{32}$

(GATE 2011, 2 Marks)

Solution: Here, total outcomes $= 2^5 = 32$.

The probability of getting zero heads when three coins are tossed = 1/32 (when all five are tails).

Hence, the probability of getting at least one head when three coins are tossed = 1 − 1/32 = 31/32.

Ans. (d)

53. Consider a random variable X that takes values +1 and −1 with probability 0.5 each. The values of the cumulative distribution function $F(x)$ at $x = -1$ and +1 are

(a) 0 and 0.5 (b) 0 and 1
(c) 0.5 and 1 (d) 0.25 and 0.75

(GATE 2012, 1 Mark)

Solution: The probability distribution table of the random variable is

x	−1	+1
$P(x)$	0.5	0.5

The cumulative distribution function $F(x)$ is the probability up to x as follows:

x	−1	+1
$F(x)$	0.5	1.0

Hence, the values of cumulative distributive function are 0.5 and 1.0.

Ans. (c)

54. The annual precipitation data of a city is normally distributed with mean and standard deviation as 1000 mm and 200 mm, respectively. The probability that the annual precipitation will be more than 1200 mm is

(a) <50% (b) 50%
(c) 75% (d) 100%

(GATE 2012, 1 Mark)

Solution: The annual precipitation is normally distributed with $\mu = 1000$ mm and $\sigma = 200$ mm.

$$P(x > 1200) = P\left(z > \frac{1200 - 1000}{200}\right)$$
$$= P(z > 1)$$

where z is the standard normal variate.

In normal distribution,

$$P(-1 < z < 1) \approx 0.68$$

($\simeq$ 68% of data is within one standard deviation of mean)

$$P(0 < z < 1) = \frac{0.68}{2} = 0.34$$

So, $P(z > 1) = 0.5 - 0.34 = 0.16 \simeq 16\%$

which is < 50%.

Ans. (a)

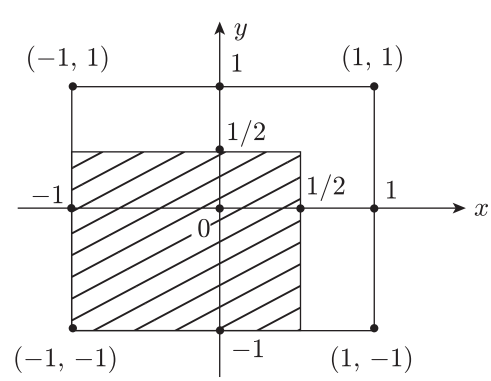
55. Two independent random variables X and Y are uniformly distributed in the interval [−1, 1]. The probability that max[X, Y] is less than 1/2 is

(a) 3/4 (b) 9/16
(c) 1/4 (d) 2/3

(GATE 2012, 1 Mark)

Solution: $-1 < x < 1$ and $-1 \le y \le 1$ is the entire rectangle.

The region in which maximum of (x, y) is less than $\frac{1}{2}$ is shown below as the shaded region inside the rectangle.



$$P\left(\max|x,y|<\frac{1}{2}\right)=\frac{\text{Area of shaded region}}{\text{Area of entire rectangle}}$$

$$=\frac{\frac{3}{2}\times\frac{3}{2}}{2\times 2}=\frac{9}{16}$$

Ans. (b)

56. A fair coin is tossed till a head appears for the first time. The probability that the number of required tosses is odd is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) $\frac{2}{3}$ (d) $\frac{3}{4}$

(GATE 2012, 2 Marks)

Solution: Probability that the number of toss is odd = Probability of the number of toss is 1, 3, 5, 7, …

Probability that the number of toss is 1 = Probability of getting head in first toss = 1/2

Probability that the number of toss is 3 = Probability of getting tail in first toss, tail in second toss and head in third toss

$$=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$

Probability that the number of toss is 5 = P(T, T, T, T, H) = $\left(\frac{1}{2}\right)^5=\frac{1}{32}$

So, probability that the number of tosses are odd

$$=\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots$$

Sum of infinite geometric series with

$$a=\frac{1}{2}\quad\text{and}\quad r=\frac{1}{4}=\frac{\frac{1}{2}}{1-\frac{1}{4}}=\frac{\frac{1}{2}}{\frac{3}{4}}=\frac{2}{3}$$

Ans. (c)

57. Suppose a fair six-sided die is rolled once. If the value on the die is 1, 2 or 3, the die is rolled a second time. What is the probability that the total sum of values that turn up is at least 6?

(a) 10/21 (b) 5/12
(c) 2/3 (d) 1/6

(GATE 2012, 2 Marks)

Solution: $P(\text{sum} > 6) = P(6 \text{ on first throw}) + P(1, 5) + P(1, 6) + P(2, 4) + P(2, 5) + P(2, 6) + P(3, 3) + P(3, 4) + P(3, 5) + P(3, 6)$

$$=\frac{1}{6}+\frac{9}{36}=\frac{15}{36}=\frac{5}{12}$$

Ans. (b)

58. In an experiment, positive and negative values are equally likely to occur. The probability of obtaining at most one negative value in five trials is

(a) 1/32 (b) 2/32
(c) 3/32 (d) 6/32

(GATE 2012, 2 Marks)

Solution: As negative and positive are equally likely, the distribution of number of negative values is binomial with $n = 5$ and $p = 1/2$.

Let X represents the number of negative values in 5 trials.

P(at most 1 negative value) = $P(x\leq 1)$

$$= P(x = 0) + P(x=1)$$

$$= {}^5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{5}\right)^5+{}^5C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4=\frac{6}{32}$$

Hence, required probability is $\frac{6}{32}$.

Ans. (d)

59. A box contains 4 red balls and 6 black balls. Three balls are selected randomly from the box one after another, without replacement. The probability that the selected set contains 1 red ball and 2 black balls is

(a) 1/20 (b) 1/12
(c) 3/10 (d) 1/2

(GATE 2012, 2 Marks)

Solution: P(1 red and 2 black)

$$=\frac{{}^4C_1\times{}^6C_2}{{}^{10}C_3}=\frac{60}{120}=\frac{1}{2}$$

Ans. (d)

60. A continuous random variable X has a probability density function $f(x)=e^{-x}, 0<x<\infty$. Then $P\{X>1\}$ is

(a) 0.368 (b) 0.5
(c) 0.632 (d) 1.0

(GATE 2013, 1 Mark)

Solution:

$$P=\int_1^\infty f(x)dx=\int_1^\infty e^{-x}dx=e^{-x}\Big|_\infty^1=e^{-1}=0.368$$

Ans. (a)

61. Suppose p is the number of cars per minute passing through a certain road junction around

5 PM, and p has Poisson distribution with mean 3. What is the probability of observing fewer than 3 cars during any given minute in this interval?

(a) $8/(2e^3)$ (b) $9/(2e^3)$

(c) $17/(2e^3)$ (d) $26/(2e^3)$

(GATE 2013, 1 Mark)

Solution: Poisson formula for ($p = x$) is given as $\frac{e^{-x}\lambda^x}{x!}$.

λ: mean of Poisson distribution = 3 (given)

Probability of observing fewer than 3 cars = ($p = 0$) + ($p = 1$) + ($p = 2$)

$$= \frac{e^{-3}\lambda^0}{0!} + \frac{e^{-3}\lambda^1}{1!} + \frac{e^{-3}\lambda^2}{2!} = \frac{17}{2e^3}$$

Ans. (c)

62. Let X be a normal random variable with mean 1 and variance 4. The probability $P\{X < 0\}$ is

(a) 0.5

(b) greater than zero and less than 0.5

(c) greater than 0.5 and less than 1.0

(d) 1.0

(GATE 2013, 1 Mark)

Solution:

$$P(x<0) = P\left(\frac{x-\mu}{\sigma} < \frac{0-\mu}{\sigma}\right) - P(Z < -0.5)$$

$$= P(Z > 0.5) = 0.5 - P(0 < Z < 0.5)$$

Hence, the probability is greater than zero and less than 0.5.

Ans. (b)

63. Find the value of λ such that function $f(x)$ is valid probability density function,

$$f(x) = \lambda(x-1)(2-x) \text{ for } 1 \le x \le 2$$
$$= 0 \text{ otherwise}$$

(GATE 2013, 2 Marks)

Solution: $$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$f(x) = \begin{Bmatrix} \lambda(-x^2+3x-2) & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{Bmatrix}$$

$$\therefore \int_1^2 \lambda(-x^2+3x-2)dx = 1$$

$$\Rightarrow \lambda\left[-\frac{x^3}{3} + 3\frac{x^2}{2} - 2x\right]_1^2 = 1$$

$$\Rightarrow \lambda\left[-\left(\frac{8}{3}-\frac{1}{3}\right) + \frac{3}{2}(4-1) - 2(2-1)\right] = 1$$

$$\Rightarrow \lambda\left[-\frac{7}{9} + \frac{9}{2} - 2\right] = 1$$

$$\Rightarrow \lambda\left[\frac{-14+27-12}{6}\right] = 1$$

$$\Rightarrow \lambda = \frac{6}{1} = 6$$

$$\lambda = 6$$

64. The security system at an IT office is composed of 10 computers of which exactly four are working. To check whether the system is functional, the officials inspect four of the computers picked at random (without replacement). The system is deemed functional if at least three of the four computers inspected are working. Let the probability that the system is deemed functional be denoted by p then $100p =$ ________.

(GATE 2014, 1 Mark)

Solution: If p is the probability that system is functional means at least three systems are working, p will be calculated as:

p for at least 3 systems are working

$$= \frac{\text{All 4 systems working} + \text{3 systems working}}{\text{Number of ways to pick 4 systems from 10}}$$

p that 3 systems are working

= (Number of systems not selected)

×(Number of ways to pick 3 from 4 selected systems)

$= 6 \times 4 \times 3 \times 2 \times 4$

Therefore,

p for at least 3 systems are working

$$= \frac{(4\times3\times2\times1)+(6\times4\times3\times2\times4)}{10\times9\times8\times7} = 0.1192$$

So, $100p = 11.92$

Ans. 11.92

65. Each of the nine words in the sentence '**The quick brown fox jumps over the lazy dog**' is written on a separate piece of paper. These nine pieces of paper are kept in a box. One of the pieces is drawn at random from the box. The *expected* length of the word drawn is ________. (The answer should be rounded to one decimal place.)

(GATE 2014, 1 Mark)

Solution: The given 9 words are as follows:

The, Quick, Brown, Fox, Jumps, Over, The, Lazy, Dog

Words with length 3 are THE, FOX, THE, DOG (4)

Words with length 4 are OVER, LAZY (2)

Words with length 5 are QUICK, BROWN, JUMPS (3)

Probability for drawn 3 length words = 4/9

Probability for drawn 4 length words = 2/9

Probability for drawn 5 length words = 3/9

Expected word length

$$= \left\{\left(3\times\frac{4}{9}\right)+\left(4\times\frac{2}{9}\right)+\left(5\times\frac{3}{9}\right)\right\} = 3.88$$

Ans. 3.88

66. In a housing society, half of the families have a single child per family, while the remaining half have two children per family. The probability that a child picked at random has a sibling is ________.

(GATE 2014, 1 Mark)

Solution: Let E_1 = one child family, E_2 = two children family and A = picking a child. Then by Bayes' theorem, required probability is given by

$$P(E_2/A) = \frac{\frac{1}{2}x}{\frac{1}{2}\cdot\frac{x}{2}+\frac{1}{2}x} = \frac{2}{3} = 0.667$$

where x is the number of families.

Ans. 0.667

67. Let X_1, X_2 and X_3 be independent and identically distributed random variables with the uniform distribution on [0, 1]. The probability $P\{X_1$ is the largest$\}$ is ________.

(GATE 2014, 1 Mark)

Solution: Given that three random variables X_1, X_2, and X_3 are uniformly distributed on [0, 1], we have the following possible values:

$X_1 > X_2 > X_3$
$X_1 > X_3 > X_2$
$X_2 > X_1 > X_3$
$X_2 > X_3 > X_1$
$X_3 > X_1 > X_2$
$X_3 > X_2 > X_1$

Since all the three variables are identical, the probabilities for all the above inequalities are same. Hence, the probability that X_1 is the largest is $P\{X_1$ is the largest$\} = 2/6 = 1/3 = 0.33$

Ans. 0.33

68. Let X be a random variable which is uniformly chosen from the set of positive odd numbers less than 100. The expectation, $E[X]$, is ________.

(GATE 2014, 1 Mark)

Solution: $X = 1, 3, 5,..., 99 \Rightarrow n = 50$ (number of observations)

Therefore

$$E(x) = \frac{1}{n}\sum_{i=j}^{n} x_i = \frac{1}{50}[1+3+5+\cdots+99]$$
$$= \frac{1}{50}\cdot(50)^2 = 50$$

Ans. 50

69. An unbiased coin is tossed an infinite number of times. The probability that the fourth head appears at the tenth toss is

(a) 0.067 (b) 0.073 (c) 0.082 (d) 0.091

(GATE 2014, 1 Mark)

Solution: P[fourth head appears at the tenth toss] = P[getting 3 heads in the first 9 tosses and one head at tenth toss]

$$= \left[{}^9C_3\cdot\left(\frac{1}{2}\right)^9\right]\left[\frac{1}{2}\right] = \frac{21}{256} = 0.082$$

Ans. (c)

70. Let X be a zero mean unit variance Gaussian random variable. $E[|X|]$ is equal to ________.

(GATE 2014, 1 Mark)

Solution:

$$X - N(0.1) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, -\infty < x < \infty$$

Therefore

$$E\{|x|\} = \int_{-\infty}^{\infty} |x|\, f(x)$$
$$= \frac{1}{\sqrt{2\pi}}x^2\int_0^{\infty} xe^{-x^2/2}dx = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{-u}du$$
$$= \sqrt{\frac{2}{\pi}} = 0.797 \simeq 0.8$$

Ans. 0.8

71. Consider a dice with the property that the probability of a face with n dots showing up is proportional to n. The probability of the face with three dots showing up is ________.

(GATE 2014, 1 Mark)

Solution: $P(n) = K\cdot n$ where n = 1 to 6. We know that,

$$\sum_n P(x) = 1 \Rightarrow K[1+2+3+4+5+6] = 1$$

$$\Rightarrow K = \frac{1}{21}$$

Therefore, required probability is $P(3) = 3K = \dfrac{1}{7}$.

72. Lifetime of an electric bulb is a random variable with density $f(x) = kx^2$, where x is measured in years. If the minimum and maximum lifetimes of bulb are 1 and 2 years, respectively, then the value of k is ________.

(GATE 2014, 1 Mark)

Solution: Density of the random variable is $f(x) = kx^2$. Since $f(x)$ is a PDF

$$f(x) = \begin{cases} kx^2, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\int_1^2 f(x)dx = 1 \Rightarrow k\left[\frac{x^3}{3}\right]_1^2 = 1 \Rightarrow k = \frac{3}{7} = 0.428 \simeq 0.43$$

73. A box contains 25 parts of which 10 are defective. Two parts are being drawn simultaneously in a random manner from the box. The probability of both the parts being good

(a) $\dfrac{7}{20}$ (b) $\dfrac{42}{125}$ (c) $\dfrac{25}{29}$ (d) $\dfrac{5}{9}$

(GATE 2014, 1 Mark)

Solution: The probability of selecting two parts from 25 is ${}^{25}C_2$. The probability of selecting two good parts is ${}^{(25-10)}C_2$.

So, the required probability is $\dfrac{{}^{15}C_2}{{}^{25}C_2} = \dfrac{105}{300} = \dfrac{7}{20}$

Ans. (a)

74. A group consists of equal number of men and women. Of this group, 20% of the men and 50% of the women are unemployed. If a person is selected at random from this group, the probability of the selected person being employed is ________.

(GATE 2014, 1 Mark)

Solution: Let M be the event of person selected is a man, W be the event of person selected is a woman, E be the event of person selected being employed and U be the event of person selected being unemployed.

Given that $P(\mathrm{M}) = 0.5$, $P(\mathrm{W}) = 0.5$, $P(\mathrm{U/M}) = 0.2$ and $P(\mathrm{U/W}) = 0.5$

The probability of selecting an unemployed person is

$P(\mathrm{U}) = P(\mathrm{M}) \times (P(\mathrm{U/M}) + P(\mathrm{W}) \times (P(\mathrm{U/W}) = 0.5 \times 0.2 + 0.5 \times 0.5 = 0.35$

So, the probability of selecting an employed person $= 1 - P(U) = 1 - 0.35 = 0.65$

Ans. 0.65

75. A nationalized bank has found that the daily balance available in its savings account follows a normal distribution with a mean of ₹ 500 and a standard deviation of ₹ 50. The percentage of savings account holders, who maintain an average daily balance more than ₹ 500 is ________.

(GATE 2014, 1 Mark)

Solution: Given that mean (μ) = ₹ 500 and standard deviation (σ) = ₹ 50

If x follows the normal distribution, then the standard normal variable,

$$z = \frac{x-\mu}{\sigma} = \frac{500-500}{50} = 0$$

Hence, if $x > 500$ then $z > 0$ and so the percentage of savings account holders, who maintain an average daily balance more than ₹ 500 is ≈50%.

Ans. 50%

76. The probability density function of evaporation E on any day during a year in a watershed is given by

$$f(E) = \begin{cases} \dfrac{1}{5} & 0 \le E \le 5\,\text{mm/day} \\ 0 & \text{otherwise} \end{cases}$$

The probability that E lies in between 2 and 4 mm/day in a day in the watershed is (in decimal) ________.

(GATE 2014, 1 Mark)

Solution: The probability density function of E on any day is given by

$$f(E) = \begin{cases} \dfrac{1}{5} & 0 \le E \le 5 \text{ mm/day} \\ 0 & \text{otherwise} \end{cases}$$

$$P(2 < E < 4) = \int_2^4 f(E)dE = \int_2^4 \frac{1}{5}dE = \frac{1}{5}[E]_2^4$$

$$= \frac{1}{5}(4-2) = \frac{2}{5} = 0.4$$

Ans. 0.4

77. A fair (unbiased) coin was tossed four times in succession and resulted in the following outcomes:

(i) Head, (ii) Head, (iii) Head, (iv) Head.

The probability of obtaining a 'Tail' when the coin is tossed again is

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{4}{5}$ (d) $\frac{1}{5}$

(GATE 2014, 1 Mark)

Solution: Since the coin is unbiased, the probability of getting a Tail is $\frac{1}{2}$.

Ans. (b)

78. If $\{x\}$ is a continuous, real-valued random variable defined over the interval $(-\infty, +\infty)$ and its occurrence is defined by the density function given as $f(x) = \frac{1}{\sqrt{2\pi + b}} e^{\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$ where 'a' and 'b' are the statistical attributes of the random variable $\{x\}$. The value of the integral $\int_{-\infty}^{a} \frac{1}{\sqrt{2\pi + b}} e^{\frac{1}{2}\left(\frac{x-a}{b}\right)^2} dx$ is

(a) 1 (b) 0.5 (c) π (d) $\frac{\pi}{2}$

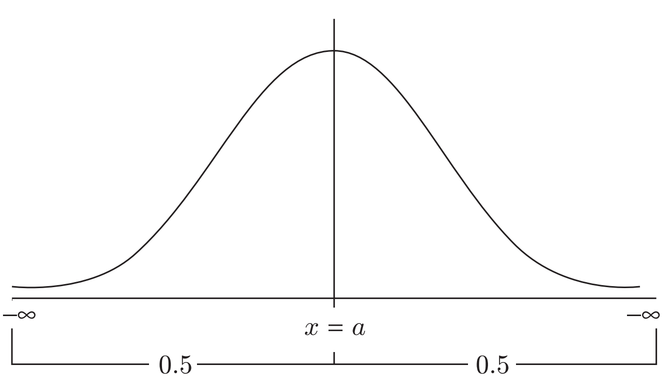
(GATE 2014, 1 Mark)

Solution: We have

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{-\infty}^{a} f(x)dx = 0.5$$



Ans. (b)

79. Given that x is a random variable in the range $[0, \infty]$ with a probability density function $\frac{e^{\frac{x}{2}}}{K}$, the value of the constant K is ________.

(GATE 2014, 1 Mark)

Solution: We have,

$$\int_{0}^{\infty} \frac{e^{-x/2}}{K} = 1$$

$$\Rightarrow \frac{1}{K} \cdot \frac{e^{-x/2}}{(-1/2)}\Bigg|_0^\infty = 1$$

$$\Rightarrow \frac{-2}{K}(0-1) = 1 \Rightarrow \frac{2}{K} = 1 \Rightarrow K = 2$$

80. Four fair six-sided dice are rolled. The probability that the sum of the results being 22 is $X/1296$. The value of X is ________.

(GATE 2014, 2 Marks)

Solution: Sum 22 comes with following possibilities:
$6 + 6 + 6 + 4 = 22$ (numbers can be arranged in 4 ways)
$6 + 6 + 5 + 5 = 22$ (numbers can arranged in 6 ways)
Therefore, the total ways $X = 4 + 6 = 10$

Ans. 10

81. The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2, 3 or 5 is ________.

(GATE 2014, 2 Marks)

Solution: Let A event $\rightarrow$ Divisible by 2, B event $\rightarrow$ Divisible by 3, C event $\rightarrow$ Divisible by 5. Then,
$P(A' \cap B' \cap C') = 1 - P(A \cup B \cup C)$
$P(A) = 50/100$, $P(B) = 33/100$, $P(C) = 20/100$
$P(A \cap B) = 16/100$, $P(B \cap C) = 6/100$,
$P(A \cap C) = 10/100$, $P(A \cap B \cap C) = 3/100$
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$
$P(A \cup B \cup C) = (50 + 33 + 20 - 16 - 6 - 10 + 3)/100 = 74/100$
$P(A' \cap B' \cap C') = 1 - 74/100 = 26/100 = 0.26$

Ans. 0.26

82. Let S be a sample space and two mutually exclusive events A and B be such that $A \cup B = S$. If $P(.)$ denotes the probability of the event, the maximum value of $P(A)P(B)$ is ________.

(GATE 2014, 2 Marks)

Solution: Maximum value of $P(A)\ P(B) = ?$

$$P(A)P(B) = P(A)[1 - P(A)]$$

Let $P(A) = x$

$$f(x) = x - x^2$$

$$f'(x) = 1 - 2x = 0$$

So, $x = 1/2$ and $f''(x) = -2$ and x is maximum at 1/2. Hence

$$P(A)P(B) = 0.25$$

Ans. 0.25

83. A fair coin is tossed repeatedly till both head and tail appear at least once. The average number of tosses required is ________.

(GATE 2014, 2 Marks)

Solution: Let the first toss be head. Let x denotes the number of tosses (after getting first head) to get first tail.
We can summarize the event as:

Event (After getting the first H)	x	Probability, $p(x)$
T	1	1/2
HT	2	1/2 × 1/2 = 1/4
HHT and so on …	3	1/8

$$E(x) = \sum_{x=1}^{\infty} xp(x) = 1\times\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + \cdots$$

Let,

$$S = 1\times\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + \cdots \qquad \text{(i)}$$

$$\Rightarrow \frac{1}{2}S = \frac{1}{4} + 2\times\frac{1}{8} + 3\times\frac{1}{16} + \cdots \qquad \text{(ii)}$$

Subtracting Eq. (ii) from Eq. (i) we get

$$\left(1 - \frac{1}{2}\right)S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$\Rightarrow \frac{1}{2}S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \Rightarrow S = 2 \Rightarrow E(x) = 2$$

Hence, the expected number of tosses (after first head) to get first tail is 2 and same can be applicable if first toss results in tail.
Thus, the average number of tosses is 1 + 2 = 3.

Ans. 3

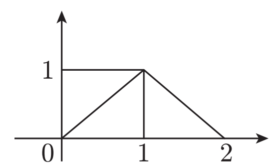
84. Let x_1, x_2 and x_3 be independent and identically distributed random variables with the uniform distribution on [0, 1]. The probability $P\{x_1 + x_2 \le x_3\}$ is ________.

(GATE 2014, 2 Marks)

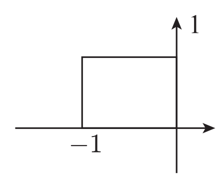
Solution: Given x_1, x_2 and x_3 be independent and identically distributed with uniform distribution on [0, 1]. Let $z = x_1 + x_2 = x_3$. Then

$$P\{x_1 + x_2 \le x_3\} = P\{x_1 + x_2 - x_3 \le 0\}$$
$$= P\{z \le 0\}$$

Let us find probability density function of random variable z.
Since z is summation of three random variables x_1, x_2 and $-x_3$, overall PDF of z is convolution of the PDF of x_1, x_2 and $-x_3$.
PDF of $\{x_1 + x_2\}$ is



PDF of $\{-x_3\}$ is



$$P\{z \le 0\} = \int_{-1}^{0} \frac{(z+1)^2}{2}dz = \left.\frac{(z+1)^3}{6}\right|_{-1}^{0} = \frac{1}{6} = 0.16$$

Ans. 0.16

85. Parcels from sender S to receiver R pass sequentially through two post offices. Each post office has a probability 5 of losing an incoming parcel, independently of all other parcels. Given that a parcel is lost, the probability that it was lost by the second post office is ________.

(GATE 2014, 2 Marks)

Solution: Parcel will be lost if
(a) It is lost by the first post office
(b) It is passed by first post office but lost by the second post office

$$P(\text{parcel is lost}) = \frac{1}{5} + \frac{4}{5}\times\frac{1}{5} = \frac{9}{25}$$

P(parcel lost by second post office if it passes first post office) = P(Parcel passed by first post office) × P(Parcel lost by second post office) $= \frac{4}{5}\times\frac{1}{5} = \frac{4}{25}$

P(parcel lost by second post office | parcel lost) = $\dfrac{4/25}{9/25} = \dfrac{4}{9} = 0.44$

Ans. 0.44

86. A fair coin is tossed n times. The probability that the difference between the number of heads and tails is $(n-3)$ is

(a) 2^{-n} (b) 0 (c) ${}^{n}C_{n-3}2^{-n}$ (d) 2^{-n+3}

(GATE 2014, 2 Marks)

Solution: Let X be the difference between number of heads and tails.

Take $n = 2 \Rightarrow S = \{HH, HT, TH, TT\}$ and $X = -2$, 0, 2. Here, $n - 3 = -1$ is not possible.

Take $n = 3 \Rightarrow S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ and $X = -3, -1, 1, 3$.

Here, $n - 3 = 0$ is not possible.

Similarly, if a coin is tossed n times, then the difference between heads and tails is $n - 3$, which is possible.

Thus, required probability is 0.

Ans. (b)

87. Let X be a random variable with probability density function

$$f(x) = \begin{cases} 0.2, & \text{for } |x| \le 1 \\ 0.1, & \text{for } 1 < |x| \le 4 \\ 0, & \text{otherwise} \end{cases}$$

The probability $P(0.5 < x < 5)$ is ________.

(GATE 2014, 2 Marks)

Solution: The probability density function of the random variable is

$$f(x) = \begin{cases} 0.2, & \text{for } |x| \le 1 \\ 0.1, & \text{for } 1 < |x| \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} P(0.5 < x < 5) &= \int_{0.5}^{5} f(x)\,dx \\ &= \int_{0.5}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx + \int_{4}^{5} f(x)\,dx \\ &= (0.2)(x)_{0.5}^{1} + (0.1)(x)_{1}^{4} + 0 \\ &= 0.1 + 0.3 = 0.4 \end{aligned}$$

88. The mean thickness and variance of silicon steel laminations are 0.2 mm and 0.02, respectively. The varnish insulation is applied on both die sides of the laminations. The mean thickness of one side insulation and its variance are 0.1 mm and 0.01, respectively. If the transformer core is made using 100 such varnish coated laminations, the mean thickness and variance of the core, respectively, are

(a) 30 mm and 0.22 (b) 30 mm and 2.44
(c) 40 mm and 2.44 (d) 40 mm and 0.24

(GATE 2014, 2 Marks)

Solution: Mean thickness of laminations = 0.2 mm

Mean thickness of one side varnish = 0.1 mm

Mean thickness of varnish including both sides = 0.2 mm

Therefore, mean thickness of one lamination varnished on both sides = 0.4 mm

Thickness of the stack of 100 such laminations = $0.4 \times 100 = 40$ mm

If there are 100 laminations with mean thickness = 0.2 mm and variance = 0.1 mm, then we can write the following expressions:

$$(d_1 - 0.2)^2 + (d_2 - 0.2)^2 + \ldots + (d_{100} - 0.2)^2/100 = 0.02$$
$$(x_1 - 0.1)^2 + (x_2 - 0.1)^2 + \ldots + (x_{100} - 0.1)^2 = 0.01$$

Assuming all thicknesses to be equal to d and each side lamination varnish thickness to be equal to x, the two equations reduce to:

$$d = 0.3414 \text{ and } x = 0.2$$

Each lamination thickness therefore including steel and two-side varnish coating thicknesses = 0.3414 + 0.4 = 0.7414 mm.

Variance of overall core is therefore given by

$$100 \times (0.7414 - 0.4)^2/100 = 0.1166$$

Therefore, none of the given answers matches the correct answer.

89. In the following table, x is a discrete random variable and $p(x)$ is the probability density. The standard deviation of x is

x	1	2	3
$P(x)$	0.3	0.6	0.1

(a) 0.18 (b) 0.36 (c) 0.54 (d) 0.6

(GATE 2014, 2 Marks)

Solution: From the given table, we have

Mean (μ) $= \sum xP(x) = 1 \times 0.3 + 2 \times 0.6 + 3 \times 0.1 = 1.8$

$\sum x^2P(x) = 1 \times 0.3 + 4 \times 0.6 + 9 \times 0.1 = 3.6$

Variance $= \sum x^2P(x) - \left[\sum xP(x)\right]^2 = 3.6 - (1.8)^2 = 0.36$

Standard deviation $= \sqrt{\text{Variance}} = \sqrt{0.36} = 0.6$

Ans. (d)

90. Consider an unbiased cubic dice with opposite faces coloured identically and each face coloured red, blue or green such that each colour appears only two times on the dice. If the dice is thrown thrice, the probability of obtaining red colour on top face of the dice at least twice is ______.

(GATE 2014, 2 Marks)

Solution: The probability of each colour appearing only two times on the dice is $p = 2/6 = 1/3$ and the probability of obtaining red colour at the top face is $q = 1-(1/3)$.

Therefore, the required probability of obtaining red colour on top face of the dice at least twice is

$$p(x \geq 2) = {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 + {}^3C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 = \frac{7}{27}$$

91. A machine produces 0, 1 or 2 defective pieces in a day with associated probability of 1/6, 2/3 and 1/6, respectively. The mean value and the variance of the number of defective pieces produced by the machine in a day, respectively, are

(a) 1 and 1/3 (b) 1/3 and 1
(c) 1 and 4/3 (d) 1/3 and 4/3

(GATE 2014, 2 Marks)

Solution: From the question, we have

x	0	1	2
$P(x)$	1/6	2/3	1/6

where x is the number of defective pieces.

Mean (μ) $= \sum xP(x) = 0 \times (1/6) + 1 \times (2/3) + 2 \times (1/6) = 1$

$\sum x^2P(x) = 0 \times (1/6) + 1 \times (2/3) + 4 \times (1/6) = 4/3$

Variance $= \sum x^2P(x) - \left[\sum xP(x)\right]^2 = 4/3 - (1)^2 = 1/3$

Ans. (a)

92. The number of accidents occurring in a plant in a month follows Poisson distribution with mean as 5.2. The probability of occurrence of less than 2 accidents in the plant during a randomly selected month is

(a) 0.029 (b) 0.034 (c) 0.039 (d) 0.044

(GATE 2014, 2 Marks)

Solution: For Poisson distribution, $P(\lambda) = \dfrac{e^{-\lambda}\lambda^n}{n!}$. Given that $\lambda = 5.2$ and $n < 2$, so

$$P(\lambda) = \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-1}\lambda^1}{1!} = e^{-5.2}(1+5.2)$$
$$= 0.00552 \times 6.2 = 0.034$$

Ans. (b)

93. A traffic office imposes on an average 5 number of penalties daily on traffic violators. Assume that the number of penalties on different days is independent and follows a Poisson distribution. The probability that there will be less than 4 penalties in a day is ______.

(GATE 2014, 2 Marks)

Solution:

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \lambda = 5$$
$$P(x<4) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$$
$$= \frac{e^{-5}5^0}{0!} + \frac{e^{-5}5^1}{1!} + \frac{e^{-5}5^2}{2!} + \frac{e^{-5}5^3}{3!}$$
$$= e^{-5}\left[1 + 5 + \frac{25}{2} + \frac{125}{6}\right] = 0.265$$

Ans. 0.265

94. An observer counts 240 vehicles/h at a specific highway location. Assume that the vehicle arrival at the location is Poission distributed, the probability of having one vehicle arriving over a 30-second tome interval is ______.

(GATE 2014, 2 Marks)

Solution: Average number of vehicles per hour,

$$\lambda = 240/\text{h} = \frac{240}{60}/\text{min} = 4/\text{min} = 2/30\,\text{s}$$

$$P(x=1) = \frac{e^{-2}\cdot 2}{1!} = 0.27$$

Ans. 0.27

95. Consider the following two normal distributions

$$f_1(x) = \exp\left(-\pi x^2\right)$$

$$f_2(x) = \frac{1}{2\pi}\exp\left\{-\frac{1}{4\pi}\left(x^2 + 2x + 1\right)\right\}$$

If μ and σ denote the mean and standard deviation, respectively, then

(a) $\mu_1 < \mu_2$ and $\sigma_1^2 < \sigma_2^2$

(b) $\mu_1 < \mu_2$ and $\sigma_1^2 > \sigma_2^2$

(c) $\mu_1 > \mu_2$ and $\sigma_1^2 < \sigma_2^2$

(d) $\mu_1 > \mu_2$ and $\sigma_1^2 > \sigma_2^2$

(GATE 2014, 2 Marks)

Solution: We are given that

$$f_1(x) = e^{-\pi x^2}$$

Comparing with $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, we get

$$\mu_1 = 0 \text{ and } \sigma_1 = \frac{1}{\sqrt{2\pi}}$$

Now, $f_2(x) = \dfrac{1}{2\pi} e^{-\frac{1}{4\pi}\left(x^2+2x+1\right)} = \dfrac{1}{2\pi} e^{-\frac{1}{4\pi}(x+1)^2}$

Comparing with $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ we get

$$\mu_2 = -1 \text{ and } \sigma_2 = \sqrt{2\pi}$$

So

$$\mu_1 > \mu_2 \text{ and } \sigma_1^2 < \sigma_2^2$$

Ans. (c)

96. In rolling of two fair dice, the outcome of an experiment is considered to be the sum of the numbers appearing on the dice. The probability is highest for the outcome of ________.

(GATE 2014, 2 Marks)

Solution: We can form the table as follows:

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Where x is a random variable and denotes the sum of the numbers appearing on the dice.

$P(x)$ = corresponding probabilities

Thus, the probability is highest for the outcome '7,' i.e. $\frac{6}{36}$.

Ans. 0.1667

97. Suppose A and B .are two independent events with probabilities $P(A) \neq 0$ and $P(B) \neq 0$. Let $\bar{A}$ and $\bar{B}$ be their complements. Which of the following statement is FALSE?

(a) $P(A \cap B) = P(A)P(B)$

(b) $P(A/B) = P(A)$

(c) $P(A \cup B) = P(A) + P(B)$

(d) $P(\bar{A} \cap \bar{B}) = P(\bar{A}) + P(\bar{B})$

(GATE 2015, 1 Mark)

Solution: We know that A and B are independent, then

$$P(A \cap B) = P(A) \cdot P(B)$$

A and B are independent then $P(A/B) = P(A)$ and $P(B/A) = P(B)$.

Also, if A and B are independent then $\bar{A}$ and $\bar{B}$ are also independent, that is,

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$$

Thus, (a), (b) and (d) are correct. Hence, (c) is false.

Ans. (c)

98. The variance of the random variable X with probability density function $f(x) = \frac{1}{2}\,|\,x\,|\,e^{-|x|}$ is ________.

(GATE 2015, 1 Mark)

Solution: Given that $f(x) = \frac{1}{2}|x|e^{-|x|}$ is probability density function of random variable X.

$$V(x) = E(x^2) - \{E(x)\}^2$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{\infty} \frac{1}{2}|x|e^{|x|}x dx = 0$$

($\because$ the function is odd)

$$E(x)^2 = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{\infty} x^2 \frac{1}{2}|x|e^{-|x|}dx$$

$$= \frac{2}{3}\int_{2}^{\infty} x^3 e^{-x}dx \ (\because \text{ function is even})$$

$$= 3! = 6$$

Ans. 6

99. A random variable X has the probability density function $f(x)$ as given below:

$$f(x) = \begin{cases} a + bx & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If the expected value $E[X] = 2/3$, then $Pr[X < 0.5]$ is ______.

(GATE 2015, 1 Mark)

Solution: We have

$$f(x) = \begin{cases} a + bx & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

So, $$\int_0^1 (a + bx)dx = 1$$

$$a + \frac{b}{2} = 1 \Rightarrow 2a + b = 2 \qquad \text{(i)}$$

Given that,

$$E[X] = \frac{2}{3} = \int_0^1 x[a + bx]dx$$

$$\frac{2}{3} = \frac{a}{2} + \frac{b}{3} \Rightarrow 3a + 2b = 4 \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we get $a = 0$ and $b = 2$

$$\Pr[X < 0.5] = \int_0^{0.5} f(x)dx = 2\int_0^{0.5} xdx = 0.25$$

Ans. 0.25

100. Two players, A and B, alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is

(a) 5/11 (b) 1/2 (c) 7/13 (d) 6/11

(GATE 2015, 1 Mark)

Solution: Probability of getting 6 $= \dfrac{6}{36} = \dfrac{1}{6}$

That is, probability of A wins the game $= \dfrac{1}{6}$

Probability of A not wins the game $= 1 - \dfrac{1}{6} = \dfrac{5}{6}$

Probability of B wins the game $= \dfrac{1}{6}$

Probability of B not winning the game $= \dfrac{5}{6}$

If A starts the game, probability that A wins the game $= P(A) + P(\bar{A})P(\bar{B})P(A) + P(\bar{A})P(\bar{B})P(\bar{A}).$
$P(\bar{B})P(A) + \cdots$

$$= \frac{1}{6} + \frac{5}{6}\frac{5}{6}\frac{1}{6} + \frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{1}{6} + \cdots$$

$$= \frac{1}{6}\left[1 + \frac{5}{6}\frac{5}{6} + \frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6} + \cdots\right] = \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right]$$

$$= \frac{1}{6}\left[\frac{1}{1 - \left(\frac{5}{6}\right)^2}\right] = \frac{1}{6} \times \frac{36}{11} = \frac{6}{11}$$

Ans. (d)

101. If $P(X) = 1/4$, $P(Y) = 1/3$, and $P(X \cap Y) = 1/12$, the value of $P(Y/X)$ is

(a) $\dfrac{1}{4}$ (b) $\dfrac{4}{25}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{29}{50}$

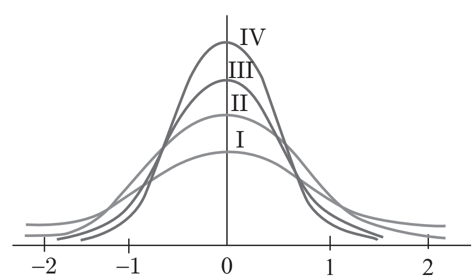
(GATE 2015, 1 Mark)

Solution:

$$P(Y/X) = \frac{P(X \cap Y)}{P(X)} = \frac{1/12}{1/4} = \frac{1}{3}$$

Ans. (c)

102. Among the four normal distribution with probability density functions as shown below, which one has the lowest variance?



(a) I (b) II (c) III (d) IV

(GATE 2015, 1 Mark)

Solution: The correct answer is option (d).

Ans. (d)

103. Consider the following probability mass function (p.m.f) of a random variable X.

$$p(x, q) = \begin{cases} q, & \text{if } X = 0 \\ 1 - q, & \text{if } X = 1 \\ 0, & \text{otherwise} \end{cases}$$

If $q = 0.4$, the variance of X is ______.

(GATE 2015, 1 Mark)

Solution: We are given,

$$p(x,q)=\begin{cases} q, & \text{if } X=0 \\ 1-q, & \text{if } X=1 \\ 0, & \text{otherwise}\end{cases}$$

Given $q = 0.4$, hence

$$p(x,q)=\begin{cases} 0.4, & \text{if } X=0 \\ 0.6, & \text{if } X=1 \\ 0, & \text{otherwise}\end{cases}$$

X	0	1
$p(X=x)$	0.4	0.6

Required value $= V(X) = E(X^2) - \{E(X)\}^2$

$$E(X) = \sum x_i p_i = 0\times 0.4 + 1\times 0.6 = 0.6$$
$$E(X^2) = \sum x_i^2 p_i = 0^2\times 0.4 + 1^2 \times 0.6 = 0.6$$
$$V(X) = 0.6 - (0.6)^2 = 0.6 - 0.36 = 0.24$$

Ans. 0.24

104. Let X and Y denote the sets containing 2 and 20 distinct objects, respectively, and F denote the set of all possible functions defined from X to Y. Let f be randomly chosen from F. The probability of f being chosen from F. The probability of f being one-to-one is ________.

(GATE 2015, 2 Marks)

Solution: $|X| = 2, |Y| = 20$

Number of functions from X to Y is 20^2, that is 400 and number of one-one functions from X to Y is $20\times 19 = 380$.

Therefore, probability of a function f being one-one is $\dfrac{380}{400} = 0.95$.

Ans. 0.95

105. A random binary wave $y(t)$ is given by

$$y(t) = \sum_{n=-\infty}^{\infty} X_n p(t - nT - \phi)$$

where $p(t) = u(t) - u(t - T)$, $u(t)$ is the unit-step function and ϕ is an independent random variable with uniform distribution in $[0, T]$. The sequence $\{X_n\}$ consists of independent and identically distributed binary valued random variables with $P\{X_n = +1\} = P\{X_n = -1\} = 0.5$ for each n.

The value of the autocorrelation $R_{yy}\left(\dfrac{3T}{4}\right) \triangleq E\left[y(t)y\left(t-\dfrac{3T}{4}\right)\right]$ equals ________.

(GATE 2015, 2 Marks)

Solution: We are given,

$$y(t) = \sum_{n=-\infty}^{\infty} X_n p(t - nT - \phi) \quad R_{yy(z)} = \left[1 - \frac{|\tau|}{T}\right]$$

$$R_{yy}\left(\frac{3T}{4}\right) = \left[1 - \frac{3\pi/4}{\pi}\right] = \frac{1}{4} = 0.25$$

Ans. 0.25

106. The chance of a student passing an exam is 20%. The chance of a student passing the exam and getting above 90% marks in it is 5%. Given that a student passes the examination, the probability that the student gets above 90% marks is

(a) $\dfrac{1}{18}$ (b) $\dfrac{1}{4}$
(c) $\dfrac{2}{9}$ (d) $\dfrac{5}{18}$

(GATE 2015, 2 Marks)

Solution: Let A be the event that a student passes the exam and B be the event that student gets above 90% marks.

We are given,

$$P(A) = 0.2; P(A\cap B) = 0.05$$

Required probability is $P(B/A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{0.05}{0.2} = \dfrac{1}{4}$

Ans. (b)

107. The probability of obtaining at least two 'SIX' in throwing a fair dice 4 times is

(a) 425/432 (b) 19/144
(c) 13/144 (d) 125/432

(GATE 2015, 2 Marks)

Solution: From the given data, $n = 4$ and $p = 1/6$

$$\Rightarrow q = 1 - \frac{1}{6} = \frac{5}{6}$$

$$p(x\geq 2) = 1 - p(x<2) = 1 - [p(x=0) + p(x=1)]$$
$$= 1 - \left[4C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4 + 4C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3\right] = \frac{19}{144}$$

Ans. (b)

108. For probability density function of a random variable, x is

$$f(x) = \frac{x}{4}(4 - x^2) \text{ for } 0 \leq x \leq 2$$
$$= 0 \text{ otherwise}$$

The mean μ_x of the random variable is ________.

(GATE 2015, 2 Marks)

Solution: We are given

$$f(x) = \frac{x}{4}(4 - x^2),\ 0 \leq x \leq 2$$

Mean $= \mu_x = E(x)$

$$= \int_0^2 x f(x)dx = \int_0^2 x\left(\frac{x}{4}\right)(4 - x^2)dx$$

$$= \frac{1}{4}\int_0^2 (4x^2 - x^4)dx = \frac{1}{4}\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{1}{4}\left[4.\frac{8}{3} - \frac{32}{5}\right] = \frac{32}{4}\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= 8\left[\frac{2}{15}\right] = \frac{16}{15} = 1.0667$$

Ans. 1.0667

109. The probability that a thermistor randomly picked up from a production unit is defective is 0.1. The probability that out of 10 thermistors randomly picked up 3 are defective is

(a) 0.001 (b) 0.057 (c) 0.107 (d) 0.3

(GATE 2015, 2 Marks)

Solution: Let p be probability of a thermistor is defective = 0.1

$$q = 1 - p = 1 - 0.1 = 0.9$$
$$n = 10$$

Let X be the random variable which is number of defective pieces.
Required probability $= P(x = 3)$

$$= {}^{10}C_3 p^3 q^7 = {}^{10}C_3 (0.1)^3 (0.9)^7 = 0.057$$

Ans. (b)

110. The probability-density function of a random variable x is $p_x(x) = e^{-x}$ for $x \geq 0$ and 0 otherwise. The expected value of the function $g_x(x) = e^{3x/4}$ is ________.

(GATE 2015, 2 Marks)

Solution: Probability density function of x is given as

$$p_x(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and $\qquad g_x(x) = e^{\frac{3x}{4}}$

$$E[g_x(x)] = \int_0^\infty f_x(x) g_x(x) dx$$

$$= \int_0^\infty e^{-x} e^{\frac{3x}{4}} dx = \int_0^\infty e^{\frac{3x}{4} - x} dx = \int_0^\infty e^{-\frac{x}{4}} dx$$

$$= \left[\frac{e^{-x/4}}{-1/4}\right]_0^\infty = -4(e^{-\infty} - 1) = -4(0 - 1) = 4$$

Ans. 4

111. A probability density function on the interval $[a,\ 1]$ is given by $1/x^2$ and outside this interval the value of the function is zero. The value of a is ________.

(GATE 2016, 1 Mark)

Solution: It is given that

$$f(x) = \begin{cases} 1/x^2, & \text{for } a \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

That is,

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_a^1 \frac{1}{x^2} dx = 1$$

$$\Rightarrow \left[-\frac{1}{x}\right]_a^1 = 1$$

$$\Rightarrow \frac{1}{a} - 1 = 1$$

$$\Rightarrow a = \frac{1}{2} = 0.5$$

Ans. 0.5

112. Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is of Type 2 is 0.4. The probability that an LED bulb chosen uniformly at random lasts more than 100 hours is ________.

(GATE 2016, 1 Mark)

Solution:

(i) **Type 1 LED bulb:** The probability that the bulb is of Type 1 and it lasts more than 100 hours is

$$\frac{1}{2} \times 0.7$$

(ii) **Type 2 LED bulb:** The probability that the bulb is of Type 2 and it lasts for more than 100 hours is

$$\frac{1}{2} \times 0.4$$

Therefore, the probability that an LED bulb chosen uniformly at random and lasts more than 100 hours is

$$\left(\frac{1}{2} \times 0.7\right) + \left(\frac{1}{2} \times 0.4\right) = 0.55$$

Ans. 0.55

113. The second moment of a Poisson-distributed random variable is 2. The mean of the random variable is ________.

(GATE 2016, 1 Mark)

Solution: For this Poisson distribution,

$$\text{Mean} = \text{Variance} = 2$$

However, we know that variance is

$$\sigma^2 = m_2 - m_1^2 \qquad \text{(i)}$$

where m_2 is the second central moment and m_1 is the mean. From Eq. (i), we get

$$\lambda = 2 - \lambda^2$$

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow (\lambda + 2)\ (\lambda - 1) = 0$$

Therefore,

$$\lambda = -2 \text{ or } \lambda = 1$$

Since variance (λ) is positive, $\lambda = 1$.

Ans. 1

114. The probability of getting a 'Head' in a single toss of a biased coin is 0.3. The coin is tossed repeatedly till a 'Head' is obtained. If the tosses are independent, then the probability of getting 'Head' for the first time in the fifth toss is ________.

(GATE 2016, 1 Mark)

Solution: We have

$$P(\text{Head}) = 0.3 = p \text{ and } P(\text{Tail}) = 0.7 = q$$

The probability of getting 'head' for the first time in the fifth toss:

$$P(\text{Tail}) \cdot P(\text{Tail}) \cdot P(\text{Tail}) \cdot P(\text{Tail}) \cdot P(\text{Head}) = (0.7)^4$$
$$(0.3) = 0.07203$$

Ans. 0.07203

115. Consider a Poisson distribution for the tossing of a biased coin. The mean for this distribution is μ. The standard deviation for this distribution is given by

(a) $\sqrt{\mu}$ (b) μ^2 (c) μ (d) $1/\mu$

(GATE 2016, 1 Mark)

Solution: For Poisson distribution,

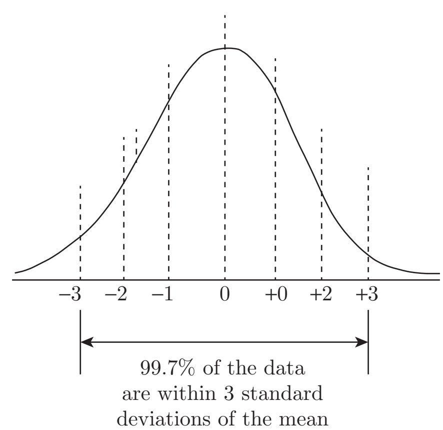
$$\sigma^2 = \mu$$
$$\Rightarrow \sigma = \sqrt{\mu}$$

Ans. (a)

116. The area (in percentage) under standard normal distribution curve of random variable Z within limits from −3 to +3 is ______.

(GATE 2016, 1 Mark)

Solution: A standard normal curve (as shown in figure) has 68% are in limits −1 to +1, 95% area is limits −2 to +2 and 99.7% area in limits −3 to +3.



Ans. 99.7

117. Type II error in hypothesis testing is

(a) acceptance of the null hypothesis when it is false and should be rejected
(b) rejection of the null hypothesis when it is true and should be accepted
(c) rejection of the null hypothesis when it is false and should be rejected
(d) acceptance of the null hypothesis when it is true and should be accepted

(GATE 2016, 1 Mark)

Solution: In hypothesis testing, type II error has acceptance of null hypothesis when it is false and should be rejected.

Ans. (a)

118. X and Y are two random independent events. It is known that $P(X) = 0.40$ and $P(X \cup Y^C) = 0.7$. Which one of the following is the value of $P(X \cup Y)$?

(a) 0.7 (b) 0.5 (c) 0.4 (d) 0.3

(GATE 2016, 1 Mark)

Solution: $P(X) = 0.4$ and $P(X \cup Y^C) = 0.7$

$$P(X) + P(Y^C) - \{P(X \cap Y^C)\} = 0.7$$

$$P(X) + P(Y^C) - P(X)P(Y^C) = 0.7$$

$$0.4 + P(Y^C) - 0.4\ P(Y^C) = 0.7$$

$$0.4 + 0.6\ P(Y^C) = 0.7$$

$$P(Y^C) = \frac{0.7 - 0.4}{0.6}$$

$$P(Y^C) = 0.5$$

Now,

$$P(Y^C) = 1 - P(Y)$$

Therefore,

$$P(Y) = 1 - 0.5 = 0.5$$

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

$$= 0.4 + 0.5 - 0.4 \times 0.5$$

$$= 0.7$$

Ans. (a)

119. Consider the following experiment.

Step 1. Flip a fair coin twice.
Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.
Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.
Step 4. If the outcomes are (TAILS, TAILS), then go to Step 1.

The probability that the output of the experiment is Y is (up to two decimal places) ________.

(GATE 2016, 2 Marks)

Solution: The probability, that the output of the given experiment is Y, can be written as follows:

$$P(Y) = \left(\frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4} \times \frac{1}{4}\right)$$

$$= \frac{1/4}{1 - (1/4)} = \frac{1}{3} = 0.33$$

1st time	HEADS	HEADS	TAILS	TAILS
2nd time	HEADS	TAILS	HEADS	TAILS

$\underbrace{\text{HEADS, HEADS} \quad \text{HEADS, TAILS}}_{N}$ $\quad$ $\underset{\downarrow}{\text{TAILS, HEADS}} \; Y$

Ans. 0.33

120. Two random variables x and y are distributed according to

$$f_{x,y}(x, y) = \begin{cases} (x + y), & 0 \le x \le 1 \quad 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

The probability $P(X+Y\le 1)$ is ________.

(GATE 2016, 2 Marks)

Solution: We have

$$P(x + y \le 1) = \int_{x=0}^{1} \int_{y=0}^{1} f_{xy}(x,y)\, dx\, dy$$

$$\Rightarrow \int_{x=0}^{1} \int_{0}^{1-x} (x + y) dx\, dy = \int_{x=0}^{1} \left(xy + \frac{y^2}{2} \right)_0^{1-x} dx$$

$$= \int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2} \right) dx$$

$$= \frac{x}{2} - \frac{x^3}{6} \Bigg|_0^1 = 0.33$$

Ans. 0.33

121. Candidates were asked to come to an interview with 3 pens each. Black, blue, green and red were the permitted pen colours that the candidate could bring. The probability that a candidate comes with all 3 pens having the same colour is ________.

(GATE 2016, 2 Marks)

Solution: Let Black = a
Blue = b
Green = c
Red = d

For all three pens to be of the same colour, four cases are possible:

aaa

bbb

ccc

ddd

Let first pen be a.

Next, choose 1 from 4 = ${}^4C_1$

Next, choose 1 from 4 = $^4C_1$

$$= a \cdot {}^4C_1 \cdot {}^4C_1 = 16 \text{ cases}$$

Similarly

$$b \cdot {}^4C_1 \cdot {}^4C_1 = 16 \text{ cases}$$
$$c \cdot {}^4C_1 \cdot {}^4C_1 = 16 \text{ cases}$$
$$d \cdot {}^4C_1 \cdot {}^4C_1 = 16 \text{ cases}$$

Total 64 cases. Therefore, probability $= \dfrac{4}{64} = \dfrac{1}{6}$ or 0.167

Ans. 0.167

122. Let the probability density function of a random variable, X, be given as

$$f_X(x) = \frac{3}{2}e^{-3x}u(x) + ae^{4x}u(-x)$$

where $u(x)$ is the unit step function.

Then the value of 'a' and Prob$\{X \le 0\}$, respectively, are

(a) $2, \dfrac{1}{2}$ (b) $4, \dfrac{1}{2}$

(c) $2, \dfrac{1}{4}$ (d) $4, \dfrac{1}{4}$

(GATE 2016, 2 Marks)

Solution: For the given function:

$$f(x) = \begin{cases} ae^{4x}, & x < 0 \\ \dfrac{3}{2}e^{-3x}, & x \ge 0 \end{cases}$$

$$\int_{\infty}^{+\infty} f_x(x) = 1$$

Therefore,

$$\int_{-\infty}^{0} ae^{4x}dx + \int_{0}^{\infty} \frac{3}{2}e^{-3x}dx = 1$$

$$\left.\frac{ae^{4x}}{4}\right|_{-\infty}^{0} + \frac{3}{2}\left(\frac{-1}{3}\right)e^{-3x}\Big|_{0}^{\infty} = 1$$

$$\frac{a}{4} + \frac{1}{2} = 1 \Rightarrow a = 2$$

Therefore,

$$P(X \le 0) = \int_{-\infty}^{0} 2e^{4x}dx = \frac{1}{2}$$

Ans. (a)

123. The probability that a screw manufactured by a company is defective is 0.1. The company sells screws in packets containing 5 screws and gives a guarantee of replacement if one or more screws in the packet are found to be defective. The probability that a packet would have to be replaced is ______.

(GATE 2016, 2 Marks)

Solution: P(screw is defective) = 0.1

P(that a packet would have to be replaced) = P(one screw is found defective) + P(two screws defective) + $\cdots$ + P(five screws are defective)

$$= 1 - P(\text{no screw is defective})$$
$$= 1 - 0.9 \times 0.9 \times 0.9 \times 0.9 \times 0.9$$
$$= 0.4095$$

Ans. 0.4095

124. Three cards were drawn from a pack of 52 cards. The probability that they are a king, a queen, and a jack is

(a) $\dfrac{16}{5525}$ (b) $\dfrac{64}{2197}$ (c) $\dfrac{3}{13}$ (d) $\dfrac{8}{16575}$

(GATE 2016, 2 Marks)

Solution: Total ways in which three cards can be drawn

$$= {}^{52}C_3$$

Number of ways in which a king, a queen and a jack can be drawn

$$= {}^4C_1 \times {}^4C_1 \times {}^4C_1$$

Therefore, the required probability

$$= \frac{4 \times 4 \times 4}{\dfrac{52 \times 51 \times 50}{6}}$$

$$= \frac{384}{132600} = \frac{16}{5525}$$

Ans. (a)

125. Probability density function of a random variable X is given below

$$f(x) = \begin{cases} 0.25 & \text{if } 1 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

$P(X \le 4)$ is

(a) $\dfrac{3}{4}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{4}$ (d) $\dfrac{1}{8}$

(GATE 2016, 2 Marks)

Solution: $P(X \le 4) = \int_0^4 f(x)\,dx$

$$= \int_0^1 f(x)\,dx + \int_1^4 f(x)\,dx$$

$$= 0 + \int_1^4 0.25\,dx = [0.25x]_1^4$$

$$= 0.25\,[4-1] = 0.25 \times 3$$

$$= 0.75 = \frac{3}{4}$$

Ans. (a)

126. If $f(x)$ and $g(x)$ are two probability density functions.

$$f(x) = \begin{cases} \dfrac{x}{a} + 1 & : \; -a \le x < 0 \\ -\dfrac{x}{a} + 1 & : \; 0 \le x \le a \\ 0 & : \; \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} -\dfrac{x}{a} & : \; -a \le x < 0 \\ \dfrac{x}{a} & : \; 0 \le x \le a \\ 0 & : \; \text{otherwise} \end{cases}$$

Which one of the following statements is true?

(a) Mean of $f(x)$ and $g(x)$ are same: Variance of $f(x)$ and $g(x)$ are same
(b) Mean of $f(x)$ and $g(x)$ are same: Variance of $f(x)$ and $g(x)$ are different
(c) Mean of $f(x)$ and $g(x)$ are different: Variance of $f(x)$ and $g(x)$ are same
(d) Mean of $f(x)$ and $g(x)$ are different: Variable of $f(x)$ and $g(x)$ are different

(GATE 2016, 2 Marks)

Solution: Mean of $f(x)$ = Mean of $g(x)$
Variance of $f(x) \neq$ Variance of $g(x)$

Ans. (b)

127. Let X be a Gaussian random variable with mean 0 and variance σ^2. Let $Y = \max(X, 0)$, where max (a, b) is the maximum of a and b. The median of Y is ____________.

(GATE 2017, 1 Mark)

Solution: As X is a Gaussian random variable, the distribution of x is $N(0, \sigma^2)$.

Given $Y = \max(X, 0)$

$$= \begin{cases} X, & \text{if } 0 < X < \infty \\ 0, & \text{if } -\infty < X \le 0 \end{cases}$$

Therefore, median remains at 0, as it is a positional average.

Hence, median $(Y) = 0$.

Ans. (0.0)

128. An urn contains 5 red balls and 5 black balls. In the first draw, one ball is picked at random and discarded without noticing its colour. The probability to get a red ball in the second draw is

(a) $\frac{1}{2}$ (b) $\frac{4}{9}$ (c) $\frac{5}{9}$ (d) $\frac{6}{9}$

(GATE 2017, 1 Mark)

Solution: Let the first ball be red. Then P(red) in the second draw = 4/9.
Let the first ball be black. Then P(red) in the second draw = 5/9.
Therefore, probability

$$= \frac{1}{2} \times \frac{4}{9} + \frac{1}{2} \times \frac{5}{9} = \frac{1}{2}$$

Ans. (a)

129. Two resistors with nominal resistance values R_1 and R_2 have additive uncertainties ΔR_1 and ΔR_2, respectively. When these resistances are connected in parallel, the standard deviation of the error in the equivalent resistance R is

(a) $\pm\sqrt{\left\{\dfrac{\partial R}{\partial R_1}\Delta R_1\right\}^2 + \left\{\dfrac{\partial R}{\partial R_2}\Delta R_2\right\}^2}$

(b) $\pm\sqrt{\left\{\dfrac{\partial R}{\partial R_2}\Delta R_1\right\}^2 + \left\{\dfrac{\partial R}{\partial R_1}\Delta R_2\right\}^2}$

(c) $\pm\sqrt{\left\{\dfrac{\partial R}{\partial R_1}\right\}^2 \Delta R_2 + \left\{\dfrac{\partial R}{\partial R_2}\right\}^2 \Delta R_1}$

(d) $\pm\sqrt{\left\{\dfrac{\partial R}{\partial R_1}\right\}^2 \Delta R_1 + \left\{\dfrac{\partial R}{\partial R_2}\right\}^2 \Delta R_2}$

(GATE 2017, 1 Mark)

Solution:

$$\sigma_{\text{res}} = \sqrt{\left(\frac{\partial R}{\partial R_1}\right)^2 \sigma_1^2 + \left(\frac{\partial R}{\partial R_2}\right)^2 \sigma_2^2}$$

$$= \sqrt{\left(\frac{\partial R}{\partial R_1}\right)^2 \Delta R_1^2 + \left(\frac{\partial R}{\partial R_2}\right)^2 \Delta R_2^2}$$

Ans. (a)

130. A six-face dice is rolled a large number of times. The mean value of the outcome is ____.

(GATE 2017, 1 Mark)

Solution: The probabilities corresponding to the outcomes are 1/6, 2/6, 3/6, 4/6, 5/6, 6/6.

Therefore, the mean value of the outcome is 21/6 = 3.5.

Ans. 3.5

131. Two coins are tossed simultaneously. The probability (upto two decimal points accuracy) of getting at least one head is ______.

(GATE 2017, 1 Mark)

Solution: Total number of outcomes when two coins are tossed is 4 and sample space is

$$S = \{HH, HT, TH, TT\}$$
$$n(S) = 4$$

Favourable outcomes for existence of at least one head (H) are only three. Thus, required probability is 3/4 = 0.75.

Ans. 0.75

132. A sample of 15 data is as follows: 17, 18, 17, 17, 13, 18, 5, 5, 6, 7, 8, 9, 20, 17, 3. The mode of the data is

(a) 4 (b) 13 (c) 17 (d) 20

(GATE 2017, 1 Mark)

Solution: The mode is the value of the data which occurred most of time. For the given data, mode is 17.

Ans. (c)

133. The number of parameters in the univariate exponential and Gaussian distributions, respectively, are

(a) 2 and 2 (b) 1 and 2
(c) 2 and 1 (d) 1 and 1

(GATE 2017, 1 Mark)

Solution: Normal or Gaussian distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

There are two parameters, i.e. μ and σ, in Gaussian distribution.
The PDF of an exponential distribution is

$$f(x) = \lambda e^{-\lambda x}; \qquad x = 0$$

where $\lambda > 0$ is the parameter of distribution.

Therefore, the total parameters are

Exponential = 1

Gaussian = 2

Ans. (b)

134. Vehicles arriving at an intersection from one of the approach roads follow the Poisson distribution. The mean rate of arrival is 900 vehicles per hour. If a gap is defined as the time difference between two successive vehicle arrivals (with vehicles assumed to be points), the probability (up to four decimal places) that the gap is greater than 8 seconds is _________.

(GATE 2017, 1 Mark)

Solution: Probability of time headway being greater than 8 s is

$$P(n \geq 8) = e^{-8\lambda} = e^{-8\times\frac{900}{360}} = e^{-2} = 0.1353$$

Ans. (0.1353)

135. A two-faced fair coin has its faces designated as head (H) and tail (T). This coin is tossed three times in succession to record the following outcomes: H, H, H. If the coin is tossed one more time, the probability (up to one decimal place) of obtaining H again, given the previous realizations of H, H and H, would be ______.

(GATE 2017, 1 Mark)

Solution:

$$\text{Probability of getting (H, H, H, H)} = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{16}$$

$$\text{Probability of getting (H, H, H)} = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$$

The given condition is getting next H after (H, H, H). Therefore,

$$\text{Required probability} = \frac{\frac{1}{16}}{\frac{1}{8}} = 0.5$$

Ans. (0.5)

136. If a random variable X has a Poisson distribution with mean 5, then the expectation $E[(X + 2)^2]$ equals ______.

(GATE 2017, 2 Marks)

Solution:

In Poisson distribution, if X is a random variable, then

$$E(X) = \text{Mean} = \lambda = 5$$

and

$$E(X) = V(X) = 5$$

Now,

$$E(X^2) = V(X) + (E(X))^2 = 5 + 5^2 = 30$$

Therefore,

$$E[(X+2)^2] = E(X^2 + 4X + 4)$$
$$= E(X^2) + 4E(X) + 4$$
$$= 30 + 4 \times 5 + 4 = 54$$

Ans. (54.0)

137. A person decides to toss a fair coin repeatedly until he gets a head. He will make at most 3 tosses. Let the random variable Y denote the number of heads. The value of var(Y), where var {•} denotes the variance, equals

(a) $\frac{7}{8}$ (b) $\frac{49}{64}$ (c) $\frac{7}{64}$ (d) $\frac{105}{64}$

(GATE 2017, 2 Marks)

Solution:

No. of Head (Y)	0	1
$P(Y)$	$\left(\frac{1}{2^3}\right)$	$\frac{7}{2^3}$

$$E(Y) = 0 \times \frac{1}{8} + 1 \times \frac{7}{8} = \frac{7}{8}$$

$$E(Y^2) = \frac{7}{8}$$

$$\text{Var}\{Y\} = \frac{7}{8} - \left(\frac{7}{8}\right)^2 = \frac{7}{64}$$

Ans. (c)

138. For the function $f(x) = a + bx$, $0 \le x \le 1$, to be valid probability density function, which one of the following statements is correct?

(a) $a = 1, b = 4$ (b) $a = 0.5, b = 1$
(c) $a = 0, b = 1$ (d) $a = 1, b = -1$

(GATE 2017, 2 Marks)

Solution: For probability density function, $f(x)$, to be valid, we have

$$\int_{-\alpha}^{\alpha} f(x) = 1$$

$$\Rightarrow \int_{-\alpha}^{\alpha} (a + bx) = 1$$

$$\Rightarrow \int_{0}^{1} (a + bx)dx = 1$$

$$\Rightarrow \left[ax + \frac{bx^2}{2}\right]_0^1 = 1$$

$$\Rightarrow a + \frac{b}{2} = 1$$

For equation to be satisfied, $a = 0.5$, $b = 1$.

Ans. (b)

CHAPTER 6

NUMERICAL METHODS

INTRODUCTION

The limitations of analytical methods have led the engineers and scientists to evolve graphical and numerical methods. However, numerical methods can be derived which are more accurate. These have recently become highly important due to increasing demand for numerical answers to various problems.

In this chapter, we shall discuss some numerical methods for the solution of linear and algebraic and transcendental equations. The chapter also covers numerical integration and numerical solutions for ordinary differential equations.

NUMERICAL SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Linear equations occur in various mathematical problems and though the system of equations can be solved by Cramer's rule or by Matrix method, these methods are very tedious and lengthy. Therefore, the scope of a mistake increases while solving the problem. Hence, there exist other numerical methods of solution which are well-suited for computing machines. This section explains the various numerical methods of solution.

Gauss Elimination Method

The Gauss elimination method is a basic method of obtaining solutions. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution.

Now, consider the following equations:

$$a_1x + b_1y + c_1z = d_1 \tag{1}$$

$$a_2x + b_2y + c_2z = d_2 \tag{2}$$

$$a_3x + b_3y + c_3z = d_3 \tag{3}$$

Step 1: Eliminate x from Eqs. (2) and (3)

Assuming $a_1 \neq 0$, we eliminate x from Eq. (2) by subtracting (a_2/a_1) times Eq. (1) from Eq. (2). Similarly, we eliminate x from Eq. (3) by subtracting (a_3/a_1) times Eq. (1) from Eq. (3). Hence, the new set of equations is given by

$$a_1x + b_1y + c_1z = d_1 \tag{4}$$

$$b_2'y + c_2'z = d_2' \tag{5}$$

$$b_3'y + c_3'z = d_3' \tag{6}$$

Here, Eq. (4) is called the *pivotal equation* and a_1 is called the *first pivot*.

Step 2: Eliminate y from Eq. (3)

Assuming $b_2 \neq 0$, we eliminate y from Eq. (3), by subtracting (b_3'/b_2') times Eq. (2) from Eq. (3). Hence, the new set of equations is given by

$$a_1x + b_1y + c_1z = d_1 \tag{7}$$

$$b_2'y + c_2'z = d_2' \tag{8}$$

$$c_3''z = d_3'' \tag{9}$$

Now, Eq. (8) is called the *pivotal equation* and b_2' is called the *new pivot*.

Step 3: Evaluate the unknowns

The values of x, y and z are found from Eqs. (7), (8) and (9) by back substitution.

Matrix Decomposition Methods (LU Decomposition Method)

In this section, we will discuss some more numeric methods for solving linear systems of n equations in n unknowns $x_1, \ldots, x_n$,

$$AX = B \tag{10}$$

where

$$A = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is the $n \times n$ coefficient matrix, $X = [x_1 \ldots x_n]$ and $B = [b_1 \ldots b_n]$. We discuss two related methods, namely Doolittle's and Crout's that are modifications of the Gauss elimination, which require fewer arithmetic operations. They use the idea of the LU-factorization of A. This method is based on the fact that every matrix can be expressed as the product of a lower and an upper triangular matrix, provided all the principal minors are non-singular.

An LU-factorization of a given square matrix A is of the form

$$A = LU \tag{11}$$

where L is the lower triangular and U is the upper triangular. For example, say

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Since $AX = B$ and $A = LU$,

$$LUX = B \tag{12}$$

$A = LU$ reduces to

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Putting $UX = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, we get

$$LY = B \tag{13}$$

The equivalent system is

$$l_{11}y_1 = b_1 \tag{14}$$

$$l_{21}y_1 + l_{22}y_2 = b_2 \tag{15}$$

$$l_{31}y_1 + l_{32}y_2 + l_{33}y_3 = b_3 \tag{16}$$

This can be solved by forward substitution for y_1, y_2 and y_3. Now, using $UX = Y$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, we get

$$u_{11}x_1 + u_{12}x_2 + l_{13}x_3 = y_1 \tag{17}$$

$$u_{22}x_2 + u_{23}x_3 = y_2 \tag{18}$$

$$u_{33}x_3 = y_3 \tag{19}$$

This can be solved by backward substitution for x_1, x_2 and x_3.

Doolittle's Method

Now, we can conclude that L and U in Eq. (11) can be computed directly, without solving simultaneous equations (thus, without using the Gauss elimination method). Once we have Eq. (11), we can use it for solving $AX = B$ in two steps, simply by noting that $AX = LUX = B$ may be written as

$$LY = B \quad (20)$$

$$UX = Y \quad (21)$$

and solving first Eq. (20) for Y and then Eq. (21) for X.

Here we require that L have main diagonal 1, …, 1 then this is called *Doolittle's method*. Both systems in Eqs. (20) and (21) are triangular, so we can solve them as in the back substitution for the Gauss elimination.

Crout's Reduction

A similar method is obtained from Eq. (11) if U (instead of L) is required to have main diagonal 1, …, 1, and this method is called *Crout's method*. The rest of the steps for obtaining the solution are similar to those in Section 6.2.2.

Gauss–Jordan Method

Gauss–Jordan method is a modification of the Gauss elimination method. In this method, the elimination of the unknowns is performed not only in the equations below the primary diagonal, but also in the equations above it. Hence, the system is ultimately reduced to a diagonal matrix form; that is, each equation involving only one unknown. From these equations, x, y and z are obtained easily.

Iterative Methods of Solution

The methods described till now are known as direct methods as they give exact solutions. In this section, we discuss the iterative methods, in which we start from an approximation to the true solution and obtain better approximations from a computation cycle repeated until we obtain the desired accuracy.

Gauss–Jacobi Method

An iteration method is called a method of simultaneous corrections if no component of an approximation $x^{(m)}$ is used until all the components of $x^{(m)}$ have been computed. A method of this type is the Jacobi iteration which involves not using improved values until a step has been completed and then replacing $x^{(m)}$ by $x^{(m+1)}$ at once, directly before the beginning of the next step.

The Jacobi iteration method can be explained by the following steps:

Step 1: Consider the system of equation as

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \rightarrow \quad 0$$

If we assume $|a_1| \geq |b_1| + |c_1|$; $|b_2| \geq |a_2| + |c_2|$; $|c_3| \geq |a_3| + |b_3|$ to be true, then this iterative method can be used for the above system to calculate x, y and z.

Step 2: Suppose

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z) \quad (22)$$

$$y = \frac{1}{b_1}(d_2 - a_2x) \quad (23)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \quad (24)$$

Step 3: If $x^{(0)}, y^{(0)}, z^{(0)}$ are the initial values of x, y, z, respectively, then

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}) \quad (25)$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)}) \quad (26)$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)}) \quad (27)$$

Step 4: Similarly, we can conclude that

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1y^{(r)} - c_1z^{(r)}) \quad (28)$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2x^{(r)} - c_2z^{(r)}) \quad (29)$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3x^{(r)} - b_3y^{(r)}) \quad (30)$$

This process is continued till the convergence is obtained.

Gauss–Seidel Method

The Gauss–Seidel iteration is a modification of the Gauss–Jacobi iteration method. It is a method of successive corrections because for each component we successively replace an approximation of a component by a corresponding new approximation as soon as the latter has been computed. It can also be shown that the Gauss–Seidel method converges twice as fast as the Jacobi method.

The Gauss–Seidel iteration method can be explained by the following steps:

Step 1: Consider the system of equation as

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \rightarrow \quad 0$$

If we assume $|a_1| \geq |b_1| + |c_1|$; $|b_2| \geq |a_2| + |c_2|$; $|c_3| \geq |a_3| + |b_3|$ to be true, then the iterative method can be used for the above system to find the values of x, y and z.

Step 2: Suppose

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \tag{31}$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \tag{32}$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \tag{33}$$

Step 3: We start with the initial values $y^{(0)}$, $z^{(0)}$ for y and z, respectively, and get $x^{(1)}$ from the first equation:

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)}) \tag{34}$$

Step 4: For the second equation, substitute $z^{(0)}$ for z and $x^{(1)}$ for x instead of $x^{(0)}$.

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)}) \tag{35}$$

Step 5: Now substitute $x^{(1)}$ for x and $y^{(1)}$ for y in the third equation.

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)}) \tag{36}$$

Step 6: In finding the values of the unknowns, we use the latest available values on R.H.S. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the r^{th} iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)}) \tag{37}$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)}) \tag{38}$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)}) \tag{39}$$

This process is continued till the convergence is assumed.

NUMERICAL SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

In scientific and engineering work, a frequently occurring problem is to find the roots of equations of the form

$$f(x) = 0$$

There are some numerical methods for the solutions of equations of the above form, where $f(x)$ is algebraic or transcendental or combinations of both. These methods are discussed in this section.

Bisection Method

This is one of the simplest methods and convergence is guaranteed but slow; that is, the solution of $f(x)$ can always be obtained. This method is also known as the *method of successive bisection*. This method is based on the principle that if $f(x)$ is continuous between x_0 and x_1, and $f(x_0), f(x_1)$ are of opposite signs, then there exists at least one root between $x = x_0$ and $x = x_1$.

We begin the iterative cycle by choosing two trial points x_0 and x_1, which enclose the actual root. Then $f(x_0)$ and $f(x_1)$ are of opposite signs. The interval (x_0, x_1) is bisected and its midpoint x_2 is obtained as

$$x_2 = (x_1 + x_0)/2 \tag{40}$$

If $f(x_2) = 0$, then x_2 itself is the root. If not, the root lies either between x_0 and x_2 or between x_1 and x_2. That is, if $f(x_0)$ and $f(x_1)$ are of opposite signs, then root lies between x_0 and x_2; otherwise if $f(x_1)$ and $f(x_2)$ are of opposite signs, the root lies between x_1 and x_2. The new interval for searching the root is therefore (x_0, x_1) in the first case and (x_1, x_2) in the second case, which is much less compared to the first interval (x_0, x_1). This process is illustrated in Fig. 1. This new interval will be the beginning interval for the next iteration, and the processes of bisection and finding another such x_2 are repeated.

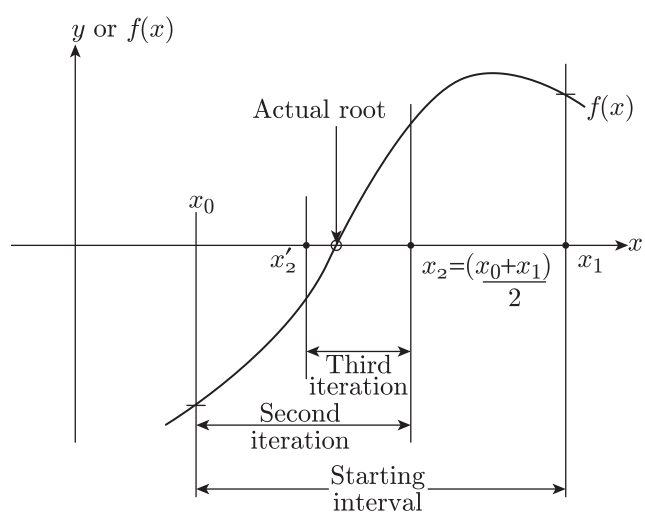


Figure 1 Illustration of bisection method.

Regula–Falsi Method (Method of False Position Method)

The Regula–Falsi method is better than the bisection method in the sense that it guarantees convergence at a faster rate (that is, solution can be obtained in less number of iterations, all other parameters being same).

As before, the iterative procedure is started by choosing two values x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Then the two points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$ are joined by a straight line. The intersection of this line with the x-axis gives x_2. If $f(x_2)$ and $f(x_0)$ are of opposite signs, then replace x_1 by x_2; otherwise, replace x_0 by x_2. This yields a new set of values for x_0 and x_1. The present range is much smaller than the range or interval between the first chosen set of x_0 and x_1. The convergence is thus established, and the iterations are carried over with the new set of x_0 and x_1. Another x_2 is found by the intersection of the straight line joining the new $f(x_0)$ and $f(x_1)$ points with x-axis. Each new or successive interval is smaller than the previous interval, and it is guaranteed to converge to the root.

The procedure is illustrated in Fig. 2.

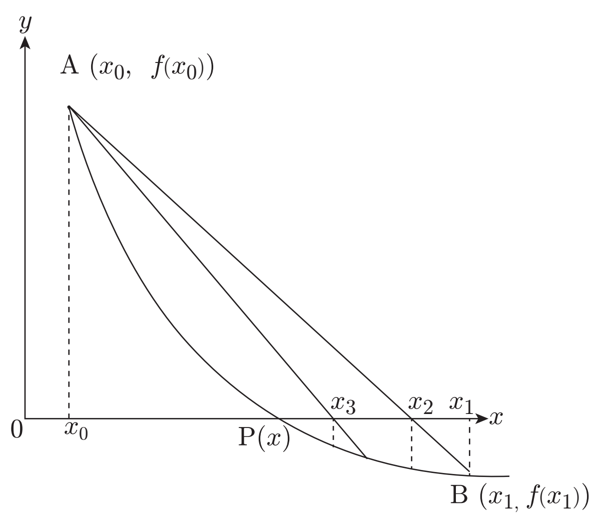


Figure 2 | Illustration of Regula–Falsi method.

From Fig. 2, it can be seen (from the equation of the straight line) that

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \tag{41}$$

where $x_2 = x_0 - \dfrac{x_0 - x_1}{f(x_1) - f(x_0)} f(x_0)$

which is an approximation to the root.

Newton–Raphson Method

Newton's method, also known as Newton–Raphson method, is another iteration method for solving equations $f(x) = 0$, where f is assumed to have a continuous derivative f'. The underlying idea is that we approximate the graph of f by suitable tangents. The Newton–Raphson method has second-order convergence.

Using an approximate value x_0 obtained from the graph of f, we let x_1 be the point of intersection of the x-axis and the tangent to the curve of f at x_0 (Fig. 3). Then

$$\tan\beta = f'(x_0) = \frac{f(x_0)}{x_0 - x_1} \tag{42}$$

Hence,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{43}$$

In the second step, we compute

$$x_2 = \frac{x_1 - f(x_1)}{f'(x_1)}$$

in the third step x_3 from x_2 again by the same formula, and so on.

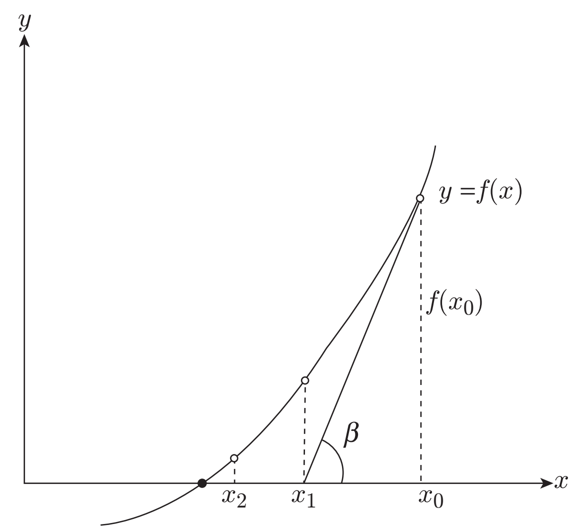


Figure 3 | Newton–Raphson method.

Secant Method

Another method for finding the roots using a succession of roots of secant lines to better approximate a root of a function f is called secant method. The secant method can be thought of as a finite difference approximation of Newton–Raphson method.

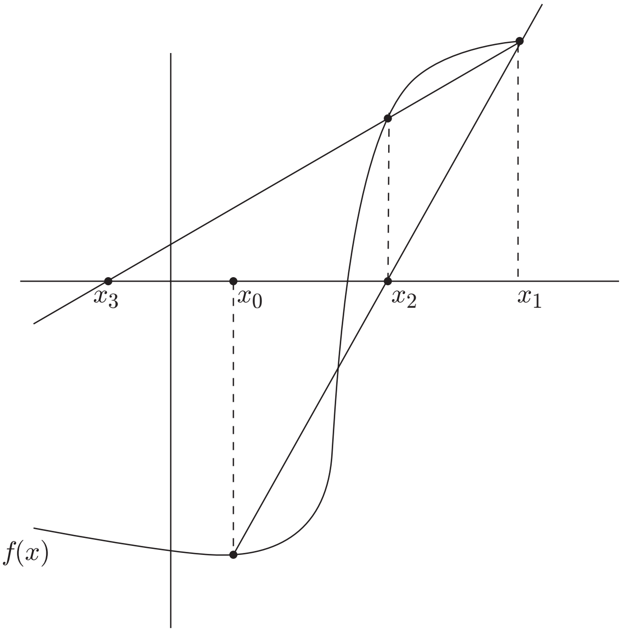


Figure 4 | Secant method.

Consider Fig. 4. The slope of line from points (x_0, $f(x_0)$) to (x_1, $f(x_1)$) can be given by the following equation.

$$y = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1) + f(x_1)$$

We now calculate the solution of x by putting $y = 0$.

$$x = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

We use the new value of x as x_2 and repeat the same process using x_1 and x_2 instead of x_0 and x_1. This process is continued in the same manner unless we obtain $x_n = x_{n-1}$.

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$x_3 = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

Hence, we obtain the generalized solution for the secant method as

$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \tag{44}$$

Jacobian

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \partial u / \partial x & \partial u / \partial y \\ \partial v / \partial x & \partial v / \partial y \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called *numerical differentiation*. The problem of numerical integration is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits.

Numerical integration means the numeric evaluation of integrals

$$J = \int_a^b f(x)dx \tag{45}$$

where a and b are given and f is a function given analytically by a formula or empirically by a table of values. Geometrically, J is the area under the curve of f between a and b (Fig. 5).

We know that if f is such that we can find a differentiable function F whose derivative is f, then we can evaluate J by applying the familiar formula,

$$J = \int_a^b f(x)dx = F(b) - F(a) \text{ [where } F'(x) = f(x)] \tag{46}$$

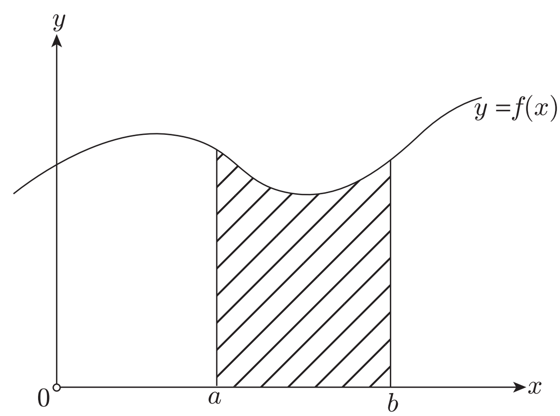


Figure 5 | Geometric interpretation of a definite integral.

Newton–Cotes Formulas (General Quadrature)

Let $y_0, y_1, y_2, \ldots, y_n$ be the values of $y = f(x)$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$, which are equally spaced with interval as h. Then by Newton's forward interpolation formula,

$$y = y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \tag{47}$$

where $u = (x - x_0)/h$. Now,

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_0+nh} y\,dx$$

$$= \int_0^h \left[y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots \right] h\,du$$

On replacing y by the interpolating polynomial in Eq. (47) and putting $dx = h\,du$, we get

$$\int_{x_0}^{x_n} y\,dx = h\int_0^n \left[y + u\Delta y_0 + \frac{u^2-u}{2}\Delta^2 y_0 + \left(\frac{u^3-3u^2+2u}{6}\right)\Delta^3 y_0 + \cdots \right] du$$

$$= h\left[y_0 u + \frac{u^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left[\frac{u^4}{4} - u^3 + u\Delta^3 y_0 + \cdots \right]_0^n \right.$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^3 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 - n^2\right)\Delta^3 y_0 + \cdots \right] \tag{48}$$

Rectangular Rule

Numerical integration methods are obtained by approximating the integrand f by functions that can easily be integrated. The simplest formula, the rectangular rule, is obtained if we subdivide the interval of integration $a \le x \le b$ into n subintervals of equal length $h = (b - a)/n$ and in each subinterval approximate f by the constant $f(x_j{}^*)$, which is the value of f at the midpoint $(x_j{}^*)$ of the jth subinterval (Fig. 6). Then f is approximated by a step function. The n rectangles in Fig. 6 have the areas $f(x_1{}^*)h, \ldots, f(x_n{}^*)h$, and the rectangular rule is

$$J = \int_a^b f(x)dx \approx h\left[f(x_1{}^*) + f(x_2{}^*) + \cdots + f(x_n{}^*)\right]$$

$$\left(h = \frac{b-a}{n}\right) \tag{49}$$

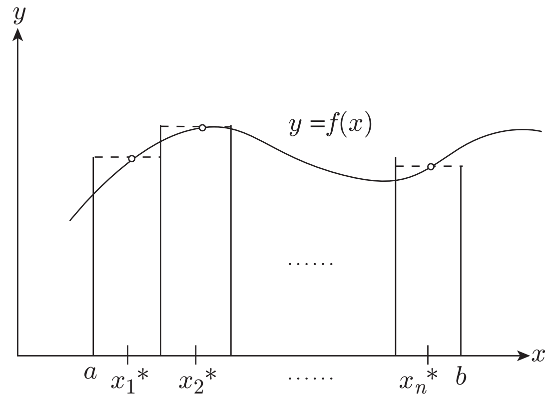


Figure 6 | Rectangular rule.

Trapezoidal Rule

The trapezoidal rule is generally more accurate. We obtain it if we take the same subdivision as before and approximate f by a broken line of segments (chords) with endpoints $[a, f(a)]$, $[x_1, f(x_1)]$, …, $[b, f(b)]$ on the curve of f (Fig. 7). Then the area under the curve of f between a and b is approximated by n trapezoids of areas

$$\frac{1}{2}\left[f(a) + f(x_1)\right]h,\ \frac{1}{2}\left[f(x_1) + f(x_2)\right]h,\ \ldots,$$
$$\frac{1}{2}\left[f(x_{n-1}) + f(b)\right]h$$

By taking their sum, we obtain the trapezoidal rule

$$\int_a^b f(x)dx = b\left[\frac{1}{2}f(a) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right] \tag{50}$$

where $h = (b-a)/n$, as in Eq. (49). The x_j's and a and b are called nodes.

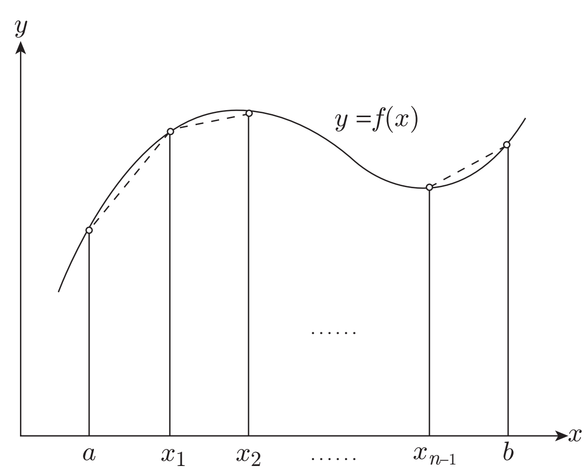


Figure 7 | Trapezoidal rule.

Now, let us derive the trapezoidal rule. By putting $n = 1$ in Eq. (48), we get

$$\int_{x_0}^{x_0+h} f(x) = \int_{x_0}^{x_1} ydx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

(since higher order derivatives do not exist)

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] - \frac{h}{2}[y_0 + y_1] \qquad (51)$$

So

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_0+nh} ydx$$

$$= \int_{x_0}^{x_1} ydx + \int_{x_1}^{x_2} ydx + \cdots + \int_{x_{n-1}}^{x_n} ydx \qquad (52)$$

Here $x_i - x_{i-1} = h, \quad i = 0, 1, 2, ..., n$. So we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \cdots + (y_{n-1} + y_n)]$$

$$= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

$= h/2$[(sum of first and last ordinates) + 2(sum of the remaining ordinates)] (53)

Equation (53) is called the trapezoidal rule.

Simpson's Rule

Simpson's One-Third (1/3) Rule

To apply this rule, the number of intervals should be even. That is, the no of ordinates must be odd. The error in Simpson's one third rule is of the order h^4.

Putting $n = 2$ in Eq. (48), we get

$$\int_{x_0}^{x_0+2h} ydx = h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left[\frac{8}{3} - \frac{4}{2}\right]\Delta^3 y_0 + \cdots\right]$$

(higher order derivatives do not exist)

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}y_2 - \frac{2}{3}y_1 + \frac{1}{3}y_0\right]$$

$$= \frac{h}{3}(y_2 + 4y_1 + y_0) \qquad (54)$$

Therefore,

$$\int_{x_0}^{x_n} ydx = \int_{x_0}^{x_0+2h} ydx + \int_{x_2}^{x_0+4h} ydx + \cdots + \int_{x_{n-2}}^{x_n} ydx$$

$$= \frac{4}{3}(y_2 + 4y_1 + y_0) + \frac{4}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{4}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{4}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots) + 4(y_1 + y_3 + \cdots)]$$

$= \frac{4}{3}$[(sum of first and last ordinates) + 2(sum of remaining odd ordinates) + 4(sum of even ordinates)] (55)

Simpson's Three-Eighth (3/8) Rule

This rule is applicable if n (the number of intervals) is a multiple of 3; that is, $y(x)$ is a polynomial of degree three. Putting $n = 3$ in Eq. (48), we get

$$\int_{x_0}^{x_0+3h} ydx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(\frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4} - 27 + 9\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}(y_3 + 3y_2 + 3y_1 + y_0) \qquad (56)$$

Therefore,

$$\int_{x_0}^{x_n} ydx = \int_{x_0}^{x_0+3h} ydx + \int_{x_3}^{x_0+6h} ydx + \cdots + \int_{x_{n-3}}^{x_n} ydx$$

$$= \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)]$$

$$= \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-2})] \qquad (57)$$

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATION (O.D.E.)

Analytical methods of solutions are applicable only to a few cases and hence are used very rarely. The differential equations which appear in the physical problems frequently do not belong to any of these familiar types and hence one has to resort to numerical methods.

A number of numerical methods are available for the solution of first-order differential equations of the form

$$\frac{dy}{dx} = f(x, y), \text{ given that } y(x_0) = y_0$$

These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution or as a set of values of x and y. Methods such as Picard and Taylor series belong to the former class and methods such as Euler, Runge–Kutta, Milne, etc., belong to the latter one.

This section contains the most popular used methods, namely Picard's, Euler's, modified Euler's and Runge–Kutta method of fourth order.

Picard's Method

This method is also known as Picard's *method of successive approximations*. Consider

$$\frac{dy}{dx} = f(x, y) \text{ and } y = y_0 \text{ at } x = x_0 \tag{58}$$

Integrating the given equation, we get

$$y = \int_{x_0}^{x} f(x, y)dx + C$$

As $y = y_0$ at $x = x_0$, we have $y_0 = C$. Therefore,

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx + C \tag{59}$$

Since y is a function of x, Eq. (58) can also be expressed as

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y)dx$$

Equation (58) can be solved by successive approximation method (also known as the method of iteration).

The first approximation of y is obtained by putting y_0 for y on the right-hand side of Eq. (58). Then $y = y^{(6.1)}$ is the value of y and is given by

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx \tag{60}$$

As y_0 is a constant, the integral on the right-hand side of Eq. (60) can be solved, which yields $y^{(6.1)}$. This is again substituted on the right-hand side of Eq. (60) to get the second approximation to y, which is $y^{(6.2)}$.

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx$$

Proceeding in this way, we get $y^{(3)}, y^{(4)}, \ldots, y^{(n-1)}$ and $y^{(n)}$, where

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx \tag{61}$$

with $n = 1, 2, 3$, etc., and when $x = x_0$, we have $y = y_0$ as the initial condition, or $y^{(0)} = y_0$ becomes the initial or starting value for this iterative method. We thus proceed from $y^{(0)}$ to $y^{(n)}$ till two successive approximations are sufficiently close. This method is not of much practical use.

Euler's Method

In Taylor series method, we obtain approximate solution of the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

as a power series in x and the solution can be used to compute y numerically using specified value of x near x_0. Let $y_1 = y(x_i)$, where $x_1 = x_0 + h$. Then $y_1 = y(x + h)$. Then by Taylor series,

$$y_1 = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \cdots \tag{62}$$

Neglecting the terms with h^2 and higher powers of h in Eq. (62), we get

$$y_1 = y_0 + hf(x_0, y_0) \tag{63}$$

Expression from Eq. (63) gives an approximate value of y at $x_1 = x_0 + h$.

Similarly, we get

$$y_2 = y_1 + hf(x_1, y_1) \text{ for } x_2 = x_1 + h$$

For any n,

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \text{ for } n = 1, 2, 3, \ldots \tag{64}$$

Figure 8 shows the Euler's method.

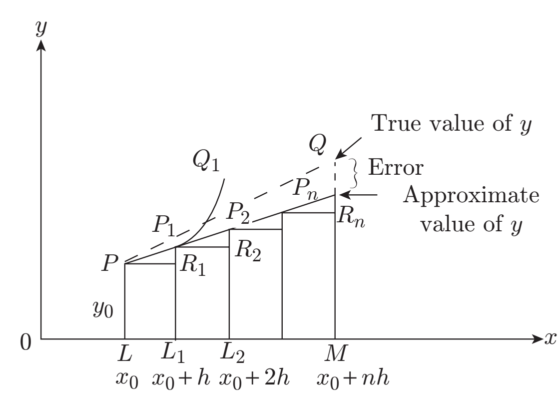


Figure 8 Illustration of Euler's method.

Modified Euler's Method

We know by Eq. (63) that

$$y_1 = y_0 + hf(x_0, y_0)$$

Now, in modified Euler's method, we first compute $y_1^{(1)} = y_0 + hf(x_0, y_0)$ by Euler's method and $y_1 = y(x_1)$ is given by

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] \tag{65}$$

This formula is known as improved Euler's formula or modified Euler's formula. Now y_{n+1} for $x_{n+1} = x_n + h$ is computed as follows:

$$y(x+h) = y(x) + h\left[f\left(x + \frac{h}{2}, y + \frac{h}{2}f(x,y)\right)\right] \tag{66}$$

or

$$y_{(n)} = y_{n-1} + h\left[f\left(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{h}{2}f(x_{n-1}, y_{n-1})\right)\right] \tag{67}$$

where $y_n = y(x_{n-1} + h)$, h is the step size and $n = 1$, 2, 3, …, etc.

Runge–Kutta Method

A method of great practical importance and much greater accuracy than that of the improved Euler's method is the Runge–Kutta method. In each step, we first compute four auxiliary quantities k_1, k_2, k_3, k_4 and then the new value y_{n+1}. This method needs no special starting procedure and does not require a lot of storage and repeatedly uses the same straightforward computational procedure. Hence, the method is well suited to the computer. Moreover, it is numerically stable.

The use of the previous methods to solve the differential equation numerically is restricted due to either slow convergence or labor involved, especially in Taylor's series method. But in Runge–Kutta methods, the derivatives of higher order are not required and we require only two given functional values at different points.

The fourth-order Runge–Kutta method algorithm is mostly used for solving problems unless otherwise mentioned. It is given as follows:

$$k_1 = h + (x, y) \tag{68}$$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k}{2}\right) \tag{69}$$

$$k_2 = hf\left(x + \frac{h}{2}, y + k\right) \tag{70}$$

$$k_4 = hf(x + h, y + k_3) \tag{71}$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{72}$$

$$y(x+h) = y(x) + \Delta y \tag{73}$$

Euler's Predictor–Corrector Method

If x_{n-1} and x_n are two consecutive points, then $x_n = x_{n-1} + h$. Hence, by Euler's method, we know that

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \text{ for } n = 1, 2, 3, \ldots \tag{74}$$

The modified Euler's method gives

$$y_{(n)} = y_{n-1} + h\left\{f\left[x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{h}{2}f(x_{n-1}, y_{n-1})\right]\right\} \tag{75}$$

where $y_n = y(x_{n-1} + h)$, h is the step size and $n = 1$, 2, 3, …, etc.

The value of y_n is first estimated by using Eq. (74), then this value is inserted on the right-hand side of Eq. (75), giving a better approximation of y_n. This value of y_n is again substituted in Eq. (75) to find a still better approximation of y_n. This step is repeated till two consecutive values of y_n are equal. This method is known as predictor–corrector method. Equation (74) is called the predictor and Eq. (75) is called the corrector of y_n.

ACCURACY AND PRECISION

Accuracy describes how close the measured value of a quantity is to its true or actual value. For example, if a weighing machine measures the weight of a person, who actually weighs 72 kg, as let us say 71.9 kg, the measurement can be considered fairly accurate.

Precision is defined as the ability of a measurement to be consistently reproduced and the number of significant digits to which a value has been reliably measured. If on making several measurements, the measured value matches the actual value while the actual values held constant, then the measurement is considered as precise also in addition to being accurate. As far as the number of significant digits with reference to precision is concerned, the number 71.9135 is more precise than 71.91. It may be mentioned that a measurement can be accurate and imprecise at the same time. Similarly, measurements can be precise and inaccurate, inaccurate and imprecise and accurate and precise. The following examples illustrate this further.

1. A temperature sensor is used for measuring the temperature inside a refrigerator whose temperature is maintained constant at 8°C. Ten measurements are made with the sensor. The results are 8.1°C, 7.9°C, 7.8°C, 8.2°C, 8.1°C, 7.9°C, 8.2°C, 8.1°C, 7.9°C and 8.1°C. The measurements show a tendency towards a particular temperature value, which is also very close to the true value. This is an example of good accuracy and good precision.
2. In the above example, if the results of 10 measurements were 8.2°C, 7.9°C, 7.5°C, 8.7°C, 7.3°C, 8.1°C, 7.9°C, 8.6°C, 7.4°C and 7.2°C, the measurement is not only inaccurate but also imprecise.
3. Again, in the above example, if the measurements were 8.1°C, 8.2°C, 8.3°C, 7.8°C, 7.7°C, 8.2°C, 8.3°C, 7.8°C and 8.4°C, they can be considered fairly accurate but lack precision.
4. Yet in another case, in the above example, if the measurements were 7.1°C, 7.2°C, 6.9°C, 7.0°C, 7.1°C, 7.2°C, 6.9°C, 7.1°C, 7.0°C and 7.1°C, they are fairly precise but highly inaccurate.

To sum up, for evaluating the accuracy of a measurement, the measured value must be compared to the true value. To evaluate the precision of a measurement, you must compare the values of two or preferably more measurements.

CLASSIFICATION OF ERRORS

Classification of errors is important for understanding of error analysis. Errors are broadly classified as

1. Systematic or determinate errors
2. Random or indeterminate errors

Systematic errors are those that can be avoided or whose magnitude can be determined. Systematic errors are further classified as (a) operational errors, (b) instrument errors, (c) errors of method and (d) additive or proportional error.

Operational errors are those for which the individual analyst is responsible. These are not connected with the procedure. Instrument errors are the errors that occur due to the instrument used for making a measurement. Errors of method occur due to the method. These errors are difficult to correct. Proportional error is a type of systematic error that changes as the variable being observed changes, although the change is predictable.

Random errors, as the name suggests, are random in nature and very difficult to predict. Random errors are statistical fluctuations (in either direction) in the measured data due to precision limitations of a measurement device. Random errors are also called statistical errors because they can be eliminated in a measurement by statistical means due to their random nature.

Also, the error can be specified as an absolute error or a relative error. The absolute error is the measure of uncertainty in the quantity and has the same units as the quantity itself.

For example, if the weight of a commodity is specified as 535 ± 20 grams, then 20 grams is the absolute error. The relative error, also called fractional error, is obtained by dividing absolute error by the quantity itself and is a dimensionless quantity. When expressed as a percentage, it is multiplied by 100. For example, the floor area of a room when specified as 10 ± 0.5 m^2 has an absolute error of 0.5 m^2 and relative error of 0.05 or 5%. Relative error is usually more significant than the absolute error. For example, a 1 mm error in a skate wheel is more serious than a 1 mm error in the diameter of a truck tyre.

SIGNIFICANT FIGURES

Significant figures of a measured quantity are the meaningful digits in it. Following are the important facts about significant figures:

1. Any digit that is not zero is significant. For example, 346 has three significant figures and 2.5478 has five significant figures.
2. Zeros between non-zero digits are significant. For example, 3046 has four significant figures.
3. Zeros to the left of first non-zero digit are not significant. For example, 0.000346 has only three significant figures.
4. For numbers with decimal points, zeros to the right of a non-zero digit are significant 23.00 and 10.0430, respectively, have four and three significant figures.
5. For numbers without decimal points, trailing zeros may or may not be significant. For example, 500 and 500.5, respectively, have one and four significant figures. Even 500. has three significant figures due to addition for a decimal point.
6. Exact numbers have an infinite number of significant digits. For example, the number representing π has infinite number of significant digits.

With reference to significant figures, while expressing uncertainty, last significant figure in any result should be of the same order by magnitude, that is, in the same decimal position as the uncertainty. For example, 9.81 ± 0.03, 15.2 ± 1.3 and 7 ± 1 are correct representation; 9.81 ± 0.03567 is not correct.

MEASURING ERROR

There are several different ways in which the distribution of the measured values of a repeated experiment can be specified. Errors may be measured as maximum error, probable error, average deviation, mean and standard deviation.

Maximum error is given by

$$\Delta x_{\max} = \frac{x_{\max} - x_{\min}}{2}$$

Virtually, no measurements should ever fall outside $(\bar{x} \pm \Delta x_{\max})$, where $\bar{x}$ is the mean. Probable error, Δx_{prob}, specifies the range $\bar{x} \pm \Delta x_{\text{prob}}$, which contains 50% of the measured values.

Average deviation is the average of the deviations from the mean and is given by

$$\frac{\sum_{k=1}^{N} |x_k - \bar{x}|}{N}$$

Mean is given by

$$\frac{1}{N}\sum_{k=1}^{N} x_k$$

Standard deviation is given by

$$\sigma_x = \left[\frac{1}{N}\sum_{k=1}^{N}(x_k - \bar{x})^2\right]^{1/2}$$

The square of standard deviation is called variance.

Mean and standard deviation are important parameters to estimate random error. If a measurement, which is subject to only random fluctuations, is repeated many times, then approximately 68% of the measured values will fall in the range $\bar{x} \pm \sigma_x$, where $\bar{x}x$ and σ_x are, respectively, mean and standard deviation of measured quantity x. If the measurement is repeated more number of times, we become sure that $\bar{x}$ represents the true value of the quantity. A useful parameter, in this regard, is the standard deviation $\sigma_{\bar{x}}$ of the mean and is given by $(\sigma_x/\sqrt{N})$. $\sigma_{\bar{x}}$ is a good estimate of uncertainty in $\bar{x}$. The precision of measurement increases in proportion to $\sqrt{N}$, as we increase the number of measurements.

PROPAGATION OF ERRORS

In real-life experimental measurements, there may be more than one parameter that contributes to the final result. In such cases, you may arrive at the final result using some formula. This calculated result or the propagated error can be obtained from a few simple rules provided that the errors in the measured quantities are random in nature and are independent, that is, if one quantity is measured as being larger than it actually is, another quantity is still just as likely to be smaller or larger. These rules are summarized as follows and do not apply to systematic errors.

1. If $z = x \pm y$, then $\Delta z = [(\Delta x)^2 + (\Delta y)^2]^{1/2}$
   Here, Δz is the propagated error.
2. If $z = ax \pm b$, then $\Delta z = a\Delta x$
3. If $z = cxy$, then $\Delta z/z = [(\Delta x/x)^2 + (\Delta y/y)^2]^{1/2}$
4. If $z = c\dfrac{y}{x}$, then $\Delta z/z = [(\Delta x/x)^2 + (\Delta y/y)^2]^{1/2}$
5. If $z = cx^a$, then $\Delta z/z = a(\Delta x/x)$
6. If $z = cx^a y^b$, then $\Delta z/z = [(a\Delta x/x)^2 + (b\Delta y/y)^2]^{1/2}$
7. If $z = \sin x$, then $\Delta z/z = \Delta x \cot x$
8. If $z = \cos x$, then $\Delta z/z = \Delta x \tan x$
9. If $z = \tan x$, then $\Delta z/z = \Delta x/\sin x \cos x$

In the above formulas, errors in a, b and c are considered negligible.

METHOD OF LEAST SQUARE APPROXIMATION

The principle of least squares provides an elegant procedure for fitting a unique curve to a given data. The procedure or method is illustrated as follows with the help of an example.

Let us suppose that it is required to fit a curve given by $y = a + bx + cx^2$ (a parabola) to a given set of observations defined by (x_1, y_1), (x_2, y_2), …, (x_4, y_4). Now for any x_i, the observed value is y_i and the expected value is $a + bx_i + cx_1^2$.

The error is $e_i = y_i - (a + bx_i + cx_i^2)$.

The sum of squares of these errors (E) is given by

$$\begin{aligned} E &= e_1^2 + e_2^2 + e_3^2 + e_4^2 \\ &= [y_1 - (a + bx_1 + cx_1^2)]^2 + [y_2 - (a + bx_2 + cx_2^2)]^2 \\ &\quad + \cdots + [y_4 - (a + bx_4 + cx_4)]^2 \end{aligned}$$

For E to be minimum,

$$\frac{\partial E}{\partial a} = \frac{\partial E}{\partial b} = \frac{\partial E}{\partial c} = 0$$

This leads to the following three equations:

$$\begin{aligned} &2[y_1 - (a + bx_1 + cx_1^2) - 2[y_2 - (a + bx_2 + cx_2^2)] \\ &\quad - \cdots - 2[y_4 - (a + bx_4 + cx_4^2)] = 0 \end{aligned}$$

$$-2x_1[y_1-(a+bx_1+cx_1^2)]-2x_2[y_2-(a+bx_2+cx_2^2)]$$
$$-\cdots-2x_4[y_4-(a+bx_4+cx_4^2)]=0$$
$$-2x_1^2[y_1-(a+bx_1+cx_1^2)]-2x_2^2[y_2-(a+bx_2+cx_2^2)]$$
$$-\cdots-2x_4^2[y_4-(a+bx_4+cx_4^2)]=0$$

These equations can be further simplified to

$$\sum y_i = 4a + b\sum x_i + c\sum x_i^2 \tag{76}$$

$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 + c\sum x_i^3 \tag{77}$$

$$\sum x_i^2 y_i = a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 \tag{78}$$

Equations (76), (77) and (78) are known as normal equations. These equations are solved simultaneously to determine a, b and c. Values of a, b and c are substituted in the equation of the curve to be filled (in this case, $y = a + bx + cx^2$) to get the desired curve of best fit. The procedure is summarized as follows:

1. Substitute the observed set of values in the equation of desired curve of fit.
2. Form normal equations for each constant.
3. Solve normal equations as simultaneous equations to determine values of constants.
4. Substitute the values of constants in the equation of curve to be fitted to obtain the desired curve of best fit.

LAGRANGE POLYNOMIALS

Lagrange polynomial is a polynomial $P(x)$ of degree $\leq (n-1)$ that passes through the n points $(x_1, y = f(x_1))$, $(x_2, y_2 = f(x_2))$, … $(x_n, y_n = f(x_n))$. It is given by

$$P(x) = \sum_{j=1}^{n} P_j(x)$$

where $P_j(x) = y_j \prod_{\substack{k=1\\k\neq j}}^{n} \dfrac{x-x_k}{x_j-x_k}$.

It can also be written as

$$P(x) = \frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x-x_n)}y_1 + \frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2 + \cdots + \frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}y_n$$

While using Lagrange polynomials for interpolation, there is a trade-off between having a better fit and a smooth-fitting function. More the number of data points, higher is the degree of Lagrange polynomial. Higher degree polynomial produces greater oscillation between data points. As a result, a higher degree interpolation may be a poor predictor of function between data points, though the accuracy at the data points will be higher.

For example, for $n = 3$,

$$P(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_1)(x-x_2)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}y_3$$

Now

$$P'(x) = \frac{2x-x_2-x_3}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{2x-x_1-x_3}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{2x-x_1-x_2}{(x_3-x_1)(x_3-x_2)}y_3$$

It can be seen from the following equations that function $P\ (x)$ passes through the points $(x_i,\ y_i)$

$$P(x_1) = \frac{(x_1-x_2)(x_1-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x_1-x_1)(x_1-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x_1-x_1)(x_1-x_2)}{(x_3-x_1)(x_3-x_2)}y_3 = y_1$$

$$P(x_2) = \frac{(x_1-x_2)(x_2-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x_2-x_1)(x_2-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x_2-x_1)(x_2-x_2)}{(x_3-x_1)(x_3-x_2)}y_3 = y_2$$

$$P(x_3) = \frac{(x_3-x_2)(x_3-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x_3-x_1)(x_3-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x_3-x_1)(x_3-x_2)}{(x_3-x_1)(x_3-x_2)}y_3 = y_3$$

Generalizing to arbitrary (n),

$$P(x_j) = \sum_{k=1}^{n} P_k(x_j) = \sum_{k=1}^{n} \delta j_k y_k = y_j$$

Lagrange interpolating polynomials can also be written using Lagrange's fundamental interpolating polynomials.

Let $$\pi(x) \equiv \prod_{k=1}^{n}(x-x_k)$$

$$\pi(x_j) = \prod_{k=1}^{n}(x_j-x_k)$$

$$\pi'(xj) = \left[\frac{d\pi}{dx}\right]_{x=x_j} = \prod_{\substack{k=1\\k\neq j}}^{n}(x_j-x_k)$$

So that $\pi(x)$ is an nth degree polynomials with zeros at $x_1, x_2, \ldots, x_n$. Fundamental polynomials are defined by

$$\pi_\nu(x) = \frac{\pi(x)}{\pi'(x_\nu)(x - x_\nu)}$$

which satisfy $\pi_\nu(x_\mu) = \delta_{\nu\mu}$.

$(\delta_{\nu\mu})$ is the Kronecker delta

Let $y_1 = P(x_1)$, $y_2 = P(x_2)$, …, $y_n = P(x_n)$, then

$$P(x) = \sum_{k=1}^{n} \pi_k(x) y_k = \sum_{k=1}^{n} \frac{\pi(x)}{(x - x_k)\pi'(x_k)} y_k$$

This is the unique Lagrange interpolating polynomial assuming the values y_k at x_k.

NUMERICAL DIFFERENTIATION

Numerical differentiation is the process of determining the numerical value of a derivative of a function at some assigned value of x from the given set of values (x_i, y_i). The interpolation formula to be used depends on the assigned value of x at which it is desired to find (dy/dx). Three commonly used formulas include the following:

1. **Newton's forward interpolation formula:** It is used if the values of x are equispaced and it is desired to find (dy/dx) near the beginning of the table.
2. **Newton's backward interpolation formula:** It is used if it is desired to find (dy/dx) near the end of the table.
3. **Stirling's or Bessel's formula:** It is used when (dy/dx) is desired to be computed near the middle of the table.

Newton's Forward Formula

Let the function $y = f(x)$ take values $y_1, y_2, y_3, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of (x). If p is any real number and it is required to evaluate $f(x)$ for $x = x_0 + ph$, we can write

$$y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^{-p} y_0$$

E is defined as

$$E^p f(x) = f(x + ph) \qquad (\because E = 1 + \Delta)$$

Therefore,

$$y_p = \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right] y_0$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

Differentiating both sides with respect to p, we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!}\Delta^3 y_0 + \cdots$$

Since, $p = \dfrac{x - x_0}{h}$,

Therefore, $\dfrac{dp}{dx} = \dfrac{1}{h}$

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!}\Delta^3 y_0 + \cdots\right]$$

At $x = x_0$, $p = 0$. Substituting $p = 0$, we get

$$\left[\frac{dy}{dx}\right]_{x_0}$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dp}\left[\frac{dy}{dp}\right]\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2 - 36p + 22}{4!}\Delta^4 y_0 + \cdots\right]$$

Substituting $p = 0$, we get

$$\left[\frac{d^2 y}{dx^2}\right]_{x_0}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right]$$

Similarly, $\left[\dfrac{d^3 y}{dx^3}\right] = \dfrac{1}{h^3}\left[\Delta^3 y_0 - \dfrac{3}{2}\Delta^4 y_0 + \cdots\right]$

Newton's Backward Formula

In this case, we will write the following expression for y_p:

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n$$

$$(\because E^{-1} = 1 - \nabla)$$

$$y_p = \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots\right] y_n$$

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

Differentiating with respect to (p), we get

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots$$

Since $p = (x - x_n)/h$, $dp/dx = 1/h$.

Now

$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$$
$$= \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots\right]$$

At $x = x_n$, $p = 0$. Substituting $p = 0$, we get

$$\left[\frac{dy}{dx}\right]_{x_0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

Differentiating again with respect to (x), we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left[\frac{dy}{dx}\right]\frac{dp}{dx}$$
$$= \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

Substituting $p = 0$, we get

$$\left[\frac{d^2y}{dx^2}\right]_{x_0}$$
$$= \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\Delta^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right]$$

Similarly,

$$\left[\frac{d^2y}{dx^3}\right]_{x_0} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

Stirling's or Bessel's Formula

Stirling's formula is the mean of Gauss' forward interpolation formula and Gauss' back interpolation formula.

Gauss' formula is written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Gauss' back interpolation formula is given by

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)(p)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)(p)(p-1)}{4!}\Delta^4 y_{-2}$$

Taking the mean of the two, we get

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\times\left[\frac{\Delta^2 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

This is known as Gauss' backward interpolation formula; is also known as Stirling formula.

In the case of central difference equation, the equation takes the form:

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2+1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0$$

or

$$\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2})$$

or $$\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\delta y_0$$

$$\frac{1}{2}(\Delta^2 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^2 y_{1/2} + \delta^3 y_{-1/2}) = \mu\delta^3 y_3$$

SOLVED EXAMPLES

1. Solve the following set of equations using Gauss elimination method:

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4$$

Solution: We have

$$x + 4y - z = -5 \quad (1)$$
$$x + y - 6z = -12 \quad (2)$$
$$3x - y - z = 4 \quad (3)$$

Now, performing Eq. (2) – Eq. (1) and Eq. (3) – 3 × Eq. (1) to eliminate x from Eq. (2) and (3), we get

$$-3y - 5z = -7 \quad (4)$$
$$-13y + 2z = 19 \quad (5)$$

Now, eliminating y by performing Eq. (5) – $\frac{13}{3}$ × Eq. (4), we get

$$\frac{71}{3}z = \frac{148}{3} \quad (6)$$

Now, by back substitution, we get

$$z = \frac{148}{71} = 2.0845$$

From Eq. (6), we get

$$y = \frac{7}{3} - \left(\frac{5}{3}\right)\left(\frac{148}{71}\right) = -\frac{81}{71} = -1.1408$$

From Eq. (i), we get

$$x = -5 - 4\left(-\frac{81}{71}\right) + \left(\frac{148}{71}\right) = \frac{117}{71} = 1.6479$$

Hence, $x = 1.6479, y = -1.1408, z = 2.0845$.

2. Solve the following system of equations by Crout's method:

$$x + y + z = 3$$
$$2x - y + 3z = 16$$
$$3x + y - z = -3$$

Solution: We choose $u_{ii} = 1$ and write

$$A = LU$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Equating, we get

$$l_{11} = 1, l_{21} = 2, \; l_{31} = 3$$

$$l_{11}u_{12} = 1 \Rightarrow u_{12} = 1, \quad l_{11}u_{13} = 1 \Rightarrow u_{13} = 1$$
$$l_{21}u_{12} + l_{22} = -1 \Rightarrow l_{22} = -3,$$
$$l_{31}u_{13} + l_{22}u_{23} = 3 \Rightarrow u_{23} = -\frac{1}{3},$$
$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = -1 \Rightarrow l_{33} = -\frac{14}{3}$$

Thus, we get

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & -\frac{14}{3} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

The given system is $AX = B$. This gives

$$LUX = B \quad (1)$$

Let $UX = Y$, so from Eq. (1), we have

$$LY = B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & -\frac{14}{3} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ -3 \end{bmatrix}$$

which gives

$$y_1 = 3$$
$$3y_1 - 3y_2 = 16 \Rightarrow 9 - 3y_2 = 16 \Rightarrow y_2 = -\frac{10}{3}$$
$$3y_1 - 2y_2 - \frac{14}{3}y_3 = -3 \Rightarrow 9 + \frac{20}{3} - \frac{14}{3}y_3$$
$$= -3 \Rightarrow y_3 = 4$$

$$\text{Now, } UX = Y \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{10}{3} \\ 4 \end{bmatrix}$$

which gives

$$x + y + z = 3$$
$$z = 4$$
$$y - \frac{1}{3}z = -\frac{10}{3} \Rightarrow y = -\frac{10}{3} + \frac{4}{3} = -2$$
$$x - 2 + 4 = 3 \Rightarrow x = 1$$

By back substitution, $x = 1, y = -2, z = 4$.

3. Solve the following system of equations using Doolittle's method:

$$8x_2 + 2x_3 = -7$$
$$3x_1 + 5x_2 + 2x_3 = 8$$
$$6x_1 + 2x_2 + 8x_3 = 26$$

Solution: The decomposition is obtained from

$$A = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

By comparing, we get

$$a_{11} = 3 = 1 \times u_{11} = u_{11}$$
$$a_{21} = 0 = l_{21}u_{11}, \quad l_{21} = 0$$
$$a_{31} = 6 = l_{31}u_{11} = l_{31} \times 3, \quad l_{31} = 2$$

$$a_{12} = 5 = 1 \times u_{12} = u_{12}$$
$$a_{22} = 8 = l_{21}u_{12} + u_{22}, \quad u_{22} = 8$$
$$a_{32} = 2 = l_{31}u_{12} + l_{32}u_{22}$$
$$= 2 \times 5 + l_{32} \times 8, \quad l_{32} = -1$$
$$a_{13} = 2 = 1 \times u_{13} = u_{13}$$
$$a_{23} = 2 = l_{23}u_{13} + u_{23}, \quad u_{23} = 2$$
$$a_{33} = 8 = l_{31}u_{13} + l_{32}u_{23} + u_{33}$$
$$= 2 \times 2 - 1 \times 2 + u_{33} \Rightarrow u_{33} = 6$$

Using these values, we get

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}; U = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

We first solve $LY = B$, determining values of y_1, y_2 and y_3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}$$

By solving the above equation, we get

$$y_1 = 8$$
$$y_2 = -7$$
$$2y_1 - y_2 + y_3 = 26 \Rightarrow 2(8) - (-7) + y_3 = 26$$
$$y_3 = 3$$

Hence, $Y = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$

Then we solve $UX = Y$, determining values of x_1, x_2 and x_3,

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$$

So,

$$x_3 = \frac{1}{2}$$
$$8x_2 + 2x_3 = -7 \Rightarrow x_2 = -1$$
$$2x_1 - 5x_2 + 2x_3 = 8 \Rightarrow x_1 = 4$$

Hence, $X = \begin{bmatrix} 4 \\ -1 \\ 1/2 \end{bmatrix}$.

4. Apply Gauss–Jordan method to solve the following set of equations:

$$x + y + z = 9$$
$$2x - 3y + 4z = 13$$
$$3x + 4y + 5z = 40$$

Solution: We are given that

$$x + y + z = 9 \quad (1)$$
$$2x - 3y + 4z = 13 \quad (2)$$
$$3x + 4y + 5z = 40 \quad (3)$$

Performing Eq. (2) – 2 × Eq. (1) and Eq. (3) – 3 × Eq. (1) to eliminate x from Eq. (2) and Eq. (3), we get

$$x + y + z = 9 \quad (4)$$
$$-5y + 2z = -5 \quad (5)$$
$$y + 2z = 13 \quad (6)$$

Performing Eq. (4) + $\frac{1}{5}$ × Eq. (5) and Eq. (6) – $\frac{1}{5}$ × Eq. (5) to eliminate y from Eq. (4) and Eq. (6), we get

$$x + \frac{7}{5}z = 8 \quad (7)$$
$$-5y + 2z = -5 \quad (8)$$
$$\frac{12}{5}z = 12 \quad (9)$$

Performing Eq. (7) + $\frac{7}{12}$ × Eq. (9) and Eq. (8) – $\frac{5}{6}$ × Eq. (9) to eliminate y from Eq. (7) and Eq. (8), we get

$$x = 1$$
$$-5y = -15 \Rightarrow y = 3$$
$$\frac{12}{5}z = 12 \Rightarrow z = 5$$

Hence, the solution is $x = 1,\ y = 3,\ z = 5$.

5. Solve the following system of equations using Gauss–Seidel iterative method:

$2x + 10y + z = 51;\ 10x + y + 2z = 44;$
$x + 2y + 10z = 61$

Solution: The given system is not diagonally dominant. Hence, rearrange the equations as follows to make them diagonally dominant:

$$10x + y + 2z = 44$$
$$2x + 10y + z = 51$$
$$x + 2y + 10z = 61$$

Now, Gauss–Seidel's iterative method can be applied. From the above equations, we get

$$x = \frac{1}{10}(44 - y - 2z) \qquad (1)$$

$$y = \frac{1}{10}(51 - 2x - z) \qquad (2)$$

$$z = \frac{1}{10}(61 - x - 2y) \qquad (3)$$

The first approximation is obtained by putting $y = z = 0$ in Eq. (1),

$$x^{(1)} = \frac{1}{10}(44 - 0 - 0) = 4.4$$

Now, putting $x = 4.4$ and $z = 0$ in Eq. (2), we get

$$y^{(1)} = \frac{1}{10}[51 - (2)(4.4) - 0] = 4.2$$

Again putting $x = 4.4$ and $y = 4.22$ in Eq. (3), we get

$$z^{(1)} = \frac{1}{10}[61 - 4.4 - (2)(4.2)] = -2.9753$$

The second approximation is given by

$$x^{(2)} = \frac{1}{10}[44 - 4.2 - 2(-2.9753)] = 3.0148$$

$$y^{(2)} = \frac{1}{10}[51 - (2)(3.0148) - (-2.9753)] = 4.01544$$

$$z^{(2)} = \frac{1}{10}[61 - 3.0148 - 2(4.01544)] = 4.995432$$

Similarly, the third approximation is given by

$$x^{(3)} = \frac{1}{10}[44 - 4.01544 - 2(4.995432)]$$
$$= 2.9993696$$

$$y^{(3)} = \frac{1}{10}[51 - (2)(2.9993696) - (4.995432)]$$
$$= 4.00058288$$

$$z^{(3)} = \frac{1}{10}[61 - 2.9993696 - 2(4.00058288)]$$
$$= 4.999946464$$

The fourth approximation is given by

$$x^{(4)} = \frac{1}{10}[44 - 4.00058288 - 2(4.999946464)]$$
$$= 2.9993696 \approx 3$$

$$y^{(4)} = \frac{1}{10}[51 - (2)(3) - (4.995432)] = 4.00058288 \approx 4$$

$$z^{(4)} = \frac{1}{10}[61 - 3 - 2(4)] = 4.999946464 \approx 5$$

Hence, after four iterations, we obtain $x = 3, y = 4, z = 5$.

6. Solve the following system of equations by Gauss–Jacobi method:

$$10x - 5y - 2z = 3; \quad 4x - 10y + 3z = -3;$$
$$x + 6y + 10z = -3$$

Solution: Here, we see that the diagonal elements are dominant.
Hence, the coefficient matrix

$$\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix}$$

is diagonally dominant since

$$|\,10\,| > |-5\,| + |-2\,|, \quad |-10\,| > |\,4\,| + |\,3\,|, \quad |\,10\,| > |\,1\,| + |\,6\,|$$

Therefore, the iteration process can be applied.

Solving for x, y, z, we get

$$x = \frac{1}{10}(3 + 5y + 2z) \qquad (1)$$

$$y = \frac{1}{10}(3 + 4x + 3z) \qquad (2)$$

$$z = \frac{1}{10}(-3 - x - 6y) \qquad (3)$$

The first iteration is given by substituting $x = 0$, $y = 0$ and $z = 0$. Therefore,

$$x^{(1)} = \frac{1}{10}[3 + 5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3 + 4(0) + 3(0)] = 0.3$$

$$z^{(1)} = \frac{1}{10}[-3 - (0) - 6(0)] = -0.3$$

The second iteration is given by

$$x^{(2)} = \frac{1}{10}[3 + 5(0.3) + 2(-0.3)] = 0.39$$

$$y^{(2)} = \frac{1}{10}[3 + 4(0.3) + 3(-0.3)] = 0.33$$

$$z^{(2)} = \frac{1}{10}[-3 - (0.3) - 6(0.3)] = -0.51$$

The third iteration is given by

$$x^{(3)} = \frac{1}{10}[3 + 5(0.33) + 2(-0.51)] = 0.363$$

$$y^{(3)} = \frac{1}{10}[3 + 4(0.39) + 3(-0.51)] = 0.303$$

$$z^{(3)} = \frac{1}{10}[-3 - (0.39) - 6(0.33)] = -0.537$$

The fourth iteration is given by

$$x^{(4)} = \frac{1}{10}[3 + 5(0.303) + 2(-0.537)] = 0.3441$$

$$y^{(4)} = \frac{1}{10}[3 + 4(0.363) + 3(-0.537)] = 0.2841$$

$$z^{(4)} = \frac{1}{10}[-3 - 0.363 - 6(0.303)] = -0.5181$$

The fifth iteration is given by

$$x^{(5)} = \frac{1}{10}[3+5(0.284)+2(-0.5181)] = 0.33843$$

$$y^{(5)} = \frac{1}{10}[3+4(0.3441)+3(-0.5181)] = 0.2822$$

$$z^{(5)} = \frac{1}{10}[-3-(0.3441)-6(0.2841)] = -0.50487$$

The sixth iteration is given by

$$x^{(6)} = \frac{1}{10}[3+5(0.2822)+2(-0.50487)] = 0.340126$$

$$y^{(6)} = \frac{1}{10}[3+4(0.33843)+3(-0.50487)] = 0.283911$$

$$z^{(6)} = \frac{1}{10}[-3-(0.33843)-6(0.2822)] = -0.503163$$

The seventh iteration is given by

$$x^{(7)} = \frac{1}{10}[3+5(0.283911)+2(-0.503163)] = 0.3413229$$

$$y^{(7)} = \frac{1}{10}[3+4(0.340126)+3(-0.503163)] = 0.2851015$$

$$z^{(7)} = \frac{1}{10}[-3-(0.340126)-6(0.283911)] = -0.5043592$$

The eighth iteration is given by

$$x^{(8)} = \frac{1}{10}[3+5(0.2851015)+2(-0.5043592)]$$
$$= 0.34167891$$

$$y^{(8)} = \frac{1}{10}[3+4(0.3413229)+3(-0.5043592)]$$
$$= 0.2852214$$

$$z^{(8)} = \frac{1}{10}[-3-(0.3413229)-6(0.2851015)]$$
$$= -0.50519319$$

The ninth iteration is given by

$$x^{(9)} = \frac{1}{10}[3+5(0.2852214)+2(-0.50519319)]$$
$$= 0.341572062$$

$$y^{(9)} = \frac{1}{10}[3+4(0.34167891)+3(-0.50519319)]$$
$$= 0.285113607$$

$$z^{(9)} = \frac{1}{10}[-3-(0.34167891)-6(0.2852214)]$$
$$= -0.505300731$$

Hence, the values correct to three decimal places are $x = 0.342, y = 0.285, z = -0.505$.

7. Find a positive root of $x^3 - x - 1 = 0$ correct to two decimal places by bisection method.

Solution: Let

$$f(x) = x^3 - x - 1 = 0$$

Using $x_0 = 1$ and $x_1 = 2$ in the given function, we get

$$f(1) = (1)^3 - 1 - 1 = -1 < 0 \text{ and}$$

$$f(2) = (2)^3 - 2 - 1 = 5 > 0.$$

Therefore, one root lies between 1 and 2.

By bisection method, the next approximation x_2 is given by

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1+2}{2} = 1.5$$

$$\Rightarrow f(1.5) = (1.5)^3 - (1.5) - 1 = 0.875 > 0.$$

Therefore, the root lies between 1 and 1.5.

The next approximation x_3 is given by

$$x_3 = \frac{x_1 + x_2}{2} = \frac{1+1.5}{2} = 1.2$$

$$\Rightarrow f(1.25) = (1.25)^3 - (1.25) - 1 = -0.2968 < 0$$

Therefore, the root lies between 1.25 and 1.5.

The next approximation x_4 is given by

$$x_4 = \frac{x_2 + x_3}{2} = \frac{1.25+1.5}{2} = 1.375$$

$$\Rightarrow f(1.375) = (1.375)^3 - (1.375) - 1 = 0.224 > 0.$$

Therefore, the root lies between 1.25 and 1.375.

The next approximation x_5 is given by

$$x_5 = \frac{x_3 + x_4}{2} = \frac{1.25+1.375}{2} = 1.3125$$

$$\Rightarrow f(1.3125) = (1.3125)^3 - (1.3125) - 1 = -0.0515 < 0$$

Therefore, the root lies between 1.3125 and 1.375.

The next approximation x_6 is given by

$$x_6 = \frac{x_4 + x_5}{2} = \frac{1.375+1.3125}{2} = 1.34375$$

$$\Rightarrow f(1.34375) = (1.34375)^3 - (1.34375) - 1$$
$$= 0.0826 > 0$$

Therefore, the root lies between 1.3125 and 1.34375.

The next approximation x_7 is given by

$$x_7 = \frac{x_5 + x_6}{2} = \frac{1.3125+1.34375}{2}$$
$$= 1.3281 \Rightarrow f(1.3281)$$
$$= (1.3281)^3 - (1.3281) - 1 = 0.01447 > 0$$

Therefore, the root lies between 1.3125 and 1.3281.

The next approximation x_8 is given by

$$x_8 = \frac{x_5 + x_7}{2} = \frac{1.3125+1.3281}{2} = 1.32$$

$$\Rightarrow f(1.32) = (1.32)^3 - (1.32) - 1 = -0.0187 < 0$$

Hence, the root is 1.32.

8. By using Regula–Falsi method, find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 and 3. Carry out three approximations.

Solution: Let $f(x) = x^4 - x - 10$. Given $x_0 = 1.8$, $x_1 = 2$. Then

$$f(x_0) = f(1.8) = -1.3 < 0 \text{ and}$$

$$f(x_1) = f(2) = 4 > 0$$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs, $f(x) = 0$ has a root between x_0 and x_1. The first-order approximation of this root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.8)(4) - 2(-1.3)}{4 - (-1.3)}$$
$$= \frac{7.2 + 2.6}{5.3} = \frac{9.8}{5.3} = 1.849$$
$$\Rightarrow f(x_2) = (1.849)^4 - 1.849 - 10 = -0.161$$

Now $f(x_2)$ and $f(x_1)$ are of opposite signs. Hence, the root lies between x_2 and x_1. The second-order approximation of the root is given by

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(2)(-0.161) - 1.849(4)}{-0.161 - 4}$$
$$= \frac{7.7182}{4.161} = 1.8549$$
$$\Rightarrow f(x_3) = (1.8549)^4 - (1.8549) - 10 = -0.019$$

Now $f(x_3)$ and $f(x_2)$ are of same sign. Hence, the root does not lie between x_2 and x_3. But $f(x_3)$ and $f(x_1)$ are of opposite signs. So the root lies between x_3 and x_1 and the third-order approximation of the root is

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$
$$= \frac{1.849(-0.019) - 1.8549(-0.161)}{-0.019 + 0.161}$$
$$= \frac{0.2635}{0.142} = 1.8557$$

9. Using Newton–Raphson method, find a positive root of $x^4 - x - 9 = 0$.

Solution: Let $f(x) = x^4 - x - 9$.

Now,

$$f(0) = -9 < 0$$
$$f(1) = -9 < 0$$
$$f(2) = 5 > 0$$

Therefore, the root lies between 1 and 2. Now,

$$f(1.5) = (1.5)^4 - (1.5) - 9 = -5.4375$$
$$f(1.75) = (1.75)^4 - (1.75) - 9 = -1.3711$$
$$f(1.8) = (1.8)^4 - (1.8) - 9 = 0.3024$$
$$f(1.9) = (1.9)^4 - (1.9) - 9 = 2.1321$$
$$f(2) = (2)^4 - (2) - 9 = 5$$

There is a change of sign between 1.75 and 1.8. Hence, the root lies between 1.75 and 1.8.

By Newton–Raphson method, we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Since $f(x)$ and $f'(x)$ have same sign at 1.8, we choose 1.8 as a starting point, that is $x_0 = 1.8$.

$$x_1 = x_0 - \frac{f(x)}{f'(x)} = 1.8 - \frac{f(1.8)}{f'(1.8)}$$
$$= 1.8 - \frac{0.3024}{22.328} = 1.8 - 0.0135 = 1.7865$$

The second approximation is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.7865 - \frac{(1.7865)^4 - 1.7865 - 9}{4(1.7865)^3 - 1}$$
$$= 1.7865 + \frac{0.6003}{21.807} = 1.814$$

The third approximation is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.814 - \frac{(1.814)^4 - 1.814 - 9}{4(1.814)^3 - 1}$$
$$= 1.814 - \frac{0.014}{22.8766} = 1.8134$$

The fourth approximation is given by

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.8134 - \frac{(1.8134)^4 - 1.8134 - 9}{4(1.8134)^3 - 1}$$
$$= 1.8134 - \frac{0.000303}{22.8529} = 1.8134$$

Since $x_3 = x_4 = 1.8134$, the desired root is 1.8134.

10. Compute $\int_0^{\pi/2} \sin x \, dx$ using Simpson's three-eighth rule of integration, taking $h = \frac{\pi}{18}$.

Solution: The values of $f(x) = \sin x$ are as follows:

x	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$
$f(x)$	0	0.17365	0.342020	0.5	0.64279
x	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{9\pi}{18} = \frac{\pi}{2}$
$f(x)$	0.76604	0.86603	0.93969	0.984808	1

By Simpson's 3/8 rule,

$$\int_0^{\pi/2} \sin x\, dx = \frac{3h}{8}\big[(y_0 + y_9) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8) + 2(y_3 + y_6)$$

$$= \frac{3}{8}\left(\frac{\pi}{18}\right)\big[(0+1) + 3(0.17365 + 0.342020 + 0.64279 + 0.76604 + 0.93969 + 0.984808) + 2(0.5 + 0.86603)\big]$$

$$= 1.06589044$$

11. The table below shows the temperature $f(t)$ as a function of time:

t	1	2	3	4	5	6	7
$f(t)$	81	75	80	83	78	70	60

Use Simpson's 1/3 method to estimate $\int_1^7 f(t)\, dt$.

Solution: Say $f(t) = y$. Also we have taken $h = 1$. By Simpson's 1/3 rule,

$$\int_1^7 f(t)\, dt = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(81 + 60) + 4(75 + 83 + 70) + 2(80 + 78)]$$

$$= \frac{1}{3}[141 + 912 + 316] = \frac{1369}{3} = 456.3333.$$

12. Evaluate $\int_0^{\pi/2} e^{\sin x} dx$ taking $h = \pi/6$.

Solution: Let $y = e^{\sin x}$. Given $h = \pi/6$. Length of interval is $\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}$. The table can be constructed as follows:

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6} = \frac{\pi}{3}$	$\frac{3\pi}{6} = \frac{\pi}{2}$
$y = e^{\sin x}$	1 (y_0)	1.6487 (y_1)	2.3774 (y_2)	2.71828 (y_3)

Here $n = 3$. Using trapezoidal rule, we get

$$\int_0^{\pi/2} e^{\sin x} dx = \frac{h}{2}[(y_0 + y_3) + 2(y_1 + y_2)]$$

$$= \frac{\pi}{2}[(1 + 2.71828) + 2(1.6487 + 2.3774)]$$

$$= \frac{\pi}{2}[(11.77048)] = 3.0815$$

13. Evaluate $\int_0^2 \frac{dx}{x^2 + x + 1}$ to three decimals, dividing the range of integration into 8 equal parts.

Solution: Dividing the interval (0, 2) into 8 equal parts, each of width

$$h = \frac{2 - 0}{8} = \frac{1}{4}$$

Therefore, the values of $f(x) = 1/(x^2 + x + 1)$ are as follows:

x	0	1/4	1/2	3/4	1
$f(x)$	1	0.76190	0.57143	0.43243	0.3333
x	5/4	3/2	7/4	2	
$f(x)$	0.2623	0.215	0.17204	0.1429	

By trapezoidal rule,

$$\int_0^2 \frac{dx}{x^2 + x + 1} = \frac{1}{8}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{1}{8}[(1 + 0.1429) + 2(0.76190 + 0.57143 + 0.43243 + 0.2623 + 0.2105 + 0.17204 + 0.3333)] = 0.8288$$

14. Evaluate $\int_0^1 \frac{1}{1+x} dx$ by

(a) trapezoidal rule
(b) Simpson's 1/3 rule
(c) Simpson's 3/8 rule

Solution: We divide the interval [0, 1] into six (multiple of 3) subintervals. Let $y = 1/(1 + x)$.

The table is constructed as follows:

x	0	1/6	2/6	3/6
$y = \dfrac{1}{1+x}$	1 (y_0)	0.8571 (y_1)	0.75 (y_2)	0.6666 (y_3)
x	4/6	5/6	1	
$y = \dfrac{1}{1+x}$	0.6 (y_4)	0.5454 (y_5)	0.5 (y_6)	

(a) By trapezoidal rule

$$\int_0^1 \frac{1}{1+x}\,dx$$

$$= \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{12}[(1+0.5)+2(0.8571+0.75+0.6666$$
$$+0.6+0.5454)] = 0.69485$$

(b) By Simpson's 1/3 rule

$$\int_0^1 \frac{1}{1+x}\,dx$$

$$= \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{18}[(1+0.5)+4(0.8571+0.6666+0.5454)$$
$$+2(0.75+0.6)] = 0.6931$$

(c) By Simpson's 3/8 rule

$$\int_0^1 \frac{1}{1+x}\,dx$$

$$= \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)]$$

$$= \frac{3}{(6)(8)}[(1+0.5)+3(0.8571+0.75+0.6$$
$$+0.5454)+2(0.6666)]$$

$$= \frac{1}{16}[1.5+8.2575+1.3332] = \frac{11.0907}{16}$$
$$= 0.6932$$

15. Find the value of y at $x = 0.1$ by Picard's method, given that $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, $y(0) = 1$.

Solution: Here

$$f(x,y) = \frac{y-x}{y+x},\ x_0 = 0 \text{ and } y_0 = 1$$

By Picard's method,

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx = y_0 + \int_0^x \frac{y-x}{y+x}\,dx \qquad (1)$$

For the first approximation, in the integrand on the R.H.S. of Eq. (1), y is replaced by its initial value which is 1. Therefore,

$$y^{(1)} = 1 + \int_0^x \frac{1-x}{1+x}\,dx = 1 + \int_0^x \left(-1+\frac{2}{1+x}\right)dx$$

$$= 1 + \left[-x + 2\log(1+x)\right]_0^x$$

$$= 1 + \left[-x + 2\log(1+x)\right] - \left[0 + 2\log(1+0)\right]$$

$$= 1 - x + 2\log(1+x) \qquad (2)$$

For the second approximation, from Eq. (1),

$$y^{(2)} = 1 + \int_0^x \frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x}\,dx$$

$$= 1 + \int_0^x \frac{1-2x+2\log(1+x)}{1+2\log(1+x)}\,dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1+2\log(1+x)}\right]dx$$

$$= 1 + x - 2\int_0^x \frac{x}{1+x\log(1+x)}\,dx$$

which is very difficult to integrate.

Hence, we use the first approximation, Eq. (2), itself as the value of y.

$$y(x) = y^{(1)} = 1 - x + 2\log(1+x)$$

Putting $x = 0.1$, we obtain

$$y(0.1) = 1 - 0.1 + 2\log(1.1) - 1 - 0.1 + 0.1906203$$
$$= 1.0906204 = 1.0906 \text{ (Correct to 4 decimals)}$$

16. Using Euler's method, compute $y(0.04)$ for the differential equation $\dfrac{dy}{dx} = -y;\ y(0) = 1$.
Take $h = 0.01$.

Solution: We have

$$x_0 = 0,\ y_0 = 1,\ h = 0.01$$

and $$\frac{dy}{dx} = f(x,y) = -y \quad h = 0.01$$

First approximation:

$$y_1 = y(0.01) = y_0 + hf(x_0, y_0)$$
$$= 1 + (0.01)(-1) = 0.99$$

Second approximation:

$$y_2 = y(0.02) = y_1 + hf(x_1, y_1)$$
$$= 0.99 + (0.01)(0.99) = 0.9999$$

Third approximation:

$$y_3 = y(0.03) = y_2 + h\,f(x_2, y_2)$$
$$= 0.9999 + (0.01)(-0.9999) = 1.009899$$

Fourth approximation:

$$y_4 = y(0.04) = y_3 + h\,f(x_3, y_3)$$
$$= 1.009899 + (0.01)(-0.009899) = 1.01999799$$

17. Find $y(0.1)$ using Euler's modified formula given that $\dfrac{dy}{dx} = x^2 - y, y(0) = 1.$

Solution: We have

$$\frac{dy}{dx} = f(x, y) = x^2 - y, x_0 = 0, y_0 = 1 \text{ and } h = 0.1$$

Now to find $y_1 = y(0.1)$, we have

$$f(x_0, y_0) = f(0,1) = 0 - 1 = -1$$

Using Euler's formula, we get

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.1)(-1) = 0.9$$

Now $x_1 = 0.1$ and

$$f\left(x_1, y_1^{(0)}\right) = f(0.1, 0.9) = (0.1)^2 - 0.9 = -0.89$$

First approximation:

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$
$$= 1 + \frac{0.1}{2}\left[-1 + f(0.1, -0.89)\right] = 1 + \frac{0.1}{2}(-1 + 0.9)$$
$$= 1 - 0.005 = 0.995$$

Second approximation:

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$
$$= 1 + \frac{0.1}{2}\left[-1 + f(0.1, 0.995)\right]$$
$$= 1 + \frac{0.1}{2}(-1 - 0.985) = 1 - 0.09925$$
$$= 0.90075$$

Third approximation:

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$
$$= 1 + \frac{0.1}{2}\left[-1 + f(0.1, 0.90075)\right]$$
$$= 1 + \frac{0.1}{2}(-1 - 0.89075) = 0.90546$$

Fourth approximation:

$$y_1^{(4)} = y_0 + \frac{h}{4}\left[f(x_0, y_0) + f\left(x_1, y_1^{(3)}\right)\right]$$
$$= 1 + \frac{0.1}{2}\left[-1 + f(0.1, 0.90546)\right]$$
$$= 1 + \frac{0.1}{2}(-1 - 0.89546)$$
$$= -1 - 0.09477 = 0.90523$$

Fifth approximation:

$$y_1^{(5)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(4)}\right)\right]$$
$$= 1 + \frac{0.1}{2}\left[-1 + f(0.1, 0.90523)\right]$$
$$= 1 + \frac{0.1}{2}(-1 - 0.89523)$$
$$= -1 - 0.09476 = 0.90523$$

Since $y_1^{(4)} = y_1^{(5)}$, we have

$$y_1 = y(0.1) = 0.90523$$

18. Use Runge–Kutta method of fourth order, solve the following differential equation:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2} \text{ with } y(0) = 1 \text{ at } x = 0.2,\ 0.4.$$

Solution: Here, $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}, x_0 = 0, y_0 = 1$ and $h = 0.2$

By Runge–Kutta method,

$$k_1 = hf(x_0, y_0) = hf(0,1) = 0.2$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 = \frac{k_1}{2}\right) = hf(0.1, 1.1) = 0.19672$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf(0.1, 1.09836) = 0.239279$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = hf(0.2, 1.239279) = 0.29916$$

Therefore,

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.239279) + 0.29916]$$
$$= 0.228526$$

and $y_1 = y(0.2) = y_0 + \Delta y = 1 + 0.228526 = 1.228526$

Also $x_1 = x_0 + h = 0.2, y_1 = 1.228526$ and $h = 0.2$

Again, we calculate

$$k_1 = hf(x_1, y_1) = hf(0.2, 1.228526) = 0.19045$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = hf(0.3, 1.32375)$$
$$= 0.18046$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = hf(0.3, 1.31876)$$
$$= 0.180318$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = hf(0.4, 1.40884)$$
$$= 0.170161$$

Therefore,

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.19045 + 2(0.18046) + 2(0.180318) + 0.170161]$$
$$= 0.18036115$$
$$y_2 = y(0.4) = y_1 + \Delta y = 1.40889.$$

19. If $\frac{dy}{dx} = x + y^2$, use Runge–Kutta method of fourth order to find an approximate value of y for $x = 0.2$ given that $y = 1$ when $x = 0$. (Take $h = 0.1$.)

Solution: We are given that

$f(x, y) = x + y^2$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

By Runge–Kutta method, we have

$$k_1 = hf(x_0, y_0) = h\left(x_0 + y_0^2\right) = 0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\left[x_0 + \frac{h}{2} + \left(y_0 + \frac{k_1}{2}\right)^2\right]$$
$$= 0.11525$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\left[x_0 + \frac{h}{2} + \left(y_0 + \frac{k_2}{2}\right)^2\right]$$
$$= 0.116857$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = h\left[x_0 + h + (y_0 + k_3)^2\right]$$
$$= 0.134737$$

Therefore,

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.1 + 2(0.11525) + 2(0.116857) + 0.134737]$$
$$= 0.116492$$

Thus, $y_1 = y(0.1) = y_0 + \Delta y = 1.116492$ and $x_1 = x_0 + h = 0.1$.

Now, $x_1 = 0.1, y_1 = 1.116492$ and $h = 0.1$

Again,

$$k_1 = hf(x_1, y_1) = h\left(x_1 + y_1^2\right) = 0.134655$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = h\left[x_1 + \frac{h}{2} + \left(y_1 + \frac{k_1}{2}\right)^2\right]$$
$$= 0.155143$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = h\left[x_1 + \frac{h}{2} + \left(y_1 + \frac{k_2}{2}\right)^2\right]$$
$$= 0.157579$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = h\left[x_1 + h + (y_1 + k_3)^2\right]$$
$$= 0.1823257$$

Therefore,

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.134655 + 2(0.155143) + 2(0.157579) + 0.1823257]$$
$$= 0.15707$$

Hence, $y_2 = y(0.2) = y_1 + \Delta y = 1.273562$ and $x_2 = x_1 + h = 0.2$.

20. Given $y' = x^2 - y$, $y(0) = 1$, find correct to four decimal places the value of $y(0.1)$, by using Euler's method.

Solution: We are given that

$x_0 = 0, y_0 = 1, h = 0.1$ and $f(x, y) = x^2 - y$.

By Euler's method, we have

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Therefore,

$$y_1 = y(0.1) = y_0 + hf(x_0, y_0) = 1 + (0.1)f(0, 1)$$
$$= 1 + (0.1)(0 - 1) = 1 - 0.1 = 0.9$$

Hence, the solution is $y(0.1) = 0.9$.

21. While measuring a certain parameter, the mean was determined to be 25. The measurements were repeated four times and the mean in the other three measurements were observed to be 26, 24 and 27. Determine the resultant value.

(a) 25 ± 1

(b) 25 ± 2

(c) 25.5 ± 1

(d) 25.5 ± 0.6

Solution: The deviations of mean in the four measurements are 0, 1, 1 and 2. Sum of squares of these deviations is 6. The standard deviation is $\sqrt{6/4} = 1.22$. The mean is 25.5; standard deviation of the mean is $1.22/\sqrt{4} = 0.61$. The result therefore is 25.5 ± 0.6.

22. A certain velocity was measured by measuring displacement x over a period of time t. The two quantities were measured as 5.1 ± 0.4 m and 0.4 ± 0.1 s. What would be the uncertainty in the velocity?

Solution: We have that

Velocity, $v = \dfrac{x}{t}$

Therefore, uncertainty in velocity

$$\delta v = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta t}{t}\right)^2} = \sqrt{\left(\frac{0.4}{5.1}\right)^2 + \left(\frac{0.1}{0.4}\right)^2}$$
$$= 3.34 \text{ m/s}$$

23. An object is released from rest in a free fall. At a certain point during its free fall, its velocity was measured as 5 ± 0.2 m/s. How long it has been in free fall given that $g = 9.81$ m/s^2?

Solution: Time t is expressed as $t = -v/g = -(1/g) \times v$; g is constant.

Therefore,

$$\delta t = \left|-\frac{1}{g}\right| \times \delta v = \frac{1}{9.81} \times 0.2 = 0.0.0204 \cong 0.02$$

24. Determine the equation of the desired straight line, when a straight line given by $y = a + bx$, is fitted to the below-mentioned data using method of least squares.

x	1	2	3	4	5
y	2	5	3	8	7

Solution: The equation of least square line is $y = a + bx$.

Normal equation for a is $\Sigma y = na + b\Sigma x$, where n is number of observations = 5.

Normal equation for b is $\Sigma xy = a\Sigma x + b\Sigma x^2$.

Now $\Sigma x = 15$, $\Sigma x^2 = 55$, $\Sigma xy = 88$ and $\Sigma y = 25$.

Therefore, the normal equations become

$$5a + 15b = 25 \text{ and } 15a + 55b = 88$$

Solving these equations simultaneously, we get $a = 1.1$ and $b = 1.3$.

Therefore, equation of least square line is $y = 1.1 + 1.3x$.

25. In equation obtained in Question 24, if $\hat{y}$ are the trend values, then $\Sigma(y - \hat{y})$ would be equal to

Solution: Trend values are 2.4 (for $x = 1$), 3.7 (for $x = 2$), 5.0 (for $x = 3$), 6.3 (for $x = 4$) and 7.6 (for $x = 5$). Corresponding values of $(y - \hat{y})$ are -0.4, $+1.3$, -2, $+1.7$ and -0.6. Therefore, $\sum y - \hat{y} = -0.4 + 1.3 - 2 + 1.7 - 0.6 = 0$.

26. If a second-degree parabola, defined by $y = a + bx + cx^2$, is to be made to fit to the below-mentioned data, then determine the expression for best-fit curve.

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Solution: The parabola of fit is given by $y = a + bx + cx^2$.

Assume the following substitution to simplify calculations $u = x - 2$ and $v = y$.

The equation of parabola becomes $v = A + Bu + Cu^2$.

The normal equations are

$$\Sigma v = 5A + B\Sigma u + C\Sigma u^2$$

which gives $5A + 10C = 12.9$ (1)

(Σu and Σu^2 can be computed from given data).

$$\Sigma uv = A\Sigma u + B\Sigma u^2 + c\Sigma u^3$$

which gives $10B = 11.3$ (2)

$$\Sigma u^2 v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4$$

which gives $10A + 34C = 33.5$ (3)

Now B can be computed as $B = 11.3/10 = 1.13$.

A and C can be determined by simultaneous solution of Eqs. (1) and (3).

Multiplying Eq. (1) by 2, we get

$$10A + 20C = 25.8 \tag{4}$$

Subtracting Eq. (4) from Eq. (3), we get

$$14C = 7.7;\ C = \frac{7.7}{14} = 0.55$$

Substituting the value of C in Eq. (3), we get

$$10A + 34 \times 0.55 = 33.5$$

$$A = \frac{33.5 - 34 \times 0.55}{10} = 1.48$$

Substituting the values of A, B and C, we get

$$v = 1.48 + 1.13u + 0.55u^2$$

Now $u = x - 2$ and $v = y$.
Therefore,

$$\begin{aligned} y &= 1.48 + 1.13\,(x - 2) + 0.55\,(x - 2)^2 \\ &= 1.48 + 1.13x - 2.26 + 0.55x^2 - 2.2x + 2.2 \\ &= 1.42 - 1.07x + 0.55x^2 \end{aligned}$$

27. Consider the following data:

X	1	1.3	1.6	1.9	2.2
$f(x)$	0.1411	−0.6878	−0.9962	−0.5507	−0.3115

where $f(x) = \sin 3x$.

Use Lagrange polynomials to estimate $f(1.5)$

Solution: As a first step, we find Lagrange polynomials $L_k(x)$ for $k = 1, 2, 3, 4$ and 5.

$$L_1(x) = \frac{(x-1.3)(x-1.6)(x-1.9)(x-2.2)}{(1-1.3)(1-1.6)(1-1.9)(1-2.2)}$$

$$L_2(x) = \frac{(x-1)(x-1.6)(x-1.9)(x-2.2)}{(1.3-1)(1.3-1.6)(1.3-1.9)(1.3-2.2)}$$

$$L_3(x) = \frac{(x-1)(x-1.3)(x-1.9)(x-2.2)}{(1.6-1)(1.6-1.3)(1.6-1.9)(1.6-2.2)}$$

$$L_4(x) = \frac{(x-1)(x-1.3)(x-1.6)(x-2.2)}{(1.9-1)(1.9-1.3)(1.9-1.6)(1.9-2.2)}$$

$$L_5(x) = \frac{(x-1)(x-1.3)(x-1.6)(x-1.9)}{(2.2-1)(2.2-1.3)(2.2-1.6)(2.2-1.9)}$$

Now

$$P(x) = 0.1411 \times L_1(x) - 0.6878 \times L_2(x) - 0.9962 \times L_3(x) - 0.5507 \times L_4(x) + 0.3115 \times L_5(x)$$

For $x = 1.5$,

$$L_1(x) = \frac{(0.2)(-0.1)(-0.4)(-0.7)}{(-0.3)(-0.6)(-0.9)(-1.2)} = -\frac{56 \times 10^{-4}}{1944 \times 10^{-4}} = -0.0288$$

$$L_2(x) = \frac{(0.5)(-0.1)(-0.4)(-0.7)}{(0.3)(-0.3)(-0.6)(-0.9)} = -\frac{140 \times 10^{-4}}{486 \times 10^{-4}} = 0.288$$

$$L_3(x) = \frac{(0.5)(0.2)(-0.4)(-0.7)}{(0.6)(0.3)(-0.3)(-0.6)} = -\frac{280 \times 10^{-4}}{324 \times 10^{-4}} = 0.864$$

$$L_4(x) = \frac{(0.5)(0.2)(-0.1)(-0.7)}{(0.9)(0.6)(0.3)(-0.3)} = -\frac{70 \times 10^{-4}}{486 \times 10^{-4}} = -0.144$$

$$L_5(x) = \frac{(0.5)(0.2)(-0.1)(-0.4)}{(1.2)(0.9)(0.6)(0.3)} = -\frac{40 \times 10^{-4}}{1944 \times 10^{-4}} = 0.020$$

$$f(1.5) = \sin(4.5) \cong -0.9975$$

$$\begin{aligned} P\,(1.5) &= -0.1411 \times 0.0288 - 0.6878 \times 0.288 - 0.9962 \times 0.864 - 0.5507 \times (-0.144) + 0.3115 \times 0.020 \\ &\cong -0.0040 - 0.1980 - 0.8607 + 0.0793 + 0.0062 \\ &\cong -0.9772 \end{aligned}$$

Therefore, $P(1.5)$ very closely approximates $f(1.5)$. In fact in the interval $x = 1$ to 2.2, $P\,(x)$ approximates $f(x)$.

28. The following data give the velocity of a particle for 20 s at an interval of 5 s. Find the initial acceleration using the entire data.

Time (t) in seconds	0	5	10	15	20
Velocity (v) in m/s	0	3	14	69	228

Solution: The difference table can be drawn from the given data as follows:

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
		3			
5	3		8		
		11		36	
10	14		44		24
		55		60	
15	69		104		
		159			
20	228				

The aim is to find initial acceleration, that is, dv/dt at $t = 0$. Therefore, we will use Newton's forward formula.

$$\begin{aligned} \left[\frac{dv}{dt}\right]_{t=0} &= \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \cdots\right] \\ &= \frac{1}{5}\left[3 - \frac{8}{2} + \frac{36}{3} - \frac{24}{4}\right] \\ &= \frac{1}{5}(3 - 4 + 12 - 6) = 1 \text{ m/s} \end{aligned}$$

29. In a given machine, a slider moves along a straight rod. The distance x that it moves along the rod for different values of t is given below. Find the acceleration and velocity of the slider when $t = 0.3$ s.

t	x	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$	$\Delta^5 x$	$\Delta^6 x$
0.0	30.13						
		1.49					
0.1	31.62		− 0.24				
		1.25		− 0.24			
0.2	32.87		− 0.48		0.26		
		0.77		0.02		− 0.27	
0.3	33.64		− 0.46		−0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		− 0.45		0.01		
		− 0.14		0.02			
0.5	33.81		− 0.43				
		− 0.57					
0.6	33.24						

t	0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Solution: The difference table is given as follows:

Since velocity and acceleration are to be determined near the middle of the table, we will use Stirling's formula.

$$\left[\frac{dx}{dt}\right]_{t_0} = \frac{1}{h}\left[\frac{\Delta x_0 + \Delta x_{-1}}{2}\right] - \frac{1}{6}\left[\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right] + \frac{1}{30}\left[\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right] + \cdots$$

and $$\left[\frac{d^2x}{dt^2}\right]_{t_0} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} \cdots\right]$$

It is given that $h = 0.1$, $t_0 = 0.3$.

From the difference table,

$\Delta x_0 = 0.31$, $\Delta x_{-1} = 0.77$, $\Delta^2 x_{-1} = -0.46$, $\Delta^3 x_{-1} = 0.01$, $\Delta^3 x_{-2} = 0.02$, $\Delta^5 x_{-2} = 0.2$ and $\Delta^5 x_{-3} = 0.2$

Substituting these values in expressions for velocity and acceleration, we get

$$\left[\frac{dx}{dt}\right]_{0.3} = \frac{1}{0.1}\left[\frac{0.31 + 0.77}{2} - \frac{1}{6}\left(\frac{0.01 + 0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02 - 0.27}{2}\right) - \cdots\right]$$

$$= 5.33$$

$$\left[\frac{d^2x}{dt^2}\right]_{0.3} = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \cdots\right]$$

$$= -45.6$$

30. Obtain Newton's forward interpolating polynomial $P_5(x)$ for the following tabular data. What would be the interpolated value of the function at $x = 0.0045$?

X	0	0.001	0.002	0.003	0.004	0.005
Y	1.121	1.123	1.1255	1.127	1.128	1.1285

Solution: The forward difference table is drawn as follows:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
0	1.121	0.002	0.005		0.002	−0.0025
				−0.0015		
0.001	1.123	0.0025	−0.0010	0.0005	−0.0005	
0.002	1.1255	0.0015	−0.0005	0.0		
0.003	1.127	0.001	−0.0005			
0.004	1.128	0.005				
0.005	1.1285					

Now $p = \dfrac{x - x_0}{h} = \dfrac{0.0045 - 0}{0.001} = 4.5$

$$P_5(x) = 1.121 + 0.002 \times 4.5 + \frac{(4.5)(4.5-1)}{2!} \times 0.0005 + \frac{4.5(4.5-1)(4.5-2)}{3!} \times (-0.0015) + \frac{4.5(4.5-1)(4.5-2)(4.5-3)}{4!} \times (0.002) + \frac{4.5(4.5-1)(4.5-2)(4.5-3)(4.5-4)}{5!} \times (-0.0025)$$

$$= 1.121 + 0.002 \times 4.5 + \frac{0.0005}{2} \times 4.5 \times 3.5 - \frac{0.0015}{6} \times 4.5 \times 3.5 \times 2.5 + \frac{0.002}{24} \times 4.5 \times 3.5 \times 2.5 \times 1.5 - \frac{0.0025}{120} \times 4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5$$

$$= 1.12840045$$

PRACTICE EXERCISE

1. Solve the following systems of linear equation:

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

(a) $x_1 = 3/2,\ x_2 = 1/2,\ x_3 = -1$
(b) $x_1 = 1,\ x_2 = 1/2,\ x_3 = -1/2$
(c) $x_1 = 1,\ x_2 = 1,\ x_3 = -1/2$
(d) $x_1 = 1,\ x_2 = 2,\ x_3 = -1/2$

2. Solve the following system using Crout's decomposition method: $3x - y + 2z = 12, x + 2y + 3z = 11,$ $2x - 2y - z = 3.$

(a) $x = 3,\ y = 1,\ z = 2$
(b) $x = 3,\ y = 1,\ z = 2$
(c) $x = 3,\ y = 1,\ z = 2$
(d) $x = 3,\ y = 1,\ z = 2$

3. Solve the following system of equations using Gauss elimination method:

$$10a - 7b + 3c + 5d = 6, -6a + 8b - c - 4d = 5,$$

$$3a + b + 4c + 11d = 2, 5a - 9b - 2c + 4d = 7$$

(a) $a = 2,\ b = 4,\ c = -1,\ d = 1$
(b) $a = 1,\ b = 3,\ c = 2,\ d = 1$
(c) $a = 5,\ b = 4,\ c = -7,\ d = 1$
(d) $a = 3,\ b = 4,\ c = -7,\ d = -1$

4. Use Gauss–Seidel iterative method to solve the following system of simultaneous equation:

$$9x + 4y + z = -17; x - 2y - 6z; x + 6y = 4$$

Perform four iterations.

(a) $x = 2, y = -3, z = -2$
(b) $x = -2, y = 1, z = -2$
(c) $x = -2, y = 1, z = -3$
(d) $x = -1, y = -1, z = -4$

5. Solve the following system of equations by using Gauss–Jacobi method (correct to three decimal places):

$$8x - 3y + 2z = 20;\ 4x + 11y - z = 33;$$
$$6x + 3y + 12z = 35$$

(a) $x = 3.102, y = 2.122, z = 1.052$
(b) $x = 3.123, y = 2.102, z = 0.901$
(c) $x = 2.999, y = 1.912, z = 1.101$
(d) $x = 3.016, y = 1.986, z = 0.912$

6. Find a real root of the equation $x^3 - 6x - 4 = 0$ by bisection method.

(a) 2.0575 (b) 2.71875
(c) 3.0252 (d) 2.85578

7. Find a root of the equation $x^3 - 4x - 9 = 0$ by bisection method correct to three decimal places.

(a) 2.706 (b) 2.890
(c) 2.656 (d) 2.901

8. Use Regula–Falsi method to find the smallest positive root of the following equation correct to four significant digits $x^3 - 5x + 1 = 0.$

(a) 0.1626 (b) 0.2016
(c) 0.1832 (d) 0.2805

9. Find an approximate root of $x \log_{10} x - 1.2 = 0$ by Regula–Falsi method.

(a) 2.1550 (b) 3.0125
(c) 2.4022 (d) 2.7405

10. Find the real root of the equation $x\log_{10}x = 4.77$ correct to four decimal places using Newton–Raphson method.

(a) 5.5152 (b) 4.4784
(c) 6.0831 (d) 7.0125

11. Find the real root of the equation $3x = \cos x + 1$ using Newton–Raphson method.

(a) 0.5887 (b) 0.6255
(c) 0.6071 (d) 0.7015

12. Compute the value of $\int_{0.2}^{1.4}\left(\sin x - \log x = e^{x}\right)dx$ using Simpson's 3/8 rule.

(a) 4.053 (b) 3.755
(c) 5.052 (d) 2.784

13. Compute the value of $\int_{0}^{0.6} e^{-x^2}dx$ using Simpson's 1/3 rule by taking seven ordinates.

(a) 0.6012 (b) 0.5351
(c) 0.1255 (d) 0.4292

14. Evaluate $\int_{1}^{2}\frac{dx}{x}$ by Simpson's 1/3 rule with four strips.

(a) 0.55253 (b) 0.72456
(c) 0.59874 (d) 0.69326

15. Evaluate $\int_{0}^{6}\frac{e^{x}}{1+x}dx$ approximately using Simpson's 3/8 rule on integration.

(a) 0.653556 (b) 0.553313
(c) 0.801256 (d) 0.701652

16. Evaluate $\int_{0.6}^{2.0} y\,dx$ using trapezoidal rule.

(a) 8.112 (b) 10.416
(c) 9.845 (d) 1.565

17. Evaluate $\int_{0}^{6}\frac{dx}{1+x^2}$ by using trapezoidal rule.

(a) 5.1414 (b) 3.4422
(c) 1.4108 (d) 2.3432

18. If $\frac{dy}{dx} = 1 + y^2, y(0) = 1$, find $y(0.4)$ by using Euler's method. Take $h = 0.02$.

(a) 1.992 (b) 1.564
(c) 2.014 (d) 1.664

19. Given that $y' = 1 - y, y(0)$, find the value of $y(0.2)$ using modified Euler's method.

(a) 0.07111 (b) 0.22345
(c) 0.18098 (d) 0.56522

20. Find $y(1)$ and $y(2)$ using Runge–Kutta fourth-order formula $y' = x^2 - y$ and $y(0) = 1$.

(a) 0.8213 (b) 0.9052
(c) 0.5212 (d) 1.2428

21. Five measurements were made of the weight of certain article. The results obtained were 9.8 kg, 9.9 kg, 10.2 kg, 10.1 kg and 9.9 kg. If the true weight is 10 kg, then the measurement can be termed as

(a) accurate and precise
(b) accurate but lacking precision
(c) inaccurate but precise
(d) inaccurate and imprecise

22. In the above question, if the true value of the weight were 12 kg, then the measurement can be termed as

(a) accurate and precise
(b) precise but inaccurate
(c) inaccurate and imprecise
(d) accurate but imprecise

23. The uncertainty in a certain measurement was observed to be 5% (relative error). The measurement was repeated 25 times. The uncertainty would now be (assume errors are random)

(a) 5% only (b) 0.2%
(c) 1% (d) None of these

24. While measuring a certain length, whose true value is 100 m, the measured value was observed to be 98 m. Relative error in this case is

(a) 2 m
(b) 0.02
(c) Indeterminate from given data
(d) None of these

25. Which one of the following error types is random?

(a) Errors in the calibration of measuring instruments
(b) Parallax error
(c) Bias of the person making measurement
(d) AC noise causing voltmeter needle to fluctuate

26. An experiment resulted in the standard deviation of the quantity under measurement to be equal to 0.2.

The experiment was repeated 16 times. The standard deviation of the mean is equal to

(a) 3.2
(b) 0.0125
(c) Indeterminate from given data since mean value is not given
(d) 0.05

27. A quantity R depends upon quantities X, Y and Z. The value of R is given by $R = X + Y - Z$. The uncertainty δR in the value of R in terms of uncertainties δX, δY and δZ, respectively, in the values of X, Y and Z is given by

(a) $\delta R = \delta X + \delta Y - \delta Z$
(b) $(\delta R)^2 = (\delta X)^2 + (\delta Y)^2 + (\delta Z)^2$
(c) $(\delta R)^2 = (\delta X)^2 + (\delta Y)^2 - (\delta Z)^2$
(d) $\delta R = \delta X + \delta Y + \delta Z$

28. The starting and finishing positions in a 100 m race were measured as 0 ± 0.05 m and 100 ± 0.15 m. The error in the measured track length would be

(a) 0.151 m
(b) 0.2 m
(c) Indeterminate from given data
(d) None of these

29. Use least square method to determine the equation of the line of best fit for the below given data.

x	8	2	11	6	5	4	12	9	6	1
y	3	10	3	6	8	12	1	4	9	14

(a) $y = 1.1 - 14x$
(b) $y = 14 - 1.1x$
(c) $y = 14 + 1.1x$
(d) $y = -1.1 + 14x$

30. If $f(x)$ is known at the following data points, find $f(0.5)$ using Newton's forward different formula.

x_i	0	1	2	3	4
f_i	1	7	23	55	109

(a) 3.125
(b) 3.215
(c) 2.315
(d) 2.325

31. If it is desired to find derivative of a function for a given data near the beginning of the table, one would preferably employ

(a) Stirling's formula
(b) Newton's backward formula
(c) Newton's forward formula
(d) None of these

32. A function $f(x)$ is defined for five different values of (x) at $x = 1, 2, 3, 4$ and 5. It is desired to determine $f'(x)$ at $x = 3$. One of the following formulas would preferably be used for computation:

(a) Stirling's formula
(b) Newton's backward formula
(c) Newton's forward formula
(d) None of these

ANSWERS

1. (b) **2.** (a) **3.** (c) **4.** (c) **5.** (d) **6.** (b) **7.** (a)
8. (b) **9.** (d) **10.** (c) **11.** (c) **12.** (a) **13.** (b) **14.** (d)
15. (d) **16.** (b) **17.** (c) **18.** (a) **19.** (c) **20.** (d) **21.** (a)
22. (b) **23.** (c) **24.** (b) **25.** (d) **26.** (d) **27.** (b) **28.** (a)
29. (b) **30.** (a) **31.** (c) **32.** (a)

EXPLANATIONS AND HINTS

1. (b) We choose $u_{ii} = 1$ and write

$$A = LU$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

By equating, we get

$l_{11} = 1,\ l_{21} = 4,\ l_{31} = 3$ and

$l_{11}u_{12} = 1 \Rightarrow u_{12} = 1, \quad l_{11}u_{13} = 1 \Rightarrow u_{13} = 1$

$l_{21}u_{12} + l_{22} = 3 \Rightarrow 4 + l_{22} = 3, \quad l_{22} = 3 \Rightarrow l_{22} = -1$

$l_{21}u_{13} + l_{22}u_{23} = -1 \Rightarrow 4 + (-1)u_{23} = -1 \Rightarrow u_{23} = 5$

$l_{31}u_{12} + l_{32} = 5 \Rightarrow 3 + l_{32} = 5 \Rightarrow l_{32} = 2$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 3 \Rightarrow 3 + 2.5 + l_{33} = 3 \Rightarrow l_{33} = -10$

Thus, we get $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

The given system is $AX = B \Rightarrow LUX = B$. (1)

Let UX = Y. So from Eq. (1), we have

$$LY = B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

By solving, we get $y_1 = 1$

$$4y_1 - y_2 = 6 \Rightarrow y_2 = -2$$
$$3y_1 + 2y_2 - 10y_3 = 4 \Rightarrow y_3 = \frac{-1}{2}$$

Now,

$$\text{UX} = \text{Y} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}$$

By solving, we get

$$x_3 = \frac{-1}{2}$$
$$x_2 + 5x_3 = -2 \Rightarrow x_2 - \frac{5}{2} = -2 \Rightarrow x_2 = \frac{1}{2}$$
$$x_1 + x_2 + x_3 = 1 \Rightarrow x_1 + \frac{1}{2} - \frac{1}{2} = 1 \Rightarrow x_1 = 1$$

Hence, the solution of the system of equations is $x_1 = 1, x_2 = \frac{1}{2}$ and $x_3 = -\frac{1}{2}$.

2. (a) We choose $u_{ii} = 1$ and write

$$A = LU$$

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

By equating, we get $l_{11} = 3,\ l_{21} = 1,\ l_{31} = 2$ and

$$l_{21}u_{12} = -1 \Rightarrow u_{12} = \frac{-1}{3}$$
$$l_{11}u_{13} = 2 \Rightarrow u_{13} = \frac{2}{3}$$
$$l_{21}u_{12} + l_{22} = 2 \Rightarrow l_{22} = \frac{7}{3}$$
$$l_{31}u_{12} + l_{32} = -2 \Rightarrow l_{32} = \frac{-4}{3}$$
$$l_{21}u_{13} + l_{22}u_{23} = 3 \Rightarrow u_{23} = 1$$
$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = -1 \Rightarrow l_{33} = -1$$

Thus, we get

$$A = LU = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 7/3 & 0 \\ 2 & -4/3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The given system is $AX = B \Rightarrow LUX = B$. Let $UX = Y$, so $LY = B$.

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 1 & 7/3 & 0 \\ 2 & -4/3 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \\ 2 \end{bmatrix}$$

By solving, we get $3y_1 = 12 \Rightarrow y_1 = 4$

$$y_1 + \frac{7}{3}y_2 = 11 \Rightarrow y_2 = 3$$
$$2y_1 - \frac{4}{3}y_2 - y_3 = 2 \Rightarrow y_3 = 2$$

Now,

$$\text{UX} = \text{Y} \Rightarrow \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

By solving, we get $z = 2$

$$y + z = 3 \Rightarrow y = 1$$
$$x - \frac{y}{3} + \frac{2}{3}z = 4 \Rightarrow x = 3$$

Hence, the solution is $x = 3,\ y = 1,\ z = 2$.

3. (c) We have

$$10a - 7b + 3c + 5d = 6 \quad (1)$$
$$-6a + 8b - c - 4d = 5 \quad (2)$$
$$3a + b + 4c + 11d = 2 \quad (3)$$
$$5a - 9b - 2c + 4d = 7 \quad (4)$$

To eliminate a, let us perform Eq. (2) $-\left(\frac{-3}{5}\right)$Eq. (1), Eq. (3) $-\left(\frac{3}{10}\right)$Eq. (1) and Eq. (4) $-\left(\frac{1}{2}\right)$Eq. (1). Therefore, we get

$$3.8b + 0.8c - d = 8.6 \quad (5)$$

$$3.1b + 3.1c + 9.5d = 0.2 \quad (6)$$

$$-5.5b - 3.5c + 1.5d = 4 \quad (7)$$

To eliminate b, let us perform Eq. (6) $-\left(\frac{3.1}{3.8}\right)$Eq. (5) and Eq. (7) $-\left(\frac{-5.5}{3.8}\right)$Eq. (5). Therefore, we get

$$2.4473684c + 10.315789d = -6.8157895 \quad (8)$$

$$-2.3421053c + 0.0526315d = 16.447368 \quad (9)$$

To eliminate c, let us perform Eq. (9) $-\left(\frac{-2.3421053}{2.4473684}\right)$Eq. (8). Therefore, we get

$$9.9249319d = 9.9245977 \quad (10)$$

From Eq. (10), we have

$$d = 1$$

Putting the value of d in Eq. (9), we get

$$c = -7$$

Putting the value of c and d in Eq. (7), we get

$$b = 4$$

Putting the value of b, c and d in Eq. (4), we get

$$a = 5.$$

4. (c) The given system is not diagonally dominant. Hence, rearranging the equations as

$$9x + 4y + z = -17; \; x + 6y = 4; \; x - 2y = 14$$

Now, Gauss–Seidel's iterative method can be applied.

From the above equations, we get

$$x = \frac{1}{9}(-17 - 4y - z) \quad (1)$$

$$y = \frac{1}{6}(4 - x) \quad (2)$$

$$z = \frac{1}{6}(x - 2y - 14) \quad (3)$$

Starting with $y = 0 = z$, we obtain

$$x^{(1)} = -\frac{17}{9} = -1.8888$$

Now, putting $x = -1.888$ and $z = 0$ in Eq. (2), we get

$$y^{(1)} = \frac{1}{6}[4 - (-1.8888)] = 0.9815$$

Again putting $x = -1.888$ and $y = 0.9815$ in Eq. (3), we get

$$z^{(1)} = \frac{1}{6}[-1.8888 - 2(0.9815) - 14] = -2.9753$$

The first iteration is given by

$$x^{(1)} = -1.8888$$
$$y^{(1)} = 0.9815$$
$$z^{(1)} = -2.9753$$

The second iteration is given by

$$x^{(2)} = \frac{1}{9}(-17 - 4y^{(1)} - z^{(1)}) = -1.9945$$

$$y^{(2)} = \frac{1}{6}(4 - x^{(2)}) = 0.9991$$

$$z^{(2)} = \frac{1}{6}(x^{(2)} - 2y^{(2)} - 14) = -2.9988$$

The third iteration is given by

$$x^{(3)} = \frac{1}{9}(-17 - 4y^{(2)} - z^{(2)}) = -1.9997$$

$$y^{(3)} = \frac{1}{6}(4 - x^{(3)}) = 0.9999$$

$$z^{(3)} = (x^{(3)} - 2y^{(3)} - 14) = -2.9999$$

The fourth iteration is given by

$$x^{(4)} = \frac{1}{9}(-17 - 4y^{(3)} - z^{(3)}) = -1.9999 \simeq -2;$$

$$y^{(4)} = \frac{1}{6}(4 - x^{(4)}) = 0.9999 = 1$$

$$z^{(4)} = (x^{(4)} - 2y^{(4)} - 14) = -2.9999 = -3$$

Hence, after four iterations, we obtain $x = -2, y = 1, z = -3$.

5. (d) Since the diagonal elements are dominant in the coefficient matrix, we write x, y, z as follows:

$$x = \frac{1}{8}(20 + 3y - 2z) \quad (1)$$

$$y = \frac{1}{11}(33 - 4x + z) \quad (2)$$

$$z = \frac{1}{12}(35 - 6x - 3y) \quad (3)$$

Gauss–Jacobi method:

Let the initial values be $x = 0$, $y = 0$, $z = 0$

Using the values $x = 0$, $y = 0$, $z = 0$ in Eqs. (1), (2), (3), we get the first iteration as

$$x^{(1)} = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$

$$y^{(1)} = \frac{1}{11}[33 - 4(0) + 0] = 3.0$$

$$z^{(1)} = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.916666$$

The second iteration is given by

$$x^{(2)} = \frac{1}{8}[20 + 3(3.0) - 2(2.916666)] = 2.895833$$

$$y^{(2)} = \frac{1}{11}[33 - 4(2.5) + (2.916666)] = 2.356060$$

$$z^{(2)} = \frac{1}{12}[35 - 6(2.5) - 3(3.0)] = 0.916666$$

The third iteration is given by

$$x^{(3)} = \frac{1}{8}[20 + 3(2.356060) - 2(0.916666)] = 3.154356$$

$$y^{(3)} = \frac{1}{11}[33 - 4(2.895833) + (0.916666)] = 2.030303$$

$$z^{(3)} = \frac{1}{12}[35 - 6(2.895833) - 3(2.356060)] = 0.879735$$

The fourth iteration is given by

$$x^{(4)} = \frac{1}{8}[20 + 3(2.030303) - 2(0.879735)] = 3.041430$$

$$y^{(4)} = \frac{1}{11}[33 - 4(3.154356) + (0.879735)] = 1.932937$$

$$z^{(4)} = \frac{1}{12}[35 - 6(3.154356) - 3(2.030303)] = 0.831913$$

The fifth iteration is given by

$$x^{(5)} = \frac{1}{8}[20 + 3(1.932937) - 2(0.831913)] = 3.016873$$

$$y^{(5)} = \frac{1}{11}[33 - 4(3.041430) + (0.831913)] = 1.969654$$

$$z^{(5)} = \frac{1}{12}[35 - 6(3.041430) - 3(1.932939)] = 0.912717$$

The sixth iteration is given by

$$x^{(6)} = \frac{1}{8}[20 + 3(1.969654) - 2(0.912717)] = 3.010441$$

$$y^{(6)} = \frac{1}{11}[33 - 4(3.016873) + (0.912717)] = 1.985930$$

$$z^{(6)} = \frac{1}{12}[35 - 6(3.016873) - 3(1.969654)] = 0.915817$$

The seventh iteration is given by

$$x^{(7)} = \frac{1}{8}[20 + 3(1.985930) - 2(0.915817)] = 3.015770$$

$$y^{(7)} = \frac{1}{11}[33 - 4(3.010441) + (0.915817)] = 1.988550$$

$$z^{(7)} = \frac{1}{12}[35 - 6(3.010441) - 3(1.985930)] = 0.914964$$

The eighth iteration is given by

$$x^{(8)} = \frac{1}{8}[20 + 3(1.988550) - 2(0.914964)] = 3.016946$$

$$y^{(8)} = \frac{1}{11}[33 - 4(3.015770) + (0.914964)] = 1.986535$$

$$z^{(8)} = \frac{1}{12}[35 - 6(3.015770) - 3(1.988550)] = 0.911644$$

The ninth iteration is given by

$$x^{(9)} = \frac{1}{8}[20 + 3(1.986535) - 2(0.911644)] = 3.017039$$

$$y^{(9)} = \frac{1}{11}[33 - 4(3.016946) + (0.911644)] = 1.985805$$

$$z^{(9)} = \frac{1}{12}[35 - 6(3.016946) - 3(1.986335)] = 0.911560$$

The tenth iteration is given by

$$x^{(10)} = \frac{1}{8}[20 + 3(1.985805) - 2(0.911560)] = 3.016786$$

$$y^{(10)} = \frac{1}{11}[33 - 4(3.017039) + (0.911560)] = 1.985764$$

$$z^{(10)} = \frac{1}{12}[35 - 6(3.017039) - 3(1.985805)] = 0.911696$$

In 8^{th}, 9^{th} and 10^{th} iteration, the values of x, y and z are same correct to three decimal places. Hence, we stop at this level.

6. (b) Let

$$f(x) = x^3 - 6x - 4$$

Now,

$$f(2) = 8 - 12 - 4 = -8 < 0,$$

$$f(3) = 27 - 18 - 4 = 5 > 0$$

Therefore, one root lies between 2 and 3. Consider $x_0 = 2$ and $x_1 = 3$. By bisection method, the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = 2.5 \Rightarrow f(x_2) = (2.5)^3 - 6(2.5) - 4$$

$$= -3.375 < 0 \text{ and } f(3) > 0.$$

Therefore, the root lies between 2.5 and 3. The next approximation x_3 is given by

$$x_3 = \frac{2.5 + 3}{2} = \frac{5.5}{2} = 2.75 \Rightarrow f(x_3) = (2.75)^3$$

$$- 6(2.75) - 4 = 0.2968 > 0 \text{ and } f(2.5) < 0.$$

Therefore, the root lies between 2.5 and 2.75. The next approximation x_4 is given by

$$x_4 = \frac{2.5 + 2.75}{2} = \frac{5.25}{2} = 2.625 \Rightarrow f(x_4) = (2.625)^3$$

$$- 4 = -1.6621 < 0 \text{ and } f(2.75) > 0.$$

Therefore, the root lies between 2.625 and 2.75. The next approximation x_5 is given by

$$x_5 = \frac{2.625 + 2.75}{2} = 2.6875 \Rightarrow f(x_5) = (2.6875)^3$$

$$- 6(2.6875) - 4 = -0.7141 < 0 \text{ and } f(2.75) > 0.$$

Therefore, the root lies between 2.6875 and 2.75. The next approximation x_6 is given by

$$x_6 = \frac{2.6875 + 2.75}{2} = 2.71875$$

This is the approximate value of the root of the given equation.

7. (a) Let $f(x) = x^3 - 4x - 9$

Now, $f(2) = -9$ and $f(3) = 6$.

Since $f(2)$ is negative and $f(3)$ is positive, a root lies between 2 and 3.

Therefore, the first approximate of the root is

$$x_1 = \frac{1}{2}(2 + 3) = 2.5$$

Thus, $f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375 < 0$

Hence, the root lies between x_1 and 3. Thus, the second approximation to the root is

$$x_2 = \frac{1}{2}(2.5 + 3) = 2.75$$

Thus, $f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969 > 0$

Hence, the root lies between x_1 and x_2. Thus, the third approximation to the root is

$$x_3 = \frac{1}{2}(2.5 + 2.75) = 2.625$$

Thus, $f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121 < 0$

Hence, the root lies between x_2 and x_3. Thus, the fourth approximation to the root is

$$x_4 = \frac{1}{2}(2.5 + 2.625) = 2.6875$$

Thus,

$f(x_4) = (2.6875)^3 - 4(2.6875) - 9 = -0.3391 < 0$

Hence, the root lies between x_2 and x_4. Thus, the fifth approximation to the root is

$$x_5 = \frac{1}{2}(2.75 + 2.6875) = 2.71875$$

Thus,

$f(x_5) = (2.71875)^3 - 4(2.71875) - 9 = 0.2209 > 0$

Hence, the root lies between x_4 and x_5. Thus, the sixth approximation to the root is

$$x_6 = \frac{1}{2}(2.71875 + 2.6875) = 2.703125$$

Thus,

$f(x_6) = (2.703125)^3 - 4(2.703125) - 9 = -0.0611 < 0$

Hence, the root lies between x_5 and x_6. Thus, the seventh approximation to the root is

$$x_7 = \frac{1}{2}(2.71875 + 2.703125) = 2.71094$$

Thus,

$f(x_7) = (2.71094)^3 - 4(2.71094) - 9 = 0.0795 > 0$

Hence, the root lies between x_6 and x_7. Thus, the eighth approximation to the root is

$$x_8 = \frac{1}{2}(2.71094 + 2.703125) = 2.70703$$

Thus,

$f(x_8) = (2.70703)^3 - 4(2.70703) - 9 = 0.00879 > 0$

Hence, the root lies between x_6 and x_8. Thus, the ninth approximation to the root is

$$x_9 = \frac{1}{2}(2.70703 + 2.703125) = 2.70508$$

Thus,

$f(x_9) = (2.70508)^3 - 4(2.70508) - 9 = -0.0260 < 0$

Hence, the root lies between x_8 and x_9. Thus, the tenth approximation to the root is

$$x_{10} = \frac{1}{2}(2.70703 + 2.70508) = 2.70605$$

Thus,

$f(x_{10}) = (2.70605)^3 - 4(2.70508) - 9 = -0.0086 < 0$

Hence, the root lies between x_8 and x_{10}. Thus, the eleventh approximation to the root is

$$x_{11} = \frac{1}{2}(2.70703 + 2.70605) = 2.70654$$

Thus,

$f(x_{11}) = (2.70654)^3 - 4(2.70654) - 9 = 0.000216 > 0$

Hence, the root lies between x_{10} and x_{11}. Thus, the twelfth approximation to the root is

$$x_{12} = \frac{1}{2}(2.70654 + 2.70605) = 2.706295$$

Hence, the root is 2.706.

8. (b) Consider

$$f(x) = x^3 - 5x + 1 = 0$$

Let $x_0 = 0.2016$ and $x_1 = 0.2017$. Then

$$f(x_0) = f(0.2016) = 0.000935 > 0 \text{ and}$$
$$f(x_1) = f(0.2017) = -0.0002943 < 0.$$

Hence, the root lies between 0.2016 and 0.2017. The first approximation to the root is given by

$$\begin{aligned} x_2 &= x_1 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0) \\ &= 0.2016 - \left\{\frac{0.2017 - 0.2016}{f(0.2017) - f(0.2016)}\right\} f(0.2016) \\ &= 0.201641984 \end{aligned}$$

Hence, the root correct to four decimal places is 0.2016.

9. (d) Let

$$f(x) = x\log_{10} x - 1.2$$

$$\Rightarrow f(1) = -1.2 < 0;\ f(2) = 2\times 0.30103 - 1.2 = -0.597940;$$

$$f(3) = 3\times 0.47712 - 1.2 = 0.231364 > 0$$

Hence, the root lies between 2 and 3. Therefore, the next approximation is given by

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = \frac{2\times 0.23136 - 3\times(-0.59794)}{0.23136 + 0.59794}$$

$$= 2.721014 \Rightarrow f(x_1) = (2.7210)\log_{10}(2.7210) - 1.2$$

$$= -0.017104$$

The root lies between 2.721014 and 3. Therefore, the next approximation is given by

$$x_2 = \frac{x\times f(3) - 3\times f(2)}{f(3) - f(x_1)}$$

$$= \frac{2.721014\times 0.231364 - 3\times(-0.017104)}{0.23136 + 0.017104}$$

$$= \frac{0.68084}{0.2486} = 2.740211$$

$$\Rightarrow f(x_2) = f(2.7402) = 2.7402\times\log(2.7402) - 1.2$$

$$= 0.00038905$$

Therefore, the root lies between 2.740211 and 3. The next approximation is given by

$$x_3 = \frac{2.7402\times f(3) - 3\times f(2.7402)}{f(3) - f(2.7402)}$$

$$= \frac{2.7402\times 0.23136 + 3\times 0.00038905}{0.23136 + 0.00038905}$$

$$= \frac{063514}{0.23175} = 2.740627$$

$$\Rightarrow f(x_3) = (2.7406)\log_{10}(2.7406) - 1.2$$

$$= 0.00011998$$

Therefore, the root lies between 2.740211 and 2.740627. The next approximation is given by

$$x_4 = \frac{2.7402\times f(2.7406) - 2.7406\times f(2.7402)}{f(2.7406) - f(2.7402)}$$

$$= \frac{2.7402\times 0.00011998 + 2.7406\times 0.00038905}{0.00011998 + 0.00038905}$$

$$= \frac{0.0013950}{0.00050903} = 2.7405$$

Hence, the root is 2.7405.

10. (c) Let

$$f(x) = x\log_{10} x - 4.77$$

Now,

$$f(6) = -0.1010 < 0$$

$$f(7) = 1.1456 > 0$$

So, a root of $f(x)$ lies between 6 and 7. Let us take $x_0 = 6.5$, we get

$$f'(x) = \log_{10} x + \log_{10} e = \log_{10} x + 0.43429$$

The iteration formula is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \left\{\frac{x_i\log_{10} x_i - 4.77}{\log_{10} x_i + 0.43429}\right\}$$

$$= \frac{0.43429x_i + 4.77}{\log_{10} x_i + 0.43429}$$

Putting $n = 0, 1, 2, 3, \ldots$, we get

$$x_1 = \frac{0.43429x_0 + 4.77}{\log_{10} x_0 + 0.43429} = 6.08792861$$

$$x_2 = \frac{0.43429x_1 + 4.77}{\log_{10} 6.0879286 + 0.43429} = 6.083173814$$

$$x_3 = \frac{0.43429x_1 + 4.77}{\log_{10} 6.08317381 + 0.43429} = 6.083173439$$

Hence, the required root is 6.0831.

11. (c) Let $f(x) = 3x - \cos x - 1$

$$f(0) = 3(0) - \cos(0) - 1 = -2 < 0 \quad \text{and}$$
$$f(1) = 3(1) - \cos 1 - 1 = 1.4597 > 0$$

Hence, a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Thus, we take $x_0 = 0.6$. Also,

$$f'(x) = 3 + \sin x$$

Using Newton's iteration formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n\sin x_n + \cos x_n + 1}{3 + \sin x_n} \quad (1)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0}$$

$$= \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3\sin(0.6)}$$

$$= \frac{0.6(0.5729) + 0.82533 + 1}{3 + 0.5729} = 0.6071$$

Putting $n = 1$ in Eq. (1), the second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$

$$= \frac{(0.6071)\sin(0.6071) + \cos(0.6071) + 1}{3\sin(0.6071)}$$

$$= \frac{(0.6071)(0.57049) + 0.8213 + 1}{3 + 0.57049} = 0.6071$$

Since $x_1 = x_2$, the desired root is 0.6071.

12. (a) Let $y = \sin x - \log x + e^x$ and $h = 0.2, n = 6$.

The values of y are given as follows:

x	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y	3.0295 y_0	2.7975 y_1	2.8976 y_2	3.1660 y_3	3.5597 y_4	4.698 y_5	4.4042 y_6

By Simpson's 3/8 rule, we have

$$\int_{0.2}^{1.4} y\,dx = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3}{8}(0.2)\left[7.7336 + 2(3.1660) + 3(13.3247)\right] = 4.053$$

$$\therefore \int_{0.2}^{1.4} \left(\sin x - \log x + e^x\right) dx = 4.053.$$

13. (b) Say $y = f(x) = e^{-x^2}$

Since we have to divide the interval into six parts, width $h = \dfrac{0.6 - 0}{6} = 0.1$. The values of y are given as follows:

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
y	1 y_0	0.9900 y_1	0.9608 y_2	0.9139 y_3	0.8521 y_4	0.7788 y_5	0.6977 y_6

By Simpson's 1/3 rule, we have

$$\int_0^{0.6} e^{-x^2} dx = \frac{h}{3}\left\{[y_0 + y_6] + 4[y_1 + y_3 + y_5 + 2(y_2 + y_4)]\right\}$$

$$= \frac{0.1}{3}\left\{[1 + 0.6977] + 4[0.99 + 0.9139 + 0.7788 + 2(0.9608 + 0.8521)]\right\}$$

$$= \frac{0.1}{3}(1.6977 + 10.7308 + 3.6258)$$

$$= \frac{0.1}{3}(16.0543) = 0.5351$$

$$\therefore \int_0^{0.6} e^{-x^2} dx = 0.5351.$$

14. (d) Dividing the interval (1, 2) into four equal parts, each of width

$$h = \frac{2-1}{4} = 0.25$$

Therefore, the values of $y = f(x) = 1/x$ are as follows:

x	1	1.25	1.50	1.75	2.00
$f(x)$	1	0.8	0.6667	0.57143	0.5

By Simpson's 1/3 rule, we have

$$\int_1^2 \frac{dx}{x} = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$= \frac{(0.25)}{3}[(1 + 0.5) + 4(0.8 + 0.57143) + 2(0.6667)]$$

$$= 0.69326$$

$$\therefore \int_1^2 \frac{dx}{x} = 0.69326.$$

15. (d) Divide the given integral of integration into six equal subintervals. The arguments are 0, 1, 2, 3, 4, 5, 6 and $h = 1$. We have

$$f(x) = \frac{e^x}{1+x}$$

So

$$y_0 = f(0) = 1, y_1 = f(1) = \frac{e}{2}, \; y_2 = f(2) = \frac{e^2}{3},$$

$$y_3 = f(3) = \frac{e^3}{4}, \; y_4 = f(4) = \frac{e^4}{5}, \; y_5 = f(5) = \frac{e^5}{6},$$

$$y_6 = f(6) = \frac{e^6}{7}$$

The table is as follows:

x	0	1	2	3	4	5	6
y	1 (y_0)	$\frac{e}{2}$ (y_1)	$\frac{e^2}{3}$ (y_2)	$\frac{e^3}{4}$ (y_3)	$\frac{e^4}{5}$ (y_4)	$\frac{e^5}{6}$ (y_5)	$\frac{e^6}{7}$ (y_6)

Applying Simpson's three-eighth rule, we have

$$\int_0^6 \frac{e^x}{1+x}dx = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$= \frac{3}{8}\left[\left(1+\frac{e^6}{7}\right)+3\left(\frac{e}{2}+\frac{e^2}{3}+\frac{e^4}{5}+\frac{e^5}{6}\right)+2\frac{e^3}{4}\right]$$

$$= \frac{3}{8}\{[1+57.6327]+3[1.3591+2.463+10.9196$$
$$+24.7355+2(5.0214)\}$$

$$= 0.701652$$

$$\therefore \int_0^6 \frac{e^x}{1+x}dx = 0.701652.$$

16. (b) Here $h = 0.2$.

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1.23	1.58	2.03	4.32	6.25	8.38	10.23	12.45

By trapezoidal rule,

$$\int_{0.6}^{2.0} y\,dx = \frac{h}{2}[(y_0+y_7)+2(y_1+y_2+y_3+y_4+y_5+y_6)]$$

$$= \frac{0.2}{2}[(1.23+12.45)+2(1.58+2.03+4.32$$
$$+6.25+8.38+10.23+12.45)]$$

$$= (0.1)[13.68+90.48] = 10.416$$

$$\therefore \int_{0.6}^{2.0} y\,dx = 10.416.$$

17. (c) Divide the interval (0, 6) into six parts, each of width $h = 1$. The value of $f(x) = \dfrac{1}{1+x^2}$ are given as follows:

x	0	1	2	3	4	5	6
$y = f(x)$	1 (y_0)	0.5 (y_1)	0.2 (y_2)	0.1 (y_3)	0.05884 (y_4)	0.0385 (y_5)	0.027 (y_6)

By trapezoidal rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{2}[(1+0.027)+2(0.5+0.2+0.1$$
$$+0.0588+0.0385)]$$

$$= 1.4108$$

$$\therefore \int_0^6 \frac{1}{1+x^2} = 1.4108.$$

18. (a) We have $x_0 = 0$, $y_0 = 1$ and $h = 0.2$.

$$\frac{dy}{dx} = f(x,y) = 1+y^2$$

First approximation

$$y_1 = y(0.2) = y_0 + hf(x_0,y_0) = y_0 + h\left(1+y_0^2\right)$$

$$= 1+(0.2)(1+1) = 1.4$$

Second approximation

$$y_2 = y(0.4) = y_1 + h\ f(x_1,y_1) = y_1 + h\left(1+y_1^2\right)$$

$$= 1.4+(0.2)(1+1.96) = 1.992$$

Hence, $y(0.4) = 1.992$.

19. (c) Given

$f(x,y) = 1-y,\ x_0 = 0,\ y_0 = 0,\ x_1 = 0.1.$

By modified Euler's method, we get

$$y_{n+1} = y_n + hf\left[x_n+\frac{h}{2}, y_n+\frac{1}{2}hf(x_n,y_n)\right]$$

$$y_1 = y_0 + hf\left[x_0+\frac{h}{2}, y_0+\frac{h}{2}f(x_0,y_0)\right]$$

$$y_1 = 1+(0.1)f\left[0+\frac{0.1}{2}, 0+\frac{0.1}{2}f(0,0)\right]$$

$$y_1 = 0.095$$

$$y_2 = y_1 + hf\left[x_1+\frac{h}{2},\ y_1+\frac{h}{2}f(x_1,y_1)\right]$$

$$= (0.095)+(0.1)f\left[0.1+\frac{0.1}{2}, 0.095\right.$$
$$\left.+\frac{0.1}{2}f(0.1,0.095)\right]$$

$$y_2 = 0.18098$$

Hence, the value of $y(0.2)$ is 0.18098.

20. (d) Here $x_0 = 0, y_0 = 1, h = 0.2, f(x_0,y_0) = 1$

$$k_1 = hf(x_0,y_0) = (0.2)(1) = 0.2$$

$$k_2 = hf\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_1\right) = 0.2\times f(0.1,1.1)$$

$$= 0.2400$$

$$k_3 = hf\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_2\right) = 0.2\times f(0.1,1.12)$$

$$= 0.2440$$

$$k_4 = hf(x_0+h, y_0+k_3) = 0.2\times f(0.2,1.244)$$

$$= 0.2888$$

$$\therefore k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$
$$= \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888)$$
$$= \frac{1}{6}(1.4568) = 0.2468$$

Hence, the required approximate value of y is 1.2428.

21. (a) All measured values tend to be close to the actual value, therefore it is accurate. All measured values are close to each other, therefore it is precise too.

22. (b) The measured values are far from true value. However, all measured values are close to each other. So it is pre use but inaccurate.

23. (c) If the errors are random, uncertainty would reduce by a factor $\sqrt{N}$, where N is number of measurements.

Therefore, $\sqrt{25} = 5 \;\; (\because \mathrm{N} = 25)$

Now uncertainty would be $\frac{5\%}{5} = 1\%$

24. (b) Relative error is the ratio of measured error to the true value.

$$\frac{100-98}{100} = \frac{2}{100} = 0.02$$

25. (d) The first three cases represent systematic errors. The error due to AC noise is equally likely to occur in both positive and negative directions.

26. (d) Standard deviation of mean is given by

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} = \frac{0.2}{\sqrt{16}} = 0.05$$

27. (b) According to the laws relating to propagation of errors,

If $Z = X \pm Y$

Then $\Delta Z = \left[(\Delta X)^2 + (\Delta Y)^2\right]^{\frac{1}{2}}$

Extending this further,

If $R = X \pm Y \pm Z$

Then $\Delta R = \left[(\Delta X)^2 + (\Delta Y)^2 + (\Delta Z)^2\right]^{\frac{1}{2}}$

Therefore, $\delta R = \left[(\delta X)^2 + (\delta Y)^2 + (\delta Z)^2\right]^{\frac{1}{2}}$

or $(\delta R)^2 = (\delta X)^2 + (\delta Y)^2 + (\delta Z)^2$

28. (a) Error $= \sqrt{(0.15)^2 + (0.05)^2}$

$$= \sqrt{225 \times 10^{-4} + 25 \times 10^{-4}} = \sqrt{250} \times 10^{-2}$$
$$= 0.151$$

29. (b) Let the equation of the line be $y = ax + bx$.
Normal equation for a is $\sum y = 10a + b\sum x$
Normal equation for b is $\sum xy = a\sum x + b\sum x^2$.

Now $\quad \sum x = 64$
$\sum y = 70$
$\sum xy = 317$
$\sum x^2 = 528$

Therefore, the normal equations are

$$10a + 64b = 70 \qquad (1)$$

and $$64a + 528b = 317 \qquad (2)$$

Multiplying Eq. (1) by 6.4, we get

$$64a + 409.6b = 448 \qquad (3)$$

Subtracting Eq. (3) from Eq. (2), we get

$$121.1b = -131$$

Therefore, $b = -\frac{131}{121.1} \cong -1.1$

Substituting the value of b in Eq. (1), we get $a \cong 14$; therefore, the equation of the line is $y = (14 - 1.1x)$.

30. (a) The forward difference table is drawn as follows:

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	1				
		6			
1	7		10		
		16		6	
2	23		16		0
		32		6	
3	55		22		
		54			
4	109				

From Newton's forward difference formula,

$$f(x) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0$$

$$p = \frac{x - x_0}{h};\; h = 1 \text{ and } x = 0.5$$

Therefore,

$$p = \frac{0.5 - 0}{1} = 0.5$$

$\Delta f_0 = 6,\ \Delta^2 f_0 = 10,\ \Delta^3 f_i = 6 \ldots$

(from difference table).

Therefore,

$$f(0.5) = 1 + 0.5 \times 6 + \frac{0.5(0.5-1)}{2} \times 10 + \frac{0.5(0.5-1)(0.5-2)}{6} \times 6$$
$$= 1 + 3 + (-1.25) + 0.375$$
$$= 4.375 - 1.25 = 3.125$$

and hence the answer.

31. (c) Newton's forward formula is employed to find derivative of a function for a given data near the beginning of the table.

32. (a) Since derivative is desired to be computed at the centre of the given data table, Stirling's formula will be used.

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. The accuracy of Simpson's rule quadrature for a step size h is

(a) $O(h^2)$ (b) $O(h^3)$

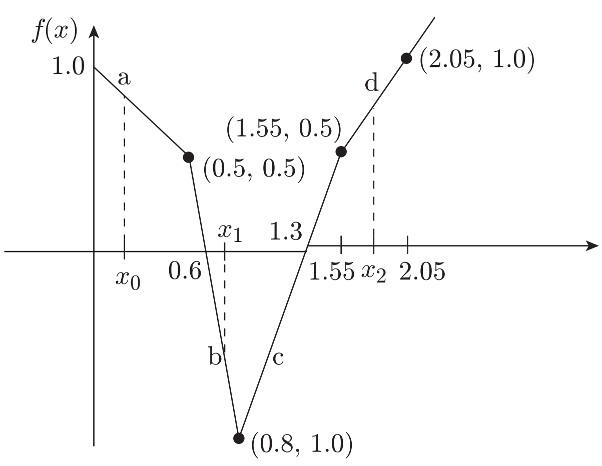
(c) $O(h^4)$ (d) $O(h^5)$

[GATE 2003, 1 Mark]

Solution: Option (c).

Ans. (c)

2. A piecewise linear function $f(x)$ is plotted using thick solid lines in the figure below (the plot is drawn to scale).



If we use the Newton–Raphson method to find the roots of $f(x) = 0$ using x_0, x_1 and x_2, respectively, as initial guesses, the roots obtained would be

(a) 1.3, 0.6 and 0.6, respectively
(b) 0.6, 0.6 and 1.3, respectively
(c) 1.3, 1.3 and 0.6, respectively
(d) 1.3, 0.6 and 1.3, respectively

[GATE 2003, 2 Marks]

Solution: Starting from x_0, slope of line

$$a = \frac{1-0.5}{0-0.5} = -1$$

y-intercept = 1

Equation of a is $y = mx + c = -x + 1$

This line will cut x-axis (i.e. $y = 0$), at $x = 1$

Since $x = 1 > x = 0.8$, a perpendicular at $x = 1$ will cut the line c and not line b.

Therefore, the root will be 1.3.

Starting from x_1, the perpendicular at x_1 is cutting line b and the root will be 0.6.

Starting from x_2,

$$\text{slope of line, } d = \frac{1-0.5}{2.05-1.55} = 1$$

Equation of d is given by

$$y - 0.5 = 1(x - 1.55)$$

i.e.

$$y = x - 1.05$$

This line will cut x-axis at $x = 1.05$.

Since $x = 1.05$ is greater than $x = 0.8$, the perpendicular at $x = 1.05$ will cut line c and not line b. The root will be therefore equal to 1.3.

So starting from x_0, x_1 and x_2, the roots will be 1.3, 0.6 and 1.3, respectively.

Ans. (d)

3. With a 1 unit change in b, what is the change in x in the solution of the system of equations $x + y = 2, 1.01x + 0.99y = b$?

(a) Zero (b) 2 units
(c) 50 units (d) 100 units

[GATE 2005, 2 Marks]

Solution: Given that $x + y = 2$ (1)

$1.01\ x + 0.99\ y = b$ (2)

Multiplying Eq. (1) by 0.99 and subtracting it from Eq. (2), we get

$$(1.01 - 0.99)x = b - 2 \times 0.99$$

$$0.02x = b - 1.98$$

$$\Rightarrow x = \frac{1}{0.02}b - \frac{1.98}{0.02}$$

$$\Rightarrow \frac{\Delta x}{\Delta b} = \frac{dx}{db} = \frac{1}{0.02} = 50$$

So, for 1 unit change in b, change in x would be 50 units.

Ans. (c)

Statement for Linked Answer Questions 4 and 5

Given $a > 0$, we wish to calculate the reciprocal value $\frac{1}{a}$ by Newton–Raphson method for $f(x) = 0$.

4. The Newton–Raphson algorithm for the function will be

(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$ (b) $x_{k+1} = \left(x_k + \frac{a}{2}x_k^2\right)$

(c) $x_{k+1} = 2x_k - ax_k^2$ (d) $x_{k+1} = x_k - \frac{a}{2}x_k^2$

[GATE 2005, 2 Marks]

Solution: To calculate $\frac{1}{a}$ using Newton–Raphson method, set up the equation as $x = \frac{1}{a}$.

Therefore, $$\frac{1}{x} = a$$

$$\Rightarrow \quad \frac{1}{x} - a = 0$$

i.e. $$f(x) = \frac{1}{x} - a = 0$$

Now, $$f'(x) = -\frac{1}{x^2}$$

$$f(x_k) = \frac{1}{x_k} - a$$

$$f'(x_k) = -\frac{1}{x_k^2}$$

For Newton–Raphson method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow \quad x_{k+1} = x_k - \frac{(1/x_k - a)}{-\frac{1}{x_k^2}}$$

Simplifying the above equation, we get

$$x_{k+1} = 2x_k - ax_k^2$$

Ans. (c)

5. For $a = 7$ and starting with $x_0 = 0.2$, the first two iterations will be

(a) 0.11, 0.1299 (b) 0.12, 0.1392
(c) 0.12, 0.1416 (d) 0.13, 0.1428

[GATE 2005, 2 Marks]

Solution: For $a = 7$, the iteration equation is given by

$$x_{k+1} = 2x_k - 7x_k^2$$

with $$x_0 = 0.2$$

$$x_1 = 2x_0 - 7x_0^2 = 2 \times 0.2 - 7(0.2)^2 = 0.12$$

and $x_2 = 2x_1 - 7x_1^2 = 2 \times 0.2 - 7(0.12)^2 = 0.1392$

Ans. (b)

6. Starting from $x_0 = 1$, one step of Newton–Raphson method in solving the equation $x^3 + 3x - 7 = 0$ gives the next value (x_1) as

(a) $x_1 = 0.5$ (b) $x_1 = 1.406$

(c) $x_1 = 1.5$ (d) $x_1 = 2$

[GATE 2005, 2 Marks]

Solution: Using Newton–Raphson method, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad (1)$$

Given function is $f(x) = x^3 + 3x - 7$

and $$f'(x) = 3x^2 + 3$$

Putting $x_0 = 1$, we get

$$f(x_0) = f(1) = (1)^3 + 3 \times (1) - 7 = -3$$

$$f'(x_0) = f'(1) = 3 \times (1)^2 + 3 = 6$$

Substituting $x_0, f(x_0)$ and $f'(x_0)$ values into Eq. (1), we get

$$x_1 = 1 - \left(\frac{-3}{6}\right) = 1.5$$

Ans. (c)

7. Match List I with List II and select the correct answer using the codes given below the lists:

List I	List II
A. Newton–Raphson method	1. Solving non-linear equations
B. Runge–Kutta method equations	2. Solving simultaneous linear equations
C. Simpson's rule equations	3. Solving ordinary differential
D. Gauss elimination	4. Numerical integration
	5. Interpolation
	6. Calculation of Eigenvalues

Codes:

	A	B	C	D
(a)	6	1	5	3
(b)	1	6	4	3
(c)	1	3	4	2
(d)	5	3	4	1

[GATE 2005, 2 Marks]

Ans. (c)

8. The differential equation $(dy/dx) = 0.25\,y^2$ is to be solved using the backward (implicit) Euler's method with the boundary condition $y = 1$ at $x = 0$ and with a step size of 1. What would be the value of y at $x = 1$?

(a) 1.33 (b) 1.67
(c) 2.00 (d) 2.33

[GATE 2006, 1 Mark]

Solution: We have

$$\frac{dy}{dx} = 0.25y^2\,(y = 1 \text{ at } x = 0)$$
$$h = 1$$

Iterative equation for backward (implicit) Euler's methods for the above equation is given by

$$y_{k+1} = y_k + h\,f(x_{k+1}, y_{k+1})$$
$$y_{k+1} = y_k + h \times 0.25y_{k+1}^2$$
$$\Rightarrow \quad 0.25h\,y_{k+1}^2 - y_{k+1} + y_k = 0$$

Putting $k = 0$ in the above equation, we get

$$0.25h\,y_1^2 - y_1 + y_0 = 0$$

Since $y_0 = 1$ and $h = 1$

$$0.25h\,y_1^2 - y_1 + 1 = 0$$
$$\Rightarrow \quad y_1 = \frac{1 \pm \sqrt{1-1}}{2 \times 0.25} = 2$$
$$\Rightarrow \quad y_1 = 2$$

Ans. (c)

9. A 2nd degree polynomial, $f(x)$ has values of 1, 4 and 15 at $x = 0$, 1 and 2, respectively. The integral $\int_0^2 f(x)\,dx$ is to be estimated by applying the trapezoidal rule to this data. What is the error (defined as "true value – approximate value") in the estimate?

(a) $-\frac{4}{3}$ (b) $-\frac{2}{3}$
(c) 0 (d) $\frac{2}{3}$

[GATE 2006, 2 Marks]

Solution: We are given that $f(x) = 1, 4, 15$ at $x = 0$, 1 and 2, respectively.

$\int_0^2 f(x)\,dx = \frac{h}{2}(f_1 + 2f_2 + f_3)$ (three-point trapezoidal rule)

Here, $h = 1$.

$$\therefore \quad \int_0^2 f(x)\,dx = \frac{1}{2}(1 + 2 \times 4 + 15) = 12$$

Therefore, approximate value by trapezoidal rule = 12

Since $f(x)$ is a second-degree polynomial, let

$$f(x) = a_0 + a_1x + a_2x^2$$
$$f(0) = 1$$
$$\Rightarrow \quad a_0 + 0 + 0 = 1$$
$$\Rightarrow \quad a_0 = 1$$
$$f(1) = 4$$
$$\Rightarrow \quad a_0 = a_1 + a_2 = 4$$
$$\Rightarrow \quad 1 + a_1 + a_2 = 4$$
$$\Rightarrow \quad a_1 + a_2 = 3 \qquad (1)$$
$$f(2) = 15$$
$$\Rightarrow \quad a_0 + 2a_1 + 4a_2 = 15$$
$$\Rightarrow \quad 1 + 2a_1 + 4a_2 = 15$$
$$\Rightarrow \quad 2a_1 + 4a_2 = 14 \qquad (2)$$

Solving Eqs. (1) and (2), $a_1 = -1$ and $a_2 = 4$

$$\therefore \quad f(x) = 1 - x + 4x^2$$

Now, the exact value of

$$\int_0^2 f(x)\,dx = \int_0^2 (1 - x + 4x^2)\,dx = \left[x - \frac{x^2}{2} + \frac{4x^2}{3}\right]_0^2 = \frac{32}{3}$$

$$\text{Error} = \text{Exact} - \text{Approximate value} = \frac{32}{3} - 12 = -\frac{4}{3}$$

Ans. (a)

10. Given that one root of the equation $x^3 - 10x^2 + 31x - 30 = 0$ is 5, the other two roots are

(a) 2 and 3 (b) 2 and 4
(c) 3 and 4 (d) −2 and −3

[GATE 2007, 2 Marks]

Solution: We have

$$f(x) = x^3 - 10x^2 + 31x - 30 = 0$$

Since 5 is a root of $f(x)$, it is divisible by $x - 5$. Now dividing $f(x)$ by $x - 5$, we get

$$\begin{array}{r|l} & x^2 - 5x + 6 \\ x - 5 & x^3 - 10x^2 + 31s - 30 \\ & x^3 - 5x^2 \\ \hline & -5x^2 + 31x - 30 \\ & -5x^2 + 25x \\ \hline & 6x - 30 \\ & 6x - 30 \\ \hline & 0 \end{array}$$

$\therefore \quad x^3 - 10x^2 + 31x - 30 = 0$

$\Rightarrow \quad (x-5)(x^2 - 5x + 6) = 0$

Roots of $x^2 - 5x + 6$ are 2 and 3.

Hence, the other two roots are 2 and 3.

Ans. (a)

11. The following equation needs to be numerically solved using the Newton–Raphson method:

$$x^3 + 4x - 9 = 0$$

The iterative equation for this purpose is (k indicates the iteration level)

(a) $x_{k+1} = \dfrac{2x_k^3 + 9}{3x_k^2 + 4}$ (b) $x_{k+1} = \dfrac{3x_k^3 + 4}{2x_k^2 + 9}$

(c) $x_{k+1} = x_k - 3x_k^2 + 4$ (d) $x_{k+1} = \dfrac{4x_k^2 + 3}{9x_k^2 + 2}$

[GATE 2007, 2 Marks]

Solution: We have

$$f(x) = x^3 + 4x - 9 = 0$$

$$f'(x) = 3x^2 + 4$$

Newton–Raphson equation for iteration is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$f(x_k) = x_k^3 + 4x_k - 9$$

$$f'(x_k) = 3x_k^2 + 4$$

$$x_{k+1} = x_k - \frac{(x_k^3 + 4x_k - 9)}{(3x_k^2 + 4)}$$

$$x_{k+1} = \frac{(3x_k^3 + 4x_k) - (x_k^3 + 4x_k - 9)}{3x_k^2 + 4}$$

$$x_{k+1} = \frac{2x_k^3 + 9}{3x_k^2 + 4}$$

Ans. (a)

12. The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using the Newton–Raphson method. If $x = 2$ is taken as the initial approximation of the solution, then the next approximation using this method will be

(a) $\dfrac{2}{3}$ (b) $\dfrac{4}{3}$

(c) 1 (d) $\dfrac{3}{2}$

[GATE 2007, 2 Marks]

Solution: Here, $x_0 = 2$

We are given

$$f(x) = x^3 - x^2 + 4x - 4$$

Then, $\quad f'(x) = 3x^2 - 2x + 4$

Also, $\quad f(x_0) = f(2) = 8$

$$f'(x_0) = f'(2) = 12$$

$$\Rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8}{12} = \frac{4}{3}$$

Ans. (b)

13. Consider the series $x_{n+1} = \dfrac{x_n}{2} + \dfrac{9}{8x_n}$, $x_0 = 0.5$ obtained from the Newton–Raphson method.

The series converges to

(a) 1.5 (b) $\sqrt{2}$
(c) 1.6 (d) 1.4

[GATE 2007, 2 Marks]

Solution: We are given that

$$x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n}, x_0 = 0.5$$

As $n \to \infty$, when the series converges

$$x_{n+1} = x_n = \alpha = \text{ root of equation}$$

$$\alpha = \frac{\alpha}{2} + \frac{9}{8\alpha}$$

$$\Rightarrow \alpha = \frac{4\alpha^2 + 9}{8\alpha}$$

$$\Rightarrow \quad 8\alpha^2 = 4\alpha^2 + 9$$

$$\Rightarrow \quad \alpha^2 = \frac{9}{4}$$

$$\Rightarrow \alpha = \frac{3}{2} = 1.5$$

Ans. (a)

14. A calculator has accuracy up to 8 digits after decimal place. The value of $\int_0^{2\pi} \sin x\, dx$ when evaluated using this calculator by trapezoidal method with 8 equal intervals, to 5 significant digits is

(a) 0.00000 (b) 1.0000
(c) 0.00500 (d) 0.00025

[GATE 2007, 2 Marks]

Solution: We know that

$$h = \frac{2\pi - 0}{8} = \frac{\pi}{4}$$

We can tabulate as follows:

i	x	$y = \sin x$
0	0	$y_0 = \sin(0) = 0$
1	$\frac{\pi}{4}$	$y_1 = \sin\left(\frac{\pi}{4}\right) = 0.70710$
2	$\frac{\pi}{2}$	$y_2 = \sin\left(\frac{\pi}{2}\right) = 1$
3	$\frac{3\pi}{4}$	$y_3 = \sin\left(\frac{3\pi}{4}\right) = 0.7010$
4	π	$y_4 = \sin(\pi) = 0$
5	$\frac{5\pi}{4}$	$y_5 = \sin\left(\frac{5\pi}{4}\right) = -0.70710$

(*Continued*)

i	x	$y = \sin x$
6	$\frac{6\pi}{4}$	$y_6 = \sin\left(\frac{6\pi}{4}\right) = -1$
7	$\frac{7\pi}{4}$	$y_7 = \sin\left(\frac{7\pi}{4}\right) = -0.70710$
8	2π	$y_8 = \sin\left(\frac{8\pi}{4}\right) = 0$

Trapezoidal rule is given by

$$\int_{x_0}^{x_0+nh} f(x) \cdot dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

$$\int_0^{2\pi} \sin x \cdot dx = \frac{\pi}{8} \times \left[(0+0) + 2\ (0.70710 + 1 + 0.70710 + 0 - 0.70710 - 0.70710)\right]$$

$$= 0.00000$$

Ans. (a)

15. The differential equation $(dx / dt) = \left[(1-x) / \tau\right]$ is discretised using Euler's numerical integration method with a time step $\Delta T > 0$. What is the maximum permissible value of ΔT to ensure stability of the solution of the corresponding discrete time equation?

(a) 1 (b) $\tau/2$
(c) τ (d) 2τ

[GATE 2007, 2 Marks]

Solution: Here, $\frac{dx}{dt} = \frac{1-x}{\tau}$

Here, $f(x, y) = \frac{1-x}{\tau}$

The Euler's method equation is given by

$$x_{i+1} = x_i + h f(x_i, y_i)$$

$$\Rightarrow \quad x_{i+1} = x_i + h\left(\frac{1-x_i}{\tau}\right)$$

$$\Rightarrow \quad x_{i+1} = \left(1 - \frac{h}{\tau}\right)x_i + \frac{h}{\tau}$$

For stability $\left|1 - \frac{h}{\tau}\right| < 1$

$$\Rightarrow \quad -1 \le 1 - \frac{h}{\tau} \le 1$$

Since $h = \Delta T$. Here,

$$-1 \le 1 - \frac{\Delta T}{\tau} < 1$$

$$\Rightarrow \quad \Delta T < 2\tau$$

So, the maximum permissible value of ΔT is 2τ.

Ans. (d)

16. Equation $e^x - 1 = 0$ is required to be solved using Newton's method with an initial guess $x_0 = -1$. Then, after one step of Newton's method, estimate x_1 of the solution will be given by

(a) 0.71828 (b) 0.36784
(c) 0.20587 (d) 0.00000

[GATE 2008, 2 Marks]

Solution: Here $f(x) = e^x - 1$

$$f'(x) = e^x$$

The Newton–Raphson iterative equation is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$f(x_i) = e^{xi} - 1$$

$$f'(x_i) = e^{xi}$$

$$x_{i+1} = x_i - \frac{e^{xi} - 1}{e^{xi}}$$

$\therefore$

i.e. $$x_{i+1} = \frac{x_i\, e^{x_i} - \left(e^{x_i} - 1\right)}{e^{x_i}} = \frac{e^{x_i}\left(x_i - 1\right) + 1}{e^{x_i}}$$

Now putting $i = 0$, we get

$$x_1 = \frac{e^{x_0}\left(x_0 - 1\right) + 1}{e^{x_0}}$$

Putting $x_0 = -1$ as given, $x_1 = \left[e^{-1}(-2) + 1\right] / e^{-1}$
$= 0.71828$

Ans. (a)

17. Let $x^2 - 117 = 0$. The iterative steps for the solution using Newton–Raphson method is given by

(a) $x_{k+1} = \dfrac{1}{2}\left(x_k + \dfrac{117}{x_k}\right)$

(b) $x_{k+1} = x_k - \dfrac{117}{x_k}$

(c) $x_{k+1} = x_k - \dfrac{x_k}{117}$

(d) $x_{k+1} = x_k - \dfrac{1}{2}\left(x_k + \dfrac{117}{x_k}\right)$

[GATE 2008, 2 Marks]

Solution: We have

$$f(x) = x^2 - 117 = 0$$

The Newton–Raphson method is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$= x_k - \frac{x_k^2 - 117}{2x_k}$$

$$= \frac{1}{2}\left[x_k + \frac{117}{x_k}\right]$$

Ans. (a)

18. The recursion relation to solve $x = e^{-x}$ using Newton–Raphson method is

(a) $x_{n+1} = e^{-x_n}$

(b) $x_{n+1} = x_n - e^{-x_n}$

(c) $x_{n+1} = \left(1 + x_n\right)\dfrac{e^{-x_n}}{1 + e^{-x_n}}$

(d) $x_{n+1} = \dfrac{x_n^2 - e^{-x_n}\left(1 + x_n\right) - 1}{x_n - e^{-x_n}}$

[GATE 2008, 2 Marks]

Solution: The given equation is

$$x = e^{-x}$$

The equation can be rewritten as

$$f(x) = x - e^{-x} = 0$$

$$f'(x) = 1 + e^{-x}$$

The Newton–Raphson iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_n) = x_n - e^{-x_n}$$

Here $$f'(x_n) = 1 + e^{-x_n}$$

$\therefore$ The Newton–Raphson iterative formula is given by

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}} = \frac{e^{-x_n} x_n + e^{-x_n}}{1 + e^{-x_n}}$$

$$= \left(1 + x_n\right)\frac{e^{-x_n}}{1 + e^{-x_n}}$$

Ans. (c)

19. The Newton–Raphson iteration $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{R}{x_n}\right)$ can be used to compute the

(a) square of R (b) reciprocal of R
(c) square root of R (d) logarithm of R

[GATE 2008, 2 Marks]

Solution: We have

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

We have convergence at $x_{n+1} = x_n = \alpha$

$$= \frac{1}{2}\left(\alpha + \frac{R}{\alpha}\right)$$

$$2\alpha = \alpha + \frac{R}{\alpha}$$

$$2\alpha = \frac{\alpha^2 + R}{\alpha}$$

$$\Rightarrow \quad \alpha^2 = R$$

$$\alpha = \sqrt{R}$$

Hence, this iteration will compute the square root of R.

The correct choice is option (c).

Ans. (c)

20. The minimum number of equal length subintervals needed to approximate $\int_1^2 xe^x dx$ to an accuracy of at least $1/3 \times 10^{-6}$ using the trapezoidal rule is

(a) $1000e$ (b) 1000
(c) $100e$ (d) 100

[GATE 2008, 2 Marks]

Solution: Here, the function being integrated is

$$f(x) = xe^x$$

$$f'(x) = xe^x + e^x = e^x(x+1)$$

$$f'(x) = xe^x + e^x + e^x = e^x(x+2)$$

Since both e^x and x are increasing functions of x, the maximum value of $f''(\alpha)$ in interval $1 \le \alpha \le 2$ occurs at $\alpha = 2$.

So, $$\max\left|f''(\alpha)\right| = e^2(2+2) = 4e^2$$

Truncation error for trapezoidal rule = TE (bound)

$$= \frac{h^3}{12}\max\left|f''(\alpha)\right| \times N_i$$

where N_i is number of subintervals

$$N_i = \frac{b-a}{h}$$

$$\therefore \quad T_{\in(\text{bound})} = \frac{h^3}{12}\max\left|f''(\alpha)\right| \times \frac{b-a}{h}$$

$$= \frac{h^2}{12}(b-a)\max\left|f''(\alpha)\right|, \quad 1 \le \xi \le 2$$

$$= \frac{h^2}{12}(2-1)\left(4e^2\right) = \frac{h^2}{3}e^2$$

Now putting $T_{\in(\text{bound})} = \frac{1}{3} \times 10^{-6}$, we get

$$\frac{h^3}{3}e^2 = \frac{1}{3} \times 10^{-6}$$

$$\Rightarrow \quad h^2 = \frac{10^{-6}}{e^2}$$

$$\Rightarrow \quad h = \frac{10^{-3}}{e}$$

Now, number of intervals $= N_i = \frac{b-a}{h} = \frac{2-1}{\left(10^{-3}/e\right)}$
$= 1000\ e$

Ans. (a)

21. In the solution of the following set of linear equations by Gauss elimination using partial pivoting $5x + y + 2z = 34$, $4y - 3z = 12$ and $10x - 2y + z = -4$, the pivots for elimination of x and y are

(a) 10 and 4 (b) 10 and 2
(c) 5 and 4 (d) 5 and -4

[GATE 2009, 2 Marks]

Solution: The equations are

$$5x + y + 2z = 34$$
$$0x + 4y - 3z = 12$$
and $$10x - 2y + z = -4$$

The augmented matrix for Gauss elimination is $\left[\begin{array}{ccc|c} 5 & 1 & 2 & 34 \\ 0 & 4 & -3 & 12 \\ 10 & -2 & 1 & -4 \end{array}\right]$.

Now we exchange row 1 with row 3. Therefore,

$$\left[\begin{array}{ccc|c} 5 & 1 & 2 & 34 \\ 0 & 4 & -3 & 12 \\ 10 & -2 & 1 & -4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 10 & -2 & 1 & -4 \\ 0 & 4 & -3 & 12 \\ 5 & 1 & 2 & 34 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 10 & -2 & 1 & -4 \\ 0 & 4 & -3 & 12 \\ 5 & 1 & 2 & 34 \end{array}\right] \xrightarrow{R_3 - \frac{5}{10}R_1} \left[\begin{array}{cccc} 10 & -2 & 1 & -4 \\ 0 & 4 & -3 & 12 \\ 0 & 2 & 3/2 & 34 \end{array}\right]$$

Now to eliminate y, we need to compare the elements in the second column at and below the diagonal element.

Since $a_{22} = 4$ is already larger in the absolute value compared to $a_{32} = 2$, the pivot element for eliminating y is $a_{22} = 4$.

Therefore, the pivots for eliminating x and y are 10 and 4, respectively.

Ans. (a)

22. Newton–Raphson method is used to compute a root of the equation $x^2 - 13 = 0$ with 3.5 as the initial value. The approximation after one iteration is

(a) 3.575 (b) 3.677
(c) 3.667 (d) 3.607

[GATE 2010, 1 Mark]

Solution: The equation is $f(x) = x^2 - 13 = 0$.

Newton–Raphson iteration equation is

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right]$$

Now $$f(x_0) = x_0^2 - 13$$

$$\Rightarrow f'(x_0) = 2x_0$$

$$\therefore \quad x_1 = x_0 - \left[\frac{x_0^2 - 13}{2x_0}\right] = \frac{x_0^2 + 13}{2x_0}$$

Putting $x_0 = 3.5$ (as given), we get

$$x_1 = \frac{3.5^2 + 13}{2 \times 3.5} = 3.607$$

Ans. (d)

23. The table below gives values of a function $F(x)$ obtained for values of x at intervals of 0.25.

x	0	0.25	0.5	0.745	1.0
$F(x)$	1	0.9412	0.8	0.64	0.50

The value of the integral of the function between the limits 0 to 1 using Simpson's rule is

(a) 0.7854 (b) 2.3562
(c) 3.1416 (d) 7.5000

[GATE 2010, 2 Marks]

Solution: We know that

$$I = \frac{1}{3}h\left(f_0 + 4f_i + 2f_2 + 4f_3 + f_4\right)$$

$$= \frac{1}{2} \times 0.25(1 + 4 \times 0.9412 + 2 \times 0.8 + 4 \times 0.64 + 0.5)$$

$$= 0.7854$$

Ans. (a)

24. Torque exerted on a flywheel over a cycle is listed in the table.

Angle (degree)	0	60	120	180	240	300	360
Torque (Nm)	0	1066	−323	0	323	−355	0

Flywheel energy (in Joule per unit cycle) using Simpson's rule is

(a) 542 (b) 993
(c) 1444 (d) 1986

[GATE 2010, 2 Marks]

Solution: Flywheel energy $= \int_0^{2\pi} T(\theta)d\theta$, where $T(\theta)$ is torque exerted.

The integral by using Simpson's 1/3 rule is given by

$$l = \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6\right)$$

$$h = 60 \text{ degrees} = \frac{\pi}{3} \text{ radius}$$

$$l = \frac{1}{3} \times \frac{\pi}{3} \times [0 + 4 \times 1066 + 2(-323) + 4(0) +$$

$$2(323) + 4(-355) + 0] = 993$$

Ans. (b)

25. Consider a differential equation $\frac{dy(x)}{dx} - y(x) = x$ with the initial conditions $y(0) = 0$. Using Euler's first-order method with a step size of 0.1, the value of $y(0.3)$ is

(a) 0.01 (b) 0.031
(c) 0.0631 (d) 0.1

[GATE 2010, 2 Marks]

Solution: We have

$$\frac{dy}{dx} - y = x, \quad y(0) = 0$$

Step size $= h = 0.1$

Euler's first-order formula is

$$y_{i+1} = y_i + h\,f(x_i, y_i)$$
$$y_1 = y_0 + h\,f(x_0, y_0)$$

Here, $x_0 = 0, y_0 = y(x_0) = y(0) = 0$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$
$$f(x,y) = \frac{dy}{dx} = y + x$$

$$\Rightarrow \quad y_1 = y_0 + h\,f(x_0, y_0)$$
$$= 0 + 0.1 \times f(0,0) = 0 + 0.1 \times (0+0) = 0$$

Now, $\quad x_1 = 0.1,\ y_1 = 0$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.1 = 0.2$$
$$\Rightarrow y_2 = y_1 + h\,f(x_1, y_1)$$
$$= 0 + 0.1 \times f(0.1, 0) = 0 + 0.1(0.1 + 0) = 0.01$$

Now, $\quad x_2 = 0.2,\ y_2 = 0.01$

$$x_3 = x_0 + 3h = 0 + 3 \times 0.1 = 0.3$$
$$\Rightarrow \quad y_3 = y_2 + h\,f(x_2, y_2)$$
$$= 0.01 + 0.1 \times f(0.2, 0.01) = 0.01 + 0.1(0.2 + 0.01)$$
$$= 0.031$$

Therefore, at $x_3 = 0.3,\ y_3 = 0.031$.

Hence, the correct answer is option (b).

Ans. (b)

26. Roots of the algebraic equation $x^3 + x^2 + x + 1 = 0$ are

(a) $(+1, +j, -j)$ (b) $(+1, -1, +1)$

(c) $(0, 0, 0)$ (d) $(-1, +j, -j)$

[GATE 20011, 1 Mark]

Solution: We have

$$x^3 + x^2 + x + 1 = 0$$

–1 is one of the roots since

$$(-1)^3 + (-1)^2 + (-1) + 1 = 0$$

By polynomial division, we have

$$\frac{x^3 + x^2 + x + 1}{[x - (-1)]} = x^2 + 1$$

$$\Rightarrow \quad x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$$

Therefore, roots are $(-1, +j, -j)$.

Ans. (d)

27. The matrix $[A] = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decomposed into a product of a lower triangular matrix $x[L]$ and an upper triangular matrix $[U]$. The properly decomposed $[L]$ and $[U]$ matrices, respectively, are

(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

[GATE 2011, 2 Marks]

Solution: Let us use Crout's decomposition method. By putting $u_{11} = 1$ and $u_{22} = 1$, we get

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

$$l_{11} = 2, l_{11}u_{12} = 1$$
$$\Rightarrow u_{12} = \frac{1}{2} = 0.5$$
$$l_{21} = 4, l_{21}u_{12} + l_{22} = -1$$
$$\Rightarrow 4 \times \frac{1}{2} + l_{22} = -1 \Rightarrow l_{22} = -3$$

$$\therefore \quad \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

Ans. (d)

28. The square root of number N is to be obtained by applying the Newton–Raphson iterations to the equation $x^2 - N = 0$. If i denotes the iteration index, the correct iterative scheme will be

(a) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$ (b) $x_{i+1} = \frac{1}{2}\left(x_i^2 + \frac{N}{x_i^2}\right)$

(c) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N^2}{x_i}\right)$ (d) $x_{i+1} = \frac{1}{2}\left(x_i - \frac{N}{x_i}\right)$

[GATE 2011, 2 Marks]

Solution: We have

$$f(x) = x^2 - N = 0$$

Using Newton–Raphson iterations, we get

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \left(\frac{x_i^2 N}{2x_i}\right)$$

$$= \frac{x_i^2 N}{2x_i} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

Ans. (a)

29. A numerical solution of the equation $f(x) = x + \sqrt{x} - 3 = 0$ can be obtained using Newton–Raphson method. If the starting value is $x = 2$ for the iteration, the value of x that is to be used in the next step is

(a) 0.306 (b) 0.739
(c) 1.694 (d) 2.306

[GATE 2011, 2 Marks]

Solution: We have

$$f(x) = x + \sqrt{x} - 3 = 0$$

Using Newton–Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Putting $x = 2$, we get

$$f(x_0) = 2 + \sqrt{2} - 3 = \sqrt{2} - 1$$

Now $$f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

$$\Rightarrow f'(x_0) = 1 + \frac{1}{2\sqrt{2}}$$

Then, $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{\sqrt{2} - 1}{1 + \frac{1}{2\sqrt{2}}}$$

$$\Rightarrow \quad x_1 = 1.694$$

Ans. (c)

30. The integral $\int_1^3 \frac{1}{x}\,dx$, when evaluated by using Simpson's 1/3 rule on two equal subintervals each of length 1, equals

(a) 1.000 (b) 1.098
(c) 1.111 (d) 1.120

[GATE 2011, 2 Marks]

Solution: We have

$$I = \int_1^3 \frac{1}{3}\,dx$$

x	$f(x) = \frac{1}{x}$
1	1
2	1/2
3	1/2

$$\Rightarrow \quad I = \frac{h}{3}(t_0 + 4f_1 + f_2)$$

$$= \frac{1}{3}\left(1 + 4 \times \frac{1}{2} \times \frac{1}{3}\right) = 1.111$$

Ans. (c)

31. Solution of the variables x_1 and x_2 for the following equations is to be obtained by employing the Newton–Raphson iterative method:

Equation (i) $10x_2 \sin x_1 - 0.8 = 0$

Equation (ii) $10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$

Assuming the initial values $x_1 = 0.0$ and $x_2 = 1.0$, the Jacobian matrix is

(a) $\begin{bmatrix} 10 & -0.8 \\ 0 & -0.6 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -0.8 \\ 10 & -0.6 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & 0 \\ 10 & -10 \end{bmatrix}$

[GATE 2011, 2 Marks]

Solution:

$$u(x_1, x_2) = 10x_2 \sin x_1 - 0.8 = 0$$

$$v(x_1, x_2) = 10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$$

The Jacobian matrix is

$$\begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 10x_2 \cos x_1 & 10 \sin x_1 \\ 10x_2 \sin x_1 & 20x_2 - 10 \cos x_1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Ans. (b)

32. The estimate of $\int_{0.5}^{1.5} \frac{dx}{x}$ obtained using Simpson's rule with three-point function evaluation exceeds the exact value by

(a) 0.235 (b) 0.068
(c) 0.024 (d) 0.012

[GATE 2012, 1 Mark]

Solution: The exact value of $\int_{0.5}^{1.5} \frac{dx}{x} = [\log x]_{0.5}^{1.5}$

$$= \log(1.5) - \log(0.5) = 1.0986$$

Applying value using Simpson's 1/3 rule, we get

$$I = \frac{h}{3}[f(0) + 4f(1) + f(2)] \qquad (1)$$

Number of intervals = 2

Hence, $h = \frac{1.5 - 0.5}{2} = 0.5$

We have $f(0) = 1/0.5$, $f(1) = 1$ and $f(2) = 1/1.5$

Substituting these values in Eq. (1), we get

$$I = \frac{h}{3}\left[\frac{1}{0.5} + 4(1) + \frac{1}{1.5}\right] = 1.1111$$

Hence, the estimate exceeds the exact value by

$$1.1111 - 1.0986 = 0.012499 = 0.012$$

Ans. (d)

33. The bisection method is applied to compute a zero of the function $f(x) = x^4 - x^3 - x^2 - 4$ in the interval [1, 9]. The method converges to a solution after ________ iterations.

(a) 1 (b) 3
(c) 5 (d) 7

[GATE 2012, 2 Marks]

Solution: We have

$$f(x) = x^4 - x^3 - x^2 - 4$$

and it has a zero at 2 since

$$2^4 - 2^3 - 2^2 - 4 = 16 - 8 - 4 - 4 = 0$$

Using bisection method, we get $x_0 = 1$ and $x_1 = 9$. After one iteration, since $f(x_1)f(x_2) > 0, x_2$ replaces x_1 and $x_0 = 1$ and $x_1 = 3$. After third iteration, $x_2 = \frac{1+3}{2} = 2$ which is exactly a root. Therefore, the method converges exactly to the root in 3 iterations.

Ans. (b)

34. The error in $\left.\frac{d}{dx}f(x)\right|_{x=x_0}$ for a continuous function estimated with $h = 0.03$ using the central difference formula, $\left.\frac{d}{dx}f(x)\right|_{x=x_0} = \frac{f(x_0+h) - f(x_0-h)}{2h}$, is 2×10^{-3}. The value of x_0 and $f(x_0)$ are 19.78 and 500.01, respectively. The corresponding error in the central difference estimate for $h = 0.02$ is approximately

(a) 1.3×10^{-4} (b) 3.0×10^{-4}
(c) 4.5×10^{-4} (d) 9.0×10^{-4}

[GATE 2012, 2 Marks]

Solution: Error in central difference formula is $O(h^2)$

This means,

$$\text{error} \propto h^2$$

If error for $h = 0.03$ is 2×10^{-3}, error for $h = 0.02$ is approximately

$$2 \times 10^{-3} \times \frac{(0.02)^2}{(0.03)^2} \simeq 9 \times 10^{-4}$$

Ans. (d)

35. Match the correct pairs:

Numerical Integration Scheme	Order of Fitting Polynomial
P. Simpson's 3/8 rule	1. First
Q. Trapezoidal rule	2. Second
R. Simpson's 1/3 rule	3. Third

(a) P-2, Q-1, R-3 (b) P-3, Q-2, R-1
(c) P-1, Q-2, R-3 (d) P-3, Q-1, R-2

[GATE 2013, 1 Mark]

Solution: Option (d).

Ans. (d)

36. When the Newton–Raphson method is applied to solve the equation $f(x) = x^3 + 2x - 1 = 0$, the solution at the end of the first iteration with the initial guess values as $x_0 = 1.2$ is

(a) −0.82 (b) 0.49
(c) 0.705 (d) 1.69

[GATE 2013, 2 Marks]

Solution: We have

$$f(x) = x^3 + 2x - 1 = 0$$

Using Newton–Raphson method, we have

$$f'(x_1) = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad (1)$$

$$\therefore\ f'(x) = 3x^2 + 2$$

$$\Rightarrow\ f'(x_0) = 3(1.2)^2 + 2 = 6.32$$

Now $f(x_0) = (1.2)^3 + 2 \times 1.2 - 1 = 3.128$

Substituting these values in Eq. (1), we get

$$f'(x_1) = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.2 - \frac{3.728}{6.32}$$

$$= 0.705$$

Ans. (c)

37. Calculate the magnitude as the error (correct to two decimal places) in the estimation of the following integral using Simpson's 1/3 rule.

$$\int_0^4 \left(x^4 + 10\right) dx$$

Take the step length as 1.

[GATE 2013, 2 Marks]

Solution: Using Simpson's rule, we have

x	0	1	2	3	4
y	10	11	26	91	266

$$\int_0^4 \left(x^4 + 10\right) dx = \frac{1}{3}\left[(10 + 266) + 2(26) + 4(11 + 91)\right]$$

$$= 245.33$$

The value of integral is given by

$$\int_0^4 \left(x^4 + 10\right) dx = \left[\frac{x^5}{5} + 10x\right]_0^4 = \frac{4^5}{5} + 10 \times 4 = 244.8$$

Therefore, magnitude of error $= 245.33 - 244.8 = 0.53$

38. While numerically solving the differential equation $\frac{dy}{dx} + 2xy^2 = 0$, $y(0) = 1$ using Euler's predictor–corrector (improved Euler–Cauchy) with a step size of 0.2, the value of y after the first step is

(a) 1.00 (b) 1.03
(c) 0.97 (d) 0.96

[GATE 2013, 2 Marks]

Solution: We have

$$\frac{dy}{dx} + 2xy^2 = 0$$

$$\Rightarrow \frac{dy}{dx} = -2xy^2$$

After one iteration, we have

$$y_1' = y_0 + h\left[-2x_0y_0^2\right] = 1 + 0.2\left[-2 \times 0 \times 1^2\right]$$

$$= 1 + 0 = 1$$

Therefore,

$$y_1 = y_0 + \frac{1}{2} \times (0.2)\left(-2 \times x_0 \times y_0^2 - 2x_1y_1'\right)$$

$$= 1 + 0.1\left\{-\left(2 \times 0 \times 1^2\right) - \left[(2 \times 0.2 \times 1)\right]\right\}$$

$$= 1 + 0.1[-0 - 0.4] = 1 - 0.04 = 0.96$$

Ans. (d)

39. Match the application to appropriate numerical method.

Application	Numerical Method
P1: Numerical integration	M1: Newton–Raphson method
P2: Solution to a transcendental equation	M2: Runge–Kutta method
P3: Solution to a system of linear equations	M3: Simpson's 1/3-rule
P4: Solution to a differential equation	M4: Gauss elimination method

(a) P1 – M3, P2 – M2, P3 – M4, P4 – M1
(b) P1 – M3, P2 – M1, P3 – M4, P4 – M2
(c) P1 – M4, P2 – M1, P3 – M3, P4 – M2
(d) P1 – M2, P2 – M1, P3 – M3, P4 – M4

[GATE 2014, 1 Mark]

Solution: P1 – M3, P2 – M1, P3 – M4, P4 – M2

Ans. (b)

40. The definite integral $\int_1^3 (1/x)dx$ is evaluated using the trapezoidal rule with a step size of 1. The correct answer is

[GATE 2014, 1 Mark]

Solution: In the given integral $\int_1^3 (1/x)dx$, let $y = 1/x$ and given that $h = 1$, so the number of subintervals for the interval of integration, $n = (3 - 1)/1 = 2$.
So the values of x are 1, 1 + 1, 1 + 2(1) for which the corresponding values of y are 1, 0.5, 0.33.
According to trapezoidal rule,

$$\int_a^b f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + \cdots + y_{n-1})]$$

Therefore,

$$\int_1^3 \frac{1}{x}\,dx = \frac{1}{2}[(1 + 0.33) + 2 \times 0.5] = 1.165$$

Ans. 1.165

41. In the Newton–Raphson method, an initial guess of $x_0 = 2$ is made and the sequence $x_0, x_1, x_2, \ldots$ is obtained for the function $0.75x^3 - 2x^2 - 2x + 4 = 0$ Consider the statements

(I) $x_3 = 0$.
(II) The method converges to a solution in a finite number of iterations.

Which of the following is TRUE?

(a) Only I (b) Only II
(c) Both I and II (d) Neither I nor II

[GATE 2014, 2 Marks]

Solution: According to Newton–Raphson method:

$$x_1 = x_0 - f_0/f_0'$$

$$\begin{aligned} f(x) &= 0.75x^3 - 2x^2 - 2x + 4 \\ f'(x) &= 2.25x^2 - 4x - 2 \\ x_0 &= 2,\ f_0 = -2,\ f_0' = -1 \\ x_1 &= 2 - (-2)/(-1) = 0 \\ f_1 &= 4,\ f_1' = -2 \\ x_2 &= 0 - (4)/(-2) = 2 \\ f_2 &= -2,\ f_2' = -1 \\ x_3 &= 2 - (-2)/(-1) = 0 \end{aligned}$$

Hence, (I) is true. Also the value diverges, thus option (II) is false.

Ans. (a)

42. With respect to the numerical evaluation of the definite integral, $K = \int_a^b x^2\,dx$, where a and b are given, which of the following statements is/are TRUE?

(I) The value of K obtained using the trapezoidal rule is always greater than or equal to the exact value of the definite integral.
(II) The value of K obtained using the Simpson's rule is always equal to the exact value of the definite integral.

(a) I only (b) II only
(c) Both I and II (d) Neither I nor II

[GATE 2014, 2 Marks]

Solution:

$$\int_0^1 x^2\,dx = \frac{x^3}{3} = \frac{1}{3} = 0.33333$$

As $N = 4$, we have

x	0	0.25	0.5	0.75	1
$y = x^2$	0	0.0625	0.25	0.5625	1
	y_0	y_1	y_2	y_3	y_4

Trapezoidal rule:

$$\int_0^1 x^2\,dx = \frac{0.25}{2}[(0+1) + 2(0.0625 + 0.25 + 0.5625)]$$
$$= 0.34375$$

Simpson's 1/3 rule:

$$\int_0^1 x^2\,dx = \frac{0.25}{2}[(0+1) + 2(0.25) + 4(0.0625 + 0.5625)]$$
$$= 0.3333$$

Ans. (c)

43. The function $f(x) = e^x - 1$ is to be solved using Newton–Raphson method. If the initial value of x_0 is taken as 1.0, then the absolute error observed at 2^{nd} iteration is ________.

[GATE 2014, 2 Marks]

Solution: Clearly, $x = 0$ is a root of the equation $f(x) = e^x - 1 = 0$.

Differentiating, we get

$$f'(x) = e^x \text{ and } x_0 = 1.0$$

Using Newton–Raphson method, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(e-1)}{e} = \frac{1}{e}$$

and $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{e} - \frac{\left(e^{1/e} - 1\right)}{e^{1/e}}$$
$$= \frac{1}{e} + \frac{1}{e^{1/e}} - 1 = 0.37 + 0.69 - 1 = 0.06$$

Therefore, absolute error at 2^{nd} iteration is $|0 - 0.06| = 0.06$.

Ans. 0.06

44. Using the trapezoidal role, and dividing the interval of integration into three equal subintervals, the definite integral $\int_{-1}^{1} |x|\,dx$ is ________.

[GATE 2014, 2 Marks]

Solution: In the given integral $\int_{-1}^{1} |x|\,dx$, let $y = |x|$ and given that the number of subintervals for the interval of integration, $n = 3$.

Now, $h = \frac{1-(-1)}{3} = \frac{2}{3}$

So the values of x are -1, $-1 + (2/3)$, $-1 + 2(2/3)$, $-1 + 3(2/3)$ and so on for which the corresponding values of y are 1, 1/3, 1/3, 1.

x	-1	$-\frac{1}{3}$	$\frac{1}{3}$	1
$y = \|x\|$	1	$\frac{1}{3}$	$\frac{1}{3}$	1

According to trapezoidal rule,

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + \cdots + y_{n-1})\right]$$

Therefore,

$$\int_{-1}^{1} |\, x \,|\, dx = \frac{1}{3}\left[(1+1) + 2\left(\frac{1}{3} + \frac{1}{3}\right)\right] = \frac{10}{9} = 1.11$$

Ans. 1.11

45. The value of $\int_{2.5}^{4} \ln(x)\, dx$ calculated using the trapezoidal rule with five subintervals is ________.

[GATE 2014, 2 Marks]

Solution: In the given integral, $\int_{2.5}^{4} \ln(x)\, dx$, let $y = \ln(x)$ and given that the number of subintervals for the interval of integration, $n = 5$. Now

$$h = \frac{4 - 2.5}{5} = \frac{1.5}{5} = 0.3$$

So the values of x are 2.5, 2.5 + (0.3), 2.5 + 2(0.3), 2.5 + 3(0.3), 2.5 + 4(0.3), 2.5 + 5(0.3) and so on for which the corresponding values of y are 0.91, 1.03, 1.13, 1.22, 1.31, 1.38.

x	2.5	2.8	3.1	3.4	3.7	4.0
$y = \ln x$	0.91	1.03	1.13	1.22	1.31	1.38

According to trapezoidal rule,

$$\int_a^b f(x)\, dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + \cdots + y_{n-1})\right]$$

Therefore,

$$\int_{2.5}^{4} \ln(x)\, dx = 0.25\left[(0.91 + 1.38) + 2(1.03 + 1.13 + 1.22 + 1.31)\right] = 1.745$$

Ans. 1.745

46. The real root of the equation $5x - 2\cos x - 1 = 0$ (up to two decimal accuracy) is ________.

[GATE 2014, 2 Marks]

Solution: Let $y = 5x - 2\cos x - 1$, so $dy/dx = 5 + 2\sin x$.
We have $y(0) = -3$, $y(1) = 2.9$, $x_0 = 1$ rad $= 57.32°$
By Newton–Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{2x_n \sin x_n + 2\cos x_n + 1}{5 + 2\sin x}$$

The roots are found as $x_1 = 0.5632$, $x_2 = 0.5425$

Ans. 0.5425

47. Consider an ordinary differential equation $\frac{dx}{dt} = 4t + 4$. If $x = x_0$ at $t = 0$, the increment in x calculated using Runge–Kutta fourth-order multi-step method with a step size of $\Delta t = 0.2$ is

(a) 0.22 (b) 0.44
(c) 0.66 (d) 0.88

[GATE 2014, 2 Marks]

Solution: The differential equation is $\frac{dx}{dt} = 4t + 4$.
At $t = 0$, $x = x_0$ and Δt or $h = 0.2$.
Using Runge–Kutta fourth-order multi-step method, we have

$$K_1 = f(t_0, x_0) = |dy/dx|_{t=0} = 4$$
$$K_2 = f(t_0 + 0.1,\ x_0 + 0.1K_1) = 4(0.1) + 4 = 4.4$$
$$K_3 = f(t_0 + 0.1,\ x_0 + 0.1K_2) = 4(0.1) + 4 = 4.4$$
$$K_4 = f(t_0 + 0.2,\ x_0 + 0.2K_3) = 4(0.2) + 4 = 4.8$$

Now, $$x(0.2) = x_0 + \frac{\Delta t}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= x_0 + 0.033(4 + 8.8 + 8.8 + 4.8) = x_0 + 0.88$$

So, the increment in x is 0.88.

Ans. (d)

48. The iteration step in order to solve for the cube roots of a given number N using the Newton–Raphson's method is

(a) $x_{k+1} = x_k + \frac{1}{3}\left(N - x_k^3\right)$

(b) $x_{k+1} = \frac{1}{3}\left(2x_k + \frac{N}{x_k^2}\right)$

(c) $x_{k+1} = x_k - \frac{1}{3}\left(N - x_k^3\right)$

(d) $x_{k+1} = \frac{1}{3}\left(2x_k - \frac{N}{x_k^2}\right)$

[GATE 2014, 2 Marks]

Solution: Newton's formula for finding $\sqrt[m]{N}$ is

$$x_{k+1} = \frac{1}{m}\left[(m-1)x_k + \frac{N}{x_k^{m-1}}\right]$$

For $m = 3$,

$$x_{k+1} = \frac{1}{3}\left[2x_k + \frac{N}{x_k^2}\right]$$

Ans. (b)

49. The Newton−Raphson method is used to solve the equation $f(x) = x^3 - 5x^2 + 6x - 8 = 0$. Taking the initial guess as $x = 5$, the solution obtained at the end of the first iteration is ______.

[GATE 2015, 1 Mark]

Solution: We are given

$$f(x) = x^3 - 5x^2 + 6x - 8$$
$$x_0 = 5$$
$$f'(x) = 3x^2 - 10x + 6$$

By Newton–Raphson method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 - \frac{f(5)}{f'(5)} = 5 - \frac{22}{31}$$
$$= 5 - 0.7097$$
$$= 2.4903$$

Ans. 2.4903

50. Using a unit-step size, the value of integral $\int_1^2 x \ln x dx$ by trapezoidal rule is ______ .

[GATE 2015, 1 Mark]

Solution: We can form the table,

$\boldsymbol{x}$	1	2
$\boldsymbol{y = x \ln x}$	0	$2\ln 2$

By trapezoidal rule,

$$\int_1^2 x \ln x dx = \frac{1}{2}[0 + 2\ln 2] = \ln 2 = 0.69$$

51. Simpson's $\frac{1}{3}$ rule is used to integrate the function $f(x) = \frac{3}{5}x^2 + \frac{9}{5}$ between $x = 0$ and $x = 1$ using the least number of equal sub-intervals. The value of the integral is ______ .

[GATE 2015, 1 Mark]

Solution: We are given a function

$$f(x) = \frac{3}{5}x^2 + \frac{9}{5}$$

We can formulate the table as follows:

$\boldsymbol{x}$	0	$\frac{1}{2}$	1
$\boldsymbol{y = f(x) = \frac{3}{5}x^2 + \frac{9}{5}}$	$\frac{9}{5}$	$\frac{39}{20}$	$\frac{12}{5}$

$$\int_0^1 y dx = \frac{\left(\frac{1}{2}\right)}{2}\left[\left(\frac{9}{5} + \frac{12}{5}\right) + 4\left(\frac{39}{20}\right)\right] = 3$$

Ans. 3

52. In Newton–Raphson iterative method, the initial guess value (X_{ini}) is considered as zero while finding the roots of the equation:

$$f(x) = -2 + 6x - 4x^2 + 0.5x^3$$

The correction, Δx, to be added to X_{ini} in the first iteration is ______ .

[GATE 2015, 1 Mark]

Solution: We have,

$$f(x) = -2 + 6x - 4x^2 + (0.5)x^3$$

Differentiating $f(x)$, we get

$$f'(x) = 6 - 8x + 1.5x^2$$

Using $x_0 = 0$, we get

$$f(0) = -2 \text{ and } f'(0) = 6$$

By Newton–Raphson method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-2)}{6} = \frac{2}{6} = 0.3333$$

$$\Delta x = x_1 - x_0 = 0.3333 - 0 = 0.3333$$

Ans. 0.3333

53. The velocity v (in kilometer/minute) of a motorbike which starts from rest is given at fixed intervals of time t (in minutes) as follows:

$\boldsymbol{t}$	2	4	6	8	10	12	14	16	18	20
$\boldsymbol{v}$	10	18	25	29	32	20	11	5	2	0

The approximate distance (in kilometers) rounded to two places of decimals covered in 20 min using Simpson's 1/3rd rule is ________.

[GATE 2015, 2 Marks]

Solution: Let s be the distance covered in 20 min, then by Simpson's 1/3rd rule.
Here length of each sub-interval is $h = 2$.

$$s = \int_0^{20} v dt = \frac{2}{3}[(0+0)+4(10+25+32+11+2) + 2(18+29+20+5)] = 309.33 \text{ km}$$

Ans. 309.33

54. The value of function (x) at five discrete points are given below:

x	0	0.1	0.2	0.3	0.4
$f(x)$	0	10	40	90	160

Using trapezoidal rule with step size of 0.1, the value of $\int_0^{0.4} f(x)dx$ is ________ .

[GATE 2015, 2 Marks]

Solution: We are given

x	0	0.1	0.2	0.3	0.4
$f(x)$	0	10	40	90	160
	y_0	y_1	y_2	y_3	y_4

$$\int_0^{0.4} f(x)dx = \int_0^{0.4} ydx = \frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)]$$
$$= \frac{0.1}{2}[(0+160)+2(10+40+90)] = 22$$

Ans. 22

55. Newton–Raphson method is used to find the roots of the equation, $x^3 + 2x^2 + 3x - 1 = 0$. If the initial guess is $x_0 = 1$, then the value of x after second iteration is ________ .

[GATE 2015, 2 Marks]

Solution: By Newton–Raphson method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{5}{10} = \frac{1}{2}$$

where

$$f(x) = x^3 + 2x^2 + 3x - 1 \Rightarrow f(1) = 5$$
$$f'(x) = 3x^2 + 4x + 3 \Rightarrow f'(1) = 10$$

$$x_1 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.3043$$

Ans. 0.3043

56. The integral $\int_{x_1}^{x_2} x^2 dx$ with $x_2 > x_1 > 0$ is evaluated analytically as well as numerically using a single application of the trapezoidal rule. If I is the exact value of the integral obtained analytically and J is the approximate value obtained using the trapezoidal rule, which of the following statement is correct about their relationship?

(a) $J > 1$
(b) $J < 1$
(c) $J = 1$
(d) Insufficient data to determine the relationship

[GATE 2015, 2 Marks]

Solution: We know that the approximated value of $\int_a^b f(x)dx$ obtained by trapezoidal rule is always greater than the analytical value. Thus, $J > 1$, where J is the approximate value and 1 is the analytical value.

Ans. (a)

57. The quadratic equation $x^2 - 4x + 4 = 0$ is to be solved numerically, starting with the initial guess $x_0 = 3$. The Newton–Raphson method is applied once to get a new estimate and then the secant method is applied once using in the initial guess and this new estimate. The estimated value of the root after the application of the secant method is ________.

[GATE 2015, 2 Marks]

Solution: We have,

$$f(x) = x^2 - 4x + 4$$
$$x_0 = 3$$
$$f'(x) = 2x - 4$$

By Newton–Raphson method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{1}{2} = 2.5$$

For secant method, let $x_0 = 2.5$ and $x_1 = 3$
By secant method,

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 3 - \frac{(3-2.5)}{f(3) - f(2.5)} f(3)$$

$$= 3 - \frac{0.5}{1-(0.25)} \times 1 = 3 - \frac{0.5}{0.75} = 3 - 0.6667 = 2.333$$

Ans. 2.333

58. For step-size, $\Delta x = 0.4$, the value of following integral using Simpson's 1/3 rule is ________ .

$$\int_0^{0.8} (0.2 + 25x - 200x^2 + 975x^3 - 900x^4 + 400x^5)\, dx$$

[GATE 2015, 2 Marks]

Solution: We are given,

$h = \Delta x = 0.4$

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

$x_0 = 0, x_n = 0.8 \Rightarrow n = \dfrac{0.8-0}{0.4} = 0$

x	0	0.4	0.8
$y = f(x)$	0.2	24.456	−126.7444

Applying Simpson's 1/3 rule,

$$\int_0^{0.8} f(x)dx = \frac{0.4}{3}[(0.2 - 136.744) + 4(24.456)] = -3.8293$$

Ans. −3.8293

59. The solution of the non-linear equation

$$x^3 - x = 0$$

is to be obtained using Newton–Raphson method. If the initial guess is $x = 0.5$, the method converges to which one of the following values:

(a) −1 (b) 0
(c) 1 (d) 2

[GATE 2015, 2 Marks]

Solution: We are given the equation,

$$f(x) = x^3 - x,\ f'(x) = 3x^2 - 1$$

Initial guess = 0.5 = x_0

First iteration,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow x_1 = 0.5 - \frac{(0.5)^3 - 0.5}{3(0.5)^2 - 1}$$

$$= 0.5 - \frac{(-0.375)}{(-0.25)} \Rightarrow x_1 = -1$$

Solution to above equations are 0.1 and −1 but after first iteration if it is converging to −1.

Ans. (a)

60. Consider the first-order initial value problem $y' = y + 2x - x^2$, $y(0) = 1$, $(0 \le x < \infty)$ with exact solution $y(x) = x^2 + e^x$. For $x = 0.1$, the percentage difference between the exact solution and the solution obtained using a single iteration of the second-order Runge–Kutta method with step-size $h = 0.1$ is ________.

[GATE 2016, 1 Mark]

Solution: It is given that

$$y_1 = y + 2x - x^2 \Rightarrow f(x, y) = y + 2x - x^2$$

From Runge–Kutta second-order method, we have

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k_1)$. It is given that $x_0 = 0$, $y_0 = 1$, $h = 0.1$. Then,

$$k_1 = (0.1)(y_0 + 2x_0 - x_0^2)$$
$$= (0.1)(1) = 0.1$$
$$k_2 = 0.1f(0 + 0.1, 1 + 0.1)$$
$$= (0.1)f(0.1, 1.1)$$
$$= (0.1)(1.1 + 2(0.1) - 0.01) = 0.129$$

Therefore,

$$y_1 = y(0.1) = y_0 + \frac{1}{2}(0.1 + 0.129) = 1.1145 \quad (1)$$

However, the exact solution is

$$y(x) = x^2 + e^x$$
$$y(0.1) = (0.1)^2 + e^{0.1} = 1.1152 \quad (2)$$

The percentage difference is

$$\left(\frac{1.1152 - 1.1145}{1.1152}\right)\% = 6.28 \times \frac{1}{100} = 0.0628$$

Ans. 0.0628

61. Solve the equation $x = 10 \cos(x)$ using the Newton–Raphson method. The initial guess is $x = \pi/4$. The value of the predicted root after the first iteration, up to second decimal, is ______.

[GATE 2016, 1 Mark]

Solution: Given that

$$f(x) = x - 10 \cos x$$
$$\Rightarrow f'(x) = 1 + 10 \sin x$$

Now,

$$x_0 = \frac{\pi}{4}$$

Thus,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= x_0 - \frac{x - 10\cos(x)}{1 + 10\sin(x)}$$

$$= \frac{\pi}{4} - \frac{(\pi/4) - 10\cos(\pi/4)}{1 + 10\sin(\pi/4)}$$

$$= 1.56$$

Ans. 1.56

62. Numerical integration using trapezoidal rule gives the best result for a single variable function, which is

(a) linear (b) parabolic
(c) logarithmic (d) hyperbolic

[GATE 2016, 1 Mark]

Solution: Trapezoidal rule gives the best result for a linear function.

Ans. (a)

63. The root of the function $f(x) = x^3 + x - 1$ obtained after first iteration on application of Newton–Raphson scheme using an initial guess of $x_0 = 1$ is

(a) 0.682 (b) 0.686
(c) 0.750 (d) 1.000

[GATE 2016, 1 Mark]

Solution: According to Newton–Raphson scheme,

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n)}$$

$$\Rightarrow x_1 = x_0 = -\frac{f(x_0)}{f'(x_0)} = 1 - \frac{(1+1-1)}{[3x(1)^2 + 1]} = 1 - \frac{1}{4}$$

$$\Rightarrow x_1 = \frac{3}{4} = 0.75$$

Ans. (c)

64. Newton–Raphson method is to be used to find root of equation $3x - e^x + \sin x = 0$. If the initial trial value for the root is taken as 0.333, the next approximation for the root would be ______.
(*Note: Answer up to three decimal digits.*)

[GATE 2016, 1 Mark]

Solution: As we know that $x_1 = x_0 - \dfrac{f(0)}{f'(0)}$

Let $f(x) = 3x - e^x + \sin x$

Now, $f(0.333) = -0.07$

and $f'(0.333) = +2.55$

Therefore,

$$x_1 = 0.333 - \frac{(-0.07)}{(+2.55)}$$

$$\Rightarrow x_1 = 0.360$$

Ans. 0.360

65. Gauss–Seidel method is used to solve the following equations (as per the given order):

$$x_1 + 2x_2 + 3x_3 = 5$$
$$2x_1 + 3x_2 + x_3 = 1$$
$$3x_1 + 2x_2 + x_3 = 3$$

Assuming initial guess as $x_1 = x_2 = x_3 = 0$, the value of x_3 after the first iteration is ______.

[GATE 2016, 2 Marks]

Solution: The values after first iteration are given by

$$x_1^{(1)} + 0 + 0 = 5$$

$$\Rightarrow x_1^{(1)} = 5$$

$$2x_1^{(1)} + 3x_2^{(1)} + 0 = 1$$

$$\Rightarrow 2 \times 5 + 3x_2^{(1)} + 0 = 1$$

$$\Rightarrow x_2^{(1)} = -3$$

$$3x_1^{(1)} + 2x_2^{(1)} + x_3^{(1)} = 3$$

$$\Rightarrow 3 \times 5 + 2 \times (-3) + x_3^{(1)} = 3$$

$$\Rightarrow x_3^{(1)} = -6$$

Ans. −6

66. The error in numerically computing the integral $\int_0^\pi (\sin x + \cos x)dx$ using the trapezoidal rule with three intervals of equal length between 0 and π is ______.

[GATE 2016, 2 Marks]

Solution: Exact value of the integral is calculated as

$$I = \int_0^x (\sin x + \cos x)dx = [-\cos x + \sin x]_0^\pi$$

$$I = (\sin \pi - \cos \pi) - (\sin 0 - \cos 0)$$
$$= 0 - (-1) - (0 - 1) = +1 + 1 = 2$$

Integral using trapezoidal rule with three intervals

$$I' = \left(\frac{\pi - 0}{2\times 3}\right)\left[f(0) + 2\left[f\left(\frac{\pi}{3}\right) + f\left(\frac{2\pi}{3}\right)\right] + f(\pi)\right]$$

$$= \frac{\pi}{6}[1 + 2[1.366 + 0.366] + (-1)]$$

$$= 1.813$$

Therefore,

$$\text{Error} = 2 - 1.813 = 0.187$$

Ans. 0.187

67. Starting with $x = 1$, the solution of the equation $x^3 + x = 1$, after two iterations of Newton–Raphson's method (up to two decimal places) is ________.

[GATE 2017, 2 Marks]

Solution: $F(x) = x^3 + x - 1 = 0$ (i)

By Newton–Raphson's iterative method, we have

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \quad \text{(ii)}$$

Given

$$x_0 = 1 \Rightarrow F(x_0) = 1 + 1 - 1 = 1$$

$$F'(x) = 3x^2 + 1$$

$$F'(x_0) = 3 + 1 = 4$$

So, from Eq. (ii), we have

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

$$x_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

Therefore, from Eq. (i),

$$F(x_1) = \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right) - 1 = 0.1718$$

$$\Rightarrow F'(x_1) = 3\left(\frac{3}{4}\right)^2 + 1 = 2.6875$$

Therefore,

$$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$$

$$x_2 = \frac{3}{4} - \frac{0.1718}{2.6875} = 0.686$$

Ans. (0.68)

68. Only one of the real roots of $f(x) = x^6 - x - 1$ lies in the interval $1 \le x \le 2$ and bisection method is used to find its value. For achieving an accuracy of 0.001, the required minimum number of iterations is ______. (Give the answer up to two decimal places.)

[GATE 2017, 2 Marks]

Solution: $f(x) = x^6 - x - 1;\ 1 \le x \le 2;\ \varepsilon = 0.001$

Number of iterations of bisection method $= \dfrac{|b-a|}{2^h} < \varepsilon$

$$\Rightarrow \frac{2-1}{2^n} < 0.001 \Rightarrow 2^n > 1000$$

$$\Rightarrow n > \log_2 1000$$

$$\Rightarrow n > 9.96 \Rightarrow n = 10$$

Ans. (10)

69. $P(0, 3)$, $Q(0.5, 4)$ and $R(1, 5)$ are three points on the curve defined by $f(x)$. Numerical integration is carried out using both trapezoidal rule and Simpson's rule within limits $x = 0$ and $x = 1$ for the curve. The difference between the two results will be

(a) 0 (b) 0.25 (c) 0.5 (d) 1

[GATE 2017, 2 Marks]

Solution: Let $x = 0$, 0.5, 1 and $y = 3$, 4, 5. Using Trapezoidal rule,

$$\int_0^1 f(x)\,dx = \frac{0.5}{2}\left[(3+5) + 2(4)\right]$$

$$= \frac{0.5}{2} \times 16$$

$$= 4$$

Using Simpson's rule,

$$\int_0^1 f(x)\,dx = \frac{0.5}{3}\left[(3+5) + 0 + 4(4)\right]$$

$$= \frac{0.5}{3} \times 24$$

$$= 4$$

The difference between the two results is zero.

Ans. (a)

CHAPTER 7

MATHEMATICAL LOGIC

INTRODUCTION

Mathematical logic is a subfield of mathematics that explores the applications of logic to mathematics. It is closely linked to the foundations of mathematics, metamathematics and theoretical computer science.

STATEMENTS

Atomic Statements

The simple statements without any connectives are called atomic or primary statements.

Molecular Statements

The statements formed by joining two or more statements through connectives are called molecular or compound statements.

Truth Table

A table showing the truth value of a statement formula for every possible combination of truth values of component statements is called its truth table.

Truth Values

A statement is a declarative sentence which has one and only one of the two possible values called truth values. The two truth values are TRUE and FALSE. These can also be denoted as T and F or 1 and 0, respectively.

As there are only two possible truth values for all the statements, the logic is called two-valued logic.

For example, consider the statement $10 + 10 = 100$.

Now, the statement mentioned above has a truth value, which depends on the system. If we consider the statement in context of a binary system, then it is true. On the contrary, if we consider the statement in context of a decimal system, then it is false.

CONNECTIVES

New sentences can be formed from given sentences using phrases or logical operators called connectives.

A symbol or word used to connect two or more sentences in a grammatically valid way is called a logical connective.

The logical relations when expressed by phrases are called propositional connectives. The sentence obtained will be a proposition only when the new sentence has a truth value T or F but not both.

Types of Connectives

Some of the commonly used connectives are explained in this section. They are described as follows:

1. **Negation (NOT):** If X is a proposition, then negation X or NOT X (denoted by $\rceil X$) is a proposition whose value is T when the truth value of X is F and F when truth value of X is T. Table 1 depicts the truth table for negation.

Table 1 | Truth table for negation

X	$\rceil X$
T	F
F	T

2. **Conjunction (AND):** If X and Y are two propositions, then conjunction of X and Y or X AND Y (denoted by $X \wedge Y$) is T only when the truth value of both X and Y is T. For all other values, X AND Y is F. Table 2 depicts the truth table for conjunction.

Table 2 | Truth value for conjunction

X	Y	$X \wedge Y$
T	T	T
T	F	F
F	T	F
F	F	F

3. **Disjunction (OR):** If X and Y are two propositions, then disjunction of X and Y or X OR Y (denoted by $X \vee Y$) is T when the truth value of either X or Y is T and F when the truth value of both X and Y is F. Table 3 depicts the truth table for disjunction.

Table 3 | Truth table for disjunction

X	Y	$X \vee Y$
T	F	T
T	F	T
F	T	T
F	F	F

4. **Implication (IF … THEN …):** If X and Y are two propositions, then "IF X THEN Y" denoted by $X \rightarrow Y$ is a proposition whose value is T when the truth value of X is F and whose value is equal to the value of Y when value of X is T. Table 4 depicts the truth table for implication.

Table 4 | Truth table for implication

X	Y	$X \rightarrow Y$
T	T	T
T	F	F
F	T	T
F	F	T

5. **IF AND ONLY IF:** If X and Y are two statements, then X IF AND ONLY IF Y, denoted by $X \rightleftarrows Y$, is a statement whose truth value is T when the truth values of X and Y are same and F when the truth values of X and Y are different. Table 5 depicts the truth table for IF AND ONLY IF.

Table 5 | Truth table for IF AND ONLY IF

X	Y	$X \rightleftarrows Y$
T	T	T
T	F	F
F	T	F
F	F	T

Table 6 summarizes all the connectives discussed thus far.

Table 6 Representation and meaning of logical connectives

Connective	Symbol	Read as	Denoted as
Negation	$\rceil$ or $\neg$ or $\sim$	NOT X	$\rceil X$
Conjunction	$\wedge$	X AND Y	$X \wedge Y$
Disjunction	$\vee$	X OR Y (or both)	$X \vee Y$
Implication	$\rightarrow$	IF X THEN Y	$X \rightarrow Y$
IF AND ONLY IF	$\rightleftarrows$	X IF AND ONLY IF	$X \rightleftarrows Y$

For example

Translate the following sentences into propositional forms:

(a) If my course is complete and I have time, then I will go to play.
(b) I do not have time and I will not go to play.
(c) My course is not complete and I do not have time.
(d) I will go to play only if my course is not complete.

Solution: Let X be proposition "My course is complete."

Y be proposition "I have time."

Z be proposition "I will go to play."

Then

(a) $X \wedge Y \rightarrow Z$ (b) $\rceil Y \wedge \rceil Z$
(c) $\rceil X \wedge \rceil Y$ (d) $Z \rightleftarrows \rceil X$

WELL-FORMED FORMULAS

A propositional variable is a variable which can take the value T or F. A propositional variable is not a proposition but can be replaced by a proposition.

A well-formed formula (WFF), or simply formula, can be defined in the following ways:

1. X is a WFF if it is a propositional variable.
2. If X is a WFF, then $\rceil X$ is also a WFF.
3. If X and Y are WFFs, then $(X \wedge Y)$, $(X \vee Y)$, $(X \rightarrow Y)$ and $(X \rightleftarrows Y)$ are all WFFs.
4. A combination of symbols is a WFF if it is obtained by finite amount of applications of conditions (1)–(3).

If we have a WFF involving n propositional constants, we have 2^n possible combinations of truth values of propositions replacing the variables.

For example,

Obtain the truth table for $= (X \vee Y) \wedge (X \rightarrow Y)$.

X	Y	$X \vee Y$	$X \rightarrow Y$	A
T	T	T	T	T
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

Construct the truth table for $\alpha = (X \wedge Y) \rightleftarrows ((X \vee Z) \rightarrow (Z \vee Y))$.

X	Y	Z	$(X \wedge Y)$	$Z \vee Y$	$(X \vee Z) \rightarrow (Z \vee Y)$	$(X \wedge Y)$	α
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	F	F
T	F	F	T	F	F	F	T
F	T	T	T	T	T	F	F
F	T	F	F	T	T	F	F
F	F	T	T	T	T	F	F
F	F	F	F	F	T	F	F

Duality Law

Two WFFs A and A' are said to be duals of each other if either one can be obtained from the other by interchanging $\wedge$ to $\vee$ and T to F.

Also, if we have A and A' as duals of each other and $X_1, X_2 \ldots X_n$ are their propositional variables, then using De Morgan's law, we can show that

$$\rceil A\ (X_1, X_2 \ldots X_n) = A'\ (\rceil X_1, \rceil X_2 \ldots \rceil X_n)$$

For example,

Find the dual of $A = (X \wedge Y) \vee (X \wedge Z)$.

Let us say A' is the dual of A.

Therefore, $A' = (X \wedge Y) \vee (X \wedge Z)$.

Equivalent Well-Formed Formula

A WFF whose value is T for all possible assignments of truth values to the propositional variables is called a tautology or a universally true formula.

For example,

Show that $\alpha = ((X \rightarrow Y) \wedge (Y \rightarrow Z)) \rightarrow (X \rightarrow Z)$ is a tautology.

X	Y	Z	$X \to Y$	$Y \to Z$	$(X \to Y) \wedge (Y \to Z)$	$(X \to Z)$	α
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

As for all possible values of X, Y and Z we have $\alpha = \text{T}$, α is a tautology.

A WFF whose value is F for all possible assignments of truth values to the propositional variables is called a contradiction.

For example,

Show that $\alpha = (X \wedge Y) \wedge \neg Y$ is a contradiction.

X	Y	$X \wedge Y$	$\neg Y$	α
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	F	T	F

As for all possible values of X and Y, the value of α is F, $\alpha = (X \wedge Y) \wedge \neg Y$ is a contradiction.

Two WFFs, say α and β, in propositional variables X_1, X_2, ..., X_n are said to be logically equivalent if the formula $\alpha \rightleftarrows \beta$ is a tautology. Two equivalent WFFs are denoted as

$$\alpha \equiv \beta$$

In a simpler language, α and β are equivalent if their truth table is same for all possible values of the proportional variables.

For example,

Show that $(X \vee (Y \wedge Z)) \equiv (X \vee Y) \wedge (X \vee Z)$.

Let $\alpha = X \vee (Y \wedge Z)$ and $\beta = (X \vee Y) \wedge (X \vee Z)$.

X	Y	Z	$Y \wedge Z$	α	$X \vee Y$	$X \vee Z$	β
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

As the truth table for α and β is same, $\alpha \equiv \beta$.

LOGICAL IDENTITIES

There are certain laws or identities which are used for deducing other equivalences. Some of the important identities are:

1. Idempotent laws

$$X \wedge X \equiv X \text{ and } X \vee X \equiv X$$

2. Commutative laws

$$X \vee Y \equiv Y \vee X \text{ and } X \wedge Y \equiv Y \wedge X$$

3. Absorption laws

$$X \vee (X \vee Y) = X \text{ and } X \wedge (X \wedge Y) \equiv X$$

4. Associative laws

$$X \vee (Y \vee Z) \equiv (X \vee Z) \vee Z \text{ and } X \wedge (Y \wedge Z) \equiv (X \wedge Y) \wedge Z$$

5. Distributed laws

$$X \vee (Y \wedge Z) \equiv (X \vee Y) \wedge (X \vee Z) \text{ and } X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$$

6. De Morgan's laws

$$\neg(X \vee Y) = \neg X \wedge \neg Y,\ \neg(X \wedge Y) = \neg X \vee \neg Y$$

7. Involution laws

$$X = \neg(\neg X)$$

8. $X \vee T \equiv T$, $\quad X \vee F = X$

9. $X \wedge T \equiv X$, $\quad X \wedge F = F$

10. $X \wedge \neg X = F$, $X \vee \neg X = T$

11. $(X \to Y) \wedge (\neg Y) \equiv \neg X$

12. $X \to Y \equiv \neg Y \to \neg X$

13. $X \to Y \equiv (\neg X \vee Y)$

For example,

Using the identities described before, show that $(X \wedge Y) \vee (X \wedge \neg Y) \equiv X$.

$$\begin{aligned} \text{L.H.S.} &= (X \wedge Y) \vee (X \wedge \neg Y) \\ &= X \wedge (Y \vee \neg Y) \quad \text{[By Distributive law]} \\ &= X \wedge \text{T} \quad \text{[Using Identity 8]} \\ &= X \quad \text{[Using Identity 9]} \\ \text{R.H.S.} &= X \end{aligned}$$

Normal Form

We know that various WFFs having two propositional variables, say X and Y, are equivalent if they have the same truth table. Also, we know that the number of distinct truth tables for formulas in X and Y is 2^4. Hence, any formula in X and Y is equivalent to one of the 16 equivalent formulas.

A method to reduce a given formula into an equivalent form is called a normal form. When referring to a normal form, we use "product" for conjunction (or AND), "sum" for disjunction (or OR) and "literal" for X or $\rceil X$, where X is a propositional variable.

A product of literals is called an elementary product.

A sum of literals is called an elementary sum.

For example, $\rceil X \wedge Y$ is elementary product.

$\rceil X \vee Y$ is elementary sum.

Disjunction Normal Form

A formula which is a sum of elementary products is in a disjunctive normal form.

For example, $X \vee (Y \wedge Z) \vee (\rceil Y \wedge \rceil Z)$ is in disjunctive normal form.

The following steps are used to obtain a disjunctive normal form from a given formula.

1. Eliminate "$\to$" using logical identity.
2. Use De Morgan's laws to eliminate $\rceil$ (negation) before sums or products.
3. Apply distributive laws successively to obtain disjunctive normal form by eliminating product of sums.

For example,

Obtain a disjunctive normal form from the formula

$$\alpha = X \vee (Y \vee (Y \to \rceil Z))$$

$$\alpha = X \vee (Y \vee (\rceil Y \to \rceil Z)) = X \vee Y \vee \rceil Y \vee \rceil Z$$

Principal Disjunctive Normal Form

A min term for propositional variables X and Y is given by $X \wedge Y$, $\rceil X \wedge Y$, $X \wedge \rceil Y$ and $\rceil X \wedge \rceil Y$. The number of min terms of n variable is 2^n.

If a formula is a sum of min terms, then it is said to be in principal disjunctive normal form.

The following steps are used to obtain a principal disjunctive normal form from a given formula:

1. Follow steps (1)–(3) mentioned in Section 7.5.2 to obtain a disjunctive normal form.
2. Remove contradicting elementary products such as $X \wedge \rceil X$.
3. If X_i and $\rceil X_i$ are missing in the normal formula (say α), then replace α with $(\alpha \wedge X_i) \vee (\alpha \wedge \rceil X_i)$.
4. Repeat step (3) successively until all elementary products are reduced to sum of min terms. Apply idempotent laws to avoid repetition of min terms.

For example,

Obtain the principle disjunctive form of $\alpha = (\rceil X \vee \rceil Z) \to (\rceil X \wedge Y)$.

$$\begin{aligned}\alpha &= (\rceil X \vee \rceil Z) \to (\rceil X \wedge Y)\\ &= (\rceil(\rceil X \vee \rceil Z)) \vee (\rceil X \wedge Y)\\ &= (X \wedge Z) \vee (\rceil X \wedge Y)\end{aligned}$$

Now, the above formula is in disjunctive normal form.

$$\begin{aligned}\alpha = {}&(X \wedge Z \wedge Y) \vee (X \wedge Z \wedge \rceil Y) \vee\\ &(\rceil X \wedge Y \wedge Z) \vee (\rceil X \wedge Y \wedge \rceil Z)\end{aligned}$$

Hence, α is now in principle disjunctive normal form.

Functionally Complete Set of Connectives

Any set of connectives which can be used to express every formula as an equivalent formula containing the same connectives from the set is called functionally complete set of connectives.

For example, $\{\wedge, \rceil\}$ and $\{\vee, \rceil\}$ are functionally complete.

Some Other Connectives

The other connectives used are described as follows:

1. **Exclusive OR:** It is denoted by $\bar{\vee}$. If X and Y are two propositions, then $X \bar{\vee} Y$ is T when either X or Y is T but not both. The truth table for exclusive OR is given in Table 7.

Table 7 Truth table for exclusive OR

X	Y	$X \bar{\vee} Y$
T	T	F
T	F	T
F	T	T
F	F	F

2. **NAND:** NAND is denoted by $\uparrow$. If X and Y are two propositions, then $X \uparrow Y$ is T for all values expect when both X and Y are T. The connective NAND is logically equivalent to the negation of conjunction, that is, their truth tables are same. The truth table for NAND is given in Table 8.

Table 8 | Truth table for NAND

X	Y	$X \uparrow Y$
T	T	F
T	F	T
F	T	T
F	F	T

3. **NOR:** NOR is denoted by $\downarrow$. If X and Y are two propositions, then $X \downarrow Y$ is T when the truth value of both X and Y is F and F for all other values. The connective NOR is logically equivalent to the negation of disjunction, that is, their truth tables are same. The truth table for NOR is given in Table 9.

Table 9 | Truth table for NOR

X	Y	$X \downarrow Y$
T	T	F
T	F	F
F	T	F
F	F	T

Conjunction and disjunction of X and Y can be expressed by NOR connective as follows:

$$X \wedge Y = (X \downarrow X) \downarrow (Y \downarrow Y)$$
$$X \vee Y = (X \downarrow Y) \downarrow (X \downarrow Y)$$

Conjunction and disjunction of X and Y can be expressed by NAND connective as follows:

$$X \wedge Y = (X \uparrow Y) \uparrow (X \uparrow Y)$$
$$X \vee Y = (X \uparrow X) \uparrow (Y \uparrow Y)$$

Conjunctive Normal Form

A formula which is a product of elementary sums is in a conjunctive normal form.

For example, $X \wedge (X \vee Y) \wedge (Y \wedge \rceil Z)$ is in conjunctive normal form.

Let us say α is in disjunctive normal form, then $\rceil\alpha$ is in conjunctive normal form.

Hence, to obtain conjunctive form of any formula (α), we first obtain disjunctive normal form of $\rceil\alpha$ using the steps mentioned before and then find its negation. The resultant is a conjunctive normal form.

Principal Conjunctive Normal Form

A max term for propositional variables X and Y is given by $X \vee Y$, $\rceil X \vee Y$, $X \vee \rceil Y$ and $\rceil X \vee \rceil Y$. The number of max terms of n variables is 2^n.

If a formula is a product of max terms, then it is said to be in principal conjunctive normal form. To obtain a principal conjunctive normal form of any formula, say α, we construct a principal disjunctive normal form of $\rceil\alpha$ and then apply negation on the result.

For example,

Find the principal conjunctive normal form of $\alpha = X \vee (Y \rightarrow Z)$.

We first calculate principal disjunctive normal form of $\rceil\alpha$.

Thus,
$$\begin{aligned} \rceil\alpha &= \rceil(X \vee (Y \rightarrow Z)) \\ &= \rceil(X \vee (\rceil Y \vee Z)) \\ &= \rceil X\ (\rceil(\rceil Y \wedge Z)) \\ &= \rceil X \wedge Y \wedge \rceil Z \end{aligned}$$

The above equation is the principal disjunction normal form of $\rceil\alpha$.

Hence, principal conjunction normal form of α is given by

$$\rceil(\rceil X \wedge Y \wedge \rceil Z)$$

PROPOSITIONAL CALCULUS

Propositional calculus is a formula in which formulae of a language are interpreted to represent propositions. Certain propositions are assumed to be T and based on these assumptions, other propositions are derived. A system of inference rules allow certain formulas to be derived, which are called theorems.

The propositions that are assumed to be true are called hypotheses or premises.

The proposition that is derived using a system of inference rules is called a conclusion.

The process of deriving a conclusion on the basis of assumption of hypotheses is called a valid argument.

Rules of Inference

In mathematical logic, the rule of inference is a logical form consisting of a function which takes premises, studies its syntax and returns a set of conclusions. The action of rule of inference is purely syntactic and does not need to preserve any semantic property.

The rules of inference for valid arguments are given in Table 10.

Table 10 | Rules of inference

S. no.	Rule	Definition	Notation
1.	Modus ponens	$\dfrac{X \to Y, X}{\therefore Y}$	$((X \to Y) \wedge X) \to Y$
2.	Modus tollens	$\dfrac{X \to Y, \rceil Y}{\therefore \rceil X}$	$((X \to Y) \wedge \rceil Y) \to \rceil X$
3.	Biconditional introduction	$\dfrac{X \to Y, Y \to X}{\therefore X \leftrightarrow Y}$	$((X \to Y) \wedge (Y \to X)) \to (X \leftrightarrow Y)$
4.	Biconditional elimination	$\dfrac{(X \leftrightarrow Y)}{\therefore (X \to Y)}$	$(X \leftrightarrow Y) \to (X \to Y)$
		$\dfrac{(X \leftrightarrow Y)}{\therefore (Y \to X)}$	$(X \leftrightarrow Y) \to (Y \to X)$
5.	Disjunction introduction	$\dfrac{X}{\therefore X \vee Y}$	$X \to (X \vee Y)$
6.	Disjunction elimination	$\dfrac{X \to Y, Z \to Y, X \vee Z}{\therefore Y}$	$(((X \to Y) \wedge (Z \to Y)) \wedge (X \vee Z)) \to Y$
7.	Disjunction syllogism	$\dfrac{X \vee Y, \rceil X}{\therefore Y}$	$((X \vee Y) \wedge \rceil X) \to Y$
8.	Conjunction introduction	$\dfrac{Y}{\therefore X \wedge Y}$	$Y \to X \wedge Y$
9.	Simplification	$\dfrac{X \wedge Y}{\therefore Y}$	$(X \wedge Y) \to Y$
10.	Hypothetical syllogism	$\dfrac{X \to Y, Y \to Z}{\therefore X \to Z}$	$((X \to Y) \wedge (Y \to Z)) \to (X \to Z)$
11.	Constructive dilemma	$\dfrac{X \to Y, Z \to W, X \vee Z}{\therefore Y \vee W}$	$(((X \to Y) \wedge (Z \to W)) \wedge (X \vee Z)) \to (Y \vee W)$
12.	Destructive dilemma	$\dfrac{X \to Y, Z \to W, \rceil Y \vee \rceil W}{\therefore \rceil X \vee \rceil Z}$	$(((X \to Y) \wedge (Z \to W)) \wedge (\rceil Y \vee \rceil W)) \to (\rceil X \vee \rceil Z)$
13.	Absorption	$\dfrac{X \to Y}{\therefore X \to (X \wedge Y)}$	$(X \to Y) \leftrightarrow (X \to (X \wedge Y))$

For example,

Conclude W from the following premises:

$$X \to Y$$
$$X \to Z$$
$$\rceil(Y \wedge Z)$$
$$W \vee X$$

Let us consider the first two hypotheses/premises

$$X \to Y \qquad (1)$$
$$X \to Z \qquad (2)$$

Applying conjunction rule in Eqs. (1) and (2) gives

$$(X \to Y) \wedge (X \to Z) \qquad (3)$$

The third hypothesis is given by

$$\rceil(Y \wedge Z) \tag{4}$$

Applying De Morgan's law in Eq. (4) gives

$$\rceil Y \vee \rceil Z \tag{5}$$

Applying destructive dilemma in Eqs. (3) and (5) gives

$$\rceil X \vee \rceil X = \rceil X \tag{6}$$

The fourth hypothesis is given by

$$W \vee X \tag{7}$$

Applying disjunctive syllogism in Eqs. (6) and (7) gives

$$W$$

PREDICATE CALCULUS

Predicate calculus or first-order calculus deals with sentences having a common feature called predicates.

Let us consider the following two propositions:

1. Ram is a cricketer.
2. Rahul is a cricketer.

Now, there is no direct relation between the two propositions. However, we know that they have something in common, that is both Ram and Rahul share the property of being a cricketer. Hence, the above propositions can be replaced by a single statement "X is a cricketer." Here, X is a variable and "is a cricketer" is called a predicate.

Predicates

Predicates are functions of zero or more variables. They can be T or F depending on the values of their arguments. Predicates are denoted by $P(x)$ where P denotes the common property and x is the variable.

It should be noted that $P(x)$ is not a proposition since it involves a variable x and hence we cannot calculate a truth value for $P(x)$. However, if we replace x by a constant or an individual value, then we get a proposition.

The universe for a sentence involving a predicate is the set of all possible values that the variable of the predicate can take. For example, universe for $P(x)$: "x is an odd number" can be taken as the set of all odd integers.

Quantifier

The phrases "for all," "for any," "every" and "for each" are called universal quantifiers. Universal quantifiers are denoted by $\forall$.

For example,

"All students are sitting" can be denoted by $\forall x\ c(x)$, where $c(x)$ denotes "x is sitting" and takes the universe of discourse as a set of all students.

The phrases "there exists," "for some" and "for at least one" are called existential quantifiers. Existential quantifiers are denoted by $\exists$.

For example,

"Some students are sitting" can be denoted by $\exists\ x\ c(x)$, where $c(x)$ denotes "X is sitting" and takes the universe of discourses as a set of all students.

WFF for predicate calculus can be formed using the following rules:

1. $P(x_1, x_2, \ldots, x_n)$ is a WFF if P is a predicate involving n variables $x_1, x_2, \ldots, x_n$.
2. If α is a WFF, then $\rceil\alpha$ is also a WFF.
3. If α and β are WFFs, then $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$ and $\alpha \rightleftarrows \beta$ are also WFFs.
4. $\forall X(\alpha)$ and $\exists X(\alpha)$ are WFFs if α is a WFF and X is a variable.
5. An array is a WFF if and only if is obtained by finite amount of applications of rules (1)–(4) mentioned above.

Consider α and β as two predicate formulas of variables $x_1, x_2, \ldots, x_n$ in the universe of discourse (U), then α and β are equivalent (i.e. $\alpha \equiv \beta$) over U for every possible value to each variable in α and β, if they have the same truth values.

If $\forall x\ P(x)$ or $\exists x\ P(x)$ occurs as part of α, where α is a predicate formula, then such a part is called an x-bound part of α, and occurrence of x is called bound occurrence of x.

If a predicate variable in α is free in any part of α, then it is not a bounded occurrence and the occurrence is called free.

For example,

In $\alpha = \exists x\ P(x, Y)$, the occurrence of x is a bound occurrence and the occurrence of y is a free occurrence.

Predicate formulas where all the variables are quantified are propositional formulas. Therefore, all the rules of inference for propositional calculus are applicable to predicate calculus. Table 11 shows the rules of inference for addition and deletion of quantifiers.

Table 11 Rules of inference for addition and deletion of quantifiers

S. no.	Rule of Inference	Definition	Condition
1.	Universal generalization	$\dfrac{P(x)}{\forall x P(x)}$	x should not be free in any of the given premises.
2.	Universal instantiation	$\dfrac{\forall x P(x)}{\therefore P(c/x)}$	$P(x/c)$ is the result of substituting c for all occurrences of x in P.
3.	Existential generalization	$\dfrac{P(c)}{\therefore \exists x P(x)}$	c replaces all free instances of x within $P(x)$.
4.	Existential instantiation	$\dfrac{\exists x P(x)}{\therefore P(c)}$	Where c is any arbitrary element for which $P(c)$ is true.

Satisfiable, Unsatisfiable and Valid Formulas

A predicate formula having truth value of resulting proposition as T for some assignment of values to predicate variables is called satisfiable predicate formula.

A predicate formula having truth value of resulting proposition as F for all possible assignment of values to predicate variables is called unsatisfiable predicate formula.

A predicate formula having truth value of resulting proposition as T for all possible assignment of values to predicate variables is called valid predicate formula.

SOLVED EXAMPLES

1. Conclude $\rceil X$ from $\rceil X \vee Y$, $\rceil(Y \wedge \rceil Z)$ and $\rceil Z$.

Solution: We have two premises

$$\rceil Z \quad \text{(i)}$$
$$\rceil(Y \vee \rceil Z) \quad \text{(ii)}$$

Applying De Morgan's law in Eq. (ii) gives

$$\rceil Y \vee Z \quad \text{(iii)}$$

Applying disjunction syllogism in Eqs. (i) and (iii) gives

$$\rceil Y \quad \text{(iv)}$$

We have another premise,

$$\rceil X \vee Y \quad \text{(v)}$$

Applying disjunction syllogism in Eqs. (iv) and (v) gives $\rceil X$, which is the required result.

2. Find the equivalent formula for $(X \rightarrow Y)$ containing $\uparrow$ only.

Solution: We know that

$$X \rightarrow Y = \rceil X \vee Y$$

Also, $X \vee Y = (X \uparrow Y) \uparrow (Y \uparrow Y)$ and $X \uparrow X = \rceil(X \wedge X) = \rceil X$

Therefore, $\rceil X \vee Y = (\rceil X \uparrow \rceil X) \uparrow (Y \uparrow Y)$
$$= \rceil(\rceil X) \uparrow (\rceil Y) = X \uparrow \rceil Y$$

3. Prove that $[\rceil, \rightarrow]$ is functionally complete.

Solution: Let us consider $X \wedge Y \equiv \rceil(\rceil X \vee \rceil Q)$, hence $\{\rceil, \vee\}$ is functionally complete.
Now, $X \vee Y = \rceil X \rightarrow Y$, hence we can replace $\vee$ with $\rceil$ and $\rightarrow$.

Hence, $\{\rceil, \vee\} = \{\rceil, \rightarrow\}$

Thus, $\{\rceil, \rightarrow\}$ is functionally complete.

4. Show that truth value of the following formula is independent of its components.

$$(X \rightarrow Y) \rightleftarrows (\rceil X \vee Y)$$

Solution: Consider

$$(X \rightarrow Y) \rightleftarrows (\rceil X \vee Y)$$

Now, $\rceil X \vee Y = X \rightarrow Y$

Therefore, $(X \rightarrow Y) \rightleftarrows (X \rightarrow Y)$.

Also, truth value of $X \rightleftarrows X$ is T regardless of value X.

Therefore, $(X \rightarrow Y) \rightleftarrows (X \rightarrow Y)$ has a truth value of T for all values of X and Y. Hence, it is a tautology.

5. Show that $\{\vee, \wedge\}$ is functionally incomplete.

Solution: Let us consider $(\neg X \vee X = \mathrm{T})$. This formula cannot be written as a combination of $\wedge$ and $\vee$ alone. Hence, $\{\vee, \wedge\}$ is functionally incomplete.

6. Show that $[(\neg X \vee Y) \rightarrow (Y \rightarrow X)]$ is not a tautology.

Solution: Let us simplify the above expression

$$
\begin{aligned}
&[(\neg X \vee Y) \rightarrow (Y \rightarrow X)] \\
&= [(\neg X \vee Y) \rightarrow (\neg Y \vee X)] \\
&= [\neg(\neg X \vee Y) \vee (\neg Y \vee X)] \\
&= [(X \wedge \neg Y) \vee (\neg Y \vee X)] \\
&= [\neg X \vee Y]
\end{aligned}
$$

The above expression is F for $X = 1$ and $Y = 0$. Hence, it is not a tautology.

7. Show that proposition Z is a logical consequence of the formula

$$X \wedge (X \rightarrow (Y \vee Z)) \wedge (Y \rightarrow \neg X)$$

Solution: If we want to show that any formula B is a logical consequence of formula A, then $A \rightarrow B$ is a tautology. Now if $A = X \wedge (X \rightarrow (Y \vee Z)) \wedge (Y \rightarrow \neg X)$ and $B = Z$, consider the following truth table:

X	Y	Z	$Y \vee Z$	$X \rightarrow (Y \vee Z)$	$\neg X$	$Y \rightarrow \neg X$	A	$A \rightarrow B$
T	T	T	T	T	F	F	F	T
T	T	F	T	T	F	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	F	F	F	T	F	T
F	T	T	T	T	T	T	F	T
F	T	F	T	T	T	T	F	T
F	F	T	T	T	T	T	F	T
F	F	F	T	T	T	T	F	T

8. Obtain principal disjunctive normal from of

$$\alpha = X \vee (\neg X \wedge \neg Y \wedge \neg Z)$$

Solution: As we can see, $\alpha = X \vee (\neg X \wedge \neg Y \wedge \neg Z)$ is already in principal disjunctive normal form. However, we have to add the missing terms.

Now, $X = (X \wedge Y) \vee (X \vee \neg Y)$

$$= (X \wedge Y \wedge Z) \vee (X \wedge Y \wedge \neg Z) \vee (X \wedge \neg Y \wedge Z) \vee (X \wedge \neg Y \wedge \neg Z)$$

Substituting the value of X in α gives

$$\alpha = (X \wedge Y \wedge Z) \vee (X \wedge Y \wedge \neg Z) \vee (X \wedge \neg Y \wedge Z) \vee (X \wedge \neg Y \wedge \neg Z) \vee (\neg X \wedge \neg Y \wedge \neg Z)$$

9. Check the validity of the following argument:

If Jack has completed B.Sc. or M.Sc., then he is assured a good job. If Jack is assured of a good job, he is content. Jack is not content, so Jack has not completed M.Sc.

Solution: The propositions can be formed as follows:

W denotes "Jack has completed B.Sc."

X denotes "Jack has completed M.Sc."

Y denotes "Jack is assured of a good job."

Z denotes "Jack is content."

The given premises are:

$$(W \vee X) \rightarrow Y \tag{1}$$

$$Y \rightarrow Z \tag{2}$$

$$\neg Z \tag{3}$$

Applying hypothetical syllogism in Eqs. (1) and (2) gives

$$(W \vee X) \rightarrow Z \tag{4}$$

Applying modus tollens in Eqs. (3) and (4) gives

$$\neg(W \vee X) \tag{5}$$

Applying De Morgan's law in Eq. (5) gives

$$\neg W \wedge \neg W \tag{6}$$

Simplification of Eq. (6) gives

$$\neg X$$

Thus, the argument is valid.

10. Discuss the validity of the following arguments:

All graduates are educated. Garima is a graduate. Therefore, Garima is educated.

Solution: Let $A(x)$ denotes "x is a graduate"
$B(x)$ denotes "x is educated"
G denotes "Garima".

The premises are:

$$\forall x\, (A(x) \rightarrow B(x)) \tag{1}$$

$$A(G) \tag{2}$$

The conclusion is $B(R)$.

Now, applying universal instantiation in Eq. (1) gives

$$A(R) \rightarrow B(R) \tag{3}$$

Applying modus ponens in Eqs. (2) and (3) gives

$$B(R)$$

Thus, the conclusion is valid.

PRACTICE EXERCISE

1. Find an equivalent formula for $(X \rightarrow Y) \vee Z$ containing $\uparrow$ only.

(a) $((X \uparrow (Y \uparrow Y)) \uparrow (X \uparrow (Y \uparrow Y)) \uparrow (Z \uparrow Z))$
(b) $(X \uparrow Y) \uparrow ((X \uparrow X) \uparrow Y) \uparrow (Y \uparrow Z)$
(c) $(X \uparrow X) \uparrow (X \uparrow (Y \uparrow Y)) \uparrow (Y \uparrow Z)$
(d) $((X \uparrow X) \uparrow Y) \uparrow (X \uparrow (Y \uparrow Y)) \uparrow (X \uparrow Y \uparrow Z)$

2. Find an equivalent for $(X \rightarrow \rceil Y)$ containing $\downarrow$ only.

(a) $(X \downarrow \rceil Y) \downarrow (\rceil X \downarrow Y)$ (b) $(\rceil X \downarrow Y) \downarrow (\rceil X \downarrow \rceil Y)$
(c) $(\rceil X \downarrow \rceil Y) \downarrow (\rceil X \downarrow \rceil Y)$ (d) $(X \downarrow Y) \downarrow (\rceil X \downarrow \rceil Y)$

3. Substitution instance of a tautology is a

(a) contradiction
(b) tautology
(c) can be (a) or (b)
(d) none of the above

4. Express $X \uparrow Y$ in terms of $\downarrow$ only.

(a) $((X \downarrow Y) \downarrow (X \downarrow Y))$
(b) $((X \downarrow X) \downarrow (Y \downarrow Y))$
(c) $((X \downarrow Y) \downarrow (Y \downarrow Y)) \downarrow ((X \downarrow Y) \downarrow (X \downarrow X))$
(d) $((X \downarrow X) \downarrow (Y \downarrow Y)) \downarrow ((X \downarrow X) \downarrow (Y \downarrow Y))$

5. $\{\uparrow\}$ connective is

(a) functionally complete
(b) functionally incomplete
(c) cannot say
(d) none of the above

6. $\rceil(X \rightarrow Y)$ is equivalent to

(a) $X \wedge \rceil Y$ (b) $\rceil X \wedge Y$
(c) $\rceil X \vee Y$ (d) $X \vee Y$

7. The translation of the sentence "Some cats are white but all swans are white" using the following notations:

$A(x)$: x is white

$B(x)$: x is cat

$C(x)$: x is swan

$(\forall x)$: for all x

$(\exists x)$: for some x

(a) $(\exists x)(B(x) \wedge A(X)) \wedge (\forall x)(C(x) \rightarrow A(x))$
(b) $(\exists x)(B(x) \wedge A(x)) \wedge (\forall x)(C(x) \wedge A(x))$
(c) $(\exists x)(B(x) \rightarrow A(x)) \wedge (\forall x)(C(x) \wedge A(x))$
(d) $(\exists x)(B(x) \rightarrow A(x)) \wedge (\forall x)(C(x) \rightarrow A(x))$

8. The negation of statement $\forall x \, \exists y, P(x, Y)$ is given by

(a) $\exists x \forall y, P(x, y)$ (b) $\exists x \forall y, \rceil P(x, y)$
(c) $\forall x \forall y, \rceil P(x, y)$ (d) $\exists x \exists y, \rceil P(x, y)$

9. $x \rightarrow (y \rightarrow z)$ is equivalent to

(a) $(x \vee y) \rightarrow z$ (b) $(x \wedge y) \rightarrow z$
(c) $(x \vee y) \rightarrow \rceil z$ (d) $(x \wedge y) \rightarrow \rceil z$

10. In predicate logic, $\rceil \forall P(x)$ is equivalent to

(a) $\exists x \, P(x)$ (b) $\forall x \, \rceil P(x)$
(c) $\exists x \, \rceil P(x)$ (d) $\rceil \forall x \, P(x)$

11. Which of the following is/are tautologies?

(a) $(X \vee Y) \leftrightarrow Y$ (b) $(X \vee (X \rightarrow Y)) \rightarrow X$
(c) $((X \vee Y) \wedge X) \rightarrow Y$ (d) $((X \vee Y) \wedge \rceil X) \rightarrow Y$

12. Which one of the following is a tautology?

(a) $X \rightarrow (X \vee Y)$ (b) $X \wedge \rceil Y$
(c) $X \rightarrow (X \wedge Y)$ (d) $(X \vee Y) \rightarrow (X \wedge Y)$

13. Which one of the following is a declarative statement?

(a) It is beautiful
(b) It is not beautiful
(c) He says "It is beautiful"
(d) It may or may not be beautiful

14. Evaluate $(X \vee Y) \wedge (X \rightarrow Z) \wedge (Y \rightarrow Z)$.

(a) X (b) Y
(c) Z (d) $Z \wedge (Y \vee X)$

15. If X is a tautology and Y is any other formula, then $(X \vee Y)$ is a

(a) tautology (b) contradiction
(c) well-formed formula (d) none of these

16. Which one of the following is a functionally complete set?

(a) $\{\downarrow, \wedge\}$ (b) $\{\neg \wedge \vee\}$
(c) $\{\uparrow\}$ (d) $\{\rightarrow, \wedge\}$

17. What is the conclusion of the following premises?

$A \rightarrow B,\ B \rightarrow C,\ C \rightarrow (D \wedge E),\ A$

(a) E (b) D
(c) C (d) $A \rightarrow B$

18. For a given formula α, the truth table is given below. Find the principal disjunctive form.

X	Y	Z	α
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

(a) $(\neg X \wedge Y \wedge \neg Z) \vee (X \wedge \neg Y \wedge Z) \vee (\neg X \wedge \neg Y \wedge Z) \vee (\neg X \wedge \neg Y \wedge \neg Z)$
(b) $(\neg X \vee Y \vee \neg Z) \wedge (X \vee \neg Y \vee Z) \wedge (\neg X \vee \neg Y \vee Z) \wedge (\neg X \vee \neg Y \vee \neg Z)$
(c) $(X \wedge Y \wedge \neg Z) \vee (X \wedge \neg Y \wedge \neg Z) \vee (\neg X \wedge Y \wedge Z) \vee (\neg X \wedge \neg Y \vee \neg Z)$
(d) $(X \vee Y \vee \neg Z) \wedge (X \vee \neg Y \vee \neg Z) \wedge (\neg X \vee Y \vee Z) \wedge (\neg X \vee \neg Y \vee \neg Z)$

19. $(X \wedge Y) \wedge \neg Y$ is a

(a) tautology (b) well-formed formula
(c) contradiction (d) none of these

20. Which one of the following is True?

$$\alpha = (X \vee \neg Y) \wedge (Y \rightarrow Z) \vee (Z \vee X)$$

(a) α is tautology
(b) α is a contradiction
(c) If X is True, Y is True and Z is False, then α is True
(d) If X is True, Y is False and Z is False, then α is True

ANSWERS

1. (a)	**5.** (a)	**9.** (b)	**13.** (c)	**17.** (b)
2. (c)	**6.** (a)	**10.** (c)	**14.** (d)	**18.** (c)
3. (b)	**7.** (d)	**11.** (d)	**15.** (a)	**19.** (c)
4. (d)	**8.** (b)	**12.** (a)	**16.** (c)	**20.** (d)

EXPLANATIONS AND HINTS

1. (a) We know that $X \uparrow X = \neg(X \wedge X) = \neg X$

Also $X \wedge Y = (X \uparrow Y) \uparrow (X \uparrow Y)$

and $X \vee X = (X \uparrow X) \uparrow (Y \uparrow Y)$

Hence, $(X \rightarrow X) \vee X = \neg X \vee Y \vee Z = (\neg X \vee Y) \vee Z$

$= ((\neg X \vee Y) \uparrow (\neg X \vee Y)) \uparrow (Z \uparrow Z)$
$= ((\neg X \uparrow \neg X) \uparrow (Y \uparrow Y)) \uparrow ((\neg X \uparrow \neg X) \uparrow (Y \uparrow Y)) \uparrow (Z \uparrow Z)$
$= ((X \uparrow (Y \uparrow Y)) \uparrow (X \uparrow (Y \uparrow Y)) \uparrow (Z \uparrow Z)$

2. (c) We know that $X \downarrow X = \neg X$

Also $X \vee Y = (X \downarrow Y) \downarrow (X \downarrow Y)$

$X \wedge Y = (X \downarrow X) \downarrow (Y \downarrow Y)$

Hence, $(X \rightarrow \neg Y) = \neg X \vee \neg Y$

$= (\neg X \downarrow \neg Y) \downarrow (\neg X \downarrow \neg Y)$

3. (b) The structure of tautology will not change after substitution of a formula. Hence, answer is tautology.

4. (d) We know that

$$X \uparrow Y = \rceil(X \wedge Y)$$
$$= (X \wedge Y) \downarrow (X \wedge Y)$$
$$= ((X \downarrow X) \downarrow (Y \downarrow Y)) \downarrow ((X \downarrow X) \downarrow (Y \downarrow Y))$$

5. (a) We know that

$$X \wedge Y = (X \uparrow Y) \uparrow (X \uparrow Y)$$
$$X \wedge Y = (X \uparrow X) \uparrow (Y \uparrow Y)$$
$$\rceil X = X \uparrow X$$

Hence, $\{\uparrow\}$ is functionally complete since any formula consisting of $\wedge$, $\vee$ and $\rceil$ can be formed consisting of $\uparrow$ alone.

6. (a) We have $\rceil(X \rightarrow Y)$

$$(X \rightarrow Y) = \rceil X \vee Y$$
$$\rceil(X \rightarrow Y) = \rceil(\rceil X \vee Y)$$

Applying De Morgan's law, we get

$$X \wedge \rceil Y$$

7. (d) "Some cats are white" can be denoted by

$$\exists x\ (B(x) \rightarrow A(x))$$

"All Swans are white" can be denoted by

$$\forall x\ (C(x) \rightarrow A(x))$$

Hence, joining the two notations, we get

$$\exists x\ (B(x) \rightarrow A(x)) \wedge \forall x\ (C(x) \rightarrow A(x))$$

8. (b) Negation of the statement $\forall x$, $\exists y$, $P(x,\ Y)$ is given by

$$\exists x\ \forall y,\ \rceil P(x,\ Y)$$

9. (b) We have, $x \rightarrow (y \rightarrow z)$ (1)

Now, we know that

$$x \rightarrow y = \rceil x \vee y$$

Applying the identity in Eq. (i), we get

$$x \rightarrow (\rceil y \vee z)$$
$$= (x \rightarrow \rceil y) \vee (x \rightarrow z)$$

Applying the identity again, we get

$$\rceil x \vee \rceil y \vee \rceil x \vee z$$

Now, using idempotent laws, we know that

$$\rceil x \vee \rceil x = \rceil x$$

Hence, $\rceil x \vee \rceil y \vee z$

Applying De Morgan's law,

$$\rceil(x \wedge y) \vee z$$
$$= (x \wedge y) \rightarrow z$$

10. (c) $\rceil\forall P(x)$ is equivalent to $\exists x \rceil P(x)$.

11. (d) Consider $(X \vee Y) \leftrightarrow Y$

By assigning truth values, we get

X	Y	$X \vee Y$	$(X \vee Y) \leftrightarrow Y$
T	T	T	T
T	F	T	F
F	T	T	T
F	F	F	T

Hence, it is not a tautology.

Consider $(X \vee (X \vee Y)) \rightarrow X$

Simplifying the above equation gives

$$((X \vee X)(X \vee Y)) \rightarrow X$$
$$= (X \rightarrow (X \vee Y)) \rightarrow X$$
$$= (\rceil X \vee (X \vee Y)) \rightarrow X$$
$$= (T \vee Y) \rightarrow X$$
$$= T \vee X = X$$

By assigning truth values, we get

X	$T \uparrow X$
T	T
F	F

Hence, it is not a tautology.

Consider $((X \vee Y) \vee X) \rightarrow Y$

X	Y	$(X \vee Y)$	$(X \vee Y) \wedge X$	$((X \vee Y) \vee X) \uparrow Y$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	F	T

Hence, it is not a tautology.

Consider $((X \vee Y) \wedge \rceil X) \rightarrow Y$

Simplifying the above equation gives

$$(X \wedge \rceil X \vee (Y \vee \rceil X)) \rightarrow Y$$
$$= (Y \wedge \rceil X) \rightarrow Y$$
$$= \rceil(Y \wedge \rceil X) \vee Y$$
$$= \rceil Y \vee X \vee Y \quad \text{(By applying De Morgan's law)}$$

$= T \vee X \qquad (Y \vee \rceil Y = T)$

$= T \qquad (T \vee X = T)$

Hence, it is a tautology.

12. (a) Consider

$$X \rightarrow (X \vee Y)$$

By assigning truth values, we get

X	Y	$X \vee Y$	$X \uparrow (X \vee Y)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Hence, it is a tautology.

Note: Similarly, the rest of the three options can be proved to be not a tautology by assigning truth values.

13. (c) ‘He says “It is beautiful.”’ is a declarative statement.

14. (d) We have

$$(X \vee Y) \wedge (X \rightarrow Z) \wedge (Y \rightarrow Z) \qquad (1)$$

$$X \rightarrow Z = \rceil X \vee Z \qquad (2)$$

and $$Y \rightarrow Z = \rceil Y \vee Z \qquad (3)$$

Substituting Eqs. (2) and (3) in Eq. (1) gives

$$(X \vee Y) \wedge (\rceil X \vee Z) \wedge (\rceil Y \vee Z)$$

$$= ((X \wedge Z) \wedge (\rceil X \wedge Y) \vee (Y \wedge Z))(\rceil Y \vee Z)$$

$$= (X \wedge Z \wedge \rceil Y) \vee (X \wedge Z)$$

$$\vee (\rceil X \wedge Y \wedge Z) \vee (Y \wedge Z)$$

$$= Z \wedge ((X \wedge \rceil Y) \vee X \vee (\rceil X \wedge Y) \vee (Y))$$

$$= Z \wedge (Y \vee X)$$

15. (a) As X is a tautology, means X is true for all possible inputs. So, $X \vee Y$ is also true for values since disjunction of any variable with a true value is always true.

16. (c) Because all formulas using $\wedge$, $\vee$ and $\rceil$ can be formed using $\uparrow$, $\{\uparrow\}$ is a functionally complete set.

17. (b) We have

$$A \rightarrow B \qquad (1)$$

$$B \rightarrow C \qquad (2)$$

$$C \rightarrow (D \wedge E) \qquad (3)$$

$$A \qquad (4)$$

Applying modus ponens in Eqs. (4) and (1) gives

$$B \qquad (5)$$

Applying modus ponens in Eqs. (5) and (2) gives

$$C \qquad (6)$$

Applying modus ponens in Eqs. (6) and (3) gives

$$D \wedge E \qquad (7)$$

Simplification of Eq. (7) gives

$$D$$

18. (c) We have to write the formula for values when α is True.

The min terms corresponding to rows when α is True are $X \wedge Y \wedge Z$, $X \wedge \rceil Y \wedge \rceil Z$, $\rceil X \wedge Y \wedge Z$ and $\rceil X \wedge \rceil Y \wedge \rceil Z$.

Principal disjunctive normal form of α is

$$(X \wedge Y \wedge Z) \vee (X \wedge \rceil Y \wedge \rceil Z)$$
$$\vee (\rceil X \wedge Y \wedge Z) \vee (\rceil X \wedge \rceil Y \wedge \rceil Z)$$

19. (c) We have

$$(X \wedge Y) \wedge \rceil Y$$

By assigning truth values and forming truth table, we get

X	Y	$\rceil Y$	$X \wedge Y$	$(X \wedge Y) \wedge \rceil Y$
T	T	F	T	F
T	F	T	F	F
F	T	F	F	F
F	F	T	F	F

Hence, the formula is a contradiction because value is False for all possible values of X and Y.

20. (d) α is not a tautology because for $X =$ F, $Y =$ T and $Z =$ F, $\alpha =$ False.

α is not a tautology because for $X =$ F, $Y =$ T and $Z =$ T, $\alpha =$ True.

Now, for $X =$ True, $Y =$ False and $Z =$ False, $\alpha =$ True.

Hence, the correct option is “If X is True, Y is False and Z is False, then α is True.”

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. Let $(S, \leq)$ be a partial order with two minimal elements a and b, and a maximum element c.

Let $P : S \to [\text{True, False}]$ be a predicate defined on S.

Suppose that $P(a) = \text{True}, P(b) = \text{False}$ and $P(x) \Rightarrow P(y)$ for all $x, y \in S$ satisfying $x \leq y$,

where $\Rightarrow$ stands for logical implication.

Which one of the following statements CANNOT be true?

(a) $P(x)$ = True for all $x \in S$ such that $x \neq b$
(b) $P(x)$ = False for all $x \in S$ such that $x \neq a$ and $x \neq c$
(c) $P(x)$ = False for all $x \in S$ such that $b \leq x$ and $x \neq c$
(d) $P(x)$ = False for all $x \in S$ such that $a \leq x$ and $a \leq x$

(GATE 2003, 2 Marks)

Solution: $(S, \leq)$ is partial order in which S is reflexive, symmetric and transitive.
We have,

$$P(a) = \text{True}, P(b) = \text{False}$$

$$P(x) \Rightarrow P(y)$$

If $P(x)$, then it is given that $P(Y)$ and $a, y \in S$ satisfies $x < y$

Therefore, $-P(x) \cup P(y)$

$$\Rightarrow \text{False} \cup \text{False} \Rightarrow \text{False}$$

Hence, $P(x) = \text{False}$ for all $x \in S$, such that $b \leq x$ and $x \neq c$.

Ans. (c)

2. Consider the following formula α and its two interpretations I_1 and I_2

$\alpha : (\forall x)[Px \Leftrightarrow (\forall y)[Qxy \Leftrightarrow \neg Qyy]] \Rightarrow (\forall x)[\neg P]$

I_1: Domain: the set of natural numbers

P_x = 'x is a prime number'

Qxy = 'y divides x'

I_2: Same as I_1 except that Px = 'x is a composite number'.

Which one of the following statements is true?

(a) I_1 satisfies α, I_2 does not
(b) I_2 satisfies α, I_1 does not
(c) Neither I_1 nor I_2 satisfies α
(d) Both I_1 and I_2 satisfy α

(GATE 2003, 2 Marks)

Solution: We have

$$\alpha = (\forall x)[Px \Leftrightarrow (\forall y)[Qxy \Leftrightarrow \neg Qyy]]$$
$$\Rightarrow (\forall x)[\neg P]$$

I_1: Domain: the set of natural numbers

P_x = 'x is a prime number'
= 2, 3, 5, 7, 11, 13, …

Qxy = 'y divides x'
= 2, 3, 5, 7, 11, 13, …

I_2: Px = 'x is a composite number means number is not a prime number'.

= 4, 6, 8, 9, 10, 12, 14, …

Qxy = 2, 3, 5, 7, 11, 13, …

Thus, I_1 satisfies α, I_2 does not.

Ans. (a)

3. Consider the following logic program P

$$A(x) \leftarrow B(x, y), C(y)$$
$$\leftarrow B(x, x)$$

Which one of the following first-order sentences is equivalent to P?

(a) $(\forall x)[(\exists y)B(x,y) \wedge C(y) \Rightarrow A(x)] \wedge \neg(\exists x)[B(x,x)]$

(b) $(\forall x)[(\forall y)[B(x,y) \wedge C(y)] \Rightarrow A(x) \wedge \neg(\exists x)[B(x,x)]$

(c) $(\forall x)[(\exists y)[B(x,y) \wedge C(y)] \Rightarrow A(x)] \wedge \neg(\exists x)[B(x,x)]$

(d) $(\forall x)[(\forall y)[B(x,y) \wedge C(y)] \Rightarrow A(x)] \wedge \neg(\exists x)[B(x,x)]$

(GATE 2003, 2 Marks)

Solution: We have

$$A(x) \leftarrow B(x,y), C(y)$$
$$\leftarrow B(x,x)$$

Now, we can write

$$B(x,x) \to B(x,y), C(y) \to A(x)$$

$B(x,\ x) \to \rceil\,[B\ (x,\ y),\ C\ (y)]\ \cup A(x)$

$B(x,\ x)\ \cup\ \rceil\,[B\ (x,\ y),\ C\ (y)]\ \cup A(x)$

Also, we can write

$\forall\ (\rceil B(x,\ x)\ \cup\ \rceil(B\ (x,\ y)),\ C\ (y))\ \cup A(x)$

Rewriting option (c),

$$(\forall x)\rceil[(\exists y)[B(x,y) \wedge C(y)]]\cup\ A(x)\cup\ \rceil(\exists(x)[B(x,x)])$$

$$\Rightarrow (\forall x)\ [(\forall y)\ \rceil[B(x,y) \wedge C(y)] \cup A(x) \cup\ (\forall x)\ [\rceil B(x,\ x)]$$

$$\Rightarrow (\forall xy)\ \ [\rceil B(x,\ Y) \cup \rceil C(y)\ \cup A(x)\ \rceil B(x,\ x)]$$

$$\Rightarrow \forall\ [\rceil(B(x,\ y),\ C(y))\ \cup A(x)\ \rceil B(x,\ x)]$$

Thus, the correct answer is (c).

Ans. (c)

4. The following resolution rule is used in logic programming:

Derive clause $(P \vee Q)$ from clauses $(P \vee R)$, $(Q \vee \neg R)$

Which one of the following statements related to this rule is FALSE?

(a) $((P\vee\ R) \wedge (Q\vee \neg R)) \Rightarrow (P\vee\ Q)$ is logically valid

(b) $(P\vee\ Q) \Rightarrow ((P\vee\ R) \wedge (Q\vee \neg R))$ is logically valid

(c) $(P \vee Q)$ is satisfiable if and only if $(P \vee R) \wedge (Q \vee \neg R)$ is satisfiable

(d) $(P\vee\ Q) \Rightarrow$ FALSE if and only if both P and Q are unsatisfiable

(GATE 2003, 2 Marks)

Solution: If $P =$ True, $Q =$ False, and $R =$ True, then

$$(T\cup\ F) \Rightarrow ((T \cup\ T) \wedge (F\cup\ F))$$
$$T \Rightarrow (T \wedge F)$$
$$T \Rightarrow F$$
$$\therefore\ F \cup F = \text{False (is not valid)}$$

Hence, "$(P\vee\ Q) \Rightarrow ((P\vee\ R) \wedge (Q\vee \neg R))$ is logically valid" is the incorrect statement.

Ans. (b)

5. Identify the correct translation into logical notation of the following assertion.

Some boys in the class are taller than all the girls

Note: taller $(x,\ y)$ is true if x is taller than y.

(a) $(\exists x)(\text{boy}(x) \to (\forall y)(\text{girl}(y) \wedge \text{taller}(x,y)))$

(b) $(\exists x)(\text{boy}(x) \wedge (\forall y)(\text{girl}(y) \wedge \text{taller}(x,y)))$

(c) $(\exists x)(\text{boy}(x) \to (\forall y)(\text{girl}(y) \to \text{taller}(x,y)))$

(d) $(\exists x)(\text{boy}(x) \to (\forall y)(\text{girl}(y) \wedge \text{taller}(x,y)))$

(GATE 2004, 1 Mark)

Solution:

$(\exists x)(\text{boy}(x) \to (\forall y)(\text{girl}(y) \to \text{taller}(x,y)))$

Ans. (c)

6. The following propositional statement is

$$(P \to (Q\vee\ R)) \to ((P \wedge Q) \to R)$$

(a) satisfiable but not valid
(b) valid
(c) contradiction
(d) none of the above

(GATE 2004, 2 Marks)

Solution: Let us form the truth table for the given statement.

P	Q	R	$(P \to (Q\vee\ R)) \uparrow ((P \wedge Q) \to R)$
F	F	F	T
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	F
T	T	T	T

Hence, we can see from the truth table that the statement is true for some values and false for some. Hence, it is satisfiable but not valid.

Ans. (a)

7. Let P, Q and R be three atomic propositional assertions. Let X denotes $(P \vee Q) \to R$ and Y denotes $(P \to R) \vee (Q \to R)$.

Which one of the following is a tautology?

(a) $X \equiv Y$
(b) $X \to Y$
(c) $Y \to X$
(d) $\rceil Y \to X$

(GATE 2005, 2 Marks)

Solution: Let us form the truth table for the statement.

P	Q	R	X	Y	$Y \uparrow X$	$X \uparrow Y$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	F	T	F	T
F	T	T	T	T	T	T
T	F	F	F	T	F	T
T	F	T	T	T	T	T
T	T	F	F	F	T	T
T	T	T	T	T	T	T

Hence, from the truth table, we can see that $X \rightarrow Y$ is a tautology.

Ans. (b)

8. What is the first-order predicate calculus statement equivalent to the following?

Every teacher is liked by some student

(a) $\forall(x)[\text{teacher}(x) \rightarrow \exists(y)[\text{student}(y) \rightarrow \text{likes}(y,x)]]$

(b) $\forall(x)[\text{teacher}(x) \rightarrow \exists(y)[\text{student}(y) \wedge \text{likes}(y,x)]]$

(c) $\exists(y)\forall(x)[\text{teacher}(x) \rightarrow [\text{student}(y) \wedge \text{likes}(y,x)]]$

(d) $\forall(x)[\text{teacher}(x) \wedge \exists(y)[\text{student}(y) \rightarrow \text{likes}(y,x)]]$

(GATE 2005, 2 Marks)

Solution: For the statement "Every teacher is liked by some student", the logical expression is given by

$$\forall(x)\left[\text{teacher }(x) \rightarrow \exists(y)\left[\text{student}(y) \wedge \text{likes}(y,x)\right]\right]$$

where likes (y, x) means y likes x, such that y represents the student and x represents the teacher.

Ans. (b)

9. Which one of the first-order predicate calculus statements given below correctly expresses the following English statement?

Tigers and lions attack if they are hungry or threatened.

(a) $\forall(x)[(\text{tiger}(x) \wedge \text{lion}(x)) \rightarrow ((\text{hungry}(x) \vee \text{threatened}(x)) \rightarrow \text{attacks}(x))]$

(b) $\forall(x)[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow ((\text{hungry}(x) \vee \text{threatened}(x)) \wedge \text{attacks}(x))]$

(c) $\forall(x)[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow (\text{attacks}(x) \rightarrow (\text{hungry}(x) \vee \text{threatened}(x)))]$

(d) $\forall(x)[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow ((\text{hungry}(x) \vee \text{threatened}(x)) \rightarrow \text{attacks}(x))]$

(GATE 2006, 2 Marks)

Solution: The given statement can be interpreted as follows:
Consider x to be a lion or a tiger. If x is hungry or is threatened, it will attack.

$$P(x) = \text{lion}(x) \vee \text{tiger}(x)$$
$$Q(x) = \text{hungry}(x) \vee \text{threatened}(x)$$
$$R(x) = \text{attacks}(x)$$
$$P(x) \rightarrow (Q(x) \rightarrow R(x))$$

Expanding, we get

$$\forall(x)[(\text{tiger}(x) \vee \text{lion}(x)) \rightarrow ((\text{hungry}(x) \vee \text{threatened}(x)) \rightarrow \text{attacks}(x))]$$

Ans. (d)

10. Consider the following propositional statements:

$$P_1 : (((A \wedge B) \rightarrow C)) \equiv ((A \rightarrow C) \wedge (B \rightarrow C))$$

$$P_2 : (((A \vee B) \rightarrow C)) \equiv ((A \rightarrow C) \vee (B \rightarrow C))$$

Which one of the following is true?

(a) P_1 is a tautology, but not P_2
(b) P_2 is a tautology, but not P_1
(c) P_1 and P_2 are both tautologies
(d) Both P_1 and P_2 are not tautologies

(GATE 2006, 2 Marks)

Solution: Let us make the truth table for the given statements.

A	B	C	$A \vee B$	$A \wedge B$	$(A \wedge B)\uparrow C$	$(A \vee B)\uparrow C$	$((A\uparrow C)\wedge(B\uparrow C))$	$((A\uparrow C)\vee(B\uparrow C))$
F	F	F	F	F	T	T	T	T
F	F	T	F	F	T	T	T	T
F	T	F	T	F	F	T	F	T
F	T	T	T	F	T	T	T	T
T	F	F	T	F	F	T	F	T
T	F	T	T	F	T	T	T	T
T	T	F	T	T	F	F	F	F
T	T	T	T	T	T	T	T	T

Hence, from the truth table, we can say that both P_1 and P_2 are not tautologies.

Ans. (d)

11. Let *Graph*(x) be a predicate which denotes that x is a graph. Let *Connected*(x) be a predicate which denotes that x is connected. Which of the following first-order logic sentences DOES NOT represent the statement: "Not every graph is connected"?

(a) $\neg\forall x(\text{Graph}(x) \Rightarrow \text{Connected}(x))$

(b) $\exists x(\text{Graph}(x) \wedge \neg\,\text{Connected}(x))$

(c) $\neg\forall x(\neg\,\text{Graph}(x) \vee \text{Connected}(x))$

(d) $\forall x(\text{Graph}(x) \Rightarrow \neg\,\text{Connected}(x))$

(GATE 2007, 2 Marks)

Solution: $\forall x(\text{Graph }(x) \Rightarrow \neg\text{Connected }(x))$ represents "for every x if x is a graph then it is not connected".

Ans. (d)

12. P and Q are two propositions. Which of the following logical expressions are equivalent?

I. $P \vee \sim Q$

II. $\sim(\sim P \wedge Q)$

III. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

IV. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$

(a) Only I and II
(b) Only I, II and III
(c) Only I, II and IV
(d) All of I, II, III and IV

(GATE 2008, 2 Marks)

Solution: Let us construct the truth table.

P	Q	I	II	III	IV
F	F	T	T	T	T
F	T	F	F	F	F
T	F	T	T	T	T
T	T	T	T	T	T

Hence, all the four expressions are equivalent.

Ans. (d)

13. Which one of the following is the most appropriate logical formula to represent the statement:

"Gold and silver ornaments are precious"

The following notations are used:

$G(x)$: x is a gold ornament.

$S(x)$: x is a silver ornament.

$P(x)$: x is precious.

(a) $\forall x(P(x) \rightarrow (G(x) \wedge S(x)))$

(b) $\forall x((G(x) \wedge S(x)) \rightarrow P(x))$

(c) $\exists x((G(x) \wedge S(x)) \rightarrow P(x))$

(d) $\forall x((G(x) \vee S(x)) \rightarrow P(x))$

(GATE 2009, 2 Marks)

Solution: We are given that

"Gold and silver ornaments are precious"

Hence, we use

$$\forall x((G(x) \vee S(x)) \rightarrow P(x))$$

Ans. (d)

14. Suppose the predicate $F(x, y, t)$ is used to represent the statement that person x can fool person y at time t. Which one of the statements below expresses best the meaning of the formula $\forall x\,\exists y\,\exists t(\neg F(x,y,t))$?

(a) Everyone can fool some person at some time
(b) No one can fool everyone all the time
(c) Everyone cannot fool some person all the time
(d) No one can fool some person at some time

(GATE 2010, 2 Marks)

Solution: We are given that $F(x, y, t)$ is a predicate.

$$\forall x\ \exists y\ \exists t(\neg F(x,y,t))$$
$$\Rightarrow \forall x \neg(\forall y \forall t F(x,y,t))$$

Hence, we can conclude that "No one can fool everyone all the time."

Ans. (d)

15. Which one of the following options is CORRECT given three positive integers x, y and z and a predicate

$$P(x) = \neg(x=1) \wedge \forall y(\exists z(x = y * z)$$
$$\Rightarrow (y = x) \vee (y = 1)$$

(a) $P(x)$ being true means that x is a prime number
(b) $P(x)$ being true means that x is a number other than 1
(c) $P(x)$ is always true irrespective of the value of x
(d) $P(x)$ being true means that x has exactly two factors other than 1 and x

(GATE 2011, 2 Marks)

Solution: Consider the following table of precedence of logical operators:

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

The predicate is evaluated as

$$P(x) = \neg(x=1) \wedge \forall y\ (\exists z(x = y * z)$$
$$\Rightarrow (y = x) \vee (y = 1))$$

$P(x)$ being true means $x \neq 1$ and for all y if there exists a z such that $x = y * z$ then y must be x (i.e. $z = 1$) or y must be 1 (i.e. $z = x$).

Hence, only x has two factors, 1 and x itself.

Thus, this predicate defines the prime number.

Ans. (d)

16. What is the correct translation of the following statement into mathematical logic?

"Some real numbers are rational"

(a) $\exists x\left(\text{real}(x) \vee \text{rational}(x)\right)$

(b) $\forall x\left(\text{real}(x) \rightarrow \text{rational}(x)\right)$

(c) $\exists x\left(\text{real}(x) \wedge \text{rational}(x)\right)$

(d) $\exists x\left(\text{rational}(x) \rightarrow \text{real}(x)\right)$

(GATE 2012, 1 Mark)

Solution: The translation of each option is as follows:

Option (a): There exists an x which is either real or rational and can be both.

Option (b): All real numbers are rational.

Option (c): There exists a real number which is rational.

Option (d): There exists some number which is not rational or which is real.

Ans. (c)

17. Consider the following logical inferences:

I_1: If it rains, then the cricket match will not be played.

The cricket match was played.

Inference: There was no rain.

I_2: If it rains, then the cricket match will not be played.

It did not rain.

Inference: The cricket match was played.

Which of the following is TRUE?

(a) Both I_1 and I_2 are correct inferences.
(b) I_1 is correct but I_2 is not a correct inference.
(c) I_1 is not correct but I_2 is a correct inference.
(d) Both I_1 and I_2 are not correct inferences.

(GATE 2012, 1 Mark)

Solution: For I_1, since we know that the cricket match was played, we can infer that there was no rain. Hence, inference 1 is correct.

For I_2, since we know that it did not rain. However, this only confirms that the match can be played and tells nothing about whether the match was played or not. Hence, inference 2 is incorrect.

Therefore, I_1 is correct and I_2 is incorrect inference.

Ans. (b)

18. Which one of the following is NOT logically equivalent to $\neg\exists x(\forall y(\alpha) \wedge \forall z(\beta))$?

(a) $\forall x(\exists x(\neg\beta) \rightarrow \forall y(\alpha))$

(b) $\forall x(\forall z(\beta) \rightarrow \exists y(\neg\alpha))$

(c) $\forall x(\forall y(\alpha) \rightarrow \exists z(\neg\beta))$

(d) $\forall x(\exists y(\neg\alpha) \rightarrow \exists z(\neg\beta))$

(GATE 2013, 2 Marks)

Solution: We have $\neg\exists x\left(\forall y(\alpha)\wedge\forall z(\beta)\right)$,

Using the identity $\neg(p\wedge q)\equiv p\Rightarrow\neg q$, we get

$$\neg\exists x\left(\forall y(\alpha)\wedge\forall z(\beta)\right)\equiv\forall\times\left[\forall y(\alpha)\to\exists z(\neg\beta)\right]$$

Using the identity $p\Rightarrow q\equiv\neg q\Rightarrow\neg p$, we get

$$\neg\exists x\left(\forall y(\alpha)\wedge\forall z(\beta)\right)\equiv\forall\times\left[\forall z(\beta)\to\exists y(\neg\alpha)\right]$$

Hence, the correct options are (a) and (d).

Ans. (a) and (d)

19. Consider the statement: "Not all that glitters is gold". Predicate glitters(x) is true if x glitters and predicate gold(x) is true if x is gold. Which one of the following logical formulae represents the above statement?

(a) $\forall x$; glitters(x) $\Rightarrow\neg$ gold(x)
(b) $\forall x$; gold(x) $\Rightarrow$ glitters(x)
(c) $\exists x$; gold(x) $\wedge\neg$ glitters(x)
(d) $\exists x$; glitters(x) $\wedge\neg$ gold(x)

(GATE 2014, 1 Mark)

Solution: 'Not all that glitters is gold' means there exist some glitters that are not gold. So $\exists x$ is used along with AND operator.
$\exists$x; glitters(x)$\wedge\neg$gold(x)

Ans. (d)

20. Consider the following statements:

P: Good mobile phones are not cheap
Q: Cheap mobile phones are not good
L: P implies Q
M: Q implies P
N: P is equivalent to Q

Which one of the following about L, M, and N is CORRECT?

(a) Only L is TRUE.
(b) Only M is TRUE.
(c) Only N is TRUE.
(d) L, M and N are TRUE.

(GATE 2014, 1 Mark)

Solution: P: Good mobile phones are not cheap
Q: Cheap mobile phones are not good.
P $\to$ Q and Q $\to$ P
Hence, P $\leftrightarrow$ Q. So, L, M and N are true.

Ans. (d)

21. The CORRECT formula for the sentence, 'not all rainy days are cold' is

(a) $\forall d$ (Rainy(d) $\wedge\sim$ Cold(d))
(b) $\forall d$ (~Rainy(d) $\to$ Cold(d))
(c) $\exists d$ (~Rainy(d) $\to$ Cold(d))
(d) $\exists d$ (Rainy(d) $\wedge\sim$ Cold(d))

(GATE 2014, 2 Marks)

Solution: 'Not all rainy days are cold'

$$\sim\forall d\left(\text{rainy}(d)\to\text{cold}(d)\right)\to\exists d\left(\text{rainy}(d)\wedge\sim\text{cold}(d)\right)$$

Ans. (d)

22. Which one of the following is NOT equivalent to $p\leftrightarrow q$?

(a) $(\rceil p\vee q)\wedge(p\vee\rceil q)$

(b) $(\rceil p\vee q)\wedge(q\to p)$

(c) $(\rceil p\wedge q)\vee(p\wedge\rceil q)$

(d) $(\rceil p\wedge\rceil q)\vee(p\wedge q)$

(GATE 2015, 1 Mark)

Solution:

$$p\leftrightarrow q\equiv(p\to q)\wedge(q\to p)$$
$$\equiv\left(\rceil p\vee q\right)\wedge\left(\rceil p\vee q\right)\quad(\because p\to q\equiv p\vee q)$$
$$\equiv\left(\rceil p\wedge\rceil q\right)\vee\left(q\wedge\rceil q\right)\vee\left(\rceil p\wedge p\right)\vee(q\wedge p)$$
(using distributive laws)
$$\equiv\left(\rceil p\wedge\rceil q\right)\vee(q\wedge p)$$
(using complement and commutative laws)

Thus, $p\leftrightarrow q$ is not equivalent to $(\rceil p\wedge q)\vee(p\wedge\rceil q)$.

Ans. (c)

23. Consider the following two statements:

(i) If a candidate is known to be corrupt, then he will not be elected.
(ii) If a candidate is kind, he will be elected.
Which one of the following statements follows from (i) and (ii) as per sound inference rules of logic?

(a) If a person is known to be corrupt, he is kind.
(b) If a person is not known to be corrupt, he is not kind.
(c) If a person is kind, he is not known to be corrupt.
(d) If a person is not kind, he is not known to be corrupt.

(GATE 2015, 1 Mark)

Solution: Let
p: candidate known to be corrupt
q: candidate will be elected
r: candidate is kind

Then,

$$s_1 = p \to\sim q$$
$$= q \to\sim p \ \text{(contrapositive rule)}$$

And $s_2 : r \to q \Rightarrow r \to\sim p$ (transitive rule)

That is if a person is kind, he is not known to be corrupt.

Therefore, option (c) is correct.

Ans. (c)

24. Which one of the following well-formed formulae is a tautology?

(a) $\forall x\, \exists y\ R(x,y) \leftrightarrow \forall x R(x,y)$

(b) $\forall x\,[\exists y\ R(x,y) \to S(x,y)] \to \forall x \exists y S(x,y)$

(c) $[\forall x\, \exists y\ \{P(x,y) \to R(x,y)\}] \leftrightarrow$
$[\forall x \exists y\ \{\neg P(x,y) \vee R(x,y)\}]$

(d) $\forall x \forall y P(x,y) \to \forall x \forall y P(x,y)$

(GATE 2015, 1 Mark)

Solution: Since $P \to R = \neg P \vee R$

$$[\forall x \exists y \{P(x,y) \to R(x,y)\}] \leftrightarrow [\forall x \exists y \{\neg P(x,y) \vee R(x,y)\}]$$

Ans. (c)

25. Consider the following grammar G

S → F | H

F → p | c

H → d | c

where S, F and H are non-terminal symbols, and p, d and c are terminal symbols. Which of the following statement(s) is/are correct?

S1: LL(1) can parse all strings that are generated using grammar G
S2: LR(1) can parse all strings that are generated using grammar G

(a) only S1
(b) only S2
(c) Both S1 and S2
(d) Neither S1 nor S2

(GATE 2015, 2 Marks)

Solution: Neither S1 nor S2 is correct.

Ans. (d)

26. Let p, q, r, s represent the following propositions.

p: $x \in \{8, 9, 10, 11, 12\}$

q: x is a composite number

r: x is a perfect square

s: x is a prime number

The integer $x \geq 2$ which satisfies $\neg\,((P \Rightarrow q) \wedge (\neg\, r V \neg s))$ is __________.

(GATE 2016, 1 Mark)

Solution: We have

$$\neg((P \Rightarrow q) \wedge (\neg r V \neg s)) = \neg(\neg p v q)) \vee (\neg(\neg r V \neg S)$$
$$= (p \wedge \neg q) \vee (r \wedge s)$$

which can be read as follows:

(i) $x \in \{8, 9, 10, 11, 12\}$ AND (x is not a composite number) OR (x is a perfect square and x is a prime number)

Now, x is a perfect square and x is a prime number can never be true as every square has at least 3 factors 1, x and x^2. Thus, the second condition due to AND can never be true. This means the first condition has to be true.

Now, $x \in \{8, 9, 10, 11, 12\}$ AND (x is not a composite number).

However, here only 11 is not composite so the correct answer is $x = 11$.

Ans. 11

27. Consider the following expressions:

(i) False
(ii) Q
(iii) True
(iv) $P \vee Q$
(v) $\neg Q \vee P$

The number of expression given above that are logically implied by $P \wedge (P \Rightarrow Q)$ __________.

(GATE 2016, 1 Mark)

Solution: We know that $P \to Q$. Therefore,

$$\frac{P}{Q} \quad \text{(modus ponens)}$$

$$Q \vee P \text{ (by addition)}$$

$P \wedge (P \rightarrow Q)$ (by applying simplification and then addition)

Therefore, Q, $P \vee Q$, $\neg Q \vee P$ can be logically implied.

Ans. 3

28. Which one of the following well-formed formulae in predicate calculus is **NOT** valid?

(a) $(\forall x p(x) \Rightarrow \forall x q(x)) \Rightarrow (\exists x \neg p(x) \vee \forall x q(x))$

(b) $(\exists x p(x) \vee \exists x q(x)) \Rightarrow \exists x (p(x) \vee q(x))$

(c) $\exists x (p(x) \wedge q(x)) \Rightarrow (\exists x p(x) \wedge \exists x q(x))$

(d) $\forall x (p(x) \vee q(x)) \Rightarrow (\forall x p(x) \vee \forall x q(x))$

(GATE 2016, 2 Marks)

Solution:

(a) Since L.H.S. $\Leftrightarrow$ R.H.S., the well-formed formula is valid.

(b) Same as option (a).

(c)

i.	$\exists_x \{p(x) \wedge q(x)\}$	Premise
ii.	$p(a) \wedge q(a)$	(i) E.S.
iii.	$p(a)$	(ii) Simplification
iv.	$q(a)$	(ii) Simplification
v.	$\exists_x p(x)$	(iii) E.G.
vi.	$\exists_x q(x)$	(iv) E.G.
vii.	$\exists_x p(x) \wedge \exists_x q(x)$	(v), (vi) Conjunction

Hence, option (c) is proved to be valid.

(d) Let $p(x)$: x be a flower and $q(x)$: x be an animal. Now, let U={Tiger, Lion, Rose} be the universe of discourse. It is obvious that the antecedent of the implication is true but the consequent is false. Hence, option (d) is proved to be not valid.

Ans. (d)

29. The Boolean expression $\overline{(a + \overline{b} + c + \overline{d}) + (b + \overline{c})}$ simplifies to

(a) 1 (b) $\overline{a, b}$

(c) a, b (d) 0

(GATE 2016, 2 Marks)

Solution: The given function

$$F = \overline{(a + \overline{b} + c + \overline{d}) + (b + \overline{c})}$$

can be simplified using De Morgan's theorem as

$$F = \overline{(a + \overline{b} + c + \overline{d})} \cdot (\overline{b + \overline{c}})$$
$$= (\overline{a} \cdot b \cdot \overline{c} \cdot d) \cdot (\overline{b}c)$$
$$= 0$$

Ans. (d)

30. The statement $(\neg p) \Rightarrow (\neg q)$ is logically equivalent to which of the below statements?

I. $p \Rightarrow q$ II. $q \Rightarrow p$

III. $(\neg q) \vee p$ IV. $(\neg p) \vee q$

(a) I only (b) I and IV only

(c) II only (d) II and III only

(GATE 2017, 1 Mark)

Solution: From the given statement, we have

$$\neg p \Rightarrow \neg q \equiv \neg(\neg p) \vee \neg q \equiv p \vee \neg q$$

Now, considering the statements given in the options, we have

I. $p \Rightarrow q \equiv \neg p \vee q$, False

II. $q \Rightarrow p \equiv \neg q \vee p$, True

III. $(\neg q) \vee p \equiv \neg q \vee p$, True

IV. $(\neg p) \vee q \equiv \neg p \vee q$, False

Therefore, only II and III are correct.

Ans. (d)

31. Consider the first-order logic sentence $F : \forall x (\exists y R(x, y))$. Assuming non-empty logical domains, which of the below sentences are implied by F?

I. $\exists y (\exists x R(x, y))$

II. $\exists y (\forall x R(x, y))$

III. $\forall y (\exists x R(x, y))$

IV. $\neg \exists x (\forall y \neg R(x, y))$

(a) IV only (b) I and IV only

(c) II only (d) II and III only

(GATE 2017, 1 Mark)

Solution: Given $F : \forall x (\exists y R(x, y))$

Now, considering the sentences given in the options, we have

I. $\exists y(\exists x R(x,y))$

$\forall x(\exists y R(x,y)) \rightarrow \exists y(\exists x R(x,y))$ is true, since $\exists y(\exists x R(x,y)) \equiv \exists x(\exists y R(x,y))$.

II. $\exists y(\forall x R(x,y))$

$\forall x(\exists y R(x,y)) \rightarrow \exists y(\forall x R(x,y))$ is false, since $\exists y$ when it is outside is stronger when it is inside.

III. $\forall y(\exists x R(x,y))$

$\forall x(\exists y R(x,y)) \rightarrow \forall y(\exists x R(x,y))$ is false, since $R(x, y)$ may not be symmetric in x and y.

IV. $\neg\exists x(\forall y \neg R(x,y))$

$\forall x(\exists y R(x,y)) \rightarrow \neg\exists x(\forall y \neg R(x,y))$ is true, since $\neg(\exists x \forall y \neg R(x,y)) \equiv \forall x \exists y R(x,y)$

So, IV will reduce to $\forall x(\exists y R(x,y)) \rightarrow \forall x(\exists y R(x,y))$ which is trivially true.

Therefore, only I and IV are correct.

Ans. (b)

32. Let p, q and r denote the statements "*It is raining*," "*It is cold*," and "*It is pleasant*," respectively. Then the statement "*It is not raining and it is pleasant, and it is not pleasant only if it is raining and it is cold*" is represented by

(a) $(\neg p \wedge r) \wedge (\neg r \rightarrow (p \wedge q))$

(b) $(\neg p \wedge r) \wedge ((p \wedge q) \rightarrow \neg r)$

(c) $(\neg p \wedge r) \vee ((p \wedge q) \rightarrow \neg r)$

(d) $(\neg p \wedge r) \vee (r \rightarrow (p \wedge q))$

(GATE 2017, 1 Mark)

Solution: It is not raining and it is pleasant: $\neg p \wedge r$.

It is not pleasant only if it is raining and it is cold: $\neg r \rightarrow (p \wedge q)$.

It is not raining and it is pleasant, and it is not pleasant only if it is raining and it is cold = $(\neg p \wedge r) \wedge (\neg r \rightarrow (p \wedge q))$.

Ans. (a)

33. Let p, q and r be propositions and the expression $(p \rightarrow q) \rightarrow r$ be a contradiction. Then, the expression $(r \rightarrow p) \rightarrow q$ is

(a) a tautology

(b) a contradiction

(c) always TRUE when p is FALSE

(d) always TRUE when q is TRUE

(GATE 2017, 2 Marks)

Solution: Given that

$$(p \rightarrow q) \rightarrow r = \text{False}$$
$$\Rightarrow \sim(\sim p \vee q) \vee r = \text{False}$$
$$\Rightarrow (p \wedge \sim q) \vee (r) = \text{False}$$

This is possible when

$$p \wedge \sim q = \text{False}$$

and

$$r = \text{False}$$

We have to check

$$(r \rightarrow p) \rightarrow q$$
$$\Rightarrow \sim(\sim r \vee p) \vee q$$
$$\Rightarrow (r \wedge \sim p) \vee q$$

Given r is false

$$\Rightarrow r \wedge \sim p \rightarrow \text{False}$$
$$\Rightarrow (r \rightarrow p) \rightarrow q \equiv q$$

So, $(r \rightarrow p) \rightarrow q$ is true only if q is true.

Ans. (d)

CHAPTER 8

SET THEORY AND ALGEBRA

INTRODUCTION TO SET THEORY

A set is a collection of distinct objects, considered as a single entity of size equal to the number of distinct objects.

For example: A is a set of sports played by a student.

$A = \{$Cricket, Football, Tennis, Basketball, Squash$\}$.

A set can be represented in two ways:

1. **Set-builders form:** In this representation, we first write a variable (say x or y) and then state the properties of that variable.
   For example: $A = \{x : x \text{ is a prime number and } 1 < x < 100\}$.
2. **Tabular or roaster form:** In this representation, all the elements of the set are listed between the braces { } separated by commas.
   For example: $A = \{1, 2, 3, 5, 10, 20\}$

Subsets and Supersets

If we have two sets A and B, then A is said to be a subset of B if every element of A is an element of B. It is denoted by $A \subseteq B$.

For example: If $A = \{1, 2\}$ and $B = \{0, 1, 2, 3\}$, then

$$A \subseteq B$$

If every element of A is an element of B and B has at least one element which is not an element of A, then A is said to be a proper subset of B. It is denoted by $A \subset B$.

For example: If $A = \{1, 2, 3\}$ and $B = \{1, 3, 2, 4\}$, then $A \subset B$.

If B contains all the elements of A, then B is said to be the superset of A. It is denoted by $B \supseteq A$.

Equal Sets

Two sets A and B are said to be equal, if and only if every element of A is a member of B and every element of B is a member of A.

For example: If $A = \{0, 1, 2\}$ and $B = \{2, 0, 1\}$, then $A = B$.

Comparable Sets

Two sets A and B are comparable if any one of the following conditions is met:

1. $A \subset B$
2. $B \subset A$
3. $A = B$

Similarly, if neither of the conditions mentioned above is met, then A and B are said to be incomparable.

Universal Set

Any set that contains all the sets under consideration is called a universal set. It is denoted by S or U.

For example: If $A = \{0, 1, 2\}$, $B = \{1, 2, 3, 4\}$ and $C = \{5, 6, 9\}$, then $S = \{0, 1, 2, 3, 4, 5, 6, 9\}$.

Power Set

The collection or family of all subsets of A is called the power set of A and is denoted by $P(A)$. If A has n elements, then $P(A)$ has 2^n elements. Mathematically,

$$P(A) = \{x : x \subseteq A\}$$

Hence, $x \in P(A)$

$$\Rightarrow x \subseteq A$$

For example: If $A = \{0, 1\}$, then

$$P(A) = \{\phi, \{0\}, \{1\}, \{0,1\}\}$$

Types of Sets

The different types of sets are described as follows:

1. **Empty set (or null set):** A set having zero elements is called an empty or null set. It is denoted by ϕ.
   For example: $A = \{x : x$ is an odd number divisible by 2$\}$
2. **Singleton set:** A set having a single element is called a singleton set.
   For example: $A = \{2\}$
3. **Pair set:** A set having two elements is called a pair set.
   Example: $A = \{0, 1\}$
4. **Finite set:** A set having finite number of elements is called a finite set.
   For example: $A = \{0, 1, 2, 3, 5, 7\}$
5. **Infinite set:** A set having infinite number of elements is called an infinite set.
   For example: $A = \{x : x$ is a set of all the even numbers$\}$

OPERATIONS ON SETS

1. **Union of sets:** The union of two sets, A and B, is the set of all elements which are present either in A or B. It is denoted by $A \cup B$.
   For example: If $A = \{0, 1\}$ and $B = \{1, 2, 3\}$
   $$A \cup B = \{0, 1, 2, 3\}$$
2. **Intersection of sets:** The intersection of two sets, A and B, is the set of all elements which are present in A and B. It is denoted by $A \cap B$.
   For example: If $A = \{0, 1, 2\}$ and $B = \{1, 2, 5\}$
   $$A \cap B = \{1, 2\}$$
3. **Difference of sets:** The difference of two sets, A and B, is the set of all elements of A which are not in B.
   $$A - B = \{x : x \in A \text{ and } x \notin B\}$$
   Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$.
   For example: If $A = \{1, 2, 5, 6, 7, 9\}$ and $B = \{2, 3, 4, 5\}$
   $$A - B = \{1, 6, 7, 9\}$$
   $$B - A = \{3, 4\}$$
4. **Disjoint sets:** Two sets A and B which have no common element are called disjoint sets. For disjoint sets, $A \cap B = \phi$.
   For example: $A = \{0, 1, 2\}$ and $B = \{5, 6\}$ are disjoint sets.
5. **Symmetric difference of sets:** It is defined as a set of elements that are either a member of A or B but not both. It is denoted by $A \oplus B$. Mathematically,
   $$A \oplus B = (A \cup B) - (A \cap B)$$
   $$= (A - B) \cup (B - A)$$
6. **Complement of set:** A set having all the elements of the universal sets which are not elements of A is called complement of A. It is denoted by A^c or A^{-1} or $A°$. Symbolically,
   $$A^{-1} = U - A$$
   $$= \{x : x \in U \text{ and } x \in A\}$$

IMPORTANT LAWS AND THEOREMS

1. Idempotent laws
   $$A \cup A = A,\ A \cap A = A$$
2. Identity laws
   $$A \cup \phi = A,\ A \cap \phi = \phi$$

3. Commutative laws

$$A \cup B = B \cup A,\ A \cap B = B \cap A$$

4. Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C),\ A \cap (B \cap C) = (A \cap B) \cap C$$

5. Distribution laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

6. De Morgan's laws

$$(A \cup B)' = A' \cap B',\ (A \cap B)' = A' \cup B'$$

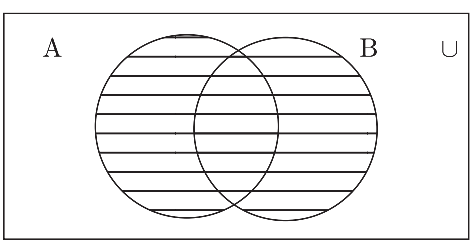
VENN DIAGRAMS

The diagrams drawn to represent sets are called Venn–Euler diagrams or simply Venn diagrams. In Venn diagrams, the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle. If a set A is a subset of B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then to represent A and B we draw two intersection circles. Two disjoints sets are represented by two non-intersecting circles.

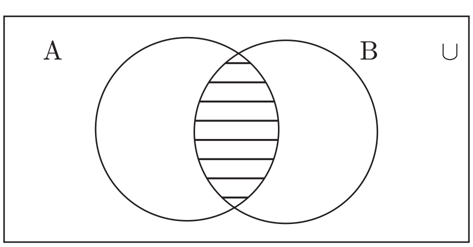
Operations on Sets Using Venn Diagrams

This section discusses the most commonly used operations performed on sets using Venn diagrams.

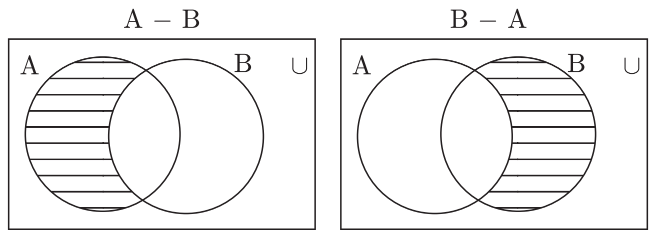
1. Union of sets: $A \cup B$



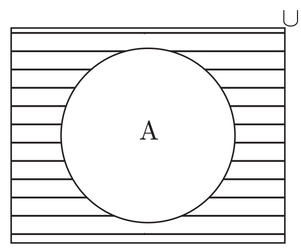
2. Intersection of sets: $A \cap B$



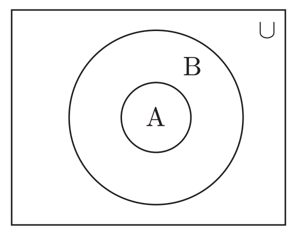
3. Difference of sets: $A - B$ and $B - A$



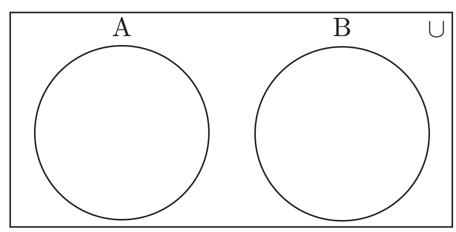
4. Complement of a set



5. Subset: $A \subseteq B$



6. Disjoint sets



APPLICATION OF SETS

Let A, B and C be finite sets and U be a finite universal set.

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
   Note: If A and B are disjoint sets, then $n(A \cap B) = 0$.
2. $n(A - B) = n(A) - n(A \cap B)$
3. Number of elements which belong to exactly one of A or $B = n(A) + n(B) - 2n(A \cap B)$
4. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
5. Number of elements in exactly two of the sets A, B, $C = n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

6. Number of elements in exactly one of the sets A, B and C
$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
7. $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$
8. $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

CARTESIAN PRODUCT OF SETS

Ordered pair is a pair of elements x and y of some sets, which is ordered in the sense that $(x, y) \neq (y, x)$ for $x \neq y$. It is denoted by (x, y).

Here, x is known as first coordinate and y is known as second coordinate of the ordered pair (x, y).

Let us say we have two sets A and B, then Cartesian product of these two sets is the set of all those pairs whose first coordinate is an element of A and second coordinate is an element of B. The set is denoted by $A \times B$.

Symbolically,

$$A \times B = \{(x, y) : x \in A \wedge y \in B\}$$

RELATIONS

If we have two sets A and B, then a relation R between two sets is a subset of $A \times B$.

Thus, $R \subseteq A \times B$.

If R is a relation from a non-empty set A to another non-empty set B and if $(a, b) \in R$, then $a\ R\ b$ is read as 'a is related to b by the relation R'. Similarly, if $(a, b) \notin R$, then we write $a \not{R}\ b$, which is read as 'a is not related to b by the relation R'.

If A and B are two non-empty finite sets consisting of x and y elements, respectively, then $A \times B$ consists of xy ordered pairs. Hence, the total number of subsets of $A \times B$ is 2^{xy}. Also, since each subset defines a relation between A and B, the total number of relations is 2^{xy}.

Now, consider R as a relation from set A to set B. The set of all first coordinates of the ordered pairs belonging to R is called the domain of R.

The set of all second coordinates of the ordered pairs in R is called the range of R.

If A is a non-empty set, then relation from A to itself (i.e. a subset of $A \times A$) is called a relation on set A.

Types of Relations

The different types of relations are as follows:

1. **Inverse relation:** If R is a relation from a set A to a set B, then inverse of R is a relation from B to A. It is denoted by R^{-1}.
It is defined by
$$R^{-1} = \{(x, y) : (x, y) \in R\}$$
It is to be noted that if R is a relation from A to B, then domain of R is the range of R^{-1} and range of R is the domain of R^{-1}.
For example: Let $A = \{1, 2\}$, $B = \{0, 3, 4\}$ and $R = \{(1, 0), (1, 3), (1, 4), (2, 0)\}$ then R^{-1} is given by
$$R^{-1} = \{(0, 1), (3, 1), (4, 1), (0, 2)\}$$
2. **Identity relation:** A relation R in a set A is an identity relation if
$$I_A = \{(x, x) : x \in A\}$$
For example: Let $A = \{0, 1, 2\}$, then
$$I_A = \{(0, 0), (1, 1), (2, 2)\}$$
3. **Universal relation:** If R is a relation in a set A, then $R = A \times A$ is known as a universal relation.
For example: If $A = \{0, 1, 2\}$, then universal relation is given by
$$R = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$$
4. **Void relation:** If R is a relation in a set A, then R is a void relation if $R = \phi$.
For example: Let $A = \{1, 3, 5\}$ and R be defined as $x\ R\ y$ if x divides y, then $R = \phi \subset A \times A$ is a void relation.

Properties of Relation

The properties of relation are as follows:

1. **Reflexive and anti-reflexive relation:** Let R be a relation in a set A, then R is a reflexive relation if $\forall x \in A$, $(x, x) \in R$. Thus, R is reflexive if $x\ R\ x$ exists for all $x \in A$.
R is an anti-reflexive relation if $\forall x \in A$, $(x, x) \notin R$.
For example: Let $A = \{0, 1, 2\}$, then the relation
$R_1 = \{(0, 0), (0, 1), (1, 1), (2, 2)\}$ is reflexive.
$R_2 = \{(0, 0), (0, 1), (2, 2)\}$ is not reflexive since $(1, 1) \notin R_2$.
$R_3 = \{(0, 1), (0, 2), (1, 2), (2, 0)\}$ is anti-reflexive.
$R_4 = \{(0, 1), (1, 1), (1, 2)\}$ is not anti-reflexive since $(1, 1) \in R_4$.
2. **Symmetric and anti-symmetric relation:** Let R be a relation in a set A, then R is said to be symmetric relation if $\forall x\, \forall y \in A$, $(x, y) \in R \to (y, x) \in R$.
R is anti-symmetric relation if $\forall x\, \forall y \in A$, $(x, y) \in R \wedge (y, x) \in R \to x = y$.

For example: Let $A = \{0, 1, 2\}$, then the relation

$R_1 = \{(1, 2), (2, 1)\}$ is symmetric.

$R_2 = \{(1, 2), (2, 1), (0, 1)\}$ is not symmetric since R_2 does not contain $(1, 0)$.

$R_3 = \{(1, 1), (2, 2)\}$ is anti-symmetric.

$R_4 = \{(1, 2), (2, 1), (1, 0)\}$ is not anti-symmetric since R_4 contains $(1, 2)$ and $(2, 1)$ but $1 \neq 2$.

3. **Transitive relations:** Let R be a relation in set A, then R is a transitive relation if

   $\forall x\, \forall y\, \forall z \in A, (x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R$

   A relation R in a set A is not transitive if

   $\forall x\, \forall y\, \forall z \in A, (x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \notin R$

   For example: Let $A = \{0, 1, 2\}$, then

   $R_1 = \{(0, 1), (1, 2), (0, 2)\}$ is transitive.

   $R_2 = \{(0, 1), (1, 2)\}$ is not transitive since $(0, 2) \notin R_2$.

4. **Partial order:** Let R be a relation in set A, then R is partial order if

   (a) R is reflexive.
   (b) R is anti-symmetric.
   (c) R is transitive.

   For example: Let $A = \{0, 1, 2\}$, then

   $R = \{(0, 0), (0, 1), (0, 2), (1, 1), (1, 2), (2, 2)\}$ is partial order relation.

5. **Equivalence relation:** Let R be a relation in set A, then R is an equivalence relation if

   (a) R is reflexive.
   (b) R is symmetric.
   (c) R is transitive.

   For example: Let $A = \{0, 1, 2\}$, then

   $R = \{(0, 0), (1, 1), (2, 2)\}$ is an equivalence relation.

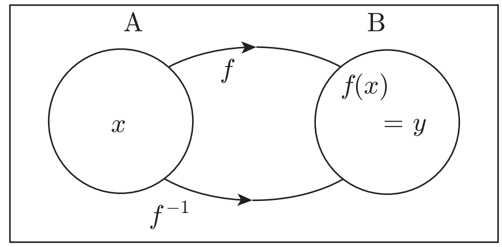
FUNCTIONS

Let A and B be two non-empty sets. A relation f from A to B is called a function, if

1. For each $x \in A$, there exists $y \in B$ such that $(x, y) \in f$.
2. $(x, y) \in f$ and $(x, z) \in f \Rightarrow y = c$.

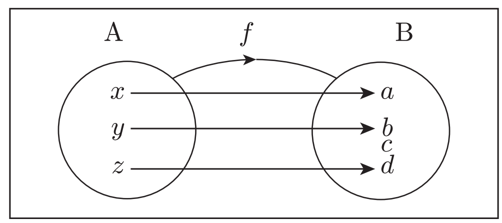
   Thus, a non-empty subset of f from $A \times B$ is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.

   If $(x, y) \in f$, then y is called the image of x under f and x is called a pre-image of f.



Let A and B be two non-empty sets. Then a function f from set A to set B is a correspondence which associates elements of set A to elements of set B such that

1. All elements of set A are associated to elements in set B.
2. An element of set A is associated to a unique element in set B.



Let f: $A \rightarrow B$ be a function from set A to set B, then set A is known as the domain of f and set B is known as the co-domain of f. The set of all the F-images of elements of A is known as the range of f and is denoted by $f(A)$. Thus,

$$f(A) = \{f(x) : x \in A\} = \text{Range of } f$$

Two functions f and g are said to be equal if

1. domain of f = domain of g,
2. co-domain of f = co-domain of g, and
3. $f(x) = g(x)$, for every x belonging to their common domain.

A function f: $A \rightarrow B$ is called a real-valued function if B is a subset of R, i.e. the set of all real numbers.

Types of Functions

The different types of functions are as follows:

1. **Into function:** If $f(A)$ is a proper subset of B, then f: $A \rightarrow B$ is called an into function.

   For example: Let f: $z \rightarrow z$ (set of integers) be defined by $f(x) = 4x$, $x \in z$ is an into function.

2. **Onto (surjective) function:** If $f(A) = B$, then f: $A \rightarrow B$ is called an onto or surjective function.

   For example: Let f: $z \rightarrow z$ be defined by $f(x) = x + 2$, $x \in z$. Then every element y in the co-domain set z has pre-image $y - 2$ in the domain set z. Therefore, $f(z) = z$ and f is an onto function.

3. **Constant function:** If k is a fixed real number, then $f(x)$ given by $f(x) = k$, $x \in R$ is called a constant function.

   For example: Let f: $R \to R$ be defined by $f(x) = 10$. Hence, f is a constant function.

4. **Identity function:** The function f that associates each real number to itself is called the identity function. It is denoted by I.
   Hence, identity function for I: $R \to R$ can be defined as

$$I(x) = x \text{ for all } x \in R$$

5. **Injective function:** If a function f: $A \to B$ has a distinct value for each district argument, then it is called an injective function. It is also called one-to-one function.
   Thus, f: $A \to B$ is injective if $x_1 \neq x_2$ in A implies $f(x_1) \neq f(x_2)$ in B.

   For example: Let f: $R \to R$ be defined by $f(x) = 3x + 1$, $x \in R$, then for x_1, $x_2 \in R$ and $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

6. **Bijective function:** If f is both injective and surjective, then f: $A \to B$ is called a bijective function.

   For example: Let f: $z \to z$ be defined by $f(x) = x + 2$, $x \in 1 \in z$.

7. **Many-to-one function:** If two or more distinct elements in A have the same image, i.e. if $f(x) = f(y)$ implies $x \neq y$, then f: $A \to A$ is called many-to-one function.

8. **Inverse function:** Let f: $A \to B$ be an onto function. Then f^{-1}: $B \to A$, which associates to each element $y \in B$ the element $x \in A$, such that $f(x) = y$ is called an inverse function of f: $A \to B$.

   For example: Let f: $A \to B$ is defined by

$$f(x) = \frac{x-2}{x-3}$$

   Then f^{-1}: $B \to A$ is given by

$$y = \frac{x-2}{x-3} \quad \Rightarrow \quad x = \frac{2-3y}{1-y}$$

$$\text{or} \quad f^{-1}(x) = \frac{2-3x}{1-x}$$

Compositions of Functions

Let f: $A \to B$ and g: $B \to C$, then composite of functions f and g is a mapping of $A \to C$ given by (fog): $A \to C$ such that

$$(gof)(x) = g\ f(x) = g[f(x)] = \forall x \in A$$

Evidently, domain of g is equal to the co-domain of f.

Properties of Composites of Mappings

1. If f: $A \to B$ is a bijective function, then

$$fof^{-1} = I_B \text{ and } f^{-1}of = I_A$$

   where I_A and I_B are identity mapping of sets A and B, respectively.

2. If f: $A \to B$, g: $B \to C$ and h: $C \to D$, then

$$ho(goh) = (hog)of$$

3. If f: $A \to B$ and g: $B \to C$ are injective functions, then (fog): $A \to C$ is injective.
4. If f: $A \to B$ and g: $B \to C$ are two functions such that fog: $A \to C$ is injective. Then f is injective.
5. If f: $A \to B$ and g: $B \to C$ are surjective functions, then (fog): $A \to C$ is surjective.
6. If f: $A \to B$ and g: $B \to C$ are two functions such that fog: $A \to C$ is surjective, then f is surjective.
7. If f: $A \to B$ and g: $B \to C$ are bijective functions, then fog: $A \to C$ is bijective.

INTRODUCTION TO ALGEBRA

Sets with an algebraic operation are called groups.

A binary operation on a non-empty set is a calculation that combines two elements of the set to produce another element of the set.

If * be a binary operation on a non-empty set A and $B \subseteq A$, then B is said to be closed with respect to the binary operation. Table 1 represents the common notations for standard sets.

Table 1 Some commonly used notations for standard sets

N	Set of all natural numbers
Z	Set of all integers
Q	Set of all rational numbers
R	Set of all real numbers
C	Set of all complex numbers
A^*	Set of all non-zero numbers of A where $A \subseteq R$
A^+	Set of all positive number of A where $A \subseteq R$
A^-	Set of all negative number of A where $A \subseteq R$

Table 2 represents the symbols used to depict the usual operations.

Table 2 | Symbols of usual operations

Addition	$+$
Subtraction	$-$
Multiplication	$\cdot$
Division	$\div$

If * is a binary operation on a non-empty finite set A, then the elements of A after operation can be represented by binary operation tables called Cayley binary operation tables.

For example: Write down binary operation table for usual multiplication on the set $A = \{1, 2, 3\}$.

$\cdot$	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

A binary operation * is associative of $(x * y) * z = x * (y * z)$, $\forall x, y, z \in A$ where A is a non-empty set.

For example: $+$ and $\cdot$ are associative binary operations on N, Z, R, Q and C.

A binary operation * is commutative if $(x * y) = (y * x)$, $\forall x, y \in A$ where A is a non-empty set.

For example: $+$ and $\cdot$ are commutative binary operations on N, Z, R, Q and C.

If A is a non-empty set and B is a set of binary operations on A, then algebraic system/algebraic structure is defined by (A, B).

For example: $(R, +)$ is an algebraic system.

SEMIGROUPS

A semigroup is an algebraic structure consisting of a set together with an associative binary operation. If A is a non-empty set and * is a binary operation on A, then algebraic system $(A, *)$ is called a semigroup if

$$x * (y * z) = (x * y) * z, \forall x, y, z \in A$$

For example: $(z, +)$ is a semigroup.

Consider $(A, *)$ to be a semigroup, then an element $x \in A$ is called a left identity of A if $x * y = y$, $\forall y \in A$.

Consider $(A, *)$ to be a semigroup, then an element $x \in A$ is called a right identity of A if $y * x = y$, $\forall y \in A$.

Consider $(A, *)$ to be a semigroup, then an element $x \in A$ is called a two-sided identity of A if $y * x = x * y$, $\forall y \in A$.

A semigroup with an identity element is called a monoid. For example, $(z, +)$ is a monoid.

Consider $(A, *)$ to be a semigroup with identity z.

If there exists an element $y \in A$ such that $y * x = z$, then $x \in A$ is called left invertible and y is called the left inverse of x.

If there exists an element $y \in A$ such that $x * y = z$, then $x \in A$ is called right invertible and y is called the right inverse of x.

If x is both left invertible and right invertible, then $x \in A$ is called invertible.

If x is invertible and $x \in A$, then the element which is both left and right inverse of x is called inverse of x.

For example: 1 is invertible in the semigroup $(N, \cdot)$.

If $x, y, z \in A$, $x * y = x * z$

$$\Rightarrow y = z$$

Then $(A, *)$ is called a left cancellative semigroup.

If $x, y, z \in A$, $y * x = z * z$

$$\Rightarrow y = z$$

Then $(A, *)$ is called a right cancellative semigroup.

If a semigroup $(A, *)$ is both left cancellative and right cancellative semigroups, then it is called cancellative semigroup.

For example: $(z, *)$ and $(Q, *)$ are cancellative semigroups.

Some Important Theorems

1. Consider $(A, *)$ to be a semigroup. If a and b are left and right identities, respectively, then $a = b$.
2. If $(A, *)$ is a semigroup with identity, then identity in A is unique.
3. Consider $(A, *)$ to be a semigroup with identity a and $x \in A$. Now, if y is a left inverse and z is a right inverse of x, then $y = z$.
4. Consider $(A, *)$ to be a semigroup with identity a. If x is an invertible element, then inverse of x is unique.
5. Consider $(A, *)$ to be a semigroup with identity a. If x is an invertible element, then x^{-1} is also invertible.
$$(x^{-1})^{-1} = x$$
6. Consider $(A, *)$ to be a semigroup with identity a. If x and y are two invertible elements and $x, y \in A$, then $x * y$ is also invertible.
$$(x * y)^{-1} = y^{-1} * x^{-1}$$

GROUP

A semigroup satisfying associativity, identity and invertibility is called a group.

Say A be a non-empty set and * be a binary operation, then $(A, *)$ is called a group if

1. $x * (y * z) = (x * y) * z, \forall x, y, z \in A$
2. $\exists e \in A \Rightarrow x * e = e * x, \forall x \in A$ where e is identity element of the group.
3. $x \in A \Rightarrow \exists y \in A \Rightarrow x * y = y * x = e$ where y is the inverse of x in the group.

A group $(A, *)$ is called an abelian or commutative group if

$$x * y = y * x, \forall x, y \in A$$

For example: $(R, +)$ and $(C, +)$ are abelian groups.

Consider $(A, \cdot)$ to be a group. An element $x \in A$ is idempotent if $a^2 = a$.

If A is a finite set, then a group $(A, \cdot)$ is called a finite group. The number of different elements in A is called the order of the finite group, $O(A)$.

If A is an infinite set, then a group $(A, \cdot)$ is called an infinite group.

The order of an infinite group is clearly ∞.

If A is a finite set containing n elements, then group of all bijections on A is called a permutation or symmetric group. It is denoted by P_n or S_n.

Some Important Theorems

1. Identity element in a group is unique.
2. Inverse of every element in a group is unique.
3. If $(A, \cdot)$ is a group and $x \in A$, then $(x^{-1})^{-1} = x$.
4. If $(A, \cdot)$ is a group and $x, y \in A$, then $(x, y)^{-1} = y^{-1}x^{-1}$.
5. If $(A, \cdot)$ is a group, then

   (a) $\forall x, y, z \in A, \quad xy = xz$
   $\Rightarrow x = z$

   (b) $\forall x, y, z \in A, \quad yx = zx$
   $\Rightarrow x = z$

6. If $(A, \cdot)$ is a group, $x \in A$ and $m, n \in z$, then

   (a) $x^m \cdot x^n = x^{m+n} = x^n \cdot x^m$
   (b) $(x^m)^n = x^{mn}$

RESIDUE CLASSES

Two integers are congruent 'modulo n' if they differ by an integral multiple of the integer n. Hence, if n is a positive integer and $x, y \in z$, then x is called congruent to y modulo n when n divides $x - y$. It is denoted by $x \equiv y \pmod{n}$.

For example: 6 divides $40 - 4 \Rightarrow 40 \equiv 4 \pmod 6$.

Also, $x \equiv y \pmod n$ if x and y have the same remainder when divided by n.

The congruence relation $x \equiv y \pmod n$ on a set separates the set into n equivalence classes called residue classes modulo n.

The residue class containing an integer x is denoted by $[x]$ or $\bar{x}$.

Residue Class Addition

Residual class addition or addition modulo n is denoted as $\oplus$ and defined on Z_n as $\bar{x} \oplus \bar{y} = \overline{x+y}, \forall \bar{x}, \bar{y} \in Z_n$.

If $\bar{x}, \bar{y} \in Z_n$, then $\bar{x} + \bar{y} = \bar{r}$, where r is remainder of $x + y$ when divided by n.

$(Z_n, \oplus)$ is an abelian group.

Residual Class Multiplication

Residual class multiplication or multiplication modulo n is defined on Z_n as $\bar{x}, \bar{y} = \overline{xy}\ \forall\ \bar{x}, \bar{y} \in Z_n$. It is also denoted by $\otimes$.

If $\bar{x}, \bar{y} \in Z_n$, then $\bar{x} \otimes \bar{y} = \bar{r}$ where r is remainder of xy when divided by n.

$(Z_n, \otimes)$ is a commutative semigroup with identity.

$(Z_n^*, \otimes)$ is an abelian group if n is prime.

PARTIAL ORDERING

As already stated, a relation R is called partial order if it is

1. Reflexive, $x\ R\ x, \forall x \in A$
2. Anti-symmetric, $(x\ R\ y) \wedge (y\ R\ x) \Rightarrow x = y$
3. Transitive, $(x\ R\ y) \wedge (y\ R\ z) \Rightarrow x\ R\ z$.

A partial order is denoted by $\leq$. A set with partial order $\leq$ is called a partially ordered set or poset.

A poset $(A, \leq)$ is called totally ordered set, if $\forall x, y \in A$, $(x \leq y) \vee (y \leq x)$.

If $x \in (A, \leq)$, then any element $y \in (A, \leq)$ is called cover of x if and only if

1. $x \leq y$.
2. There exists no element $z \in A$ such that $x \leq z$ and $z \leq y$.

Hasse Diagram

A Hasse diagram is a type of mathematical diagram used to represent a finite poset.

Consider $(A, \leq)$ to be a partial ordered set. Then to draw the Hasse diagram, all the elements of A are represented as a vertex in the plane and a line segment is drawn that goes upward from x to y whenever y covers x.

y is the least element of A if $y \leq x$ for all $x \in A$.

y is the greatest element of A if $x \leq y$ for all $x \in A$.

Consider $x \subseteq A$, then an element $x \in A$ is called upper bound of A if $y \leq x$ for all $y \in X$.

Consider $x \subseteq A$, then an element $x \in A$ is called lower bound of A if $x \leq y$ for all $y \in X$.

Least upper bound (L.U.B. or supremum) of A is the lowest upper bound belonging to A. Hence, if x and y are upper bounds of A, then x is the L.U.B. if $x \leq y$.

Greatest lower bounds (G.L.B. or infimum) of A is the highest lower bound belonging to A. Hence, if x and y are lower bounds of A, then y is the G.L.B. if $x \leq y$.

LATTICE

A lattice is a partially ordered set in which every two elements belonging to the poset have an L.U.B. (supremum) and a G.L.B. (infimum). The G.L.B. and L.U.B. are also called meet and join, respectively. Figures 1 and 2 show an example of a lattice and a non-lattice structure, respectively.

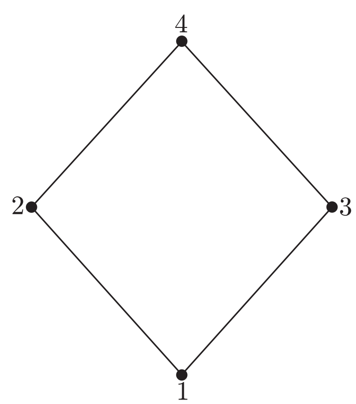


Figure 1 | Lattice.

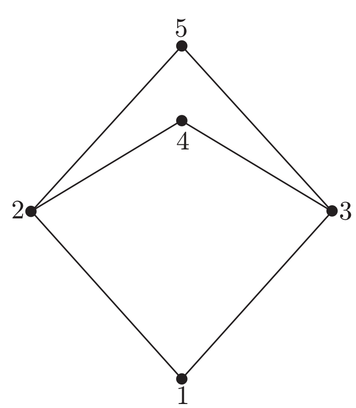


Figure 2 | Non-lattice.

If $(A, \leq)$ is a lattice, then $(A, \geq)$ is also a lattice, we define $\geq$ as:

$$(x \leq y) \Rightarrow y \geq x$$

Also, G.L.B. and L.U.B. are interchanged if we change $\leq$ with $\geq$.

In other words, the operations of meet and join of $(A, \leq)$ become the operations of join and meet of $(A, \geq)$. $(A, \leq)$ and $(A, \geq)$ are called duals of each other. The dual of any poset can be obtained by turning the poset upside down. We denote $x * y$ = meet of x and y = G.L.B. and $x \oplus y$ = join of x and y = L.U.B.

Sublattice

Consider we have a lattice $(A, *, \oplus)$ and $B \leq A$, then $(B, *, \oplus)$ is a sublattice of $(A, *, \oplus)$ if B is closed under operations $*$ and $\oplus$.

A lattice whose non-empty subsets have an L.U.B. and a G.L.B. is called a complete lattice. Every finite lattice is a complete lattice.

Bounds

As already defined before, bounds are the extreme elements of lattices, i.e. the greatest and the least. They are denoted by 1 and 0, respectively.

Consider $(L, *, \oplus, 0, 1)$ to be a bounded lattice, then x and y are complements of each other if $x * y = 0$ and $x \oplus y = 1$. Consider Fig. 3, it has two complements, 2 and 3.

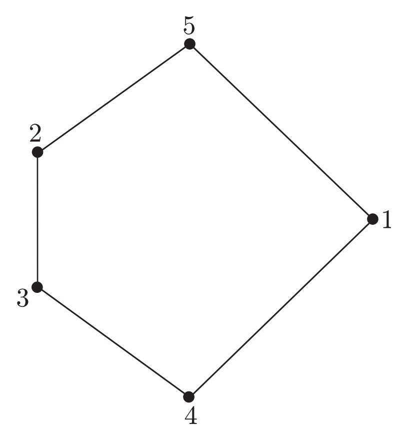


Figure 3 | Example depicting bounds.

A complemented lattice is a bounded lattice in which every element x has at least one complement y such that

$$x \oplus y = 1$$

and $$x * y = 0$$

A relatively complemented lattice is characterized by the property that for every element x in an interval $[w, z]$ there is an element y such that

$$x \oplus y = z$$

and $$x * y = w$$

Such an element is called a complement of x relative to the interval.

A distributive lattice is a lattice in which the operations of join and meet distribute over each other. A lattice $(A, *, \oplus)$ is distributive if

$$x * (y \oplus z) = (x * y) \oplus (x * z)$$

$$x \oplus (y * z) = (x \oplus y) * (x \oplus z)$$

Figures 4 and 5 show the examples of distributive lattices and non-distributive lattices, respectively.

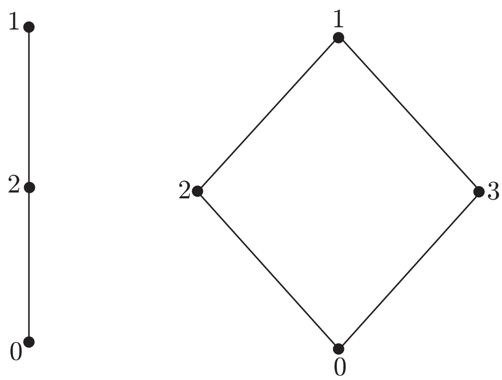


Figure 4 | Distributive lattices.

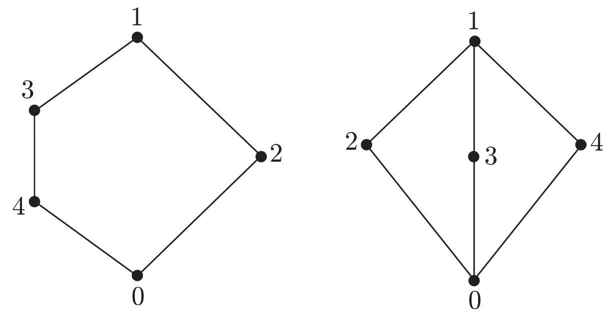


Figure 5 | Non-distributive lattices.

In a bounded distributive lattice, elements which have complements form a sub-lattice.

A modular lattice is a lattice which fulfills the following condition:

$$x \oplus (y * z) = (x \oplus y) * z, \text{ if } x \leq z$$

It is to be noted that every distributive lattice is modular but the converse is not true.

BOOLEAN ALGEBRA OF LATTICES

Consider we have a lattice $(b, *, \oplus)$ with two binary operations $*$ and $\oplus$. Let 0 and 1 be the least and greatest elements, respectively. Some common results for lattices are given in Table 3.

Table 3 | Some common results for lattices

Condition	Property	Illustration
If x, y and z belong to set B.	Idempotent	$x * x = x$, $x \oplus x = x$
	Associativity	$x * (y * z) = (x * y) * z$, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
	Commutativity	$x * y = y * x$, $x \oplus y = y \oplus x$
	Absorption	$x * (x \oplus y) = x$, $x \oplus (x * y) = x$
	Identity	$x * 0 = 0$, $x * 1 = x$
		$x \oplus 1 = 1$, $x \oplus 0 = x$
	Distributivity	$x \oplus (y * z) = (x \oplus y) * (x \oplus z)$
		$x * (y \oplus z) = (x * y) \oplus (x \oplus z)$
	Complements	$x * x' = 0$, $x \oplus x' = 1$
	De Morgan's	$(x * y)' = x' \oplus y'$, $(x \oplus y)' = x' * y'$
If B is uniquely complemented.		$x \leq y \Leftrightarrow x * y = x$, $x \oplus y = y$
		$x \leq y \Leftrightarrow x * y' = 0$
		$y' \leq x' \Leftrightarrow x' \oplus y = 1$

SOLVED EXAMPLES

1. Let $A = \{-2, -1, 0, 1, 2\}$, $B = \{0, 1, 4\}$ and $f: A \to B$ is defined as $f(x) = x^2$ is a function. Is the function one-to-one or bijection?

Solution: We have

$$f(x) = x^2$$

Now, $f(-2) = 4, f(-1) = 1, f(0) = 0, f(1) = 1, f(2) = 4$.

$f: A \to B$ is an onto function since $f(A) = B$, hence $f: A \to B$ is not one-to-one and therefore not a bijection.

2. If $A = \{\text{a, b, c}\}$, $B = \{2, 1, 0\}$ and $f: A \to B$, find whether the function is one-to-one, onto or bijection? Given that $f = \{(a, 1), (b, 0), (c, 2)\}$.

Solution: We have

$$f = \{(a,1), (b,0), (c,2)\}$$

Different elements in A have different f-images in B and hence f is one-to-one.

$$f(A) = \{1, 0, 2\} = B$$

Hence, $f: A \to B$ is onto.

By definition, $f: A \to B$ is a bijection, since it is one-to-one and onto.

3. Find the inverse function of $f(x) = 4x + 7$.

Solution: We have

$$f(x) = 4x + 7 = y$$

$$\Rightarrow x = \frac{y-7}{4}$$

$$\therefore f^{-1}(y) = \frac{y-7}{x}$$

and $$f^{-1}(x) = \frac{x-7}{y}$$

4. Form the binary operation table for a set $A = \{1, a, a^2\}$ under usual multiplication.

Solution: The table can be formed as follows:

.	1	a	a^2
1	1	a	a^2
a	a	a^2	a^3
a^2	a^2	a^3	a^4

5. Verify if $a * b = ab + 1$ is a binary operation on N. Also verify whether they are associative and commutative.

Solution: We know a and b are natural numbers, then $ab + 1$ is also a natural number. Therefore, $a * b$ is a binary operation on N.

Now, $a * b = ab + 1 = ba + 1 = b * a$.

Hence, $*$ is commutative.

$$(a * b) * c = (ab + 1) * c = (ab + 1)c + 1$$
$$= abc + c + 1$$
$$a * (b * c) = a * (bc + 1) = a(bc + 1) + 1$$
$$= abc + a + 1$$

Thus,

$$(a * b) * c \neq a * (b * c)$$

Hence, $*$ is not associative.

6. If A is a non-empty set and $P(A)$ is a power set of A, then show that $(P(A), \cup)$, $(P(A), \cap)$ are monoids.

Solution: Let $X, Y \in P(A)$.

Now, $X \subseteq A$, $Y \subseteq A$ and hence, $X \cup Y \subseteq A$, $X \cap Y \subseteq A$.

Hence, $X \cup Y \in P(A)$, $X \cap Y \in P(A)$.

Also, $(X \cup Y) \cup Z = X \cap (Y \cap Z)$

$(X \cap Y) \cap Z = X \cap (Y \cap Z)$

$\forall X, Y, Z \in P(A)$

Again, $X \cup \phi = \phi \cup X = X$

and $X \cap A = A \cap X = X$.

Hence, $(P(A), \cup)$ and $(P(A), \cap)$ are monoids.

7. Consider $P = Q - \{1\}$, and if $*$ is defined on P as $x * y = x + y - xy$, prove that $(P, *)$ is an abelian group.

Solution: We can see that P is non-empty.

Now, let $x, y \in P$.

$$x \in P \Rightarrow x \in Q, x \neq 1$$

Similarly, $y \in P \Rightarrow y \in Q, y \neq 1$

Thus,

$x, y \in Q \Rightarrow x + y \in Q$, $xy \in Q$ and $x + y - xy \in Q$

$$\Rightarrow x * y \in Q$$

Consider $x * y = 1$

$$\Rightarrow x + y - xy = 1$$
$$\Rightarrow x - 1 + y - xy = 0$$
$$\Rightarrow (x - 1) - y(x - 1) = 0$$
$$\Rightarrow (x - 1)(1 - y) = 0$$
$$\Rightarrow x = 1 \text{ or } y = 1$$

Now, consider $x * y \neq 1$

$$x * y \in Q \Rightarrow x * y \in P$$

Therefore, * is a binary operation of P.

Let $x, y, z \in P$.

$$x * (y * z) = x * (y + z - yz) = x * ((y + z) - yz) - x(y + z - yz)$$
$$= x + y + z - yz - xy - xz + xyz$$
$$(x * y) * z = (x + y - xy) * z = x + y - xy + z - (x + y - xy)z$$
$$= x + y + z - xy - xz - yz + xyz$$

$\therefore x * (y * z) = (x * y) * z$

Hence, * is associative.

Let $x \in P$ and y be x^{-1}, then

$$x * y = 0$$
$$\Rightarrow x + y - xy = 0$$
$$\Rightarrow y(x - 1) = x \Rightarrow y = \frac{x}{x-1} \in P$$

Now, say $x, y \in P$, then $x * y = x + y - xy = y + x - yx = y * x$

Thus, * is commutative.

Let e be identity element of P, then

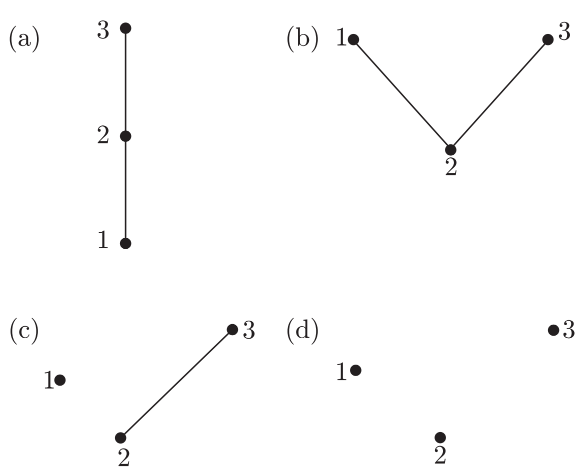
$$x * e = x, \forall x \in P$$
$$\Rightarrow x * e = x$$
$$\Rightarrow x + e - xe = x$$
$$\Rightarrow e(1 - x) = 0$$
$$\Rightarrow e = 0 \text{ (since } x \neq 1)$$

Therefore, $x = 0 \in P$ is the identity element of P.

Hence, $(P, *)$ is an abelian group.

8. Show that a lattice with less than four elements is always a chain.

Solution: If a lattice contains 1 or 2 elements, then it is a chain. Now, let us draw all the possible combinations with three elements.



The combination (a) is a lattice and is a chain. However, the combinations (b), (c) and (d) are not lattice because no two elements have G.L.B or L.U.B.

9. Prove that $x * (x' \oplus y) = x * y$.

Solution: We know that

$$x * (x' \oplus y) = (x * x') \oplus (x * y)$$
$$= 0 \oplus (x * y)$$
$$= x * y = \text{R.H.S.}$$

Hence proved.

10. Show that the relation R on the set $A = \{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ is symmetric but neither reflexive nor transitive.

Solution: As (1, 1), (2, 2) and (3, 3) do not belong to R, R is not reflexive.

Clearly, $(1, 2) \in R$ and $(2, 1) \in R$. So, R is symmetric.

As $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$, R is not transitive.

11. Let a relation R on the set R of real number be defined as

$$(a, b) \in R_1 \Leftrightarrow 1 + ab > 0 \text{ for all } a, b \in R$$

Show that R_1 is reflexive and symmetric but not transitive.

Solution: Let a be an arbitrary element of R. Then,

$$a \in R$$

$\Rightarrow 1 + a \cdot a = 1 + a^2 > 0$ [$\because a^2 > 0$ for all $a \in R$]

$\Rightarrow (a, a) \in R_1$

Thus, $(a, a) \in R_1$ for all $a \in R$. Hence, R_1 is reflexive on R.

Now, let $(a, b) \in R$. Then,

$$(a, b) \in R_1$$

$\Rightarrow 1 + ab > 0$ and $1 + ba > 0$ [$\because ab = ba \; \forall a, b \in R$]

$\Rightarrow (b, a) \in R_1$

Thus, $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ for all $a, b \in R$. So, R_1 is symmetric on R.

Now, we observe that $(1, 1/2) \in R_1$ and $(1/2, -1) \in R_1$ but $(1, -1) \notin R_1$ because $1 + 1 \times (-1) = 0 \not> 0$.

Thus, R_1 is not transitive on R.

12. Prove that the relation R on the set Z of all integers defined by $(x, y) \in R \Leftrightarrow x - y$ is divisible by n is an equivalence relation on Z.

Solution: For any $a \in N$, we have

$$a - a = 0 = 0 \times n$$

$a - a$ is divisible by n

$\Rightarrow (a, a) \in R$

Thus, $(a, a) \in R$ for all $a \in z$. So, R is reflexive on Z.

Let $(a, b) \in R$. Then,

$(a, b) \in R$

$\Rightarrow (a - b)$ is divisible by n

$\Rightarrow (a - b) = np$ for some $p \in Z$

$b - a = n(-P)$

$\Rightarrow b - a$ is divisible by n

$\Rightarrow (b, a) \in R$

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in Z$.

So, R is symmetric on Z.

Now, let $a, b, c \in Z$ such that $(a, c) \in R$ and $(b, a) \in R$. Then, $(a, b) \in R$

$\Rightarrow (a - b)$ is divisible by $n \Rightarrow a - b = np$ for some $p \in z$.

$\Rightarrow (b, c) \in R$

$\Rightarrow (b - c)$ is divisible by $n \Rightarrow b - c = nq$ for some $q \in z$.

$\therefore (a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a - b) = np$ and $(b - c) = nq$

$\Rightarrow (a - b) + (b - c) = np + nq$

$\Rightarrow a - c = n(p + q)$

Hence, $a - c$ is divisible by n.

$$\Rightarrow (a, c) \in R$$

Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in Z$.

So, R is a transitive relation on Z.

Therefore, R is an equivalence relation on Z.

13. Show that the function f: $R \to R$ defined by $f(x) = 3x^3 + 5$ for all $x \in R$ is a bijection.

Solution: Let $x, y \in R$.

Then, $f(x) = f(y) \Rightarrow 3x^3 + 5 = 3y^3 + 5 \Rightarrow x^3 = y^3$ $\Rightarrow x = y$

Hence, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R$.

Therefore, f is an injective map.

Now, let y be an arbitrary element of R. Then,

$$f(x) = y \Rightarrow 3x^3 + 5 = y \Rightarrow x^3 = \frac{y-5}{3}$$

$$\Rightarrow x = \left(\frac{4-5}{3}\right)^{1/3}$$

Hence, for all $y \in R$ (co-domain) there exists $x = \left(\frac{y-5}{3}\right)^{1/3} \in R$ (domain) such that

$$f(x) = f\left(\left(\frac{y-5}{3}\right)^{1/3}\right) = 3\left[\left(\frac{y-5}{3}\right)^{1/3}\right]^3 + 5$$

$$= y - 5 + 5 = y$$

This shows that every element in the co-domain has its pre-image in the domain. So, f is a surjection. Hence, f is a bijection.

14. If the function f: $R \to R$ be given by $f(x) = x^2 + 2$ and g: $R \to R$ be given by $g(x) = \frac{x}{x-1}$. Find fog and gof.

Solution: Clearly, range f = domain g and range g = domain f.

Hence, fog and gof both exist.

Now,

$$(fog)(x) = f(g(x)) = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2$$

$$= \frac{x^2}{(x-1)^2} + 2$$

and $$(gof)(x) = g(f(x)) = g(x^2 + 2)$$

$$= \frac{x^2+2}{(x^2+2)-1} = \frac{x^2+2}{x^2+1}$$

Hence, gof: $R \to R$ and fog: $R \to R$ are given by

$$(gof)(x) = \frac{x^2+2}{x^2+1} \text{ and } (fog)(x) = \frac{x^2}{(x-1)^2} + 2$$

15. Consider f, g and h functions from R to R. Show that

$$(f + g)oh = foh + goh$$

Solution: Since f, g and h are functions from R to R,

$$(f + g)oh: R \to R \text{ and } foh + goh: R \to R$$

Now,

$((f + g)oh)(x) = (f + g)(h(x))$

$\Rightarrow [(f + g)oh](x) = f[h(x)] + g[h(x)]$

$\Rightarrow [(f + g)oh](x) = foh(x) + goh(x)$, for all $x \in R$

Thus, $(f + g)oh = foh + goh$.

16. If f: $R \to R$ is a bijection given by $f(x) = x^3 + 3$, find $f^{-1}(x)$.

Solution: Let $f(x) = y$. Then,

$$f(x) = y \Rightarrow x^3 + 3 = y \Rightarrow x = (y - 3)^{1/3}$$

$$\Rightarrow f^{-1}(y) = (y - 3)^{1/3}$$

Thus, f^{-1}: $R \rightarrow R$ is defined as $f^{-1}(x) = (x - 3)^{1/3}$ for all $x \in R$.

17. Show that function f: $R \rightarrow R$ given by $f(x) = x^2 + 1$ is not invertible.

Solution: We have

$$f(x) = x^2 + 1$$

Clearly, $-2 \neq 2$ but $f(-2) = f(2) = 5$.

So, f is not a one-to-one function.

Hence, f is not invertible.

18. For any two sets A and B, prove that $(A \cap B) \cup (A - B) = A$.

Solution:

$$\begin{aligned}(A \cap B) \cup (A - B) &= (A \cap B) \cup (A \cap B')\\ &= X \cup (A \cap B'), \text{ where } X = A \cap B\\ &= (X \cup A) \cap (X \cup B')\\ &= A \cap (A \cup B')\end{aligned}$$

$$\left[\begin{aligned}&\because X \cup A = (A \cap B) \cup A = A. \text{ Since, } A \cap B \subset A\\ &X \cup B' = (A \cap B) \cup B' = (A \cup B') \cap (B \cup B')\end{aligned}\right]$$

$$\begin{aligned}&= (A \cup B') \cap U\\ &= A \cup B'\\ &= A \quad [\because A \subset A \cup B']\end{aligned}$$

19. Let R be the relation on the set N of natural numbers defined by $R = \{(a, b)$: $a + 3b = 12$, $a \in N$, $b \in N\}$. Find R, domain and range of R.

Solution: We have

$$a + 3b = 12$$
$$a = 12 - 3b$$

Putting $b = 1, 2, 3$, we get $a = 9, 6, 3$, respectively.

For $b = 4$, we get $a = 0 \notin N$. Also, for $b > 4$, $a \notin N$.

$\therefore R = \{(9, 1), (6, 2), (3, 3)\}$

Domain of $R = \{9, 6, 3\}$

Range of $R = \{1, 2, 3\}$

20. In a survey of 2000 consumers, 1720 consumers liked X and 1450 consumers liked Y. What is the least number that must have liked both the products?

Solution: Let U be set of all consumers, A be set of consumers who liked X and B be set of consumers who liked Y.

We are given that

$$n(U) = 2000,\ n(A) = 1720,\ n(B) = 1450$$

$$\begin{aligned}n(A \cup B) &= n(A) + n(B) - n(A \cap B)\\ &= 1720 + 1450 - n(A \cap B)\\ &= 3170 - n(A \cap B)\end{aligned}$$

Since $A \cup B \subset U$

$$\begin{aligned}n(A \cup B) &\leq n(U)\\ 3170 - n(A \cap B) &\leq 2000\\ n(A \cap B) &\geq 1170\end{aligned}$$

Hence, the least value of consumer who liked both the products is 1170.

21. Let F be a collection of all functions f: $\{1, 2, 3\} \rightarrow \{1, 2, 3\}$.

If f and $g \in F$, define an equivalence relation $\sim$ by $f \sim g$ if and only if $f(3) = g$.

(a) Find the equivalence classes defined by $\sim$.
(b) Find the number of elements in each equivalence class.

Solution:
(a) There are three equivalence classes, $A = \{f \in F$, $f(3) = i\}$, $i = 1, 2$.
(b) Consider $f(3) = 1$.

$f(3)$ can take any of the three values: 1, 2 or 3. Similarly, $f(1)$ and $f(2)$ can also take any of the three values.

Hence, there are $3 * 3 = 9$ functions in each equivalence class.

22. Mr. X claims the following:

If a relation R is both symmetric and transitive, then R is reflexive. For this, Mr. X offers the following proof.

"From xRy, using symmetry we get yRx. Now because R is transitive xRy and yRx together imply xRx. Therefore, R is reflexive."

What is the flaw in Mr. X's proof?

Solution: R is an empty set, and hence Mr. X's proof becomes invalid.

23. Prove that $P(A \cap B) = P(A) \cap P(B)$.

Solution: Say $A = \{0, 1, 2\}$ and $B = \{1, 2\}$

Now, $P(A) = [\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}, \{\phi\}]$

$$P(B) = [\{1\}, \{2\}, \{1, 2\}, \{\phi\}]$$

$$P(A) \cap P(B) = [\{1\}, \{2\}, \{1, 2\}, \{\phi\}]$$

$$A \cap B = \{1, 2\}$$
$$P(A \cap B) = [\{1\}, \{2\}, \{1, 2\}, \{\phi\}]$$

Thus, $P(A \cap B) = P(A) \cap P(B)$

Hence proved.

24. Consider the function h: $N \times N \to N$, so that $h(x, y) = (2x + 1)2^y - 1$ where $N = \{0, 1, 2, \ldots\}$ is a set of natural numbers:

(a) Prove that function h is an injective function.
(b) Prove that function h is a subjective function.

Solution:

(a) We have h: $N * N \to N$, $h(x, y) = (2x + 1)2^y - 1$

Let $x_1 = x_2$ and $y_1 = y_2$

$$\therefore 2x_1 = 2x_2$$
$$\Rightarrow 2x_1 + 1 = 2x_2 = 1 \qquad (1)$$

Also, $2^{y_1} = 2^{y_2}$ (2)

Multiplying Eqs. (1) and (2), we get

$$(2x_1 + 1)2^{y_1} = (2x_2 + 1)2^{y_2}$$
$$\Rightarrow (2x_1 + 1)2^{y_1} - 1 = (2x_2 + 1)2^{y_2} - 1$$

$\therefore h(x_1, y_1) = h(x_2, y_2)$ for $x_1 = x_2$, $y_1 = y_2$

Thus, h is injective.

(b) Let $h(x, y) = 0$

Hence, $(2x_1 + 1)2^{y_1} - 1 = 0$

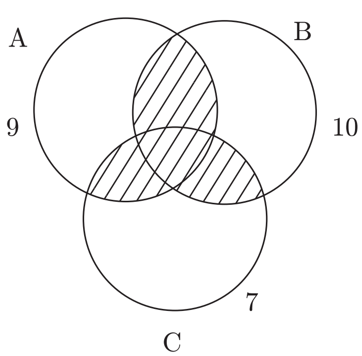
$\Rightarrow (2x_1 + 1)2^{y_1} = 1 \Rightarrow (2x_1 + 1)2^{y_1} = 2^0 \times 1$

$\Rightarrow 2^{y_1} = 2^0$ [$\because$ $(2 \times 1 + 1)$ is odd]

$\Rightarrow y_1 = 0$, $x_1 = 0$. Therefore, h is surjective.

25. Out of a group of 21 persons, 9 eat vegetables, 10 eat fish and 7 eat eggs, 5 eat all three. How many persons eat at least two out of the three dishes?

Solution: Let A represents a person eating vegetables, B represents a person eating fish and C represents a person eating eggs.



The shaded region represents a person eating at least two items. The shaded region can be expressed as

$$R = (A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C) \cup (A \cap B \cap C)$$

Since all above regions are mutually disjoint,

$$P(R) = P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C) \qquad (1)$$

We are given that

$$P(A \cap B \cap C) = \frac{5}{21},\ P(A) = \frac{9}{21},\ P(B) = \frac{10}{21},$$
$$P(C) = \frac{7}{21}$$

Also, $P(A \cup B \cup C) = P(A) + P(B) + P(C) + P(A \cap B \cap C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$

26. Let G_1 and G_2 be subgroups of a group G.

(a) Show that $G_1 \cap G_2$ is also a subgroup of G.
(b) Is $G_1 \cup G_2$ always a subgroup of G?

Solution:

(a) Let G_1 and G_2 be two subgroups of G. Then, $G_1 \cap G_2 \neq \phi$ since identity elements is common to both G_1 and G_2.

Now, $a \in G_1 \cap G_2$, $b \in G_1 \cap G_2$

$$\Rightarrow ab^{-1} \in G_1 \cap G_2$$
$$a \in G_1 \cap G_2 \Rightarrow a \in G_1 \text{ and } a \in G_2$$
$$b \in G_1 \cap G_2 \Rightarrow b \in G_1 \text{ and } b \in G_2$$

But G_1 and G_2 are subgroups. Therefore,

$$a \in G_1,\ b \in G_1$$
$$\Rightarrow ab^{-1} \in G_2,\ a \in G_2,\ b \in G_2$$
$$ab^{-1} \in G_2$$

Also, $ab^{-1} \in G_1$, $ab^{-1} \in G_2$

$$\Rightarrow ab^{-1} \in G_1 \cap G_2$$

Thus, $a \in G_1 \cap G_2$, $b \in G_1 \cap G_2$

and $ab^{-1} \in G_1, G_2$

Hence, $G_1 \in G_2$ is a subgroup of G.

(b) No, it is not necessary that $G_1 \cup G_2$ is a subgroup of G.

A group must satisfy four conditions: closure, associative, identity and inverse.

However, it does not necessarily fulfill the inverse condition.

27. Let $(A, *)$ be a semigroup. Furthermore, for every a and b in A, if $a \neq b$, then $a * b \neq b * a$.

(a) Show that for every a in A, $a * a = a$.
(b) Show that for every a and b in A, $a * b * a = a$.

(c) Show that for every a, b and c in A, $a * b * c = a * c$.

Solution:

(a) Because A is a semigroup,

$$(a * b) * c = a * (b * c) \quad (1)$$

Now, putting $b = a$ and $c = a$, we get

$(a * a) * a = a * (a * a)$ and A is not abelian.

$$\Rightarrow a * a = a \quad (2)$$

(b) Let $b \in A$, then

$$b * b = b$$

Multiplying both the sides of above equation by a, we get

$$a * (b * b) = a * b \quad (3)$$

By associativity, we have

$$(a * b) * b = a * b$$

$$\Rightarrow a * b = a$$

Thus, $a * b * a = (a * b) * a$

Using Eq. (2), we get

$$a * b * a = a * a \quad (4)$$
$$= a$$

(c) $a * b * c = (a * b) *c$
$= a * c$ [using Eq. (4)]

28. Let G be a finite group and H be a subgroup of G, for $a \in G$, define $aH = \{ah|h \in H\}$

(a) Show that $|aH| = |H|$.
(b) Show that for every pair of elements a, $b \in G$, either $aH = bH$ or aH and bH are disjoint.
(c) Use the above conditions to argue that the order of H must divide the order of G.

Solution:

(a) Assume that $|aH| \neq |H|$
Hence, $|aH| < |H|$ for two distinct values of h_1 and h_2

Now, $a * h_1 = a * h_2$

where h_1, $h_2 \in H$ and * is the operation defined in the group.

$\Rightarrow a = a * h_2 * h_1^{-1}$ [inverse property of groups]

$\Rightarrow a^{-1} * a = h_2 * h_1^{-1}$

$\Rightarrow 1 = h_2 * h_1^{-1}$

$\Rightarrow h_1 = h_2$

$\therefore |aH| = |H|$

Hence proved.

(b) Consider that $a * H$ and $b * H$ are not disjoint. Then there exists f such that $f \in a * H$ and $f \in b * H$.

i. e. $f = a * h_1 = b * h_2$

where h_1, $h_2 \in H$

$$\therefore a = b * h_2 * h_1^{-1}$$

Let $x \in a * H$

or $x = a * h_3$ (for $h_3 \in H$)

$$\Rightarrow x = b * (h_2 * h_1^{-1} * h_3)$$

$$\Rightarrow x = b * (h_4) \quad \text{(for } h_4 \in H\text{)}$$

Because groups are closed under * operation, they are identical.

(c) Because $a * H$ and $b * H$ are either disjoint or identical for any a and $b \in G$ and cardinality of all sets is same (since $|H|$), there exists some partitioning of the elements in G into sets having $|H|$ elements each. Also, each element of G belongs to some $a * H$. Hence, the order of any subgroup of a finite group divides the order of the group.

29. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $\otimes$ denotes multiplication modulo 8, i.e. $x \otimes y = (xy) \bmod 8$

(a) Prove that $(\{0, 1\}, \otimes)$ is not a group.
(b) Write three distinct groups $(G, \otimes)$ where $G \subset S$ and G has two elements.

Solution:

(a) $\otimes$ does not have an inverse, hence $(\{0, 1\}, \otimes)$ is not a group.

(b) $\{1, 3\}$, $\{1, 5\}$, $\{1, 7\}$

30. Let $(\{p * q\}, *)$ be a semigroup where $p * p = q$. Show that

(a) $p * q = q * p$ (b) $q * p = q$

Solution: Let $S = (\{p, q\}, *)$ be a semigroup.

Because S is a semigroup, it is closed and associative, i.e.

$$x * y \in S, x \vee y \in \mathrm{S}$$

$$x * (y * z) = (x * y) * z, \forall x, y, z \in S$$

(a) $p * q = p * (p * p)$
$= (p * p) * p$
$= q * p$ $(\because\ p * p = q)$

(b) Because S is closed, $p * q \in S = \{p, q\}$.

Thus, $p * q = p$ or $p * q = q$

Case 1: Let $p * q = p$

$$q * q = q * (p * p)$$
$$= (q * p) * p = (p * q) * p = p * p = q$$

Case 2: Let $p * q = q$

$$q * q = (p * p) * q$$
$$= p * (p * q) = p * q = q$$

PRACTICE EXERCISE

1. If f: $R \to R$ and g: $R \to R$ be two functions such that $fog(x) = \sin x^2$ and $gof(x)\ \sin^2 x$, then find $f(x)$ and $g(x)$.

(a) $f(x) = x^2$ and $g(x) = \sin x$
(b) $f(x) = \sin x$ and $g(x) = x^2$
(c) $f(x) = \sin x$ and $g(x) = \sin x^2$
(d) $f(x) = \cos x$ and $g(x) = \sin x \cos x$

2. If f: $R \to R$ and g: $R \to R$ be functions defined by

$$f(x) = x^2 + 1 \text{ and } g(x) = \sin x,$$

then find fog and gof.

(a) $fog = \sin^2 x + 1$, $gof = \sin x^2 + 1$
(b) $fog = \sin x^2 + 1$, $gof = \sin^2 x + 1$
(c) $fog = (\sin x + 1)^2$, $gof = \sin x + 1$
(d) $fog = \sin x + 1$, $gof = (\sin x + 1)^2$

3. Let $A = \{\phi, \{\phi\}, 1, \{1, \phi\}, 2\}$. Which of the following is FALSE?

(a) $\phi \in A$ (b) $\{\phi\} \in A$
(c) $\{1\} \in A$ (d) $\{2, \phi\} \subset A$

4. Two finite sets have x and y elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the values of x and y.

(a) $x = 6$, $y = 3$ (b) $x = 4$, $y = 2$
(c) $x = 2$, $y = 4$ (d) $x = 3$, y $= 6$

5. A school awarded 38 prizes in football, 15 in basketball and 20 in cricket. If these prizes went to a total of 58 students and only 3 students got prizes in all three sports, how many received medals in exactly two of the three sports?

(a) 18 (b) 15
(c) 12 (d) 9

6. In a class of 35 students, 17 have taken mathematics, 10 have taken mathematics but not computers. Find the students who have taken both mathematics and computers.

(a) 11 (b) 7
(c) 9 (d) 13

7. If R is an equivalence relation on a set A, then R^{-1} is

(a) reflexive (b) symmetric
(c) transitive (d) all of these

8. Number of subsets of a set of order 3 is

(a) 3 (b) 6
(c) 8 (d) 9

9. If A and B are any two sets and $A \cup B = A \cap B$, then

(a) $A = \phi$ (b) $B = \phi$
(c) $A = B$ (d) none of these

10. Which of the following sets are null sets?

(a) { } (b) $\{\phi\}$
(c) $\{0\}$ (d) $\{0, \phi\}$

11. If relation R is defined on N by $R = \{(a, b)$: a divides b; $a, b \in N\}$. Then R is

(a) reflexive (b) symmetric
(c) transitive (d) reflexive and transitive

12. If X and Y are two sets, then $X \cap (Y \cap X)^C$ equals

(a) ϕ (b) Y
(c) X (d) $X \cap Y^C$

13. Let X be any non-empty set. The identity function on set A is

(a) injective (b) surjective
(c) bijective (d) none of these

14. The function f: $N \to N$ defined by $f(n) = 2n + 3$ is

(a) injective and surjective
(b) injective and not surjective
(c) not injective and surjective
(d) not injective and not surjective

15. The number of distinct relations on a set of three elements is

(a) 8 (b) 9
(c) 128 (d) 512

16. $A - (B \cup C)$ is equal to

(a) $(A - B) \cup (A - C)$ (b) $(A - B - C)$
(c) $(A - B) \cap (A - C)$ (d) $(A - B) \cup C$

Common Data for Questions 17 and 18: Let * be a Boolean operation defined as

$$A * B = AB + \bar{A}\bar{B}$$
$$C = A * B$$

17. $A * A$ is equal to

(a) 0 (b) 1
(c) A (d) $\bar{A}$

18. $C * A$ is equal to

(a) A (b) B
(c) AB (d) $\bar{A}B$

Common Data for Questions 19–21: The following equations are given

$$AB + \bar{A}C = 1$$
$$AC + B = 0$$

19. Value of A will be

(a) 0 (b) 1
(c) ∞ (d) Indeterminant

20. Value of B will be

(a) 0 (b) 1
(c) ∞ (d) Indeterminant

21. Value of C will be

(a) 0 (b) 1
(c) ∞ (d) Indeterminant

Common Data for Questions 22–23: In a survey on beverages, 80 people like Coca Cola, 60 like Pepsi, 50 like Fanta, 30 like Coca Cola and Pepsi, 20 like Pepsi and Fanta, 15 like Coca Cola and Fanta and 10 like all the three drinks.

22. How many people like at least one drink?

(a) 135 (b) 30
(c) 10 (d) 45

23. How many people like at least two drinks?

(a) 135 (b) 30
(c) 10 (d) 45

24. Let A and B be two sets. f: $A \times B \to B \times A$ defined by $f(a, b) = (b, a)$ is

(a) bijective
(b) injective but not surjective
(c) surjective but not injective
(d) neither surjective nor injective

25. The domain and range are same for

(a) constant function
(b) identity function
(c) absolute value function
(d) greatest integer function

26. If X and Y have 3 and 6 elements each, then minimum number of elements in $A \cup B$ is

(a) 3 (b) 6
(c) 9 (d) 18

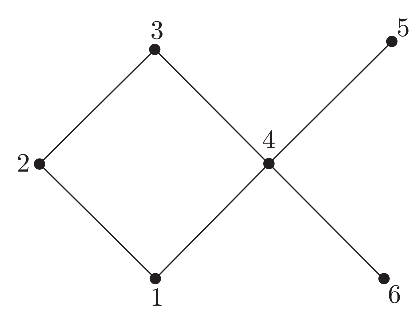
27. If X and Y have 3 and 6 elements each, then maximum number of elements in $A \cap B$ is

(a) 3 (b) 6
(c) 9 (d) 18

28. Hasse diagrams are drawn for

(a) partially ordered sets
(b) lattices
(c) Boolean algebra
(d) none of these

29. Maximal and minimal elements of the poset are



(a) maximal 5, 6; minimal 2
(b) maximal 5, 6; minimal 1
(c) maximal 3, 5; minimal 1, 6
(d) none of these

30. The universal relation $A \times A$ on A is

(a) all equivalence relation
(b) anti-symmetric
(c) a partial ordering relation
(d) not symmentric and not anti-symmetric

31. Let $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and relation I be a partial ordering on D_{30}. The L.U.B. of 10 and 15 is

(a) 6 (b) 10
(c) 15 (d) 30

32. Let D_{30} =\{1, 2, 3, 5, 6, 10, 15, 30\} and relation I be a partial ordering on D_{30}. The G.L.B. of 10 and 15 is

(a) 10 (b) 6
(c) 5 (d) 1

33. If the function f: $[1, \infty] \to [1, \infty]$ defined by $f(x) = 2x(x-1)$ is invertible, find $f^{-1}(x)$.

(a) $\dfrac{1+\sqrt{1+2\log_2 x}}{4}$ (b) $\dfrac{1-\sqrt{1+2\log_2 x}}{4}$

(c) $\dfrac{1+\sqrt{1+4\log_2 x}}{2}$ (d) $\dfrac{1-\sqrt{1+4\log_2 x}}{2}$

34. On Q, the set of all rational numbers, a binany operation * is defined by

$$a * b = \frac{ab}{5}, \text{ for all } a, b \in Q$$

(a) 1 (b) 10
(c) 5 (d) none of these

35. If $f(x)$ and $g(x)$ are defined on domains A and B, respectively, then domain of $f(x) + g(x)$ is

(a) $A + B$ (b) $A - B$
(c) $A \cup B$ (d) $A \cap B$

36. The binary relation $S = \phi$ (empty set) on set $A = \{1, 2, 3\}$ is

(a) neither reflexive nor symmetric
(b) symmetric and reflexive
(c) transitive and reflexive
(d) transitive and symmetric

37. Let $X = \{2, 3, 6, 12, 24\}$ and $\leq$ be the partial order defined by $X \leq Y$ if X divides Y. Number of edges in the Hasse diagram of $(X, \leq)$ is

(a) 3 (b) 4
(c) 5 (d) none of these

38. Let $P(S)$ denote the power set of S. Which of the following is always true?

(a) $P(P(S)) = P(S)$
(b) $P(S) \cap S = P(S)$
(c) $P(S) \cap P(P(S)) = [\phi]$
(d) $S \in P(S)$

39. The number of binary relation on a set with n elements is

(a) n^2 (b) 2^n
(c) 2^{n^2} (d) none of these

40. If A is a finite set with n elements, then number of elements in the largest equivalence relation of A is

(a) 1 (b) n
(c) $n + 1$ (d) n^2

41. The number of functions from an m element set to an n element set is

(a) $m + n$ (b) m^n
(c) n^m (d) $m * n$

42. The number of equivalence relations of the set \{1, 2, 3, 4\} is

(a) 4 (b) 15
(c) 16 (d) 24

43. If A and B are sets and A^C and B^C denote the complements of the sets A and B, then set $(A-B) \cup (B-A) \cup (A \cap B)$ is equal to

(a) $A \cup B$ (b) $A^C \cup B^C$
(c) $A \cap B$ (d) $A^C \cap B^C$

44. Let R be a non-empty relation on a collection of sets defined by $A\ R\ B$ if and only if $A \cap B = \phi$. Then

(a) R is reflexive and transitive
(b) R is symmetric and not transitive
(c) R is an equivalence relation
(d) R is not reflexive and not symmetric

45. If R is a symmetric and transitive relation on a set A, then

(a) R is reflexive and hence an equivalence relation
(b) R is reflexive and hence a partial order
(c) R is not reflexive and hence not an equivalence relation
(d) none of these

46. The number of elements in the power set $P(S)$ of the set $S = \{[\phi], 1, \{2, 3\}\}$ is

(a) 2 (b) 4
(c) 8 (d) none of the these

47. Let S be an infinite set and $S_1, S_2, S_3, \ldots, S_n$ be set such that

$$S_1 \cup S_2 \cup S_3 \cup \ldots S_n = S, \text{ then}$$

(a) at least one of the set S_i is a finite set
(b) not more than one of the set S_i is a finite set
(c) at least one of the set S_i is an infinite set
(d) none of the these

48. If A is a finite set of size n, then number of elements in the power set of $A \times A$ is

(a) 2^{2^n} (b) 2^{n^2}
(c) 2^{2n} (d) none of these

49. If A and B are sets with cardinalities m and n, respectively, then the number of one-to-one mappings from A to B, when $m < n$, is

(a) m^n (b) ${}^nP_m$
(c) ${}^nC_m$ (d) none of these

50. Consider the following relations:

$R_1(a, b)$ if $(a + b)$ is even over the set of integers

$R_2(a, b)$ if $(a + b)$ is odd over the set of integers

$R_3(a, b)$ if $a \cdot b > 0$ over the set of non-zero rational numbers

$R_4(a, b)$ if $|a - b| \leq 2$ over the set of natural numbers

Which of the following statement is correct?

(a) R_1 and R_2 are equivalence relations, R_3 and R_4 are not
(b) R_1 and R_3 are equivalence relations, R_2 and R_4 are not
(c) R_1 and R_4 are equivalence relations, R_2 and R_3 are not
(d) R_1, R_2, R_3 and R_4 are all equivalence relations

51. Consider the following statements:

S_1: There exist infinite sets A, B and C such that $A \cap (B \cup C)$ is finite.

S_2: There exist two irrational numbers x and y such that $(x + y)$ is rational. Which of the following is true about S_1 and S_2?

(a) Only S_1 is correct
(b) Only S_2 is correct
(c) Both S_1 or S_2 are correct
(d) None of S_1 and S_2 are correct

52. Let f: $A \rightarrow B$ be a function, and let E and F be subsets of A. Consider the following statements about images.

S_1: $f(E \cup F) = f(E) \cup f(F)$

S_2: $f(E \cap F) = f(E) \cap f(F)$

Which of the following is true about S_1 and S_2?

(a) Only S_1 is correct
(b) Only S_2 is correct
(c) Both S_1 and S_2 are correct
(d) None of S_1 and S_2 are correct

53. Let R_1 and R_2 be two equivalence relations on a set. Consider the following statements:

S_1: $R_1 \cup R_2$ is an equivalence statement.

S_2: $R_1 \cap R_2$ is an equivalence statement.

(a) Both statements are true
(b) S_1 is true but S_2 is not true
(c) S_2 is true but S_1 is not true
(d) S_1 and S_2 are false

54. The binary relation $r = \{(1, 1), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}$ on the set $A = \{1, 2, 3, 4\}$ is

(a) reflexive, symmetric and transitive
(b) neither reflexive nor irreflexive but transitive
(c) irreflexive, symmetric and transitive
(d) irreflexive and anti-symmetric

55. In a room containing 28 people, there are 18 who speak English, 15 who speak Hindi and 22 who speak Kannad, 9 who speak both Hindi and English, 11 who speak both Hindi and Kannad and 13 who speak both Kannad and English. How many speak all the three languages?

(a) 6 (b) 7
(c) 8 (d) 9

56. Let x and y be sets and $|x|$ and $|y|$ are their respective cardinalities. It is given that there are exactly 97 functions from x to y. From this, one can conclude that

(a) $|x| = 1$, $|y| = 97$ (b) $|x| = 97$, $|y| = 1$
(c) $|x| = 97$, $|y| = 97$ (d) none of these

57. Let R denote the set of real numbers and f: $R \times R \rightarrow R \times R$ be a bijective function defined by $f(x, y) = f(x + y, x - y)$. The inverse of f is given by

(a) $f^{-1}(x,y) = \left[\dfrac{1}{(x+y)}, \dfrac{1}{(x-y)}\right]$

(b) $f^{-1}(x,y) = (x - y, x + y)$

(c) $f^{-1}(x,y) = \left[\dfrac{x+y}{2}, \dfrac{x-y}{2}\right]$

(d) $f^{-1}(x,y) = [2(x - y), 2(x + y)]$

58. Which of the following is true?

(a) The set of all rational negative numbers forms a group under multiplication
(b) The set of all non-singular matrices forms a group under multiplication

(c) The set of all matrices forms a group under multiplication
(d) Both (b) and (c)

59. Let $(Z, *)$ be an algebraic structure, where Z is the set of integers and the operation $*$ is defined by $n * m = \text{maximum } (n, m)$. Which of the following statement is true for $(Z, *)$?

(a) $(Z, *)$ is a monoid
(b) $(Z, *)$ is an abelian group
(c) $(Z, *)$ is a group
(d) None of these

60. Which of the following statement is false?

(a) The set of rational numbers is an abelian group under addition
(b) The set of rational integers is an abelian group under addition
(c) The set of rational numbers forms an abelian group under multiplication
(d) None of these

61. Let A be the set of all non-singular matrices over real numbers and let * be the matrix multiplication operator. Then

(a) A is close under * but $<A, *>$ is not a semigroup.
(b) $<A, *>$ is a semigroup but not a monoid.
(c) $<A, *>$ is a monoid but not a group.
(d) $<A$, *ø is a group but not an abelian group.

62. If a group $(G, 0)$ is known to be abelian, then which one of the following option is true for G?

(a) $g = g^{-1}$ for every $g \in G$
(b) $g = g^2$ for every $g \in G$
(c) $(goh)^2 = g^2 oh^2$ for every $g, h \in G$
(d) G is of finite order

63. Match the following:

A. Groups	1. Associativity
B. Semigroups	2. Identity
C. Monoids	3. Commutative
D. Abelian groups	4. Left inverse

Codes

	A	B	C	D
(a)	4	1	2	3
(b)	3	1	4	1
(c)	2	3	1	4
(d)	1	2	3	4

64. The simultaneous equations on the Boolean variables x, y, z and w

$$x + y + z = 1, \qquad xy = 0$$
$$xz + w = 1, \qquad xy + \bar{z} \cdot \bar{w} = 0$$

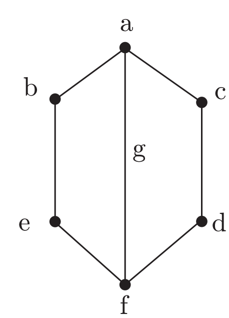
have the following solution for x, y, z and w, respectively

(a) 0 1 0 0
(b) 1 1 0 1
(c) 1 0 1 1
(d) 1 0 0 0

65. Let L be a set with a relation R which is transitive, anti-symmetric and reflexive and for any two elements $a, b \in L$. Let least upper bound (a, b) and greatest lower bound (a, b) exist. Which of following option(s) is/are true?

(a) L is a poset
(b) L is a Boolean algebra
(c) L is a lattice
(d) Both (a) and (c)

66. In the lattice defined by the Hasse diagram, how many complements does the element "e" have?



(a) 2 (b) 3
(c) 0 (d) 1

67. A partial order $\leq$ is defined on the set

$$S = \{x, a_1, a_2, a_3, \ldots, a_n, y\}$$

as $x \leq a_i$ for all i and $a_i \leq y$ for all i, where $n \geq 1$. The number of total order on the set S which contains partial order $\leq$ is

(a) 1 (b) n
(c) $n + 2$ (d) $n!$

68. What values of A, B, C and D satisfy the following simultaneously Boolean equations?

$$\overline{A} + AB = 0, \;\; AB = AC, \;\; AB + A\overline{C} + CD = \overline{CD}$$

(a) $A = 1, B = 0, C = 0, D = 1$
(b) $A = 1, B = 1, C = 0, D = 0$
(c) $A = 1, B = 0, C = 1, D = 1$
(d) $A = 1, B = 0, C = 0, D = 0$

ANSWERS

1. (b)	**13.** (c)	**25.** (b)	**37.** (b)	**49.** (b)	**61.** (d)
2. (a)	**14.** (b)	**26.** (b)	**38.** (d)	**50.** (b)	**62.** (c)
3. (c)	**15.** (d)	**27.** (a)	**39.** (c)	**51.** (a)	**63.** (a)
4. (a)	**16.** (c)	**28.** (a)	**40.** (d)	**52.** (a)	**64.** (b)
5. (d)	**17.** (b)	**29.** (c)	**41.** (c)	**53.** (c)	**65.** (d)
6. (b)	**18.** (b)	**30.** (a)	**42.** (b)	**54.** (b)	**66.** (b)
7. (d)	**19.** (a)	**31.** (d)	**43.** (a)	**55.** (a)	**67.** (d)
8. (c)	**20.** (a)	**32.** (b)	**44.** (b)	**56.** (a)	**68.** (a), (d)
9. (c)	**21.** (b)	**33.** (c)	**45.** (d)	**57.** (c)	
10. (a)	**22.** (a)	**34.** (c)	**46.** (c)	**58.** (b)	
11. (d)	**23.** (d)	**35.** (d)	**47.** (c)	**59.** (d)	
12. (a)	**24.** (a)	**36.** (d)	**48.** (b)	**60.** (d)	

EXPLANATIONS AND HINTS

1. (b) We have

$$fog(x) = \sin x^2$$
$$gof(x) = \sin^2 x$$
$$f(g(x)) = \sin (x^2) \text{ and } g(f(x)) = (\sin x)^2$$
$$\Rightarrow f(x) = \sin x \text{ and } g(x) = x^2$$

2. (a) We have

$$f(x) = x^2 + 1$$
$$g(x) = \sin x$$

Now, $x^2 \geq 0$ for all $x \in R$

$$\Rightarrow x^2 + 1 \geq 1 \text{ for all } x \in R$$
$$\Rightarrow f(x) \geq 1 \text{ for all } x \in R$$

Range of $f = [1, \infty]$.

Also, $-1 \leq \sin x \leq 1$ for all $x \in R$.

Range of $g = [-1, 1]$.

Clearly, range of $f \subseteq$ domain of g and range of $g \subseteq$ domain of f.

So, gof: $R \to R$ and fog: $R \to R$ are given by

$$gof(x) = g(f(x)) = g(x^2 + 1) = \sin x^2 + 1$$

and $fog(x) = f(g(x)) = f(\sin x) = \sin^2 x + 1$

3. (c) It is clear that ϕ and $\{\phi\} \in A$

However, $\{1\} \notin A$.

Also, $\{2, \phi\}$ is a subset of A.

Hence, $\{1\} \notin A$ is a false statement.

4. (a) Let A and B be two sets having x and y elements, respectively. The number of subsets of $A = 2^x$.

Similarly, the number of subsets of $B = 2^y$.

We know that

$$2^x + 2^y = 56$$
$$2^y\,(2^{x-y} - 1) = 2^3(2^3 - 1) = 56$$
$$\therefore 2^3 = 2^y \Rightarrow y = 3$$
$$x - y = 3$$
$$x = 3 + 3 = 6$$
$$\Rightarrow x = 6,\ y = 3$$

5. (d) Suppose A, B and C denote the set of students who got prizes in football, basketball and cricket, respectively.

Then, we have

$n(A) = 38$, $n(B) = 15$, $n(C) = 20$, $n(A \cup B \cup C) = 58$ and $n(A \cap B \cap C) = 3$

Also, we know that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$\Rightarrow 59 = 38 + 15 + 20 - n(A \cap B) - n(A \cap C) - n(B \cap C) + 3$$

$$\Rightarrow n(A \cap B) + n(A \cap C) + n(B \cap C) = 18$$

Now, number of men who received medals in exactly two sports

$$= n(A \cap B) + n(A \cap C) + n(B \cap C) - 3n(A \cap B \cap C)$$
$$= 18 - (3 \times 3) = 9$$

6. (b) Suppose A denotes the set of students who have taken mathematics and B denotes the set of students who have taken computers.

Then, $n(A \cup B) = 35$, $n(A) = 17$ and $n(A - B) = 10$.

$$n(A) = n(A - B) + n(A \cap B)$$
$$\Rightarrow 17 = 10 + n(A \cap B)$$
$$\Rightarrow n(A \cap B) = 17 - 10 = 7$$

7. (d) We know that R is an equivalence relation on A, then by definition R^{-1} is also an equivalence relation.

Hence, R^{-1} is reflexive, symmetric and transitive.

8. (c) Number of subsets of a set of order $3 = 2^2 = 8$.

9. (c) If the union and intersection of any two sets is same, then the sets are equal. Hence, $A = B$.

10. (a) All the other options apart from the first one have elements, hence they are not null or empty sets.

11. (d) We have

$$R = \{(a, b): a \text{ divides } b;\ a, b \in N\}$$

Now, for any $a \in R$, we have

a divides a

$$\Rightarrow (a, a) \in R \text{ for all } a \in R$$
$$\Rightarrow R \text{ is reflexive.}$$

For any $a, b \in R$, if a divides b, then it is not necessary that b divides a

$$\Rightarrow R \text{ is not symmetric.}$$

Let $(a, b) \in R$ and $(b, c) \in 0$. Then,

$$a \text{ divides } b \text{ and } b \text{ divides } c$$
$$\Rightarrow a \text{ divides } c \Rightarrow (a, c) \in R$$

Thus, R is transitive.

12. (a) We have

$$X \cap (Y \cup X)^C$$
$$= X \cap (Y^C \cap X^C) \quad \text{[Applying De Morgan's law]}$$
$$= X \cap Y^C \cap X^C$$
$$= \phi \cap Y^C \ [\because X \cap X^C = \phi]$$
$$= \phi \ [\because \phi \cap A = \phi]$$

13. (c) The identity function I_A: $X \to X$ is defined as

$$I_X(a) = a \text{ for all } a \in X.$$

Let a and b be any two elements of x. Then

$$I_X(a) = I_x(b) \Rightarrow a = b$$

Hence, I_x is an injective mapping.

Let $b \subset X$. Then, there exists $a = b \in X$ such that

$$I_X(a) = a = b$$

Hence, I_X is a surjective mapping.

Therefore, I_X: $X \to X$ is a bijection.

14. (b) We have

f: $N \to N$ defined by $f(n) = 2n + 3$

Let x and y be any two elements of N. Then

$$f(x) = f(y)$$
$$\Rightarrow 2x + 3 = 2y + 3 \Rightarrow 2x = 2y \Rightarrow x = y$$

Hence, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in N$.

So, f is injective.

Now, let y be an arbitrary element of N. Then

$$f(x) = y$$
$$2x + 3 = y$$
$$x = \frac{y-3}{2}$$

Now, $\frac{y-3}{2}$ does not belong to N for all values of $y \in N$.

Hence, f is not surjective.

Therefore, f is injective and not surjective.

15. (d) Total elements on set = 3

$$\text{Total distinct relations} = 2^{3\times3} = 2^9 = 512$$

16. (c) De Morgan's law can also be expressed as

$$A - (B \cup C) = (A - B) \cap (A - C)$$

17. (b) $A * A = A \cdot A + \bar{A} \cdot \bar{A}$

$$= A + \bar{A} = 1$$

18. (b) $C = A * B$

$= AB + \bar{A}\cdot\bar{B}$

Then,

$$C * A = (AB + \bar{A}\cdot\bar{B})\cdot A + (\overline{AB + \bar{A}\cdot\bar{B}})\cdot\bar{A}$$
$$= (A\cdot A\cdot B + \bar{A}\cdot\bar{B}\cdot A) + (\overline{AB}\cdot\overline{\bar{A}\cdot\bar{B}})\cdot\bar{A}$$
$$= AB + 0 + ((\bar{A}+\bar{B})\cdot(A+B)\cdot\bar{A})$$
$$= AB + (\bar{A}\cdot A + \bar{A}B + \bar{B}A + \bar{B}\cdot B)\bar{A}$$
$$= AB + (\bar{A}\cdot B\cdot\bar{A} + \bar{B}\cdot A\cdot\bar{A})$$
$$= AB + (\bar{A}\bar{B}) = B(A+\bar{A})$$
$$= B$$

19. (a) We have two Boolean equations,

$$AB + \bar{A}C = 1 \quad (1)$$
$$AC + B = 0 \quad (2)$$

From Eq. (2),

$AC = 0$ and $B = 0$

Using these in Eq. (1), we get

$\bar{A}C = 1$ and $AC = 0$

$\Rightarrow \bar{A}C + AC = 1 + 0 \Rightarrow C(A + \bar{A}) = 1 \Rightarrow C = 1$

$\therefore A(1) = 0$

$\Rightarrow A = 0$

20. (a) As already calculated in the previous question, $B = 0$.

21. (b) As already calculated in Question 19, $C = 1$.

22. (a) Let A, B and C denote the sets of people who like Coca Cola, Pepsi and Fanta, respectively.

Thus, we know that

$n(A) = 80$, $n(B) = 60$, $n(C) = 50$, $n(A\cap B) = 30$, $n(A\cap C) = 20$,

$n(B\cap C) = 15$, $n(A\cap B\cap C) = 10$

Therefore, $n(A\cup B\cup C) = 80 + 60 + 50 - 30 - 20 - 15 + 10 = 135$

23. (d) Total people who like at least two drinks

$= n(A\cap B) + n(A\cap C) + n(B\cap C) - 2n(A\cap B\cap C)$

$= 30 + 20 + 15 - 2\ (10)$

$= 45$

24. (a) Let (a_1, b_1) and $(a_2, b_2) \in A\times B$ such that

$$f(a_1, b_1) = f(a_2, b_2)$$
$$\Rightarrow (b_1, a_1) = (b_2, a_2)$$
$$\Rightarrow b_1 = b_2 \text{ and } a_1 = a_2$$
$$\Rightarrow (a_1, b_1) = (a_2, b_2)$$

Thus, $f(a_1, b_1) = f(a_2, b_2) \Rightarrow (a_1, b_1) = (a_2, b_2)$ for all (a_1, b_1) $(a_2, b_2) \in A\times B$

So, f is injective.

Let (b, a) be an arbitrary element of $B\times A$. Then, $b\in B$ and $a\in A$.

$$\Rightarrow (a, b)\in A\times B$$

Thus, for all $(b, a)\in B\times A$ there exists $(a, b)\in A, B$ such that

$$f(a, b) = (b, a)$$

Therefore, f: $A\times B\to B\times A$ is surjective.

Hence, f is bijective.

25. (b) By definition, the domain and range of an identity function is same.

26. (b) For any two sets, X and Y, minimum number of elements of $X\cup Y$ is the elements in the bigger set.

Therefore, answer is 6.

27. (a) For any two sets, X and Y, maximum number of elements of $X\cap Y$ is the elements in the smaller set.

Hence, answer is 3.

28. (a) By definition, Hasse diagrams are used to represent partially ordered sets.

29. (c) It is clear from the poset that 3 and 5 are the maximal, and 1 and 6 are the minimal.

30. (a) The universal relation $A\times A$ on A is an equivalence relation.

31. (d) The only upper bound of 10 and 15 is 30. Hence, L.U.B. is 30.

32. (b) The greatest lower bound of 10 and 15 is 6.

33. (c) We have

$$(fof^{-1})(x) = x \text{ for all } x\in[1,\infty)$$

$\Rightarrow (fof^{-1})(x) = x$

$\Rightarrow 2^{f^{-1}(x)\{f^{-1}(x)-1\}} = x$

$\Rightarrow f^{-1}(x)\{f^{-1}(x)-1\} = \log_2 x$

$\Rightarrow \{f^{-1}(x)\}^2 - f^{-1}(x) - \log_2 x = 0$

$$\Rightarrow f^{-1}(x) = \frac{1\pm\sqrt{1+4\log_2 x}}{2} = \frac{1+\sqrt{1+4\log_2 x}}{2}$$
$$\left[\because f^{-1}(x)\in(1,\infty), f^{-1}(x)\ge 1\right]$$

34. (c) Let e be identity element. Then

$$a * e = a = e * a \text{ for all } a\in Q$$

$$\Rightarrow \frac{ae}{5} = a \text{ and } \frac{ea}{5} = a \text{ for all } a \in Q$$

$$\Rightarrow e = 5$$

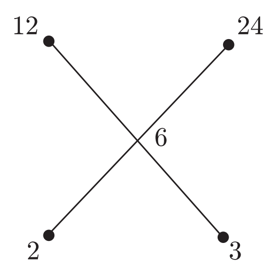
35. (d) If $f(x)$ and $g(x)$ have domains A and B, respectively, then domain of $f(x) + g(x)$ is $A \cap B$.

36. (d) In mathematics, a binary relation S over a set A is transitive if an element a is related to an element b, b is in turn related to an element c, and then a is also related to c.

A binary relation S over a set A is symmetric if it holds for all a and b in A that if a is related to b then b is related to a.

Hence, S is transitive and symmetric; however, it is not reflexive since (1, 1), (2, 2) and (3, 3) $\notin S$.

37. (b) The Hasse diagram for $(X, \leq)$ is given by



The number of edges in the Hasse diagram is 4.

38. (d) Let $S = \{a_1, a_2, \ldots, a_n\}$ where n is a finite number, then the power set of S is given by

$$P(S) = \{a_1, a_2, \ldots, a_n, \{a_1, a_2\}, \ldots, \{a_1, a_2, \ldots, a_n\}\}$$

Hence, $S \in P(S)$ is always true.

39. (c) The number of distinct relations are given by

$$2^{n.n} = 2^{n^2}$$

40. (d) Largest equivalence relation can include all the n elements of A and the relation will be with the elements of A itself. Hence, the largest equivalent relation has $n * n = n^2$.

41. (c) Functions are defined from m to n elements

Hence, the total number of functions $= n^m$.

42. (b) We just need to calculate the number of ways of placing the four elements of our set into the following sized bins:

$$4,\ 3 + 1,\ 2 + 2,\ 2 + 1 + 1,\ 1 + 1 + 1 + 1$$

There is just one way to put four elements into a bin of size 4.

These are four ways to assign the four elements into one bin of size 3 and of size 1.

There are three ways to assign the four elements into one bin of size 2. In this, two of the elements are related to each other and the other two are related to themselves.

There are six ways to assign four elements into one bin of sizes 2, 1 and 1.

There is just one way to assign the four elements into bins of sizes 1, 1, 1 and 1.

Total equivalence relations = 1 + 4 + 3 + 6 + 1 = 15.

43. (a) We know that

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Hence,

$$(A - B) \cup (B - A) \cup (A \cap B) = (A \cup B) - (A \cap B) \cup (A \cap B)$$

Let $X = (A \cup B), Y = (A \cap B)$

Also, $(X - Y) \cup Y = X \cup Y$

Substituting values of X and Y, we get

$$(A - B) \cup (B - A) \cup (A \cap B) = (A \cup B) \cup (A \cap B) = A \cup B$$

44. (b) Solution: We know that $A\ R\ B$ if and only if $A \cap B = \phi$.

$$A \cap A \neq \phi$$

Hence, R is not reflexive.

$$A \cap B = B \cap A$$

Hence, R is symmetric.

If $A \cap B$ and $B \cap C$, then it is not necessary that $A \cap C$.

Hence, R is not transitive.

Therefore, R is symmetric and not transitive.

45. (d) We know that R is symmetric and transitive on set A. However, it does not say anything about reflexive. Hence, the correct option is (d).

46. (c) The power set $P(S)$ for $S = \{\{\phi\}, 1, \{2, 3\}\}$ is

$\{\phi, 1, \{2, 3, 2, 3\}\} = \{[\phi], [1], [1, 2], [1, 3], [1, 2, 3], [2], [3], [2, 3]\}$

Hence, $P(S)$ contains 8 elements.

47. (c) We know that the union of two sets contains the elements of the bigger set.

Hence, if S_i is the union of n sets and its value is infinity, then the value of at least one set is infinity. Hence, the correct option is (c).

48. (b) Total elements in $A = n$.

Total elements in $A \times A = n^2$.
Total elements in power set of $A \times A = 2^{n^2}$.

49. (b) We are given that A has m elements and B has n elements. Also, $n > m$.

Now, the one-to-one mappings have a distinct value for each district argument.

Therefore, the total number of one-to-one mappings is ${}^n p_m$.

50. (b) $R_1(a, b)$ if $(a + b)$ is even over set of integers.

We know that

$a + a = 2a$ which is even. Hence, $(a, a) \in R_1$ for $a \in I$

Therefore, R_1 is reflexive.

If $a + b$ is even, then $b + a$ is also even. Hence, $(b, a) \in R_1$

Therefore, R_1 is symmetric.

If $a + b$ is even and $b + c$ is even, then $a + c$ is also even. Hence, $(a, b) \in R_1$, $(b, c) \in R_1$ and $(a, c) \in R_1$.

Therefore, R_1 is transitive.

Hence, R_1 is an equivalence relation.

$R_2(a, b)$ if $(a + b)$ is odd over set of integers.

However, $(a + a)$ is not odd. Hence, $(a, a) \notin R_1$. Therefore, R_1 is not reflexive and hence not an equivalence relation.

$R_3(a, b)$ if $(a, b) > 0$ over set of non-zero rational numbers.

If a is non-zero, then $a^2 > 0$ for all values of a being non-zero rational numbers. Hence, $(a, a) \in R_3$.

Thus, R_3 is reflexive.

If $a \cdot b > 0$, then $b \cdot a > 0$. Hence, $(b, a) \in R_3$.

Thus, R_3 is symmetric.

If $a \cdot b > 0$ and $b \cdot c > 0$, then $(a \cdot c) > 0$. Hence, $(a, c) \in R_3$.

Therefore, R_3 is transitive.

Hence, R_3 is an equivalence relation.

$R_4(a, b)$ if $|a - b| \leq 2$ over set of natural numbers.

$|a - a| = 0$, i.e. $|a - a| \leq 2$. Hence, $(a, a) \in R_4$ for $a \in N$

Thus, R_4 is reflexive.

Also, $|a - b| = |b - a|$. Hence, R_4 is symmetric.

However, if $|a - b| \leq 2$ and $|b - c| \leq 2$, then it is not necessary that $|a - c| \leq 2$. Hence, R_4 is not an equivalence relation.

We conclude that R_1 and R_3 are equivalence relations and R_2 and R_4 are not equivalence relations.

51. (a) S_1: There exist infinite sets A, B and C such that $A \cap (B \cup C)$ is finite.

The union of two infinite sets is always infinite. However, the intersection of two infinite sets can be finite.

Example: $A = \{0\} \cup \{3, 4, 5, \ldots\}$

$B = \{1\} \cup \{3, 4, 5, \ldots\}$

$C = \{2\} \cup \{3, 4, 5, \ldots\}$

$A \cap (B \cup C) = \{3, 4, 5, \ldots\}$

S_1 is true.

S_2: There exist two irrational numbers x and y such that $(x + y)$ is rational.

Sum of two irrational numbers is always irrational. Hence, S_2 is false.

52. (a) S_1 is true since both $f(E \cup F)$ and $f(E) \cup f(F)$ will contain exactly the same images of all elements in E and F.

Now, let f be a constant function such that its range is $\{1\}$. Now, let E and F be two partitions of set A, then $(A \cap B) = \phi$ and $f(\phi)$ is not defined, but $f(E) \cap f(F)$ is $\{1\}$. Hence, S_2 is false.

53. (c) Consider R_1 and R_2 to be relations on set A.

$\therefore R_1 \subseteq A \times A$ and $R_2 \subseteq A \times A$

$R_1 \cap R_2 \subseteq \text{A} \times \text{A}$

$\therefore R_1 \cap R_2$ is also a relation on A.

Let $a \in A$

$\Rightarrow (a, a) \in R_1$ and $(a, a) \in R_2$ [$\because R_1$ and R_2 are reflexive]

$\Rightarrow (a, a) \in R_1 \cap R_2$

Hence, $R_1 \cap R_2$ is reflexive.

Let $a, b \in A$ such that $(a, b) \in R_1 \cap R_2$. Then

$(a, b) \in R_1 \cap R_2$

$\Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2 \Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2$ [$\because$ R_1 and R_2 are symmetric]

$\Rightarrow (b, a) \in R_1 \cap R_2$

Hence, $R_1 \cap R_2$ is symmetric.

Let $a, b, c \in A$ such that $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$. Then,

$(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$

$\{((a, b) \in R_1 \text{ and } (a, b) \in R_2)\}$ and $\{((b, c) \in R_1$ and $(b, c) \in R_2)\}$

$\{(a, b) \in R_1, (b, c) \in R_1\}$ and $\{(a, b) \in R_2, (a, b) \in R_2\}$

$\Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2 \Rightarrow (a, c) \in R_1 \cap R_2$

Thus, $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2 \Rightarrow (a, c) \in R_1 \cap R_2$

So, $R_1 \cap R_2$ is transitive on A.

Hence, $R_1 \cap R_2$ is an equivalence relation on A.

Now, let $A = a, b, c$ and R_1 and R_2 be two relations on A given by

$R_1 = \{(a, a), (b, b), (c, c), (a, b)\ (b, a)\}$ and $R_2 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$

R_1 and R_2 both are equivalence relations on A. But $R_1 \cap R_2$ is not transitive because $(a, b) \in R_1 \cup R_2$ and $(b,c) \in R_1 \cup R_2$ but $(a, c) \notin R_1 \cup R_2$.

Hence, $R_1 \cup R_2$ is not an equivalence relation on A.

Therefore, S_1 is false and S_2 is true.

54. (b) The relation is not reflexive because it does not contain the entry (4, 4). It is not irreflexive because it contains the entries (1, 1), (2, 2) and (3, 3). Hence, the only possible answer is (b).

However, it can also be proved that the relation r is transitive since (2, 3), (3, 4) and (2, 4) $\in r$.

Hence, r is neither reflexive, nor irreflexive but transitive.

55. (a) We know that

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$28 = 18 + 15 + 22 - 9 - 11 - 13 + n(A \cap B \cap C)$

$\Rightarrow n(A \cap B \cap C) = 28 - 55 + 33 = 6$

Hence, the total number of people who speak all the three languages = 6.

56. (a) We know that there are exactly 97 functions from x to y. Now, the total number of element is $x = 1$ and $y = 97$.

57. (c) $f^{-1}(x,y) = \left[\dfrac{x+y}{2}, \dfrac{x-y}{2}\right]$

58. (b) Set of all rational negative numbers does not form a group under multiplication because it is not closed and does not contain an identity element.

Set of all non-singular matrices forms a group under multiplication because it is associative, identity element exists and is invertible.

However, set of all matrices does not form a group under multiplication since all matrices are not invertible.

Hence, the correct option is (b).

59. (d) Z is closed and associative, but it does not have an identity element. Hence, it is just a semigroup. Thus, none of the options are true.

60. (d) For a group to be abelian, it should be commutative. Now, addition of numbers and integers is commutative. Also, multiplication of rational numbers is commutative. Hence, all the statements are true.

61. (d) A is a group because $a*(b*c) = (a*b)*c$ for every $a, b, c \in A$ and every element of A is invertible as A is a set of non-singular matrices. However, A is not an abelian group because multiplication of matrices is not commutative. Hence, $<A, *>$ is a group but not an abelian group.

62. (c) Let us assume that there exists two consecutive integers, n and $n + 1$, such that

$(goh)^n = g^n oh^n$ and $(goh)^{n+1} = g^{n+1} oh^{n+1}$ for every $g, h \in G$.

Then,

$$g^{n+1}oh^{n+1} = (goh)^{n+1} = (goh)^n\ goh = g^n oh^n goh$$

Hence,

$$g^{n+1}oh^{n+1} = g^n oh^n\,(goh)$$

$$\Rightarrow goh^n = h^n og$$

Taking $n = i$ and $n = i + 1$, we get $goh^i = h^i og$ and by taking $n = i + 1$ and $n = i + 2$, we have $goh^{i+1} = h^{i+1} og$.

Hence, this shows that $goh^{i+1} = h^{i+1} oa \Rightarrow goh = hog$. Thus, G is abelian; hence,

$$(goh)^2 = g^2 oh^2 \text{ for every } g, h \in G.$$

63. (a) By definition, a semigroup with identity is a monoid.

A semigroup follows associativity and an abelian group follows commutative.

Then the correct matching would be $A = 4$, $B = 1$, $C = 2$ and $D = 3$.

64. (b) The given equations are

$$x + y + z = 1 \quad (1)$$

$$xy = 0 \quad (2)$$

$$xz + w = 1 \quad (3)$$

$$xy + \bar{z} \cdot \bar{w} = 0 \quad (4)$$

For Eq. (2) to be true, both x and y cannot be 1. Hence, answer cannot be (b).

Substituting the value of Eq. (2) in Eq. (4), we get

$$\bar{z} \cdot \bar{w} = 0$$

Hence, z and w cannot be 0. Therefore, answer cannot be (a) and (d).

Thus, the answer is (b).

65. (d) Because L is a set with a relation R which is transitive, anti-symmetric and reflexive for any two elements a, $b \in L$, it is a partially ordered set or a poset.

Because L.U.B. (least upper bound) and G.L.B. (greatest lower bound) exist, L is a lattice. Hence, the correct option is (d).

66. (b) e has three complements a, c, d.

67. (d) The total order on the set S which contains partial order $\leq$ is given by

$$\begin{aligned}&(a_1,a_1),(a_1,a_2),\ldots,(a_1,a_n)\\&(a_2,a_2),(a_2,a_3),\ldots,(a_1,a_n)\\&\vdots\\&(a_n,a_n)\end{aligned}$$

Hence, total orders are $n(n-1)(n-2)\ldots n = n!$

68. (a), (d) We are given

$$\overline{A} + AB = 0 \qquad (1)$$

$$AB = AC \qquad (2)$$

$$AB + A\overline{C} + CD = \overline{CD} \qquad (3)$$

Considering Eq. (1),

$$\overline{A} + AB = 0$$

$$\Rightarrow \overline{A} + B = 0 \qquad [\because \overline{X} + Y = 0] \quad (4)$$

Considering Eq. (2),

$$AB = AC$$

$$\Rightarrow B = C \qquad (5)$$

Now, because all the options have $A = 1$, we substitute $A = 1$ in Eq. (4),

$$\overline{1} + B = 0 \Rightarrow B + 0 = 0 \Rightarrow B = 0$$

Substituting value of B in Eq. (5), we get

$$C = 0$$

Substituting values of A, B and C in Eq. (3), we get

$$1.0 + 1.\overline{0} + CD = \overline{CD} \Rightarrow \overline{CD} = 1 + CD = 1$$

For $C = 0$ and $D = 0$, 1, the Boolean equations will be satisfied. Hence, the correct options are (a) and (d).

Ans. (a), (d)

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. Consider the binary relation:

$$S = \{(x, y) | y = x + 1 \text{ and } x, y \in \{0, 1, 2\}\}$$

The reflexive transitive closure of S is

(a) $\{(x, y) | y > x \text{ and } x, y \in \{0, 1, 2\}\}$
(b) $\{(x, y) | y \geq x \text{ and } x, y \in \{0, 1, 2\}\}$
(c) $\{(x, y) | y < x \text{ and } x, y \in \{0, 1, 2\}\}$
(d) $\{(x, y) | y \leq x \text{ and } x, y \in \{0, 1, 2\}\}$

(GATE 2004, 1 Mark)

Solution: Reflexive closure of a relation R on set X is the smallest reflexive relation which contains R. So to find reflexive closure of R, we just take its union with set $\{(a,a)\, \forall a \in X\}$.

We are given $S = \{(0, 1), (1, 2)\}$

We calculate the reflexive by taking its union with set $\{(0, 0), (1, 1), (2, 2)\}$. Thus, $S_R = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2)$.

The transitive closure is also defined in the similar way, i.e. smallest transitive relation which contains S. So we try to make S_R transitive now. For this, we have to see where does it violate property of transitivity, and then add appropriate pair. So here we see that we have (0, 1) and (1, 2), but not (0, 2), so we add that to make relation $S_{RT} = \{(0, 0), (0, 1), (0, 2), (1, 1), (1, 2), (2, 2)\}$.

Now, we can see that S_{RT} is transitive. Among the given options, only (b) matches S_{RT}, hence it is the correct answer.

Ans. (b)

2. The inclusion of which of the following options sets into

$$S = \{\{1, 3\}, \{1, 2, 3\}, \{1, 3, 5\}, \{1, 2, 4\}, \{1, 2, 3, 4, 5\}\}$$

is necessary and sufficient to make S a complete lattice under the partial order defined by set containment?

(a) {1} (b) {1}, {2, 3}
(c) {1}, {1, 3} (d) {1}, {1, 3}, {1, 2, 3, 4}, {1, 2, 3, 5}

(GATE 2004, 2 Marks)

Solution: A lattice is complete if every subset of partial order set has a supremum and infimum element. For example, here we are given a partial order set S. Now it will be a complete lattice whatever the subset we choose, it has a supremum and infimum element. Here, supremum element will be the union of all sets in the subset we choose and infimum element will be the intersection of all the sets in the subset we choose.

Thus, we take the two subsets, {1, 3, 5} and {1, 2, 4} and find their intersection, i.e. {1} which is not present in S.

Ans. (a)

3. Let A, B and C be non-empty sets and let

$X = (A - B) - C$ and $Y = (A - C) - (B - C)$

Which one of the following is TRUE?

(a) $x = y$ (b) $x \subset y$
(c) $y \subset x$ (d) None of these

(GATE 2005, 1 Mark)

Solution: We know that $(A - B) = A \cap B'$, where B' is complement of set B.

So, $X = (A - B) - C = (A \cap B') \cap C' = A \cap B' \cap C'$

Similarly,

$$
\begin{aligned}
Y &= (A - C) - (B - C) \\
&= (A \cap C') \cap (B \cap C')' \\
&= (A \cap C') \cap (B' \cup C) \\
&= (A \cap C' \cap B') \cup (A \cap C' \cap C) \\
&= (A \cap C' \cap B') \cup (A \cap \phi) \\
&= (A \cap C' \cap B') \cup \phi = (A \cap C' \cap B') = X
\end{aligned}
$$

Hence, the answer is (a).

Ans. (a)

4. Let X, Y, Z be sets of sizes x, y and z, respectively. Let $W = X \times Y$ and E be the set of all subsets of W. The number of functions from Z to E is

(a) $Z^{2^{xy}}$ (b) $Z \times 2^{xy}$
(c) $Z^{2^{x+y}}$ (d) Z^{xyz}

(GATE 2006, 1 Mark)

Solution: Number of elements in $W = 2^{xy}$

Number of elements in $Z = z$

Number of functions from Z to $E = z^{2xy}$

Ans. (a)

5. If P, Q, R are subsets of the universal set U, then $(P \cap Q \cap R) \cup (P^C \cap Q \cap R) \cup Q^C \cup R^C$ is

(a) $Q^C \cup R^C$ (b) $P \cup Q^C \cup R^C$
(c) $P^C \cup Q^C \cup R^C$ (d) U

(GATE 2008, 1 Mark)

Solution: We are given

$$
\begin{aligned}
&(P \cap Q \cap R) \cup (P^C \cap Q \cap R) \cup Q^C \cup R^C \\
&= (P \cup P^C) \cap (Q \cap R) \cup (Q^C \cup R^C) \\
&= U \cap ((Q \cap R) \cup (Q \cap R)^C) \\
&= U \cap U = U
\end{aligned}
$$

Ans. (d)

6. Which one of the following in NOT necessarily a property of a Group?

(a) Commutativity
(b) Associativity
(c) Existence of inverse for every element
(d) Existence of indentity

(GATE 2009, 1 Mark)

Solution: By definition in a group, commutativity is not necessary.

Ans. (a)

7. Consider the binary relation $R = \{(x, y), (x, z), (z, y), (z, y)\}$ on the set $\{x, y, z\}$.

Which one of the following is TRUE?

(a) R is symmetric but NOT antisymmetric
(b) R is NOT symmetric but antisymmetric
(c) R is both symmetric and antisymmetric
(d) R is neither symmetric nor antisymetric

(GATE 2009, 1 Mark)

Solution: A relation is symmetric if for each (x, y) in R, (y, x) is also in R. This relation R in question is not symmetric as (x, y) is present but (y, x) is absent.
A relation is anti-symmetric if for each pair of (x, y) and (y, x) in R, $x = y$. Here R is not anti-symmetric as for (x, z) and (z, x), $x \neq z$.

Ans. (d)

8. What is the possible number of reflexive relations on a set of 5 elements?

(a) 2^{10} (b) 2^{15}
(c) 2^{20} (d) 2^{25}

(GATE 2010, 1 Mark)

Solution: Consider a table of size 5×5 in which each possible pair is listed. In a reflexive relation, we must include all 5 diagonal elements. So from rest of the 20 elements, we have a choice whether to include them or not. So we have 2^{20} possible reflexive relations.

Ans. (c)

9. Consider the set $S = \{1, \omega, \omega^2\}$, where ω and ω^2 are cube roots of unity. If * denotes the multiplication operation, the structure $(S, *)$ forms

(a) A group (b) A ring
(c) An integral domain (d) A field

(GATE 2010, 1 Mark)

Solution: The answer is 'a group' since the other three options (b), (c) and (d) require two operations over structure.

Also, we can test the set for all the properties of a group, i.e. closure, associativity, identity element and inverse element to find that it fulfills all the properties.

Ans. (a)

10. Let X and Y be finite sets and $f : X \to Y$ be a function. Which one of the following statements is TRUE?

(a) For any subsets A and B of X, $|f(A \cup B)| = |f(A)| + |f(B)|$
(b) For any subsets A and B of X, $f(A \cap B) = f(A) \cap f(B)$
(c) For any subsets A and B of X, $|f(A \cap B)| = \min\{|f(A)|, |f(B)|\}$
(d) For any subsets S and T of Y, $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

(GATE 2014, 1 Mark)

Solution: We have $f : X \to Y$ defined by $f(a) = 1$, $f(b) = 1$, $f(c) = 2$, where

$$X = \{a, b, c\}, Y = \{1, 2\}$$

Let $A = \{a, c\}$, $B = \{b, c\}$ be subsets of X then

$$|f(A \cup B)| = 2;\ |f(A)| = 2;\ |f(B)| = 2$$

$$f(A \cap B) = \{2\};\ f(A) = \{1, 2\};\ f(B) = \{1, 2\}$$
$$f(A) \cap f(B) = \{1, 2\}$$
$$|f(A \cap B)| = 1$$

Therefore, options (a), (b) and (c) are not true. Hence, option (d) is true.

Ans. (d)

11. Let G be a group with 15 elements. Let L be a subgroup of G. It is known that $L \neq G$ and that the size of L is at least 4. The size of L is ________.

(GATE 2014, 1 Mark)

Solution: According to Lagrange's theorem, the order of subgroup divides order of group. Order of group is 15.

3, 5, 15 divide 15. According to the given condition, size should be atleast 4 and cannot be equal to that of group. So, 5 is the order of subgroup.

Ans. 5

12. If V_1 and V_2 are 4-dimensional subspaces of a 6-dimensional vector space V, then the smallest possible dimension of $V_1 \cap V_2$ is ________.

(GATE 2014, 1 Mark)

Solution: Let V be $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. Then

$V_1 = \{e_1, e_3, e_5, e_6\}$
$V_2 = \{e_2, e_3, e_4, e_5\}$

Smallest possible set of $V_1 \cap V_2 = 2$.

Ans. 2

13. Consider the following relation on subsets of the set S of integers between 1 and 2014. For two distinct subsets U and V of S we say $U < V$ if the minimum element in the symmetric difference of the two sets is in U.

Consider the following two statements:

S1: There is a subset of S that is larger than every other subset.

S2: There is a subset of S that is smaller than every other subset.

Which one of the following is CORRECT?

(a) Both S1 and S2 are true
(b) S1 is true and S2 is false
(c) S2 is true and S1 is false
(d) Neither S1 nor S2 is true

(GATE 2014, 2 Marks)

Solution: Both S1 and S2 are true.

Ans. (a)

14. Consider the set of all functions f: $\{0, 1, \ldots, 2014\} \to \{0, 1, \ldots, 2014\}$ such that $f(f(i)) = i$, for $0 \le f \le 2014$. Consider the following statements.

P. For each such function it must be the case that for every i, $f(i) = i$.

Q. For each such function it must be the case that for some i, $f(i) = i$.

R. Each such function must be onto.

Which one of the following is CORRECT?

(a) P, Q and R are true
(b) Only Q and R are true
(c) Only P and Q are true
(d) Only R is true

(GATE 2014, 2 Marks)

Solution: Let us consider a function as

$$f(0) = 1, f(1) = 0, f(2) = 3, f(3) = 2, \ldots,$$
$$f(2012) = 2013, f(2013) = 2012 \text{ and } f(2014) = 2014$$

Clearly,

$$f\big(f(i)\big) = i \text{ for } 0 \le i \le 2014$$

Here, $f(i) \ne i$ for every i and $f(i) = i$ for some i.
Also, f is onto. Hence, only Q and R are true.

Ans. (b)

15. There are two elements x, y in a group $(G,*)$ such that every element in the group can be written as a product of some number of x's and y's in some order. It is known that

$$x \times x = y \times y = x \times y \times x \times y = y \times x \times y \times x = e$$

where e is the identity element. The maximum number of elements in such a group is ________.

(GATE 2014, 2 Marks)

Solution: We are given,
$x \times x = e \Rightarrow x$ is its own inverse.
$y \times y = e \Rightarrow y$ is its own inverse.
$(x \times y) \times (x \times y) = e \Rightarrow (x \times y)$ is its own inverse.

$(y \times x) \times (y \times x) = e \Rightarrow (y \times x)$ is its own inverse.
Also,

$x \times x \times e = e \times e$ can be rewritten as follows:

$x \times y \times y \times x = e \times y \times y \times e = e \quad [\because y \times y = e]$

$(x \times y) \times (y \times x) = e$ shows that $(x \times y)$ and $(y \times x)$ are each other's inverse and we already know that $(x \times y)$ and $(y \times x)$ are inverse of its own.

As per $(G,*)$ to be group any element should have only one inverse element. (unique)

This process $x \times y = y \times x$ (is one element)

So, the element of such group are 4 which are $\{x, y, e, x \times y\}$.

Ans. 4

16. For a set A, the power set of A is denoted by 2^A. If $A = \{5, \{6\}, \{7\}\}$, which of the following options are TRUE?

(i) $\phi \in 2^A$

(ii) $\phi \subseteq 2^A$

(iii) $\{5, \{6\}\} \in 2^A$

(iv) $\{5, \{6\}\} \subseteq 2^A$

(a) (i) and (iii) only
(b) (ii) and (iii) only
(c) (i), (ii) and (iii) only
(d) (i), (ii) and (iv) only

(GATE 2015, 1 Mark)

Solution: $2^A \to$ Power set of A, that is set of all subsets of A.
Since empty set is a subset of every set, therefore

$$\phi \subseteq 2^A \text{ and } \phi \in 2^A$$

Since, $\{5, \{6\}\} \subseteq A$ and $5 \notin 2^A$

Thus, $\{5, \{6\}\} \in 2^A$ and $\{5, \{6\}\} \subseteq 2^A$

Hence, (i), (ii) and (iii) are true.

Ans. (c)

17. Suppose U is the power set of the set $S = \{1, 2, 3, 4, 5, 6\}$. For any $T \in U$, let $|T|$ denote the number of elements in T and T' denote the complement of T. For any $T, R \in U$, let $T \backslash R$ be the set of all elements in T which are not in R. Which one of the following is true?

(a) $\forall X \in U (|X| = |X'|)$

(b) $\exists X \in U \ \exists Y \in U (|X| = 2, |Y| = 5 \text{ and } X \cap Y = 0)$

(c) $\forall X \in U \ \forall Y \in U (|X| = 2, |Y| = 3 \text{ and } X \backslash Y = 0)$

(d) $\forall X \in U \ \forall Y \in U (X \backslash Y = Y' \backslash X')$

(GATE 2015, 2 Marks)

Solution: $X \backslash Y = X - Y = X \cap Y'$ and

$$Y' \backslash X' = Y' - X' = Y' \cap (X')' = Y' \cap X = X \cap Y'$$

Therefore, $X \setminus Y = Y' \setminus X', \forall X, Y \in U$

Ans. (d)

18. Let R be a relation on the set of ordered pairs of positive integers such that $[(p, q), (r, s)] \in R$, if and only if $p - s = q - r$. Which one of the following is true about R?

(a) Both reflexive and symmetric
(b) Reflexive but not symmetric
(c) Not reflexive but symmetric
(d) Neither reflexive nor symmetric

(GATE 2015, 2 Marks)

Solution:
Since $p - q \neq q - p$

Thus, $(p,q)R(p,q)$

$$\Rightarrow R \text{ is not reflexive.}$$

Let $(p,q)R(r,s)$ then $p - s = q - r$

$$\Rightarrow r - q = s - p$$
$$\Rightarrow (r,s)R(p,q)$$

Thus, R is symmetric.

Ans. (c)

19. A function $f : \mathbb{N}^+ \to \mathbb{N}^+$, defined on the set of positive integers N^+, satisfies the following properties:

$$f(n) = f(n/2), \quad \text{if } n \text{ is even}$$
$$f(n) = f(n+5), \quad \text{if } n \text{ is odd}$$

Let $R = \{i \mid \exists j: f(j) = i\}$ be the set of distinct values that f takes. The maximum possible size of R is ________.

(GATE 2016, 2 Marks)

Solution: It is given that

$$f(n) = f(n/2), \quad \text{if } n \text{ is even}$$
$$f(n) = f(n+5), \quad \text{if } n \text{ is odd}$$

Let us use the definition of function to show the following:

(i) $f(1) = f(2) = f(3) = f(4) = f(6) = f(7) = f(8) = f(9) = \cdots$

(ii) $f(5) = f(10) = f(15) = f(20) = \cdots$

Hence, we conclude that the range of $f(n)$ comprises two distinct components.

Ans. 2

20. The number of integers between 1 and 500 (both inclusive) that are divisible by 3 or 5 or 7 is ________.

(GATE 2017, 2 Marks)

Solution: The number of integers divisible by 3 or 5 or 7 is given by

$$n(3 \vee 5 \vee 7) = n(3) + n(5) + n(7) - n(3 \wedge 5) - n(5 \wedge 7) - n(3 \wedge 7) + n(3 \wedge 5 \wedge 7)$$

$$= \left[\frac{500}{3}\right] + \left[\frac{500}{5}\right] + \left[\frac{500}{7}\right] - \left[\frac{500}{15}\right] - \left[\frac{500}{35}\right] - \left[\frac{500}{21}\right] + \left[\frac{500}{105}\right]$$

$= 166 + 100 + 71 - 33 - 14 - 23 + 4$
$= 271$

Ans. (271.0)

CHAPTER 9

COMBINATORY

INTRODUCTION

The selection of objects from a given set where the order of arrangement matters is known as *permutation*.

The selection of objects from a given set where the order of arrangements does not matter is known as *combination*.

For example, suppose we have three objects x, y and z and we have to choose two of them. Then, the two objects can be selected using permutations in six different ways, i.e., xy, yx, xz, zx, yz and zy. Also, the two objects can be selected using combinations in three different ways, i.e., xy, yz and xz.

If one operation can be performed in m different ways and if corresponding to each way of performing this operation, there are n different ways of performing the second operation, the two operations can be performed in $(m \times n)$ ways.

COUNTING

There are two fundamental principles of counting, which are *principle of addition* and *principle of multiplication*. The two principles form the basis of *permutations* and *combinations*.

Fundamental Principle of Addition

If there are two tasks such that they can be performed independently in m and n ways, respectively, then either of the two jobs can be performed in $(m + n)$ ways.

Fundamental Principle of Multiplication

If there are two tasks such that one of them can be completed in m ways and after its completion the other one can be completed in n ways, then the two tasks in succession can be completed in $(m \times n)$ ways.

PERMUTATIONS

As already stated, arranging or selecting objects where the order of arrangement matters is called *permutation*. The generalized formula of n different things taken r things at a time is given by

$$^{n}\mathrm{P}_{r} = \mathrm{P}(n, r) = \frac{n!}{(n-r)!}$$

Conditional Permutations

Permutations where repetition of items are allowed, distinction between some of the items are ignored or a particular item occurs in every arrangement are called *permutation under certain conditions*.

1. The number of permutations of n different objects taken r at a time, when a particular object is to be always included in each arrangement, is given by $r \cdot {}^{n-1}\mathrm{P}_{r-1}$.
2. The number of permutations of n distinct objects taken r at a time, when a particular object is not taken in any arrangement, is given by ${}^{n-1}\mathrm{P}_{r}$.
3. The number of permutations of n different objects taken r at a time in which two specified objects always occur together is given by $2!(r-1)\,{}^{n-2}\mathrm{P}_{r-2}$.

Permutations When all Objects are Not Distinct

Here, we discuss the permutations of a given number of objects when all objects are not distinct. The number of mutually distinguishable permutations of n things, taken all at a time, of which p are of one kind and q of another such that $p + q = n$, is given by:

$$\frac{n!}{p!q!}$$

Now, the following cases arise:

1. The number of permutations of n things, of which p_1 are alike of one kind, p_2 are alike of second kind and so on such that $p_1 + p_2 + \cdots + p_r = n$, is given by
$$\frac{n!}{p_1!\,p_2!\ldots p_r!}$$
2. The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining are all distinct, is given by
$$\frac{n!}{p!q!}$$
3. Suppose there are r things to be arranged allowing repetitions. Let $p_1, p_2, \ldots, p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times and so on. Then the total number of permutations of these r objects to the above condition is
$$\frac{(p_1 + p_2 + \cdots + p_r)!}{p_1!\,p_2!\,p_3!\ldots p_r!}$$

Circular Permutations

In all the cases discussed so far, we have considered the permutations to be linear, i.e., objects are arranged in a straight line.

Now, we will discuss the arrangements along a circle. Such arrangements are called *circular permutations*. In a circular permutation, there is neither a beginning nor the end. It should be noted that there is no difference between clockwise and counterclockwise arrangement of objects.

The number of ways of arranging n distinct objects along a round table is $(n-1)!$.

The number of ways of arranging n persons along a round table so that no person has the same two neighbours is $\frac{1}{2}(n-1)!$.

COMBINATIONS

Each of different selections made by taking some or all objects, irrespective of the order of arrangements is called *combinations*. The generalized formula of n different things taken r things at a time is given by

$$^{n}\mathrm{C}_{r} = \mathrm{C}(n, r) = \frac{n!}{r!(n-r)!}$$

Properties of Combinations

Some important properties of combinations are given as follows:

1. For $0 \le r \le n$,
$$^{n}\mathrm{C}_{r} = {}^{n}\mathrm{C}_{n-r}$$
2. If n and r are positive integers such that $r \le n$, then
$$^{n}\mathrm{C}_{r} + {}^{n}\mathrm{C}_{r-1} = {}^{n+1}\mathrm{C}_{r}$$
3. If n and r are positive integers such that $1 \le r \le n$, then
$$^{n}\mathrm{C}_{r} = \frac{n}{r} \times {}^{n-1}\mathrm{C}_{r-1}$$
4. If $1 \le r \le n$, then
$$n \cdot {}^{n-1}\mathrm{C}_{r-1} = (n-r+1) \cdot {}^{n}\mathrm{C}_{r-1}$$
5. If $^{n}\mathrm{C}_{x} = {}^{n}\mathrm{C}_{y}$, then
$$x = y \Rightarrow x + y = n$$

6. If n is an even natural number, then the greatest of the values ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ is given by ${}^nC_{n/2}$.

7. If n is an odd natural number, then the greatest of the values ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ is given by ${}^nC_{(n-1)/2} = {}^nC_{(n+1)/2}$.

Conditional Combinations

1. The number of ways in which $(m + n)$ things can be divided in the two groups containing m and n things, respectively, is given by $C(m+n, n) = \dfrac{(m+n)!}{m!n!}$.

If $m = n$, then two cases arise:

(a) When distinction can be made between the groups, then the required number of ways $= \dfrac{(2m)!}{(m!)^2}$.

(b) When no distinction can be made, then the number of ways $= \dfrac{(2m)!}{2!(m!)^2}$.

2. The total number of ways in which a selection can be made out of $p + q + r$ things of which p are of one kind, q are of another kind and remaining r are different, then the required number of ways is given by

$$(p+1)(q+1)2^r - 1$$

3. The number of combinations of n different things taken r at a time, when p particular things are always included, is given by

$${}^{n-p}C_{r-p}$$

4. The number of combinations of n different things, taken r at a time, when p particular things are always to be excluded, is given by

$${}^{n-p}C_r$$

GENERATING FUNCTIONS

A generating function is a formal power series in one indeterminate, whose coefficients encode information about a sequence of number a_n that is indexed by the natural numbers. It can also be defined as a powerful tool for solving counting problems. Generally, in counting problems, we are often interested in counting the number of objects of 'size n', which we denote by a_n. By varying n, we get different values of a_n. In this way, we get a sequence of real numbers

$$a_0, a_1, a_2, \ldots$$

Now, from the sequence, we can define a power series which in some sense can be regarded as an *infinite degree polynomial*.

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots$$

The above $G(x)$ is called the generating function for the sequence $a_0, a_1, a_2, \ldots$

Ordinary Generating Function

The ordinary generating function of a sequence a_n is given by

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n$$

If a_n is the probability mass function of a discrete random variable, then its ordinary generating function is called probability generating function.

The ordinary generating function of a two-dimensional array $a_{m,n}$ (where n and m are natural numbers) is

$$G(a_{m,n}; x, y) = \sum_{m,n=0}^{\infty} a_{m,n} x^m y^n$$

Exponential Generating Function

The exponential generating function of a sequence a_n is given by

$$\mathrm{EG}(a_n; x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

Exponential generating functions are preferred over ordinary generating functions for combinatorial enumeration problems involving labelled objects.

Poisson Generating Function

The Poisson generating function of a sequence a_n is given by

$$\mathrm{PG}(a_n; x) = \sum_{n=0}^{\infty} a_n e^{-x} \frac{x^n}{n!} = e^{-x}\mathrm{EG}(a_n; x)$$

RECURRENCE RELATION

A recurrence relation is a sequence that gives us a connection between two consecutive terms. The two terms can be given as U_n and U_{n-1} or U_n and U_{n+1}. It defines a sequence once one or more initial terms are given, and each further term of the sequence is defined as a function of the preceding terms.

Some of the common types of recurrence relations are logistic map, Fibonacci numbers and binomial coefficients.

Logistic Map

Logistic map is of the form

$$x_{n+1} = rx_n(1 - x_n)$$

with a given constant r; given the initial term x_0, each subsequent term is determined by this relation. Solving a recurrence relation means obtaining a closed-form solution: a non-recursive function of n.

Fibonacci Numbers

Fibonacci numbers have a linear, homogeneous recurrence relation with constant coefficients. They are defined using the following recurrence relation:

$$f_n = f_{n-1} + f_{n-2}$$

with $f_0 = 0$ and $f_1 = 1$. Hence, we can deduce that recurrence yields the equations

$$f_2 = f_1 + f_0$$
$$f_3 = f_2 + f_1$$
$$f_4 = f_3 + f_2$$

and so on.

Binomial Coefficients

A binomial coefficient is an example of a multi-dimensional recurrence relation which counts the number of ways of selecting i out of a set of n elements. They can be computed by the recurrence relation

$$\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$$

With the base cases $\binom{n}{0} = \binom{n}{n} = 1.$

SUMMATION

The operation of adding a sequence of numbers to obtain their sum or total is called summation. For finite sequences, summation always produces a well-defined sum. However, for the summation of an infinite sequence of values (also called series), we often use a limit to define its value.

Summations are represented as $\sum_{a=1}^{n} f(a)$ and it can be expanded as

$$\sum_{a=1}^{n} f(a) = f(1) + f(2) + f(3) + \cdots + f(n)$$

Some of the common summation rules that we use are given as follows:

$$\sum_{a=1}^{n} a = \frac{n(n+1)}{2}$$

$$\sum_{a=1}^{n} a^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{a=1}^{n} a^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{a=1}^{n} i = ni, \text{ where } i \text{ is a constant}$$

ASYMPTOTIC ANALYSIS

Asymptotic analysis is a method of describing the limiting behaviour. In applied mathematics, asymptotic analysis is used to build numerical methods to approximate equation solutions. It finds a number of applications in different branches of science, which includes *analysis of algorithms* in computer science; *statistical mechanics* for studying the behaviour of very large physical systems and in accident analysis to identify for example the cause of crash through count modeling with large number of crash counts in a given time and space. One use of asymptotic analysis is evident from the following example.

Let us consider a function $f(n)$ given by $f(n) = n^3 + 2n$. We want to analyze the properties of this function when (n) becomes very large. As is evident from the function; the term $2n$ becomes insignificant as compared to n^3 as n becomes large. The function $f(n)$ in this case is said to be asymptotically equivalent to n^3 as $n \to \infty$. This is written symbolically as $f(n) \sim n^3$.

In general, for given functions f and g of a natural number variable n,

$$f \sim g(\text{as } n \to \infty)$$

If and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

This relation is an equivalence relation on the set of functions of n. The equivalence class of f informally consists of all functions g which are approximately equal to f in a relative sense, in the limit.

Asymptotic expansion of a function $f(x)$ is expressed in terms of a series such that taking any initial partial sum provides an asymptotic formula for f and successive terms provide an increasingly accurate description of the order of growth of f. The partial sums do not necessarily converge. Mathematically, it is expressed as follows.

$$f \sim g_1$$
$$f \sim g_1 + g_2$$

and $f \sim g_1 + \cdots + g_k$

The requirement that the successive sums improve the approximation may then be expressed as:

$$f - (g_1 + \cdots + g_k) = o(g_k)$$

In case the asymptotic expansion does not converge, for any particular value of the argument, there will be a particular partial sum which provides the best approximation and adding additional terms will decrease the accuracy. However, this optimal partial sum will usually have more terms as the argument approaches the limit value.

SOLVED EXAMPLES

1. In a class of 20 boys and 18 girls, the teacher wants to select one student as the class monitor. In how many ways the teacher can make this selection.

Solution: The teacher can choose a student as class monitor by choosing a boy in 20 ways or by choosing a girl in 18 ways. Hence, the teacher can choose a class monitor in 38 ways.

2. In a class of 20 boys and 18 girls. The teacher wants to select one boy and one girl as class monitor. In how many ways the teacher can make the selection.

Solution: The teacher can choose a boy in 20 ways and a girl in 18 ways. The teacher can select one boy and one girl as class monitor in $20\times 18 = 360$ ways.

3. What is the number of different permutations of APPLE?

Solution: We have to choose five letter words from the letters A, P, P, L and E.

Hence, total arrangements $= {}^5P_5 = 5!$.

However, letter P is used twice and hence the total number of arrangements $= \dfrac{5!}{2!} = 5 \times 4 \times 3 = 60$.

4. Find the number of permutations formed by using all the letters of the word 'ENGINEERING' when all E's come together.

Solution: Since we want all the E's to come together, we treat it as a single letter.

Therefore, total number of arrangements $= 9!$.

However, N, I and G are repeated 3, 2 and 2 times, respectively.

Thus, total number of arrangements $= \dfrac{9!}{3!\times 2!\times 2!}$ $=15120$.

5. Sonakshi has six friends. In how many ways can she invite one or more of them to dinner?

Solution: The order of inviting friends does not matter. Hence, we use the formula of combination.

Sonakshi can invite the friends in ${}^6C_1$ ways.

Similarly, Sonakshi can invite one or more of them in ${}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$ ways

$= 6 + 15 + 20 + 15 + 6 + 1 = 63$

6. How many different committees of 5 members can be formed from 6 men and 4 women on which exactly 3 men and 2 women serve?

Solution: We know that

Total members in the committee = 5 (3 men and 2 women)

3 men out of 6 men can be selected in ${}^6C_3$ ways

Similarly, 2 women out of 4 women can be selected in ${}^4C_2$ ways

Hence, the committee can be formed in ${}^6C_3 \times {}^4C_2$ ways

$$\frac{6!}{3!\times 3!}\times\frac{4!}{2!\times 2!} = 5\times 4\times 3\times 2 = 120$$

7. If $P(n-1, 3) : P(n, 4) = 1:9$, then find n.

Solution: We have $P(n-1, 3) : P(n, 4) = 1:9$

$$\therefore \frac{\dfrac{(n-1)!}{(n-1-3)!}}{\dfrac{n!}{(n-4)!}} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{(n-4)!}\times\frac{(n-4)!}{n!} = \frac{1}{9}$$

$$\frac{(n-1)!}{n!} = \frac{1}{9} \Rightarrow \frac{1}{n} = \frac{1}{9} \Rightarrow n = 9$$

8. If ${}^nP_r = 720$ and ${}^nC_r = 120$, then find the value of r.

Solution: We know that

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$$\Rightarrow 120 = \frac{720}{r!}$$

$$\Rightarrow r! = 6$$

Now, $3! = 3 \times 2 = 6$

Hence, $r! = 3! \Rightarrow r = 3$.

9. How many triangles can be formed by joining the vertices of a hexagon?

Solution: Total vertices of a hexagon = 6

Triangle is formed by selecting a group of 3 vertices.

Hence, this can be done in ${}^6C_3$ ways.

Number of triangles $= {}^6C_3 = \dfrac{6!}{3!3!} = 20$.

10. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

Solution: Any number greater than a million will contain all seven digits. Also, 2 and 3 are repeated twice and thrice, respectively.

Number of arrangements $= \dfrac{7!}{2!\times 3!} = 420$.

However, arrangements also include 0 at the millionth place.

These 6 digits can be arranged in $\dfrac{6!}{2!\times 3!} = 60$ ways.

Hence, the required numbers = 420 − 60 = 360.

11. If $f(x) = x^3 + 3x + 2$, what would $f(x)$ be for $x \to \infty$.

Solution: $f(x) \sim x^3$ as the remaining terms are insignificant as x tends to infinity.

Also, $(x^3 + 3x + 2)/x^3 = (1 + 3/x^2 + 2/x^3)$, which equals 1 as $x \to \infty$

PRACTICE EXERCISE

1. In how many ways can 5 red and 4 white balls be drawn from a bag containing 10 red and 8 white balls?

(a) ${}^{10}C_5 \times {}^{8}C_4$ (b) ${}^{8}C_5 \times {}^{10}C_4$
(c) ${}^{18}C_9$ (d) ${}^{10}C_9$

2. The number of triangles that can be formed by joining the vertices of an octagon is

(a) 28 (b) 40
(c) 20 (d) 56

3. In an examination, there are four multiple questions and each question has four choices. The number of ways in which a student can fail to get all answers correct is

(a) 216 (b) 215
(c) 81 (d) none of these

4. The number of ways to arrange the letters of the word 'CHEESE' is

(a) 60 (b) 64
(c) 120 (d) 6

5. In a class of 10 students, there are 3 girls A, B and C. In how many different ways can they be arranged in a row such that no two of the three girls are seated together?

(a) 336 × 7! (b) 336 × 8!
(c) 176 × 7! (d) 56 × 8!

6. If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, find the value of r.

(a) 51 (b) 49
(c) 45 (d) 41

7. If ${}^{n}C_{r-1} = 36$, ${}^{n}C_{r} = 84$ and ${}^{n}C_{r+1} = 126$, then find ${}^{r}C_2$.

(a) 10 (b) 3
(c) 1 (d) 9

Common Data for Q. 8 and Q. 9 We have a word 'ALLAHABAD' and we make different words from the letters.

8. In how many of them vowels occupy the even positions?

(a) 60 (b) 45
(c) 54 (d) 36

9. In how many of them both 'L' do not come together?

(a) 5440 (b) 7560
(c) 1680 (d) 5880

Common Data for Q. 10 to Q. 12 We have to choose a cricket team of 11 players out of 15.

10. In how many ways can it be done if there is no restriction on the selection?

(a) 1250 (b) 1175
(c) 1365 (d) 1100

11. In how many ways can it be done if a particular player is always chosen?

(a) 1000 (b) 1500
(c) 1001 (d) 1221

12. In how many ways can it be done if a particular player is never chosen?

(a) 520 (b) 364
(c) 770 (d) 216

13. If 6 identical coins are arranged in a row, then the number of arrangements in which 4 are heads and 2 are tails are

(a) 4 (b) 6
(c) 12 (d) 15

14. A polygon has 44 diagonals. What is the number of sides of the polygon?

(a) 8 (b) 9
(c) 10 (d) 11

15. A box contains 5 black and 6 white balls. In how many ways can 6 balls be selected so that there are at least 2 balls of each colour?

(a) 425 (b) 525
(c) 388 (d) 510

16. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

(a) 25000 (b) 26400
(c) 25200 (d) 27400

Common Data for Q. 17 and Q. 18 We have to form four-letter words using the letters of the word 'FAILURE'.

17. In how many ways can it be done if 'F' is included in each other.

(a) 180 (b) 360
(c) 480 (d) 540

18. In how many ways can it be done if 'F' is not included in any word.

(a) 180 (b) 360
(c) 480 (d) 540

19. If ${}^{n}P_4 = 20 \times {}^{n}P_2$, then what is the value of n?

(a) 7 (b) 5
(c) 3 (d) 1

20. How many different signals can be given using any number of flags from four flags of different colours?

(a) 105 (b) 35
(c) 64 (d) 25

21. How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed?

(a) 30 (b) 60
(c) 75 (d) 50

22. How many four-letter words can be formed out of the letters of the word 'LOGARITHMS' if repetition of letters is not allowed?

(a) 1080 (b) 4220
(c) 5040 (d) 3660

23. How many words can be formed from the letters of the word 'DAUGHTER' so that the vowels are always together?

(a) 2400 (b) 2150
(c) 3600 (d) 4320

24. How many words can be formed from the letters of the word 'DAUGHTER' so that the vowels are not together?

(a) 3600 (b) 36000
(c) 4032 (d) 40320

25. How many different words can be formed from the letters of the word 'MISSISSIPPI'?

(a) 45480 (b) 27600
(c) 34650 (d) 69300

26. A man has 7 relatives, 4 of them are ladies and 3 are gentlemen. His wife also has 7 relatives, 3 of them are ladies and 4 are gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of the man's relatives and 3 of the wife's relatives?

(a) 485 (b) 375
(c) 505 (d) 420

27. Find the value of n, if ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in A.P.

(a) 7 (b) 7/2
(c) 3 (d) 3/2

28. From 4 mangoes, 5 oranges and 6 apples, how many selections of fruits can be made by taking at least one of them?

(a) 210 (b) 254
(c) 209 (d) 201

29. How many words can be formed from the letters of the word 'TRIANGLE' if each word begins with 'T' and ends with 'E'?

(a) 120 (b) 40320
(c) 660 (d) 720

30. Five boys and five girls form a line with the boys and girls standing at alternate places. Find the number of ways of making the line.

(a) $5 \times 5!$ (b) $(5!)^2$
(c) $(5!)^3$ (d) $2 \times (5!)^2$

31. If $f(n) = n^3 + 2n$, then one of the following expressions is correct.

(a) $f(n) \sim n^3$ (b) $f(n) \sim 2n$
(c) $f(n) \sim n(n^2 + 2)$ (d) None of these

ANSWERS

1. (a)	**5.** (a)	**9.** (d)	**13.** (d)	**17.** (c)	**21.** (b)	**25.** (c)	**29.** (d)
2. (d)	**6.** (d)	**10.** (c)	**14.** (d)	**18.** (b)	**22.** (c)	**26.** (a)	**30.** (d)
3. (b)	**7.** (b)	**11.** (c)	**15.** (a)	**19.** (a)	**23.** (d)	**27.** (b)	**31.** (a)
4. (c)	**8.** (a)	**12.** (b)	**16.** (c)	**20.** (c)	**24.** (b)	**28.** (c)	

EXPLANATIONS AND HINTS

1. (a) 5 red balls can be selected from 10 red balls in $^{10}C_5$ ways. Similarly, 4 white balls can be selected from 8 white balls in $^{8}C_4$ ways. Therefore, number of ways in which 5 red balls and 4 white balls can be selected would be equal to:

$$^{10}C_5 \times {}^{8}C_4$$

2. (d) Total vertices required to draw a triangle = 3.

Total vertices of octagon = 8.

Total triangles that can be drawn from octagon

$$= {}^{8}C_3 = \frac{8!}{5! \times 3!} = \frac{8 \times 7 \times 6}{3 \times 2} = 56.$$

3. (b) Each multiple question has 4 options. Hence, an answer can be chosen in 4 ways.

Now, there are 4 questions and 4 options can be chosen in 4 × 4 × 4 × 4 = 256 ways.

Also there is only one combination when all the combinations are correct.

Hence, the number of ways in which a student can fail to get all the answers correct = 216 − 1 = 215.

4. (c) Since order of letters does not matter, we can form $^{6}P_6$ = 6! words from the letters of 'CHEESE'.

However, letter 'E' is repeated thrice.

Therefore, total different words from letters 'C, H, S, and E' $= \frac{6!}{3!} = 12.$

5. (a) We know that

Total number of boys = 7

Total number of girls = 3

Seven boys can be arranged in a row $= {}^{7}P_7 = 7!$ ways.

Now, we have 8 places to arrange 3 girls $= {}^{8}P_3$.

Hence, the number of arrangements $= {}^{8}P_3 \times 7! = 336 \times 7!$.

6. (d) We know that

$$^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$$

$$\Rightarrow \frac{56!}{(56-r-6)!} : \frac{54!}{(54-r-3)!} = 30800 : 1$$

$$\Rightarrow \frac{56!}{(50-r)!} : \frac{54!}{(51-r)!} = 30800 : 1$$

$$\Rightarrow \frac{56!}{(50-r)!} \times \frac{(51-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow 56 \times 55 \times (51 - r) = 30800$$

$$\Rightarrow 51 - r = 10$$

$$\Rightarrow r = 41$$

7. (b) We know that

$$\frac{^{n}C_r}{^{n}C_{r+1}} = \frac{r+1}{n-r}$$

$$\Rightarrow \frac{r+1}{n-r} = \frac{84}{126} = \frac{2}{3} \Rightarrow (2n - 5r) = 3 \qquad (1)$$

Now,

$$\Rightarrow \frac{^{n}C_{r-1}}{^{n}C_r} = \frac{r}{n-(r-1)} = \frac{36}{84}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{36}{84} \Rightarrow \frac{r}{n-r+1} = \frac{3}{7}$$

$$\Rightarrow 3n - 10r = -3 \qquad (2)$$

Multiplying Eq. (1) by 2 and subtracting Eq. (2) from Eq. (1), we get

$$4n - 3n = 6^3 - (-3)$$

$$\Rightarrow n = 9$$

Substituting the value of n in Eq. (2), we get

$$27 - 10r = -3$$

$$r = \frac{30}{10} = 3$$

Therefore, $^{r}C_2 = {}^{3}C_2 = \frac{3!}{2!\ 1!} = 3$

8. (a) There are 9 letters in 'ALLAHABAD' out of which 4 are A's, 2 are L's and rest are all distinct.

Hence, the total number of words $= \frac{9!}{4!\ 2!} = 7560.$

There are 4 vowels and all are 'A'. There are 4 even places and the vowels can occupy the even places in 1 way.

Now, 5 letters can be arranged in 5! ways. Also, 'L' is repeated twice. Hence, letters can be arranged in $\frac{5!}{2!}$ ways.

Hence, the total number of words in which vowels occupies the even places $= \frac{5!}{2!} \times \frac{4!}{4!} = 60.$

9. (d) Considering both 'L' together and treating them as one letter, and that 'A' repeats 4 times, the letters can be arranged in $\frac{8!}{4!}$ ways = 1680.

Number of words in which both 'L' do not come together

= Total words − Number of words in which both 'L' come together

= 7560 − 1680 = 5880.

10. (c) The total number of ways of selecting 11 players out of 15 is

$$^{15}C_{11} = {}^{15}C_4 = \frac{15\times14\times13\times12}{4\times3\times2\times1} = 1365$$

11. (c) If a particular player is always chosen, then 10 players are selected out of the remaining 14 players.

Hence, remaining number of ways = $^{14}C_{10} = {}^{14}C_4$ $= 1001$.

12. (b) If a particular player is never chosen, then 11 players are selected out of the remaining 14 players.

Hence, required number of ways = $^{14}C_{11} = {}^{14}C_3$ $= 364$.

13. (d) Total coins = 6

Now, they can be arranged in 6! ways.

However, we also know that 4 are heads and 2 are tails.

Hence, coins can be arranged in $\frac{6!}{4!\times 2!}$ = 15 ways.

14. (d) Suppose there are n sides of a polygon.

We know that the number of diagonals of n-sided polygon is $\frac{n(n-3)}{2}$.

Now, $\frac{n(n-3)}{2} = 44 \Rightarrow n^2 - 3n = 88$

$\Rightarrow (n-11)(n+8) = 0 \Rightarrow n = 11 \quad (\because n > 0)$

15. (a) The selection of 6 balls consisting of at least 2 balls each from 5 black balls and from 6 white balls can be made in the following way by selecting 2 black balls out of 5 and 4 white balls out of 6. This can be done in $^5C_2 \times {}^6C_4$ ways.

Selecting 3 black balls out of 5 and 3 white balls out of 6 = $^5C_3 \times {}^6C_3$ ways.

Selecting 4 black balls out of 5 and 2 white balls out of 6 = $^5C_4 \times {}^6C_2$ ways.

Hence, total number of arrangements = $^5C_2 \times {}^6C_4 + {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2$

$$= \frac{5!}{3!\times 2!}\times\frac{6!}{4!\times 2!} + \frac{5!}{3!\times 2!}\times\frac{6!}{3!\times 3!} + \frac{5!}{4!\times 1!}\times\frac{6!}{4!\times 2!}$$

= 10 × 15 + 10 × 20 + 5 × 15 = 425

16. (c) We can choose 3 consonants out of 7 in $^7C_3$ ways. Similarly, we can choose 2 vowels out of 4 in $^4C_2$ ways.

Hence, there are $^7C_3 \times {}^4C_2$ ways of arranging 3 consonants and 2 vowels. Now, each group contains 5 letters and can be arranged in 5! ways.

Hence, the required number of words = $({}^7C_3 \times {}^4C_2)\times 5! = \frac{7!}{4!\times 3!}\times\frac{4!}{2!\times 2!}\times 5! = 25200.$

17. (c) We know that there are 7 letters in the word 'FAILURE'. Also, 'F' is included in all the 4-letter words. Now, the rest of 3 letters can be selected from the remaining 6 letters in $^6C_3$ ways.

Now, the 4-letter word can be arranged in 4! ways.

Hence, the total number of words = $^6C_3 \times 4! = \frac{6!}{3!\times 3!}\times 4! = 480.$

18. (b) Now, 'F' is not included. Hence, the 4 letters from the remaining 6 letters can be selected in $^6C_4$ ways.

Also, the 4-letter word can be arranged in 4! ways.

Hence, the total number of words = $^6C_4 \times 4! = \frac{6!}{4!\times 2!}\times 4! = 360.$

19. (a) We know that

$^nP_4 = 20 \times {}^nP_2$

$\Rightarrow \frac{n!}{(n-4)!} = 20 \times \frac{n!}{(n-2)!}$

$\Rightarrow (n-4)! \times 20 = (n-2)!$

$\Rightarrow (n-2)(n-3) = 20$

$\Rightarrow n^2 - 5n + 6 = 20 \Rightarrow n^2 - 5n - 14 = 0$

$\Rightarrow (n-7)(n+2) = 0 \Rightarrow n = 7 \quad (\because n > 0)$

20. (c) Signals can be made using 1, 2, 3 or 4 flags.

Hence, the total number of signals

$$\begin{aligned} &= {}^4P_1 + {}^4P_2 + {}^4P_3 + {}^4P_4 \\ &= 4 + (4\times 3) + (4!) + 4! \\ &= 4 + 12 + 24 + 24 \\ &= 64 \end{aligned}$$

21. (b) All numbers between 100 and 1000 are three digit numbers.

Now, we have to find permutations of five digits 1, 2, 3, 4, 5 taken three at a time. Hence, the required number of numbers is

$$^5P_3 = \frac{5!}{2!} = 5\times4\times3 = 60$$

22. (c) We know that there are 10 letters in the word 'LOGARITHMS' and all are distinct.

Hence, the total number of 4-letter words = $^{10}P_4 = \frac{10!}{6!} = 10 \times 9 \times 8 \times 7 = 5040.$

23. (d) There are 8 letters in the word 'DAUGHTER', 3 vowels 'A, U, E' and 5 consonants 'D, G, H, T, R'.

Now, since vowels come together, we consider them as one letter. Therefore, the 6 letters can be arranged in ${}^6P_6 = 6!$ ways.

The vowels can be arranged in 3! ways.

Hence, the required number of ways = 6! × 3! = 720 × 6 = 4320.

24. (b) Total number of words using letters of the word 'DAUGHTER' is ${}^8P_8 = 8! = 40320$.

Total number of words in which vowels are together = 4320.

Total number of words in which vowels are not together = 40320 − 432 = 36000.

25. (c) Total letters in the word 'MISSISSIPPI' are 11.

However, 'S' is repeated 4 times, 'I' is repeated 4 times and 'P' is repeated 2 times.

Hence, total number of words $= \dfrac{11!}{4!\times 4!\times 2!} = 34650.$

26. (a) Let L and G denote ladies and gentlemen, respectively.

Now, they need to invite 3 ladies and 3 gentlemen. Also, they need 3 relatives each from the man's and the wife's relatives. This can be done in the following ways:

- 3 L from the man's relatives and 3 G from the wife's relatives
- 3 G from the man's relatives and 3 G from the wife's relatives
- 2 L, 1 G from the man's relatives and 1 L, 2 G from the wife's relatives.
- 1 L, 2 G from the man's relatives and 2 L, 1 G from the wife's relatives.

Therefore, the required number of ways of inviting a dinner party = ${}^4C_3 \times {}^4C_3 + {}^4C_2 \times {}^3C_1 \times$

${}^3C_1 \times {}^4C_2 + {}^4C_1 \times {}^3C_2 \times {}^3C_2 \times {}^4C_3 + {}^3C_3 \times {}^3C_3$

= 4 × 4 + 6 × 3 × 3 × 6 + 4 × 3 × 3 × 4 + 1 × 1

= 16 + 324 + 144 + 1 = 485

27. (b) We know that ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in A.P.

$$\Rightarrow 2\times {}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$$

$$\Rightarrow 2\times \frac{(2n)!}{(2n-2)!2!} = \frac{(2n)!}{(2n-1)!} + \frac{(2n)!}{(2n-3)!3!}$$

$$\Rightarrow 2\times \frac{2n(2n-1)}{2} = (2n) + \frac{(2n)(2n-1)(2n-2)}{3!}$$

$$\Rightarrow 4n^2 - 2n = 2n + \frac{n(2n-1)(2n-2)}{3}$$

$$\Rightarrow 4n - 2 = 2 + \frac{(2n-1)(2n-2)}{3}$$

$$\Rightarrow 12n - 6 = 6 + 4n^2 - 2n - 4n + 2$$

$$\Rightarrow 4n^2 - 18n + 14 = 0$$

$$\Rightarrow 2n^2 - 9n + 7 = 0$$

$$\Rightarrow n = \frac{-(-9) \pm \sqrt{(-9)^2 - 456}}{4}$$

$$\Rightarrow \frac{9\pm 5}{4} = \frac{7}{2}, 1$$

Now, $n \neq 1$ since ${}^2C_3$ is not possible.

$\therefore n = 7/2$.

28. (c) Four mangoes can be selected in 5 ways. This also includes the case when no mango is selected. Similarly, 5 oranges and 6 apples can be dealt in 6 and 7 ways, respectively.

Therefore, the total number of ways of selecting a fruit = 210.

However, this also includes the case when no fruit is selected.

Hence, at least one of the fruit is selected in (210 − 1) = 209 ways.

29. (d) If 'T' and 'E' are fixed at the first and last spots, then there are 6 letters remaining which can be arranged in ${}^6P_6$ ways.

Therefore, the total number of words which begin with 'T' and end with 'E' = 6! = 720.

30. (d) Five boys can be arranged in a line in ${}^5P_5 = 5!$ ways.

Now, girls are arranged in alternating places to boys. If a girl is placed before a boy then they can be arranged in 5! ways.

Similarly, if a girl is placed after a boy then they can be arranged in 5! ways.

Total ways of arranging girls = 5! + 5!.

Total ways of arranging students = 5!(5! + 5!) = $2\times(5!)^2$.

31. (a) Concept based.

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. The minimum number of comparisons required to find the minimum and the maximum of 100 numbers is ________.

(GATE 2014, 2 Marks)

Solution: Number of comparisons for finding minimum and maximum of n numbers $= \frac{3}{2}n - 2$

Here $n = 100$, so $(300/2) - 2 = 148$

Ans. 148

2. The number of distinct positive integral factors of 2014 is ________.

(GATE 2014, 2 Marks)

Solution: $2014 = 2 \times 19 \times 53$ (product of prime factors)

Number of distinct positive integral factors of 2014 $= (1 + 1) \times (1 + 1) \times (1 + 1) = 8$

Ans. 8

3. The number of divisors of 2100 is ________.

(GATE 2015, 1 Mark)

Solution: Let $N = 2100 = 2^2 + 3 \times 5^2 \times 7$ (i.e. product of primes)

Then the number of division of 2100 is given by

$$(2+1)(1+1)(2+1)(1+1) = (3)(2)(3)(2) = 36$$

Ans. 36

4. An unordered list contains n distinct elements. The number of comparisons to find an element in this list that is neither maximum nor minimum is

(a) $O(n \log n)$
(b) $O(n)$
(c) $O(\log n)$
(d) $O(1)$

(GATE 2015, 1 Mark)

Solution: Consider first three element of the list, at least one of them will be neither minimum nor maximum. Therefore, the correct answer is option (d).

Ans. (d)

5. The number of four-digit numbers having their digits in non-decreasing order (from left to right) constructed by using the digits belonging to the set {1, 2, 3} is ________.

(GATE 2015, 1 Mark)

Solution: Four-digit numbers with first digit as 1: 1111, 1112. 1113, 1122, 1123, 1133, 1222, 1223, 1233, 1333, that is 10.

Four-digit numbers with first digit as 2: 2222, 2223, 2233, 2333, that is 4.

Four-digit numbers with first digit 3: 3333, that is 1.

Therefore, total number of four digit numbers = 10 + 4 + 1 = 15.

Ans. 15

6. Consider the equality $\sum_{i=0}^{n} i^3 = X$ and the following choices for X:

(i) $\theta(n^4)$ (ii) $\theta(n+5)$ (iii) $O(n^5)$ (iv) $\Omega(n^3)$

(a) Only (i)
(b) Only (ii)
(c) (i) or (iii) or (iv) but not (ii)
(d) (ii) or (iii) or (iv) but not (i)

(GATE 2015, 1 Mark)

Solution: X = sum of the cubes of first n natural numbers $= \dfrac{n^2(n+1)^2}{4}$ which is $\theta(n^4), O(n^5)$ and $\Omega(n^3)$.

Ans. (c)

7. Consider the equation $(43)_x = (y3)_8$ where x and y are unknown. The number of possible solutions is ________.

(GATE 2015, 1 Mark)

Solution: We have,

$$(43)_x = (y3)_8$$

$$\Rightarrow 3 + 4x = 3 + 8y \Rightarrow 4x = 8y$$
$$\Rightarrow x = 2y$$
$$\Rightarrow x \geq 5 \text{ and } y \leq 7$$

Thus, five solutions are possible which are (14, 7), (12, 6), (10, 5), (8, 4) and (6, 3).

Ans. 5

8. Let a_n be the number of n-bit strings that do NOT contain two consecutive 1s. Which one of the following is the recurrence relation for a_n?

(a) $a_n = a_{n-1} + 2a_{n-2}$
(b) $a_n = a_{n-1} + a_{n-2}$
(c) $a_n = 2a_{n-1} + a_{n-2}$
(d) $a_n = 2a_{n-1} + 2a_{n-2}$

(GATE 2016, 1 Mark)

Solution: Let us consider a_n be the number of n-bit strings that do not contain two consecutive 1's. Also, let us develop a recurrence relation for a_n.

(i) Let us assume 1-bit strings 0, 1: That is, $a_1 = 2$.

(ii) Let us also assume 2-bit strings 00, 01, 10, 11: In this case, 00, 01, 10 do not have two consecutive 1s. Therefore, $a_2=3$.

(iii) Now, let us assume 2-bit strings as shown below:

$$00 \begin{cases} 0 \\ 1 \end{cases}$$

$$01 \begin{cases} 0 \\ 1 \end{cases}$$

$$10 \begin{cases} 0 \\ 1 \end{cases}$$

In this case, only the five strings 000, 001, 010, 100 and 101 do not have two consecutive 1s; therefore, $a_3=5$. Therefore, the three numbers $a_1=2$, $a_2=3$ and $a_2=3$ satisfy the following recurrence relation:

$$a_n=a_{n-1}+a_{n-2}$$

Ans. (b)

9. The spot speeds (expressed in km/hr) observed at a road section are 66, 62, 45, 79, 32, 51, 56, 60, 53, and 49. The median speed (expressed in km/hr) is ______.

(Note: Answer with one decimal accuracy.)

(GATE 2016, 1 Mark)

Solution: Writing spot speeds in ascending order is

32, 45, 49, 51, 53, 56, 60, 62, 66, 78 (total 10 terms)

Median will be given as
(i) Middle term for odd no. of terms.
(ii) Average of middle two terms for even no. of terms.

Since even number of terms hence median will be $\dfrac{53+56}{2}=54.5$

Ans. 54.5

10. The coefficient of x^{12} in $(x^3+x^4+x^5+x^6+\cdots)^3$ is __________.

(GATE 2016, 2 Marks)

Solution: We have

$$\begin{aligned}(x^3+x^4+x^5+x^6+\cdots)^3 &= [x^3(1+x^1+x^2+\cdots)]^3 \\ &= x^9(1+x+x^2+\cdots)^3 \\ &= \frac{x^9}{(1-x)^3} \\ &= x^9\sum_{r=0}^{\infty}(3-1)+{}^{r}C_r x^r\end{aligned}$$

Substituting $r=3$ in the above equation, we get coefficient of x^{12} in $(x^3+x^4+x^5+x^6+\cdots)^3$ as follows:

$${}^{3+2}C_3 = {}^5C_3 = {}^5C_2 = 10$$

Ans. 10

11. Consider the recurrence relation $a_1=8$, $a_n=6n^2+2n+a_{n-1}$. Let $a_{99}=K\times10^4$. The value of K is __________.

(GATE 2016, 2 Marks)

Solution: The recurrence relation is given by

$$a_n-a_{n-1}=6n^2+2n(1)$$

Let the complementary function be C_1 and the particular solution be

$$(An^2+BnC)n$$

On substituting in Eq. (1), we get $A=2$; $B=4$; $C=2$. The solution is

$$a_n=C_1+2n^3+4n^2+2n$$

It is given that $a_1=8$; therefore,

$$8=C_1+8\Rightarrow C_1=0$$

Also,

$$\begin{aligned}a_{99}&=2[(99^3)+2(99^2)+99]\\&=2[(100-1)^3+2(100-1)^2+(100-1)]\\&=10^4(198)\end{aligned}$$

That is,

$$a_{99}=10^4(198)\Rightarrow K=198$$

Ans. 198

12. The value of the expression 13^{99} (mod 17), in the range 0 to 16, is __________.

(GATE 2016, 2 Marks)

Solution: By Fermat's theorem; $a^{p-1}=d$ (mod p). It is a fact that 17 is a prime number and it is not a divisor of 13. Therefore,

$$13^{16}=1 \text{ (mod } p)$$

$$13^{99}=(13)^{96}\times(13)^3=(13^{16})^6\times2197=1^6\times2197 \text{ (mod 17)}$$

By modulo arithmetic, we get

$$(1^6\times2197) \bmod 17=4$$

Ans. 4

CHAPTER 10

GRAPH THEORY

INTRODUCTION

Graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects. A graph may be undirected, i.e., there is no distinction between the two vertices associated with each edge or its edges may be directed from one vertex to another.

Graphs are used in many practical problems and can be used to model many types of relations and processes in physical, biological and information systems.

FUNDAMENTAL CONCEPTS OF GRAPH

A graph is formed by vertices or nodes which are connected by edges. Mathematically, a graph is a pair of sets given by $G = [V(G), E(G)]$, where $V(G)$ is a finite set of vertices and $E(G)$ is the set of edges formed by pairs of vertices. $E(G)$ is a multi-set, i.e., its elements can occur more than once so that every element has a multiplicity.

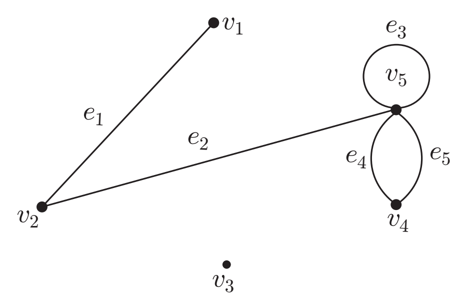


Figure 1 | Graph showing vertices and edges.

In Fig. 1, we have $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ for the vertices and $E(G) = \{(v_1, v_2), (v_2, v_5), (v_5, v_5), (v_5, v_4), (v_5, v_4)\}$ for the edges.

Common Terminologies

Two vertices joined by an edge are called the end vertices of that edge. In Fig. 1, v_1 and v_2 are the end vertices of e_1. Similarly, v_2 and v_5 are the end vertices of e_2.

If two or more edges have the same end vertices, then these edges are called parallel. In Fig. 1, e_4 and e_5 are parallel edges because they have the same end vertices, v_4 and v_5.

An edge joining a vertex to itself is called a loop. A loop has only one end vertex. In Fig. 1, e_3 is a loop with end vertex v_5.

A graph with no edges is called an empty graph. For an empty graph, $E(G)$ is empty.

A graph with no vertices is called a null graph. For a null graph, $V(G)$ and $E(G)$ are empty.

Edges are adjacent if they share a common end vertex. Two vertices are adjacent if they are connected by an edge. In Fig. 1, e_1 and e_2 are adjacent edges and v_1 and v_2 are adjacent vertices.

A graph which has no parallel edges or loops is called a simple graph. Figure 2 depicts a simple graph.

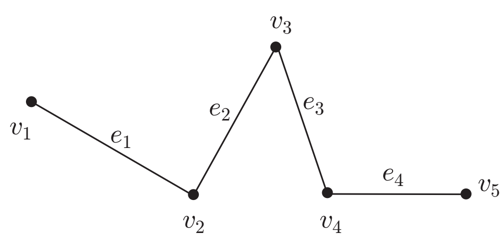


Figure 2 | Simple graph.

A vertex which is not the end of any edge is called an isolated vertex. In Fig. 1, v_3 is the isolated vertex.

Degree of a Vertex

Degree of a vertex in a graph is equal to the number of edges which are incident on that vertex or the number of edges which contain that vertex as an endpoint. It is represented as $\deg(v)$ or $d(v)$.

The vertex is said to be even or odd according to the value of $d(x)$.

A vertex whose degree is 1 is called a pendant vertex. Evidently, an edge that has a pendant vertex as an end vertex is a pendant vertex.

Also, an isolated vertex has a degree 0.

In Fig. 1, v_2 has a degree of 2 because it is the endpoint of two edges, e_1 and e_2.

Theorem 1. The graph $G = \{V(G), E(G)\}$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$, satisfies

$$\sum_{i=1}^{n} d(v_i) = 2m$$

Hence, it can be noted that every graph has an even number of vertices of odd degree.

Multigraph

A multigraph (also called pseudo graphs) also consists of sets $V(G)$ and $E(G)$. However, it may consist of parallel edges and may consist of one or more loops. An example of multigraph is shown in Fig. 3.

A multigraph is finite if the sets $V(G)$ and $E(G)$ are finite.

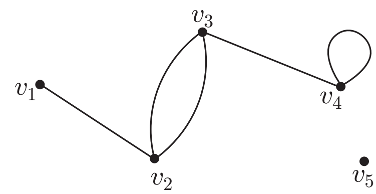


Figure 3 | Multigraph.

WALKS, PATHS AND CONNECTIVITY

A walk in the graph $G = \{V(G), E(G)\}$ consists of alternating vertices and edges of G. A walk is of the form,

$$v_{i0}, e_{j1}, v_{i1}, e_{j2}, \ldots, e_{jk}, v_{ik}$$

A walk starts and ends at a vertex. v_{i0} is the initial vertex and v_{ik} is the final vertex. A walk is open if $v_{i0} \neq v_{ik}$ and is closed if $v_{i0} = v_{ik}$.

A walk in which any edge is traversed at most once is called a trail.

A trail is a path in which any vertex is visited at most once except possibly the initial and terminal vertices when they are the same (i.e., the path is closed). A closed path is also called a circuit. The number of edges traversed in a path is called the length of a path.

A closed path whose vertices are distinct except $v_{i0} = v_{ik}$ is called a cycle. A cycle of length n is called n-cycle.

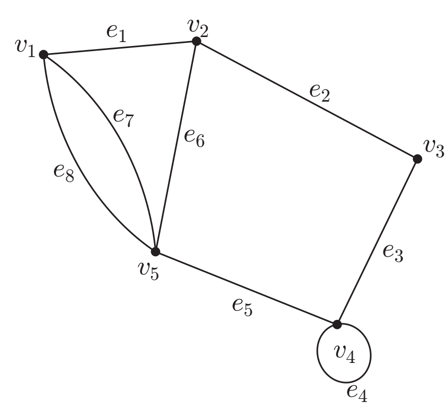


Figure 4 | Example of connectivity of graph.

In Fig. 4, walk

$$v_2, e_6, v_5, e_5, v_4, e_4, v_4, e_3, v_3$$

is open, and walk

$$v_2, e_6, v_5, e_8, v_1, e_1, v_2$$

is closed.

In Fig. 4, walk

$$v_1, e_7, v_5, e_8, v_1, e_1, v_2, e_6, v_5$$

is a trial.

In Fig. 4, walk

$$v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_5, v_5$$

is a path, and walk

$$v_1, e_1, v_2, e_6, v_5, e_8, v_1$$

is a closed path or circuit.

A graph is connected if all the vertices are connected to each other.

SUBGRAPH

A subgraph of a graph G is a graph whose $V(G)$ is a subset of G and whose adjacency relation is a subset of that of G restricted to this subset.

If from a graph G with vertex set $V(G)$, a subset $U(G)$ of $V(G)$ is deleted and all the edges have a vertex in $U(G)$ as an endpoint, then $G-U(G)$ is called vertex-deleted subgraph.

If a subset $U(G)$ of $E(G)$ is deleted from G, then $G-U(G)$ denotes the subgraph of G with vertex set $V(G)$ and edge set $E(G)-U(G)$, then $G-U(G)$ is called edge-deleted subgraph.

The subgraph G_1 of a graph G is a component of G if

1. G_1 is connected.
2. Either G_1 is trivial or G_1 is not trivial, and G_1 is the subgraph induced by those edges of G that have one end vertex in G_1.

Theorem 1. If graph G has a vertex v that is connected to a vertex of the component G_1 of G, then v is also a vertex of G_1.

Theorem 2. Every vertex or edge of G belongs to exactly one component of G.

Theorem 3. A graph is circuit less when there is at most one path between any two given vertices and there are no loops.

Theorem 4. If G is a connected graph and $k \geq 2$ is the maximum path length, then any two paths in G with length k share at least one common vertex.

TYPES OF GRAPHS

Complete Graph

A graph in which every vertex is connected to all the other vertices of the graph is called a complete graph. A complete graph of n vertices is given by K_n. Figure 5 gives an example of complete graphs.

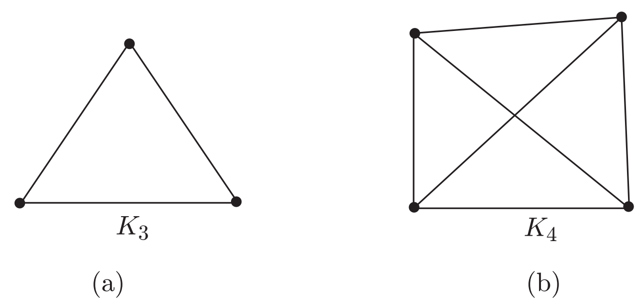


Figure 5 Complete graphs K_3 and K_4 with vertices 3 and 4, respectively.

Number of edges, m, in a complete graph K_n with n vertices is given by

$$m = \frac{n(n-1)}{2}$$

Regular Graph

A regular graph is a graph whose every vertex has the same degree. Figure 6 shows example of regular graphs.

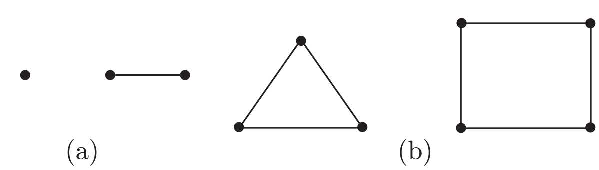


Figure 6 (a) 0-regular graph and (b) 1-regular graph.

Bipartite Graph

A simple graph G is a bipartite graph if a vertex set $V(G)$ can be partitioned into two non-empty disjoint subsets V_1 and V_2 such that every edge connects a vertex V_1 and a vertex V_2. The partition $V = V_1 \cup V_2$ is called a bipartition of G. Figure 7 shows an example of bipartite graph.

A bipartite graph is complete if there is an edge from every vertex in v_1 to every vertex in v_2, denoted by $K_{m,n}$ where $m = |v_1|$ and $n = |v_2|$. Figure 8 shows an example of complete bipartite graph.

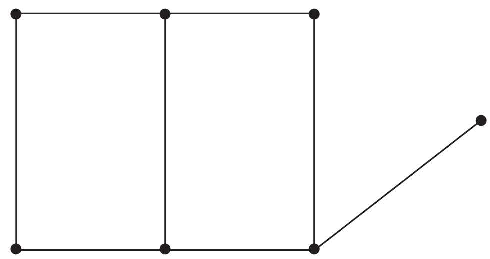


Figure 7 Bipartite graph.

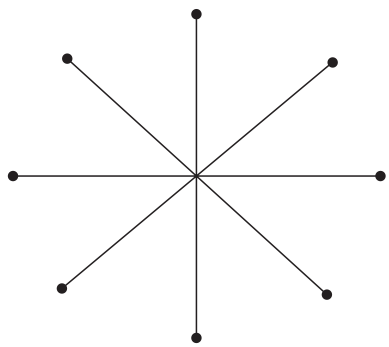


Figure 8 Complete bipartite graph.

The diameter of $k_{1,1}$ will be one because there are only two vertices and the shortest path between them is the length one. All other bipartite graphs have diameter two because any two points in either m or n will be exactly distance 2 units apart.

Tree Graph

A tree is a connected acyclic simple graph. A forest is a circuit less graph. A tree is simply a connected forest. The drawing of tree becomes complicated as the number of vertices increase. Figures 9 and 10 show examples of tree graphs.

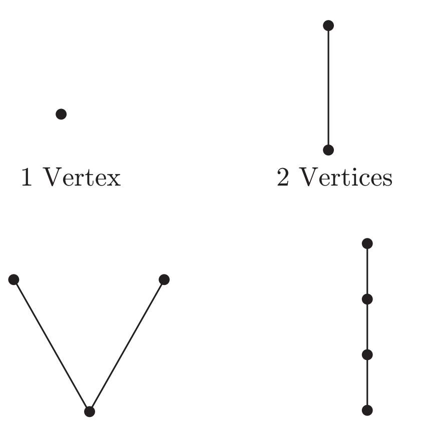


Figure 9 Tree graph.

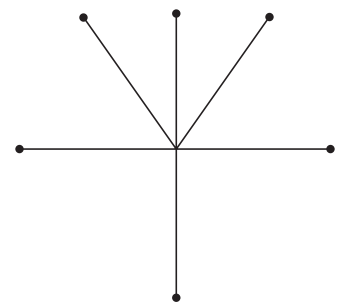


Figure 10 Forest.

A spanning tree of a connected graph is a sub-tree that includes all the vertices of the graph. If T is a spanning tree of graph G, the G-T gives cospanning tree. The edges of the spanning tree are called branches and the edges of the corresponding cospanning tree are called links or chords. Figure 11 shows examples of spanning and cospanning trees.

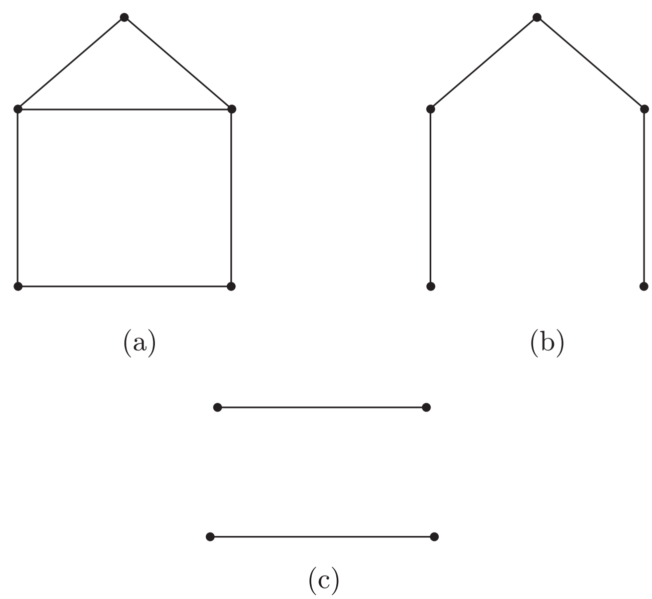


Figure 11 (a) Graph, G, (b) spanning tree of G and (c) cospanning tree.

Theorem 1. If T is a tree with at least two vertices and the longest path in T is given by $u_0, u_1, \ldots, u_n$, then both u_0 and u_1 have degree 1.

In other words, any tree T with at least two vertices has more than one vertex of degree 1.

Theorem 2. If a tree has n vertices, then it has precisely $(n - 1)$ edges.

Theorem 3. A connected graph has at least one spanning tree.

Trivial Graph

A graph with no edges is called a trivial graph.

CYCLE

A non-trivial closed trail in a graph with its origin and internal vertices distinct is called a cycle. A cycle graph with n vertices is denoted by C_n. Every vertex of a cycle has degree 2. The closed trail $C = v_1, v_2, ..., v_n$ and v_1 is cycle, if C has at least one edge and $v_1, v_2, ..., v_n$ are n distinct vertices.

A cycle with k edges is called k-cycle. A k-cycle is even or odd depending on whether k is odd or even. Two cycles with same length are isomorphic. Figure 12 shows a K-cycle of length 4.

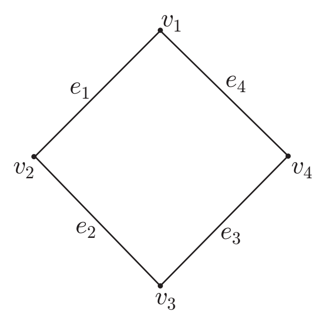


Figure 12 | Cycle of length 4.

OPERATIONS ON GRAPHS

Operations on graphs are performed to obtain new graphs. Some of the commonly used operations are:

1. **Union:** Suppose we have two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The union will be $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. If V_1 and V_2 are disjoint, their union is referred to as the disjoint union, and denoted by $G_1 + G_2$.
2. **Intersection:** Suppose we have two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The intersection will be $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$.
3. **Complement:** The complement or inverse of a graph G_1 is a graph G_2 on the same vertices such that two distinct vertices of G_2 are adjacent if and only if they are not adjacent in G_1. Figure 13 illustrates complement of a graph.

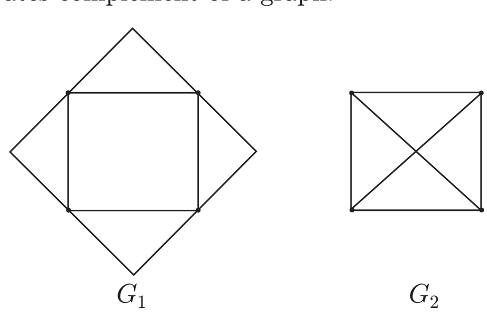


Figure 13 | Complement of G_1 given by G_2.

4. **Product of graphs:** To define product $G_1 \times G_2$ of two graphs, consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$, then u and v are adjacent in $G_1 \times G_2$ whenever
   1. $u_1 = v_1$ and u_2 is adjacent to v_2 or
   2. $u_2 = v_2$ and u_1 is adjacent to v_1.
5. **Composition of graphs:** Suppose we have two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The composition $G = G_1[G_2]$ is the graph with point vertex $V_1 \times V_2$ and $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$ whenever
   1. u_1 is adjacent to v_1 or
   2. $u_1 = v_1$ and u_2 is adjacent to v_2.

   Figure 14 illustrates composition of graphs.

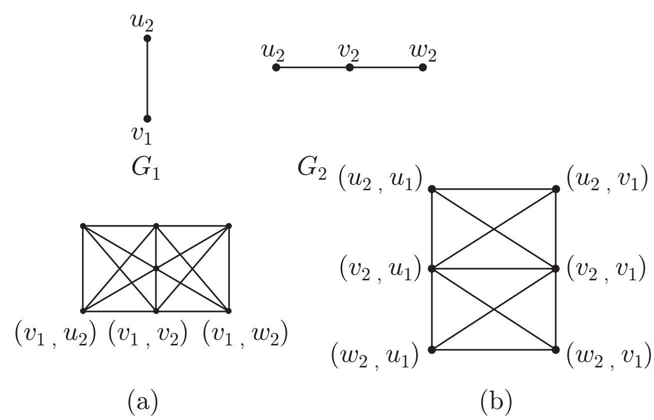


Figure 14 | (a) Composition $G_1[G_2]$ and (b) composition $G_2[G_1]$.

MATRIX REPRESENTATION OF GRAPHS

In this section, we discuss the two methods of matrix representation of a graph, namely, the adjacency matrix and the incidence matrix of the graph.

Adjacent Matrix

Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. The adjacency matrix of G is given by

$$A(G) = (a_{ij}),$$

where $(a_{ij}) = [E_G(v_i, v_j)]$ = number of edges joining the vertex v_i to v_j. If G has no loops, then all entries of the main diagonal will be 0 and if G has no parallel edges, then entries of $A(G)$ are either 0 or 1.

Theorem 1. Let A be the adjacency matrix of a graph G with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. Then the entry in position (i, j) of A^k is the number of different $(v_i - v_j)$ walks in G of length k.

Theorem 2. Let A be the adjacency matrix of a graph G with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and $B=(b_{ij})$ be the matrix. Hence,

$$B = A + A^2 + \cdots + A^{n-1}$$

Then G is a connected graph if for every pair of distinct indices i and j, we have $b_{ij} \neq 0$, i.e., b has no zero entries of the main diagonal.

The results from Theorem 1 and Theorem 2 are used to check whether a graph is connected or not.

Incidence Matrix

Let G be a graph with n vertices, $V(G)=\{v_1,v_2,\ldots,v_n\}$, and t edges, $E(G)=\{e_1,e_2,\ldots,e_t\}$. The incidence matrix $M(G)$ is the $n\times t$ matrix given by

$$M(G)=(m_{ij})$$

where m_{ij} is number of times the vertex v_i is incident with the edge e_j.

Hence,

$$m_{ij} = \begin{cases} 0, & \text{if } v_i \text{ is not an end of } e_j \\ 1, & \text{if } v_i \text{ is an end of non-loop } e_j \\ 2, & \text{if } v_i \text{ is an end of loop } e_j \end{cases}$$

The degree of v_i is given by the sum of elements in the ith row of $M(G)$.

CUTS

An edge e is called a cut edge of the graph G if by removing e, the connected component of the graph either remains unchanged or it increases exactly by 1. Cut edge is also called a bridge.

Similarly, a vertex v of a graph G is a cut vertex or an articulation vertex of G if the graph $(G-V)$ consists of a greater number of components than G.

A graph is separable if it is not connected or if there exists at least one cut vertex in the graph. Otherwise, graph is non-separable. Figure 15 shows cut-edges.

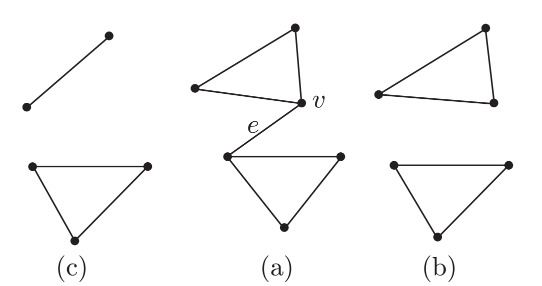


Figure 15 Graphs: (a) G, (b) $G - e$ and (c) $G - c$.

Theorem 1. Let e be a bridge of graph G, then

$$w(G) \leq w(G-e) \leq w(G)+1$$

where $w(G)$ is the number of connected components of G.

Theorem 2. An edge e of a graph G is a bridge if it is not a part of any cycle in G.

Theorem 3. Let G be a graph with n vertices and m edges, then

$$m \geq n - w(G)$$

where $w(G)$ is the number of connected components of G.

Theorem 4. The vertex v is a cut vertex of the connected graph G if and only if there exists two vertices u and w in G, such that

(a) $v \neq u, v \neq w$ and $u \neq w$, but
(b) v is on every $u - w$ path.

Theorem 5. A non-trivial simple graph has at least two vertices which are not cut vertices.

SPANNING TREES AND ALGORITHMS

As already mentioned, a spanning tree of a connected graph G is a sub-tree that includes all the vertices of that graph.

A complete tree K_n has n^{n-2} different spanning trees.

A graph in which each edge is assigned a weight, $w(e)$, is called a weighted graph. Suppose H is a subgraph of a weighted graph G and has edge set $E(H)=\{e_1,e_2,\ldots,e_k\}$, then

$$w(H) = w(e_1) + w(e_2) + \cdots + w(e_k)$$

A minimal spanning tree (MST) is a spanning tree with weight less than or equal to the weight of every other spanning tree.

In the sections that follow, we have discussed two algorithms for finding MST. It is to be noted that a necessary condition for these algorithms is that no weights should be negative.

Kruskal's Algorithm

It is a greedy algorithm in graph theory that finds a minimum spanning tree for a connected weighted graph. The algorithm gives an acyclic subgraph of G and it is a MST of G.

The following steps are involved in Kruskal's algorithm:

1. Select an edge of G, say e_1 such that $w(e_1)$ is as small as possible and e_1 is not a loop.
2. If edges $e_1, e_2, ..., e_i$ have been chosen, then choose an edge e_{i+1} that is not already chosen such that
   (a) Induced subgraph $G[e_1, e_2, ..., e_{i+1}]$ is acyclic.
   (b) $w(e_{i+1})$ is as small as possible.
3. If G has n vertices, then stop after $(n - 1)$ edges have been chosen, or else repeat step 2.

Prim's Algorithm

It is a greedy algorithm that finds a minimum spanning tree for a connected weighted undirected graph. It finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized.

The following steps are involved in Prim's algorithm:

1. Choose any vertex of G, say v_1.
2. Choose an edge $e_1 = v_1v_2$ of G such that $v_2 \neq v_1$ and e_1 has the smallest weight among the edges of G incident with v_1.
3. If edges $e_1, e_2, ..., e_i$ have been chosen involving endpoints $v_1, v_2, ..., v_{i+1}$, choose an edge $e_{i+1} = v_jv_k$ with $v_j \in \{v_i, ..., v_{i+1}\}$ and $v_k \notin \{v_1, ..., v_{i+1}\}$ such that e_{i+1} has the smallest weight among the edges of G with precisely one end in $\left[v_1, ..., v_{i+1}\right]$.
4. Stop after $n - 1$ edges have been chosen. Otherwise repeat step 3.

BINARY TREES

A binary tree is a rooted tree that is also an ordered tree, in which every node has at most two children, which are referred to as the left child and the right child.

A full binary tree or a perfect tree is a tree in which every node other than the leaves has two children. Figure 16 shows an example of a full binary tree.

Binary trees are extensively used in data representation.

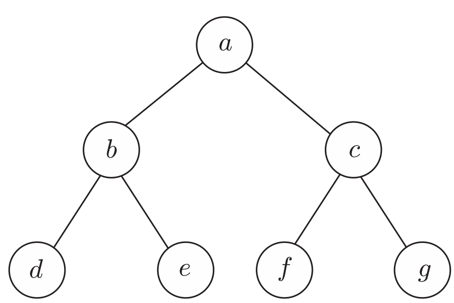


Figure 16 Full binary tree.

EULER TOURS

A tour of any graph is a closed walk of that graph in which every edge is encountered at least once.

A Euler tour of any graph is a closed walk of that graph in which every edge is encountered exactly once. Any graph G is called Eulerian or Euler if it has a Euler tour.

The necessary conditions for the existence of Eulerian circuits are that all vertices in the graph have an even degree.

Theorem 1. A connected graph G has a Euler trail if it has at most two odd vertices, i.e., it has either no vertices of odd degree or exactly two vertices of odd degree.

Königsberg Bridge Problem

The Königsberg bridge problem asks if the seven bridges of the city of Königsberg can all be traversed in a single trip without doubling back, with the additional requirement that the trip ends in the same place it began.

For the existence of Eulerian trails, it is necessary that zero or two vertices have an odd degree. This means the Königsberg graph is not Eulerian. Therefore, a solution does not exist. Figure 17 shows Euler's graphical representation of the Königsberg bridge problem.

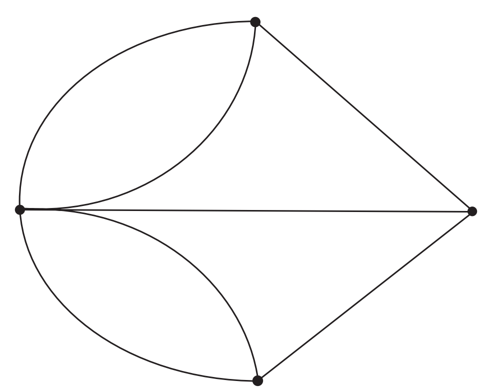


Figure 17 Euler's graphical representation of the Königsberg bridge problem.

HAMILTONIAN GRAPHS

Hamiltonian path is a path in an undirected or directed graph that visits each vertex exactly once. Such a part is a cycle and is called Hamiltonian cycle.

As already mentioned before, a Eulerian cycle uses every edge exactly once but may repeat vertices, whereas a Hamiltonian cycle uses each vertex exactly once (except for the first and the last) but may skip edges.

A simple graph G is called maximal non-Hamiltonian if it is not Hamiltonian but addition of any edge connecting two non-adjacent vertices forms a Hamiltonian graph. Figures 18 (a) and (b), respectively, show Hamiltonian and Eulerian graphs.

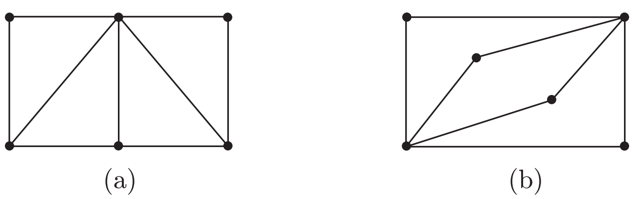


Figure 18 (a) A graph which is Hamiltonian and non-Eulerian and (b) a graph which is Eulerian and non-Hamilton.

Theorem 1. If G is a simple graph with n vertices where $n \geq 3$, and the degree $d(v) \geq \dfrac{n}{2}$ for every vertex v of G, then G is Hamiltonian.

CLOSURE OF A GRAPH

Suppose G is a simple graph. If we have two non-adjacent vertices u_1 and v_1 in G such that $d(u_1) + d(v_1) \geq n$ in G_1, join u_1 and v_1 by an edge to form the super graph G_1. Similarly, if we have two non-adjacent vertices u_2 and v_2 in G_1 such that $d(u_2) + d(v_2) \geq n$, join u_2 and v_2 by an edge to form the super graph G_2.

Continue these steps until we get the last pair of non-adjacent vertices whose degree sum is at least n. The final super graph obtained is called closure of G or $C(G)$.

It should be noted that a simple graph is Hamiltonian if its closure $C(G)$ is Hamiltonian. Figure 19 shows closure of graph.

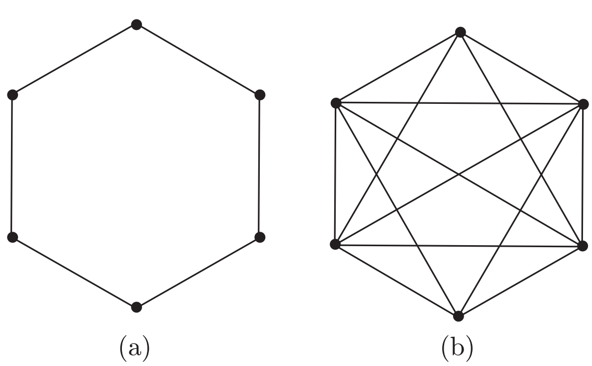


Figure 19 (a) Simple graph, G, and (b) closure of graph, $C(G)$.

GRAPH ISOMORPHISM

Suppose we have two graphs $G_1 = \{v_1, E_1\}$ and $G_2 = \{v_2, E_2\}$. G_1 and G_2 are said to be isomorphic if

1. there is a bijection (or one-to-one correspondence) f from v_1 to v_2 and
2. there is a bijection g from E_1 to E_2 that maps each edge (v, u) to $(f(v), f(u))$.

Figures 20 (a) and (b), respectively, show non-isomorphic and isomorphic graphs.

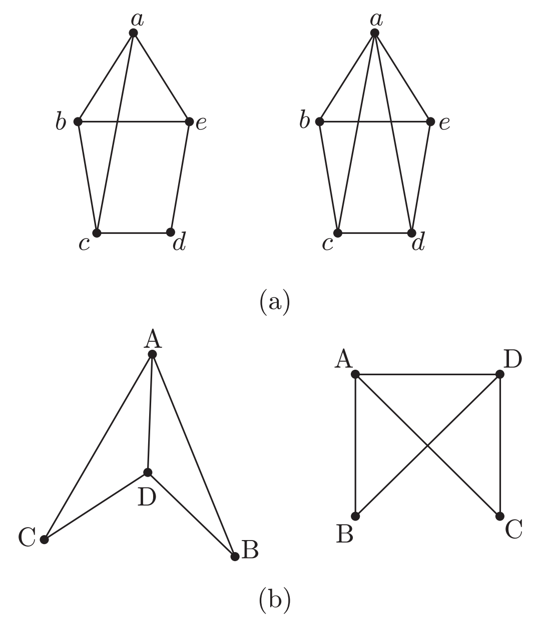


Figure 20 (a) Non-isomorphic graphs and (b) isomorphic graphs.

HOMEOMORPHIC GRAPHS

Suppose we have a graph G, we can get a new graph by dividing an edge of G with additional vertices. This is called homeomorphism and the two graphs G and H are called homeomorphic graphs.

Two graphs G and H are homeomorphic if they can be made isomorphic by inserting new vertices of degree 2 into their existing edges (Fig. 21).

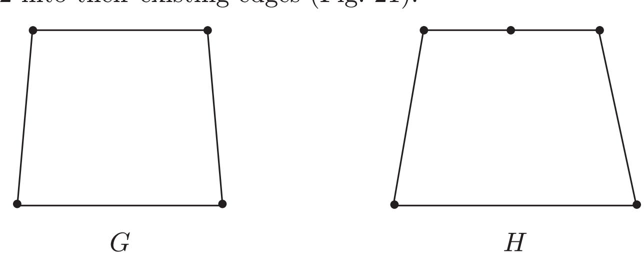
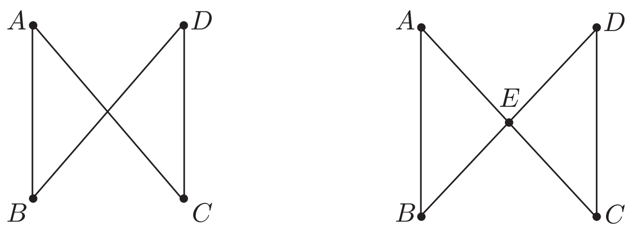


Figure 21 Homeomorphic graphs G and H.

PLANAR GRAPHS

A planar graph is a graph that can be embedded in a plane, i.e., it can be drawn on a plane in a way that its edges intersect only at their endpoints and no edges cross each other. Figures 22 (a) and (b), respectively, show examples of non-planar and planar graph.



(a) (b)

Figure 22 (a) Non-planar graph and (b) planar graph.

MATCHING

In graph theory, matching is a set of edges without common vertices. Formally, a matching M in G is a set of pairwise non-adjacent edges, i.e., no two edges share a common vertex.

In other words, a vertex is matched if it is an endpoint of one of the edges in the matching or else the vertex is unmatched.

A maximal matching M of a graph is a matching with the property that if any edge which is not in M is added to M, it is no longer a matching. In other words, M is maximal if it is not a proper subset of any other matching in graph G.

The number of edges in a largest maximum matching is called matching number of the graph.

A matching which matches as the vertices of the graph is called a perfect matching. In other words, every vertex of the graph is incident to exactly one edge of the matching.

Theorem 1. A complete matching of v_1 and v_2 in a bipartite graph exists, if and only if every subset of v vertices in v_1 is collectively adjacent to one or more vertices in v_2 for all values of v.

Theorem 2. In a bipartite graph, if a positive integer n exists such that degree of every vertex in $v_1 \geq 2 \geq$ degree of every vertex in v_2, then a complete matching v_1 into v_2 exists.

COVERING

A set of edges $E(G)$ of a graph G is set to cover G if every vertex in G is incident on at least one edge in $E(G)$. A set of edges that covers a graph G is called edge covering.

Covering exists for a graph if and only if the graph has no isolated vertex.

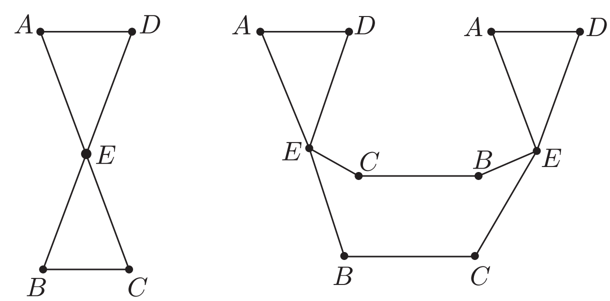
Covering of an n-vertex graph will have at least $\left[\frac{n}{2}\right]$ edges, where $[x]$ denotes the smallest integer not less than x.

If we denote remaining edges of a graph by $(G - E(G))$, the set of edges $E(G)$ is a covering if and only if, for every vertex v,

$$\text{degree of vertex in} \left(G - E\left(G\right)\right) \leq (\text{degree of vertex } v \text{ in } G)$$

A minimal covering for an n-vertex graph can contain no more than n-edges.

The number of edges in a minimal covering of the smallest size is called covering number of the graph. Figure 23 shows a graph G and its covering $C(G)$.



G $C(G)$

Figure 23 Graph G and its covering $C(G)$.

INDEPENDENT SET

A set of vertices in a graph in which no two sets are adjacent is called an independent set. In other words, it is a set of vertices such that there is no edge connecting any two vertices of that set. The size of an independent set is the number of vertices it contains.

An independent set of the largest possible size of a graph G is called a maximum independent set and this size is called independence number of G, which is denoted by $\alpha(G)$.

GRAPH COLORING

The assignment of colors to graph elements subject to certain conditions or constraints is called graph coloring. Graph coloring is a special case of graph labeling.

Coloring the graph vertices such that no two adjacent vertices have the same color is called vertex coloring.

Coloring the graph edges such that no two adjacent edges have the same color is called edge coloring.

Coloring the graph faces such that no two faces that share a boundary have the same color is called face coloring.

SOLVED EXAMPLES

1. Find the number of edges in the graphs: (a) K_8 and (b) K_{14}.

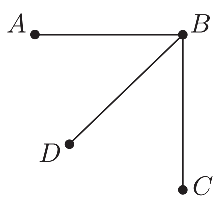
Solution: The number of edges in a complete graph is given by

$$m = \frac{n(n-1)}{2}, \text{ where } n = \text{number of vertices}$$

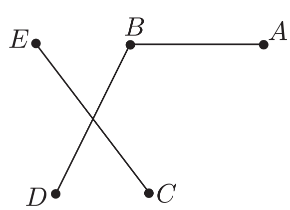
(a) For K_8, $m = \dfrac{8(7)}{2} = 28$

(b) For K_{14}, $m = \dfrac{14(13)}{2} = 91$

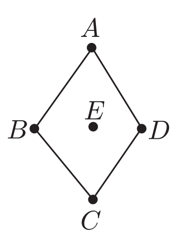
2. Explain whether the following graphs are connected or not.



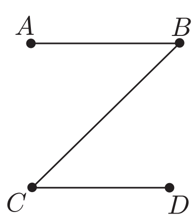
(a)



(b)



(c)



(d)

Solution:

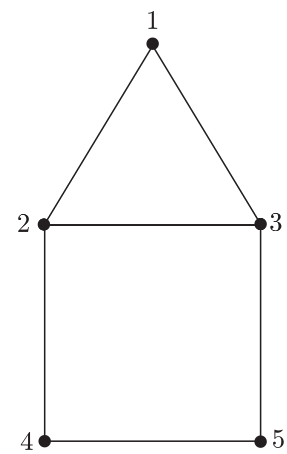
(a) Yes, it is a connected graph because all the vertices are connected to each other.

(b) No, it is not a connected graph. A, B and D are connected, and E and C are connected. However, there is no connectivity between A, B and D, and C and E.

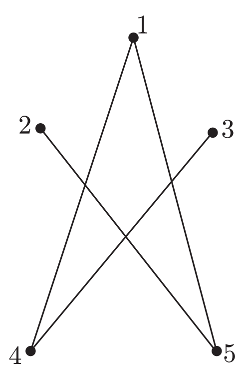
(c) No, it is not a connected graph because E is not connected to any other vertex.

(d) Yes, it is connected because all the vertices are connected to all other vertices of the graph.

3. Find the complement of the graph given below:



Solution: We have to keep in mind that the two vertices are adjacent in the complement only when they are not originally adjacent.



4. A graph G has the following adjacency matrix. Find whether it is connected.

$$A(G) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Solution: We have

$$n = 5$$

Hence, $B = A + A^2 + A^3 + A^4$

We find A^2, A^3 and A^4 using matrix multiplication,

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 5 & 4 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 3 & 3 & 0 & 0 & 6 \end{bmatrix}$$

Now,

$$B = A + A^2 + A^3 + A^4$$

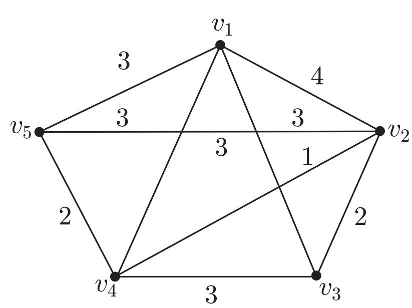
$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 5 & 4 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 3 & 3 & 0 & 0 & 6 \end{bmatrix}$$

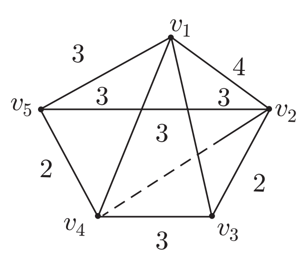
$$= \begin{bmatrix} 3 & 1 & 3 & 1 & 4 \\ 1 & 3 & 1 & 3 & 4 \\ 3 & 1 & 7 & 5 & 4 \\ 1 & 3 & 5 & 7 & 4 \\ 4 & 4 & 4 & 4 & 8 \end{bmatrix}$$

As B has no more zero entry off the main diagonal, the graph is connected.

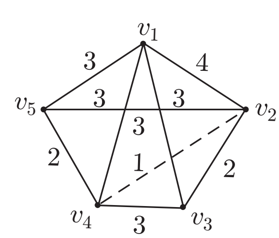
5. Apply Kruskal's algorithm to determine a minimal spanning tree of the graph G with 5 vertices.



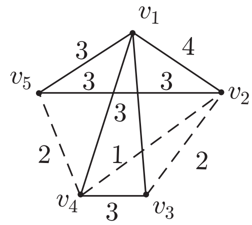
Solution: Using step 1, choose $e = v_2 v_4$ as it has minimum weight.



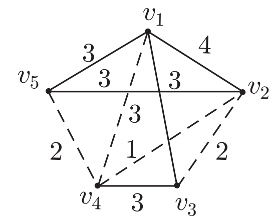
Choose $e = v_4 v_5$



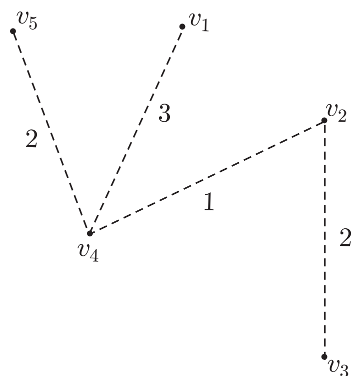
Choose $e = v_2\, v_3$



Choose $e = v_1\, v_4$



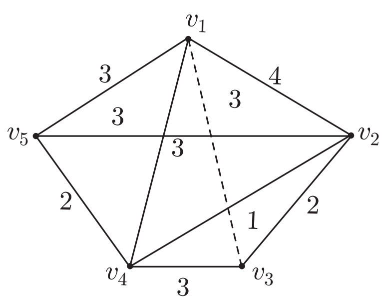
As vertices are 5, we require 4 edges. Minimal spanning tree of graph is given by



6. Apply Prim's algorithm to determine a minimal spanning tree of the graph G in the previous example.

Solution: Using step 1, we choose vertex v_1.

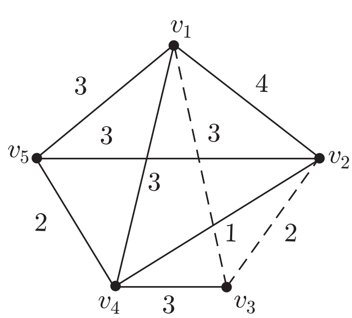
Now, edge with the smallest weight incident on v_1 is $e = v_1 v_3$.



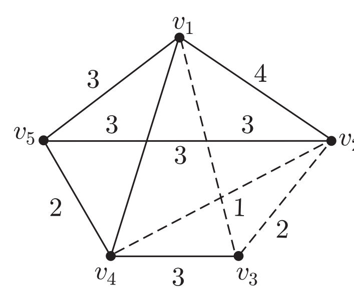
Now, we look on the weights,

$w(v_1, v_2) = 4,\ w(v_1, v_4) = 3,\ w(v_1, v_5) = 3,$
$w(v_3, v_2) = 2,\ w(v_3, v_4) = 3$

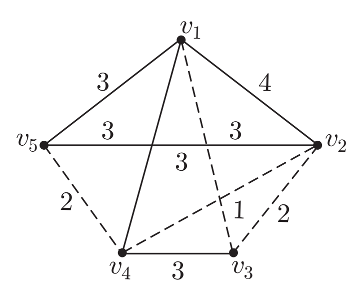
Now, we choose the minimum, $e = (v_3, v_2)$



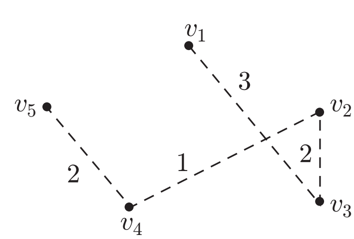
Again, $w(v_2, v_5) = 3, w(v_2, v_4) = 1$ and $w(v_3, v_4) = 3$
We choose the minimum, $e = (v_2, v_4)$



Now, $w(v_4, v_5) = 2$, we choose $e = v_4, v_5$



The minimal spanning tree is shown below:



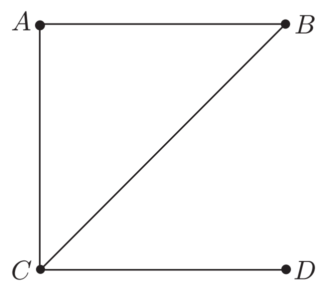
7. Suppose $G(V,E)$ has 5 vertices. Find the maximum number of m of edges in E if G is a multigraph.

Solution: As multiple edges are allured, G can have any number of edges and loops, finite or infinite. Hence, no such maximum number m exists.

8. Draw the graph for the following vertex and edge set:

$V(G) = \{A,B,C,D\}$ and $E(G) = \{\{A,B\},\{C,A\},$
$\{C,D\},\{B,C\}\}$

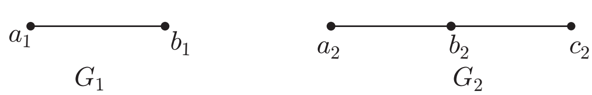
Solution:



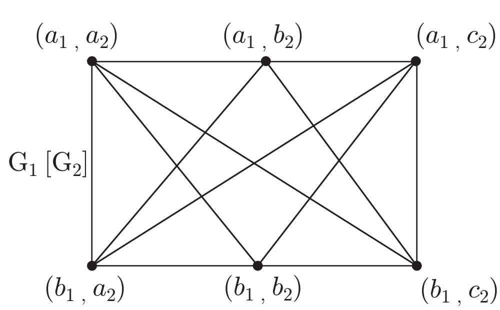
9. Show that every simple finite graph has two vertices of the same degree.

Solution: Let us assume that the graph has n vertices. Each of these vertices is connected to either of the other vertices, $0,1,2,.....,n-1$. If any of the vertices is connected to $n-1$ vertices, then it is connected to all the others, so there cannot be a vertex connected to 0 others. Thus, it is impossible to have a graph with n vertices where one vertex has degree 0 and another has degree $n-1$. Thus, the vertices can have at most $n-1$ different degrees, but since there are n vertices, at least two must have the same degree.

10. Find the composition $G_1[G_2]$, given that

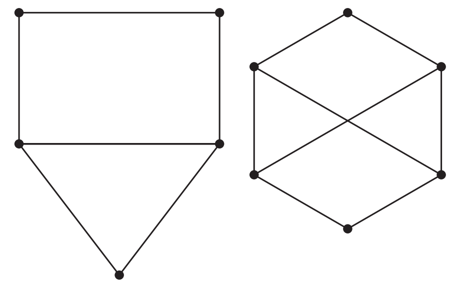


Solution:



11. Prove that in finite graphs, the number of vertices of odd degree is always even.

Solution: Consider the following two graphs with their degree sequence.



Let V_{odd} be the set of all vertices of odd degree and V_{even} be the set of all vertices of even degree.

Now, the sum of all degrees is twice the number of edges. Therefore, we have

$$\sum d(v) = 2e$$

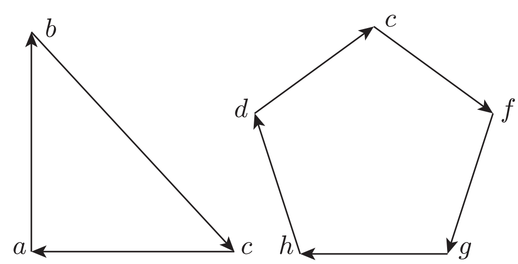
$$\Rightarrow \sum_{v=V_{\text{odd}}} d(v) = \sum_{v=V_{\text{even}}} d(v) = \text{even integer}$$

$$\Rightarrow \sum_{v=V_{\text{odd}}} d(v) + (\text{even integer}) = \text{even integer}$$

$$\Rightarrow \sum_{v=V_{\text{odd}}} d(v) = \text{even integer}$$

This is true only when V_{odd} has even number of elements. Hence, G has even number of odd degree vertices.

12. Let R be a binary relation on $A = \{a,\ b,\ c,\ d,\ e,\ f,\ g,\ h\}$ represented by the following two components. Find the smallest integers m and n such that $m < n$ and $R^m = R^n$.



Solution: Here,

$$R = \{(a,b),(b,c),(c,a),(d,e),(e,f),(f,g),(g,h),(h,d)\}$$

Say $R_1 = \{(a,b),(b,c),(c,a)\}$ and

$$R_2 = \{(d,e),(e,f),(f,g),(g,h),(h,d)\}$$

$$\therefore R = R_1 \cup R_2$$

From the digraph,

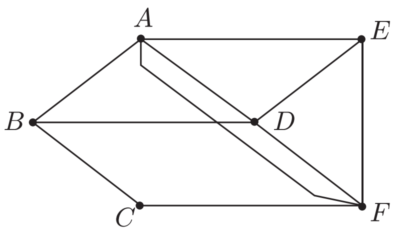
$R_1 = R_1{}^4 = R_1{}^7 = R_1{}^{10} = R_1{}^{13} = R_1{}^{16} = \cdots$

and $R_2 = R_2{}^6 = R_2{}^{11} = R_2{}^{16} = \cdots$

$\Rightarrow R = R_1^{16} \cup R_2^{16} = R^{16}$

$\Rightarrow m = 1$ and $n = 16$

13. Let G be a connected, undirected graph. A cut in G is a set of edges whose removal results in G being broken into two or more components which are not connected with each other. The size of a cut is called its cardinality. A min-cut of G is a cut in G of minimum cardinality. Consider the following graph.



(a) Which one of the following sets of edges is a cut?

(i) $\{(A, B), (E, F), (B, D), (A, E), (A, D)\}$

(ii) $\{(B, D), (C, F), (A, B)\}$

(b) What is the cardinality of min-cut in this graph?

(c) Prove that if a connected undirected graph G with n vertices has a min-cut of cardinality K, then G has at least $\left(\frac{nk}{2}\right)$ edges.

Solution:

(a) Set (ii) is a cut because on deleting BD, CF and AB, the graph gets disconnected.

(b) Cardinality of minimum edge cut in this graph is 2. The minimum edge cut is the set $\{(B, C), (C, F)\}$.

(c) If degree ¡ k, then by deleting all the edges incident on this vertex, we would be able to disconnect the graph. Hence, the degree of each vertex on this graph is at least k.

As there are n vertices and minimum degree of any vertex is k, and we know that each edge is incident on two vertices, G has at least $\left(\frac{nk}{2}\right)$ edges.

14. Let S be a set of n elements $\{1, 2, \ldots, n\}$ and G a graph with 2^n vertices, where each vertex corresponds to a distinct subset of S. Two vertices are adjacent if the symmetric difference of the corresponding sets has exactly 2 elements.

(a) Every vertex in G has the same degree. What is the degree of a vertex in G?

(b) How many connected components does G have?

Solution:

(a) $\dfrac{n(n-1)}{2}$

(b) G has two connected components.

PRACTICE EXERCISE

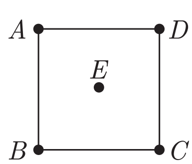
1. Find the number of edges in the graph K_6.

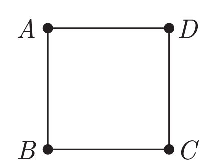
(a) 12 (b) 15
(c) 16 (d) 30

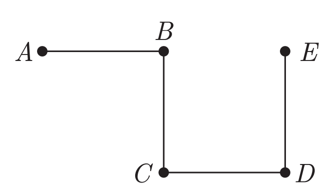
2. Find the number of edges in the graph K_{11}.

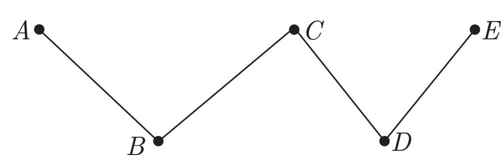
(a) 110 (b) 40
(c) 55 (d) 30

3. Which one of the following graphs is not connected?

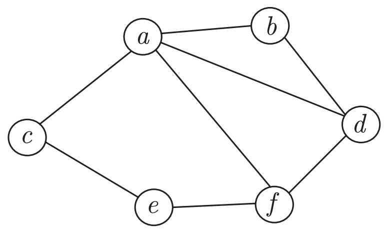
(a)


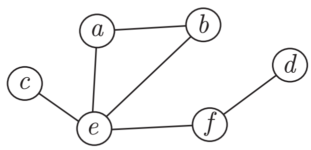
(b)


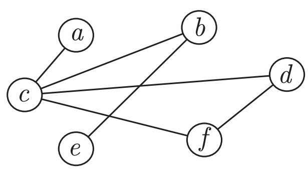
(c)


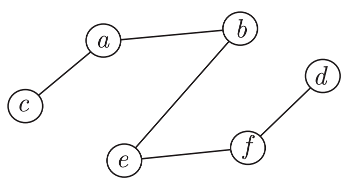
(d)


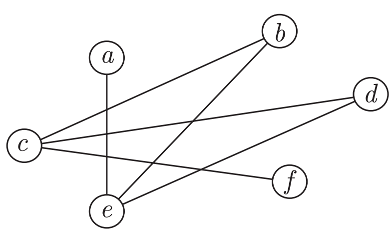
4. Which one of the following options is the complement of the following graph?



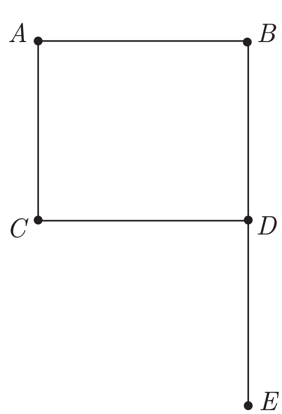
(a) 

(b) 

(c) 

(d) 

5. How can you represent the following graph?



(a) $V(G) = \{A, B, C, D\}$ and
$E(G) = \{\{A, B\}, \{C, A\}, \{C, D\}, \{B, D\}\}$

(b) $V(G) = \{A, B, C, D, E\}$ and
$E(G) = \{\{A, B\}, \{C, A\}, \{C, D\}, \{B, E\}, \{D, E\}\}$

(c) $V(G) = \{A, B, C, D, E\}$ and
$E(G) = \{\{A, B\}, \{C, A\}, \{D, C\}, \{B, D\}, \{D, E\}\}$

(d) $V(G) = \{A, B, C, E\}$ and
$E(G) = \{\{A, B\}, \{C, A\}, \{C, D\}, \{B, D\}, \{D, E\}\}$

6. Consider the adjacency matrix of the following graphs. Find which one of the following options is connected.

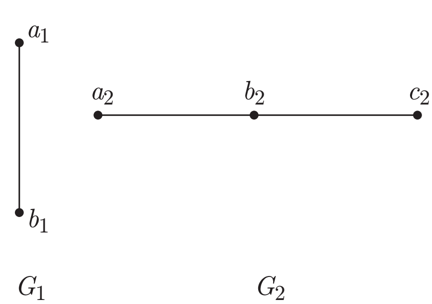
(a) $\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

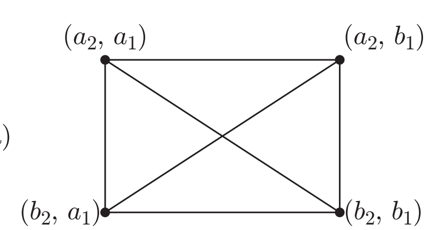
(c) $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

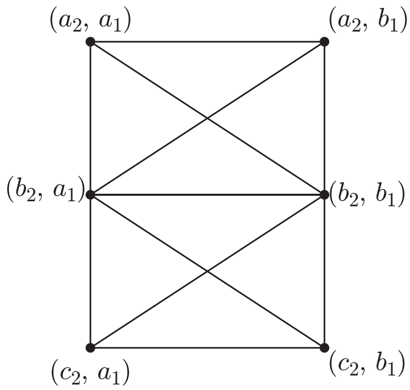
7. The number of distinct simple graphs with up to five nodes is

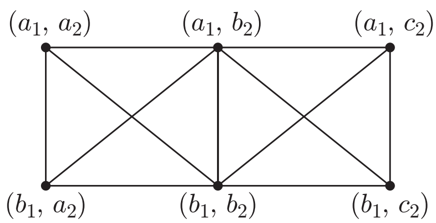
(a) 15 (b) 10
(c) 31 (d) 9

8. Find the value of $G_2[G_1]$ if G_1 and G_2 are given as follows:



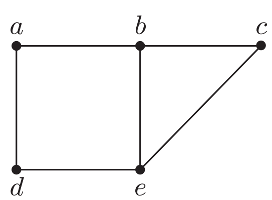
(a) 

(b) 

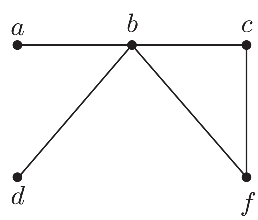
(c) 

(d) None of these

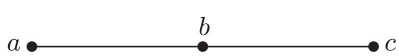
9. Find the value of $G_1 \cup G_2$ when

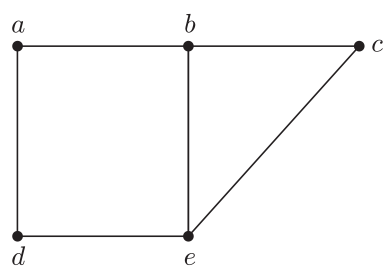


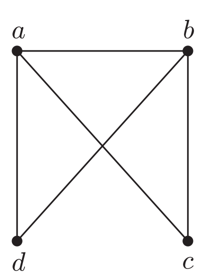
G_1

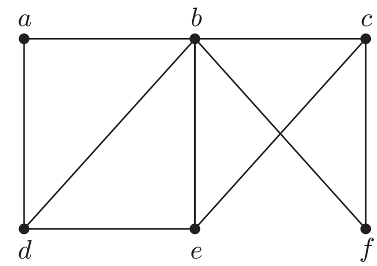


G_2

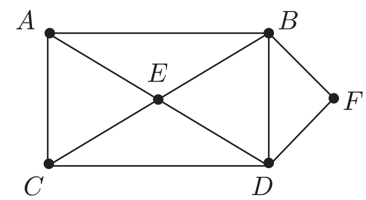
(a) 

(b) 

(c) 

(d) 

10. Find the degree of vertex of E.



(a) 1 (b) 2
(c) 3 (d) 4

11. The minimum number of edges in a connected cyclic graph on n vertices is

(a) $n - 1$ (b) n
(c) $n + 1$ (d) none of these

12. The number of distinct simple graphs with up to three nodes is

(a) 15 (b) 10 (c) 7 (d) 9

13. A graph is planar if and only if it does not contain

(a) subgraphs homomorphic to k_3 and $k_{3.3}$

(b) subgraphs isomorphic to k_5 or $k_{3.3}$

(c) subgraphs isomorphic to k_3 or $k_{3.3}$

(d) subgraphs homomorphic to k_3 or $k_{3.3}$

14. Maximum number of edges in an n-node undirected graph without self-loops is

(a) n^2 (b) $\dfrac{n(n-1)}{2}$

(c) $n - 1$ (d) $\dfrac{n(n+1)}{2}$

15. Let G be a graph with 100 vertices numbered 1 to 100. Two vertices i and j are adjacent if $|i - j| = 8$ or $|i - j| = 12$. The number of connected components in G is

(a) 8 (b) 4
(c) 12 (d) 25

16. Consider a simple connected graph G with n vertices and n edges $(n > 2)$. Then which of the following statements are TRUE?

(a) G has no cycles

(b) G has at least one cycle

(c) Graph obtained by removing any edge from G is not connected

(d) None of these

ANSWERS

1. (b)	**5.** (c)	**9.** (d)	**13.** (d)
2. (c)	**6.** (a)	**10.** (d)	**14.** (b)
3. (a)	**7.** (c)	**11.** (b)	**15.** (b)
4. (d)	**8.** (b)	**12.** (c)	**16.** (c)

EXPLANATIONS AND HINTS

1. (b) The number of edges in a complete graph is given by

$$m = \frac{n(n-1)}{2}, \text{ where } n = \text{number of vertices}$$

For K_6, $m = \dfrac{6(5)}{2} = 15.$

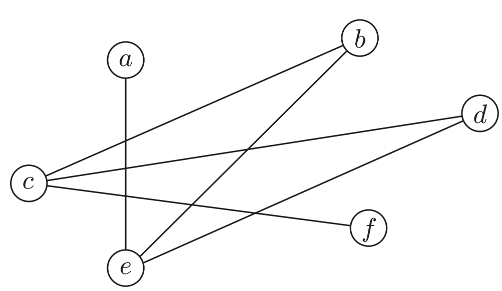
2. (c) The number of edges in a complete graph is given by

$$m = \frac{n(n-1)}{2}, \text{ where } n = \text{number of vertices}$$

For K_6, $m = \dfrac{11(10)}{2} = 55.$

3. (a) Graph (a) is not a connected graph because E is not connected to any other vertex. All the other graphs are connected because all the vertices are connected to each other.

4. (d) We have to keep in mind that the two vertices are adjacent in the complement only when they are not originally adjacent.



5. (c) The given figure has 5 vertices. Hence, the vertex set $V(G) = \{A, B, C, D, E\}$. Hence, options (a) and (d) are incorrect. The edge set $E(G) = \{\{A,B\},\{B,D\},\{D,C\},\{C,A\},\{D,E\}\}$.

6. (a) Let us consider option (a),

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

We have

$$n = 5$$

Hence, $B = A + A^2 + A^3 + A^4$

We find A^2, A^3 and A^4 using matrix multiplication,

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 2 & 0 & 3 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 3 & 1 & 2 & 0 & 3 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 3 & 5 & 6 \\ 4 & 2 & 2 & 2 & 4 \\ 1 & 2 & 0 & 4 & 5 \\ 5 & 2 & 4 & 0 & 1 \\ 6 & 5 & 5 & 1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 4 & 6 & 3 & 5 & 6 \\ 4 & 2 & 2 & 2 & 4 \\ 1 & 2 & 0 & 4 & 5 \\ 5 & 2 & 4 & 0 & 1 \\ 6 & 5 & 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 10 & 11 & 7 & 13 \\ 8 & 8 & 6 & 6 & 8 \\ 11 & 6 & 9 & 1 & 3 \\ 3 & 6 & 1 & 9 & 11 \\ 8 & 8 & 3 & 11 & 16 \end{bmatrix}$$

Now,

$$B = A + A^2 + A^3 + A^4$$

$$= \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 & 0 & 3 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 & 3 \end{bmatrix}$$

$$+ \begin{bmatrix} 4 & 6 & 3 & 5 & 6 \\ 4 & 2 & 2 & 2 & 4 \\ 1 & 2 & 0 & 4 & 5 \\ 5 & 2 & 4 & 0 & 1 \\ 6 & 5 & 5 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 17 & 10 & 11 & 7 & 13 \\ 8 & 8 & 6 & 6 & 8 \\ 11 & 6 & 9 & 1 & 3 \\ 3 & 6 & 1 & 9 & 11 \\ 8 & 8 & 3 & 11 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 18 & 16 & 13 & 23 \\ 14 & 12 & 9 & 9 & 14 \\ 14 & 9 & 11 & 6 & 9 \\ 9 & 9 & 6 & 11 & 14 \\ 16 & 15 & 9 & 14 & 21 \end{bmatrix}$$

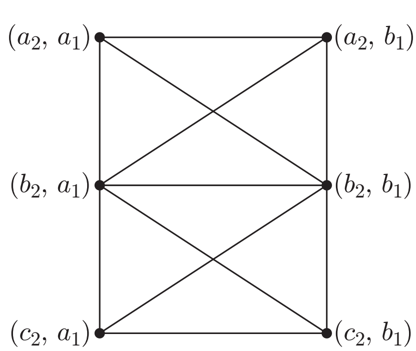
As B has no more zero entry off the main diagonal, the graph is connected.

Similarly, we can prove that the graphs given by adjacency matrices (b), (c) and (d) are not connected.

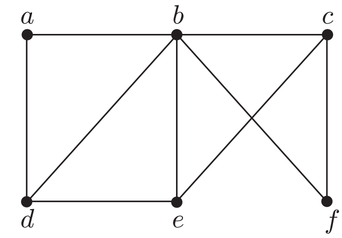
7. (c) The formula for the number of distinct simple graphs is given by $2^m - 1$, where m = number of nodes.

Therefore, the number of distinct simple graphs with three nodes = $2^5 - 1 = 31$.

8. (b)



9. (d) $G_1 \cup G_2$ should contain all the edges and vertices that are contained by G_1 and G_2.



10. (d) The vertex E is connected to A, B, C and D. Hence, the degree of the vertex E is 4.

11. (b) The minimum number of edges in a connected cyclic graph with n vertices is n.

12. (c) The formula for number of distinct simple graphs is given by $2^m - 1$, where m = number of nodes.

Therefore, number of distinct simple graphs with three nodes = $2^3 - 1 = 7$.

13. (d) A graph is planar if and only if it does not contain subgraphs homomorphic to k_3 or $k_{3.3}$.

14. (b) The edge count for each node in case of undirected graphs decreases gradually from n, $n - 1$, $n - 2$, …, 1. Hence, the maximum number of edges in an n-node undirected graph is $\frac{n(n-1)}{2}$.

15. (b) We are given that $i - j = 8$ are connected.

Hence, vertices –1, 9, 17, 25, 33, …″ are connected. Similarly,

{2, 10, 18, 26, 34, …}, {3, 11, 19, 27, 35, …},

{4, 12, 20, 28, 36, …}, {5, 13, 21, 29, 37, …},

{6, 14, 22, 30, 38, …}, {7, 15, 23, 31, 39, …} and

{8, 16, 24, 32, 40, …} are connected.

Also, $i - j = 12$ are connected. Hence,

{1, 13, 25, 37, …}, {2, 14, 26, 38, …},

{3, 15, 27, 39, …}, {4, 16, 28, 40, …},

{5, 17, 29, 41, …}, {6, 18, 30, 42, …},

{7, 19, 31, 43, …}, {8, 20, 32, 44, …},

{9, 21, 33, 45, …}, {10, 22, 34, 46, …},

{11, 23, 35, 47, …} and

{12, 24, 36, 48, …} are connected.

Now, we can see that 1, 9, 13 and 21 are connected. Hence, the graph has four connected components.

16. (c) The statement "Graph obtained by removing any edge from G is not connected" is true.

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. Let G be an arbitrary graph with n nodes and k components. If a vertex is removed from G, the number of components in the resultant graph must necessarily lie between

(a) k and n (b) $k - 1$ and $k + 1$

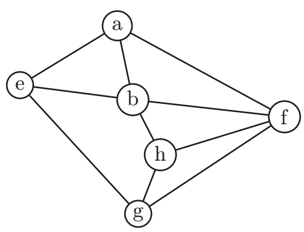
(c) $k - 1$ and $n - 1$ (d) $k + 1$ and $n - k$

(GATE 2003, 1 Mark)

Solution: Minimum: It may be possible that the removed vertex doesn't disconnect its component. Maximum: It may be possible that the removed vertex disconnects all components.

Ans. (a)

2. Consider the following graph:



Among the following sequences:

I. a b e g h f II. a b f e h g

III. a b f h g e IV. a f g h b e

Which are depth first traversals of the above graph?

(a) I, II and IV only (b) I and IV only

(c) II, III and IV only (d) I, III and IV only

(GATE 2003, 1 Mark)

Solution: For the (II) sequence, i.e. a b f e h g, it can be seen that there is no direct connection between f and e.

Ans. (d)

3. How many perfect matchings are there in a complete graph of 6 vertices?

(a) 15 (b) 24 (c) 30 (d) 60

(GATE 2003, 2 Marks)

Solution: In a perfect matching, every vertex of the graph is incident to exactly one edge of the matching. A perfect matching is therefore a matching of a graph containing $n/2$ edges, the largest possible, meaning perfect matching are only possible on graphs with an even number of vertices.

Ans. (a)

4. A graph $G = (V, E)$ satisfies $|E| \leq 3\ |V| - 6$. The min-degree of G is defined as $\min_{v \in V}\{\text{degree}(v)\}$. Therefore, min-degree of G cannot be

(a) 3 (b) 4 (c) 5 (d) 6

(GATE 2003, 2 Marks)

Solution: Let the min-degree of G is x, then G has at least $|v| * x/2$ edges.

$$|v| * x/2 <= 3 * |v| - 6$$

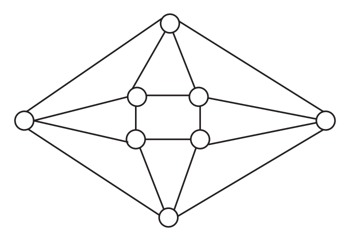
For $x = 6$, we get $0 <= 6$ Therefore, minimum degree of G cannot be 6.

Ans. (d)

5. The minimum number of colours required to colour the following graph, such that no two adjacent vertices are assigned the same colour, is

(a) 2 (b) 3

(c) 4 (d) 5



(GATE 2004, 2 Marks)

Ans. (c)

6. How many graphs on n labeled vertices exist which have at least $\dfrac{(n^2 - 3n)}{2}$ edges?

(a) ${}^{(n^2-n)/2}C_{(n^2-3n)/2}$ (b) $\sum_{k=0}^{(n^2-3n)/2} {}^{(n^2-n)}C_k$

(c) ${}^{(n^2-n)}C_n$ (d) $\sum_{k=0}^{n} {}^{(n^2-n)}C_k$

(GATE 2004, 2 Marks)

Ans. (d)

7. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be connected graphs on the same vertex set V with more than two vertices. If $G_1 \cap G_2 = (V, E_1 \cap E_2)$ is not a connected graph, then the graph $G_1 \cup G_2 = (V, E_1 \cup E_2)$

(a) cannot have a cut vertex

(b) must have a cycle

(c) must have a cut-edge (bridge)

(d) has chromatic number strictly greater than those of G_1 and G_2

(GATE 2004, 2 Marks)

Solution: We are given that G_1 and G_2 are connected graphs on the same vertex set. Hence, $G_1 \cup G_2$ has to be a cycle.

Ans. (b)

8. Let G be a simple connected planar graph with 13 vetices and 19 edges. Then, the number of faces in the planar embedding of the graph is

(a) 6 (b) 8

(c) 9 (d) 13

(GATE 2005, 1 Mark)

Solution: Applying the Euler's formula of planar graphs, we have

$v - e + f = 2$

$\Rightarrow 13 - 19 + f = 2 \Rightarrow f = 8$

Ans. (b)

9. Let G be a simple graph with 20 vertices and 100 edges. The size of the minimum vertex cover of G is 8, then the size of the maximum independent set of G is

(a) 12 (b) 8

(c) less than 8 (d) more than 12

(GATE 2005, 1 Mark)

Solution: We know that,
Number of vertices = minimum vertex cover number + size of the maximum independent set

Hence,

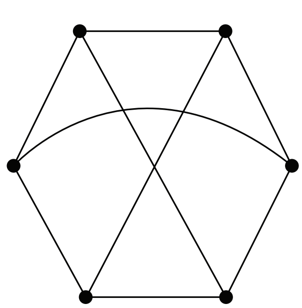
20 = 8 + size of the maximum independent set

Thus, size of the maximum independent set = 12.

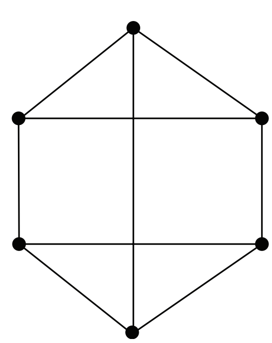
Ans. (a)

10. Which one of the following graphs is NOT planar?

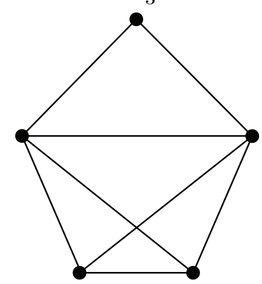
G_1:



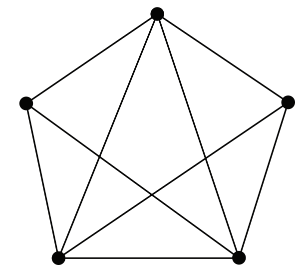
G_2:



G_3:



G_4:



(a) G_1 (b) G_2 (c) G_3 (d) G_4

(GATE 2005, 2 Marks)

Solution: A graph is planar if it can be redrawn in a plane without any crossing edges. Hence, the answer is G_1.

Ans. (a)

11. Consider the following two problems on undirected graphs:

α: Given $G(V, E)$, does G have an independent set of size $|V| - 4$?

β: Given $G(V, E)$, does G have an independent set of size 5?

Which one of the following is TRUE?

(a) α is in P and β is NP-complete

(b) α is NP-complete and β is in P

(c) Both α and β are NP-complete

(d) Both α and β are in P

(GATE 2005, 2 Marks)

Solution: We are given that the graphs have an independent set. Graph independent set decision problem is NP complete.

Ans. (c)

12. Let T be a depth first search tree in an undirected graph G. Vertices u and v are leaves of this tree T. The degrees of both u and v in G are at least 2. Which one of the following statements is true?

(a) There must exist a vertex w adjacent to both u and v in G

(b) There must exist a vertex w whose removal disconnects u and v in G

(c) There must exist a cycle in G containing u and v

(d) There must exist a cycle in G containing u and all its neighbours in G.

(GATE 2006, 2 Marks)

Solution: Since they are leaves in depth first search tree, they must be separable by a vertex.

Ans. (b)

13. Let G be the non-planar graph with the minimum possible number of edges. Then G has

(a) 9 edges and 5 vertices
(b) 9 edges and 6 vertices
(c) 10 edges and 5 vertices
(d) 10 edges and 6 vertices

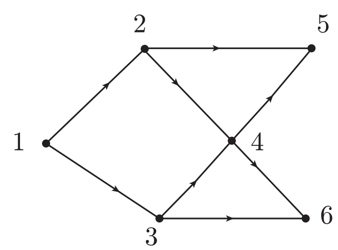
(GATE 2007, 1 Mark)

Solution: For a simple, connected, planar graph with v vertices and e edges, the following simple conditions hold:

If $v \geq 3$ then $e \leq 3v - 6$.

Ans. (b)

14. Consider the DAG with $V = \{1, 2, 3, 4, 5, 6\}$ shown below.



Which of the following is NOT a topological ordering?

(a) 1 2 3 4 5 6 (b) 1 3 2 4 5 6
(c) 1 3 2 4 6 5 (d) 3 2 4 1 6 5

(GATE 2007, 1 Mark)

Solution: In option (d), 1 appears after 2 and 3 which is not possible in topological sorting.

Ans. (d)

15. Which one of the following statements is true for every planar graph on n vertices?

(a) The graph is connected
(b) The graph is Eulerian
(c) The graph has a vertex-cover of size at most $3n/4$
(d) The graph has an independent set of size at least $n/3$

(GATE 2008, 2 Marks)

Solution: A planar graph is a graph which can be drawn on a plan without any pair of edges crossing each other.

Now, (a), (b) and (d) are false since a disconnected graph can be planar as it can be drawn on a plane without crossing edges.

Also, a Eulerian graph may or may not be planar. An undirected graph is Eulerian if all vertices have even degree.

Ans. (c)

16. G is a graph on n vertices and $2n-2$ edges. The edges of G can be partitioned into two edge-disjoint spanning trees. Which of the following is NOT true for G?

(a) For every subset of k vertices, the induced subgraph has at most $2k-2$ edges
(b) The minimum cut in G has at least two edges
(c) There are two edge-disjoint paths between every pair of vertices
(d) There are two vertex-disjoint paths between every pair of vertices

(GATE 2008, 2 Marks)

Solution: Let us take two copies of k_4 (complete graph on 4 vertices), G_1 and G_2. Let $V(G_1) = \{1, 2, 3, 4\}$ and $V(G_2) = \{5, 6, 7, 8\}$. If we construct a new graph G_3 by using these two graphs G_1 and G_2 by merging at a vertex, say (4, 5). The resultant graph is two edge connected, and of minimum degree 2 but there exists a cut vertex, the merged vertex.

Ans. (d)

17. What is the chromatic number of an n-vertex simple connected graph which does not contain any odd length cycle? Assume $n \geq 2$.

(a) 2 (b) 3 (c) $n - 1$ (d) n

(GATE 2009, 1 Mark)

Solution: A simple graph with no odd cycles is bipartite graph and a bipartite graph can be coloured using 2 colours.

Ans. (a)

18. Which one of the following is TRUE for any simple connected undirected graph with more than 2 vertices?

(a) No two vertices have the same degree
(b) At least two vertices have the same degree
(c) At least three vertices have the same degree
(d) All vertices have the same degree

(GATE 2009, 1 Mark)

Solution: Since the graph is simple, there must not be any self-loop and parallel edges. Since the graph is connected, the degree of any vertex cannot be 0.

Therefore, degree of all vertices should be from 1 to $n - 1$. So the degree of at least two vertices must be same.

Ans. (b)

19. For the composition table of a cyclic group shown below:

$*$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

Which one of the following choices is correct?

(a) a, b are generators (b) b, c are generators

(c) c, d are generators (d) d, a are generators

(GATE 2009, 2 Marks)

Solution: An element is a generator for a cyclic group if on repeated applications of it upon itself, it can generate all elements of group.

Now, considering for a, $a * a = a$, then $(a * a) * a = a * a = a$ and so on. Hence, we cannot generate any other element except a, so a is not a generator.

For b, $b * b = a$, then $(b * b) * b = a * b = b$ Then, $(b * b * b) * b = b * b = a$, and so on. Thus, we can only regenerate a and b. So b is not a generator.

Now, for c, $c * c = b$. Then, $(c * c) * c = b * c = d$. Then $(c * c * c) * c = d * c = a$. Then $(c * c * c * c) * c = a * c = c$. We see that we have generated all elements of the group. Thus, c is a generator.

Now, for d, $d * d = b$. Then, $(d * d) * d = b * d = c$. Then $(d * d * d) * d = c * d = a$. Then $(d * d * d * d) * d = a * d = d$. We see that we have generated all elements of the group. Thus, d is a generator.

Hence, c and d are generators.

Ans. (c)

20. Let $G = (V, E)$ be a graph. Define $\xi(G) = \sum_d i_d \times d$, where i_d is the number of vertices of degree d in G. If S and T are two different trees with $\xi(S) = \xi(T)$, then

(a) $|S| = 2|T|$ (b) $|S| = |T| - 1$

(c) $|S| = |T|$ (d) $|S| = |T| + 1$

(GATE 2010, 1 Mark)

Solution: The expression $\xi(G)$ is basically sum of all degrees in a tree.

Now, $\xi(G)$ and $\xi(T)$ are same for two trees, then the trees have same number of vertices.

Let it be true for n vertices. If we add a vertex, then the new vertex (it is not the first node) increases degree by 2. It doesn't matter where we add it.

Ans. (c)

21. Let G be a simple undirected planar graph on 10 vertices with 15 edges. If G is a connected graph, then the number of bounded faces in any embedding of G on the plane is equal to

(a) 3 (b) 4 (c) 5 (d) 6

(GATE 2012, 1 Mark)

Solution: We have the relation $V - E + F = 2$. Given that $V = 10$ and $E = 15$. Thus, $10 - 15 + F = 2 \Rightarrow F = 7$.

Out of 7 faces, one is an unbounded face. Hence total 6 bounded faces.

Ans. (d)

22. Let $G = (V, E)$ be a directed graph where V is the set of vertices and E the set of edges. Then which one of the following graphs has the same strongly connected components as G?

(a) $G_1 = (V, E_1)$ where $E_1 = \{(u, v) \mid (u, v) \notin E\}$
(b) $G_2 = (V, E_2)$ where $E_2 = \{(u, v) \mid (v, u) \notin E\}$
(c) $G_3 = (V, E_3)$ where $E_3 = \{(u, v) \mid$ there is a path of length ≤ 2 from u to v in $E\}$
(d) $G_4 = (V, E_4)$ where V_4 is the set of vertices in G which are not isolated

(GATE 2014, 1 Mark)

Solution: Let G be the below graph.

$$\text{A} \longrightarrow \text{B}$$

Then G_3 is a graph with below structure.

$$\begin{tikzcd} \text{A} \arrow[r] & \text{B} \arrow[l, bend right] \end{tikzcd}$$

In G, the numbers of strongly connected components are 2, whereas in G_3 it is only one.

Ans. (b)

23. Let G be a graph with n vertices and m edges. What is the tightest upper bound on the running time of Depth First Search on G, when G is represented as an adjacency matrix?

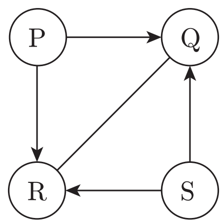
(a) $\Theta(n)$ (b) $\Theta(n+m)$ (c) $\Theta(n^2)$ (d) $\Theta(m^2)$

(GATE 2014, 1 Mark)

Solution: Tightest upper bound for running depth first search using adjacency matrix is $\Theta(V^2)$. Number of vertices here are n. So, $\Theta(n^2)$ is the correct option.

Ans. (c)

24. Consider the directed graph given below.



Which one of the following is TRUE?

(a) The graph does not have any topological ordering.
(b) Both PQRS and SRQP are topological orderings.
(c) Both PSRQ and SPRQ are topological orderings.
(d) PSRQ is the only topological ordering.

(GATE 2014, 1 Mark)

Solution: Topological order of a directed graph is the linear arrangement of vertices in which if there is an edge between u and v, then u comes before v in order. Option (c) satisfies this property, so both PSRQ and SPRQ are topological orderings.

Ans. (c)

25. The maximum number of edges in a bipartite graph on 12 vertices is ________.

(GATE 2014, 1 Mark)

Solution: The maximum number of edges in bipartite graph $= \dfrac{n^2}{4}$

For $n = 12$, maximum number of edges in bipartite graph $= \dfrac{12 \times 12}{4} = 36$ edges.

Ans. 36

26. Consider the tree arcs of a BFS traversal from a source node W in an unweighted, connected, undirected graph. The tree T formed by the tree arcs is a data structure for computing

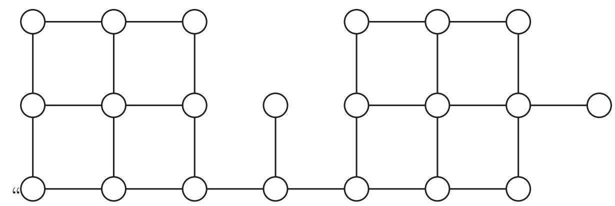
(a) the shortest path between every pair of vertices.
(b) the shortest path from W to every vertex in the graph.
(c) the shortest paths from W to only those nodes that are leaves of T.
(d) the longest path in the graph.

(GATE 2014, 1 Mark)

Solution: Application of BFS is to find shortest path from u to v. In the given statement, W is the source node. It will calculate shortest path to every node from W. So, option (b) is correct.

Ans. (b)

27. Suppose depth first search is executed on the graph below starting at some unknown vertex. Assume that a recursive call to visit a vertex is made only after first checking that the vertex has not been visited earlier. Then the maximum possible recursion depth (including the initial call) is ________.



(GATE 2014, 1 Mark)

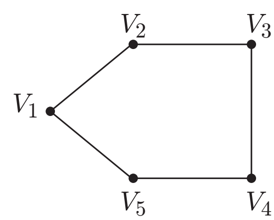
Solution: In depth first search, the nodes are traversed in depth. Maximum possible depth is 19.

Ans. 19

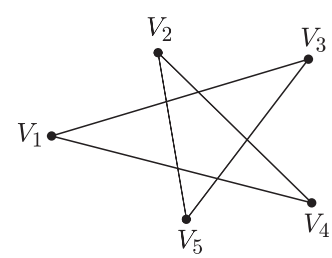
28. A cycle on n vertices is isomorphic to its complement. The value of n is ________.

(GATE 2014, 2 Marks)

Solution: Consider $n = 5$



Cycle graph



Complement of cycle graph

By checking properties, these two graphs are isomorphic.

Thus, the value of n is 5.

Ans. 5

29. If G is a forest with n vertices and k connected components, how many edges does G have?

(a) $[n/k]$
(b) $[n/k]$
(c) $n - k$
(d) $n - k + 1$

(GATE 2014, 2 Marks)

Solution: In forest, each component is a tree. There are G_1, G_2, …, G_k components. Each has $n - 1$ edges.

$$\sum_{i=1}^{k} n - i = n - k \text{ edges}$$

Ans. (c)

30. Let δ denote the minimum degree of a vertex in a graph. For all planar graphs on n vertices with $\delta \geq 3$, which one of the following is TRUE?

(a) In any planar embedding, the number of faces is at least $\dfrac{n}{2}+2$

(b) In any planar embedding, the number of faces is less than $\dfrac{n}{2}+2$

(c) There is a planar embedding in which the number of faces is less than $\dfrac{n}{2}+2$

(d) There is a planar embedding in which the number of faces is at most $\dfrac{n}{\delta+1}$

(GATE 2014, 2 Marks)

Solution: Degree ≥ 3 is of n vertices.
Means $3n \leq 2e \rightarrow e \geq 3n/2$
We know that $e = n + r - 2$
$n + r - 2 \geq 3n/2$
or $r \geq n/2 + 2$

Ans. (a)

31. Let G be a connected planar graph with 10 vertices. If the number of edges on each face is three, then the number of edges in G is ________.

(GATE 2015, 1 Mark)

Solution: By Euler's formula,
$v + f = e + 2$, where v, e and f are, respectively, number of vertices, edges and faces.

We are given $v = 10$ and number of edges on each face $= 3$.

Thus, $3f = 2e \Rightarrow f = \dfrac{2}{3}e$

Putting all the values in Euler's formula, we get

$$10+\frac{2}{3}e = e+2 \Rightarrow \frac{e}{3} = 8 \Rightarrow e = 24$$

Ans. 24

32. The minimum number of colours that is sufficient to vertex-colour any planar graph is ________.

(GATE 2016, 1 Mark)

Solution: By hit and trial approach or by 4-colour theorem, every planar graph is 4-colourable.

Ans. 4

33. Consider the weighted undirected graph with 4 vertices, where the weight of edge $\{i, j\}$ is given by the entry W_{ij} in the matrix W.

$$W = \begin{bmatrix} 0 & 2 & 8 & 5 \\ 2 & 0 & 5 & 8 \\ 8 & 5 & 0 & x \\ 5 & 8 & x & 0 \end{bmatrix}$$

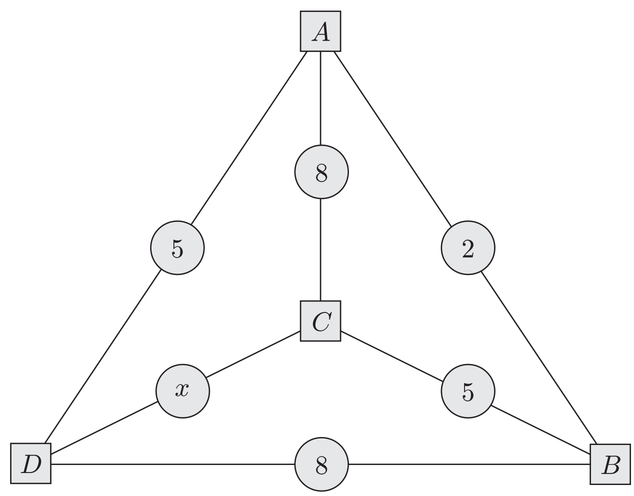
The largest possible integer value of x, for which at least one shortest path between some pair of vertices will contain the edge with weight x is ________.

(GATE 2016, 2 Marks)

Solution: Let us exclude the edge labelled x. Therefore, the shortest paths are given as follows:

AB=2; AC=7; AD=5; BC=5; BD=7; CD=12

Here, the lengths of the shortest paths can only decrease (or remain equal) if we add a new edge. The most likely is the longest shortest path: CD=12.



It is obvious that if we set x=12 on the edge joining CC and DD, there can be two shortest paths (both of length 12):

1. $C \rightarrow DC \rightarrow D$
2. $C \rightarrow B \rightarrow A \rightarrow D$

Hence, if $x \leq 12$, there can be at least one shortest path that passes through x, as requested.

We can see that a larger x (say, 13) cannot be used because all the existing shortest paths are already smaller or equal to 12.

Note: The question reads as follows: The largest possible integer value of x, for which at least one shortest path between some pair of vertices will contain the edge with weight x. The question means that "at least one shortest path", but not "every shortest path". If it is misunderstood, then it would mean as "every shortest path". Then, in such case, the answer would be 11 because we had to "beat" the shortest path $C \to B \to A \to D = 12$. However, since the question asks for "at least one" among the shortest paths, then the answer is 12 because having two shortest paths of length 12 is reasonable since one contains the edge x.

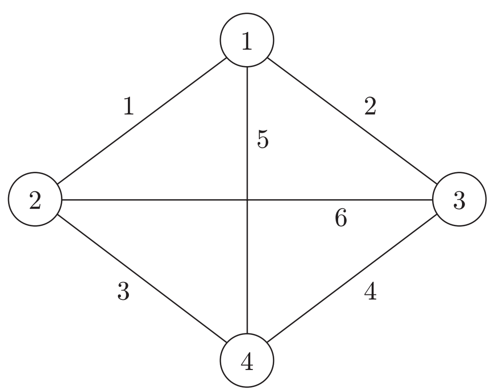
Ans. 12

34. Let G be a complete undirected graph on 4 vertices, having 6 edges with weights being 1, 2, 3, 4, 5 and 6. The maximum possible weight that a minimum weight spanning tree of G can have is ________.

(GATE 2016, 2 Marks)

Solution: We can have MST (minimum spanning tree) of 1, 2 or 3 and 1, 2 or 4. Therefore,

$$1 + 2 + 4 = 7$$



Ans. 7

35. $G = (V, E)$ is an undirected simple graph in which each edge has a distinct weight, and e is a particular edge of G. Which of the following statements about the minimum spanning trees (MSTs) of G is/are **TRUE**?

I. If e is the lightest edge of some cycle in G, then every MST of G includes e.

II. If e is the heaviest edge of some cycle in G, then every MST of G excludes e.

(a) I only (b) II only
(c) Both I and II (d) Neither I nor II

(GATE 2016, 2 Marks)

Solution: Since MST asked so if e is any heaviest edge in cycle, it obviously will not be involved. Therefore, only statement (II) is true.

Ans. (b)

36. Consider a set U of 23 different compounds in a Chemistry lab. There is a subject S of U of 9 compounds, each of which reacts with exactly 3 compounds of U. Consider the following statements:

I. Each compound in U\S reacts with an odd number of compounds.

II. At least one compound in U\S reacts with an odd number of compounds.

Which one of the above statements is **ALWAYS TRUE**?

(a) Only I (b) Only II
(c) Only III (d) None

(GATE 2016, 2 Marks)

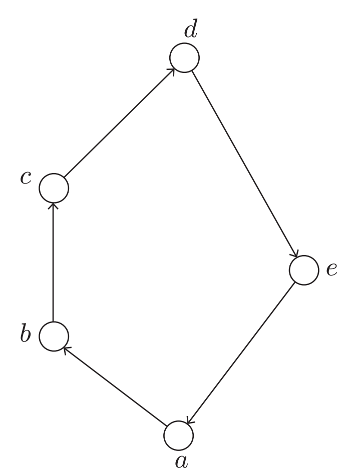
Solution: Let us denote the problem by a non-directed graph with 23 vertices (compounds). If two compounds react with each other, then there exists an edge between the corresponding vertices. In the graph, we have nine vertices with degree 3 (i.e. odd degree). From sum of degrees of vertices theorem, at least one of the remaining vertices should have odd degree.

Ans. (b)

37. Consider the set $X = \{a, b, c, d, e\}$ under the partial ordering

$R = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, b), (b, c), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)\}$

The Hasse diagram of the partial order (X, R) is shown below:



The minimum number of ordered pairs that need to be added to R to make (X, R) a lattice is ________.

(GATE 2017, 1 Mark)

Solution: As the given POSET is already a lattice, there is no need to add any ordered pair.

Ans. (0)

38. G is an undirected graph with n vertices and 25 edges such that each vertex of G has degree at least 3. Then the maximum possible value of n is ________.

(GATE 2017, 1 Mark)

Solution: As we know,

Sum of degree of n vertices = 2 × Number of edges

Since each vertex has degree at least 3. We have, from the above equation

$$\Rightarrow 2 \times 25 \geq 3 \times n$$
$$\Rightarrow n \leq 50/3$$
$$\Rightarrow n \leq 16$$

Ans. (16.0)

39. Let T be a tree with 10 vertices. The sum of the degrees of all the vertices in T is ________.

(GATE 2017, 1 Mark)

Solution: Given, $V = 10$

As we know,

Sum of degrees = 2 × Number of edges

and

$$e = V - 1$$
$$\Rightarrow e = 10 - 1$$
$$\Rightarrow e = 9$$

Therefore, sum of degrees = 2 × 9 = 18.

Ans. (18.0)

CHAPTER 11

TRANSFORM THEORY

INTRODUCTION

The chapter presents three important mathematical tools that are used extensively for analysis of signals and systems. These are Laplace transform, z-transform and Fourier transform. While Laplace transform is used to represent continuous time signals in s-domain (s being a complex variable), z-transform is used to represent discrete time signals in z-domain (z being a complex variable). Fourier transform is a frequency domain specification of a non-periodic sequence. All three are described in the following sections.

LAPLACE TRANSFORM

As outlined above, Laplace transform is used to represent continuous time signals in s-domain where s is a complex variable. Laplace transform can be used to convert integro-differential equations representing continuous time linear time invariant systems into algebraic equations thereby simplifying their analysis.

Laplace transform $X(s)$ of a continuous time signal $x(t)$ is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{1}$$

Variable s is generally complex and is given by $s = (s + j\omega)$. Laplace transform defined by Eq. (1) is also called bilateral Laplace transform. A unilateral Laplace transform is defined as

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt \tag{2}$$

The two Laplace transforms are equivalent only when $x(t) = 0$ for $t < 0$. Region of convergence (ROC) is defined with respect to Laplace transforms. The range of values of complex variables (s) for which Laplace transform converges is called the region of convergence.

Laplace Transform of Common Signals

1. Laplace transform of a unit step function $u(t)$ as shown in Fig. 1 is given by

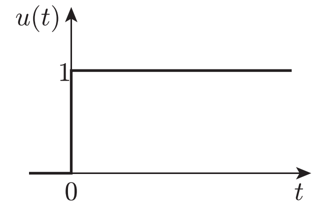


Figure 1 | Unit step function.

$$L[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0^+}^{\infty} e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big|_{0^+}^{\infty} = \frac{1}{s}, \text{Re}(s) > 0$$

$$\text{where } 0^+ = \lim_{\varepsilon\to 0}(0+\varepsilon) \tag{3}$$

2. Laplace transform of a unit impulse function as shown in Fig. 2 is given by

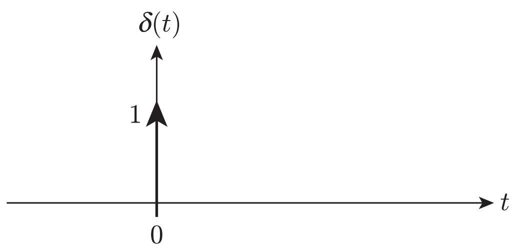


Figure 2 | Unit impulse function.

$$L[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1 \text{ for all } s \tag{4}$$

3. Laplace transform of some other common signals are given as under.

a. $$L[-u(-t)] = \frac{1}{s}\text{Re}(s) < 0 \tag{5}$$

b. $$L[tu(t)] = \frac{1}{s^2}\text{Re}(s) > 0 \tag{6}$$

c. $$L\left[t^k u(t)\right] = \frac{k!}{s^{(k+1)}}\text{Re}(s) > 0 \tag{7}$$

d. $$L\left[e^{-at}u(t)\right] = \frac{1}{(s+a)}\text{Re}(s) > -\text{Re}(a) \tag{8}$$

e. $$L\left[-e^{-at}u(-t)\right] = \frac{1}{(s+a)}\text{Re}(s) < -\text{Re}(a) \tag{9}$$

f. $$L\left[te^{-at}u(t)\right] = \frac{1}{(s+a)^2}\text{Re}(s) > -\text{Re}(a) \tag{10}$$

g. $$L\left[-te^{-at}u(-t)\right] = \frac{1}{(s+a)^2}\text{Re}(s) < -\text{Re}(a) \tag{11}$$

h. $$L\left[\cos\omega_0 t\, u(t)\right] = \frac{s}{(s^2+\omega_0^2)}\text{Re}(s) > 0 \tag{12}$$

i. $$L\left[\sin\,\omega_0 t\, u(t)\right] = \frac{\omega_0}{(s^2+\omega_0^2)}\text{Re}(s) > 0 \tag{13}$$

j. $$L\left[e^{-at}\cos\omega_0 t\, u(-t)\right]$$
$$= \frac{(s+a)}{(s+a)^2+\omega_0^2}\text{Re}(s) > -\text{Re}(a) \tag{14}$$

k. $$L\left[e^{-at}\sin\omega_0 t\; u(t)\right]$$
$$= \frac{\omega_0}{(s+a)^2+\omega_0^2}\text{Re}(s) > -\text{Re}(a) \tag{15}$$

Properties of Laplace Transform

Laplace transforms exhibit the following properties:

1. Linearity: If $X_1(s)$ and $X_2(s)$ are, respectively, Laplace transforms of $x_1(t)$ and $x_2(t)$, then

$$L\left[a_1x_1(t) + a_2x_2(t)\right] = a_1X_1(s) + a_2X_2(s)$$

If the regions of convergence of $X_1(s)$ and $X_2(s)$ are R_1 and R_2, respectively, then the region of convergence of resultant transform R is equal to intersection of R_1 and R_2.

2. Time shifting: If Laplace transform of $x(t)$ is $X(s)$, Laplace transform of $x(t-t_0)$ is given by

$$L\left[x(t-t_0)\right] \;=\; e^{-st_0}X(s) \tag{16}$$

Both transforms have same region of convergence.

3. Time scaling: If Laplace transform of $x(t)$ is $X(s)$ with a region of convergence equal to R, then Laplace transform of $x(at)$ is given by

$$L[x(at)] = \frac{1}{|\,a\,|}\times\left(\frac{s}{a}\right) \tag{17}$$

The region of convergence of resultant transform would be aR.

4. Shifting in s-domain: If Laplace transform of $x(t)$ is $X(s)$ having a region of convergence R, then

$$L\left[e^{s_0t}x(t)\right] = X(s-s_0) \tag{18}$$

Region of convergence of resultant transform $= R + \text{Re}(s_o)$

5. Time reversal: If Laplace transform of $x(t)$ is $X(s)$ with region of convergence R, then

$$L[x(-t)] = X(-s) \tag{19}$$

The transform has region of convergence $-R$.

6. **Differentiation in time domain:** If Laplace transform of $x(t)$ is $X(s)$, then

$$L\left[\frac{dx(t)}{dt}\right] = sX(s) \tag{20}$$

If R is region of convergence of X(s), then region of convergence R' of the resultant transform is expressed by $R' \supset R$.

7. **Differentiation in *s*-domain:** If Laplace of $x(t)$ is $X(s)$ with a region of convergence R, then

$$L[-tx(t)] = \frac{dX(s)}{ds} \tag{21}$$

The resultant transform also has region of convergence R.

8. **Integration in time domain:** If Laplace transform of $x(t)$ is $X(s)$, then t

$$L\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{X(s)}{s} \tag{22}$$

If R is region of convergence for $X(s)$, region of convergence R' of the resultant transform is expressed by

$$R' = R \cap \{\mathrm{Re}(s) > 0\}$$

9. **Convolution:** If $X_1(s)$ and $X_2(s)$ are Laplace transforms of $x_1(t)$ and $x_2(t)$, respectively, then

$$L[x_1(t) * x_2(t)] = X_1(s) \cdot X_2(s) \tag{23}$$

If R_1 and R_2 are regions of convergence of $X_1(s)$ and $X_2(s)$, respectively, then region of convergence of resultant transform is expressed by $R' \supset R_1 \cap R_2$. Convolution in time domain is multiplication in *s*-domain.

Inverse Laplace Transform

Inverse Laplace transform of $X(s)$ is given by

$$L^{-1}\, X(s) = x(t) = \frac{1}{2\pi j} \int_{c-j\omega}^{c+j\omega} X(s) e^{st} ds \tag{24}$$

In the above integral, the real (c) is to be selected in such a way that if region of convergence of $X(s)$ is $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, then $\sigma_1 < c < \sigma_2$

If $x(t) = x_1(t) + x_2(t) + x_3(t) + \cdots + x_n(t)$

Then, $X(s) = X_1(s) + X_2(s) + X_3(s) + \cdots + X_n(s)$

If $X(s)$ is a rational function of the form:

$$X(s) = \frac{N(s)}{D(s)} = k\frac{(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

Then inverse Laplace transform of the above function can be determined by partial fraction expansion.

Proper rational function $(m < n)$: If the poles of $X(s)$ are simple, then partial fraction expansion of $X(s)$ can be written as

$$X(s) = \frac{c_1}{(s - p_1)} + \cdots + \frac{c_n}{(s - p_n)} \tag{25}$$

where coefficients (C_k) are given by

$$C_k = (s - p_k)X(s)|s = p_k$$

If the denominator $D(s)$ has multiple roots, then expansion of $X(s)$ consists of terms of the form:

$$\frac{\lambda_1}{(s - p_i)} + \frac{\lambda_1}{(s - p_i)^2} + \cdots + \frac{\lambda_r}{(s - p_i)^r} \tag{26}$$

where $\lambda_{r-k} = \dfrac{1}{k!}\dfrac{d^k}{ds^k}[(s - p_i)^r X(s)]|s = p_i$

Improper rational function $(m \geq n)$: When $X(s)$ is an improper rational function, that is, $m \geq n$, $X(s)$ can be written in the following form:

$$X(s) = \frac{N(s)}{D(s)} = Q(s) + \frac{R(s)}{D(s)}$$

$X(s)$ is a polynomial in s with degree $(m - n)$. $R(s)$ is a polynomial in s with degree strictly less than n. The inverse Laplace transform of $X(s)$ can be determined from inverse Laplace transform of $Q(s)$ and $R(s)/D(s)$. The inverse Laplace transforms of $Q(s)$ can be determined by using transform below:

$$L\frac{d^k \delta(t)}{dt^k} = s^k,\ k = 1, 2, 3, \ldots$$

Laplace transform of $R(s)/D(s)$ can be computed by partial fraction method explained earlier. Note that $R(s)/D(s)$ is a proper rational function.

z-TRANSFORM

z-Transform is the discrete time counter part of Laplace transform. *z*-Transform is used to convert difference equations representing discrete time signals into algebraic equations. Expression of discrete time signals in the form of algebraic equations simplifies analysis of discrete time systems in the same way as the Laplace transform does to the continuous time signals in the case of continuous time systems.

The *z*-transform $X(z)$ of a discrete time signal or sequence is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{27}$$

Variable z is generally complex valued and is expressed in polar form by

$$Z = re^{j\Omega} \tag{28}$$

where r and Ω, respectively, are magnitude and angle of z.

The z-transform defined by Eq. (27) is a bilateral or two-sided transforms unilateral or one-sided z-transform is defined as follows:

$$X'(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{29}$$

$x[n]$ and $X(z)$ are said to form a transform pair denoted as

$$x[n] \leftrightarrow X(z) \tag{30}$$

Again as in the case of Laplace transform, a region of convergence is defined in the case of z-transform too. Region of convergence (ROC) in the case of z-transform is the range of value of complex variable z for which the z-transform converges. To illustrate the term *region of convergence*, let us consider

$$x[n] = a^n u[n], \text{ where } a \text{ is real}$$

z-Transform of $x[n]$ can be written as

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of z-transform,

$$\sum_{n=0}^{\infty} (az^{-1}) < \infty \text{ or } |az^{-1}| < 1$$

For convergence of z-transform,

$$\sum_{n=0}^{\infty} (az^{-1})^n < \infty \text{ or } |az^{-1}| < 1$$

This implies that $|z| > |a|$.

z-Transform of Common Sequences

1. z-Transform of a unit impulse sequence $\delta[n]$ is given by

$$X(z) = \sum_{n=0}^{\infty} \delta[n]z^{-n} = z^{-n} = 1 \text{ for all } z$$

Therefore, $$\delta[n] \leftrightarrow 1 \text{ for all } z. \tag{31}$$

2. z-Transform of a unit step sequence $u[n]$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} u[n]\, z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}, |z| > 1$$

Therefore, $$u[n] \leftrightarrow \frac{z}{z-1} \text{ for all } z. \tag{32}$$

3. Some other common z-transform pairs are given as under.

a. $$-u[-n-1] \leftrightarrow \frac{z}{z-1} \text{ for } |z| < 1 \tag{33}$$

b. $$\delta[n-m] \leftrightarrow z^{-m} \text{ for all } z \text{ except } z = 0$$
$$(\text{for } m > 0) \text{ and } \infty (\text{for } m < 0) \tag{34}$$

c. $$a^n u[n] \leftrightarrow -\frac{z}{z-a} \text{ for } |z| > |a| \tag{35}$$

d. $$-a^n u[-n-1] \leftrightarrow \frac{z}{z-a} \text{ for } |z| < |a| \tag{36}$$

e. $$na^n u[n] \leftrightarrow \frac{az}{(z-a)^2} \text{ for } |z| > |a| \tag{37}$$

f. $$-na^n u[-n-1] \leftrightarrow \frac{az}{(z-a)^2} \text{ for } |z| < |a| \tag{38}$$

g. $$(n+1)a^n u[n] \leftrightarrow \left[\frac{z}{(z-a)}\right]^2 \text{ for } |z| > |a| \tag{39}$$

h. $$(\cos\Omega_0 n)u[n] \leftrightarrow \frac{z^2 - (\cos\Omega_0)z}{z^2 - (2\cos\Omega_0)z + 1}$$
$$\text{for } |z| > |1| \tag{40}$$

i. $$(\sin\Omega_0 n)u[n] \leftrightarrow \frac{(\sin\Omega_0)z}{z^2 - (2\cos\Omega_0)z + 1}$$
$$\text{for } |z| > |1| \tag{41}$$

j. $$(r^n \cos\Omega_0 n)u[n] \leftrightarrow$$
$$\frac{z^2 - (r\cos\Omega_0)z}{z^2 - (2r\cos\Omega_0)z + r^2} \text{ for } |z| \tag{42}$$

k. $$(r^n \sin\Omega_0 n)u[n] \leftrightarrow$$
$$\frac{(r\sin\Omega_0)z}{z^2 - (2r\cos\Omega_0)z + r^2} \text{ for } |z| > r \tag{43}$$

l. $$\begin{bmatrix} a^n, \text{ for } 0 \le n \le N-1 \\ 0, \quad \text{otherwise} \end{bmatrix} \leftrightarrow$$
$$\frac{1 - a^N z^{-N}}{1 - az^{-1}} \text{ for } |z| > 0 \tag{44}$$

Properties of z-Transform

1. z-Transform exhibits the property of linearity. If $X_1(z)$ and $X_2(z)$, respectively, are z-transforms of $X_1[n]$ and $X_2[n]$, then

$$a_1x_1[n] + a_2x_2[n] \leftrightarrow a_1X_1(z) + a_2X_2(z) \qquad (45)$$

If R_1 and R_2 are ROC of $X_1(z)$ and $X_2(z)$, respectively, then ROC of $a_1X_1(z) + a_2X_2(z)$ is given by $R' \supset R_1 \cap R_2$. (a_1) and (a_2) are arbitrary constants.

2. If z-transform of $x[n]$ is $X(z)$, then z-transform of $x[n - n_0]$ is given by $z^{-n0}X(z)$

That is, $$x[n - n_0] \leftrightarrow z^{n0}X(z) \qquad (46)$$

The region of convergence is given by $R' = R \cap \{0 < |z| < \infty\}$, where R is region of convergence for $x(z)$. This is the time shifting property. As an example, z-transform of $x[n - 1]$ is given by $z^{-1}X(z)$ with region of convergence given by $R' = R \cap \{|z| < \infty\}$.

If $X(z)$ is z-transform of $x[n]$ with ROC (R), then z-transform of ${z_0}^n x[n]$ is given by

$$z_0^n x[n] \leftrightarrow x\left(\frac{z}{z_0}\right) \text{ with ROC } R = |z_0|\ R \qquad (47)$$

3. A pole (or zero) at $z = z_k$ in $X(z)$ moves to $z = z_0 z_k$ after multiplication of $x[n]$ by ${z_0}^n$. The region of convergence also gets multiplied by $|z_o|$.

As a special case of this property, following transform holds:

$$e^{j\Omega_0 n}x[n] \leftrightarrow X(e^{-j\Omega_0}z) \text{ with } R' = R \qquad (48)$$

In this special case, all poles and zeros are rotated by angle Ω_0 and the region of convergence remains unchanged.

4. According to the time reversal property of z-transform, if $X(z)$ is z-transform of $x[n]$ with ROC $= R$, then

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \text{ with ROC } R' = \frac{1}{R} \qquad (49)$$

This implies that a pole or zero at $z = z_k$ moves to $\left(\frac{1}{z_k}\right)$ after time reversal. As far as the ROC is concerned, time reversal means that a right-sided sequence becomes a left-sided sequence and vice versa.

5. In the case of multiplication of discrete sequence $x[n]$ by n, the following transform pair holds:

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \qquad (50)$$

The region of convergence remains unchanged, i.e., $R' = R$.

6. If $X(z)$ is z-transform of $x[n]$ with ROC $= R$, following transform pair holds:

$$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{1}{1 - z^{-1}}X(z) = \frac{z}{(z-1)}X(z) \qquad (51)$$

Region of convergence R' is given by $R' = R \supset \{|z| > 1\}$

$\sum_{k=-\infty}^{n} x[k]$ is the discrete time counterpart of integration in the continuous time domain.

7. According to the convolution property of z-transform,

If $X_1(z)$ is z-transform of $x_1[n]$ with ROC $= R_1$

$X_2(z)$ is z-transform of $x_2[n]$ with ROC $= R_2$

Then $$x_1[n] * x_2[n] \leftrightarrow X_1(z) \cdot X_2(z) \qquad (52)$$

And $R' \supset R_1 \cap R_2$

Inverse z-Transform

Inverse z-transform of $x(z)$ is given by

$$x[n] = \frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz \qquad (53)$$

where C is the counterclockwise contour of integration enclosing origin.

As an alternative approach, inverse z-transform of $X(z)$ may be found by expressing $x(z)$ as a sum of functions that have known inverse transforms and using linearity property. For example,

if $X(z) = X_1(z) + X_2(z) + \cdots + X_n(z)$
then $x[n] = x_1[n] + x_2[n] + \cdots + x_n[n]$
where $x_1[n] \leftrightarrow X_1(z)$
$x_2[n] \leftrightarrow X_2(z)$
.
.
.
.
$x_n[n] \leftrightarrow X_n(z)$

z-Transform as Power Series Expansion

z-Transform given by Eq. (27) earlier can also be represented by a power series where the sequence values of $x[n]$ are nothing but the coefficients of z^{-n}.

That is,

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \ldots\ldots + x[-2]z^2 + x[-1]z + x(0) \\ &\quad + x(1)z^{-1} + x[2]z^{-2} + \ldots \end{aligned} \tag{54}$$

This approach is particularly useful for finite-length sequences.

Partial Fraction Expansion

Partial fraction expansion method can be used to determine inverse z-transform when $X(z)$ is a rational function of z.

If $X(z)$ is expressed by

$$X(z) = \frac{N(z)}{D(z)} = k \cdot \frac{(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

If $n \geq m$ and all poles are simple, then

$$\begin{aligned} \frac{X(z)}{z} &= \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots + \frac{c_n}{z-p_n} \\ &= \frac{c_0}{z} + \sum_{k=1}^{n} \frac{c_k}{z-p_k} \end{aligned} \tag{55}$$

Where $c_0 = X(z)\,|_{z=0}$ and $c_k = (z-p_k)\dfrac{X(z)}{z}\,|_{z=p_k}$

Equation (56) may be written as

$$X(z) = c_0 + \sum_{k=1}^{n} c_k \frac{z}{z-p_k} \tag{56}$$

If $m > n$, then a polynomial of z must be added to the right-hand side of Eq. (57), the order of which is $(m-n)$. In that case, partial fraction expansion has the following form:

$$X(z) = \sum_{q=0}^{m-n} b_q z^q + \sum_{k=1}^{n} c_k \frac{z}{z-p_k} \tag{57}$$

In the case of multiple order poles, the partial fraction expansion of $\dfrac{X(z)}{z}$ will consists of terms of the form (assuming pole (p_i) has multiplicity of r),

$$\frac{\lambda_1}{(z-p_i)} + \frac{\lambda_2}{(z-p_i)^2} + \cdots + \frac{\lambda_r}{(z-p_i)^r}$$

where $$\lambda_{r-k} = \frac{1}{k!}\frac{d^k}{dz^k}\left[(z-p_i)^r \frac{X(z)}{z}\right]\Bigg|_{z=p_i} \tag{58}$$

Relationship between *z*-Transform and Laplace Transform

z-Transform is related to Laplace transform by following expressions.

1. $s = \dfrac{1}{T} \ln z$ (59)

2. $X(s) = X(z)|z = e^{+Ts}$ (60)

where T is the sampling period.

FOURIER TRANSFORM

Fourier transform transforms a time domain signal into its frequency domain counterpart. If $X(\omega)$ is the Fourier transform of $x(t)$, then

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{61}$$

The inverse Fourier transform is defined by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{62}$$

$x(t)$ and $X(\omega)$ form a Fourier transform pair written as

$$x(t) \leftrightarrow X(\omega)$$

$X(\omega)$ is, in general, complex and is expressed as

$$X(\omega) = |X(\omega)|e^{j\phi(w)} \tag{63}$$

$|X(\omega)|$ is known as the magnitude spectrum and $\phi(\omega)$ is called the phase spectrum of $X(\omega)$. If $X(t)$ is a real signal, we can write

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)\,e^{j\omega t}\ dt \tag{64}$$

And $X(-\omega) = X^*(\omega)$

We can also write

$$|X(-\omega)| = |X(\omega)| \text{ and } \phi(-\omega) = -\phi(\omega) \tag{65}$$

It follows from Eq. (64) that in the case of periodic signals, the amplitude spectrum $|X(\omega)|$ is an even function and phase spectrum $\phi(\omega)$ is an odd function.

Convergence of Fourier Transforms

Dirichlet conditions for convergence of Fourier transforms are defined as follows:

1. $x(t)$ is absolutely integrable, i.e. $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$.
2. $x(t)$ has a finite number of maxima and minima within any finite integer.
3. $x(t)$ has a finite number of discontinuities within any finite interval. Each of the discontinuities is finite.

Compliance of the above-mentioned Dirichlet conditions guarantees the existence of Fourier transform. It may be noted that if impulse functions are permitted in the transform, signals not satisfying these conditions can have Fourier transform.

Properties of Fourier Transform

Important properties of Fourier transforms, many of which are similar to those of Laplace transforms are briefly described as under.

1. According to linearity property, if $X_1(\omega)$ and $X_2(\omega)$ are Fourier transform of $x_1(t)$ and $x_2(t)$, respectively, then

$$a_1x_1(t) + a_2x_2(t) \leftrightarrow a_1X_1(\omega) + a_2X_2(\omega) \tag{66}$$

(a_1) and (a_2) are arbitrary constants.

2. According to the time shifting property, a shift in time causes addition of a linear phase shift in the Fourier spectrum.

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega) \tag{67}$$

3. Complex modulation in time domain causes frequency shifting in frequency domain, that is,

$$e^{j\omega_0 t} x(t) \leftrightarrow X(\omega - \omega_0) \tag{68}$$

4. Scaling in time domain causes inverse scaling in frequency domain.

$$x(at) \leftrightarrow \frac{1}{|a|} \times \left(\frac{\omega}{a}\right) \tag{69}$$

Inverse scaling of both frequency variable (ω) and amplitude scaling of $X\left(\frac{\omega}{a}\right)$ by $\frac{1}{|a|}$ takes place.

According to time reversal property,

$$x(-t) \leftrightarrow X(-\omega) \tag{70}$$

6. According to property of duality or symmetry

$$X(t) \leftrightarrow 2\pi x(-\omega) \tag{71}$$

7. Differentiation in time domain causes multiplication is frequency domain is

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega) \tag{72}$$

8. Differentiation in frequency domain causes multiplication in time domain is

$$(-jt)x(t) \leftrightarrow \frac{dX(\omega)}{d\omega} \tag{73}$$

9. Fourier transform of integration in time domain is given by

$$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \pi X(0)\delta(\omega) + \frac{1}{j\omega} X(\omega) \tag{74}$$

10. Convolution in time domain is multiplication in frequency domain

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega) \cdot X_2(\omega) \tag{75}$$

The above equation is also called *time convolution theorem*.

11. Convolution in frequency domain is multiplication in time domain this is also known as frequency convolution theorem.

$$x_1(t) \cdot x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega) \tag{76}$$

12. If $x(t)$ is real and given by $x_e(t) + x_o(t)$ where $x_e(t)$ and $x_o(t)$ are even and odd components of $x(t)$ and if

$$x(t) \leftrightarrow X(\omega) = A(\omega) + jB(\omega) \tag{77}$$

and $$X(-\omega) = X^*(\omega)$$

then $$x_e(t) \leftrightarrow A(\omega) \tag{78}$$

and $$x_o(t) \leftrightarrow j\,B(\omega) \tag{79}$$

This implies that Fourier transform of an even signal is a real function of ω and Fourier transform of an odd signal is a pure imaginary function of ω.

13. Bilateral Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)\,e^{-\sigma t}$.

SOLVED EXAMPLES

1. Determine Laplace transform of $x(t) = e^{at}u(-t)$.

Solution: Laplace transform of $x(t)$ is given by

$$X(s) = \int_{-\infty}^{\infty} e^{at}u(-t)\,e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{at}\cdot e^{-st}dt = \int_{-\infty}^{\infty} e^{-t(s-a)}dt$$

$$= -\frac{1}{(s-a)}e^{-t(s-a)}\Big|_{-\infty}^{0}$$

$$= -\frac{1}{(s-a)} \text{ for } \mathrm{Re}(s) < a$$

2. Determine the Laplace transform and region of convergence of $x(t) = e^{-3t}u(t) + e^{-4t}u(-t)$.

Solution: Laplace transform of

$$e^{-3t}u(t) = \frac{1}{(s+3)} \text{ with } \mathrm{Re}(s) > -3$$

Laplace transform of

$$e^{-4t}u(-t) = \frac{1}{(s+4)} \text{ with } \mathrm{Re}(s) < -4$$

Therefore, Laplace transform of the given function is given by

$$X(s) = \frac{1}{s+3} - \frac{1}{s+4} = \frac{1}{(s+3)(s+4)}$$

Region of convergence is given by $-3 < \mathrm{Re}(s) < -4$.

3. Determine the Laplace transform and associated region of convergence of

$$x(t) = e^{-2t}[u(t) - u(t-5)]$$

Solution: $x(t)$ can be rewritten as $x(t) = e^{-2t}u(t) - e^{-2t}u(t-5)$

$$= e^{-2t}u(t) - e^{-10}\cdot e^{-2(t-5)}u(t-5)$$

Now Laplace transform of $e^{-2t}u(t) = \dfrac{1}{s+2}$

and Laplace transform of $e^{-2(t-5)}u(t-5) =$

$e^{-5s}\times\dfrac{1}{s+2}$

therefore $X(s) = \dfrac{1}{(s+2)} - \dfrac{e^{-10}e^{-5s}}{(s+2)}$

$$= \frac{1}{(s+2)}\left[1 - e^{-5(s+2)}\right] \text{ with } \mathrm{Re}(s) > -2$$

4. Given that Laplace transform of $u(t) = \dfrac{1}{s}$, what is the Laplace transform of $\delta(t)$?

Solution: Given that $u(t) \leftrightarrow \dfrac{1}{s}$

Laplace transform of $\delta(t)$ = Laplace transform of $\dfrac{du(t)}{dt}$

Using time differentiation property, we get

$$\delta(t) \leftrightarrow s\cdot\frac{1}{s} = 1$$

5. Determine the inverse Laplace transform of

$$\frac{s}{(s^2+4)}, \mathrm{Re}(s) > 0.$$

Solution: $\cos 2t\; u(t)$

Laplace transform of $\cos\omega_0 t = \dfrac{s}{(s^2+\omega_0^2)}$ and hence the answer.

6. Determine the inverse Laplace transform and associated ROC of $\dfrac{(2s+4)}{(s^2+4s+3)}$. [Assume ROC $= -3 < \mathrm{Re}(s) < -1$]

Solution: Partial fraction expansion,

$$\frac{2s+4}{(s^2+4s+3)} = \frac{1}{(s+1)} + \frac{1}{(s+3)}$$

It is given that $\mathrm{Re}(s) > -3$

Therefore, inverse Laplace transform of

$$\frac{1}{(s+3)} = e^{-3t}u(t)$$

It is also given that $\mathrm{Re}(s) < -1$

Therefore, inverse Laplace transform of

$$\frac{1}{(s+1)} = e^{-1t}u(-t)$$

Therefore, inverse Laplace transform of $\dfrac{2s+4}{s^2+4s+3}$ is given by

$$e^{-3t}u(t) + e^{-t}u(-t).$$

7. Determine the inverse Laplace transform of $\dfrac{2s+1}{s+2}$, $\mathrm{Re}(s) > -2$.

Solution: $X(s) = \dfrac{2s+1}{s+2} = \dfrac{2(s+2)-3}{s+2} = 2 - \dfrac{3}{s+2}$

Inverse Laplace transform of $2 = 2\delta(t)$

Inverse Laplace transform of $\dfrac{3}{s+2} = 3e^{-2t}u(t)$

Therefore, $x(t) = 2\delta(t) - 3e^{-2t}u(t)$

8. Determine the z-transform and associated region of convergence of $\delta[n-3]$.

Solution: $\delta[n] \leftrightarrow 1$, for all z
Applying time shifting property,
$\delta[n-n_0] \leftrightarrow z^{-n_o}, |z| > 0$. Therefore, z-transform of $\delta[n-3] = z^{-3}$.

9. Determine the z-transform of $x[n] = a^{-n}u[-n]$.

Solution: z-Transform of $a^n u[n]$ is equal to $\dfrac{z}{z-a}$ for $|z| > a$. By using time reversal property, we get

z-Transform of $a^{-n}u[-n] = \dfrac{\frac{1}{z}}{\frac{1}{z} - a} = \dfrac{1}{1-az}$ for $|z| < \dfrac{1}{|a|}$

10. Determine the z-transform of $na^{n-1}u[n]$.

Solution: The given sequence is differential of $a^n u[n]$ with respect to a

z-Transform of $a^n u[n] = \dfrac{z}{z-a}, |z| > |a|$

Therefore, z-transform of

$$na^{n-1}u[n] = \frac{d}{da}\frac{z}{(z-a)} = \frac{z}{(z-a)^2}, |z| > |a|$$

11. Determine the inverse z-transform of $\dfrac{az}{(z-a)^2}$.

Solution: $x[n] \leftrightarrow X(z)$
By multiplication property,

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$$

Now $\qquad a^n u[n] \leftrightarrow \dfrac{z}{z-a}$

Therefore, z-transform of

$$na^n u[n] = -z\frac{d}{dz}\left(\frac{z}{z-a}\right)$$
$$= -z\left\{\frac{(z-a)-z}{(z-a)^2}\right\} = \frac{az}{(z-a)^2}$$

And hence the answer.

12. Determine the inverse z-transform of $X\left(\dfrac{z}{a}\right)$.

Solution: This is one of the properties of z-transform, which says that

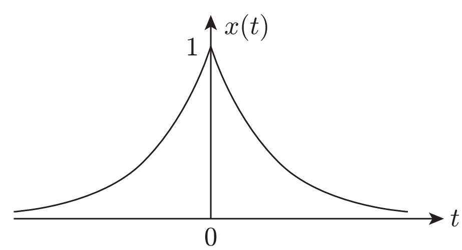
$$z_o{}^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$$

13. What is the Fourier transform of an impulse function?

Solution: Fourier transform of

$$\delta(t) \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1$$

14. Determine the Fourier transform of the signal shown in the following figure.



Solution: The given signal can be expressed as

$$x(t) = e^{-at} \text{ for } t > 0$$

and $\qquad x(t) = e^{at}$ for $t < 0$

$$\text{Therefore, } X(\omega) = \int_{-\infty}^{0} e^{at} \cdot e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t}dt$$
$$= \int_{-\alpha}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\alpha} e^{-(a+j\omega)t}dt$$
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

15. Determine the Fourier transform of $x(t) = \cos\omega_0 t$.

Solution:

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$
$$= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

Now $\qquad e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$

and $\qquad e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega + \omega_0)$

Therefore,

$$\cos\omega_0 t \leftrightarrow \frac{1}{2}\left[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right]$$

or $\qquad \cos\omega_0 t \leftrightarrow \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

16. Find inverse Fourier transform of $x(\omega) = \dfrac{1}{(a + j\omega)^2}$.

Solution:

$$X(w) = \frac{1}{(a+j\omega)^2} = \left[\frac{1}{(a+j\omega)}\right]\left[\frac{1}{(a+j\omega)}\right]$$

Now, $\dfrac{1}{(a+j\omega)} \leftrightarrow e^{-at}u(t)$

Convolution in time domain is equal to multiplication in frequency domain. Therefore,

$$\left[\frac{1}{(a+j\omega)}\right]\left[\frac{1}{(a+j\omega)}\right] \rightarrow e^{-at}u(t) * e^{-at}u(t)$$

Now $e^{-at}u(t) * e^{-at}u(t)$

$$= \int_{-\infty}^{\infty} e^{-at}u(\tau)e^{-a(t-\tau)}u(t-\tau)dz$$

$$= e^{-at}\int_0^t dz = te^{-at}u(t)$$

therefore, $te^{-at}u(t) \leftrightarrow \dfrac{1}{(a+j\omega)^2}$

17. What is the Fourier transform of a DC signal with unity strength?

Solution: $\delta(t) \leftrightarrow 1$

By principle of duality,

Fourier transform of DC signal of unity strength is given by

$2\pi\delta(-\omega) = 2\pi\delta(\omega)$ and hence the answer.

18. Give the statement of frequency convolution theorem.

Solution: According to the frequency convolution theorem, multiplication in time domain is convolution in frequency domain. That is, if $X_1(\omega)$ and $X_2(\omega)$ are Fourier transforms of $x_1(t)$ and $x_2(t)$, respectively, then according to frequency convolution theorem, $x_1(t)\cdot x_2(t) \leftrightarrow X_1(\omega) * X_2(\omega)$

PRACTICE EXERCISE

1. Which one of the following expressions is correct for determining Laplace transform of $x(t)$?

(a) $\int_{-\infty}^{\infty} x(t)e^{st}dt$ (b) $\int_{-\infty}^{\infty} x(t)e^{-st}dt$

(c) $\int_{-\infty}^{\infty} e^{-st}dt$ (d) $\int_{-\infty}^{\infty} e^{st}dt$

2. Laplace transform of $x(t) = t$ is

(a) s^2 (b) $\dfrac{2}{s^2}$ (c) $\dfrac{1}{s^2}$ (d) $\dfrac{1}{s}$

3. Inverse Laplace transform of $\dfrac{1}{(s+a)^2}$ is

(a) $te^{-at}u(t)$ (b) $e^{-at}u(t)$

(c) $tu(t)$ (d) $ae^{-at}u(t)$

4. Laplace transform of $x_1(t)$ convolving with $x_2(t)$ is

(a) $x_1(s) * x_2(s)$ (b) $x_1(s)\cdot x_2(s)$

(c) $x_1(s) / x_2(s)$ (d) None of these

5. Inverse Laplace transform of $\dfrac{10}{s^2+2s+5}$ is

(a) $5e^{-t}\cos 2t$ (b) $5e^{t}\cos 2t$

(c) $5e^{-t}\sin 2t$ (d) $5e^{t}\sin 2t$

6. Which one of the following expressions is correct for inverse Laplace transform?

(a) $x(t) = \dfrac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} X(s)e^{st}ds$

(b) $x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{st}ds$

(c) $x(t) = \dfrac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} X(s)e^{-st}ds$

(d) $x(t) = \dfrac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} X(s)e^{-st}dt$

7. z-Transform transforms

(a) Difference equations into algebraic equations
(b) Differential equations into algebraic equations
(c) Integral equations into algebraic equations
(d) Integral-differential equations into algebraic equations

8. Which one of the following expressions is correct for determining z-transform?

(a) $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^n$ (b) $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

(c) $X(z) = \sum_{n=0}^{\infty} x[n]z^n$ (d) $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$

9. z-Transform of $x[-n]$ is (assuming $x(z)$ to be the z-transform of $x[n]$)

(a) $-X(z)$ (b) $X(-z)$

(c) $X\left(\frac{1}{z}\right)$ (d) None of these

10. If $X_1(z)$ and $X_2(z)$ are, respectively, z-transforms of $x_1[n]$ and $x_2[n]$ then $x_1[n]*x_2[n]$ is

(a) $X_1(z)*X_2(z)$ (b) $X_1(z)/X_2(z)$

(c) $X_1(z)\cdot X_2(z)$ (d) $X_1(z)+X_2(z)$

11. If $X(z)$ is z-transform of $x[n]$, then z-transform of $nx[n]$ is

(a) $\frac{dX(z)}{dz}$ (b) $z\frac{dX(z)}{dz}$

(c) $\frac{d^2X(z)}{dz}$ (d) $-z\frac{dX(z)}{dz}$

12. Laplace transform and z-transform are interrelated by

(a) $s=\ln z$ (b) $s=\frac{\ln z}{T}$

(c) $s=T\ln z$ (d) $s=z$

13. Frequency convolution theorem can be expressed as

(a) $x_1(t)*x_2(t)\leftrightarrow X_1(\omega)\cdot X_2(\omega)$

(b) $x_1(t)x_2(t)\leftrightarrow\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$

(c) $x_1(\omega)*x_2(\omega)\leftrightarrow x_1(t)\cdot x_2(t)$

(d) None of these

14. Fourier transform of an impulse function is

(a) 1 (b) 0

(c) ∞ (d) $2\pi\delta(\omega)$

15. Fourier transform of $e^{-j\omega_0 t}$ is

(a) $2\pi\delta(\omega-\omega_0)$ (b) $\pi\delta(\omega-\omega_0)$

(c) $\pi\delta(\omega+\omega_0)$ (d) $2\pi\delta(\omega+\omega_0)$

ANSWERS

1. (b)	**4.** (b)	**7.** (a)	**10.** (c)	**13.** (b)
2. (c)	**5.** (c)	**8.** (b)	**11.** (d)	**14.** (a)
3. (a)	**6.** (a)	**9.** (c)	**12.** (b)	**15.** (d)

EXPLANATIONS AND HINTS

1. (b) This is by definition of Laplace transform.

2. (c) Multiplication in time domain is differentiation in s-domain. If $X(s)$ is Laplace transform of $x(t)$, then

Laplace transform of $tx(t)=-\frac{dX(s)}{ds}$

Now $x(t)=t=tu(t)$

Laplace transform of $u(t)=\frac{1}{s}$

Therefore, Laplace transform of $tu(t)=\frac{-d}{ds}\left(\frac{1}{s}\right)=\frac{1}{s^2}$

3. (a) Laplace transform of $e^{-at}u(t)=\frac{1}{s+a}$

Therefore, Laplace transform of

$$te^{-at}u(t)=\frac{-d}{ds}\left(\frac{1}{s+a}\right)=\frac{1}{(s+a)^2}$$

Therefore, inverse Laplace transform of

$$\frac{1}{(s+a)^2}=te^{-at}u(t)$$

4. (b) Convolution in time domain is multiplication in frequency domain.

5. (c) $X(s)=\frac{10}{s^2+2s+5}=\frac{2\times 5}{(s+1)^2+2^2}$

Inverse Laplace transform of

$$\frac{2}{(s+1)^2+2^2}=e^{-t}\sin 2t$$

Therefore, inverse Laplace transform of

$$\frac{10}{s^2+2s+5}=5e^{-t}\sin 2t$$

6. (a) It is by definition of inverse Laplace transform.

7. (a) What Laplace transform does to continuous time signals, z-transform does to the discrete time signals.

8. (b) It is by definition of z-transform.

9. (c) It is by the time reversal property of z-transform.

10. (c) It is the convolution property of z-transform.

11. (d) It is the multiplication property of z-transform.

12. (b) If T is the sampling time interval, then $X(s)$ becomes equal to $X(z)$ for $z = e^{+sT}$, which gives $s = \frac{1}{T}\ln z$.

13. (b) It is by the statement of multiplication property, also known as frequency convolution theorem.

14. (a) Fourier transform of impulse function

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

15. (d) It is the frequency shifting property of Fourier transform, according to which

$$e^{j\omega_0 t} x(t) \leftrightarrow X(\omega - \omega_0).$$

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. If (L) defines the Laplace transform of a function, $L[\sin(at)]$ will be equal to

(a) $\frac{a}{s^2 - a^2}$ (b) $\frac{a}{s^2 + a^2}$

(c) $\frac{s}{s^2 + a^2}$ (d) $\frac{s}{s^2 - a^2}$

(GATE 2003, 2 Marks)

Solution:

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} e^{-st} \sin(at) dt$$

$$= \frac{a}{s^2 + a^2}$$

Ans. (b)

2. Laplace transform of function $\sin \omega t$ is

(a) $\frac{s}{s^2 + \omega^2}$ (b) $\frac{\omega}{s^2 + \omega^2}$

(c) $\frac{s}{s^2 - \omega^2}$ (d) $\frac{\omega}{s^2 - \omega^2}$

(GATE 2003, 2 Marks)

Solution:

$$L \sin \omega t = \int_o^{\infty} e^{-st} \sin \omega t \, dt = \frac{\omega}{s^2 + \omega^2}$$

Ans. (b)

3. A delayed unit step function is defined as

$$u(t-a) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t > 0 \end{cases}$$

Its Laplace transform is

(a) ae^{-as} (b) $\frac{e^{as}}{s}$ (c) $\frac{e^{-as}}{a}$ (d) $\frac{e^{-as}}{s}$

(GATE 2004, 2 Marks)

Solution:

Laplace transform of $u(t-a) = \int_o^{\infty} e^{-st} u(t-a) dt$

$$= \int_o^{a} e^{-st} \cdot 0 \, dt + \int_a^{\infty} e^{-st} \cdot 1 \, dt = \frac{e^{-as}}{s}$$

Ans. (d)

4. In what range should $\text{Re}(s)$ remain so that Laplace transform of the function $e^{(a+2)t+5}$ exists.

(a) $\text{Re}(s) > a + 2$ (b) $\text{Re}(s) > a + 7$

(c) $\text{Re}(s) < 2$ (d) $\text{Re}(s) > a + 5$

(GATE 2005, 2 Marks)

Solution: $e^{(a+2)t+5} = e^{(a+2)t} \cdot e^5$

Laplace transform $= \left[\frac{1}{s-(a+2)}\right] e^5$

Therefore, Laplace transform exists for $s > (a+2)$.

Ans. (a)

5. A solution for the differential equation $\dot{x}(t) + 2x(t) = \delta(t)$ with the initial condition $x(0^-) = 0$ is

(a) $e^{-2t}u(t)$ (b) $e^{2t}u(t)$ (c) $e^{-t}u(t)$ (d) $e^{t}u(t)$

(GATE 2006, 1 Mark)

Solution: Given that $\dot{x}(t) + 2x(t) = \delta(t)$

Taking Laplace transforms on both sides, we get

$$sX(s) - x(0) + 2X(s) = 1$$

or $$X(s) = \frac{1}{s+2}$$

Therefore, $x(t) = e^{-2t}u(t)$

Ans. (a)

6. If $F(s)$ is the Laplace transform of $f(t)$, then Laplace transform of $\int_0^t f(\tau) d\tau$ is

(a) $\dfrac{F(s)}{s}$ (b) $\dfrac{F(s)}{s} - f(0)$

(c) $sF(s) - f(0)$ (d) $\int F(s)\,ds$

(GATE 2007, 2 Marks)

Solution: In general, Laplace transform of

$$\int_0^t \int_0^t \cdots \int_0^t f(t)\,dt^n = \frac{F(s)}{s^n}$$

Therefore, Laplace transform of $\int_0^t f(\tau)\,d\tau = \dfrac{F(s)}{s}$

Ans. (a)

7. Evaluate $\int_0^\infty \dfrac{\sin t}{t}\,dt$

(a) π (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{8}$

(GATE 2007, 2 Marks)

Solution:

As $\quad L(\sin mt) = \dfrac{m}{(s^2+m^2)} = f(t)$, (say)

Therefore,

$$L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s)ds = \int_s^\infty \frac{m\,ds}{s^2+m^2} = \left|\tan^{-1}\frac{s}{m}\right|_s^\infty$$

By definition, we have

$$\int_0^\infty e^{-st}\frac{\sin mt}{t}\,dt = \frac{\pi}{2}\tan^{-1}\frac{s}{m}$$

Now $\operatorname*{lt}_{s\to 0} \tan^{-1}\left(\dfrac{s}{m}\right) = 0$ if $m>0$ or π if $m<0$.

Therefore, taking limits as $s \to 0$, we get

$$\int_0^\infty \frac{\sin mt}{t}\,dt = \frac{\pi}{2} \text{ if } m>0 \text{ or } -\frac{\pi}{2} \text{ if } m<0.$$

As $m = 1$, which is > 0, the answer is $\dfrac{\pi}{2}$.

Ans. (b)

8. Inverse Laplace transform of $\dfrac{1}{s^2+s}$ is

(a) $1+e^t$ (b) $1-e^t$ (c) $1-e^{-t}$ (d) $1+e^{-t}$

(GATE 2009, 1 Mark)

Solution:

$$\frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Therefore, Laplace transform of $\dfrac{1}{s^2+s} = 1 - e^{-t}$

Ans. (c)

9. Laplace transform of $f(x) = \cosh(ax)$ is

(a) $\dfrac{a}{s^2-a^2}$ (b) $\dfrac{s}{s^2-a^2}$

(c) $\dfrac{a}{s^2+a^2}$ (d) $\dfrac{s}{s^2+a^2}$

(GATE 2009, 2 Marks)

Solution: It is a formula from the list of standard formulas in Laplace transform.

Ans. (b)

10. Laplace transform of a function $f(t)$ is $\dfrac{1}{s^2(s+1)}$. The function $f(t)$ is

(a) $t-1+e^{-t}$ (b) $t+1+e^{-t}$

(c) $-1+e^{-t}$ (d) $2t+e^t$

(GATE 2010, 2 Marks)

Solution:

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$= \frac{As(s+1) + B(s+1) + Cs^2}{s^2(s+1)}$$

Solving this, we get $A = -1, B = 1, C = 1$

Therefore, $\dfrac{1}{s^2(s+1)} = \dfrac{-1}{s} + \dfrac{1}{s^2} + \dfrac{1}{s+1}$

Therefore, inverse Laplace transform of

$$\frac{1}{s^2(s+1)} = -1 + t + e^{-t}$$

Ans. (a)

11. Given $L^{-1}\left[\dfrac{3s+1}{s^3+4s^2+(k-3)s}\right]$, if $\lim\limits_{t\to\infty} f(t) = 1$, then the value of K is

(a) 1 (b) 2 (c) 3 (d) 4

(GATE 2010, 2 Marks)

Solution:

$$\lim_{t\to\infty} f(t)dt = \lim_{s\to 0} sF(s)$$

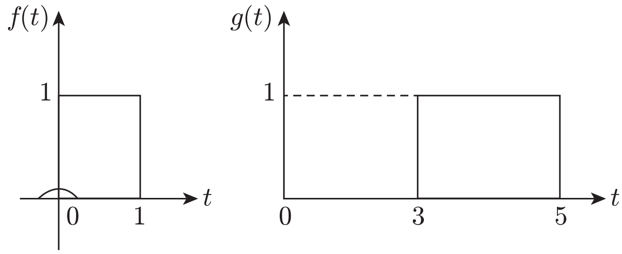
Therefore, $\lim\limits_{s\to 0} s\left[\dfrac{3s+1}{s^3+4s^2+(K-3)s}\right] = 1$ or

$$\lim_{s\to 0}\left[\frac{3s+1}{s^2+4s+K-3}\right] = 1$$

or $\dfrac{1}{K-3} = 1 \quad \Rightarrow K = 4$

Ans. (d)

Common Data Questions 12 and 13: Given $f(t)$ and $g(t)$ as shown below:



12. $g(t)$ can be expressed as

(a) $g(t) = f(2t-3)$ (b) $g(t) = f(t/2-3)$

(c) $g(t) = f(2t-3/2)$ (d) $g(t) = f(t/2-3/2)$

(GATE 2010, 2 Marks)

Solution: Let us take $g(t) = f(t/2 - 3/2)$

This gives $g(3) = f(0)$, which is true and $g(5) = f(5/2-3/2) = f(1)$, which is true.

No other answer satisfies these conditions.

Ans. (d)

13. The Laplace transform of $g(t)$ is

(a) $\frac{1}{s}\left(e^{3s} - e^{5s}\right)$ (b) $\frac{1}{s}\left(e^{-5s} - e^{-3s}\right)$

(c) $\frac{e^{-3s}}{s}\left(1 - e^{-2s}\right)$ (d) $\frac{1}{s}\left(e^{5s} - e^{3s}\right)$

(GATE 2010, 2 Marks)

Solution: Laplace transform of

$$g(t) = \int_0^3 e^{-st}\cdot 0\ dt + \int_3^5 e^{-st}\cdot 1\ dt + \int_5^\infty e^{-st}\cdot 0\ dt$$

$$= \left[\frac{e^{-st}}{s}\right]_3^5 = \left[\frac{e^{-5s} - e^{-3s}}{s}\right]$$

$$= -\frac{e^{-3s}}{s}\left(e^{-2s} - 1^3\right) = \frac{e^{-3s}}{s}\left(1 - e^{-2s}\right)$$

Ans. (c)

14. Inverse Laplace transform of $F(s) = \dfrac{1}{s(s+1)}$ is given by

(a) $f(t) = \sin t$ (b) $f(t) = e^{-t}\sin t$

(c) $f(t) = e^{-t}$ (d) $f(t) = 1 - e^{-t}$

(GATE 2012, 2 Marks)

Solution: $$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1) + Bs}{s(s+1)}$$

Solving this, $A = 1$, $B = -1$

Therefore, $\dfrac{1}{s(s+1)} = \dfrac{1}{s} - \dfrac{1}{s+1}$

Hence, inverse Laplace transform of $\dfrac{1}{s(s+1)} = 1 - e^{-t}$

Ans. (d)

15. Consider the differential equation:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y(t) = \delta(t)$$

with $y(t)\Big|_{t=0} = -2$ and $\dfrac{dy}{dt}\Big|_{t=0} = 0$

The numerical value of $\dfrac{dy}{dt}\Big|_{y=0^+}$ is

(a) -2 (b) -1 (c) 0 (d) 1

(GATE 2012, 2 Marks)

Solution: Taking Laplace transform on both sides of the given differential equation, we get

$$s_2 Y(s) - sY(0) - Y'(0) + 2[sY(s) - Y(0)] + Y(s) = 1$$

Solving this equation, we get

$$Y(s) = \frac{-(2s+3)}{(s+1)^2}$$

By partial fraction expansion, this becomes

$$= -\left[\frac{2}{(s+1)} + \frac{1}{(s+1)^2}\right]$$

Therefore, $y(t) = -\left[2e^{-t} + te^{-t}\right]u(t)$

Differentiating $y(t)$ with respect to t, we get

$$\frac{dy(t)}{dt} = -\left[-2e^{-t} + e^{-t} - te^{-t}\right]u(t)$$

$$\frac{dy(t)}{dt}\Big|_{t=0^+} = -[-2+1+0] = 1$$

Ans. (d)

16. The function $f(t)$ satisfies the differential equation:

$$\frac{d^2f}{dt^2} + f = 0$$

and the auxiliary conditions $f(0) = 0, \dfrac{df}{dt}(0) = 4.$

The Laplace transform of $f(t)$ is given by

(a) $\dfrac{2}{s+1}$ (b) $\dfrac{4}{s+1}$ (c) $\dfrac{4}{s^2+1}$ (d) $\dfrac{2}{s^2+1}$

(GATE 2013, 2 Marks)

Solution: $$\frac{d^2f}{dt} + f = 0$$

Taking Laplace transform on both sides, we get

$$s^2F(s) - sf(0) - f'(0) + F(s) = 0$$

or $s^2F(s) - 0 - 4 + F(s) = 0$

or $F(s)\left[s^2 + 1\right] = 4$

Therefore, $F(s) = \dfrac{4}{(s^2+1)}$ and hence the answer.

Ans. (c)

17. Let $X(s) = \dfrac{3s+5}{s^2+10s+21}$ be the Laplace transform of a signal $x(t)$. Then, $x(0^+)$ is

(a) 0 (b) 3 (c) 5 (d) 21

(GATE 2014, 1 Mark)

Solution: We have to find the Laplace transform of $X(s) = \dfrac{3s+5}{s^2+10s+21}$

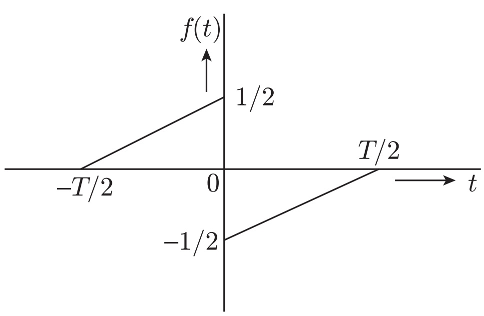
$$x[0^+] = \lim_{s\to\infty} \frac{s\cdot 3s+5}{\left(s^2+10s+21\right)}$$

[using initial value theorem]

$$= \lim_{s\to\infty} \frac{s^2\left[3+\frac{5}{s}\right]}{\left[1+\frac{10}{s}+\frac{21}{s^2}\right]} = 3$$

Ans. (b)

18. A function $f(t)$ is shown in the figure.



The Fourier transform $F(\omega)$ of $f(t)$ is

(a) real and even function of ω
(b) real and odd function of ω
(c) imaginary and odd function of ω
(d) imaginary and even function of ω

(GATE 2014, 1 Mark)

Solution: Since $f(t)$ is odd and real,

$$f(t) = -f(-t)$$

$\oint F(\omega)$ is imaginary and odd [symmetry property of Fourier transform].

Ans. (c)

19. Laplace transform of $\cos(\omega t)$ is $\dfrac{s}{s^2+\omega^2}$. The Laplace transform of $e^{-2t}\cos(4t)$ is

(a) $\dfrac{s+2}{(s-2)^2+16}$ (b) $\dfrac{s-2}{(s-2)^2+16}$

(c) $\dfrac{s-2}{(s+2)^2+16}$ (d) $\dfrac{s+2}{(s+2)^2+16}$

(GATE 2014, 1 Mark)

Solution: Let $L[f(t)] = F(a)$, then $L[e^{xt}f(t)] = F(a-x)$. Also, putting $\omega = 4$, we get

$$L[e^{-2t}\cos(4t)] = \frac{s-2}{(s-2)^2+4^2} = \frac{s-2}{(s-2)^2+16}$$

Ans. (b)

20. For the time domain function $f(t) = t^2$, which **ONE** of the following is the Laplace transform of $\int_0^t f(t)\,dt$?

(a) $\dfrac{3}{s^4}$ (b) $\dfrac{1}{4s^2}$ (c) $\dfrac{2}{s^3}$ (d) $\dfrac{2}{s^4}$

(GATE 2014, 1 Mark)

Solution: We have

$$L\left[\int_0^1 f(t)\,dt\right] = \frac{F(s)}{s} \text{ where } F(s) = L\big(f(t)\big)$$

$$\Rightarrow L\left[\int_0^1 t^2 dt\right] = \frac{\left(\frac{2}{s^3}\right)}{s} = \frac{2}{s^4} \left(\because L\left[t^2\right] = \frac{2}{s^3}\right)$$

Ans. (d)

21. For a function $g(t)$, it is given that $\int_{-\infty}^{+\infty} g(t)e^{-j\omega t}dt = \omega e^{-2\omega^2}$ for any real value ω. If $y(t) = \int_{-\infty}^{t} g(t)dt$, then $\int_{-\infty}^{+\infty} y(t)\,dt$ is

(a) 0 (b) $-j$ (c) $-\dfrac{j}{2}$ (d) $\dfrac{j}{2}$

(GATE 2014, 2 Marks)

Solution: We are given

$$\int_{-\infty}^{\infty} g(t)\cdot e^{-j\omega t}dt = \omega e^{-2\omega^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} g(t)\,dt = 0$$

$$y(t)=\int_{-\infty}^{t} g(t)\,dt \Rightarrow y(t)=g(t)*u(t)$$

[$u(t)$ in step function]

$$\Rightarrow Y(j\omega)=G(j\omega)\cdot U(j\omega)$$

$$Y(j\omega)=\int_{-\infty}^{\infty} y(t)\cdot e^{-j\omega t}dt$$

$$\Rightarrow Y(j0)=\int_{-\infty}^{\infty} y(t)\,dt=\left[\omega e^{-2\omega^2}\left[\frac{1}{j\omega}+\pi\delta(\omega)\right]\right]$$

$$\omega=0=\frac{1}{j}=-j$$

Ans. (b)

22. The unilateral Laplace transform of $f(t)$ is $\dfrac{1}{s^2+s+1}$. Which one of the following is the unilateral Laplace transform of $g(t)=t\cdot f(t)$?

(a) $\dfrac{-s}{(s^2+s+1)^2}$ (b) $\dfrac{-(2s+1)}{(s^2+s+1)^2}$

(c) $\dfrac{s}{(s^2+s+1)^2}$ (d) $\dfrac{2s+1}{(s^2+s+1)^2}$

(GATE 2014, 2 Marks)

Solution: $F(s)=\dfrac{1}{s^2+s+1}$

$$L[g(t)=t\cdot f(t)]=-\frac{d}{ds}[F(s)]$$

(using multiplication by t)

$$=\frac{2s+1}{(s^2+s+1)^2}$$

Ans. (d)

23. The result of the convolution $x(-t) * \delta(-t-t_0)$ is

(a) $x(t+t_0)$ (b) $x(t-t_0)$
(c) $x(-t+t_0)$ (d) $x(-t-t_0)$

(GATE 2015, 1 Mark)

Solution:

$$x(-t)*\delta(-t-t_0)=x(-t)*\delta(t+t_0)=x[-t(t+t_0)]$$
$$=x(-t-t_0)$$

Ans. (d)

24. The bilateral Laplace transform of a function $f(t)=\begin{cases}1 & if\ a\le t\le b\\ 0 & \text{otherwise}\end{cases}$ is

(a) $\dfrac{a-b}{s}$ (b) $\dfrac{e^s(a-b)}{s}$

(c) $\dfrac{e^{as}-e^{-bs}}{s}$ (d) $\dfrac{e^{s(a-b)}}{s}$

(GATE 2015, 1 Mark)

Solution: We are given,

$$f(t)=\begin{cases}1 & \text{if } a\le t\le b\\ 0 & \text{otherwise}\end{cases}$$

$$L\{f(t)\}=\int_0^{\infty} e^{-st}f(t)dt$$
$$=\int_0^{a} e^{-st}f(t)dt+\int_a^{\infty} e^{-st}f(t)dt+\int_b^{\infty} e^{-st}f(t)dt$$
$$=0+\int_a^{b} e^{-st}dt+0=\left.\frac{e^{-st}}{-s}\right|_a^b=\frac{-1}{s}(e^{-bs}-s^{-as})$$
$$=\frac{e^{-as}-e^{-bs}}{s}$$

Ans. (c)

25. The Laplace transform of $f(t)=2\sqrt{t/\pi}$ is $s^{-3/2}$. The Laplace transform of $g(t)=\sqrt{1/\pi t}$ is

(a) $3s^{-5/2}/2$ (b) $s^{-1/2}$
(c) $s^{1/2}$ (d) $s^{3/2}$

(GATE 2015, 1 Mark)

Solution: Given the Laplace transform of $f(t)=2\sqrt{\dfrac{t}{\pi}}$ is $s^{-3/2}$.

Given that $g(f)=\sqrt{\dfrac{1}{\pi t}}$

$$\Rightarrow g(t)=\frac{2\sqrt{t/\pi}}{2t}=\frac{f(t)}{2t}$$

$$L\{g(t)\}=L\left\{\frac{f(t)}{2t}\right\}=\frac{1}{2}\int_s^{0}\{f(t)\}ds=\frac{1}{2}\int_s^{\infty}s^{-3/2}ds$$

$$=\frac{1}{2}\left(\frac{s^{-3/2+1}}{-3/2+1}\right)_s^{\infty}=\frac{1}{2}(-2)[0-s^{-1/2}]=s^{-1/2}=\frac{1}{\sqrt{s}}$$

Ans. (b)

26. The Laplace transform of e^{i5t}, where $i=\sqrt{-1}$, is

(a) $\dfrac{s-5i}{s^2-25}$ (b) $\dfrac{s+5i}{s^2+25}$

(c) $\dfrac{s+5i}{s^2-25}$ (d) $\dfrac{s-5i}{s^2+25}$

(GATE 2015, 1 Mark)

Solution:

$$\begin{aligned} L(e^{i5t}) &= L(\cos 5t + i\sin 5t) \\ &= L(\cos 5t) + iL(\sin 5t) \\ &= \frac{s+5i}{s^2+25} \end{aligned}$$

Ans. (b)

27. Let $x(t) = \alpha s(t) + s(-t)$ with $s(t) = \beta e^{-4t}u(t)$, where $u(t)$ is a unit-step function. If the bilateral Laplace transform of $x(t)$ is $X(s) = \dfrac{16}{s^2-16} - 4 < \text{Re}\{s\} < 4$, then the value of β is ________.

(GATE 2015, 2 Marks)

Solution: We are given,

$$x(t) = \alpha s(t) + s(-t) \text{ and } s(t) = \beta e^{-4t}u(t)$$

$$x(t) = \alpha\beta e^{-4t}u(t) + \beta e^{4t}u(-t)$$

$$\alpha\beta e^{-4t}u(t) \xrightarrow{L} \frac{\alpha\beta}{s+4}$$

$$\beta e^{4t}u(-t) \xrightarrow{L} \frac{\beta}{s-4}$$

Therefore,

$$X(s) = \frac{\alpha\beta}{s+4} - \frac{\beta}{s-4}$$

$$\beta\left[\frac{\alpha(s-4)-(s+4)}{s^2-16}\right] = \frac{16}{s^2-16}; -4 < \sigma < +4$$

On solving the numerator,

$$\beta = -2$$

Ans. -2

28. The state variable representation of a system is given as $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x;\ x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ y = [0\ 1]x.$

The response $y(t)$ is

(a) $\sin(t)$ (b) $1-e^t$

(c) $1-\cos(t)$ (d) 0

(GATE 2015, 2 Marks)

Solution: $X = AX$

$$X(s) = (sI - A)^{-1}X(0)$$

$$X(s) = \begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X(s) = \begin{bmatrix} 1/s \\ 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y(t) = [0 \quad 1]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

Ans. (d)

29. Consider a signal defined by

$$x(t) = \begin{cases} e^{j\,10t} & \text{for } |t| \le 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

Its Fourier transform is

(a) $\dfrac{2\sin(\omega-10)}{\omega-10}$ (b) $2e^{j10\frac{\sin(\omega-10)}{\omega-10}}$

(c) $\dfrac{2\sin\omega}{\omega-10}$ (d) $e^{j10\omega\frac{2\sin\omega}{\omega}}$

(GATE 2015, 2 Marks)

Solution: We are given

$$x(t) = \begin{cases} e^{j10t} & \text{for } |t| \le 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

Applying Fourier transform, we get

$$X(\omega) = \int_{-1}^{1} e^{j10\cdot t} \cdot e^{-j\omega t}dt = \int_{-1}^{1} e^{j(10-\omega)t}dt$$

$$= \left.\frac{e^{j(10-\omega)t}}{j(10-\omega)}\right|_{-1}^{1} = \frac{2\sin(\omega-10)}{(\omega-10)}$$

Ans. (a)

30. The z-transform of a sequence $x[n]$ is given as $X(z) = 2z + 4 - 4/z + 3/z^2$. If $y[n]$ is the first difference of $x[n]$, then $Y(x)$ is given by

(a) $2z + 2 - \dfrac{8}{z} + \dfrac{7}{z^2} - \dfrac{3}{z^2}$

(b) $-2z + 2 - \dfrac{6}{z} + \dfrac{1}{z^2} - \dfrac{3}{z^3}$

(c) $-2z - 2 + \dfrac{8}{z} + \dfrac{7}{z^2} + \dfrac{3}{z^3}$

(d) $4z - 2 - \dfrac{8}{z} - \dfrac{1}{z^2} + \dfrac{3}{z^3}$

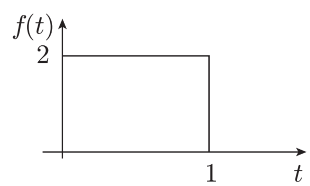
(GATE 2015, 2 Marks)

Solution:
Say, $y[n]$ is first difference of $x[n]$. So

$$g[n] = x[n] - x[n-1]$$
$$\Rightarrow Y(z) = X(1 - z^{-1}) = X(z) - z^{-1}Z(z)$$
$$Y(z) = [2z + 4 - 4z^{-1} + 3z^{-2}] - [2 + 4z^{-1} - 4z^{-2}]$$
$$= 2z + 4 - 4z^{-1} + 3z^{-2} - 2 - 4z^{-1} + 4^{-2} - 3z^{-3}$$
$$= 2z + 2 - 8z^{-1} + 7z^{-2} - 3z^{-3}$$

Ans. (a)

31. Laplace transform of the function $f(t)$ is given by $f(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$. Laplace transform of the function shown below is given by ________ .



(a) $\dfrac{1 - e^{-2s}}{s}$ (b) $\dfrac{1 - e^{-s}}{2s}$

(c) $\dfrac{2 - 2e^{-s}}{s}$ (d) $\dfrac{1 - 2e^{-s}}{s}$

(GATE 2015, 2 Marks)

Solution:

$$f(t) = \begin{cases} 2;\ 0 < t < 1 \\ 0;\ \text{otherwise} \end{cases}$$

Therefore,

$$L[f(t)] = \int_0^1 2e^{-st}dt = 2\frac{e^{-st}}{-s}\Bigg|_0^1 = \frac{2 - 2e^{-s}}{s}$$

Ans. (c)

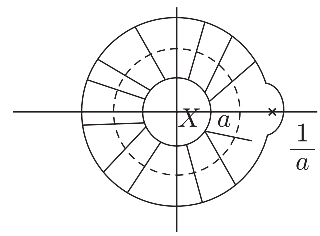
32. The z-transform of $x[n] = a^{|n|}$, $0 < |a| < 1$, is $X(z)$. The region of convergence of $X(z)$ is

(a) $|\alpha| < |z| < \dfrac{1}{|\alpha|}$ (b) $|z| > \alpha$

(c) $|z| > \dfrac{1}{|\alpha|}$ (d) $|z| < \min\left[|\alpha|, \dfrac{1}{|\alpha|}\right]$

(GATE 2015, 2 Marks)

Solution:



$$x(n) = \alpha^{|n|} = \alpha^n u(n) + \left(\frac{1}{\alpha}\right)^n u(-n-1)$$

$$|z| > \alpha;\ |z| < \frac{1}{\alpha}$$

Ans. (a)

33. Which one of the following is a property of the solutions to the Laplace equation: $\nabla^2 f = 0$?

(a) The solutions have neither maxima nor minima anywhere except at the boundaries.
(b) The solutions are not separable in the coordinates.
(c) The solutions are not continuous.
(d) The solutions are not dependent on the boundary conditions.

(GATE 2016, 1 Mark)

Solution: The solution of Laplace function is called harmonic function and the following properties: The properties are true irrespective of the number of dimensions (one-, two- or three-dimensions) in which we solve the equation for $\nabla^2 f = 0$.

1 $f(x, y, z)$ is the average of f values over a spherical surface of radius, R, which is centered at (x, y, z).

$$f(x,y,z) = \left(\frac{1}{4\pi R^2}\right) \oint_{\text{Sphere}} f da$$

This is for the case of three-dimensional solution. In case of one-dimensional solution, $f(x)$ is the average of $f(x+a)$ and $f(x-a)$. For any a,

$$f(x) = \frac{1}{2}[f(x+a) + f(x-a)]$$

Similarly, for two-dimensional solution, the value $f(x, y)$ is the average of values of f on any circle of radius R centered at the point (x, y).

2 Laplace's equation does not endure local minima or maxima. In general, the extreme values of f must occur at endpoints or boundary. Thus, we conclude that only the property in option (a) is true.

Ans. (a)

34. The Laplace transform of $f(t) = e^{2t} \sin(5t)u(t)$ is

(a) $\dfrac{5}{s^2 - 4s + 29}$ (b) $\dfrac{5}{s^2 + 5}$

(c) $\dfrac{s-2}{s^2 + 4s + 29}$ (d) $\dfrac{5}{s+5}$

(GATE 2016, 1 Mark)

Solution: For the given function $e^{2t}\sin(5t)u(t)$, the standard form of Laplace transform is

$$e^{at}\sin(bt)u(t) \xrightarrow{\text{LT}}$$

$$= \frac{b}{(s-a)^2 + b^2} = \frac{5}{(s-2)^2 + 5^2}$$

$$= \frac{5}{s^2 + 4 - 4s + 25} = \frac{5}{s^2 - 4s + 29}$$

Ans. (a)

35. Consider a continuous-time system with input $x(t)$ and output $y(t)$ given by

$$y(t) = x(t)\cos(t)$$

This system is

(a) linear and time-invariant

(b) non-linear and time-invariant

(c) linear and time-varying

(d) non-linear and time-varying

(GATE 2016, 1 Mark)

Solution: Let there be two inputs $Ax_1(t)$ and $Bx_2(t)$.

Apply $x(t) = Ax_1(t) + Bx_2(t)$

$$\begin{aligned} y(t) &= [Ax_1(t) + Bx_2(t)]\cos(t) \\ &= Ax_1(t)\cos(t) + Bx_2(t)\cos(t) \\ &= y_1(t) + y_2(t) \end{aligned}$$

Thus the function is linear.

Let input be time shifted by t_0. Then output

$$y(t) = x(t - t_0)\cos(t)$$

$y(t)$ does not shift t_0, hence it is time-varying.

Ans. (c).

36. The solution of the differential equation, for $t > 0$, $y''(t) + 2y'(t) + y(t) = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$, is [$u(t)$ denotes the unit step function],

(a) $te^{-t}\,u(t)$

(b) $(e^{-t} - te^{-t})u(t)$

(c) $(-e^{-t} + te^{-t})u(t)$

(d) $e^{-t}u(t)$

(GATE 2016, 1 Mark)

Solution: Using Laplace transforms

$$\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = 0$$

$$\Rightarrow [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + Y(s) = 0$$

$$\Rightarrow Y(s) = \frac{sY(0) + y'(0) + 2y(0)}{s^2 + 2s + 1}$$

As $y'(0) = 1, y(0) = 0$, we have

$$Y(s) = \frac{1}{(s+1)^2} \Rightarrow y(t) = t \cdot e^{-t}u(t)$$

Ans. (a)

37. If $f(t)$ is a function defined for all $t \ge 0$, its Laplace transform $F(s)$ is defined as

(a) $\int_0^\infty e^{st} f(t)\,dt$ (b) $\int_0^\infty e^{-st} f(t)\,dt$

(c) $\int_0^\infty e^{ist} f(t)\,dt$ (d) $\int_0^\infty e^{-ist} f(t)\,dt$

(GATE 2016, 1 Mark)

Solution: By definition of Laplace transform, we have

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

Ans. (b)

38. Laplace transform of $\cos(\omega t)$ is

(a) $\dfrac{s}{s^2 + \omega^2}$ (b) $\dfrac{\omega}{s^2 + \omega^2}$

(c) $\dfrac{s}{s^2 - \omega^2}$ (d) $\dfrac{\omega}{s^2 - \omega^2}$

(GATE 2016, 1 Mark)

Solution: $L(\cos \omega t) = \dfrac{s}{s^2 + \omega^2}$

Ans. (a)

39. Solutions of Laplace's equation having continuous second-order partial derivatives are called

(a) biharmonic functions
(b) harmonic functions

(c) conjugate harmonic functions
(d) error functions

(GATE 2016, 1 Mark)

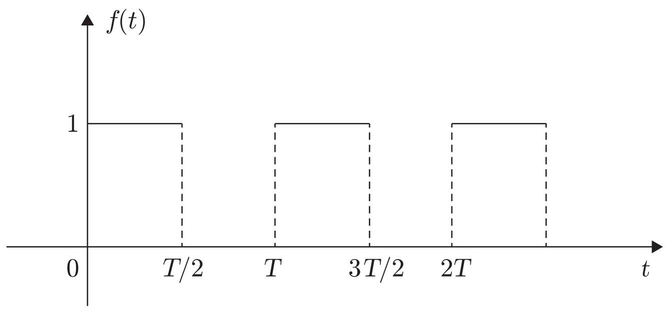
Solution: Laplace's partial differential equation is written as

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Solving this equation, we would get two harmonics that are conjugates to each other. Therefore, option (c) is correct.

Ans. (c)

40. The Laplace transform of the causal periodic square wave of period T shown in the figure below is



(a) $F(s)=\dfrac{1}{1+e^{-sT/2}}$ (b) $F(s)=\dfrac{1}{s(1+e^{-sT/2})}$

(c) $F(s)=\dfrac{1}{s(1-e^{-sT})}$ (d) $F(s)=\dfrac{1}{1-e^{-sT}}$

(GATE 2016, 2 Marks)

Solution: Laplace transform for periodic signal is given as

$$F(s)=\frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

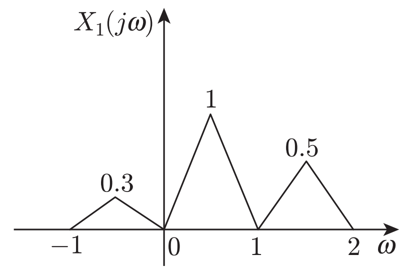
$$=\frac{1}{1-e^{-sT}}\left(\int_0^{T/2}(1)e^{-st}dt+0\right)=\frac{1}{1-e^{sT}}\left[\frac{e^{-st}}{-s}\right]_0^{T/2}$$

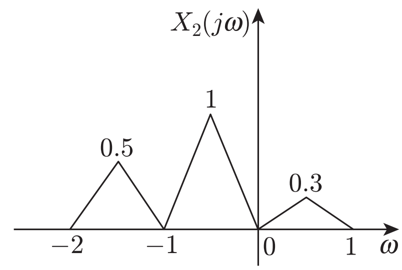
$$=\frac{1}{(e-e^{-sT})}\left(\frac{1}{s}\right)(1-e^{-sT/2})=\left[\frac{1-e^{-sT/2}}{s(1+e^{-sT/2})(1-e^{-sT/2})}\right]$$

$$\Rightarrow F(s)=\frac{1}{s(1+e^{-sT/2})}$$

Ans. (b)

41. Suppose $x_1(t)$ and $x_2(t)$ have the Fourier transforms shown as follows.





Which one of the following statements is TRUE?

(a) $x_1(t)$ and $x_2(t)$ are complex and $x_1(t)x_2(t)$ is also complex with non-zero imaginary part.

(b) $x_1(t)$ and $x_2(t)$ are real and $x_1(t)x_2(t)$ is also real.

(c) $x_1(t)$ and $x_2(t)$ are complex but $x_1(t)x_2(t)$ is real.

(d) $x_1(t)$ and $x_2(t)$ are imaginary but $x_1(t)x_2(t)$ is real.

(GATE 2016, 2 Marks)

Solution: $X_1(j\omega)$ and $X_2(j\omega)$ are not conjugate symmetric. Therefore, $x_1(t)\cdot x_2(t)$ are not real.
Fourier transform of

$$x_1(t)\cdot x_2(t)\rightarrow\frac{1}{2\pi}X_1(j\omega)*X_2(j\omega)$$

$X_1(j\omega)*X_2(j\omega)$ is conjugate symmetric, so $x_1(t)\cdot x_2(t)$ will be real.

Ans. (c)

42. The output of a continuous-time, linear time-invariant system is denoted by $\boldsymbol{T}\{x(t)\}$ where $x(t)$ is the input signal. A signal $z(t)$ is called eigen-signal of the system $\boldsymbol{T}$, when $T\{z(t)\}=\gamma z(t)$, where γ is a complex number, in general, and is called an eigenvalue of $\boldsymbol{T}$. Suppose the impulse response of the system $\boldsymbol{T}$ is real and even. Which of the following statements is TRUE?

(a) $\cos(t)$ is an eigen-signal but $\sin(t)$ is not.

(b) $\cos(t)$ and $\sin(t)$ are both eigen-signals but with different eigenvalues.

(c) $\sin(t)$ is an eigen-signal but $\cos(t)$ is not.

(d) $\cos(t)$ and $\sin(t)$ are both eigen-signals with identical eigenvalues.

(GATE 2016, 2 Marks)

Solution: Since impulse response is real and even, $H(j\omega)$ will also be real and even. Therefore,

$$H(j\omega_0) = H(-j\omega_0)$$

$$\cos(t) \text{ input } = \frac{e^{jt} + e^{-jt}}{2} \text{ input}$$

Therefore, output

$$= \frac{H(j1)e^{jt} + H(-j1)e^{-jt}}{2} = H(j1)\cos(t)$$

Similarly for $\sin(t)$,

$$\text{Output} = H(j1)\sin t$$

Output $\sin(t)$ and $\cos(t)$ are eigen-signals with same eigenvalues.

Ans. (d)

43. The Laplace transform of te^t is

(a) $\dfrac{s}{(s-1)^2}$ (b) $\dfrac{1}{(s-1)^2}$

(c) $\dfrac{1}{(s+1)^2}$ (d) $\dfrac{s}{s-1}$

(GATE 2017, 1 Mark)

Solution:

$$L\{te^t\} = \frac{1}{(s-1)^2}$$

Ans. (b)

GENERAL APTITUDE

SECTION A

VERBAL ABILITY

CHAPTER 1

ENGLISH GRAMMAR

This chapter will help improve your grammatical skills and hence the fluency with which you speak the language. It consists of basic theory of all the topics involved and multiple exercises, from very basic questions to advanced problems. The reader will be confident in solving any grammatical problem after solving the exercises given in the chapter.

ARTICLES

An *article* is a word (an adjective to be precise) that combines with a noun to indicate the type of reference being made by the noun. Articles are broadly classified into two categories:

1. **'A'** (or **'an'**) is called *indefinite article*, because it does not point out any definite or particular thing or person.
2. **'The'** is called *definite article*, because it points out some definite or particular person or thing.

Use of 'A'

1. Before a word beginning with a consonant sound: *example*, a man, a toy, a car, etc.
2. Before a word beginning with the consonant sound of 'yu', though it may begin with a vowel letter: *example*, a university, a European, a useful thing, etc.
3. Before words with the sound of 'wu', even though it may begin with a vowel letter: *example*, a one-rupee note, a one-legged dog, a one-eyed beast, etc.
4. Before a word beginning with a sounded *h*: *example*, a horse, a house, a holiday, etc.

Use of 'An'

1. Before a word beginning with a vowel sound: *example*, an apple, an octopus, an arrow, etc.
2. Before a word beginning with *h* which is not pronounced and therefore the beginning sound of the word becomes a vowel sound: *example*, an hour, an heir, an honest boy, etc.
3. Before individual letters spoken with a vowel sound: *example*, an M.L.A., an S.P., an M.P., etc.

Some common examples of the use of a (or an) are as follows:

1. Give me *a* mango.
2. I saw *an* elephant.
3. A dog is *a* faithful animal.
4. Two of *a* trade seldom agree.

5. David won *a* price of *a* hundred thousand rupees.
6. Mark always travels by *an* aeroplane.
7. What *a* long queue?
8. It is *a* rainy day.

Use of 'The'

1. When we speak of a particular person or thing, or one already mentioned or one well-known to us.

   Example:
   He is playing with *the* racquet I gave him.
   Pass me *the* cup on the table.
   Clear *the* house.

2. When a singular noun or an adjective of quality is meant to represent a whole class.

   Example:
   The lion is the kind of beasts.
   The rich are not always happy.
   The Chinese have a very high I.Q.

3. Before the names of rivers, seas, oceans, countries, important geographic names, etc.

   Example:
   The Ganga
   The Sahara desert
   The Highlands

4. Before titles of books.

   Example:
   The Gita
   The Quran
   The Ramayana

5. Before the names of ships, aeroplanes, well-known buildings and newspaper.

   Example:
   The Eiffel tower
   The Times of India
   The Red Fort

6. Before superlatives.

   Example:
   He is *the* oldest man alive.
   He is *the* youngest cricketer to score a century.
   He is *the* fastest runner in the world.

7. Before the names of important events.

   Example:
   The French Revolution
   The Jallianwala Bagh incident
   The World War

8. Before an epithet attached to a personal proper name.

   Example:
   Alexander *the* Great
   Louis *the* fifth
   Henry *the* second

9. Before a noun when special emphasis is required.

   Example:
   This is just *the* opportunity I was looking for.
   Now is *the* time to act.
   He is *the* undisputed king.

10. Before the name of a nation and sometimes communities.

    Example:
    the English
    the Hindus
    the Indians

11. Before an adjective in the comparative degree when not more than two persons or things are being compared.

    Example:
    Raj is *the* taller of the two.
    Elizabeth is *the* smarter of the two.
    Jack is *the* tougher of the two.

12. Before numeral adjectives showing order.

    Example:
    All students of *the* second year were on a holiday.
    All employees on *the* third floor are lawyers.
    All the good football teams play in *the* first division.

13. Before a Proper Noun preceded by a more or less permanent adjective.

    Example:
    The late Mr Rajesh Khanna
    The beautiful Mrs Dawson
    The clever twins

14. In other fixed phrases.

    Example:
    On *the* other hand
    To *the* contrary
    All *the* more

Exercise 1.1

Complete the following sentences using suitable definite or indefinite article in the blank.

1. I would like to buy ____ couple of shirts and ____ jeans I just tried. (a, the)
2. It has been almost ____ hour since he left. (an)

3. Jerry drew a beautiful picture of ____ mountain, similar to ____ one we went to last summer. (a, the)
4. This man is ____ Einstein of this age. (the)
5. Kshitij had ____ appointment with the doctor. (an)
6. Delhi is ____ capital of India. (the)
7. ____ apple a day, keeps ____ doctor away. (An, the)
8. ____ Americans helped ____ poor kid to cross the road. (The, a)
9. ____ fire broke out in ____ apartment taking ____ lives of ten innocent people. (A, an, the)
10. Nikhil is ____ sweet boy. (a)

NOUN

A *noun* is a word which names a person, place, animal, action, quality, feeling or anything that we can think of. Nouns are classified into five categories:

1. **Proper noun:** They are used to denote a particular person, place or thing.
   Example: India, Jack, the Ganga.
2. **Common noun:** They are used to denote a class of objects.
   Example: desk, chair, man.
   (a) Collective noun: They are used to denote several persons or things regarded as one group. *Example*: jury, army, class.
   (b) Abstract noun: They are used to denote a quality or action of a person, something which we cannot see or touch. *Example*: kindness, honesty, arrogance.
   (c) Material noun: They are used to denote a substance of which things are made. *Example*: Coal, Gold, Wool.

Use of Nouns in Singular Form

Some of the nouns which are always used in singular form are as follows:

1. Furniture
2. Fuel
3. Bread
4. Mathematics, accounts, etc.
5. Words like dozen, score, hundred, etc. when preceded by a numeral.
6. Expressions like a five-year plan, a ten-man jury, etc.

Example:
The *furniture* of my house *is* very costly.
Jack picked up a five *hundred* rupee note.
Mathematics *is* the mother of science.

Use of Nouns in Plural Form

Some of the nouns which are always used in plural form are as follows:

1. Army, police, etc.
2. People
3. Scissors
4. Trousers, pants
5. Spectacles
6. Goods

Example:
The *police have* caught the terrorist.
Where *are* my *spectacles*?
David's *trousers are* very costly.

Conversion of Nouns from Singular to Plural

Ending	Singular	Plural
um-ia	Medium	Media
on-a	Phenomenon	Phenomena
is-es	Hypothesis	Hypotheses
a-ae	Antenna	Antennae
Us-i	Radius	radii
Ex/ix-ices	Matrix	Matrices
o-i	Concerto	Concerti

Collective Nouns

Some of the most commonly used collectively nouns are given below:

1. A class of students.
2. An army of soldiers.
3. A choir of singers.
4. A crew of sailors.
5. A band of musicians.
6. A gang of thieves.
7. A group of dancers.
8. A team of players.
9. A troupe of artists/dancers.
10. A staff of employees.
11. A regiment of soldiers.
12. A panel of experts.
13. A flock of tourists.
14. A board of directors.
15. A catch of fish.
16. An army of ants.
17. A flight of birds.
18. A haul of fish.
19. A flock of sheep.
20. A herd of deer/cattle/elephants/goats/buffaloes.
21. A troop of lions.

22. A pack of wolves.
23. A litter of puppies/kittens.
24. A swarm of bees/ants/rats/flies.
25. A murder of crows.
26. A kennel of dogs.
27. A galaxy of stars.
28. A stack of wood.
29. A fleet of ships.
30. A string of pearls.
31. An album of stamps/autographs/photographs.
32. A library of books.
33. A basket of fruit.
34. A bowl of rice.
35. A pack of cards.
36. A pair of shoes.
37. A bouquet of flowers.
38. A bunch of keys.
39. A range of mountains.
40. A cloud of dust.

Exercise 1.2

A. Point out the nouns in the following sentences.

1. Mount Everest is the highest peak in the world.
2. Cheetah is a very fast animal.
3. Himani and Shivani formed a deep and lasting friendship.
4. Jawaharlal Nehru was the 1st Prime Minister of India.
5. Div saw a large tea garden in Assam.

B. Fill in the blanks using suitable nouns.

1. There is so ____ (many, much) smoke coming out of that building. (much)
2. He threw ____ (a little, some) pebbles in the river. (some)
3. Akul saw ____ (a large amount of, many) cows grazing the field. (many)
4. Nupur uses only ____ (a little, a few) cooking oil in her cooking. (a little)
5. Karan ate a ____ (group, bunch) of grapes today. (bunch)

PRONOUN

A pronoun is a word used in place of a noun.

Example:

David is a good student. David gets straight A's in all the subjects.
David is a good student. *He* gets straight A's in all the subjects.

Pronouns are classified as follows:

1. Personal
2. Reflexive and Emphatic
3. Demonstrative
4. Indefinite
5. Distributive
6. Reciprocal
7. Relative
8. Interrogative

Personal Pronoun

I, we, you, he, etc., are called *personal pronouns*. They can be used for first person, second person and third person.

First person (masculine or feminine)

	Singular	Plural
Nominative	I	We
Possessive	my, mine	our, ours
Accusative	Me	Us

Second person (masculine or feminine)

	Singular	Plural
Nominative	you	You
Possessive	Your	Yours
Accusative	You	You

Third person

	Singular			Plural
	Masculine	Feminine	Neutral	All Genders
Nominative	He	she	it	They
Possessive	His	her, hers	its	their, theirs
Accusative	Him	Her	it	Them

Reflexive and Emphatic Pronoun

When '-self' is added to my, your, him, her, it and '-selves' to our, your, etc. then such pronouns are called compound personal pronouns. They are called *reflexive pronouns* when an action done by the subject reflects upon the subject.

Example: She hurt *herself*.

If compound personal pronouns are used for the sake of emphasis, they are called *emphatic pronouns*.

Example: He *himself* admitted his guilt.

Demonstrative Pronoun

Pronouns that are used to point out the objects to which they refer are called *demonstrative pronouns.*

Example:

Both the vehicles are good, but *this* is better than *that.*

This shirt is better than *that* shirt.

My views are quite in accordance with *those* of the University Board.

Indefinite Pronoun

These are pronouns that refer to person or things in a general way, but do not refer to any person or thing in particular.

Example:

One hardly knows what to do.

Somebody has stolen his car.

Some are born with a golden spoon.

Distributive Pronoun

Each, either, neither are called *distributive pronouns* because they refer to persons or things one at a time.

Example:

Either of them couldn't pass the examination.

Neither of them were guilty of the crime they are convicted of.

Each of the girls got a chocolate.

Relative Pronoun

Words like who, whom, which, etc. that refer or relate to some noun are called *relative pronouns.* The different forms of relative pronouns are:

		Singular and Plural
Nominative	:	who
Genitive	:	whose
Accusative	:	whom

Example:

This is the boy *who* plays well.

This is the girl *whose* work is good.

This is the teacher *whom* all praise.

Interrogative Pronoun

Interrogative pronouns are used when you want to ask a question.

Example:

What are you both talking about?

Who is the culprit?

Which colour did you choose for your car?

Use of Pronouns

1. The pronoun 'One' must be followed by 'one's'.
2. 'Everyone' or 'Everybody' must be followed by 'his'.
3. 'Let' is followed by pronoun in the objective case.
4. 'But' and 'except' are followed by pronoun in the objective case.
5. Reflexive pronouns are never used with verbs keep, conceal, quality, spread, rest, stay.
6. 'Who' denotes subject and 'whom' denotes object.
7. 'Whose' is used for persons and 'which' for lifeless objects.
8. 'Which' conveys additional information and 'that' explains a certain thing.
9. 'Each other' is used for two and 'one another' for more than two.
10. When the same person is the subject and object, it is necessary to use reflexive pronouns.

Exercise 1.3

Fill in the blanks with suitable pronouns.

1. This is the boy ____ (who, which, whose) comes from England. (who)
2. My friends enjoyed ____ (himself, themselves, herself) very much at the movie. (themselves)
3. Mannan, did you do the maths homework ____ (yourself, himself, itself)? (yourself)
4. The man, ____ (who, which, whose) father is a scientist, forgot his phone. (whose)
5. Ram and Shyam haven't met ____ (ourselves, themselves, each other) for a long time. (each other)
6. What did you do with the money ____ (who, which, whose) I lent you? (which)
7. I bought ____ (me, myself, each other) a new camera. (myself)
8. John is driving ____ car. (he)
9. Susan is a good dancer. ____ is the best in the crew. (she)
10. Neha and Anisha are sisters. ____ fight a lot. (they)

ADJECTIVE

Any word which describes the quality or nature or quantity of a noun, is called an *adjective*.

Example:

Arun is a *good* boy.

I asked you to get *ten* bananas.

Adjectives can be classified as follows:

1. **Adjectives of Quality:** They are used to describe the quality of the noun.
   Example: Large, small, good, bad, etc.
2. **Adjectives of Quantity:** They are used to describe the quantity (how much) of the noun.
   Example: Little, some, more, etc.
3. **Adjectives of Number:** They are used to describe the number of the noun.
   Example: Ten apples, twenty-five tons of water, etc.
4. **Demonstrative Adjectives:** They are used to answer the question 'Which'.
   Example: This, that, these, those, etc.
5. **Interrogative Adjectives:** They are used to ask a question about the noun.
   Example: What, which, whose, etc.

Use of Adjectives

1. The adjectives ending like prior, junior, senior, superior, etc. take 'to' and not 'than' after them.
   Example: He is *senior to* me.
2. Some adjectives like unique, ideal, perfect, extreme, complete, infinite, etc. are not compared.
   Example:
   It is the most unique pen. (✗)
   It is a unique pen. (✓)
3. Double comparatives and double superlatives must not be used.
   Example:
   He is more wiser than his brother. (✗)
   He is wiser than his brother. (✓)
4. When two adjectives in superlative or comparative degree are used together, the one formed by adding 'more' or 'most' must follow the other adjective.
   Example:
   He is more intelligent and wiser than his brother. (✗)
   He is wiser and more intelligent than his brother. (✓)
5. When two changes happen together, comparative degree is used in both.
   Example: The *closer* you get to the sun, the *hotter* it gets.
6. When comparative degree is used in superlative sense, it is followed by 'any other'.
   Example:
   Sachin is better than any batsman. (✗)
   Sachin is better than any other batsman. (✓)
7. When two or more comparatives are joined by 'and', they must be in the same degree.
   Example: Jack is one of the most handsome and wisest men of his family.
8. If comparisons are made by using 'other', 'than' is used instead of 'but'.
   Example: He turned out to be no other than my old classmate.

Exercise 1.4

Point out the adjectives in the following sentences.

1. Bani got <u>good</u> marks in all the papers.
2. Jack is a <u>better</u> painter than Jill.
3. <u>This</u> is the article in question.
4. Alexander was a <u>great</u> conqueror.
5. Naina ate <u>three</u> apples.
6. <u>Which</u> place did you visit?
7. Pele is one of the <u>best</u> footballers ever.
8. <u>Who</u> was driving the car?
9. He has <u>little</u> knowledge about the accident.
10. Anupam is the <u>best</u> criminal lawyer.

PREPOSITION

A preposition is a word used to relate a noun or a pronoun to some other word in a sentence.

Example:
David is *at* the top of his career.

You should be home *by* midnight.

Prepositions can be used in the following manner:

1. Preposition of time
2. Preposition of position
3. Preposition of direction
4. Other uses of preposition

Preposition of Time

1. **'At'** is used with a definite point of time or with festivals.

Example:
I will reach there *at* 6 p.m.

2. **'In'** is used with parts of the day, months, years and with the future tense.
Example:
Rohit plays *in* the afternoon.

3. **'On'** is used with dates and days.
Example:
Inu was born *on* 30th October.

4. **'By'** refers to the latest time at which an action will be over.
Example:
The movie will be over *by* 9 p.m.

5. **'For'** is used with perfect continuous tense showing the duration of action.
Example: I have been living in my house *for* twenty years.

6. **'Since'** is used with the point of time when action begins and continuous.
Example: We haven't met *since* 5 years.

7. **'From'** refers to the starting point of action.
Example: *From* rags to riches.

Preposition of Position

1. **'At'** refers to an exact point.
Example: My cousin stayed *at* my house.

2. **'In'** refers to lager areas.
Example: Raj stays *in* Delhi.

3. **'Between'** is used for two persons or two things.
Example: Things are fine *between* Jack and Jill.

4. **'Among'** is used for more than two persons and **'amongst'** is used before the words which start with a vowel.
Example: Charlie is the shortest *amongst* us.

5. **'Above'** is used for higher than, **'under'** is used for below.
Example: The traveller rested *under* the tree.

Preposition of Direction

1. **'To'** or **'towards'** is used to express motion from one place to another.
Example: My father walked *towards* me.

2. **'At'** refers to aim on a target.
Example: The hunter aimed *at* the deer and fired.

3. **'Against'** shows pressure.
Example: The thief pressed the knife *against* her neck.

4. **'For'** denotes destination.
Example: Amit is leaving *for* Johannesburg tomorrow.

5. **'From'** denotes departure point.
Example: Nupur's father is coming back *from* Chandigarh.

6. **'Into'** denotes motion towards the inside of something.
Example: Prince fell into a hole.

7. **'Off'** refers to separation.
Example: He took the bandage *off* his hand.

Other Uses of Preposition

1. **'About'** shows nearness.
Example: I am *about* to reach.

2. **'After'** refers to sequence.
Example: She came just *after* her husband.

3. **'Across'** means from one side.
Example: Rahul shouted my name from *across* the road.

4. **'Before'** stands for 'in front of'.
Example: Aman stood *before* his father and begged for his forgiveness.

5. **'Beside'** means 'by the side of'.
Example: A couple should stand *beside* each other through tough times.

6. **'Besides'** means 'in addition to'.
Example: I cannot come, *besides* I am not even interested in the event.

Exercise 1.5

Complete the following sentences by use of the suitable prepositions in the blank.

1. Gautam reached ____ (at, on) 6 pm sharp. (at)
2. Kshitij lives ____ (at, on) my house. (at)
3. My father used to live ____ (at, in) Punch. (in)
4. The racer was quickly reaching ____ (towards, on) the finishing line. (towards)
5. The boy fell ____ (in, into) a very deep hole. (into)
6. Mahatma Gandhi was born ____ (at, on) 2nd October. (on)
7. Naina is the fairest ____ (among, amongst) all of us. (amongst)
8. We haven't met ____ (since, from) 10 years. (since)
9. We are leaving ____ (for, to) the office. (for)
10. ____ (for, from) rags to riches. (from)

VERBS

A *verb* is a word that tells us something about a person or thing that is an action being done by the person/thing, the action being done on the person/thing or state of being.

Verbs are of two types:

1. **Transitive Verbs:** They are verbs which involve a direct object.

   Example:
   The man is *reading* a *book*.
   He is *driving* a *car*.
   He *spoke* the *truth*.

In the above examples, reading, driving and spoke are the *verbs* while book, car and truth are the *objects*.

2. **Intransitive Verbs:** They are verbs which do not involve a direct object.

   Example:
   The man is *reading* quickly.
   He *drives* very cautiously.
   He *speaks* softly.

The *infinitive* is the base of a verb, often preceded by 'to'. When 'ing' are added to verbs, they are called *gerunds*.

Verbs can be used in active or passive voice. A verbs is said to be in the *active voice* when its form shows that the person (or thing) denoted by the subject does something. Similarly, a verb is said to be in the *passive voice* when its form shows that something is done to the person (or thing) denoted by the subject. *Example*:

1. Tom loves Jerry. (Active)
   Jerry is loved by Tom. (Passive)
2. Who did this? (Active)
   This was done by whom? (Passive)
3. He kicked the football. (Active)
   The football was kicked by him. (Passive)

Use of Verb

1. Singular subjects have singular verb and plural subjects have plural verb.

   Example:
   He *writes*.
   They *write*.

2. Two subjects joined by '**and**' will always have a plural verb whereas two singular subjects joined by '**or**' or '**nor**' will take a singular verb.

   Example:
   The doctor *and* nurse *work* together.

3. A singular subject and a plural subject joined by '**or**' or '**nor**' will take a singular or plural verb depending upon which subject is nearer the verb.

   Example:
   Neither Rahul nor his friends *are coming*.
   Neither his friends nor Rahul *is coming*.

4. If two subjects are joined together by '**as well as**' or '**with**' the verb will act according to the first subject.

   Example:
   Students as well as the teacher *are* playing.
   Jack with his sons is going for a drive.

5. Indefinite pronouns such as someone, somebody, each, nobody, anybody, everybody, everyone, either, neither etc. always take a singular verb.

   Example:
   Somebody *disturbs* Neena every night.
   Everybody *fails* at least once in life.

6. Indefinite pronouns which indicate more than one (several, few, both, many) always take plural verbs.

   Example:
   All the brothers are *coming* together.
   Both the books required careful *reading*.

7. Collective nouns (groups like army, jury, panel, committee, etc.) take singular verbs when the group works together and take plural verbs when the members of the group work alone.

   Example:
   The jury *has* given the verdict.
   The jury *have* argued for three hours.

8. Titles take singular verbs.

   Example:
   Angels and Demons *is* a very good book.
   The Avengers *is* a good movie.

9. If the subject begins with '**The number of**' use a singular verb and if the subject begins with '**A number of**' use a plural verb.

   Example:
   The number of balls *is* less.
   A number of balls *are* missing.

Use of Infinitives

1. Verbs such as learn, remember, promise, swear, refuse, try, care, hope, love etc. are followed by infinitive.

Example:
The policeman refused *to obey* the bizarre order of his superior.
I would love *to join* you.

2. Verbs such as order, tell, invite, oblige, allow, instruct, advise, remind, etc. are followed by object and infinitive.

   Example:
   Jake allowed his son *to play*.
   The newlywed couple invited their neighbours *to eat*.

3. Verbs or expressions like- will, can, do, must, may, let, etc. are followed by infinitive without '**to**'.

   Example:
   Let the witness be *excused*.
   He must *finish* this work himself.

4. Expressions like would rather, sooner than later, sooner than, rather than, had better, etc. are followed with infinitive without '**to**'.

   Example:
   I would rather *travel* alone.
   You had better *help* him.

5. The infinitive is used after adjectives like delight, angry, glad, excited, etc.

   Example:
   I was delighted *to hear* your result.
   I am glad *to be* back.

6. The verb '**know**' is never directly followed by the infinitive. It is followed by a conjunction and then the infinitive.

   Example:
   Do you know *how* to drive a car?
   I didn't know *how* to act until you taught me.

Use of Gerunds

1. When an action is being considered in general sense, gerund is used as subject.
   Example: Painting is his favourite hobby.

2. Gerund is used as subject in short prohibitions. *Example: Drinking* and *smoking* here is prohibited.

3. Verbs such as help, stop, detest, avoid, finish, dread, mind, prevent, dislike, risk, deny, recollect, no good, no use, suggest, etc. are followed by the gerund.
   Example: Let me know when you finish *studying*.

4. If there is a sense of dislike, hesitation, risk, etc. in a sentence, use gerund.
   Example: There is always a risk when *driving* fast.

Exercise 1.6

Fill in the blanks by choosing the suitable verb.

1. The number of balls ____ (are, is) less. (is)
2. They ____ (should be, should have) finished the job by now. (should have)
3. One should not drink and ____ (drive, drove). (drink)
4. The earth ____ (moves, moved) around the sun. (moves)
5. A number of kids ____ (are, is) arriving. (are)
6. He ____ (has fallen, fell) asleep while drinking. (fell)
7. I ____ (will, know) him from a long time. (know)
8. He ____ (is driving, was driving) fast when the accident happened. (was driving)
9. I would love ____ (to join, joining) you. (to join)
10. Everybody ____ (fails, failed) at least once in life. (fails)

TENSES

The changed forms of a verb that indicate time of the action are called *tenses* of the verb. The word tense comes from the Latin *tempus*, meaning time. Tenses are of three types which can further be classified as:

1. **Past Tense:** A verb that refers to something that has passed or already taken place is said to be in the *Past Tense*.

   Example:
   I played. (Simple Past)
   I was playing. (Past Continuous)
   I had played. (Past Perfect)
   I had been playing. (Past Perfect Continuous)

2. **Present Tense:** A verb that refers to the present or something happening at the moment is said to be in the *Present Tense*.

   Example:
   I eat. (Simple Present)
   I am eating. (Present Continuous)
   I have eaten. (Present Perfect)
   I have been eating. (Present Perfect Continuous)

3. **Future Tense:** A verb that refers to the future or something that is going to happen is said to be in the *Future Tense*.

   Example:
   I shall sleep. (Simple Future)
   I shall be sleeping. (Future Continuous)
   I shall have slept. (Future Perfect)
   I shall have been sleeping. (Future Perfect Continuous)

Use of Tenses

1. When the verb in the principal clause is in the past tense, the verbs of the subordinate clauses should be in the past tense. However, an exception can be made if the latter expresses universal or habitual truth.

   Example:
   She said that he had finished her assignment.
   He said that he had left from his office.

2. Any tense may be used in the sub-ordinate clause if it gives a comparison by using the word '**than**'.

   Example:
   Tim *preferred* sandwich more than he *prefers* burgers.
   The teacher *likes* Priya more than she *likes* Pooja.

3. Any tense can be when the subordinate clause is in a quotation.

   Example:
   I *said*, "I am going to be busy today."
   I *told* you, "I am going to Jammu tomorrow."

4. With the phrase '**as if**' and '**as though**', past tense and plural form must be used.

   Example:
   He preaches as if he *were* a saint.
   He boasts as though he *were* the best in everything.

5. Words like **usually**, **generally**, **often**, **whenever**, etc. are used in present indefinite tense.

   Example:
   I usually *eat* my lunch early.
   We will go whenever you want to *go*.

6. Present perfect tense should be used if the action began in the past and is still continuing in the present.

   Example:
   He *has been taking* wickets since 5 matches.
   She *has been preparing* for the exam since 3 months.

7. Past tense should be used with expressions like **suppose that**, **it is high time**, **it is time**, **as if**, etc.

   Example:
   It is time you *went* home.
   Suppose that I *played* with you.

Exercise 1.7

Read the following sentences and fill in the blanks using the right tense.

1. I didn't have any idea what the man ____ (say). (said)
2. You ____ (do) what you had to. (did)
3. Tara did ____ (manage) to persuade her husband for a vacation. (manage)
4. If it hadn't been for the bad weather, the plane would have ____ (take) off at 5 p.m. (taken)
5. What goes around, ____ (come) around. (comes)
6. He preaches as if he ____ (is) a saint. (were)
7. My father has just finished ____ (read) the newspaper. (reading)
8. I wish there ____ (is) no war. (was)
9. He ____ (say), "He won't be able to come to the party." (said)
10. The teacher ____ (like) Ram more than she ____ (like) Raj. (likes)

ADVERBS

A word that modifies the meaning of a verb, an adjective or another adverb is called an a*dverb*.

Example:

He strokes the horse *gently*.

The strawberries were *very* sweet.

Bolt runs *quickly*.

Adverbs can be classified into the following according to their meaning (though one adverb may belong to more than one class depending upon how and where they are used):

1. **Adverbs of Time:** They show *when*.

   Example:
   I have heard *that* song before.
   Edward is always *late* for his class.

2. **Adverbs of Frequency:** They show *how often* an action is done.

   Example:
   The policeman warned *twice* before shooting.
   Barking dogs *seldom* bite.

3. **Adverbs of Places:** They show *where* the action is to be performed or described.

   Example:
   The murder took place *here*.
   Rose left her family and went *away*.

4. **Adverbs of Manner:** They show how or in what manner the verb is used.

   Example:
   He works *diligently*.
   The soldiers fought *fiercely*.

5. **Adverbs of Degree or Quantity:** They show how much or the degree/extent to which the verb is used.

   Example:
   He was *too* careless.
   He couldn't be *more* precise.

6. **Adverbs of Affirmation and Negation:**

   Example:
   I am *certain* that he is the killer.
   I am *not sure* how long it will take to reach.

7. **Adverbs of Reason:**

   Example:
   He was *therefore* expelled from school.
   His efforts were *hence* futile.

Exercise 1.8

Point out the adverbs in the following sentences.

1. He finished his work quickly.
2. Barking dogs seldom bite.
3. James thought twice before answering.
4. Federer lost because he was too complacent.
5. The policeman was certain that the thief will steal again.
6. Roger was caught stealing and was therefore fired from the office.
7. Bill was late for his assessment.
8. The soldiers fought bravely.
9. Jennifer's husband left her and went away.
10. Jacob is a shy kid. He hardly has any friends.

PRACTICE EXERCISE

Directions for Q. 1 to Q. 15

Read each sentence and spot whether there is any error in any of the parts. There would be only one sentence which will have an error or none at all and the sentence will have a maximum of one error.

1.

(a) I will be studying till late.
(b) John bought some mangoes.
(c) He spoke as if he was an expert.
(d) No error.

2.

(a) He didn't knew what he was doing.
(b) I will pick you up at noon.
(c) Rahul is an employee.
(d) No error.

3.

(a) Where there is a will, there is a way.
(b) Dave got me a pair of trouser.
(c) It is a beautiful night.
(d) No error.

4.

(a) The more the merrier.
(b) He is much more gentler than his brother.
(c) Pass me a pair of scissors.'
(d) No error.

5.

(a) He was travelling with his wife, who was also his business partner.
(b) Fortune favours the brave.
(c) He shall not be forgiven.
(d) No error.

6.

(a) Raj is better than any player.
(b) A mother always loves her children equally.
(c) Juice is sold by the litre.
(d) No error.

7.

(a) He was raised among orphans.
(b) I will reach home at 7 o'clock.
(c) He went to a city where he was born.
(d) No error.

8.

(a) The person which sat next to me was a sportsperson.
(b) The pen which Arun carried was very expensive.
(c) He who must not be named has returned.
(d) No error.

9.

(a) God helps those, who help themselves.
(b) He is as old as I am
(c) May he rest in peace.
(d) No error.

10.

(a) He is wiser than his brother.
(b) He is wiser and more intelligent than his brother.
(c) He is more intelligent and wiser than his brother.
(d) No error.

11.

(a) We saw a movie
(b) We had a good play of cricket.
(c) We liked the movie.
(d) No error.

12.

(a) What can I do you for?
(b) What can I do for you?
(c) What shall I do?
(d) No error

13.

(a) These all mangoes are ripe.
(b) The oranges were juicy.
(c) An apple a day keeps the doctor away.
(d) No error.

14.

(a) Who are you and what do you want?
(b) Which kind of a person does that?
(c) What kind of a person does that?
(d) No error

15.

(a) I have hurt my hand.
(b) Give me little sugar.
(c) Can I borrow some money?
(d) No error.

Directions for Q. 16 to Q. 30

Each sentence is broken down into four parts. Read each sentence and spot the part which has an error. Ignore errors of punctuation.

16.

(a) Every person
(b) in the
(c) sinking boat
(d) have been saved.

17.

(a) Somebody disturb
(b) me every night
(c) while I
(d) am working.

18.

(a) He is
(b) an oldest
(c) person of
(d) the house.

19.

(a) Leo is
(b) more taller
(c) than anybody else
(d) in the class.

20.

(a) Neither Kshitij
(b) nor his friends
(c) is coming
(d) to the party.

21.

(a) I bought me
(b) a new
(c) camera for my
(d) birthday.

22.

(a) Women in
(b) India are more
(c) self employed
(d) then men.

23.

(a) He boasts
(b) as though
(c) he is the best
(d) at everything.

24.

(a) Divya didn't
(b) knew what
(c) she was
(d) doing.

25.

(a) The jury
(b) have
(c) agreed to
(d) a verdict.

26.

(a) I have been
(b) living in
(c) this house
(d) till twenty years.

27.

(a) It is the most
(b) unique object
(c) I have ever
(d) seen before.

28.

(a) I reached
(b) on 6 pm
(c) but my brother
(d) was late.

29.

(a) As of this morning,
(b) none of my friends
(c) have been able
(d) to solve the puzzle.

30.

(a) Jim is
(b) more faster
(c) than his
(d) brother Tim.

Directions for Q. 31 to Q. 40

Read the following sentences and answer accordingly; avoid mistakes of punctuation.

31. He is <u>more intelligent and wiser</u> than his elder brother. (wiser and more intelligent)

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

32. Hazards of life cannot be negated but they can be quite effortlessly evaded.

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

33. Ram has the <u>most best</u> collection of art. (best)

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

34. It's <u>a</u> most unique pen. My best friend gave it to me and didn't <u>took</u> any money for it. (the, take)

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

35. He is taller <u>then</u> his younger brother but <u>in</u> all his friends, he is <u>shorter</u>. (than, among, shortest)

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

36. I loved the dog but my father told me that we won't keeping it. (won't be keeping)

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

37. "Will you marry me?" he asked me as we both <u>eat</u> our lunch. (ate)

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

38. A beast is an aggressive creature by nature. It is a hard fact of life and undeniable.

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

39. Manoj's father is <u>a</u> M.P. He will stand for re-election after <u>a</u> year. (an, an)

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

40. The students were advised to follow the instructions <u>to</u> the examiner. (of)

(a) There is one error.
(b) There are two errors.
(c) There are more than two errors.
(d) No errors.

ANSWERS

1. (c)	**7.** (c)	**13.** (a)	**19.** (b)	**25.** (d)	**31.** (a)	**37.** (a)
2. (a)	**8.** (a)	**14.** (b)	**20.** (c)	**26.** (d)	**32.** (d)	**38.** (d)
3. (b)	**9.** (d)	**15.** (b)	**21.** (a)	**27.** (a)	**33.** (a)	**39.** (b)
4. (b)	**10.** (c)	**16.** (d)	**22.** (d)	**28.** (b)	**34.** (b)	**40.** (a)
5. (d)	**11.** (b)	**17.** (a)	**23.** (c)	**29.** (c)	**35.** (c)	
6. (a)	**12.** (d)	**18.** (b)	**24.** (b)	**30.** (b)	**36.** (a)	

CHAPTER 2

SYNONYMS

A synonym is defined as the word having the same or nearly the same meaning as another word in the same language. This chapter provides the reader with the list of some of the important synonyms which are commonly used followed by an exercise towards the end, but first we discuss some of the techniques to help solve the questions related to synonyms.

TIPS TO SOLVE SYNONYM BASED QUESTIONS

Questions based on synonyms and antonyms can be pretty difficult to answer if the word is new or unheard of. In such scenarios, there are certain tips or tricks that the student can use in order to increase his chances of getting the correct answer. Some of the common tips are given as follows:

1. Try to define the stem word. *Example*: tempestuous – related word is temper, severance – related word is sever, etc.
2. Put the word in context. Putting the word in a sentence or a phrase makes it easier to guess the meaning. *Example*: gratuitous – "gratuitous violence" – means unnecessary, requiem – "requiem for a heavyweight" means rest.
3. When you have a word, try to guess all its synonyms and from the answers see the exact word which has the same meaning and the opposite meaning for antonym.

   Example: Which of the following is a synonym of abundance?

   (a) Ampleness
   (b) Redundant
   (c) Donation
   (d) Augmentation

Now, we know that the meaning of the word "abundance" is something which is large or sufficient in quantity. Hence, we are looking for a word like affluent or ample. Hence, the correct answer is option (a), *ampleness*.

4. Test the word for positive or negative connotations.
   (a) As a matter of general notation, words that include the concept of going up are usually positive, while those that include the concept of going down are usually negative.
   (b) Any word that starts with 'de', 'dis' or 'anti' is usually negative.

   Example: Some of the positive connotations are elevate, ascend, adulation, etc.

   Some of the negative connotations are decline, suborn, derision, etc.

5. Some words can be used as both nouns and verbs. If the given word is used as a verb, then all of the answer choices will also be verbs. This helps the student to quickly determine if the word is being used in a secondary sense, as common words have different meanings, if they are used as verbs, nouns or adjectives. Hence, never overlook the rare meanings of words and always know the part of the speech the word is used in. *Example*: Some words commonly used as both nouns and verbs are curb, document, table, air, bustle, etc.

6. Use the following guessing strategies.
   (a) Eliminate answer choices that have no clear opposite.
   (b) If two (or more) choices have the same meaning, eliminate both.

   Example: What is the synonym of Jovial?

   (a) Miserable
   (b) Gloomy
   (c) Cheerful
   (d) Depressed

Now, we can see that words *miserable*, *gloomy* and *depressed* are similar in meaning. Hence, we eliminate these options and we are left with option (c) which is the right answer, *cheerful*.

A

Abandon: desert, discard, evacuate, leave.

Abduct: kidnap, snatch, seize.

Abet: aid, assist, help, support.

Abeyance: adjournment, postponement, intermission, recess.

Abide: accept, bear, endure, reside.

Abstain: avoid, decline, renounce, refrain.

Abundance: affluence, ampleness, bounty, copiousness.

Abysmal: bottomless, boundless, complete, deep.

Accede: accept, admit, comply, concede.

Acclaim: applaud, approve, celebrate, hail.

Accord: agree, concur, conform, harmonize.

Accustom: acclimatize, acquaint, adapt, familiarize.

Acrid: caustic, harsh, irritating, pungent.

Adamant: determined, firm, resolute, stubborn.

Adept: dexterous, masterful, expert, genius.

Adhere: cleave, cling, obey, observe.

Adroit: clever, cunning, ingenious, quick-witted.

Adversity: affliction, calamity, difficulty, misfortune.

Aggravate: annoy, increase, infuriate, intensify.

Alleviate: ease, lighten, mitigate, pacify.

Amenable: agreeable, favourable, manageable, obedient.

Anguish: discomfort, distress, pain, sorrow.

Arrogant: disdainful, imperious, pompous, scornful.

Astonish: amaze, confound, overwhelm, surprise.

Atrocious: awful, bad, rotten, shocking.

Audacious: courageous, daring, dauntless, rash.

Austere: forbidding, grave, grim, harsh.

Averse: antipathetic, disinclined, hostile, unwilling.

Avid: avaricious, eager, passionate, zealous.

B

Baffle: confound, confuse, deceive, muddle.

Banal: common, conventional, ordinary, plain.

Barren: depleted, desolate, unfertile, unfruitful.

Berate: censure, criticise, disapprove, reprimand.

Bastion: citadel, defence, fortress, mainstay.

Betray: abandon, deceive, deceit, fool.

Bias: bent, inclination, predisposition, unfair.

Bitter: acerb, acrid, harsh, sour.

Blaspheme: abuse, curse, profane, swear.

Blend: amalgam, combine, mix, union.

Bliss: beatitude, gladness, happiness, joy.

Bluff: betray, counterfeit, feign, mislead.

Bold: adventurous, daring, dauntless, fearless.

Bother: annoy, concern, fuss, irritate.

Breach: aperture, break, estrangement, infringement.

Brief: abstract, concise, short, summary.

Brilliant: accomplished, clever, ingenious, intelligent

Brisk: alert, fast, speedy, swift.

Brood: agonize, dwell, repine, ruminate.

Budget: allot, cost, plan, total.

C

Candid:	fair, honest, truthful, unbiased.
Caricature:	cartoon, imitation, mockery, parody.
Casual:	extempore, informal, natural, unplanned.
Catastrophe:	affliction, devastation, disaster, fiasco.
Category:	classification, department, division, group.
Cease:	culminate, desist, discontinue, stop.
Chaotic:	deranged, disordered, messy, turbulent.
Chastity:	celibacy, innocence, modesty, purity.
Cherish:	embrace, esteem, love, nourish.
Circumvent:	avoid, deceive, elude, escape.
Clan:	band, brotherhood, fraternity, gens.
Clinch:	assure, conclude, confirm, secure.
Colloquial:	conversational, demotic, familiar, informal.
Commemorate:	admire, celebrate, honour, salute.
Commotion:	agitation, disorder, disturbance, excitement.
Compensate:	balance, recompense, repay, satisfy.
Competent:	able, appropriate, capable, skill.
Complicate:	confuse, entangle, involve, muddle.
Conceive:	accept, design, grasp, plan.
Confirmation:	acknowledgement, affirmation, proof, testament.
Contour:	curve, figure, profile, relief.
Contradict:	challenge, deny, disagree, oppose.
Contribution:	augmentation, donation, grant, present.
Conviction:	assurance, certainty, confidence, persuasion.
Courteous:	affable, civilized, polite, well-mannered.
Craving:	desire, longing, need, passion.
Credulous:	accepting, confident, trustful, unsuspicious.
Creditable:	admirable, commendable, deserving, honourable.
Culminate:	climax, close, conclude, finish.
Cynical:	distrustful, sarcastic, scornful, sneering.

D

Dare:	challenge, defy, provocation, taunt.
Daunt:	alarm, frighten, intimidate, terrify.
Decay:	decline, deterioration, mortification, rot.
Decent:	chaste, honourable, noble, pure.
Deed:	achievement, act, action, exploit.
Deflate:	chasten, collapse, empty, shrink.
Designate:	label, name, select, title.
Detain:	apprehend, constrain, hold, keep.
Disclose:	announce, confess, discover, reveal.
Diligent:	active, attentive, busy, hardworking.
Dire:	alarming, appalling, cruel, disastrous.
Discreet:	careful, cautious, circumspect, diplomatic.
Dissent:	decline, differ, disagree, refuse.
Dodge:	avoid, deceive, duck, sidestep.
Dubious:	doubtful, hesitant, mysterious, uncertain.
Dwell:	abide, establish, inhabit, settle.
Dwindle:	abate, decay, diminish, weaken.

E

Eager:	anxious, earnest, enthusiastic, keen.
Eccentric:	abnormal, bizarre, erratic, idiosyncratic.
Edgy:	anxious, irritable, nervous, touchy.
Elaborate:	embellish, enhance, extravagant, overdone.
Elusive:	baffling, puzzling, subtle, tricky.
Emanate:	arise, derive, radiate, originate.
Embezzle:	forge, loot, purloin, steal.
Eminent:	distinguished, dominant, prominent, renowned.
Endure:	bear, last, persist, suffer.
Enrage:	aggravate, incite, irritate, provoke.
Essential:	basic, capital, necessary, needful.
Estimate:	evaluation, guess, predict, projection.
Exalt:	advance, dignify, elevate, honour.
Exhaust:	deplete, empty, fatigue, weakened.
Exhilarated:	animated, cheerful, euphoric, zestful.
Expedite:	accelerate, advance, dispatch, facilitate.
Explicit:	absolute, accurate, definite, specific.

F

Fame:	celebrity, honour, prominence, stardom.
Famine:	scarcity, dearth, hunger, starvation.
Fastidious:	critical, exacting, particular, picky.
Feeble:	weak, delicate, exhausted, infirm.
Fervour:	intensity, passion, seriousness, sincerity.
Feud:	argument, conflict, dispute, fight.
Fierce:	barbarous, brutal, cruel, passionate.
Fishy:	doubtful, dubious, suspect, suspicious.

Flatter: charm, compliment, glorify, praise.
Flawless: faultless, impeccable, perfect, unblemished.
Foresee: anticipate, divine, predict, prophesy.
Fretful: captious, complaining, irritable, peevish.
Frivolous: childish, foolish, impractical, juvenile.
Frugal: careful, discreet, prudent, saving.

G

Gaiety: animation, elation, exhilaration, glee.
Gaudy: brilliant, flashy, glaring, showy.
Gestation: development, evolution, formation, pregnancy.
Gloomy: cloudy, dissolute, dim, dreamy.
Goad: impulsion, incentive, irritation, motivation.
Grubby: filthy, frowzy, foul, shabby.
Guile: cunning, deceit, deception, treachery.
Gullible: credulous, foolish, naïve, trusting.

H

Habitual: accustomed, common, familiar, regular.
Haggle: bargain, barter, quarrel, wrangle.
Haphazard: accidental, arbitrary, chance, random.
Harass: annoy, attack, disturb, intimidate.
Hasty: abrupt, careless, hurried, quick.
Haughty: arrogant, egoistic, pretentious, snobbish.
Holocaust: annihilation, carnage, destruction, devastation.
Hygiene: cleanliness, sanitation, regimen, wholesomeness.
Hypocrisy: deceitfulness, duplicity, falseness, pretence.

I

Ideal: absolute, complete, optimal, perfection.
Idle: abandoned, barren, lazy, unoccupied.
Ignorant: inexperienced, stupid, unaware, unintelligent.
Illogical: incongruent, irrational, rambling, senseless.
Illustrious: brilliant, celebrated, eminent, famous.
Imitate: copy, duplicate, reflect, xerox.
Immense: endless, huge, large, mammoth.
Impartial: candid, equal, impersonal, unbiased.
Implicate: accuse, associate, blame, insinuate.
Importune: demand, insist, persuade, solicit.
Inadvertent: accidental, careless, negligent, unintentional.
Indifferent: apathetic, disinterested, dispassionate, uncaring.
Isolate: detach, part, quarantine, remove.

J

Jargon: argot, dialect, slang, vocabulary.
Jovial: affable, genial, merry, pleasant.
Judge: estimate, intermediary, referee, umpire.
Justification: absolution, excuse, reason, reply.
Juvenile: adolescent, childish, immature, naive.

K

Kernel: centre, core, crux, root.
Kindle: blaze, burn, flare, ignite.
Knack: ability, aptitude, skill, talent.
Knotty: complex, complicated, hard, problematic.
Kudos: credit, fame, glory, honour.

L

Label: brand, classify.
Labour: toil, work.
Lead: direct, proceed.
Lean: slim, thin.
Leave: abandon, desert.
Liberal: lenient, open-minded.
Limitation: boundary, constraint.
Lucid: clear, understandable.
Lucky: auspicious, fortunate.

M

Manage: administer, control.
Manipulate: control, shape.
Marginal: borderline, limited.
Match: agree, correspond.
Maze: complexity, labyrinth.
Meditate: ponder, think.
Memorial: commemoration, monument.
Mention: allude, refer.
Merge: blend, fuse.

N

Narrow: confined, slim, thin, restricted.
Nature: aspect, character, features, personality.

Negate:	contradict, countermand, nullify, refute.
Negligent:	careless, complacent, indifferent, unconcerned.
Negotiate:	bargain, deal, mediate, settle.
Noble:	aristocratic, distinguished, gentle, imperial.
Novice:	amateur, beginner, learner, rookie.
Nuisance:	annoyance, offense, problem, trouble.

O

Obedient:	amenable, devoted, faithful, loyal.
Objection:	challenge, disapproval, question, protest.
Obligatory:	binding, compulsory, imperative, required.
Observe:	monitor, notice, spot, watch.
Obvious:	conspicuous, definite, straightforward, transparent.
Offend:	anger, annoy, irritate, provoke.
Omen:	indication, premonition, sign, warning.
Omit:	cancel, exclude, remove, skip.
Opportune:	appropriate, advantageous, auspicious, timely.

P

Pacify:	appease, assuage, moderate, placate.
Paramount:	chief, leading, principal, superior.
Partisan:	biased, dogmatic, partial, prejudiced.
Passive:	inactive, lethargic, lifeless, static.
Permeate:	diffuse, disseminate, infiltrate, penetrate.
Perpetuate:	bolster, endure, preserve, secure.
Perplex:	astonish, baffle, cheat, deceive.
Persecute:	afflict, harass, torment, torture.

Q

Quash:	annihilated, crush, quench, repress.
Queer:	abnormal, anomalous, bizarre, odd.
Quibble:	complaint, criticism, object, protest.
Quiver:	shake, shiver, tremble, vibration.

R

Radiate:	effuse, emit, scatter, transmit.
Radical:	basic, essential, fundamental, natural.
Rank:	arrange, classify, organize, position.
Recalcitrant:	contrary, defiant, opposing, stubborn.
Receptacle:	box, container, holder, repository.
Reconcile:	adjust, atone, conciliate, pacify.
Regret:	affliction, anguish, dissatisfaction, remorse

S

Sanction:	authorization, approval, permission, support.
Scope:	aim, extent, opportunity, space.
Section:	area, division, field, portion.
Shallow:	depthless, hollow, superficial, trivial.
Shrewd:	careful, calculating, cunning, piercing.
Slight:	insignificant, negligible, slim, small.
Spontaneous:	extemporaneous, impulsive, instinctive, unplanned.
Stabilize:	balance, maintain, steady, sustain.

T

Tame:	amenable, domesticated, manageable, trained.
Tangle:	complication, intertwine, mess, twist.
Tendency:	bias, inclination, trend, readiness.
Term:	cycle, duration, period, terminology.
Thrift:	carefulness, conservation, prudence, stringiness.
Tumult:	agitation, commotion, confusion, uproar.
Turbulent:	disordered, disturbance, unstable, violent.

U

Ubiquitous:	everywhere, omnipresent, pervasive, universal.
Umbrage:	anger, annoyance, grudge, rage.
Unanimity:	accord, agreement, concurrence, unison.
Unequivocal:	absolute, apparent, certain, definite.
Unsavoury:	distasteful, disagreeable, repulsive, revolting.

V

Vain:	arrogant, boastful, haughty, narcissistic.
Valid:	accurate, authenticate, legitimate, tested.
Variety:	array, collection, diversity, range.
Verify:	attest, certify, prove, test.

W

Waggish:	amusing, cheerful, comical, playful.
Waive:	abandon, delay, leave, resign.
Warden:	administrator, caretaker, curator, guardian.
Wear:	attrition, damage, deterioration, loss.
Woe:	agony, anguish, grief, suffering.

X

Xenophobia: animosity, antipathy, bias, prejudice.

Y

Yen: craving, desire, lust, passion.

Yield: earnings, harvest, income, produce.

Z

Zany: comical, eccentric, funny, foolish.

Zeal: alacrity, determination, eagerness, enthusiasm.

Zenith: acme, climax, height, top.

Zest: bite, flavour, relish, taste.

PRACTICE EXERCISE

Read the words given below and choose the option that is most nearly same in meaning of the word.

1. Abase

(a) incur (b) dignify
(c) humiliate (d) estimate

2. Aberration

(a) deviation (b) abhorrence
(c) dislike (d) absence

3. Abet

(a) obedience (b) encourage
(c) conceive (d) evade

4. Abhor

(a) detest (b) love
(c) affection (d) embarrassed

5. Abnegation

(a) cause (b) selfishness
(c) expectation (d) renunciation

6. Abridge

(a) extend (b) lengthen
(c) mediate (d) shorten

7. Abysmal

(a) eternal (b) bottomless
(c) shallow (d) big

8. Acclivity

(a) elevation (b) decline
(c) report (d) climate

9. Accord

(a) direction (b) agreement
(c) massive (d) misdirection

10. Ample

(a) abundant (b) deficiency
(c) parity (d) less

11. Anguish

(a) enjoyment (b) excitement
(c) suffering (d) frustrated

12. Augment

(a) decrease (b) increase
(c) friendly (d) harvest

13. Authenticate

(a) original (b) forged
(c) fake (d) prove genuine

14. Baffle

(a) certain (b) confuse
(c) angry (d) sure

15. Baleful

(a) detest (b) evil
(c) powerful (d) pure

16. Baneful

(a) intellectual (b) thankful
(c) decisive (d) poisonous

17. Bawdy

(a) flashy (b) confident
(c) indecent (d) show-off

18. Blandishment

(a) flattery (b) love
(c) affection (d) embarrassed

19. Bluff

(a) cheat (b) toy
(c) sincere (d) fair

20. Brackish

(a) careful (b) salty
(c) chosen (d) tough

21. Brazen

(a) shameless (b) quick
(c) pleasant (d) modest

22. Callous

(a) kind (b) apathy
(c) sympathetic (d) insensitive

23. Candid

(a) vague (b) outspoken
(c) experienced (d) anxious

24. Capricious

(a) deviating (b) suspicious
(c) unpredictable (d) expected

25. Chalice

(a) cup (b) table
(c) drink (d) bottle

26. Chaste

(a) loyal (b) timid
(c) curt (d) pure

27. Chide

(a) unite (b) fear
(c) record (d) scold

28. Circumspect

(a) careless (b) cautious
(c) flashy (d) complacent

29. Coagulate

(a) thicken (b) loosen
(c) sum (d) thin

30. Comprehend

(a) remove (b) arrest
(c) understand (d) confused

31. Cordial

(a) friendly (b) hate
(c) distorted (d) disorganized

32. Decimate

(a) search (b) disgrace
(c) kill (d) collide

33. Declivity

(a) trap (b) downward slope
(c) quarter (d) decline

34. Decorous

(a) momentary (b) emotional
(c) suppressed d) proper

35. Defiance

(a) resistance (b) acceptance
(c) obedience (d) strength

36. Deluge

(a) confusion (b) deception
(c) flood (d) mountain

37. Depravity

(a) wickedness (b) sadness
(c) heaviness (d) tidiness

38. Dwindle

(a) blow (b) inhabit
(c) spin (d) lessen

39. Ecstasy

(a) speed (b) pleasure
(c) warmth (d) treasure

40. Efficacy

(a) efficient (b) proof
(c) mystery (d) recognize

41. Egregious

(a) anxious (b) shocking
(c) pious (d) sociable

42. Elated

(a) debased (b) depressed
(c) joyful (d) embarrassed

43. Emancipate

(a) liberate (b) poverty
(c) remove (d) avoid

44. Enigma

(a) joy (b) enjoyment
(c) affection (d) mystery

45. Exonerate

(a) apprehend (b) acquit
(c) punish (d) cremate

46. Expatriate

(a) attack (b) exile
(c) punish (d) exclude

47. Factitious

(a) artificial (b) true
(c) real (d) imaginary

48. Feebleminded

(a) confused (b) shrewd
(c) stupid (d) intelligent

49. Ferocious

(a) angry (b) punish
(c) shoot (d) fierce

50. Finesse

(a) smooth (b) delicate skill
(c) final (d) end

51. Futile

(a) fertile (b) useless
(c) worthy (d) uncertainty

52. Gainsay

(a) deny (b) accept
(c) profit (d) advantage

53. Garnish

(a) grand (b) smooth
(c) spoil (d) decorate

54. Gaudy

(a) flashy (b) introvert
(c) ill-mannered (d) extrovert

55. Genesis

(a) end (b) abolishment
(c) origin (d) attack

56. Gibberish

(a) nonsense (b) foreign
(c) ethical (d) angry

57. Grandeur

(a) big (b) impressiveness
(c) path (d) unimpressive

58. Gruesome

(a) long (b) confused
(c) exhausting (d) horrible

59. Havoc

(a) disorder (b) order
(c) passion (d) anger

60. Hubris

(a) chaos (b) confused
(c) arrogance d) crowd

61. Hue

(a) fight (b) aspect
(c) argument (d) opinion

62. Igneous

(a) volcanic (b) powerful
(c) mighty (d) anger

63. Imminent

(a) near at hand (b) danger
(c) important (d) relaxed

64. Immutable

(a) noisy (b) unchangeable
(c) different (d) immovable

65. Impeccable

(a) unreachable (b) stern
(c) perfect (d) impossible

66. Impediment

(a) judgement (b) improvement
(c) impeachment (d) hindrance

67. Impervious

(a) impenetrable (b) improvement
(c) confused (d) imperial

68. Imprudent

(a) lacking caution (b) cautious
(c) insensitive (d) ignorant

69. Indigence

(a) ingenious (b) poverty
(c) indecent (d) diligent

70. Inequity

(a) equality (b) property
(c) money (d) unfairness

71. Infallible

(a) perfect (b) to fall
(c) joyful (d) complacent

72. Insatiable

(a) devilish (b) complete
(c) unsatisfied (d) satanic

73. Interim

(a) meanwhile (b) international
(c) serious (d) interval

74. Irrevocable

(a) resistant (b) alterable
(c) unalterable (d) vocabulary

75. Jaunty

(a) showy (b) arrogant
(c) cheerful (d) sarcastic

76. Knack

(a) irritate (b) talent
(c) knock (d) hit

77. Larceny

(a) money (b) great
(c) large (d) theft

78. Loath

(a) soap (b) money
(c) reluctant (d) detest

79. Luminous

(a) grand (b) shining
(c) luscious (d) dull

80. Malady

(a) illness (b) holiday
(c) condition d) foul

81. Malicious

(a) hateful (b) adore
(c) delicious (d) confused

82. Manifest

(a) humane (b) document
(c) evident (d) urgent

83. Mutilate

(a) pacify (b) maim
(c) mute (d) suffer

84. Negate

(a) nullify (b) negative
(c) amplify (d) alter

85. Novelty

(a) copy (b) inspiration
(c) stupid (d) newness

86. Nugatory

(a) confused (b) mad
(c) worthy (d) futile

87. Obnoxious

(a) disastrous (b) offensive
(c) preposterous (d) irritating

88. Omniscient

(a) unknowing (b) present everywhere
(c) omnipresent (d) all knowing

89. Outlandish

(a) splendid (b) outlander
(c) bizarre (d) common

90. Outspoken

(a) arrogant (b) candid
(c) outsider (d) introvert

91. Pacify

(a) make peace (b) windy
(c) oceanic (d) aggressive

92. Parity

(a) close (b) equality
(c) double (d) inequality

93. Passive

(a) pacifist (b) inactive
(c) active (d) aggressive

94. Perdition

(a) blessings (b) path
(c) heaven (d) hell

95. Perjury

(a) testify falsely (b) theft
(c) burglary (d) pungent

96. Perturb

(a) challenge (b) angry
(c) disturb (d) perceive

97. Pinnacle

(a) peak (b) finicky
(c) depth (d) fruit

98. Premeditated

(a) medical (b) perception
(c) previous (d) pre-planned

99. Provisional

(a) vision (b) spacious
(c) temporary (d) pre-empt

100. Queer

(a) eccentric (b) casual
(c) ordinary (d) fight

101. Quench

(a) insatiable (b) satisfy
(c) quarry (d) quit

102. Quiver

(a) query (b) quarrel
(c) quibble (d) tremble

103. Rakish

(a) speedy (b) rash
(c) rake (d) jaunty

104. Ratify

(a) confused (b) rattled
(c) confirm (d) suspicious

105. Recant

(a) remember (b) disavow
(c) curse (d) canny

106. Refute

(a) refuge (b) disapprove
(c) recall (d) feud

107. Remedial

(a) reparable (b) medical
(c) irreparable (d) remove

108. Remonstrate

(a) objection (b) demonstrate
(c) substrate (d) remedy

109. Repudiate

(a) repulse (b) denounce
(c) reputed (d) disown

110. Robust

(a) force (b) strong
(c) weak (d) dust

111. Rue

(a) feud (b) fight
(c) regret (d) lucrative

112. Ruthless

(a) merciful (b) pitiless
(c) rusty (d) aggressive

113. Sagacious

(a) saga (b) salacious
(c) stupid (d) wise

114. Savour

(a) enjoy (b) save
(c) solitude (d) magnificent

115. Sedulous

(a) seldom (b) diligent
(c) confused (d) sedate

116. Serenity

(a) calmness (b) serendipity
(c) sane (d) insane

117. Severity

(a) severance (b) casualty
(c) harshness (d) obtuse

118. Shrewd

(a) angry (b) clever
(c) fish (d) foolish

119. Simulate

(a) stimulate (b) imitate
(c) begin (d) thrust

120. Skittish

(a) frivolous (b) slippery
(c) careful (d) gibberish

121. Sleight

(a) singular (b) slight
(c) fleet (d) dexterity

122. Slither

(a) tongue (b) slide
(c) dirty (d) cut

123. Snuffle

(a) sniffle (b) bark
(c) snivel (d) truffle

124. Solace

(a) peace (b) grief
(c) consolation (d) enjoyment

125. Spurious

(a) counterfeit (b) furious
(c) fight (d) spare

126. Stagnant

(a) pungent (b) motionless
(c) flow (d) foul

127. Supersede

(a) obsolete (b) previous
(c) super (d) replace

128. Synoptic

(a) singular (b) detailed
(c) hypnotic (d) summary

129. Tempestuous

(a) temporary (b) temper
(c) violent storm (d) preposterous

130. Tenacity

(a) casual (b) persistence
(c) needs (d) cruelty

131. Tentative

(a) provisional (b) confirmed
(c) certain (d) brittle

132. Timidity

(a) shyness (b) careless
(c) confident (d) boastful

133. Torrid

(a) rapid (b) passionate
(c) torpedo (d) violent

134. Tranquillity

(a) strike (b) hyper
(c) peace (d) extreme

135. Travail

(a) problematic (b) lazy
(c) trail (d) strenuous work

136. Truncate

(a) paste (b) shorten
(c) increase (d) trunk

137. Turpitude

(a) arrogance (b) guarantee
(c) turbulence (d) depravity

138. Ubiquitous

(a) unanimous (b) omnipresent
(c) doubt (d) anonymous

139. Umbrage

(a) anger (b) homage
(c) remorse (d) supportive

140. Uncanny

(a) strange (b) clear
(c) absolute (d) repulsive

141. Upshot

(a) loud (b) high
(c) outcome (d) upward

142. Waggish

(a) random (b) swiftly
(c) decent (d) mischievous

143. Weary

(a) wearing (b) tired
(c) loose (d) clothes

144. Woe

(a) anger (b) theft
(c) feud (d) sorrow

145. Wrangle

(a) noisy quarrel (b) involved
(c) intertwined (d) connected

146. Wrath

(a) disappointment (b) anger
(c) curtains (d) argument

147. Yen

(a) dreams (b) aspirations
(c) longing (d) dislikes

148. Yoke

(a) vomit (b) separate
(c) unite (d) shout

149. Zeal

(a) enthusiastic (b) seal
(c) liar (d) idle

150. Zenith

(a) wrong (b) below
(c) low (d) top

ANSWERS

1. (c)	**10.** (a)	**19.** (a)	**28.** (b)	**37.** (a)	**46.** (b)	**55.** (c)	**64.** (b)
2. (a)	**11.** (c)	**20.** (b)	**29.** (a)	**38.** (d)	**47.** (a)	**56.** (a)	**65.** (c)
3. (b)	**12.** (b)	**21.** (a)	**30.** (c)	**39.** (b)	**48.** (c)	**57.** (b)	**66.** (d)
4. (a)	**13.** (d)	**22.** (d)	**31.** (a)	**40.** (a)	**49.** (d)	**58.** (d)	**67.** (a)
5. (d)	**14.** (b)	**23.** (b)	**32.** (c)	**41.** (b)	**50.** (b)	**59.** (a)	**68.** (a)
6. (d)	**15.** (b)	**24.** (b)	**33.** (b)	**42.** (c)	**51.** (b)	**60.** (c)	**69.** (b)
7. (b)	**16.** (d)	**25.** (a)	**34.** (d)	**43.** (a)	**52.** (a)	**61.** (b)	**70.** (d)
8. (a)	**17.** (c)	**26.** (d)	**35.** (a)	**44.** (d)	**53.** (d)	**62.** (a)	**71.** (a)
9. (b)	**18.** (a)	**27.** (d)	**36.** (c)	**45.** (b)	**54.** (a)	**63.** (a)	**72.** (c)

73. (a)	**83.** (b)	**93.** (b)	**103.** (d)	**113.** (d)	**123.** (a)	**133.** (b)	**143.** (b)
74. (c)	**84.** (a)	**94.** (d)	**104.** (c)	**114.** (a)	**124.** (c)	**134.** (c)	**144.** (d)
75. (c)	**85.** (d)	**95.** (a)	**105.** (b)	**115.** (b)	**125.** (a)	**135.** (d)	**145.** (a)
76. (b)	**86.** (d)	**96.** (c)	**106.** (b)	**116.** (a)	**126.** (b)	**136.** (b)	**146.** (b)
77. (d)	**87.** (b)	**97.** (a)	**107.** (a)	**117.** (c)	**127.** (d)	**137.** (d)	**147.** (c)
78. (c)	**88.** (d)	**98.** (d)	**108.** (a)	**118.** (b)	**128.** (d)	**138.** (b)	**148.** (c)
79. (b)	**89.** (c)	**99.** (c)	**109.** (d)	**119.** (b)	**129.** (c)	**139.** (a)	**149.** (a)
80. (a)	**90.** (b)	**100.** (a)	**110.** (b)	**120.** (a)	**130.** (b)	**140.** (a)	**150.** (d)
81. (a)	**91.** (a)	**101.** (b)	**111.** (c)	**121.** (d)	**131.** (a)	**141.** (c)	
82. (c)	**92.** (b)	**102.** (d)	**112.** (b)	**122.** (b)	**132.** (a)	**142.** (d)	

CHAPTER 3

ANTONYMS

An antonym of any word has a meaning opposite to the word. An antonym is one of a pair of words with opposite meanings. A word may have more than one antonym. This chapter deals with the types of antonyms and a list of some antonym pairs followed by an exercise at the end.

GRADED ANTONYMS

Graded antonyms deal with levels of the meaning of the words, like if something is not "good", it may still not be "bad." There is a scale involved with some words, and besides good and bad there can be average, fair, excellent, terrible, poor, or satisfactory.

Examples include:

1. Fat and skinny
2. Young and old
3. Happy and sad
4. Hard and soft
5. Last and first
6. Foolish and wise
7. Fast and slow
8. Warm and cool
9. Wide and narrow
10. Abundant and scarce
11. Joy and grief
12. Dark and light
13. Dangerous and safe
14. Clever and foolish
15. Early and late
16. Empty and full
17. Smart and dumb
18. Risky and safe
19. Bad and good
20. Pretty and ugly
21. Best and worst
22. Simple and challenging
23. Soft and hard
24. Worried and calm
25. Sane and crazy
26. Rich and poor
27. Cool and hot
28. Wet and dry
29. Late and early
30. Ignorant and educated
31. Big and small
32. Optimistic and pessimistic
33. Excited and bored
34. Dull and interesting

COMPLEMENTARY ANTONYMS

Complementary antonyms have a relationship where there is no middle ground. There are only two possibilities, either one or the other.

Examples include:

1. Man and woman
2. Push and pull
3. Dead and alive
4. Off and on
5. Day and night
6. Absent and present
7. Exit and entrance
8. Sink or float
9. True or false
10. Pass and fail
11. Former and latter
12. Input and output
13. Interior and exterior
14. Exhale and inhale
15. Input and output
16. Occupied and vacant
17. Leave and arrive
18. Pre and post
19. Question and answer
20. Single and married
21. Hired and fired
22. Brother and sister
23. Before and after
24. Crooked and straight
25. Identical and different
26. Natural or artificial
27. Silence or noise
28. Identical or different
29. Yes and no
30. Wet and dry
31. Sharp and dull
32. Raise and lower
33. Fantasy and reality

RELATIONAL ANTONYMS

Relational antonyms are sometimes considered a subcategory of complementary antonyms. With these pairs, for there to be a relationship, both must exist.

Examples are:

1. Husband and wife
2. Doctor and patient
3. Buy and sell
4. Predator and prey
5. Above and below
6. Give and receive
7. Teach and learn
8. Instructor and pupil
9. Servant and master
10. Borrow and lend
11. Come and go
12. Toward and away
13. Divisor and dividend
14. Parent and child
15. East and west
16. North and south
17. Seller and buyer
18. Mother and daughter
19. Slave and master
20. Floor and ceiling
21. Front and back
22. Up and down
23. Win and lose
24. Part and whole
25. Offense and defence
26. Behind and ahead
27. Before and after
28. On or off
29. Trap and release
30. Lost and found
31. Left and right
32. Give and get
33. Employer employee

PRACTICE EXERCISE

Read the words given below and choose the option that is opposite or most nearly opposite in meaning to the word.

1. **Abrogate**

   (a) nullify (b) signify
   (c) annul (d) ratify

2. **Affable**

   (a) rude (b) ruddy
   (c) soft (d) lovable

3. **Affluence**

   (a) abundance (b) poverty
   (c) influence (d) consideration

4. **Agility**
(a) detest (b) solidity
(c) alacrity (d) stiffness

5. **Ample**
(a) cause (b) enough
(c) insufficient (d) parity

6. **Amplify**
(a) improve (b) ample
(c) increase (d) shorten

7. **Analogous**
(a) incomparable (b) comparable
(c) incapable (d) unanimous

8. **Anguish**
(a) suffering (b) comfort
(c) pain (d) irritation

9. **Anomaly**
(a) aberration (b) normality
(c) massive (d) desperation

10. **Antipathy**
(a) objection (b) detest
(c) admiration (d) dislike

11. **Aphasia**
(a) pain (b) crack
(c) necessity (d) volubility

12. **Augment**
(a) decrease (b) increase
(c) friendly (d) harvest

13. **Authenticate**
(a) original (b) investigate
(c) prove genuine (d) fake

14. **Baffle**
(a) confuse (b) enlighten
(c) angry (d) fight

15. **Baleful**
(a) detest (b) auspicious
(c) powerful (d) evil

16. **Baroque**
(a) polished (b) bizarre
(c) simple (d) constant

17. **Belittle**
(a) praise (b) forget
(c) disobey (d) criticize

18. **Bestial**
(a) flattery (b) clear
(c) refer (d) noble

19. **Blandishment**
(a) criticize (b) cheat
(c) sincere (d) embarrassed

20. **Bleak**
(a) pale (b) cheerful
(c) sudden (d) wet

21. **Blithe**
(a) shameless (b) cheerful
(c) happy (d) cheerless

22. **Brazen**
(a) shy (b) apathy
(c) unashamed (d) insensitive

23. **Candid**
(a) honest (b) outspoken
(c) deceitful (d) anxious

24. **Capricious**
(a) deviating (b) insured
(c) steadfast (d) expected

25. **Carnal**
(a) kind (b) impressive
(c) minute (d) tolerant

26. **Castigation**
(a) patience (b) commendation
(c) understatement (d) generosity

27. **Celerity**
(a) assurance (b) state
(c) acerbity (d) delay

28. **Cessation**
(a) prime (b) end
(c) beginning (d) complacent

29. **Chaste**
(a) corrupt (b) incorrupt
(c) timid (d) haste

30. **Comprehend**
(a) remove (b) arrest
(c) understand (d) misunderstand

31. **Cordial**
(a) friendly (b) harmony
(c) unfriendly (d) disorganized

32. Coy

(a) weak (b) airy
(c) immodest (d) old

33. Crux

(a) affliction (b) spark
(c) events (d) trivial point

34. Cynical

(a) gallant (b) trusting
(c) effortless (d) conclusive

35. Dauntless

(a) resistance (b) cowardly
(c) peculiar (d) solid

36. Decorous

(a) inappropriate (b) deception
(c) appropriate (d) orderly

37. Derogatory

(a) praising (b) immediate
(c) cynical (d) baleful

38. Desecrate

(a) desist (b) integrate
(c) violate (d) consecrate

39. Destitute

(a) deficient (b) dazzling
(c) affluent (d) treasure

40. Diffidence

(a) efficient (b) confidence
(c) hesitancy (d) timid

41. Dilate

(a) procrastinate (b) contract
(c) conclude (d) participate

42. Elated

(a) expand (b) joyful
(c) depressed (d) embarrassed

43. Emancipate

(a) imprison (b) poverty
(c) liberty (d) avoid

44. Enervate

(a) sputter (b) arrange
(c) scrutinize (d) strengthen

45. Ennui

(a) hate (b) excitement
(c) enmity (d) humility

46. Equivocal

(a) mistaken (b) certain
(c) azure (d) exclude

47. Erroneous

(a) artificial (b) incorrect
(c) bizarre (d) correct

48. Evasive

(a) confused (b) empty
(c) frank (d) fertile

49. Exculpate

(a) blame (b) prevail
(c) remove (d) fierce

50. Execrate

(a) love (b) hate
(c) admit (d) end

51. Exonerate

(a) pardon (b) blame
(c) fight (d) hit

52. Extraneous

(a) effective (b) modern
(c) essential (d) advantage

53. Feud

(a) agreement (b) fight
(c) spoil (d) angry

54. Frivolous

(a) confuse (b) senseless
(c) ill-mannered (d) sensible

55. Frugal

(a) frivolous (b) rude
(c) economical (d) lavish

56. Genesis

(a) beginning (b) end
(c) crux (d) big

57. Gratuitous

(a) costly (b) complimentary
(c) gratitude (d) unimpressive

58. Gregarious

(a) friendly (b) affable
(c) exhausting (d) unfriendly

59. Gullible

(a) disorder (b) gulp
(c) perceptive (d) naive

60. Gusty

(a) chaos (b) windy
(c) calm (d) ghastly

61. Hackneyed

(a) clichéd (b) original
(c) timely (d) shrewd

62. Haphazard

(a) careful (b) accidental
(c) aimless (d) destruction

63. Havoc

(a) recent (b) danger
(c) calamity (d) peace

64. Hilarity

(a) laughter (b) sad
(c) different (d) high

65. Hybrid

(a) mixture (b) economical
(c) homogeneous (d) heterogeneous

66. Hypothetical

(a) ethical (b) guessed
(c) imaginary (d) factual

67. Illusive

(a) real (b) improvement
(c) confused (d) deceptive

68. Imprudent

(a) cautious (b) careless
(c) insensitive (d) rude

69. Indigence

(a) ingenious (b) wealth
(c) indecent (d) poor

70. Infallible

(a) perfect (b) complacent
(c) joyful (d) faulty

71. Jaded

(a) fresh (b) complacent
(c) exhausted (d) faded

72. Jeopardy

(a) devilish (b) danger
(c) safety (d) courage

73. Jocular

(a) curiosity (b) serious
(c) funny (d) interval

74. Kindle

(a) kindly (b) burn
(c) hold (d) extinguish

75. Laconic

(a) short (b) compact
(c) cynical (d) wordy

76. Latent

(a) dormant (b) talent
(c) apparent (d) delay

77. Lethargic

(a) active (b) sluggish
(c) large (d) lethal

78. Levity

(a) amalgam (b) silliness
(c) leverage (d) seriousness

79. Livid

(a) gloomy (b) pale
(c) radiant (d) dull

80. Magnify

(a) diminish (b) enlarge
(c) condition (d) fire

81. Malicious

(a) friendly (b) detest
(c) delicious (d) confused

82. Mendacious

(a) destructive (b) honest
(c) menacing (d) dishonest

83. Mutilate

(a) destruct (b) hit
(c) repair (d) suffer

84. Nebulous

(a) nullify (b) cold
(c) starry (d) definite

85. Nefarious

(a) bad (b) good
(c) stupid (d) sinful

86. Obligatory

(a) oblivious (b) binding
(c) nonessential (d) essential

87. Obnoxious

(a) offensive (b) kind
(c) preposterous (d) irritating

88. Odium

(a) approval (b) noise
(c) shame (d) order

89. Outlandish

(a) outlaw (b) outlander
(c) bizarre (d) common

90. Overweening

(a) arrogant (b) modest
(c) wiener (d) avid

91. Pacify

(a) agitate (b) windy
(c) oceanic (d) make peace

92. Parity

(a) close (b) equality
(c) inequality (d) pair

93. Passive

(a) pacifist (b) inactive
(c) active (d) aggressive

94. Penchant

(a) fondness (b) path
(c) pungent (d) dislike

95. Pernicious

(a) baleful (b) harmless
(c) suspicious (d) hurtful

96. Perpetual

(a) brief (b) continual
(c) logical (d) standard

97. Petulant

(a) moody (b) potent
(c) angry (d) pleasant

98. Prelude

(a) presume (b) epilogue
(c) previous (d) aria

99. Pristine

(a) vision (b) condemned
(c) cultivated (d) pure

100. Profane

(a) moral (b) casual
(c) define (d) deceitful

101. Propitious

(a) induced (b) unfavourable
(c) prosperous (d) favourable

102. Protract

(a) extend (b) extract
(c) retract (d) shorten

103. Punitive

(a) rewarding (b) rash
(c) premature (d) emergency

104. Queer

(a) feud (b) odd
(c) normal (d) suspicious

105. Quiver

(a) vibration (b) stillness
(c) argument (d) solution

106. Ratify

(a) disagree (b) clarify
(c) rational (d) authorize

107. Restive

(a) lazy (b) nervous
(c) impatient (d) calm

108. Robust

(a) weak (b) demonstrate
(c) strong (d) heavy

109. Ruddy

(a) wan (b) deceitful
(c) pale (d) blushing

110. Ruthless

(a) heartless (b) compassionate
(c) denounce (d) rusty

111. Sadistic

(a) upset (b) fight
(c) kind (d) perverse

112. Sagacious

(a) merciful (b) foolish
(c) spacious (d) judicious

113. Salvage

(a) save (b) redeem
(c) solitude (d) endanger

114. Scurrilous

(a) decent (b) scabby
(c) active (d) savage

115. Sedulous

(a) active (b) indolent
(c) confused (d) vindictive

116. Serenity

(a) calmness (b) serendipity
(c) sane (d) disruption

117. Severity

(a) sever (b) casualty
(c) kindness (d) asperity

118. Shrewd

(a) frivolous (b) clever
(c) sharp (d) condescending

119. Skittish

(a) calm (b) nervous
(c) slippery (d) thrust

120. Solace

(a) homage (b) peace
(c) journey (d) discord

121. Specious

(a) singular (b) valid
(c) dexterity (d) misleading

122. Subdue

(a) moderate (b) subside
(c) incite (d) cut

123. Superficial

(a) analytical (b) artificial
(c) superior (d) real

124. Surmise

(a) surprise (b) knowledge
(c) guess (d) closure

125. Surreptitious

(a) honest (b) furious
(c) secretive (d) retreat

126. Synthesis

(a) amalgam (b) motionless
(c) division (d) scientific

127. Tactful

(a) obsolete (b) attack
(c) super (d) complacent

128. Tardy

(a) tired (b) punctual
(c) late (d) summary

129. Tempestuous

(a) temporary (b) stormy
(c) gentle (d) preposterous

130. Tenacity

(a) diligence (b) persistence
(c) weakness (d) cruelty

131. Tenuous

(a) substantial (b) thin
(c) strenuous (d) brittle

132. Thrive

(a) prosper (b) careless
(c) confident (d) fail

133. Timorous

(a) rapid (b) brazen
(c) torpedo (d) afraid

134. Transient

(a) temporary (b) active
(c) lasting (d) extreme

135. Travesty

(a) problematic (b) lazy
(c) spoof (d) seriousness

136. Truncate

(a) lengthen (b) shorten
(c) trump (d) trunk

137. Turbid

(a) clear (b) guarantee
(c) cloudy (d) depravity

138. Tyro

(a) unanimous (b) expert
(c) doubt (d) novice

139. Umbrage

(a) anger (b) homage
(c) happiness (d) displeasure

140. Unassuaged

(a) strange (b) clear
(c) absolute (d) content

141. Unfaltering

(a) loud (b) high
(c) unstable (d) perfect

142. Unilateral

(a) many-sided (b) swiftly
(c) descent (d) one-sided

143. Valour

(a) wearing (b) cowardice
(c) loose (d) bravery

144. Withdraw

(a) sit (b) remove
(c) endure (d) stay

145. Withstand

(a) yield (b) theft
(c) endure (d) sit

146. Wrangle

(a) feud (b) involved
(c) peace (d) connected

147. Wrath

(a) disappointment (b) anger
(c) happiness (d) argument

148. Wry

(a) straightforward (b) aspirations
(c) distorted (d) weary

149. Yen

(a) dreams (b) dislike
(c) unite (d) desire

150. Zeal

(a) enthusiastic (b) seal
(c) liar (d) lethargy

ANSWERS

1. (d)	**20.** (b)	**39.** (c)	**58.** (d)	**77.** (a)	**96.** (a)	**115.** (b)	**134.** (c)
2. (a)	**21.** (d)	**40.** (b)	**59.** (c)	**78.** (d)	**97.** (d)	**116.** (d)	**135.** (d)
3. (b)	**22.** (a)	**41.** (b)	**60.** (c)	**79.** (c)	**98.** (b)	**117.** (c)	**136.** (a)
4. (d)	**23.** (c)	**42.** (c)	**61.** (b)	**80.** (a)	**99.** (c)	**118.** (a)	**137.** (a)
5. (c)	**24.** (c)	**43.** (a)	**62.** (a)	**81.** (a)	**100.** (a)	**119.** (a)	**138.** (b)
6. (d)	**25.** (a)	**44.** (d)	**63.** (d)	**82.** (b)	**101.** (b)	**120.** (d)	**139.** (c)
7. (a)	**26.** (b)	**45.** (b)	**64.** (b)	**83.** (c)	**102.** (d)	**121.** (b)	**140.** (d)
8. (b)	**27.** (b)	**46.** (b)	**65.** (c)	**84.** (d)	**103.** (a)	**122.** (c)	**141.** (c)
9. (b)	**28.** (c)	**47.** (d)	**66.** (d)	**85.** (b)	**104.** (c)	**123.** (a)	**142.** (a)
10. (c)	**29.** (a)	**48.** (c)	**67.** (a)	**86.** (c)	**105.** (b)	**124.** (b)	**143.** (b)
11. (d)	**30.** (d)	**49.** (a)	**68.** (a)	**87.** (b)	**106.** (a)	**125.** (a)	**144.** (d)
12. (a)	**31.** (c)	**50.** (a)	**69.** (b)	**88.** (a)	**107.** (d)	**126.** (c)	**145.** (a)
13. (d)	**32.** (c)	**51.** (b)	**70.** (d)	**89.** (d)	**108.** (a)	**127.** (d)	**146.** (c)
14. (b)	**33.** (d)	**52.** (c)	**71.** (a)	**90.** (b)	**109.** (c)	**128.** (b)	**147.** (c)
15. (b)	**34.** (b)	**53.** (a)	**72.** (c)	**91.** (a)	**110.** (b)	**129.** (c)	**148.** (a)
16. (c)	**35.** (b)	**54.** (d)	**73.** (b)	**92.** (c)	**111.** (c)	**130.** (c)	**149.** (b)
17. (a)	**36.** (a)	**55.** (d)	**74.** (d)	**93.** (c)	**112.** (b)	**131.** (a)	**150.** (d)
18. (d)	**37.** (a)	**56.** (b)	**75.** (d)	**94.** (d)	**113.** (d)	**132.** (d)	
19. (a)	**38.** (d)	**57.** (a)	**76.** (c)	**95.** (b)	**114.** (a)	**133.** (b)	

CHAPTER 4

SENTENCE COMPLETION

Sentence completion problems are used to test your vocabulary skills and your reading ability. Each sentence contains one or two blanks. You are then presented with four options which may consist of words or phrases, or four pairs of words or phrases if there are two blanks in the sentence. From these choices, you need to select the most appropriate or correct words or phrases. To make the correct choice, you will need to able to understand the main idea of the sentence, the meaning of the words or phrases and the logical structure of the sentence.

The sentence completion questions do not have any predefined technique to solve them. However, certain steps can be followed in order to increase the odds in your favour to get the right answer.

TIPS TO SOLVE SENTENCE COMPLETION BASED QUESTIONS

Questions based on synonyms and antonyms can be pretty difficult to answer if the word is new or unheard of. In such scenarios, there are certain tips or tricks that the student can use in order to increase his chances of getting the correct answer. Some of the common tips are given as follows:

1. Before looking at the answer-choices, the student should think of a word that fits the sentence.
Example: The princess was popular and loved throughout her kingdom not only for her physical beauty, but also for her _____.

(a) Elegance (b) Honesty
(c) Anger (d) Arrogance

We can see that the sentence talks about the positive traits of the princess. Also, it is evident enough that the blank is not pertaining to her physical appearance and is a quality of her nature like humility, honesty, kind, etc. Thus, looking at the options, we can choose the answer to be *honest*.

2. Look for key words and phrases in the text that tells the student where the sentence is going. If the sentence is continuing along one line of though, then the student should look for a completion that carries the idea through. If the text changes direction in midstream, then look for a completion that establishes a contrast between ideas.

(a) *Contrast indicators*: These point out how the two things differ. In this type of sentence completion problem, we look for a word that has the opposite meaning (an antonym) of some key word or phrase in the sentence. Some of the common contrast indicators are but, yet, despite, although, however, etc.

Example: Although the warring parties had settled a number of disputes, past experience made them ____ to express optimism that the talks would be a success.
(a) Rash
(b) Ambivalent
(c) Scornful
(d) Reticent

In the sentence, "although" sets up a contrast between what has occurred, success on some issues, and what can be expected to occur — success for the whole talks. Hence, the parties are reluctant to express optimism. The common word "reluctant" is not offered as an answer-choice but we have a synonym which is *reticent*.

(b) *Support indicators*: These further explain what has already been said. In this type of sentence completion problem, we look for a word that has similar meaning (synonym) of some key word or phrase in the sentence. Some of the common support indicators are and, also, furthermore, likewise, moreover, etc.

Example: Travis is an opprobrious and ____ speaker, equally caustic toward friend or foe.

(a) Lofty
(b) Vituperative
(c) Retiring
(d) Laudatory

In the sentence, "and" indicates that the missing adjective is similar in meaning to "opprobrious", which is very negative. Also, vituperative which is the only negative word means abusive.

(c) *Cause and effect indicators*: These words indicate that one thing causes another to occur. Some of the common cause and effect indicators are because, for, thus, hence, therefore, etc.

Example: Because the House has the votes to override a presidential veto, the President has no choice but to ____.

(a) Object
(b) Abstain
(c) Capitulate
(d) Compromise

Since the House has the votes to pass the bill or motion, the President would be wise to compromise and make the best of the situation.

3. Words or phrases in apposition are placed next to each other, and the second word or phrases defines, clarifies or gives evidence to the first word or phrase.

Example: His novels are ____; he uses a long circumlocution when a direct coupling of a simple subject and verb would be best.

(a) Succinct (b) Prolix
(c) Vapid (d) Risque

The sentence has no linking words (such as because, although, etc.) Hence, the phrase following the semicolon is in apposition to the missing word, it defines or further clarifies the missing word. Now, writing filled with circumlocutions is aptly described as *prolix*.

4. While reading the sentence, the student should look out for adjectives/adverbs which give the idea of the sentence. After that, find out if the idea of the sentence is positive/negative. If words such as no, non, not, etc., are used then the idea of the sentence is negative. Punctuations can divide the sentence into 2 or 3 parts. If the flow of the first part of the sentence is positive and the second part if negative, then the blank must be negative to even the flow of the sentence. This would solve the sentence completion question without any understanding of the question.

Example: Because he did not want to appear ____, the junior executive refused to dispute the board's decision, in spite of his belief that the decision would impair employee morale.

(a) Contentious (b) Indecisive
(c) Solicitous (d) Steadfast

Options (c) and (d) are gone because they are positive words. Hence, *solicitous* and *steadfast* cannot be used. Also, we cannot use *indecisive* because the clue is "refused to dispute", which doesn't work with indecisive. Hence, the best answer is *contentious*.

5. Be flexible in handling multiple-blank completions. A completion for one blank might make sense in the context of the phrase or sentence in which it appears, yet not work in the context of the passage as a whole.

   Example: The attendees at the normally ____ office Christmas party were delighted by the lavish decorations and ____ atmosphere this year.

   (a) mundane, exuberant
   (b) entertaining, ornate
   (c) litigious, modish
   (d) monotonous, morose

   There seems to be a contrast between how that party was "normally" and how it was "this year". So, if the party is lavish this year, it probably has been dull or boring in the past. Also, *mundane* means dull and *exuberant* is a contrast to it. Thus, that is the correct answer.

6. If all the above techniques fail to obtain the correct answer, then the student can easily eliminate all the options that are definitely wrong or are eliminated through the positive/negative flow. This helps in increasing the probability of getting the correct answer. It is obvious that the probability of getting a right answer from 2 options is much higher than from 5 options.

   Example: Impressed by the doctor's ____, the medical students watched intently as he demonstrated the complicated strategy.

   (a) Inability
   (b) Competence
   (c) Recidivism
   (d) Importunity

   This particular question requires a positive answer since the students are impressed with a certain quality of the doctor. If we consider the four options, we can easily make out that *inability* cannot be used. Also, *recidivism* and *importunity* have no relation to the context in the sentence. Hence, the correct option is *competence*.

PRACTICE EXERCISE

Directions for Q. 1 to Q. 15

Each sentence given below consists of four options. The options may be words or phrases. Read each sentence carefully and choose the option which fills in the blank most appropriately.

1. The director is normally lauded for his exciting sci-fi films, but his latest effort was marred by its ____ effects.

   (a) electrifying
   (b) breathtaking
   (c) bland
   (d) emotive

2. Despite his long battle with illness, the boxer showed astonishing ____ in the ring.

   (a) strength
   (b) focus
   (c) hesitancy
   (d) indolence

3. Although the actress received great reviews during her press conference throughout the country, her much awaited debut film was met with uniformly ____ reviews.

   (a) electrifying
   (b) deprecating
   (c) appreciative
   (d) obsequious

4. Within hours the tsunami had ended, serving to ____ the fears of thousands of residents who couldn't evacuate in time.

   (a) aggravate
   (b) assuage
   (c) alleviate
   (d) annihilate

5. Even after the series of scams, the government had the ____ to talk about the progress of the country in their speeches.

   (a) audacity
   (b) shame
   (c) morality
   (d) strength

6. ____, he decided to pass on the assignment, his professed support notwithstanding.

   (a) presumably
   (b) self-indulgently
   (c) obscurely
   (d) unexpectedly

7. This is about _____ a psychological analysis can penetrate.

(a) the relative distances that
(b) as long as
(c) as far as
(d) just how far into the subject that

8. Jack hastened to _____ when he saw Jade trip over his untied shoelaces.

(a) assist with support
(b) gasp in concern
(c) remedy the disaster
(d) ease the pain

9. In the world of professional team sports, individual prowess has its place, but ultimately the players are valued chiefly for their _____ qualities.

(a) singular
(b) collaborative
(c) dispersive
(d) inspirational

10. After a destructive, summer-long drought, during which the crops _____, the farmers did not know whether to welcome or curse the heave, late-August rains.

(a) retracted
(b) persevered
(c) plundered
(d) languished

11. Having test-driven this car and finding its performance lacklustre at best, its maker's sanguine claims are _____.

(a) plausible
(b) understandable
(c) unfounded
(d) understated

12. Difficult as it may sometimes be, in all our dealings with both clients and competitors, we must be seen to be above _____.

(a) integrity
(b) question
(c) profit
(d) reproach

13. The highest reward for a man's toil is not what he gets for it but what he _____.

(a) becomes by it
(b) makes out of it
(c) gets for others
(d) has overcome through it

14. _____, the more they remain close to each other.

(a) the less the dynamism
(b) the more things change
(c) the more pronounced the transformation
(d) the more the merrier

15. _____ that in this chaos, the important things are not interfered with.

(a) it cannot be emphasized enough
(b) it is of cardinal important
(c) it is important
(d) it should be imminently understood

Directions for Q. 16 to Q. 30

Each sentence given below consists of two or three blanks. Read each sentence carefully and choose the option which contains the right combination of answers to fill in the blanks most appropriately.

16. Given the _____ nature of the evidence, the authorities are unlikely to present a _____ case against the defendant.

(a) superficial, effective
(b) rakish, tepid
(c) strong, convincing
(d) flimsy, convincing

17. It is hard to believe that the highly _____ game of football began many centuries ago as a rowdy _____ without rules, fought cross-country by entire villages.

(a) complex, séance
(b) structured, brawl
(c) unplanned, massacre
(d) organized, encounter

18. Ambition is a useful _____ that leads people to greatness, but it can also be _____ force.

(a) factor, an inspirational
(b) indicator, a pulsating
(c) motivator, a destructive
(d) tenet, a resisting

19. In the workplace, the employees should _____ the _____ of the company.

(a) confine to, peccadilloes
(b) object to, idiosyncrasies
(c) conform to, standards
(d) balk at, simplicity

20. The explorers were almost certainly _____ to failure from the start as the members lacked a sound sense of _____ knowledge.

(a) doomed, geographical
(b) accustomed, general
(c) immune, basic
(d) destined, abstruse

21. In our country, the challenges are to raise _____ incomes to reduce poverty and to _____ inefficient public sector enterprises.

(a) rural, restructure
(b) farm, liberalise
(c) middle-class, privatise
(d) workers, take over

22. The interest generated by the Football World Cup is _____ compared to the way cricket _____ India.

(a) unusual, grips
(b) tepid, inspires
(c) milder, fascinates
(d) mild, electrifies

23. The ideas that these companies used seem so simple with _____ that their business rivals will now _____ themselves for not thinking of them first.

(a) passage of time, curse
(b) analysis, applaud
(c) hindsight, kick
(d) technology, hit

24. Since her face was free of _____, there was no way to _____ if she appreciated what had happened.

(a) make-up, realise
(b) expression, ascertain
(c) scars, judge
(d) emotion, diagnose

25. The situation in Israel and Palestine is far more _____ than what we understand it to be. While we find those areas unstable, the people who live there are _____ it.

(a) stable, proud of
(b) simple, happy with
(c) complex, content by
(d) adverse, accustomed to

26. The development of drama over the centuries has been a _____ journey. Yet, much has remained unchanged – actors in costume still _____ the stage before audiences who willingly suspend their _____ in order to enter into the 'reality' of events created for them.

(a) implacable, straddle, interest
(b) modest, stalk, logic
(c) remarkable, strut, disbelief
(d) long, surround, understanding

27. Although the report indicated a _____ rise in obesity, many people, by _____ junk food over nutrition, continue to _____ the problem.

(a) huge, preferring, avoid
(b) disturbing, choosing, compound
(c) slow, ignoring, attenuate
(d) steep, selecting, abhor

28. The idea that the Internet is not a _____ place has become ingrained in popular culture. It has become more complicated for authorities to _____ the breaches of privacy that proliferate on a regular basis because of the increasing number of users. The average user should remain _____ the exchange of personal data over the Internet.

(a) secure, constrain, careful in
(b) sensible, castigate, indignant about
(c) reliable, relinquish, complacent in
(d) trustworthy, stop, sceptical toward

29. A dictionary that provides the _____ of words – that is, the origin and development of their meanings – offers proof of a _____ language. Over time, words not only change but sometimes even _____ their meaning.

(a) etymology, living, reverse
(b) taxonomy, faltering, exchange
(c) toxicology, nascent, amend
(d) etymology, variable, complicate

30. As a result of poor planning and disorganization, the young team _____ attacking the root of the problem. This went to until there was no other recourse left to them. They were obliged to _____ a _____, last minute solution to the problem.

(a) accelerated, envision, calculated
(b) accelerated, reject, premeditated
(c) expedited, implement, measured
(d) postponed, implement, desperate

ANSWERS

1. (c)	**7.** (c)	**13.** (a)	**19.** (c)	**25.** (d)
2. (a)	**8.** (a)	**14.** (b)	**20.** (a)	**26.** (c)
3. (b)	**9.** (b)	**15.** (c)	**21.** (a)	**27.** (b)
4. (b)	**10.** (d)	**16.** (d)	**22.** (d)	**28.** (a)
5. (a)	**11.** (c)	**17.** (b)	**23.** (c)	**29.** (a)
6. (d)	**12.** (d)	**18.** (c)	**24.** (b)	**30.** (d)

CHAPTER 5

VERBAL ANALOGIES

In verbal analogies, the student is given one or more pair(s) of related words and another word without its pair. The answer is the word(s) that has the same relationship to the word as the initial pair(s).

Verbal analogies assess the student's ability to identify the relationship between words and then to apply this verbal analogy. These questions provide excellent training in seeing relationships between concepts. From a practical standpoint, verbal analogies always appear in standardized tests like SAT, GRE and other professional exams. Also, employers may use these word comparisons on personnel and screening tests to determine an applicant's quickness and verbal acuity.

TYPES OF ANALOGIES

There are many ways of forming a relationship between pairs of words in verbal analogies. Some of the common relationships are given as follows:

1. **Things that go together analogies:** Some objects are indisputably connected to each other. These sets of objects are extensively used in modern verbal analogies. *Example*: bat/ball, salt/pepper, etc.
2. **Opposite analogies:** As the name suggests, this analogy type contains things that are opposites. This is a common analogy type which is encountered fairly often and pretty straightforward. *Example*: black/white, big/small, etc.
3. **Similar analogies:** As the name suggests, this analogy type contains things that are similar. Like opposites analogies, this is a common analogy type which is encountered fairly often and pretty straightforward. *Example*: cold/icy, hide/conceal, etc.
4. **Classification of objects:** Objects can be a classification, a group of objects to which they belong. Most objects can even be classified to several different groups. *Example*: red/colour, grapes/fruit, etc.
5. **Objects and group analogies:** These are objects which form a specifically named group when several are put together. *Example*: bird/flock, cow/herd, etc.
6. **Object and characteristic analogies:** This type of analogy contains an object and an adjective describing the object. In this analogy, a single object can have different analogies depending on the adjectives being used. *Example*: grass/green, ball/round, etc.

7. **Object and a part of the whole analogies:** This type of analogy is sometimes confused with the object and group analogy. The difference lies in the fact that in the object and part of a whole relation the object is not automatically the whole when lots of the objects are brought together. *Example*: hand/fingers, year/month, etc.
8. **Object and function analogies:** Some objects have designated functions which are inseparably connected to the concerning object. *Example*: knife/cut, book/read, etc.
9. **Object and location analogies:** In this relation objects are designated to their most logical location. However, this is not always strictly defined and the student will have to find the correct answer again by carefully analysing the analogy problem and its possible solutions. *Example*: plane/hangar, tree/forest.
10. **Performer and action analogies:** As the name suggests, one of the word is the performer and the other is the action which is performed. These questions are pretty easy to spot and answer. *Example*: magician/magic, actor/act, etc.
11. **Verb tenses analogies:** This type of analogy in which two tenses of a verb are analogous to two of the same tenses of another verb. *Example*: eat/ate, drive/drove, etc.
12. **Cause and effect analogies:** The similarity in this type of analogies derives from the cause on one side and its indisputably connected effect on the other side. The student should be careful as to not to confuse this type of analogy with the "effort and result". *Example*: fire/burn, trip/fall, etc.
13. **Effort and result analogies:** The difference between this analogy and "cause and effect" is the fact that for the effort and result connection an actual effort has to be made. *Example*: paint/painting, write/letter, etc.
14. **Things that are in sequence analogies:** This type of analogies is also easy to spot as it contains the words which are in direct sequence to each other. *Example*: Monday/Tuesday, day/night, etc.
15. **Degrees of characteristic analogies:** This type of analogies can be explained by the following scenario. In the case of warm and hot, one degree higher then warm can be hot and another degree higher can be burning. Similarly, one degree higher than cold can be freezing. *Example*: tired/exhausted, cold/freezing, etc.

TIPS TO SOLVE VERBAL ANALOGIES QUESTIONS

Some of the common methodologies to solve the questions of verbal analogies are given as follows:

1. Determine the relationship between the first pair of words.
2. Eliminate all the pairs in the answer choices that don't have the same relationship. The remaining choice is the correct answer.
3. Another strategy is to read the verbal analogies in sentences.
4. Sometimes it is difficult to identify the relationship by just looking at the analogy in the given order and hence sometimes it is helpful to switch the words to obtain a relationship.
5. Consider alternate meaning of words.

PRACTICE EXERCISE

1. If paw : cat then hoof : ?

(a) elephant (b) horse
(c) lion (d) lamb

2. If thrust : spear then

(a) mangle : iron (b) bow : arrow
(c) scabbard : sword (d) fence : epee

3. If pain : sedative then

(a) comfort : stimulant
(b) trance : narcotic
(c) grief : consolation
(d) ache : extraction

4. If light : blind then

(a) speech : dumb (b) language : deaf
(c) tongue : sound (d) voice : vibration

5. If after : before then

(a) first : second
(b) primary : secondary
(c) contemporary : historic
(d) successor : predecessor

6. If distance : mile then

(a) liquid : litre (b) bushel : corn
(c) weight : scale (d) fame : television

7. If ten : decimal then

(a) seven : septet (b) four : quartet
(c) two : binary (d) five : quince

8. If army : logistics then

(a) soldier : students (b) business : strategy
(c) team : individual (d) war : guns

9. If gravity : pull then

(a) iron : metal
(b) north pole : directions
(c) magnetism : attractions
(d) dust : desert

10. If scribble : write then stammer : ?

(a) shout (b) weep
(c) dance (d) speak

11. If sculptor : statue then poet : ?

(a) canvas (b) pen
(c) verse (d) paint

12. If ornithologist : birds then anthropologist : ?

(a) mankind (b) environment
(c) animals (d) plants

13. If calf : cow then puppy : ?

(a) horse (b) dog
(c) buffalo (d) bitch

14. If incandescent : glowing then

(a) indefatigable : untiring
(b) flash : flame
(c) tedious : bore
(d) fast : run

15. If indigent : wealthy then

(a) healthy : wealthy (b) pessimistic : optimistic
(c) indigent : rude (d) malicious : deceitful

16. If medicine : illness then

(a) hunger : thirst (b) law : anarchy
(c) love : treason (d) stimulant : sensitivity

17. If extort : obtain then

(a) pilfer : steal (b) explode : ignite
(c) plagiarize : borrow (d) consider : appeal

18. If seconds : minute then

(a) time : clock
(b) minute : speed
(c) running : athleticism
(d) fingers : hand

19. Cup is to coffee as bowl is to

(a) soup (b) dish
(c) spoon (d) food

20. Optimist is to cheerful as pessimist is to

(a) mean (b) petty
(c) helpful (d) gloomy

21. If muster : gather then

(a) Divide : unite
(b) Song : music
(c) Incongruous : asymmetrical
(d) Collate : order

22. If outmanoeuvre : strategy then

(a) Bait : lion
(b) Lure : decoy
(c) Competition : product
(d) Entice : tempt

23. If aphorism : terse then

(a) Draught : game (b) Malign : meagre
(c) Eulogy : praise (d) Holistic : cycle

24. If preen : pride then

(a) Wave : water (b) Lament : grief
(c) Incubate : nurture (d) Beauty : vanity

25. If moratorium : payment then

(a) Fraud : embezzlement
(b) Flush : anger
(c) Retort : insolent
(d) Reprieve : punishment

ANSWERS

1. (b)	**6.** (a)	**11.** (c)	**16.** (b)	**21.** (c)
2. (d)	**7.** (c)	**12.** (a)	**17.** (c)	**22.** (b)
3. (c)	**8.** (b)	**13.** (d)	**18.** (d)	**23.** (c)
4. (a)	**9.** (c)	**14.** (a)	**19.** (a)	**24.** (a)
5. (d)	**10.** (d)	**15.** (b)	**20.** (d)	**25.** (d)

EXPLANATIONS AND HINTS

1. (b) Paw is foot of Cat. Hoof is foot of Horse.
2. (d) Thrust is the act whereas spear is the object. Hence, the option which comes closest is fence : epee.
3. (c) Here, pain is the problem and sedative is the solution. Hence, the option grief : consolation is the correct answer.
4. (a) A person who is blind cannot see the light, and a person who is dumb cannot speak.
5. (d) After and before represent different time spans and hence the answer successor : predecessor.
6. (a) Distance is the object and mile is the quantity of measurement. Similarly, liquid is the object and litre is the quantity of measurement.
7. (c) Decimal number system has a base ten and a binary number system has a base two.
8. (b) Army use logistics to perform their operations. Similarly, business use strategies to perform their operations.
9. (c) Gravity causes the action of pull and similarly, magnetism causes the action of attraction.
10. (d) People scribble while writing and people stammer while speaking.
11. (c) A sculptor creates a statue like a poet creates a verse.
12. (a) Ornithologist is a study of birds and anthropologist is a study of mankind.
13. (d) Cow is the mother of a calf while bitch is the mother of a puppy.
14. (a) Incandescent and glowing are synonyms to each other. Similarly, indefatigable and untiring are synonyms of each other.
15. (b) Indigent and wealthy are antonyms of each other. Similarly, pessimistic and optimistic are antonyms of each other.
16. (b) Medicine is the solution of illness as law is the solution or answer to anarchy.
17. (c) To extort something is to obtain something by force. Plagiarism is borrowing material from another writer without giving him/her the acknowledgement.
18. (d) Minutes can be divided into seconds. Hence, seconds are a part of an object which is minutes. Similarly, fingers are a part of a hand.
19. (a) Coffee goes into a cup and soup goes into a bowl. Dish and spoon are incorrect since they are utensils. Option (d) that is, food, is too general. Hence, the correct option is bowl.
20. (d) An optimist is a person whose outlook is positive or cheerful. A pessimist is a person whose outlook is negative or gloomy.
21. (c) Muster and gather are both synonyms which mean collect. Incongruous and asymmetrical both are synonyms too which mean harmony.
22. (b) Function and purpose. The purpose of strategy is to outmanoeuvre the adversary or enemy, the purpose of a decoy is to lure and trap the enemy.
23. (c) The chief characteristic of aphorism is terseness pithy, as praise is of eulogy.
24. (a) Preen is an action that demonstrates pride. Lament is an action that signifies grief.
25. (d) Moratorium is a delay in payment whereas reprieve is a delay in punishment.

CHAPTER 6

WORD GROUPS

The questions on word groups are designed to check the vocabulary of the students and their understanding of word meanings. Hence, the questions focus on relationships between the words and the questions are framed such that the student needs to know the exact meaning of the words given, in order to select the correct answer.

Questions on word groups often require the student to have a good knowledge about synonyms, antonyms, dictionary definitions and word pairs. These questions also include words which sound similar but have different meanings. Such words are called homophones. A list of homophones is given as follows:

Ad, add
Ail, ale
Air, heir
All, awl
Allowed, aloud
Altar, alter
Arc, ark
Ate, eight
Auger, augur
Aural, oral
Away, aweigh
Axel, axle
Bail, bale
Bait, bate
Ball, bawl
Band, banned
Bard, barred
Berth, birth
Billed, build
Boarder, border
Brake, break
Call, caul
Cast, caste
Cede, seed
Ceiling, sealing
Cell, sell
Cent, scent, sent
Cereal, serial
Chord, cord
Coarse, course
Cue, queue
Currant, current
Dam, damn

Dear, deer
Desert, dessert
Deviser, divisor
Dew, due
Die, dye
Dual, duel
Faint, feint
Fair, fare
Faun, fawn
Feat, feet
Flea, flee
Flour, flower
Forth, fourth
Foul, fowl
Gamble, gambol
Genes, jeans
Groan, grown
Hail, gale
Hair, hare
Hall, haul
Hart, heart
Hear, here
Heard, herd
Hour, our
Idle, idol
Knead, need
Knight, night
Knob, nob
Knock, nock
Know, no
Lac, lack
Lade, laid
Laps, lapse
Leach, leech
Lead, led
Leak, leek
Liar, lyre
Loan, lone
Loot, lute
Made, maid
Mail, male
Main, mane
Maize, maze
Mall, maul
Manna, manner
Mantel, mantle
Marshal, martial
Meat, meet
Medal, meddle
Might, mite
Mind, mined
Naval, navel
Nun, none
Ode, owed
One, won
Packed, pact
Pain, pane
Pair, pear
Pause, paws
Pea, pee
Peace, piece
Peak, peek
Pearl, purl
Pedal, peddle
Plain, plane
Practice, practise
Principal, principle
Profit, prophet
Quarts, quartz
Rain, reign
Raise, rays
Rap, wrap
Read, reed
Real, reel
Right, write
Ring, wring
Role, roll
Rood, rude
Rough, ruff
Sale, sail
Satire, satyr
Soar, sore
Sea, see
Seam, seem
Shear, sheer
Shoe, shoo
Slay, sleigh
Sole, soul
Some, sum
Son, sun
Sort, sought
Stake, steak
Storey, story
Straight, strait
Tale, tail
Talk, torque
Team, teem
Threw, through

Throne, thrown
Tide, tied
Tire, tyre
Tough, tuff
Vain, vane, vein
Waist, waste
Wait, weight
Ware, wear, where
Watt, what
Weak, week
Weather, whether
Which, witch
Whine, wine
Wood, would
Yaw, yore, your
Yoke, yolk

PRACTICE EXERCISE

1. Which of the following words does not have a similar meaning to energize?

(a) rejuvenate (b) strengthen
(c) enervate (d) uplift

2. Which of the following words mean tuneful or marked with agreement?

(a) harmonious (b) inclusive
(c) exclusive (d) lucid

3. Which of the following words mean to accumulate or to gather?

(a) expend (b) deliberate
(c) abjure (d) amass

4. Which of the following words mean modest or lack of self-confidence?

(a) wanton (b) diffident
(c) trite (d) barren

5. Which of the following words does not have a similar meaning to comprise?

(a) compose (b) cover
(c) contain (d) encompass

6. Which of the following words does not have a similar meaning to densely populated?

(a) populace (b) crowded
(c) populous (d) packed

7. Which of the following words has/have meaning nearly opposite to pragmatic?

(a) exuberant (b) impractical
(c) philosophical (d) realistic

8. Which of the following words has/have meaning nearly opposite to absolve?

(a) pardon (b) condemn
(c) free (d) exonerate

9. Which of the following words has/have meaning nearly opposite to exigent?

(a) strenuous (b) light
(c) easy (d) difficult

10. Which of the following words has/have meaning similar to condescend?

(a) usurp (b) respectful
(c) patronize (d) contribute

11. Which of the following words has/have meaning similar to contradict?

(a) gainsay (b) disparage
(c) tarnish (d) oppose

12. Which of the following words has/have meaning similar to perplex?

(a) reiterate (b) affiliate
(c) dither (d) discomfit

13. Which of the following words has/have meaning similar to expedite?

(a) facilitate (b) disrespect
(c) exterminate (d) beckon

14. Which of the following words has/have meaning nearly opposite to fecund?

(a) abundant (b) barren
(c) unfriendly (d) productive

15. Which of the following words has/have meaning nearly opposite to malicious?

(a) vile (b) corrupt
(c) divine (d) malign

16. Which of the following statement is incorrect?

(a) I read the whole book in one day.
(b) I have a whole in my trousers.
(c) I ate whole of the pie.
(d) I will finish whole of my homework.

17. This is _____ difficult for me

(a) too (b) to
(c) both (a) and (b) (d) none of the above

18. Which of the following statement is incorrect?

(a) He sailed all the way to the shore.
(b) I bought this watch during the winter sale.
(c) I will buy a watch during the next sale.
(d) The boat will sale across the ocean.

19. Can you ____ me?

(a) here (b) hear
(c) both (a) and (b) (d) none of the above

20. Which of the following statement is incorrect?

(a) David is a good writer.
(b) You did the right thing.
(c) When you reach the crossing, turn right.
(d) Can you help him right his letter?

ANSWERS

1. (c)	**5.** (b)	**9.** (c)	**13.** (a)	**17.** (a)
2. (a)	**6.** (a)	**10.** (c)	**14.** (b)	**18.** (d)
3. (d)	**7.** (b)	**11.** (d)	**15.** (c)	**19.** (b)
4. (b)	**8.** (b)	**12.** (d)	**16.** (b)	**20.** (d)

EXPLANATIONS AND HINTS

1. (c) Rejuvenate, strengthen and uplift have a similar meaning to the word energise. However, enervate means lacking in energy. Hence, the answer is enervate.

2. (a) Harmonious means tuneful or free from disagreement.

3. (d) Amass means to accommodate or to gather.

4. (b) Diffident means to be shy, modest or having a lack of self-confidence.

5. (b) Contain, compose and encompass are synonyms of comprise. Hence, the answer is cover.

6. (a) Populous, crowded and packed are more or less synonyms of densely populated. However, populace means people living in a particular area.

7. (b) Pragmatic means being realistic or having a realistic approach to something. Hence, the word meaning opposite to it is impractical.

8. (b) Absolve means to free someone from guilt. It is clear that pardon, free and exonerate are synonyms of absolve. Hence, the answer is condemn.

9. (c) Exigent means to be pressing and demanding. Hence, the most appropriate word meaning opposite to it is easy.

10. (c) Condescend means to show that one feels superior. Hence, the word similar to it is patronize.

11. (d) To contradict is to speak against. Hence, the word similar in meaning is oppose.

12. (d) Perplex means to make somebody feel uncomfortable or uneasy. Hence, the word having a meaning similar to perplex is discomfit.

13. (a) Expedite means to cause an action. Hence, the word having a meaning similar to expedite is facilitate.

14. (b) Fecund means to be fertile. Hence, the word having a meaning opposite to fecund is barren.

15. (c) Malicious means to be immoral. Hence, vile, corrupt and malign are synonyms of malicious. The answer is divine.

16. (b) The answer is (b). The correct sentence should be, I have a *hole* in my trousers.

17. (a) This is *too* difficult for me. Hence, the answer is *too*.

18. (d) The boat will *sail* across the ocean. Hence, the answer is (d).

19. (b) Can you *hear* me? Hence, the answer is (b).

20. (d) Can you help him *write* his letter? Hence, the answer is (d).

CHAPTER 7

VERBAL DEDUCTION

Verbal deduction questions are not designed to measure a student's facility with English. These are designed to test the ability to take a series of facts expressed in words and to understand and manipulate the information to solve a specific problem.

Verbal deduction problems usually follow the pattern in which a paragraph is given and then certain conclusions are asked to be made from the information in the paragraph. Another type of problems involves a statement followed by the arguments and the student is asked to choose as to which of the arguments best support the statement.

TIPS AND TRICKS TO SOLVE VERBAL DEDUCTION QUESTIONS

Some of the common methodologies to solve the questions of verbal deduction questions are given as follows:

1. Scan for keywords in the questions. Read the comprehension or the directions given and mostly you will be able to find a hint which gives the answer. This is not something which is applicable for every question, but it's a very handy tool.
2. Make no assumptions. Verbal deduction questions are meant to be answer absolutely literally. Hence, if it isn't included in the passage then you cannot include it in your decision-making process for the questions. Don't factor in real-life intelligence that you know proves or disproves a statement.
3. If you are stuck, try starting at the end of the sentence. A great way to unravel a confusing piece of writing is to start at the end of the sentence and work backwards. For long statements that make contradictory points and circular references this can be very useful in decoding their meaning.
4. We can use the process of elimination in which we can start with eliminating the choices which we are sure are not the answer in order to increase the odds of getting the correct answer.

PRACTICE EXERCISE

Directions for Q. 1 to Q. 8

In the questions below are given two or three statements followed by conclusions. Take the given two statements to be true even if they seem to be at variance from commonly known facts. Read the conclusion and then decide which of the given conclusion(s) logically follows from the two given statements.

1. **Statement:** No women teachers can play. Some women teachers are athletes.

   Conclusions: (A) Male athletes can play.
   (B) Some athletes can play.

   (a) Only conclusion (A) follows
   (b) Only conclusion (B) follows
   (c) Neither (A) nor (B) follows
   (d) Both (A) and (B) follow

2. **Statement:** Some blankets are beds. Some pillows are blankets. All beds are pillows.

   Conclusions: (A) Some blankets are pillows.
   (B) Some pillows are beds.
   (C) Some beds are blankets.

   (a) Only conclusion (A) or (B) follows
   (b) Only conclusion (A) and either (B) or (C) follows
   (c) Only (C) and either (A) or (B) follows
   (d) All (A), (B) and (C) follow

3. **Statement:** Some lawyers are fools. Some fools are rich.

   Conclusions: (A) Some lawyers are rich.
   (B) Some rich are doctors.

   (a) Only conclusion (A) follows
   (b) Only conclusion (B) follows
   (c) Neither (A) nor (B) follows
   (d) Both (A) and (B) follow

4. **Statement:** All mangoes are golden in colour. No golden-coloured things are cheap.

   Conclusions: (A) All mangoes are cheap.
   (B) Golden-coloured mangoes are not cheap.

   (a) Only conclusion (A) follows
   (b) Only conclusion (B) follows
   (c) Neither (A) nor (B) follows
   (d) Both (A) and (B) follow

5. **Statement:** No bat is ball. No ball is wicket.

   Conclusions: (A) No bat is wicket.
   (B) All wickets are bats.

   (a) Only conclusion (A) follows
   (b) Only conclusion (B) follows
   (c) Neither (A) nor (B) follows
   (d) Both (A) and (B) follow

6. **Statement:** Some swords are sharp. All swords are rusty.

   Conclusions: (A) Some rusty things are sharp.
   (B) Some rusty things are not sharp.

   (a) Only conclusion (A) follows
   (b) Only conclusion (B) follows
   (c) Neither (A) nor (B) follows
   (d) Both (A) and (B) follow

7. **Statement:** All fishes are blue in colour. Some fishes are heavy.

   Conclusions: (A) All heavy fishes are blue in colour.
   (B) All light fishes are not blue in colour.

   (a) Only conclusion (A) follows
   (b) Only conclusion (B) follows
   (c) Neither (A) nor (B) follows
   (d) Both (A) and (B) follow

8. **Statement:** All snakes are trees. Some trees are roads. All roads are mountains.

   Conclusions: (A) Some mountains are snakes.
   (B) Some roads are snakes.
   (C) Some mountains are trees.

   (a) Only conclusion (A) follows
   (b) Only conclusion (B) follows
   (c) Only conclusion (C) follows
   (d) Conclusions (A), (B) and (C) follows

Directions for Q. 9 to Q. 16

In the questions below is given a statement followed by arguments to support or oppose it. Read the arguments and then decide which of the given argument(s) logically follows from the given statement.

9. **Statement:** Should India have no military force at all?

   Arguments: (A) No. Other countries in the world would not believe in non-violence.
   (B) Yes. Many Indians believe in non-violence.

   (a) Only argument (A) is true
   (b) Only argument (B) is true
   (c) Either (A) or (B) is true
   (d) Neither (A) nor (B) is true

10. **Statement:** Should the articles of only deserving authors be allowed to be published?

    Arguments: (A) Yes. It will save a lot of paper which is in short supply.

(B) No. It is not possible to draw a line between the deserving and the undeserving.

(a) Only argument (A) is true
(b) Only argument (B) is true
(c) Either (A) or (B) is true
(d) Neither (A) nor (B) is true

11. Statement: Should India be a permanent member of the UN's Security Council?

Arguments: (A) Yes. India is a country who loves peace and amity.

(B) No. Let us first solve the problems of our own people like poverty, malnutrition etc.

(a) Only argument (A) is true
(b) Only argument (B) is true
(c) Either (A) or (B) is true
(d) Neither (A) nor (B) is true

12. Statement: Should airlines immediately stop issuing free passes for all of its employees?

Arguments: (A) No. The employees have the highest right to travel free.

(B) Yes. This will help airlines provide better facilities.

(e) Only argument (A) is true
(f) Only argument (B) is true
(g) Either (A) or (B) is true
(h) Neither (A) nor (B) is true

13. Statement: Should a total ban be put on trapping wild animals?

Arguments: (A) Yes. People who trap the animals are making a lot of money.

(B) No. Bans on hunting and trapping are not effective.

(a) Only argument (A) is true
(b) Only argument (B) is true
(c) Either (A) or (B) is true
(d) Neither (A) nor (B) is true

14. Statement: Should cutting of trees be banned altogether?

Arguments: (A) Yes. People who trap the animals are making a lot of money.

(B) No. Bans on hunting and trapping are not effective.

(a) Only argument (A) is true
(b) Only argument (B) is true
(c) Both (A) and (B) are true
(d) Neither (A) nor (B) is true

15. Statement: Should jobs be linked with academic degrees and diplomas?

Arguments: (A) No. A very large number of persons with meagre academic qualifications will apply.

(B) No. Importance of higher education will be diminished.

(a) Only argument (A) is true
(b) Only argument (B) is true
(c) Either (A) and (B) is true
(d) Neither (A) nor (B) is true

16. Statement: Should judiciary be independent of the executive?

Arguments: (A) Yes. This would help curb the unlawful activities of the executives.

(B) No. The executive would not be able to take the bold measures.

(a) Only argument (A) is true
(b) Only argument (B) is true
(c) Either (A) and (B) is true
(d) Neither (A) nor (B) is true

17. Shoshana has never received a violation from the Federal Aviation Administration during her 20-year flying career. Shoshana must be a great pilot.

Which of the following can be said about the reasoning above?

(a) The definitions of the terms create ambiguity.
(b) The argument uses circular reasoning.
(c) The argument works by analogy.
(d) The argument is built upon hidden assumptions.

18. No national productivity measures are available for underground industries that may exist but remain unreported. On the other hand, at least some industries are run entirely by self-employed industrialists are included in national productivity measures.

Which of the following facts can be validly concluded from the information given above?

(a) There are at least some industries run entirely by self-employed industrialists that are not underground industries.
(b) There are at least some industries other than those run entirely by self-employed industrialists that are underground industries.
(c) No industries that are run entirely by self-employed industrialists operate underground.
(d) There are at least some industries run entirely by self-employed industrialists that are underground industries.

19. Abate observes that if flight 409 is canceled, then the manager could not possibly arrive in time for the meeting. But the flight was not canceled. Therefore, Abate concludes, the manager will certainly be on time. Eve replies that even if Abate's premises are true, his argument is fallacious. And therefore, she adds, the manager will not arrive on time after all.

Which of the following is the strongest thing that we can properly say about this discussion?

(a) Eve is mistaken in thinking Abate's argument to be fallacious, and so her own conclusion is unwarranted.
(b) Eve is right about Abate's argument, but nevertheless her own conclusion is unwarranted.
(c) Both (a) and (b)
(d) None of the above

20. No national productivity measures are available for underground industries that may exist but remain unreported. On the other hand, at least some industries that are run entirely by self-employed industrialists are included in national productivity measures.

From the information given above, it can be validly concluded that

(a) No industries that are run entirely by self-employed industrialists operate underground.
(b) There are at least some industries other than those run entirely by self-employed industrialists that are underground industries.
(c) Both (a) and (b).
(d) There are at least some industries run entirely by self-employed industrialists that are not underground industries.

Directions for Q. 21 to Q. 25

Glaciers begin to form where snow remains year-round and enough of it accumulates to transform into ice. New layers of snow compress the previous layers and this compression forces the icy snow to recrystallize, forming grains similar in size and shape to cane sugar. Gradually the grains grow larger and the air pockets between the grains get smaller, meaning that the snow slowly becomes denser. After about two winters, the snow turns into firn, an intermediate state between snow and ice. Over time the larger ice crystals become more compressed and even denser, this is known as glacial ice. Glacial ice, because of its density and ice crystals, often takes a bluish or even green hue.

21. Firn is less dense than ____, but more dense than ____.

(a) Glacial ice, ice crystals
(b) Ice crystals, glacial ice
(c) Ice, snow
(d) Snow, ice

22. The increase in density is caused by the grains becoming smaller.

(a) True
(b) False
(c) True only up to a point
(d) Indeterminate

23. Over time the larger ice crystals become more compressed and even denser, this is known as ____.

(a) Ice crystals
(b) Glacier ice
(c) Snow
(d) None of the above

24. Glaciers cannot form where snow does not remain all year round.

(a) True
(b) False
(c) True for North Pole and false for South Pole
(d) Indeterminate

25. Which of the following colour does glacial ice often take?

(a) Green
(b) Blue
(c) Red
(d) Green or blue

ANSWERS

1. (c)	**6.** (a)	**11.** (a)	**16.** (a)	**21.** (c)
2. (d)	**7.** (a)	**12.** (d)	**17.** (d)	**22.** (b)
3. (c)	**8.** (c)	**13.** (d)	**18.** (a)	**23.** (b)
4. (b)	**9.** (d)	**14.** (c)	**19.** (b)	**24.** (a)
5. (c)	**10.** (b)	**15.** (b)	**20.** (d)	**25.** (d)

EXPLANATIONS AND HINTS

1. (c) Since one premise is negative, the conclusion must be negative. Hence, neither (A) nor (B) follows.

2. (d) (A) is the converse of (B)'s premise, (B) is the converse of (C)'s premise and (C) is the converse of (A)'s premise and hence, all (A), (B) and (C) follow.

3. (c) Since both the premises are particular, no definite conclusion follows.

4. (b) Clearly, the conclusion must be universal negative and should not contain the middle term. So, it follows that 'No mango is cheap'. Thus, (A) does not follow. Also, since all mangoes are golden in colour, we may substitute mangoes with 'golden coloured mangoes'. Thus, (B) follows.

5. (c) Since both the premises are negative, no definite conclusions follow.

6. (a) Since one premise is particular, the conclusion must be particular and should not contain the middle term. Hence, (A) follows. Since both the premises are affirmative, the conclusion cannot be negative. Thus, (B) doesn't follow.

7. (a) Since one premise is particular, the conclusion must be particular and should not contain the middle term. So, it follows that 'Some heave things are blue in colour'. Thus, only (A) holds.

8. (c) All snakes are trees. Some trees are roads.

Since the middle term is not distributed even once in the premises, no conclusion follows.

Some trees are roads. All the roads are mountains. Since one premise is particular, the conclusion must be particular and should not contain the middle term. So, it follows that 'Some trees are mountains'. (C) is the converse of this statement and hence, it follows.

9. (d) Clearly, India (or any country for that matter) needs to maintain military force in order to defend itself against the threats of other military powers in the world. Hence, none of the arguments hold strong.

10. (b) Argument (A) does not provide a strong enough reason in support of the statement.

Also, it is not possible to analyse the really deserving and not deserving. Hence, argument (B) holds true.

11. (a) Argument (A) is strong since it talks about the characteristics of the India which are related to the statement. However, argument (B) has no connection to the statement and hence it does not hold.

12. (d) Free passes given to the airway employee is a privilege to them and not their right, hence argument (A) does not hold. Also, argument (B) is vague.

13. (d) The ban is necessary only to prevent our environment and wildlife. It has nothing to do with how much money the people who trap the animals make or how effective they are. Hence, both the arguments do not hold.

14. (c) Both the arguments (A) and (B) are strong, valid and are related to the statement. Hence, both the arguments hold.

15. (b) Delinking jobs with degrees will diminish the need for higher education as many people pursue such education for jobs. So, only argument (B) is strong.

16. (a) Clearly, independent judiciary is necessary for impartial judgements so that the executive does not take the wrong measure. Hence, only argument (A) holds.

17. (d) The fact that Shoshana never violated any rules doesn't mean she was a great pilot. It simply confirms the fact that she was disciplined. The argument is built upon hidden assumptions.

18. (a) There are at least some industries run entirely by self-employed industrialists that are not underground industries. This is the only apt conclusion from the passage.

19. (b) According to the above passage only the argument "Eve is right about Abate's argument, but nevertheless her own conclusion is unwarranted" is true.

20. (d) The only apt conclusion is "there are at least some industries run entirely by self-employed industrialists that are not underground industries."

21. (c) We know that snow cools into firn and firn cools into ice. Hence, ice is denser than firn and similarly, firn is denser than snow.

22. (b) The increase in density is caused by the grains becoming larger. Hence, the given statement is false.

23. (b) As it can be seen from the given passage, 'Over time the larger ice crystals become more compressed and even denser, this is known as glacier ice.'

24. (a) Glaciers are only formed where ice is present all year round. Hence, the statement "Glaciers cannot form where snow does not remain all year round." is true.

25. (d) It can be deducted from the passage that "glacial ice, because of its density and ice crystals, often takes a bluish or even green hue." Hence, the answer is option (d).

GENERAL APTITUDE

SECTION B

NUMERICAL ABILITY

UNIT 1: BASIC ARITHMETIC

CHAPTER 1

NUMBER SYSTEM

NUMBERS

Natural numbers include all positive integers. For example 1, 2, 3, 4, ….

A natural number larger than unity is a *prime number* if it does not have other divisors except for itself and unity. For example 2, 3, 5, 7, 11, 13, ….

All other natural numbers that are not prime numbers are *composite numbers*. For example 4, 6, 8, 9, 10, 12, ….

Some important formulas of natural numbers are as follows:

1. Sum of first n natural numbers, $S_n = \sum n = \dfrac{n(n+1)}{2}$.
2. Sum of squares of first n natural numbers, $\sum n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
3. Sum of cubes of first n natural numbers, $\sum n^3 = \left[\dfrac{n(n+1)}{2}\right]^2 = \left[\dfrac{n^2(n+1)^2}{4}\right] = [\sum n]^2$.

Classification of Numbers

Numbers can be more specifically classified into more categories than just natural numbers as depicted in Table 1.

Table 1 | Classification of numbers into various categories

Class	Symbol	Description
Natural number	N	Natural numbers are defined as non-negative counting numbers: $N = \{0, 1, 2, 3 \ldots\}$.
Integer	Z	Integers extend N by including the negative numbers: $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
Rational number	Q	A rational number is the ratio of two integers such that: $Q = \{x/y \text{ where } x, y \in Z, y \neq 0\}$ For example: 3/2, 2.5, etc.

(*Continued*)

Table 1 (Continued)

Class	Symbol	Description
Irrational number	P	Irrational numbers are those which cannot be represented as fractions: $P = \{\sqrt{2}, \sqrt{3}\}$
Real number	R	Real numbers are all the numbers on a number line. It is the union of all the rational numbers and all irrational numbers.
Imaginary number	I	An imaginary number is a number whose square is a negative real number, and is denoted by i. Hence, $i^2 = -1$. For example: $-3i$, $5i$, etc. Sometimes, j is used instead of i.
Complex number	C	A complex number consists of two parts, real and imaginary numbers. It can be represented as $a + ib$. For example: $7 + 5i$.

PROGRESSIONS

When the numbers in a sequence are related to each other either linearly or geometrically, then the sequence is a progression. This section discusses the type of progressions and the formulae used to calculate the sequences.

Arithmetic Progressions

An *arithmetic progression* (AP) is a sequence of numbers such that the difference between the consecutive terms is constant. This constant value is called common difference. In other words, any term of an AP can be obtained by adding common difference to the preceding term.

Let a be the first term, n be the number of terms, d be the common difference, a_n be the nth term, S_n be the sum of first n terms and S be the sum of the entire sequence. Then,

$$a_n = a + (n-1) \times d$$

$$S_n = \frac{n}{2} \times [2a + (n-1) \times d]$$

$$S = \frac{n}{2} \times (\text{first term } + \text{ last term})$$

AP can be represented as a, $a + d$, $a + 2d$, $[a + (n - 1)d]$. Here, quantity d is to be added to any chosen term to get the next term of the progression.

Geometric Progressions

A *geometric progression* (GP) is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio. In other words, any term of a GP can be obtained by multiplying the preceding number by the common ratio.

Let a be the first term, n be the number of terms, r be the common ratio, a_n be the nth term, S_n be the sum of the first n terms and S be the sum of the entire sequence. Then,

$$a_n = ar^{(n-1)}$$

$$S_n = \frac{a(1-r^n)}{1-r} \text{ if } r < 1$$

$$S_n = \frac{a(r^n-1)}{r-1} \text{ if } r > 1$$

$$S = \frac{r \times \text{last term} - \text{first term}}{r-1}$$

GP can be represented as a, ar, ar^2, …. Here, a is the first term and quantity r is the common ratio of the geometric progression.

Infinite Geometric Progressions

If $-1 < r < +1$ or $|r| < 1$, then sum of a geometric progression does not increase infinitely; it converges to a particular value. Such a GP is called *infinite geometric progression.*

Sum of an infinite GP, $S_\infty = \dfrac{a}{1-r}$.

AVERAGES, MEAN, MEDIAN AND MODE

A central value around which a group of values shows a tendency to concentrate is called *average*. Thus, an average is a single value that is in some way indicative of a group of values.

The following are five measures of central tendency:

1. **Arithmetic mean (AM):** The most commonly used average is the AM or simply the average.

   AM of n numbers x_1, x_2, x_3, …, x_n is denoted by $\bar{X}$ and calculated as

$$\bar{X} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

$$\Rightarrow \bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

If values $x_1, x_2, x_3, \ldots, x_n$ are assigned weights $w_1, w_2, w_3, \ldots, w_n$, respectively, then the *weighted arithmetic mean* is given as follows:

$$\text{Weighted arithmetic mean, } x_w = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

2. **Geometric mean (GM):** GM of n numbers $x_1, x_2, x_3, \ldots, x_n$ is the nth root of their products. It is a type of mean or average that indicates the central tendency or typical value of a set of numbers by using the product of their values. Geometric mean of n numbers $x_1, x_2, x_3, \ldots, x_n$ is denoted by

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

and calculated as

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} = \sqrt[n]{x_1 x_2 x_3 \ldots x_4}$$

3. **Harmonic mean (HM):** Harmonic mean is the special case of the power mean. As it tends strongly toward the least elements of the list, it may (compared to the arithmetic mean) mitigate the influence of large outliers and increase the influence of small values. Harmonic mean of n numbers $x_1, x_2, x_3, \ldots, x_n$ is calculated as

$$\text{HM} = \frac{n}{\sum_{i=1}^{n} (1/x_i)}$$

4. **Mode:** It is the number that occurs most number of times in a given set of numbers. In a given set of data, if two or more values occur the same number of times, then a unique mode does not exist. For example, suppose we have a given set of numbers 1, 2, 4, 4, 6, 5, 9, 1, 6, 6. The mode of this set is 6, since it occurs the most number of times in the set.
5. **Median:** It is the middle value of a group of numbers arranged in an ascending or descending order. If number of values n, in a given set of data is odd, then median is the $(n + 1)/2$th value. If number of values n, in a given set of data is even, then there will be two middle values say a and b, and hence median is taken as $(a + b)/2$.

For example, suppose we have a given set of numbers 31, 29, 42, 18, 50. The median of the set will be calculated by first arranging the values in ascending order, that is, 18, 29, 31, 42, 50. Now, the median will be $(5 + 1)/2$th value, 5th value, that is, 31.

Suppose we have a given set of numbers 19, 33, 22, 40, 28, 15. The median of the set will be calculated by first arranging the values in ascending order, that is, 15, 19, 22, 28, 33, 40. Now, the median will be $(22 + 28)/2 = 50/2 = 25$.

Relation between AM, GM and HM

Consider two numbers a and b.

Then,

$$\text{AM} = (a + b)/2; \text{ GM} = \sqrt{ab}; \text{ HM} = 2ab/(a + b).$$

Therefore,

$$(\text{GM})^2 = \text{AM} \times \text{HM}$$

Also,

$$\text{AM} > \text{GM} > \text{HM}$$

SOME ALGEBRAIC FORMULAS

Some of the commonly used algebraic formulae are as follows:

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$
2. $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$
3. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
4. $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2a(b + c + d) + 2b(c + d) + 2cd$
5. $(a \pm b)^3 = a^3 \pm 3ab(a \pm b) \pm b^3$
6. $a^3 \pm b^3 = (a \pm b)^3 \mp 3ab(a \pm b)$
7. $(x + a)(x + b) = x^2 + (a + b)x + ab$
8. $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$

The Remainder Theorem

The remainder theorem is useful for evaluating polynomials at a given value of x. It states that when we divide a polynomial $f(x)$ by $x - a$ then the remainder r equals $f(a)$.

The Polynomial Factor Theorem

The polynomial factor theorem is a special case of the remainder theorem. It is used to link the factors and zeros of any polynomial.

The factor theorem states that a polynomial $f(x)$ has a factor $(x-a)$ if and only if $f(a)=0$ or in other words, a is a root of $f(x)$.

BASE SYSTEM

The number of digits used to represent a number in a system is called the base system. The amount of digits is the base of that number system. For example, digital computers use binary base system which represents data using only two digits, viz. 0 and 1.

Some of the most commonly used base systems are given as follows:

1. **Binary (Base-2):** As previously mentioned, binary base system is used extensively in digital computing. In binary system, we use only two digits (0 and 1) to represent the data.
2. **Octal (Base-8):** Octal base systems use eight digits (0−7) to represent the data.
3. **Decimal (Base-10):** Decimal base systems use ten digits (0−9) to represent the data. We use the decimal system in our day-to-day life for various calculations.
4. **Hexadecimal or Hex (Base-16):** Hexadecimal base systems use sixteen digits (0−9 and A, B, C, D, E, F) to represent the data. Hexadecimal system is widely used by computer system designers and programmers.

Table 2 gives the conversion table which can be used for the conversions of one base system to another. We have to keep in mind that whenever we are converting from binary to octal, we divide the binary number into groups of 3 and while converting from binary to hexadecimal, we divide the binary number into groups of 4. If the binary number cannot be divided into groups of 3 or 4, then we add sufficient zeros before the M.S.B. (Most Significant Bit).

Table 2 | Conversion table of decimal, binary, octal and hexadecimal base systems

Decimal	Binary	Octal	Hexadecimal
0	0000	0	0
1	0001	1	1
2	0010	2	2
3	0011	3	3
4	0100	4	4
5	0101	5	5
6	0110	6	6
7	0111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Counting Trailing Zeros

Trailing zeros are a sequence of 0s in the decimal representation of a number after which no other digits follow. Trailing zeros to the right of a decimal point do not affect the value of a number and can be removed. They may be useful for indicating the number of significant figures.

The number of trailing zeros in a non-zero base b and integer n equals the exponent of the highest power of b that divides n. For example, 15000 has three trailing zeros and is therefore divisible by 1000 but not by 10^4.

INEQUATIONS

A mathematical statement that says that one expression is greater or smaller than another in value is called an *inequation* or an *inequality*. Some of the important symbols of inequality are as follows:

1. $x \neq y$ for x is not equal to y
2. $x > y$ for x is greater than y
3. $x < y$ for x is less than y
4. $x \geq y$ for x is greater than or equal to y
5. $x \leq y$ for x is less than or equal to y

Some of the important properties of inequality are as follows:

1. An inequality will still hold after each side has been increased, diminished, multiplied or divided by the same positive quantity. For example, if $a > b$ and $c > 0$, then

$$a + c > b + c$$
$$ac > bc$$
$$\frac{a}{c} > \frac{b}{c}$$

2. In an inequality, any term may be transposed from one side to the other if its sign is changed. That is, if $a - c > b$, then $a > b + c$ or $c > b - a$.
3. If sides of an inequality be multiplied by same negative quantity, then sign of inequality must be reversed. That is, if $a > b$ and $c > 0$, then $ac < bc$.
4. If $a > b$ and $a, b \geq 0$, then $a^n > b^n$ and $\frac{1}{a^n} < \frac{1}{b^n}$, or $a^{-n} > b^{-n}$, if n is a positive quantity.
5. Square of every real quantity is positive and therefore must be greater than zero, that is, for $a \neq b$, $(a - b)^2 > 0$; $a^2 + b^2 > 2ab$.
6. If sum of two positive quantities is given, their product is greatest when they are equal; and if product of two positive quantities is given, their sum is least when they are equal.
7. If $a, b, c, \ldots, k$ are n unequal quantities, then

$$\frac{(a+b+c+\cdots+k)^n}{n} > a \times b \times c \times d \times \cdots \times k$$

8. If a and b are positive and unequal, then

$$\frac{a^m b^m}{2} > \left(\frac{a+b}{2}\right)^m$$

except when m is a positive proper function.

If m is a positive integer or any negative quantity, then

$$\frac{a^m b^m}{2} > \left(\frac{a+b}{2}\right)^m$$

If m is positive and less than 1, then

$$\frac{a^m b^m}{2} < \left(\frac{a+b}{2}\right)^m$$

9. If x is positive and $a < b$, then

$$\frac{a+x}{b+x} > \frac{a}{b}$$

If x is positive and $a > b$, then

$$\frac{a+x}{b+x} < \frac{a}{b}$$

10. If x is positive and $a > b > x$, then

$$\frac{a-x}{b-x} > \frac{a}{b}$$

If x is positive and $x < a < b$, then

$$\frac{a-x}{b-x} > \frac{a}{b}$$

11. $a^2 + b^2 + c^2 \geq ab + bc + ac$
12. $a^2 b + b^2 c + c^2 a \geq 3abc$
13. $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$
14. $a^4 + b^4 + c^4 + d^4 \geq 4abcd$

QUADRATIC EQUATIONS

A quadratic equation is any equation having the form $ax^2 + bx + c = 0$, where x represents an unknown and a, b and c are constants with $a \neq 0$. If $a = 0$, then the equation is linear, not quadratic. The constants a, b and c are called, respectively, the quadratic coefficient, the linear coefficient and the constant or free term. The solution of a quadratic equation is given as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EVEN AND ODD NUMBERS

All positive integers divisible by 2 are called even numbers, while all positive integers not divisible by 2 are called odd numbers. For example 2, 4, 6, 8, … are even numbers and 1, 3, 5, 7, … are odd numbers. 0 is considered neither even nor odd.

Some important properties of even and odd numbers are as follows:

1. Even number + Even number = Even number
2. Odd number + Odd number = Even number
3. Even number + Odd number = Odd number
4. Odd number + Even number = Odd number
5. (Even number) × (Even number) = Even number
6. (Even number) × (Odd number) = Even number
7. (Odd number) × (Even number) = Even number
8. (Odd number) × (Odd number) = Odd number
9. $(\text{Even number})^{\text{Even}}$ = Even number
10. $(\text{Even number})^{\text{Odd}}$ = Even number
11. $(\text{Odd number})^{\text{Even}}$ = Odd number
12. $(\text{Odd number})^{\text{Odd}}$ = Odd number

PRIME NUMBERS AND COMPOSITE NUMBERS

Prime numbers are perfectly divisible either by 1 or by the number itself, while composite numbers are divisible by at least one more number other than 1 or by that number itself. For example 2, 3, 5, 7, 11, … are prime numbers while 4, 6, 9, 12, 14, … are composite numbers.

HCF AND LCM OF NUMBERS

HCF (highest common factor) of two or more numbers is the greatest number that perfectly divides each of them. HCF is also called GCD (greatest common divisor).

LCM (lowest common multiple) of two or more numbers is the least or a lowest number that is exactly divisible by each one of them.

CYCLICITY

Cyclicity of a number is used mainly for the calculation of unit digits.

1. Cyclicity of 1

   In 1^n, unit digit will always be 1.
2. Cyclicity of 2
$$2^1 = 2$$
$$2^2 = 4$$
$$2^3 = 8$$
$$2^4 = 16$$
$$2^5 = 32$$
   The unit digits can be 2, 4, 6 or 8 and after every four intervals it repeats.
3. Cyclicity of 3
$$3^1 = 3$$
$$3^2 = 9$$
$$3^3 = 27$$
$$3^4 = 81$$
$$3^5 = 243$$
   The unit digits can be 3, 9, 7 or 1 and after every four intervals it repeats.
4. Cyclicity of 4
$$4^1 = 4$$
$$4^2 = 16$$
$$4^3 = 64$$
   The unit digits can be 4 or 6 and after every two intervals it repeats.
5. Cyclicity of 5
$$5^1 = 5$$
$$5^2 = 25$$
$$5^3 = 125$$
   The unit digit will always be 5.
6. Cyclicity of 6
$$6^1 = 6$$
$$6^2 = 36$$
$$6^3 = 216$$
   The unit digit will always be 6.
7. Cyclicity of 7
$$7^1 = 7$$
$$7^2 = 49$$
$$7^3 = 343$$
$$7^4 = 2401$$
$$7^5 = 16807$$
   The unit digits can be 7, 9, 3 or 1 and after every four intervals it repeats itself.
8. Cyclicity of 8
$$8^1 = 8$$
$$8^2 = 64$$
$$8^3 = 512$$
$$8^4 = 4096$$
$$8^5 = 32768$$
   The unit digits can be 8, 4, 2 or 6 and after every four intervals it repeats itself.
9. Cyclicity of 9
$$9^1 = 9$$
$$9^2 = 81$$
$$9^3 = 729$$
   The unit digits can be 9 or 1 and after every two intervals it repeats itself.

TEST FOR DIVISIBILITY

One whole number is divisible by another if the remainder is zero after dividing. If one whole number is divisible by another number, then the second number is a factor of the first number. Divisibility tests are used to find the factors of large numbers.

Some basic tests for divisibility are as follows:

1. A number is divisible by 2 if the last digit is 0, 2, 4, 6 or 8.
2. A number is divisible by 3 if the sum of the digits is divisible by 3.
3. A number is divisible by 4 if the number formed by the last two digits is divisible by 4.
4. A number is divisible by 5 if the last digit is either 0 or 5.
5. A number is divisible by 6 if it is divisible by 2 and by 3.
6. The test for divisibility for 7 is slightly different. Remove the last digit, double it, subtract it from the truncated original number and continue doing this until only one digit remains. If this is 0 or 7, then the original number is divisible by 7.
7. A number is divisible by 8 if the number formed by the last three digits is divisible by 8.
8. A number is divisible by 9 if the sum of the digits is divisible by 9.
9. A number is divisible by 10 if the last digit is 0.

SOLVED EXAMPLES

1. What will be the last digit number obtained by multiplying the numbers $81 \times 82 \times 83 \times 84 \times 85 \times 86 \times 87 \times 88 \times 89$?

Solution: The last digit will be multiplication of $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$. Since, 5 and 2 are given here, their multiplication will result in zero as last digit.

2. The sum of the digit of a two-digit number is 10, while when the digits are reserved, the number decreases by 54. Find the changed number.

Solution: Let the two-digit number be *xy*. We know that $x + y = 10$. Also, when digits are reserved, the number is decreased by 54. Hence, the changed number can be 19, 28, 31 or 46.

$$91 - 19 = 82$$
$$82 - 28 = 54$$

Hence, the changed number = 28.

3. If GM = 12 and HM = 7.2, then what will be the value of AM?

Solution: We know that

$$\text{GM}^2 = \text{AM} \times \text{HM}$$

We have

$$\text{GM} = 12$$
$$\text{HM} = 7.2$$
$$\text{AM} = \frac{12 \times 12}{7.2} = 20$$

4. What is the least number which on being divided by 12, 15 and 24 will leave in each case a remainder of 5?

Solution: LCM of the three numbers 12, 15 and 124 is 120. Hence, the number that will leave remainder 5 = 120 + 5 = 125.

5. Find the number of divisors of 1420.

Solution: We know that

$$1420 = 142 \times 10 = 71 \times 2 \times 2 \times 5 = 2^2 \times 5^1 \times 71^1$$

So the number of divisors $= (2+1)(1+1)(1+1)$
$= 3 \times 2 \times 2 = 12$

6. What is the HCF of the polynomials $(x^2 - 5x + 6)$ and $(x^2 - 7x + 10)$?

Solution: We have

$$x^2 - 5x + 6 = (x-3)(x-2)$$

and $$x^2 - 7x + 10 = (x-5)(x-2)$$

Therefore,

$$\text{HCF} = (x-2)$$

7. Find the LCM of $(6x^2 + 5x - 4)$ and $(2x^2 + 7x + 3)$.

Solution: We have

$$(6x^2 + 5x - 4) = (2x-1)(3x+4)$$

and $$(2x^2 + 7x + 3) = (2x+1)(x+3)$$

Therefore,

$$\text{LCM} = (4x^2 - 1)(3x+4)(x+3)$$

8. What will be the remainder if 3^7 is divided by 8?

Solution: We know that

$$3^7 = 3^2 \times 3^2 \times 3^2 \times 3$$

When 3^2 is divided by 8, then remainder is 1. Therefore, remainder will be $1 \times 3 = 3$.

9. If $a = y + z$, $b = x + z$, $c = x + y$, then what will be the value of $a^2 + b^2 + c^2 - 2ab - 2bc + 2ca$.

Solution: We know that

$$a^2 + b^2 + c^2 - 2ab - 2bc + 2ca = (a - b - c)^2$$

Substituting values of a, b, c, we get

$$(y + z - x - z - x - y)^2 = (-2x)^2 = 4x^2$$

10. What is the smallest number which when increased by 5 is divisible by 20, 35, 25 and 30?

Solution: LCM of 20, 35, 25 and 30 is 2100. Hence, the number to which when 5 is added becomes divisible by 20, 35, 25 and 30 will be 2100 − 5 = 2095.

11. If $A381$ is divisible by 11, then what is the value of the smallest natural number A?

Solution: $A381$ is divisible by 11 if and only if $(A + 8) - (3 + 1)$ is divisible by 11. So,

$$A + 8 - (3 + 1) = 11$$
$$\Rightarrow A = 11 - 4 = 7$$

12. Find the unit digit of 2^{323}.

Solution: Here, we know that 2, 4, 8, 6 will repeat after every four intervals till 320. Next digit will be 8. So, unit digit of 2^{323} will be 8.

13. Solve the inequality $2x^2 - x < 1$.

Solution: We have

$$2x^2 - x < 1$$

or

$$2x^2 - x - 1 < 0$$

$$(2x+1)(x-1) < 0$$

Hence, either $2x+1 > 0$ and $x-1 < 0$, or $2x+1 < 0$ and $x-1 > 0$. Thus,

$$x > \frac{-1}{2} \text{ and } x < 1$$

or

$$x < \frac{-1}{2} \text{ and } x > 1$$

But $x < \frac{-1}{2}$ and $x > 1$ is not possible. Hence, the solution is $\left(\frac{-1}{2}, 1\right)$.

14. Find the unit digit of $122^{122} \times 133^{133}$.

Solution: Unit digit of 122^{122} will be 4 since cycle of 2 is 2, 4, 8, 6. Unit digit of 133^{133} will be 3 since cycle of 3 is 3, 9, 7 is repeated till 133^{132}.

So, unit digit of $122^{122} \times 133^{133} = 4 \times 3 = 12$ is 2

15. Solve the equation $ix^2 - x + 6i = 0$.

Solution: The equation is

$$ix^2 - x + 6i = 0$$

$$\Rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(i)6(i)}}{2i}$$

$$= \frac{1 \pm \sqrt{1+24}}{2i} = \frac{1 \pm 5}{2i} = \frac{6}{2i}, \frac{-4}{2i}$$

$$x = -3i,\ 2i$$

16. If the polynomial $3y^2 + y^2 + 2y + 5$ is divided by a function to give the quotient $3y - 5$ and remainder $9y + 10$, what is the function?

Solution: Say the function is $f(y)$. Thus,

$$3y^3 + y^2 + 2y + 5 = (3y-5) \times f(y) + (9y+10)$$

$$\Rightarrow 3y^3 + y^2 + 2y + 5 - 9y - 10 = f(y)(3y-5)$$

$$\Rightarrow 3y^3 + y^2 - 7y - 5 = f(y)(3y-5)$$

$$\Rightarrow f(y) = \frac{3y^2 + y^2 - 7y - 5}{(3y-5)}$$

$$= y^2 + 2y + 1$$

Thus, the function is $(y+1)^2$.

17. If $x + y + z = 0$, then what will be the value of $\frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy}$.

Solution: We are given that

$$x + y + z = 0$$

Therefore,

$$x^3 + y^3 + z^3 = 3xyz$$

Also,

$$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = \frac{x^3}{xyz} + \frac{y^3}{xyz} + \frac{z^3}{xyz}$$

$$= \frac{x^3 + y^3 + z^3}{xyz} = 3$$

18. Find the remainder of $\frac{9^{100}}{7}$.

Solution: We know that

$$\frac{9^{100}}{7} \Rightarrow \frac{(7+2)^{100}}{7} \Rightarrow \frac{2^{100}}{7}$$

$$\Rightarrow \frac{2^{99} \times 2}{7} = \frac{(2^3)^{33} \times 2}{7} \Rightarrow \frac{(7+1)^{33} \times 2}{7} = \frac{7 \times 2}{7} = 2$$

19. Is 100858 a perfect square?

Solution: A square of any number never ends up with 8. Hence, 100858 is not a perfect square.

20. Two numbers, x and y, are such that when divided by 6, they leave remainder 4 and 5, respectively. What is the remainder when $x^3 + y^3$ is divided by 6?

Solution: Remainder for $\frac{x}{6} = 4$

Remainder for $\frac{y}{6} = 5$

So the remainder for $\frac{x^3 + y^3}{6} = \frac{(4^3 + 5^3)}{6}$

$$= \frac{(64 + 125)}{6}$$

So the remainder $\left(\frac{4+5}{6}\right) = \text{Remainder}\left(\frac{9}{6}\right) = 3$

21. The remainder when $m + n$ is divided by 12 is 8, and the remainder when $m - n$ is divided by 12 is 6. If $m > n$, then what is the remainder when mn is divided by 6?

Solution: Since the remainder when $m + n$ is divided by 12 is 8, we have

$$m + n = 12p + 8 \quad (1)$$

Also, since the remainder when $m - n$ is divided by 12 is 6, we have

$$m - n = 12q + 6 \quad (2)$$

Here, p and q are integers. Adding Eqs. (1) and (2), we get

$$2m = 12p + 12q + 14$$

$$\Rightarrow m = 6p + 6q + 7 = 6(p + q + 1) + 1 = 6r + 1$$

where r is a positive integer $= p + q + 1$.

Now, subtracting Eq. (2) from Eq. (1), we get

$$2n = (12p + 8) - (12q + 6) = 12(p - q) + 2$$

$$\Rightarrow n = 6(p - q) + 1 = 6t + 1$$

where t is an integer $= p - q$

Hence, we have

$$mn = (6r + 1)(6t + 1) = 36rt + 6r + 6t + 1 =$$

$$6(6rt + r + t) + 1 \text{ by factoring out 6.}$$

Thus, the remainder is 1.

22. Let X be a four digit number with exactly three consecutive digits being same and is a multiple of 9. How many such X's are possible?

Solution: Let the four digit number be '$aaab$' or '$baaa$'.

Since the number has to be a multiple of 9, thus $3a + b$ should either be 9 or 18 or 27.

Case I: $3a + b = 9$

Possible cases are: (1116, 6111, 2223, 3222, 3330, 9000)

Case II: $3a + b = 18$

Possible cases are: (3339, 9333, 4446, 6444, 5553, 3555, 6660)

Case III: $3a + b = 27$

Possible cases are: (6669, 9666, 8883, 3888, 7776, 6777, 9990)

Hence, total number of cases = 20.

23. N is the smallest number that has 5 factors. How many factors does $(N - 1)$ have?

Solution: A number that has 5 factors has to be of the form p^4 where p is a prime number.

The smallest such number is $2^4 = 16$.

Therefore, $(N - 1) = 15$

The factors of 15 are 1, 3, 5 and 15. Hence, $(N - 1)$ has 4 factors.

24. Three gold coins of weight 780g, 840g and 960g are cut into small pieces, all of which have the equal weight. Each piece must be heavy as possible. If one such piece is shared by two persons, then how many persons are needed to give all the pieces of gold coins?

Solution: HCF of 780, 840 and 960 = 60

Thus, the total number of pieces $= \dfrac{780}{60} + \dfrac{840}{60} + \dfrac{960}{60}$ $= 13 + 14 + 16 = 43$

Therefore, total number of persons required $= 43 \times 2 = 86$

25. Convert 11001001_2 to decimal.

Solution: We are given that 11001001_2

The number can be represented in decimal $= 2^0 \times 1 + 2^3 \times 1 + 2^6 \times 1 + 2^7 \times 1 = 1 + 8 + 64 + 128 = 201$

26. Convert 1001.0010_2 to decimal.

Solution: We are given that 1001.0010_2

The number can be represented in decimal $= 2^0 \times 1 + 2^3 \times 1 + 2^{-3} \times 1 = 1 + 8 + \dfrac{1}{8} = 9.125$

27. What is the decimal value of the hexadecimal number 777?

Solution: We are given 777_{16}.

In decimal, the number can be represented as follows:

$7 \times 16^2 + 7 \times 16^1 + 7 \times 16^0 = 1792 + 112 + 7 = 1911$

28. What is the result when a decimal 156 is converted to base 16?

Solution: We are given 156_{10}.

The decimal value can be converted into binary as follows:

$$\begin{array}{r|ll}
2 & 156 & \\
2 & 78 & 0 \\
2 & 39 & 0 \\
2 & 19 & 1 \\
2 & 9 & 1 \\
2 & 4 & 1 \\
2 & 2 & 0 \\
2 & 1 & 0 \\
 & & 1
\end{array}$$

Hence, the binary conversion is 10011100_2.

Now, from the conversion table, we can convert from binary to hexadecimal.

$$\underbrace{1001}_{9}\underbrace{1100}_{C}$$

Hence, the hexadecimal value is 9C.

29. Convert the decimal 151.75 to binary.

Solution: We are given 151.75_{10}.

The decimal value can be converted into binary as follows:

$$\begin{array}{r|ll} 2 & 151 & \\ 2 & 75 & 1 \\ 2 & 37 & 1 \\ 2 & 18 & 1 \\ 2 & 9 & 0 \\ 2 & 4 & 1 \\ 2 & 2 & 0 \\ 2 & 1 & 0 \\ & & 1 \end{array}$$

Also, $0.75 = 2^{-1} \times 1 + 2^{-2} \times 1$

Hence, the binary conversion is 10010111.11_2.

30. Convert 111101000_2 to octal base system.

Solution: We are given that 111101000_2.

Diving them into pairs of 3 and converting using the conversion table, we get

$$\underbrace{111}_{7}\underbrace{101}_{5}\underbrace{000}_{0}$$

Thus, the conversion into octal base system $= 750_8$.

PRACTICE EXERCISE

1. What number should be subtracted from $x^3 + 4x^2 - 7x + 12$ if it is to be perfectly divisible by $x + 3$?

(a) 12 (b) 3 (c) 20 (d) 24

2. The product of four consecutive even numbers is always divisible by which of the following?

(a) 384 (b) 600 (c) 768 (d) 864

3. What is the minimum number of square marbles required to tile a floor of length 5 m 78 cm and width 3 m 74 cm?

(a) 176 (b) 187 (c) 540 (d) 648

4. What is the harmonic mean of two numbers whose geometric mean and arithmetic mean are 6 and 12, respectively?

(a) 2 (b) 3 (c) 5 (d) 9

5. Evaluate $\sqrt{3\sqrt{80} + \dfrac{3}{9 + 4\sqrt{5}}}$.

(a) $\sqrt{3\sqrt{5}}$ (b) 3 (c) $3\sqrt{3}$ (d) $3 + 2\sqrt{5}$

6. If $12x + 6y = 48$, and x and y are natural numbers, then what can be said about y?

(a) Even
(b) Odd
(c) Even or odd
(d) Even when x is even, and odd when x is odd

7. If $3x + 8y = 30$, x and y are natural numbers, then what can be said about x?

(a) Even
(b) Odd
(c) Even or odd
(d) Even when y is even, and odd when y is odd

8. A ball is dropped from a height of 24 m and it rebounds up to half of the distance it falls. If it continues to fall and rebound in this way, how far will it travel before coming to rest?

(a) 36 m (b) 48 m (c) 64 m (d) 72 m

9. What will be the HCF of $y^3 - 4y$ and $4y(y^3 - 8)$?

(a) $y(y + 2)$ (b) 1 (c) $y^2 - 4$ (d) $y - 2$

10. If $a = x + y$, $b = x^2 - y^2$ and $c = xy$, then what will be the value of $\dfrac{b}{ac}$?

(a) $\dfrac{1}{x} - \dfrac{1}{y}$ (b) $\dfrac{xy}{x + y}$ (c) $\dfrac{1}{y} - \dfrac{1}{x}$ (d) $x^2 + y^2$

11. When $x + y + z = 6$ and $xy + yz + zx = 9$, then what will be the value of $x^3 - y^3 - z^3 - 3xyz$?

(a) 54 (b) 15 (c) 82 (d) 38

12. Solve the equality $\dfrac{x - 1}{x + 1} \geq 1$.

(a) $(1, \infty)$ (b) $(-1, \infty)$
(c) $(-\infty, 1)$ (d) $(-\infty, -1)$

13. If $\dfrac{x - 4}{x + 6} \geq 2$, then which of the following is correct?

(a) $x \leq -16$ (b) $x \geq -16$
(c) $x \leq 8$ (d) $x \geq -8$

14. If a, b, c, d, x and y are non-zero, unequal integers and $\dfrac{a + bi}{c + di} = \dfrac{p}{q}$, then what will be the value of $\dfrac{a^2 + b^2}{c^2 + d^2}$?

(a) p/q (b) p^2/q (c) p^2/q^2 (d) 1

15. Find the unit's digit in $264^{102} + 264^{103}$.

(a) 0 (b) 2 (c) 4 (d) 6

16. What is the minimum number of square marbles required to tile a floor of length 3 m 84 cm and width 2 m 88 cm?

(a) 20 (b) 36 (c) 12 (d) 24

17. Average marks of 15 students in a class is 145, maximum marks being 150. If two lowest scores are removed, the average increases by 5. Also, two lowest scores are consecutively multiples of 9. What is the lowest score in the class?

(a) 108 (b) 100 (c) 118 (d) 96

18. A number when divided by a divisor leaves a remainder of 22. When twice the original number is divided by the same divisor, the remainder is 9. What is the value of the divisor?

(a) 14 (b) 30 (c) 27 (d) 35

19. x and y are integers and if x^2/y^3 is an even integer, then which of the following must be an even integer?

(a) $x-y$ (b) $y-1$ (c) x^2/y^4 (d) xy

20. Find the greatest number of four digits which when divided by 15, 20, 28 leaves in each case a remainder 2?

(a) 9077 (b) 9662 (c) 1090 (d) 4660

21. What is the remainder when $7^2 \times 8^2$ is divided by 6?

(a) 4 (b) 2 (c) 8 (d) 6

22. The LCM of two numbers is 12 times their HCF. The sum of HCF and LCM is 403. If one number is 93, then find the other.

(a) 134 (b) 130 (c) 128 (d) 124

23. For a number to be divisible by 88, it should be

(a) divisible by 22 and 8
(b) divisible by 11 and 8
(c) divisible by 11 and thrice by 2
(d) All of the above

24. The LCM of $(16-x^2)$ and (x^2+x-6) is

(a) $(x-3)(x+3)(4-x^2)$ (b) $4(4-x^2)(x+3)$
(c) $(16-x^2)(x^2+x-6)$ (d) $(4-x^2)(x-3)$

25. What is the remainder of $\dfrac{3^{250}}{7}$?

(a) 1 (b) 2 (c) 4 (d) 6

26. When a four digit number is divided by 85 it leaves a remainder of 39. If the same number is divided by 17, the remainder would be?

(a) 3 (b) 5 (c) 7 (d) 9

27. If $381A$ is divisible by 9, find the value of the smallest natural number A.

(a) 5 (b) 7 (c) 6 (d) 9

28. Let $N = 1421 \times 1423 \times 1425$. What is the remainder when N is divided by 12?

(a) 0 (b) 9 (c) 3 (d) 6

29. Find the unit digit of $17^{17} \times 27^{27}$.

(a) 7 (b) 5 (c) 3 (d) 1

30. What is the greatest number of four digits that when divided by any of the numbers 6, 9, 12, 17 leave a remainder of 1?

(a) 9997 (b) 9487 (c) 9895 (d) 9793

31. Convert 1100101_2 into octal base system.

(a) 145_8 (b) 340_8 (c) 257_8 (d) 150_8

32. Convert 1027_8 into decimal base system.

(a) 535 (b) 342 (c) 255 (d) 555

33. What is the binary equivalent of 192_{10}?

(a) 11001100 (b) 11000000
(c) 10100000 (d) 11000010

34. Convert 7431_8 into hexadecimal.

(a) D90 (b) 119 (c) F19 (d) A74

35. What is the decimal equivalent of 921_{16}?

(a) 1234 (b) 4991 (c) 3433 (d) 2337

ANSWERS

1. (d)	**5.** (c)	**9.** (a)	**13.** (b)	**17.** (a)	**21.** (a)	**25.** (c)	**29.** (d)	**33.** (b)
2. (a)	**6.** (c)	**10.** (c)	**14.** (c)	**18.** (d)	**22.** (d)	**26.** (b)	**30.** (d)	**34.** (c)
3. (b)	**7.** (a)	**11.** (a)	**15.** (a)	**19.** (d)	**23.** (b)	**27.** (c)	**31.** (a)	**35.** (d)
4. (b)	**8.** (d)	**12.** (d)	**16.** (c)	**20.** (b)	**24.** (c)	**28.** (c)	**32.** (a)	

EXPLANATIONS AND HINTS

1. (d) To find the number to be subtracted from $x^3+4x^2-7x+12$ to make it divisible by $x + 3$, we need to divide the two and find the remainder.

$$\begin{array}{r|l} & x^2+x-4 \\ x+3 & x^3+4x^2-7x+12 \\ & x^3+3x^2 \\ & - \\ \hline & x^2-7x \\ & x^2-3x \\ & + \\ \hline & -4x+12 \\ & -4x-12 \\ & + \\ \hline & 24 \end{array}$$

Therefore, the number to be subtracted is 24.

2. (a) The product of four consecutive numbers is always divisible by $4!$. Now, since we have four even numbers, we have an additional 2 available with each number. It is always divisible by

$$(2^4)4! = 16(24) = 384$$

3. (b) The marbles are square marbles. The length and width of the floor are 5 m 78 cm and 3 m 74 cm.

The HCF of 578 and 374 = 34

Hence, the side of the square = 34

The number of such square marbles required will be

$$\frac{578\times 374}{34} = 17\times 11 = 187$$

4. (b) We know that

$$\text{GM}^2 = \text{AM}\times\text{HM}$$

We have GM = 6 and AM = 12, hence

$$\text{HM} = \frac{6\times 6}{12} = 3$$

5. (c) We have

$$\sqrt{3\sqrt{80}+\frac{3}{9+4\sqrt{5}}}$$

$$= \sqrt{3\sqrt{80}+\frac{3}{9+4\sqrt{5}}\times\left[\frac{9-4\sqrt{5}}{9-4\sqrt{5}}\right]}$$

$$= \sqrt{3\sqrt{80}+\frac{(3\times 9-3\times 34\sqrt{5})}{9^2-(4\sqrt{5})^2}}$$

$$= \sqrt{3\sqrt{80}+\frac{27-12\sqrt{5}}{81-80}} = \sqrt{3\sqrt{16\times 5}+27-12\sqrt{5}}$$

$$= \sqrt{3\times 4\times\sqrt{5}+27-12\sqrt{5}} = \sqrt{12\sqrt{5}+27-12\sqrt{8}}$$

$$= \sqrt{27} = 3\sqrt{3}$$

6. (c) We know that

$$12x + 6y = 48$$

Now, all three terms are even. If $6y$ = even, then y can be even or odd since any number multiplied with an even number is even. Hence, y can be even or odd.

7. (a) We know that

$$3x + 8y = 30$$

$3x$ is even and $8y$ is also even. Hence, $3x$ has to be even since addition or subtraction of two even numbers is always even. Hence, x is always even.

8. (d) Total distance travelled by ball = 24 + 2(12 + 6 + 3 + 1.5 + …)

This is an infinite series, hence total distance will be

$$24+2\left[\frac{12}{1-(1/2)}\right]$$

$$= 24 + 4(12) = 24 + 48 = 72 \text{ m}$$

9. (a) We have

$$y^3-4y = y(y^2-4) = y(y+2)(y-2)$$

and $\quad 4y(y^3-8) = 4y(y+2)(y^2-2y+4)$

HCF $= y(y+2)$

10. (c) Given that $a = x+y$, $b = x^2-y^2$ and $c = xy$

$$\frac{b}{ac} = \frac{x^2-y^2}{xy(x+y)} = \frac{(x+y)(x-y)}{xy(x-y)}$$

$$= \frac{x-y}{xy} = \frac{1}{y}-\frac{1}{x}$$

11. (a) We are given that

$$x+y+z=6$$

$$xy+yz+zx=9$$

We know that

$$x^3-y^3-z^3-3xyz = (x+y+z)(x^2+y^2 +z^2-xy-yz-zx)$$

$$= 6[(x+y+z)^2-3(xy+yz+zx)]$$

$$= 6[36-3(9)] = 6[36-27] = 6\times 9 = 54$$

12. (d) We have

$$\frac{x-1}{x+1} \geq 1$$

First, we note that $x+1 \neq 0$, that is, $x \neq -1$ and for $x \neq -1$, $(x+1)^2 > 0$. Multiplying both sides by $(x+1)^2$, we get

$$(x-1)(x+1) \geq (x+1)^2$$
$$= x^2 - 1 \geq x^2 + 2x + 1$$
$$= -2x \geq 2$$
$$= x \leq -1; \quad \text{but } x \neq -1$$

Thus, the solution of the equation is $x \in (-\infty, -1)$.

13. (b) We have

$$\frac{x-4}{x+6} \leq 2$$
$$\Rightarrow x - 4 \leq 2x + 12$$
$$\Rightarrow x \geq -16$$

14. (c) We are given that

$$\frac{a+bi}{c+di} = \frac{p}{q}$$
$$\Rightarrow qa + qbi = pc + pdi$$

Equating real and imaginary parts, we get

$$qa = pc$$

and $$qb = pd$$

$$a = \frac{p}{q} \cdot c \text{ and } b = \frac{p}{q} \cdot d$$

Therefore,

$$\frac{a^2 + b^2}{c^2 + d^2} = \frac{(p^2 / q^2) \cdot c^2 + (p^2 / q^2) \cdot d^2}{c^2 + d^2} = \frac{p^2}{q^2}$$

15. (a) Required unit's digit = unit's digit in $4^{102} + 4^{103}$

Now, 4^2 gives unit digit 6 and 4^{102} gives unit digit 6.

4^{103} gives unit digit of 4.b

Hence, unit digit in $264^{102} + 264^{103}$ = unit's digit in (6 + 4) = 0

16. (c) The marbles are square marbles. Side of a square should be a factor to both 3 m 84 cm and 2 m 88 cm.

HCF of 384 and 288 = 96

Hence the number of square marbles required

$$\frac{384 \times 288}{96 \times 96} = 4 \times 3 = 12 \text{ marbles}$$

17. (a) Given that average marks of 15 students = 145

So the total marks = $145 \times 15 = 2175$

When two lowest marks are removed then average = 150

So the total marks = $150 \times 13 = 1950$

Say the two lowest scores are $9x + 9x + 9$. Therefore,

$$9x + 9x + 9 = 2175 - 1950 = 225$$
$$\Rightarrow 18x = 216$$
$$\Rightarrow x = 12$$

Thus, lowest score = $9 \times 12 = 108$

18. (d) Let the original number be a and the divisor be d. Let the quotient of the divisor "a" by "d" be "x." Therefore, $\frac{a}{d} = x$ and remainder = 22. Thus,

$$a = dx + 22$$
$$\Rightarrow 2a = 2dx + 44$$

Also, given that $\left(\frac{2dx + 44}{d}\right)$ leaves a remainder of 9.

"$2dx$" is perfectly divisible by "d" and will not leave a remainder. The remainder of 9 was obtained by 44 by d. When 44 is divided by 35, the remainder is 9. Hence, the divisor is 35.

19. (d) If $\frac{x^2}{y^3}$ = even

$x^2 = y^3$ even

Hence x^2 = even

and x is an integer

So x = even

So, only xy must be even.

20. (b) LCM of 15, 20, 28 is 420. So, the greatest four digit number divisible by 420 is 9660. Hence, required number is 9660 + 2 = 9662.

21. (a) We have

$$(7^2 \times 8^2) = (7 \times 8)^2 = 56^2$$

The number immediately before 56 that is divisible by 6 is 54. Now, writing 56^2 as $(54+2)^2$, we have

$$56^2 = (54+2)^2$$
$$= 54^2 + 2^2 + 2(2)54$$
$$= 54[54 + 2(2)] + 2^2$$
$$= 6 \times 9[54 + (2 \times 2)] + 4$$

Here, the remainder is 4.

22. (d) We know that

$$\text{LCM} = 12 \times \text{HCF}$$

Also, $$\text{LCM} + \text{HCF} = 403$$

$\Rightarrow$ 13 HCF = 403 $\Rightarrow$ HCF = 31

Thus,

$$\text{LCM} = 372$$

Also,

$$\text{HCF} \times \text{LCM} = x \times 93$$

$$\Rightarrow 31 \times 372 = 93x$$
$$\Rightarrow x = 124$$

23. (b) For a number to be divisible by 88, it should be divisible by 8 and 11 because 8 and 11 are co-prime numbers whose multiplication gives 88.

24. (c) We have $16 - x^2 = (4-x)(4+x)$

and $\quad x^2 + x - 6 = (x+3)(x-2)$

LCM will be $(16 - x^2)(x^2 + x - 6)$.

25. (c) We have to find the remainder of

$$\frac{(3^2)^{125}}{7}$$
$$= \frac{(7+2)^{125}}{7} = \frac{2^{125}}{7}$$
$$\Rightarrow \frac{(2^3)^{41} \times 2^2}{7} \Rightarrow \frac{1 \times 4}{7}$$

Hence, remainder is 4.

26. (b) We have

$$n = 85k + 39$$

Now,

$$\frac{n}{17} = \frac{85k+39}{17} = 5k + \frac{34+5}{17} = 5k + 2 + \frac{5}{17}$$

So the remainder is 5.

27. (c) Given that $381A$ is divisible by 9.

So, $3 + 8 + 1 + A = 12 + A$ is divisible by 9.

Thus, $A = 6$

28. (c) Remainder of $\left(\frac{1421 \times 1423 \times 1425}{12}\right) \Rightarrow$ Remainder of $\frac{(5 \times 7 \times 9)}{12}$ = Remainder $\frac{(35 \times 9)}{12}$

Remainder of $\frac{(11 \times 9)}{12}$ = Remainder $\frac{99}{12} = 3$

29. (d) Unit digit of 17^{17} is 7 and unit digit of 27^{27} is 3. So unit digit of $17^{17} \times 27^{27}$ will be $7 \times 3 = 21 \Rightarrow 1$

30. (d) LCM of 6, 9, 12, 17 = 612

The greatest number of four digits divisible by 612 is 9792.

To get remainder of 1, number should be 9792 + 1 = 9793

31. (a) We are given the binary integer, 1100101.

To convert it into octal, we need to be able to form groups of 3.

Hence, we rewrite the integer as 001100101.

$$\Rightarrow \underbrace{001}_{1}\underbrace{100}_{4}\underbrace{101}_{5}$$

Hence, the number in octal base system is 145_8.

32. (a) We are given the octal integer, 1027_8.

To convert it into decimal, we have

$$1 \times 8^3 + 2 \times 8 + 7 = 535$$

33. (b) Using the long division method, we have

$$\begin{array}{r|ll} 2 & 192 & \\ 2 & 96 & 0 \\ 2 & 48 & 0 \\ 2 & 24 & 0 \\ 2 & 12 & 0 \\ 2 & 6 & 0 \\ 2 & 3 & 0 \\ 2 & 1 & 1 \\ & & 1 \end{array}$$

Hence, the binary equivalent of 192 is 11000000.

34. (c) We are given an octal number, 7431_8.

Converting it into binary, we get

$$\underbrace{7}_{111}\ \underbrace{4}_{100}\ \underbrace{3}_{011}\ \underbrace{1}_{001}$$

Hence, the binary equivalent to the number = 111100011001.

Now, making groups of 4 and converting into hexadecimal, we get

$$\underbrace{1111}_{F}\underbrace{0001}_{1}\underbrace{1001}_{9}$$

Hence, the number in hexadecimal is $F19_{16}$.

35. (d) We are given the hexadecimal number, 921_{16}.

To convert it into decimal, we have

$$9 \times 16^2 + 2 \times 16 + 1 = 2337$$

Hence, the number in decimal is 2337.

CHAPTER 2

PERCENTAGE

INTRODUCTION

The term percentage is often used to avoid the fractions less than 1. Instead of treating the complete entity as 1, we treat it as 100 and take the ratios accordingly.

The term *percent* means for every hundred. A fraction of any quantity whose denominator is 100 is called percentage of that quantity, and numerator of the fraction is called rate percent. It is denoted by the symbol %.

SOME IMPORTANT FORMULAE

List of some of the common formulae used are as follows:

1. **To increase the number by a given percentage:** Say the number is x and the percentage is y, then the increased number z will be

$$z = x \times \frac{(100 + y)}{100}$$

2. **To decrease the number by a given percentage:** Say the number is x and the percentage is y, then the decreased number z will be

$$z = x \times \frac{(100 - y)}{100}$$

3. **To find the percent increase of a number:** Say the final value is x and the initial value is y, then the percent increase z will be

$$z = \frac{x - y}{y} \times 100$$

4. **To find the percent decrease of a number:** Say the final value is x and the initial value is y, then the percent decrease z will be

$$z = \frac{y - x}{y} \times 100$$

SOLVED EXAMPLES

1. A quantity $x = 240$ is increased by 50%. What is the new value?

Solution: It is given that $x = 240$, so the new value of x will be

$$240 + \frac{50}{100} \times 240 = 360$$

2. A quantity $x = 100$ is first increased by 20% and then the resultant is decreased by 10%. What is the final value?

Solution: It is given that $x = 100$, so the value of x after 20% increase will be

$$100 + \frac{20}{100} \times 100 = 120$$

Therefore, the value of x after 10% decrease will be

$$120 - \frac{10}{100} \times 120 = 108$$

3. If the value of x increases from 400 to 540, then what is the percent increase?

Solution: It is given that the initial value of $x = 400$ and the new value of $x = 540$. So the total change in value $= 540 - 400 = 140$. Therefore, the total percent increase will be

$$\frac{140}{400} \times 100 = 35\%$$

4. Gaurav scored 64, 80 and 70 in three out of his four exams. What should be his score in the fourth exam so that his overall average is 73?

Solution: It is given that the total marks of Gaurav in the three subjects $= 64 + 80 + 70 = 214$. Now, if marks in the fourth subject are x, then

$$\frac{214 + x}{4} = 73$$

$$\Rightarrow x = 73 \times 4 - 214 = 292 - 214$$

$$\Rightarrow x = 78$$

Hence, Gaurav must get 78 marks to get an overall average of 73.

5. 1200 boys and 800 girls are examined in a test; 55% of the boys and 45% of the girls fail. What percent of the total students passed?

Solution: Total number of boys who failed is

$$\frac{55}{100} \times 1200 = 660$$

Total number of girls who failed is

$$\frac{45}{100} \times 800 = 360$$

So the total number of students who failed is 1020 and the total number of students who passed is 980. It is given that total students = 2000. So the total percentage of the students who passed will be

$$\frac{980}{2000} \times 100 = 49\%$$

6. For a square with side of length 4 cm, the numerical value of perimeter is what percent of the numerical value of its area?

Solution: Perimeter of the square $= 4 \times 4 = 16$ cm and the area of the square $= 4^2 = 16\text{ cm}^2$

Now, since the numerical value of the area and perimeter is the same. Perimeter is 100% of the area of the square.

7. A quantity X is 50% more than Y. What percent is Y less than X?

Solution: We have two quantities X and Y and X is 50% more than Y. So

$$X = 1.5Y$$

$$= \frac{3}{2}Y$$

Also,

$$Y = \frac{2}{3}X$$

Therefore, the total percentage of Y of X is

$$\frac{(2/3)X}{X} \times 100 = 66.67\%$$

Hence, the total decrease of quantity Y to $X = 33.33\%$.

8. If the price of wheat rises by 5%, then by what percent must Raj reduce his consumption so as not to increase his expenditure?

Solution: Percent reduction in consumption is

$$\left(\frac{5}{100 + 5} \times 100\right) = \frac{1}{21} \times 100 = 4.762\%$$

9. The price of soda increases from ₹10 to ₹12. If it then decreases by double the total percent increase then what is the new price of the soda?

Solution: Percent increase of soda is

$$\frac{2}{10} \times 100 = 20\%$$

Now, the percent decrease is twice the percent increase. Hence, percent decrease = 40% of the new value.

So the final price of the soda will be

$$12 - \frac{40}{100} \times 12 = 12 - 4.8 = 7.2$$

Thus, the final price of soda is ₹ 7.2.

10. If the radius of a sphere is increased by 20%, then what is the percent change in the value of its area?

Solution: If the radius of the sphere initially is r, then the area of the sphere $= \pi r^2$
Now, the new radius is

$$r + \frac{20}{100} \times r = 1.2r$$

So the new area $= \pi(1.2r)^2 = 1.44r^2\pi$

Now, the percent change in value of the area is

$$\frac{0.44\pi r^2}{\pi r^2} \times 100 = 44\%$$

11. A salesman sells two kinds of trousers: cotton and woollen. A pair of cotton trousers is sold at 30% profit and a pair of woollen trousers is sold at 50% profit. The salesman has calculated that if he sells 100% more woollen trousers than cotton trousers, his overall profit will be 45%. However, he ends up selling 50% more cotton trousers than woollen trousers. What will be his overall profit?

Solution: Let the cost price of 1 cotton trouser and 1 woollen trouser be x and y, respectively.
Case I: Number of woollen trousers sold is 100% more than cotton trousers.

$$1.3x + 1.5 \times 2 \times y = 1.45(x + 2y)$$
$$\Rightarrow 0.15x = 0.1y$$
$$\Rightarrow 3x = 2y$$

Case II: Number of cotton trousers sold is 50% more than woollen trousers.

Selling price $= 1.3x + 1.5 \times \frac{2y}{3}$

$\Rightarrow$ Selling price $= 1.3x + 1.5x = 2.8x$

$\Rightarrow$ Cost price $= x + \left(\frac{2}{3}\right)y = 2x$

$$\text{Profit} = \left(\frac{2.8x - 2x}{2x}\right) \times 100 = 40\%$$

12. A candidate who gets 20% marks fails by 10 marks but another candidate who gets 42% marks gets 12% more than the passing marks. What are the maximum marks?

Solution: Total pass percentage = 42% − 12% = 30%
Let maximum marks be x.

30% of x − 20% of x = 10

10% of x = 10

$$\frac{10}{100} \times x = 10 \Rightarrow x = 100$$

Thus, maximum marks = 100

13. Two persons Raj and Ram started working for a company in similar jobs on January 1, 1991. Raj's initial monthly salary was ₹ 400, which increases by ₹ 40 after every year. Ram's initial monthly salary was ₹ 500 which increases by ₹ 20 after every six months. If these arrangements continue till December 31, 2000, find the total salary they received during that period.

Solution: Raj's salary on January 1, 1991 is ₹ 400 per month.
Total increment = ₹ 40 per annum.

We can use the formula of sum of arithmetic progression in order to calculate the salary in ten years.

$$12[2(400) + (10 - 1)40] \times \left(\frac{10}{2}\right) = ₹\ 69600$$

Ram's salary as on January 1, 1991 is ₹ 550 and his half yearly increment in his month salary is ₹ 20.

His total salary from January 1, 1991 to December 31, 2000 is given by

$$6[2(500) + (20 - 1)20] \times \left(\frac{20}{2}\right) = ₹\ 82800$$

Total salary of Raj and Ram in the ten-year period = 69600 + 82800 = ₹ 152400

14. The volume of the sphere Q is $\frac{37}{64}\%$ less than the volume of sphere P and the volume of sphere R is $\frac{19}{27}\%$ less than that of sphere Q. By what percent is the surface area of sphere R less than the surface area of sphere P?

Solution: Let the volume of sphere P be x.

Therefore, volume of sphere Q $= x - \frac{37}{64}x = \frac{27}{64}x$

The volume of R $= \frac{27}{64}x - \left(\frac{19}{27}\right)\left(\frac{27}{64}x\right) = \frac{1}{8}x$

Ratio of volume of P:Q:R $= x : \frac{27}{64}x : \frac{1}{8}x \Rightarrow 64:27:8$

Ratio of radius of P:Q:R $= \sqrt[3]{64} : \sqrt[3]{27} : \sqrt[3]{8} = 4:3:2$

The surface area of P:Q:R will be in the ratio of $16:9:4$

Surface area of R is less than the surface area of P, therefore

$16y - 4y = 12y$ where y is a constant.

Now, $\frac{12y}{16y} \times 100 = 75\%$

Thus, surface area of sphere R is less than the surface area of sphere P by 75%.

15. A textile manufacturing firm employees 50 looms. It makes fabrics for a branded company. The aggregate sales value of the output of the 50 looms is ₹ 500000 and the monthly manufacturing expense is ₹ 150000. Assume that each loom contributes equally to the sales and manufacturing expenses. Monthly establishment charges are ₹ 75000. If one loom breaks down and remains idle for one month, what is the decrease in profit?

Solution: Total profit $= 500000 - (150000 + 75000)$ $=$ ₹ 275000

Since, such loom contributes equally to sales and manufacturing expenses. But the monthly charges are fixed at ₹ 75000.

If one loom breaks down, sales and expenses will decrease.

New profit $= \left[500000 \times \left(\frac{49}{50}\right)\right] - \left[150000 \times \left(\frac{49}{50}\right)\right] - 75000 =$ ₹ 268000

Decrease in profit $= 275000 - 268000 =$ ₹ 7000

16. An electronic company has 14 machines of equal efficiency in its factory. The annual manufacturing costs are ₹ 42000 and establishment charges are ₹ 12000. The annual output of the company is ₹ 70000. The annual output and manufacturing costs are directly proportional to the number of machines. The shareholders get 12.5% profit, which is directly proportional to the annual output of the company. If 7.14% machines remain closed throughout the year, then what would be the percentage decrease in the amount of profit of the shareholders?

Solution: Initial profit $= 70000 - 42000 - 12000 = 16000$

If one of the machines is closed throughout the year, then change in profit $= \left(\frac{13}{14}\right) \times (70000 - 42000) = 14000$

Thus, decrease in profit % $= \left(\frac{16000 - 14000}{16000}\right) \times 100 = \left(\frac{2000}{16000}\right) \times 100 = 12.5\%$

17. The cost of raw material of a product increases by 30%, the manufacturing cost increases by 20% and the selling price of the product increases by 60%. The raw material and the manufacturing cost, originally, formed 40% and 60% of the total cost, respectively. If the original profit % was one-fourth the original manufacturing cost, then what is the approximation new profit percentage?

Solution: Let total initial cost of the product be x.
Manufacturing cost $= 0.6x$

Raw materials cost $= 0.4x$

Also, original selling price $= x + \frac{0.6x}{4} = 1.15x$

New raw material cost $= 0.4x + \left(\frac{30}{100}\right) \times 0.4x = 0.52x$

New manufacturing cost $= 0.6x + \left(\frac{20}{100}\right) \times 0.6x = 0.72x$

New cost of the product $= 0.52x + 0.72x = 1.24x$

New selling price $= 1.15x + \left(\frac{60}{100}\right)(1.15x) = 1.84x$

New profit percentage $= \frac{0.6x}{1.24x} \times 100 = 48.39\%$

18. The marks scored in History by P, Q, R and S form a geometric progression in that order. If the marks scored by R were $\frac{275}{9}\%$ less than the sum of the marks scored by P and Q, then marks scored by S were what percent more than the marks scored by Q? (Assuming nobody scored negative marks.)

Solution: Let the marks scored by P, Q, R and S be a, ax, ax^2 and ax^3, respectively.

We are given that marks scored by R were $\frac{275}{9}\%$ less than the marks scored by P and Q.

So, if marks scored by P and Q were 36 then those scored by R is 25.

Marks scored by P and Q $= a + ax = 36$

Marks obtained by R $= ax^2 = 25$

$$\frac{(a + ax)}{ax^2} = \frac{36}{25} \Rightarrow \frac{1 + x}{x^2} = \frac{36}{25} \Rightarrow 36x^2 - 25x - 25 = 0$$

Solving the quadratic equation, we get $x = \frac{5}{4}$.

Hence, P $= a$, Q $= \frac{5a}{4}$, R $= \frac{25a}{16}$ and S $= \frac{125a}{64}$.

Now, percentage difference between S and Q =

$$\left(\frac{125a/64 - 5a/4}{5a/4}\right)\times 100 = \left(\frac{45}{80}\right)\times 100 = 56.25\%$$

Thus, S scored 56.25% more than Q.

19. Mannan starts a month with provisions expected to last for the entire month. After few days, it is discovered that the provisions will, in fact, be shortened by 12 days and it is calculated that if the stock of provisions left is immediately tripled, it will be possible to exactly make up for the shortfall. If the stock of provisions left is doubled instead of being tripled and simultaneously the strength of Mannan is decreased by 25%, then by how many days will the provisions fall short?

Solution: At the moment the shortfall is discovered, let there be x days of provision left.

Now, $3x - x = 2x$ extra days of provisions last for the 12 additional days.

$$\Rightarrow 3x \text{ lasts for 18 days.}$$

But if the provisions are only doubled and the strength becomes $\left(\frac{3}{4}\right)^{\text{th}}$, then the provisions will last for

$$12\times\frac{4}{3} = 16 \text{ days.}$$

Thus, the provisions will fall short by $18 - 16 = 2$ days.

20. In the period from January to March, an electronics company sold 3150 units of television, having started with a beginning inventory of 2520 units and ending with an inventory of 2880 units. What was the value of order placed by the company during the three-month period? (Profits are 25% of cost price, uniformly.)

Solution: Units ordered = Units sold + Ending inventory − Beginning inventory

$$\Rightarrow 3150 + 2880 - 2520 = 3510$$

Total sales of television = 900 + 1800 + 6300 + 1050 + 2100 + 7350 + 1200 + 2400 + 8400 = ₹ 31500

Sales price per unit of television = $\left(\frac{31500}{3150}\right) = 10$

Profits are 25% of the cost price.

Sales price = Cost price + profits = Cost profit + $0.25\times$ Cost price = $1.25\times$ Cost price

Cost price per unit of television = $\frac{10}{1.25} = 8$

The value of the order placed = $3510\times 8 = 28080$

21. A test has 20 questions. If Peter gets 80% correct, how many questions did Peter missed?

Solution: The number of correct answers is 80% of 20 or $\frac{80}{100}\times 20 = 16$.

Since the test has 20 questions and he got 16 answers correct, the number of questions he missed is

$$20 - 16 = 4.$$

Thus, Peter missed 4 questions.

22. 24 students in a class took an algebra test. If 18 students passed the test, what percent do not pass?

Solution: Number of students who did pass = 24 − 16 = 6

Percent of people who did not pass = $\frac{6}{24}\times 100 = 25\%$

Therefore, 25% of students did not pass.

23. Rajeev buys good worth ₹ 6650. He gets a rebate of 6% on it. After getting the rebate, he pays sales tax @ 10%. Find the amount he will have to pay for the goods.

Solution: Rebate = $\frac{6}{100}\times 6650$ = ₹ 399

Sales tax = $\frac{10}{100}\times(6650 - 399) = \left(\frac{10}{100}\times 6251\right) =$ ₹ 625.1

Thus, final amount = ₹ (6251 + 625.10) = ₹ 6876.10

24. The population of a town increased from 175000 to 262500 in a decade. What is the average percent increase of population per year?

Solution: Increase in 10 years = (262500 − 175000) = 87500

Increase % = $\left(\frac{87500}{175000}\times 100\right)\% = 50\%$

Thus, required average = $\left(\frac{50}{10}\right) = 5\%$

25. Three candidates contested an election and received 1136, 7636 and 11628 votes, respectively. What percentage of the total votes did the winning candidate get?

Solution: Total number of votes pulled = (1136 + 7636 + 11628) = 20400

Therefore, required percentage = $\left(\frac{11628}{20400}\right)\times 100 = 57\%$

PRACTICE EXERCISE

1. Adam's scores in five math tests are 99, 85, 88, 94 and 93. What does he need to score in the next test for his final average in all the tests to be at least 92?

(a) 90 (b) 91
(c) 92 (d) 93

2. At a football game with 5400 fans, 60% of the fans are rooting for the home team. If 810 of the home team fans are students, what percent of the home team fans are not students?

(a) 75 (b) 25
(c) 80 (d) 70

3. A batsman scored 120 runs which included 3 boundaries and 8 sixes. What percent of his total score did he make by running between the wickets?

(a) 45.5 (b) 55
(c) 50 (d) 62

4. In a certain year, California produced 70% and South Carolina produced 10% of all fresh peach crops in the United States. If all the other states combined, produce 240 million pounds of fresh peach crop that year, how many million pounds of peach crop did South Carolina produce?

(a) 240 million pounds (b) 120 million pounds
(c) 100 million pounds (d) 360 million pounds

5. A scientist is studying the change in the population of tigers for a certain area. She observes a 25% increase in the population of tigers for that area. If the new population is 45 tigers, what was the previous population?

(a) 40 (b) 36
(c) 30 (d) 28

6. 1100 boys and 700 girls are examined in a test; 42% of the boys and 30% of the girls pass. What percentage of the total students failed?

(a) 62.67 (b) 58
(c) 55 (d) 37.33

7. The price of gasoline at a certain station has been increased from ₹ 72 to ₹ 76. If it is decreased by half the percent of the percent increase, what is the new price?

(a) 74.12 (b) 72.68
(c) 74 (d) 73.89

8. 10% of the voters did not cast their vote in an election between two candidates. 10% of the votes polled were found invalid. The successful candidate got 54% of the valid votes and won by a majority of 1620 votes. What was the total number of voters enrolled on the voters list?

(a) 32000 (b) 35000
(c) 25000 (d) 23000

9. While purchasing item A costing ₹ 400, Jack had to pay the sales tax at 8% and on item B costing ₹ 6400, the sales tax was 10%. What percent of the sales tax Jack had to pay, taking the two items together on an average?

(a) 7.65% (b) 9.88%
(c) 11.12% (d) 8.32%

10. A student has to obtain 33% of the total marks to pass. He got 125 marks and failed by 40 marks. What are the maximum marks a student can get?

(a) 1000 (b) 300
(c) 500 (d) 800

11. Difference of two numbers is 1400. If 10% of the larger number is equal to 15% of the smaller number, then what are the two numbers?

(a) 2100, 1400 (b) 4200, 2800
(c) 8400, 5600 (d) 5600, 4200

12. For a sphere of radius 20 cm, the numerical value of surface area is what percent of the numerical value of its volume?

(a) 33.3% (b) 15%
(c) 30% (d) 20%

13. What is the value of (x% of y) + (y% of x)?

(a) 20% of x/y (b) 2% of x/y
(c) 2% of xy (d) 20% of xy

14. At an election, where there are only two candidates, a candidate who gets 40% of the votes is rejected by a majority of 640 votes. What is the total number of votes recorded assuming that there was no void vote?

(a) 3200 (b) 2800
(c) 6400 (d) 4000

15. If the price of onion rises by 25%, then by what percent Amita must reduce her consumption of onions so as not to increase her expenditure?

(a) 22.5% (b) 25%
(c) 50/3% (d) 20%

16. A quantity X is 25% more than Y. What percent is Y less than X?

(a) 15% (b) 17.5%
(c) 20% (d) 25%

17. A quantity A is increased successively by 20%, 30%, 50%, 20% to give X and 20%, 50%, 30%, 20% to give Y. What percent is X of Y?

(a) 0% (b) 50%
(c) 100% (d) 200%

18. A quantity X is increased by 20% and the resultant is then increased by 30%. The same quantity is then first increased by 30% and the resultant is then increased by 20%. Say the results are Y_1 and Y_2 respectively. What percent is Y_1 of Y_2?

(a) 100% (b) 200%
(c) 0% (d) 50%

19. The length and breadth of a room in the shape of a cuboid is increased by 10% each and the height is decreased by 20%. What is the percent change in the volume of the cuboid?

(a) 0% (b) 3.2%
(c) 20% (d) 32%

20. The base and height of a right triangle are increased by 10% and 20%, respectively. What is the percent change in the area of the triangle?

(a) 10% (b) 15%
(c) 30% (d) 32%

21. Two students appeared at an examination. One of them secured 9 marks more than the other and his marks was 56% of the sum of their marks. What marks was obtained by them?

(a) 39, 30 (b) 41, 32
(c) 42, 33 (d) 43, 34

22. A fruit seller had some apples. He sells 40% apples and still has 420 apples. How many apples did he originally have?

(a) 588 (b) 600
(c) 672 (d) 700

23. What percent of numbers from 1 to 70 have 1 or 9 in the unit's digit?

(a) 1 (b) 14 (c) 20 (d) 21

24. If A = x% of y and B = y% of x, then which one of the following options is true?

(a) A is smaller than B
(b) A is greater than B
(c) If x is smaller than y, then A is greater than B
(d) A is the same as B

25. In a certain school, 20% of students are below 8 years of age. The number of students above 8 years of age is $\frac{2}{3}$ of the number of students of 8 years of age which is 48. What is the total number of students in the school?

(a) 100 (b) 120
(c) 80 (d) 150

26. Two numbers A and B are such that the sum of 5% of A and 4% of B is two-third of the sum of 6% of A and 8% of B. What is the ratio of A:B?

(a) 2:3 (b) 1:1
(c) 3:4 (d) 4:3

27. A student multiplied a number by $\frac{3}{5}$ instead of $\frac{5}{3}$. What is the percent error in the calculation?

(a) 34% (b) 44%
(c) 54% (d) 64%

28. In an election between two candidates, one got 55% of the total valid votes, 20% of the votes were invalid. If the total number of votes was 7500, then what is the number of valid votes that the other candidate got?

(a) 2700 (b) 2900
(c) 3000 (d) 3100

29. If 20% of $a = b$, then b% of 20 is the same as:

(a) 4% of a (b) 5% of a
(c) 20% of a (d) None of these

30. Gauri went to the stationers and bought things worth ₹25, out of which 30 paise went on sales tax on taxable purchases. If the tax rate was 6%, then what was the cost of the tax free items?

(a) ₹ 15 (b) ₹ 19.70
(c) ₹ 20 (d) ₹ 22.75

31. Two tailors X and Y are paid a total of ₹ 550 per week by their employer. If X is paid 120 percent of the sum paid to Y, how much is Y paid per week?

(a) ₹ 200 (c) ₹ 300
(b) ₹ 250 (d) ₹ 400

32. If $\left(\frac{x+y}{x-y}\right) = \frac{4}{3}$ and $x \neq 0$, then approximately what percentage of $x + 3y$ is $x - 3y$?

(a) 20% (b) 30%
(c) 40% (d) 50%

33. Intech Pvt. Ltd. generated a revenue of ₹ 1250 in 2006. This was 12.5% of its gross revenue. In 2007, the gross revenue grew by ₹ 2500. What is the percentage increase in the revenue in 2007?

(a) 12.5% (b) 20%
(c) 25% (d) 50%

34. If the price of petrol increases by 25%, by how much must a consumer cut down his consumption so that his expenditure on petrol remains constant?

(a) 16.67% (b) 20% (c) 25% (d) 33.33%

35. 30% of the men are more than 25 years old and 80% of the men are less than or equal to 50 years old. 20% of all men play football. If 20% of the men above the age of 50 play football, what percent of the football players are less than or equal to 50 years?

(a) 15% (b) 20% (c) 80% (d) 70%

36. How much flower nectar must be processed to yield 1 kg of honey, if nectar contains 50% water, and the honey obtained from this nectar contains 15% water?

(a) 1.7 kg (b) 1.9 kg (c) 2.5 kg (d) 3.33 kg

37. In an election contested by two parties, A secured 12% of the votes more than B. If B got 132000 votes, then by how many votes did it lose the election?

(a) 300000 (b) 168000
(c) 24000 (d) 36000

38. A shepherd has 1 million sheep at the beginning of year 2000. The numbers grow by $x\%$ ($x > 0$) during the year. Due to widespread illness next year, many of his sheep die. The sheep population decreases by $y\%$ during 2001 and at the beginning of 2002 the shepherd finds that he is left with 1 million sheep. Which one of the following options is correct?

(a) $x > y$ (b) $y > x$
(c) $x = y$ (d) None of these

39. If the price of petrol increases by 25% and Raj intends to spend only an additional 15% on petrol, by how much % will he reduce the quantity of petrol purchased?

(a) 13.5% (b)12% (c) 10% (d) 8%

40. A batsman scored 110 runs which included 3 boundaries and 8 sixes. What percent of his total score did he make by running between the wickets?

(a) 45% (b) $45\frac{5}{11}\%$
(c) $54\frac{6}{11}\%$ (d) 55%

ANSWERS

1. (d)	**6.** (a)	**11.** (b)	**16.** (c)	**21.** (c)	**26.** (d)	**31.** (b)	**36.** (a)
2. (a)	**7.** (d)	**12.** (b)	**17.** (c)	**22.** (d)	**27.** (d)	**32.** (c)	**37.** (d)
3. (c)	**8.** (c)	**13.** (c)	**18.** (a)	**23.** (c)	**28.** (a)	**33.** (c)	**38.** (a)
4. (b)	**9.** (b)	**14.** (a)	**19.** (b)	**24.** (d)	**29.** (a)	**34.** (b)	**39.** (d)
5. (b)	**10.** (c)	**15.** (d)	**20.** (d)	**25.** (a)	**30.** (b)	**35.** (c)	**40.** (b)

EXPLANATIONS AND HINTS

1. (d) Consider the score in the sixth math test to be x. It is given that the final average should be at least 92. So we get

$$\frac{99+85+88+94+93+x}{6} = 92$$

$$\Rightarrow 459 + x = 92 \times 6$$

$$\Rightarrow x = 552 - 459 = 93$$

So the marks in the sixth test should be 93.

2. (a) It is given that the total number of fans are 5400. So the total number of home fans will be

$$\frac{60}{100} \times 5400 = 3240$$

Total fans who are students are 810, so the number of fans that are not students will be

$$\frac{2430}{3240} \times 100 = 75\%$$

3. (c) It is given that the total runs scored = 120 and the total runs scored using boundaries and sixes $= 3 \times 4 + 8 \times 6 = 60$

So the total runs scored by running between the wickets = 120 − 60 = 60

Therefore, the total percent of runs scored by running are

$$\frac{60}{120} \times 100 = 50$$

4. (b) It is given that the total worth of production of crops of all other states is 240 million pounds. So this production constitutes = 100 − 70 − 10 = 20%. Therefore, the worth of total production is

$$240 \times \frac{100}{20} = 1200 \text{ million pounds}$$

Therefore, the production of South Carolina is

$$\frac{10}{100} \times 1200 = 120 \text{ million pounds}$$

5. (b) It is given that the new population of tigers = 45 and the percent increase = 25%

If the original population was x, then

$$x + \frac{25}{100}x = 45$$

$$\Rightarrow \frac{5}{4}x = 45$$

$$\Rightarrow x = 36$$

So the original population was 36.

6. (a) The total number of boys who passed is

$$\frac{42}{100} \times 1100 = 462$$

The total number of girls who passed is

$$\frac{30}{100} \times 700 = 210$$

So the total number of students who passed is 672 and the total number of students is 1800. Therefore, the percentage of students who passed is

$$\frac{672}{1800} \times 100 = 37.33\%$$

Hence, the percentage of students who failed is 62.67%.

7. (d) It is given that the rise in the price of gasoline is ₹ 4. So the percent rise in the price of gasoline will be

$$\frac{4}{72} \times 100 = \frac{100}{8} = \frac{50}{9}\%$$

Now, the percent decrease is half of the percent increase. Therefore, the percent decrease is

$$\frac{1}{2} \times \frac{50}{9} = \frac{25}{9}\%$$

New price of gasoline will be

$$= 76 - \left(\frac{25/9}{100} \times 76\right) = 76 - 2.11 = 73.89$$

8. (c) Let the total number of voters be x. Then, the votes polled are 90% of x.

So the valid votes = 90% of (90% of x)

Therefore,

54% of [90% of (90% of x)] − 46% of [90% of (90% of x)] = 1620

$$\Rightarrow \frac{54}{100} \times \left(\frac{90}{100} \times \frac{90}{100}x\right) - \frac{46}{100}\left(\frac{90}{100} \times \frac{90}{100}x\right) = 1620$$

$$\Rightarrow \frac{8}{100}\left(\frac{90}{100} \times \frac{90}{100}x\right) = 1620$$

$$\Rightarrow x = \frac{1620 \times 100 \times 100 \times 100}{8 \times 90 \times 90} = 25000$$

Therefore, the total number of voters on the voters list is 25000.

9. (b) It is given that the sales tax paid by Jack on item A is 8% of ₹ 400, then

$$\frac{8}{100} \times 400 = ₹ 32$$

Sales tax paid by Jack on item B is 10% of ₹ 6400, then

$$\frac{10}{100} \times 6400 = ₹ 640$$

Therefore, the total sales tax is ₹ 672 and the percent of sales tax will be

$$\frac{672}{6800} \times 100 = \frac{672}{68} = 9.88\%$$

10. (c) It is given that the total marks the student got are 125 and the minimum marks required to pass are 165

If the maximum marks are x, therefore,

$$\frac{33}{100} \times x = 165$$

$$\Rightarrow x = 165 \times \frac{100}{33} = 500$$

11. (b) Say the larger number is x and the smaller number is y. We know that $x - y = 1400$. Also,

$$\frac{10}{100}x = \frac{15}{100}y$$

$$\Rightarrow x = 1.5\ y$$

Putting this value in the first equation

$$1.5y - y = 1400$$

$$\Rightarrow 0.5y = 1400$$

$$\Rightarrow y = 2800$$

$$\Rightarrow x - 2800 = 1400$$

$$\Rightarrow x = 4200$$

Thus, the two numbers are 4200 and 2800.

12. (b) Given that radius of sphere = 20 cm

Surface area of sphere will be $4\pi r^2$

$$= 4 \times \pi \times 400$$

$$= 1600\pi \text{ cm}^2$$

Volume of sphere will be $\frac{4}{3}\pi r^3$

$$= \frac{4}{3}\pi \times 8000$$

$$= \frac{32000\pi}{3}\text{cm}^3$$

So the percentage of the surface area to volume

$$= \frac{1600\cancel{\pi}}{(3200\cancel{\pi}/3)} \times 100$$

$$= \frac{3}{20} \times 100$$

$$= 15\%$$

13. (c) $x\%$ of $y = \left(\frac{x}{100} \times y\right)$

$$y\% \text{ of } x = \left(\frac{y}{100} \times x\right)$$

Hence,

$$\left(\frac{x}{100} \times y\right) + \left(\frac{y}{100} \times x\right)$$

$$= \frac{xy}{100} + \frac{xy}{100} = \frac{2xy}{100}$$

$$= 2\% \text{ of } xy$$

14. (a) The total number of votes of the rejected candidate is 40% and the total number of votes of the winner candidate is 60%. The difference between the votes is 20%.

Also, the difference between the votes $= 640$

Say, total number of votes are x, then

$$\frac{20}{100} \times x = 640$$

$$\Rightarrow x = 640 \times \frac{100}{20} = 3200$$

Thus, total number of votes is 3200.

15. (d) Percent reduction in the consumption $= \left(\frac{25}{100+25} \times 100\right)$

$$= \frac{1}{5} \times 100 = 20\%$$

16. (c) We have two quantities X and Y and X is 25% more than Y. Hence

$$X = 1.25Y = \frac{5}{4}Y$$

Also,

$$Y = \frac{4}{5}X$$

Therefore, total percentage of Y of X will be

$$\frac{(4/5)X}{X} \times 100 = 80\%$$

Hence, the total decrease of quantity Y to X is 20%.

17. (c) Say we have a quantity A, then

$$X = (1.2)(1.3)(1.5)(1.2)A$$

and $\quad Y = (1.2)(1.5)(1.3)(1.2)A$

Now,

$$\frac{X}{Y} = \frac{(1.2)(1.3)(1.5)(1.2)\text{A}}{(1.2)(1.5)(1.3)(1.2)\text{A}} = 1$$

Hence, X is 100% of Y.

18. (a) Say we have a quantity X. After incrementing the quantity by 20%, we get $1.2X$. Now,

$$Y_1 = \frac{30}{100} \times (1.2X) + (1.2X)$$

$$\Rightarrow Y_1 = 1.56X$$

After incrementing the quantity by 30%, we get $1.3X$. Now

$$Y_2 = \frac{20}{100} \times (1.3X) + (1.3X)$$

$$\Rightarrow Y_2 = 1.56X$$

Hence, Y_1 is 100% of Y_2.

19. (b) Say length of cuboid be x, breadth be y and height be z

The volume of the cuboid $= x \cdot y \cdot z$

Now, new length of cuboid = 10% increase of $x = 1.1x$

And new breadth of cuboid = 10% increase of $y = 1.1y$

And new height of cuboid = 20% decrease of $z = 0.8z$

Now, decreased volume $= (1.1x)(1.1y)(0.8z) = 0.968xyz$

Hence, percent decrease in volume of the cuboid $= 3.2\%$

20. (d) Say base of triangle be x and height of triangle be y.

The area of the triangle $= \frac{1}{2}x \cdot y$

Now, new base of triangle = 10% increase of $x = 1.1x$

New height of triangle = 20% increase of $y = 1.2y$

Now, increased area $= \frac{1}{2}(1.1x)(1.2y) = 1.32xy$

Hence, the percent increase in the area of the triangle $= 32\%$

21. (c) Let their marks be $(x + 9)$ and x.

Then,

$$x+9=\frac{56}{100}(x+9+x)$$
$$\Rightarrow 25(x+9)=14(2x+9)$$
$$\Rightarrow 3x=99 \Rightarrow x=33$$

Hence, their marks are 42 and 33.

22. (d) Suppose originally he had x apples.

Then,

$$\frac{(100-40)}{100}\times x=420 \Rightarrow x=\left(\frac{420\times 100}{60}\right)=700$$

23. (c) Total numbers from 1 to 70 which have 1 or 9 in the unit's digit = 14 (1, 9, 11, 19, 21, 29, 31, 39, 41, 49, 51, 59, 61, 69)

Total numbers from 1 to 70 = 70

Required percentage $=\left(\frac{14}{70}\times 100\right)=20\%$

24. (d) $A = x\%$ of $y=\left(\frac{x}{100}\times y\right)=\frac{xy}{100}$

$B = y\%$ of $x=\left(\frac{y}{100}\times x\right)=\frac{xy}{100}$

Thus, A = B.

25. (a) Let the number of students be x. Then,

Number of students above 8 years of age = 80% of x.

Therefore,

$$\frac{80}{100}\times x=48+\frac{2}{3}\times 48 \Rightarrow \frac{80}{100}x=80 \Rightarrow x=100$$

26. (d) According to the data given,

$$\frac{5}{100}\text{A}+\frac{4}{100}\text{B}=\frac{2}{3}\left(\frac{6}{100}\text{A}+\frac{8}{100}\text{B}\right)$$
$$\Rightarrow \frac{\text{A}}{20}+\frac{\text{B}}{25}=\frac{\text{A}}{25}+\frac{\text{B}}{75}$$
$$\Rightarrow \left(\frac{1}{20}-\frac{1}{25}\right)\text{A}=\left(\frac{4}{75}-\frac{1}{25}\right)\text{B}$$
$$\Rightarrow \frac{1}{100}\text{A}=\frac{1}{75}\text{B}$$
$$\Rightarrow \frac{\text{A}}{\text{B}}=\frac{4}{3}$$

Required ratio = 4:3.

27. (d) Let the number be x.

Then, error $=\frac{5}{3}x-\frac{3}{5}x=\frac{16}{15}x$

Error% $=\left(\frac{16x}{15}\times\frac{3}{5x}\times 100\right)=64\%$

28. (a) Number of valid votes $=\frac{80}{100}\times 7500=6000$

Therefore, valid votes polled by other candidate $=\frac{45}{100}\times 6000=45\times 60=2700$

29. (a) We are given that

$$b=\frac{20}{100}a \Rightarrow b=\frac{1}{5}a$$

Therefore, $b\%$ of $20=\left(\frac{b}{100}\times 20\right)=\left(\frac{20}{100}a\times\frac{1}{100}\times 20\right)=\frac{4}{100}a=4\%$ of a.

30. (b) Let the amount taxable purchases be ₹ x.

Then,

$$\left(\frac{6}{100}\times x\right)=\frac{30}{100} \Rightarrow x+\left(\frac{30}{100}\times\frac{100}{6}\right)=5$$

Therefore, cost of tax free items $=[25-(5+0.30)]=$ ₹ 19.70

31. (b) Let the sum paid to Y per week be ₹ z.

Then,

$$z+\frac{120}{100}z=550$$

$$\Rightarrow \frac{11}{5}z=550 \Rightarrow z=\left(\frac{550\times 5}{11}\right)=250$$

32. (c) Dividing the numerator and the denominator of the given equation, we get

$$\left(\frac{x+y}{x-y}\right)=\frac{4}{3} \Rightarrow \left(\frac{\frac{x}{y}+1}{\frac{x}{y}-1}\right)=\frac{4}{3}$$

$$\Rightarrow \frac{3x}{y}+3=\frac{4x}{y}-4 \Rightarrow \frac{x}{y}=7$$

Now, required percentage $=\left(\frac{x-3y}{x+3y}\right)\times 100=\left(\frac{\frac{x}{y}-3}{\frac{x}{y}+3}\right)\times 100$

Putting value of $\frac{x}{y}$, we get

$$\left(\frac{7-3}{7+3}\right)\times 100=40\%$$

33. (c) We are given that Intech Pvt. Ltd. generated a revenue of ₹ 1350 in 2006.

Also, this was 12.5% of the gross revenue. Hence, if total revenue by end of 2007 is x, then

$$\frac{12.5}{100}\times x = 1250 \Rightarrow x = 10000$$

Thus, the growth is

$$\left(\frac{2500}{10000}\right)\times 100 = 25\%$$

34. (b) Let the price of petrol be ₹ 100 per litre. Let the user use 1 litre of petrol.

Therefore, his expense on petrol = $100\times 1 =$ ₹ 100

Now, the price of petrol increases by 25%. Therefore, new price of petrol = ₹ 125

It is given that the expenditure remains constant. Hence, he will be spending only ₹ 100 on petrol.

Let x be the number of litres of petrol he will use at the new price.

Thus,

$$125\times x = 100 \Rightarrow x = \frac{100}{125} = \frac{4}{5} = 0.8 \text{ litres.}$$

He has cut down his petrol consumption by 0.2 litres, i.e. 20% reduction.

35. (c) It is given that 20% of the men are above the age of 50 years. 20% of these men play football.

Thus, 20% of 20% or 4% of the total men are football players above the age of 50 years.

Thus, 16% of the men are football players below the age of 50 years.

Therefore, the percent of men who are football players and below the age of 50 = $\left(\frac{16}{20}\right)\times 100 = 80\%$

36. (a) Flower nectar contains 50% of non-water part.

In honey, this non-water part constitutes 85%.

Therefore, amount of flower nectar is x.

$$\frac{50}{100}\times x \Rightarrow \frac{85}{100}\times 1 \Rightarrow x = \frac{85}{50} = 1.7 \text{ kg}$$

37. (d) Let the percentage of total votes secured by Party A be x%.

Then, the percentage of total votes secured by Party B = $(x-12)\%$

As there are only two parties contesting in the election, the total sum of the votes secured by the two parties should be up to 100%. Thus,

$$x + x - 12 = 100 \Rightarrow 2x = 112 \Rightarrow x = 56\%$$

If Party A got 56% of the votes, then Party B got 44% of the total votes.

Say total votes are y.

Then,

$\left(\frac{44}{100}\right)\times y = 132000 \Rightarrow y = \left(\frac{132000\times 100}{44}\right) = 300000$ votes

The margin by which Party B lost the election = $\frac{12}{100}\times 300000 = 36000$

38. (a) Let the value of x be 10%.

Therefore, the number of sheep in the herd at the beginning of year 2001 will be 1 million + 10% of 1 million = 1.1 million.

In 2001, the numbers decrease by y% and at the end of the year the number of sheep in the herd = 1 million.

Thus, 0.1 million sheep have died in 2001. Hence,

$\left(\frac{0.1}{1.1}\right)\times 100 = 9.09\%$ decrease during 2001.

Thus, $x > y$.

39. (d) Let the price of 1 litre of petrol be x and let Aman initially buy y litres of petrol. Therefore, he has spent ₹ xy on petrol initially.

When the price of petrol increases by 25%, the new price per litre of petrol is $1.25x$.

Aman intends to increase the amount he spends on petrol by 15%.

Thus, he is willing to spend $xy + \frac{15}{100}xy = 1.15xy$

Let the new quantity of petrol that he can get be z.

Then, $1.25x\times z = 1.15xy \Rightarrow z = \frac{1.15y}{1.25} = 0.92y$

As the new quantity that he can buy is $0.92y$, he gets $0.08y$ lesser than what he used to get earlier.

Thus, total reduction = 8%.

40. Runs scored by boundaries = 12

Runs scored by sixes = 48

Runs scored by running between wickets

$$= 110 - (12 + 48)$$
$$= 50$$

Percentage = $\frac{50}{110}\times 100 = 45\frac{5}{11}\%$

CHAPTER 3

PROFIT AND LOSS

INTRODUCTION

The price for which the item is bought is called its *cost price* (CP).

The price for which the item is sold is called its *selling price* (SP).

If the selling price is greater than the cost price, then the difference between the selling price and cost price is called *profit*.

If the cost price is greater than the selling price, then the difference between the cost price and selling price is called *loss*.

The price on the label is called the *marked price* (MP).

The reduction made on the marked price of an item is called *discount*. When no discount is given, selling price is the same as marked price.

SOME IMPORTANT FORMULAE

Some of the common formulae used are as follows:

1. **To calculate profit:** If the cost price is CP and the selling price is SP, then the profit (if SP > CP) will be

$$\text{Profit} = \text{SP} - \text{CP}$$

$$\text{Profit\%} = \frac{\text{Profit}}{\text{CP}} \times 100$$

2. **To calculate loss:** If the cost price is CP and the selling price is SP, then the loss (if CP > SP) will be

$$\text{Loss} = \text{CP} - \text{SP}$$

$$\text{Loss\%} = \frac{\text{Loss}}{\text{CP}} \times 100$$

3. **To calculate SP:** If the cost price is CP, marked price is MP and we know the profit percentage or loss percentage, then the selling price SP will be

$$\text{SP} = \frac{100 + \text{Profit\%}}{100} \times \text{CP}$$

$$\text{SP} = \frac{100 - \text{Loss\%}}{100} \times \text{CP}$$

$$\text{SP} = \text{MP} - \text{Discount}$$

4. **To calculate CP:** If the selling price is SP and we know the profit percentage or loss percentage, then the selling price CP will be

$$\text{CP} = \text{SP} \times \left(\frac{100}{100 + \text{Profit\%}}\right)$$

$$\text{CP} = \text{SP} \times \left(\frac{100}{100 - \text{Loss\%}}\right)$$

5. **To calculate Discount%:** If the marked price is MP and we know the discount, then the discount% will be

$$\text{Discount\%} = \frac{\text{Discount}}{\text{MP}} \times 100$$

MARGIN AND MARKUP

Margin and markup are two very important terms which are used nowadays. However, they are often misunderstood and their meanings are interchanged.

Margin is defined as the percentage difference between the selling price and the profit. It is calculated by finding the net profit as a percentage of the revenue.

$$\text{Profit margin} = \frac{\text{Net profit}}{\text{Selling price}}$$

In other words, profit margin is the selling price that is turned into profit.

On the other hand, markup or profit percentage is the percentage difference between the actual cost and the profit. It is calculated by finding the profit as a percentage of cost price.

$$\text{Profit percentage} = \frac{\text{Net profit}}{\text{Cost price}}$$

In other words, profit percentage is the percentage of cost price that one gets a profit on top of cost price.

SOLVED EXAMPLES

1. Ali bought a car for ₹ 320000 and sold it for ₹ 240000. What is the loss percentage?

   Solution: Given that cost price of the car = ₹ 320000

   And selling price of the car = ₹ 240000

   So the loss = ₹ 80000

   Therefore,

$$\text{Loss\%} = \frac{80000}{320000} \times 100 = 25\%$$

2. A shopkeeper sold 20 books and used the money to buy 24 books. What is the profit percentage?

   Solution: Let the cost price of one book be ₹ 1.

   Then, SP of 20 books = CP of 24 books

   So, SP of 20 books = ₹ 24

   Hence, SP of 1 book $= \frac{24}{20}$

   So the profit $= \frac{24}{20} - 1 = \frac{4}{20}$

   Therefore,

$$\text{Profit\%} = \frac{4/20}{1} \times 100 = 25\%$$

3. Nikhil sold two laptops for ₹ 9900 each. On one laptop, he gained 10% and on the other laptop he lost 10%. What is the loss or gain percentage in the whole transaction?

   Solution: Given that selling price of the two laptops = ₹ 19800

   If the cost price of the first laptop (when gain is 10%) be x, then

$$10 = \frac{9900 - x}{x} \times 100$$
$$\Rightarrow x = 99000 - 10x$$
$$\Rightarrow 11x = 99000$$
$$\Rightarrow x = 9000$$

   If the cost price of the second laptop (when loss is 10%) be y, then

$$10 = \frac{y - 9900}{x} \times 100$$
$$\Rightarrow x = 10x - 99000$$
$$\Rightarrow 9x = 99000$$
$$\Rightarrow x = 11000$$

   Total selling price = ₹ 20000

   Total cost price = ₹ 19800

   So the loss = ₹ 200

Therefore,

$$\text{Loss\%} = \frac{200}{20000} \times 100 = 1\%$$

4. Shikha bought 150 plates for ₹ 3000 and spent ₹ 500 on transportation. After paying an amount of ₹ 1000 on their redecoration, she sold the plates at ₹ 40 each. Find the total profit and profit percent.

Solution: Total cost price of 150 plates = ₹ (3000 + 500 + 1000) = ₹ 4500
Total selling price of plates = 150 × 40 = ₹ 6000

So the total profit = ₹ (6000 − 4500) = ₹ 1500

Therefore,

$$\text{Profit\%} = \frac{1500}{4500} \times 100 = 33.3\%$$

5. A student bought a book from the bookstore for ₹ 525. If the list price of the book was ₹ 600, then what was the percentage of discount offered?

Solution: Given that price of the book = ₹ 525

List price = ₹ 600

So the discount = ₹ 75

Therefore

$$\text{Discount\%} = \frac{75}{600} \times 100 = 12.5\%$$

6. Find the single discount equivalent to two successive discounts of 40% and 50%.

Solution: Let the marked price be ₹ 100.

After the first discount, price = ₹ (100 − 40) = ₹ 60.

After the second discount, price will be

$$60 - \frac{50}{100} \times 60 = ₹\ 60 - 30 = ₹\ 30$$

Total discount = ₹ (100 − 30) = ₹ 70

Thus, the single discount% = 70%

7. A company allows a discount of 10% on its product. Find the profit% if the cost price is ₹ 4000 and listed or marked price is ₹ 5050.

Solution: It is given that the marked price of the product = ₹ 5050

And discount = 10%

So the selling price will be

$$5050 - \frac{10}{100} \times 5050 = 4545 = ₹4545$$

Now, the cost price = ₹ 4000

So the profit = ₹ 545

So the profit% will be

$$\frac{545}{4000} \times 100 = 13.625\%$$

8. The marked price of a frame is ₹ 1600, which is 20% above the cost price. It is sold at a discount of 8% on the marked price. Find the profit percent.

Solution: It is given that the marked price of the frame = ₹ 1600

$$\text{Cost price} = \frac{1800}{120} \times 100 = 1500$$

Selling price will be

$$1800 - \frac{8}{100} \times 1800 = ₹1656$$

So the profit = ₹ 156

So the profit% will be

$$\frac{156}{1500} \times 100 = 10.4\%$$

9. A store owner sold a bed for ₹ 9020 and made a profit of 10%. What would be the profit percentage if he had sold the watch for ₹ 9225?

Solution: Selling price of the bed = ₹ 9020

Profit = 10%

So the cost price of the bed = $9020 \times \frac{100}{110} = ₹8200$

Now, if the selling price = ₹ 9225

Profit = ₹ 1025

So the profit% will be

$$\frac{1025}{8200} \times 100 = 12.5\%$$

10. A florist allows a certain discount on the list price of his bouquet and still manages to get a profit of 15%. If the net cost price of the item is ₹ 1200 and list price is ₹ 2300, then what is the discount percent?

Solution: It is given that the cost price = ₹ 1200

So the selling price = $1200 \times \frac{115}{100} = ₹1380$

List price = ₹ 2300

So the discount = ₹ (2300−1380) = ₹ 920

So the discount% will be

$$\frac{920}{2300} \times 100 = 40\%$$

11. A trader mixes 26 kg of sugar at ₹ 20 per kg with 30 kg of sugar of other variety at ₹ 36 per kg and sells the mixture at ₹ 30 per kg. What is the total profit%?

Solution: Cost price of 56 kg sugar $= (26\times 20 + 30\times 36) = (520 + 1080) =$ ₹ 1600

Selling price of 56 kg sugar $= (56\times 30) =$ ₹ 1680

Therefore, profit% $= \left(\frac{80}{1600}\times 100\right) = 5\%$

12. A shopkeeper sells one transistor for ₹ 840 at a gain of 20% and another for ₹ 960 at a loss of 4%. What is the total gain or loss%?

Solution: Cost price of the first transistor $= \left(\frac{100}{120}\times 840\right) =$ ₹ 700

Cost price of the second transistor $= \left(\frac{100}{96}\times 960\right) =$ ₹ 1000

So, total cost price = (700 + 1000) = ₹ 1700

Total selling price = (840 + 960) = ₹ 1800

Thus, gain% $= \left(\frac{100}{1700}\times 100\right) = 5\frac{15}{17}\%$

13. A shopkeeper expects a gain of 22.5% on his cost price. If in a week, his sale was of ₹ 392, what was his profit?

Solution: Cost price $= \left(\frac{100}{122.5}\times 392\right) =$ ₹ 320

Therefore, profit $= (392 - 320) =$ ₹ 72

14. Sameer purchased 20 dozens of toys at the rate of ₹ 375 per dozen. He sold each one of them at the rate of ₹ 33. What was his profit%?

Solution: Cost price of 1 toy $= \left(\frac{375}{12}\right) =$ ₹ 31.25

Selling price of 1 toy = ₹ 33

So, gain = (33 − 31.25) = ₹ 1.75

Therefore, gain% $= \left(\frac{1.75}{31.25}\times 100\right) = 5.6\%$

15. The percentage profit earned by selling an article for ₹ 1920 is equal to the percentage loss incurred by selling the same article for ₹ 1280. At what price should the article be sold to make 25% profit?

Solution: Let cost price be x.
Thus,

$$\frac{1920 - x}{x}\times 100 = \frac{x - 1280}{x}\times 100$$
$$\Rightarrow 1920 - x = x - 1280$$
$$\Rightarrow 2x = 3200 \Rightarrow x = 1600$$

Thus, required selling price $= \frac{125}{100}\times 1600 =$ ₹ 2000

16. 100 apples are bought at the rate of ₹ 350 and sold at the rate of ₹ 48 per dozen. What is the profit% or loss%?

Solution: Cost price of 1 apple $= \left(\frac{350}{100}\right) =$ ₹ 3.50

Selling price of 1 apple $= \left(\frac{48}{12}\right) =$ ₹ 4.

Therefore, gain% $= \left(\frac{0.50}{3.50}\times 100\right) = \frac{100}{7} = 14\frac{2}{7}\%$

17. A vendor bought 6 sweets for a rupee. How many for a rupee must he sell to gain 20%?

Solution: Cost price of 6 sweets = ₹ 1

Selling price of 6 sweets $= \left(\frac{120}{100}\times 1\right) =$ ₹ $\frac{6}{5}$

For ₹ $\frac{6}{5}$, sweets sold = 6

For ₹ 1, sweets sold $= \left(6\times\frac{5}{6}\right) = 5$

18. The cost price of 20 articles is the same as the selling price of x articles. If the profit is 25%, then what is the value of x?

Solution: Let cost price of each article be ₹ 1.

Cost price of x articles = ₹ x

Selling price of x articles = ₹ 20

Profit = ₹ $(20 - x)$

Thus,

$$\left(\frac{20 - x}{x}\right)\times 100 = 25$$
$$\Rightarrow 2000 - 100x = 25x \Rightarrow 125x = 2000 \Rightarrow x = 16$$

19. If the cost of an item is ₹ 50 and it is sold for ₹ 62.5, what is the margin?

Solution: We are given cost price = ₹ 50

Selling price = ₹ 62.5

Hence, profit = ₹ 12.5

Therefore, margin $= \frac{12.5}{62.5}\times 100 = 20\%$

20. If the selling price of an item is 25% more than the cost price, what is the markup?

Solution: Let the cost price be x. Then,

Selling price $= x + \frac{25}{100}x = 1.25x$

Profit $= 0.25x$

Markup $= \frac{0.25x}{x}\times 100 = 25\%$

21. A dealer purchased a washing machine for ₹ 7660. He allows a discount of 12% on its marked price and still gains 10%. What is the marked price of the machine?

Solution: Cost price of the machine = ₹ 7660, and profit% = 10%

Therefore, selling price $= \left(\frac{100+10}{100}\right)\times 7600 =$ ₹ 8426

Let the marked price be x.

Then, the discount $= \frac{12}{100}x = \frac{3x}{25}$

Therefore, the selling price = marked price − discount $= \left(x - \frac{3x}{25}\right) = \frac{22x}{25}$

Also, selling price = ₹ 8426

$$\frac{22x}{25} = 8426 \Rightarrow x = \left(8426\times\frac{25}{22}\right) = 9575$$

Thus, marked price of the washing machine is ₹ 9575.

22. How much percent above the cost price should a shopkeeper mark his goods so that after allowing a discount of 25% on the market price, he gains 20%?

Solution: Let the cost price be ₹ 100.

Gain required = 20%

Therefore, selling price = ₹ 120

Let the marked price be x.

Then, discount $= \frac{25}{100}x = \frac{x}{4}$

Therefore, selling price = marked price − discount $= \left(x - \frac{x}{4}\right) = \frac{3x}{4}$

Now,

$$\frac{3x}{4} = 120 \Rightarrow x = \frac{120\times 4}{3} = 160$$

Therefore, marked price = ₹ 160.

Hence, the marked price is 60% above the cost price.

23. An agent receives $3\frac{1}{2}$% commission. What does his sales amount to if his commission is ₹ 70?

Solution: Let his sales be x. The commission of ₹ 70 is calculated as $3\frac{1}{2}$% of sales.

$\therefore 3\frac{1}{2}$% of $x = 70$

$$\frac{3.5}{100}\times x = 70 \Rightarrow x = \frac{7000}{3.5} = 2000$$

Hence, his sales amount to ₹ 2000.

24. The marked price of a cooler is ₹ 1250 and the shopkeeper allows a discount of 6% on it. What is the selling price of the cooler?

Solution: We are given that marked price = ₹ 1250 and discount% = 6%.

Discount $= \frac{6}{100}\times 1250 =$ ₹ 75

Selling price = marked price − discount = 1250 − 75 = ₹ 1175

Thus, the selling price of the fan is ₹ 1175.

PRACTICE EXERCISE

1. Rohan bought a cycle for ₹ 7000 and sold it at ₹ 6790. What was the total loss%?

(a) 12% (b) 3% (c) 30% (d) 13%

2. A man bought 100 glasses for ₹ 8500. He broke 20 glasses while carrying them. What should be the selling price of each of the remaining glasses in order to have an overall profit of 4%?

(a) 110.5 (b) 100 (c) 121.5 (d) 115

3. A shopkeeper sold 40 chairs and used the money to buy 32 chairs. What is the profit% or loss%?

(a) 20% profit (b) 10% loss
(c) 20% loss (d) 10% profit

4. Rachel bought a piano for ₹ 30000 and sold it at a profit of 2.5%. What is the selling price of the piano?

(a) ₹ 37500 (b) ₹ 32500 (c) ₹ 30750 (d) ₹ 30250

5. Arun sold two tables for ₹ 3000 each. On one table he gained 20% and on the other table he lost 20%. What is the loss or gain% in the whole transaction?

(a) 2% profit (b) 2% loss
(c) 4% profit (d) 4% loss

6. A man sells an item at 10% above its cost price. If he had bought it at 10% less and sold it at the price of ₹ 4 less, he would have made a profit of 20%. What is the cost price of the item?

(a) ₹ 216 (b) ₹ 200 (c) ₹ 400 (d) ₹ 350

7. If the cost price of 20 bags is equal to the selling price of 25 bags, what is the loss%?

(a) 40% (b) 25% (c) 20% (d) 10%

8. Mark bought paper sheets for ₹ 7000 and spent ₹ 100 on transportation. After paying an amount ₹ 750, he had made 310 boxes, which he sold at ₹ 26.50 each. Find the total profit and profit%.

(a) ₹ 365, 7.35% (b) ₹ 465, 6%
(c) ₹ 1215, 17.36% (d) ₹ 365, 4.65%

9. A computer salesman offers an 8% discount on his computer. If the marked price on the computer is ₹ 32000, what is the selling price of the computer?

(a) ₹ 30000 (b) ₹ 26780 (c) ₹ 28000 (d) ₹ 29440

10. Shruti bought a book worth ₹ 465 after getting a discount of 7%. What was the list price of the book?

(a) ₹ 472 (b) ₹ 500 (c) ₹ 535 (d) ₹ 550

11. A bed was listed at ₹ 32000. However, due to the lack of demand, it was sold at ₹ 28000. What percent of discount is offered?

(a) 12.5% (b) 6% (c) 16% (d) 8%

12. A company allows a discount of 15% on its product and still make a profit of 10%. Find the cost price if the listed or marked price is ₹ 32700.

(a) ₹ 25268.18 (b) ₹ 27894.22
(c) ₹ 26460.00 (d) ₹ 28244.10

13. After allowing a discount of 8%, there was still a gain of 5%. At what percent above the cost price was the marked price?

(a) 50% (b) 30% (c) 40% (d) 20%

14. Find the single discount equivalent to two successive discounts of 10% and 5%.

(a) 15% (b) 14.5% (c) 7.5% (d) 12.75%

15. Find the single discount equivalent to two successive discounts of 5%, 10% and 20%

(a) 15% (b) 17.5% (c) 35% (d) 31.6%

16. A shopkeeper allows successive discounts of 12% and 8%. What is the net selling price of the item whose marked price is ₹ 5800?

(a) ₹ 4640 (b) ₹ 4225.56 (c) ₹ 5215 (d) ₹ 4695.68

17. A vendor allows successive discounts of 10% and 5%. If the vendor still made a profit of 20%, then what is the net cost price of the item whose marked price is ₹ 2000?

(a) ₹ 1710 (b) ₹ 1700 (c) ₹ 1425 (d) ₹ 1550

18. Ritesh allows a certain discount on the list price of an item and still manages to get a profit of 20%. If the net cost price of the item is ₹ 6400 and list price is ₹ 10240, then what is the discount%?

(a) 25% (b) 20% (c) 30% (d) 27.5%

19. Manik sold his watch for ₹ 5040 and made a profit of 12%. What would be the profit% if he had sold the watch for ₹ 4797?

(a) 6% (b) 6.6% (c) 7% (d) 7.7%

20. Akhil sold his old phone for ₹ 1980 and had a loss of 10%. What would be the selling price of the phone if the loss% was 32.5%?

(a) ₹ 1420 (b) ₹ 1550 (c) ₹ 1640 (d) ₹ 1485

21. If the selling price of a product is doubled, then the profit triples. What is the profit%?

(a) 66.5% (b) 100% (c) 75% (d) 125%

22. In a certain store, the profit is 320% of the cost. If the cost is increased by 25% but the selling price remains constant, approximately what percentage of the selling price is the profit?

(a) 35% (b) 70% (c) 100% (d) 200%

23. When a plot is sold for ₹ 18700, the owner loses 15%. At what price must that plot be sold in order to gain 15%?

(a) ₹ 25800 (b) ₹ 22500 (c) ₹ 25300 (d) ₹ 21200

24. On selling 17 bells at ₹ 720, there is a loss equal to the cost price of 5 bells. What is the cost price of one bell?

(a) ₹ 60 (b) ₹ 50 (c) ₹ 55 (d) ₹ 45

25. Some articles were bought at 6 articles for ₹ 5 and sold at 5 articles for ₹ 6. What is the profit%?

(a) 30% (b) 33.3% (c) 35% (d) 44%

Direction for Q26–Q28: Each of the questions given below consists of a question and two or three statements. You have to decide whether the data provided in the statement(s) is/are sufficient to answer the given question.

26. A man mixes two types of salt (A and B) and sells the mixture at the rate of ₹ 17 per kg. Find his total profit%.

I. The rate of A is ₹ 20 per kg.
II. The rate of B is ₹ 13 per kg.

(a) I alone is sufficient while II alone is not sufficient to answer.
(b) II alone is sufficient while I alone is not sufficient to answer.
(c) Both I and II are not sufficient to answer.
(d) Both I and II are necessary to answer.

27. By selling a product with 20% profit, how much profit was earned?

I. The difference between cost price and selling price is ₹ 40.
II. The selling price is 120% of the cost price.

(a) I alone is sufficient while II alone is not sufficient to answer.
(b) II alone is sufficient while I alone is not sufficient to answer.
(c) Both I and II are not sufficient to answer.
(d) Both I and II are necessary to answer.

28. By selling a chair, what is the profit% achieved?

I. 5% discount is given on list price.
II. If discount is not given, 20% profit is achieved.
III. The cost profit of the articles is ₹ 5000.

(a) Only I and II (b) Only II and III (c) Only I and III (d) All I, II and III

29. A soda for ₹ 12 has a discount of 20% on it due to lack of demand. What is the selling price of the soda?

(a) ₹ 10 (b) ₹ 9.40 (c) ₹ 9.80 (d) ₹ 9.60

30. A customer is allowed a discount of 10% on the purchase of a vase. The discount is ₹ 7.40. What is the marked price of the vase?

(a) ₹ 50 (b) ₹ 56 (c) ₹ 64 (d) ₹ 74

ANSWERS

1. (b)	**4.** (c)	**7.** (c)	**10.** (b)	**13.** (c)	**16.** (d)	**19.** (b)	**22.** (b)	**25.** (d)	**28.** (a)
2. (a)	**5.** (d)	**8.** (d)	**11.** (a)	**14.** (b)	**17.** (c)	**20.** (d)	**23.** (c)	**26.** (c)	**29.** (d)
3. (c)	**6.** (b)	**9.** (d)	**12.** (a)	**15.** (d)	**18.** (a)	**21.** (b)	**24.** (a)	**27.** (a)	**30.** (d)

EXPLANATIONS AND HINTS

1. (b) Given that the total cost price of the scooter for Rohan = ₹ 7000

Selling price of the scooter = ₹ 6790

So the loss = 7000 − 6790 = ₹ 210

Therefore,

$$\text{Loss\%} = \frac{210}{7000} \times 100 = 3\%$$

2. (a) Given that the total glasses the man bought = 100

Cost price of 100 glasses = ₹ 8500

Glasses broken while carrying = 20

We know that overall profit should be 4%.

Now, the total selling price will be

$$8500\left(\frac{100+4}{100}\right) = 85 \times 104 = ₹\,8840$$

Selling price of each remaining glass $= \frac{8840}{80}$ = ₹ 110.5

3. (c) Let the cost price of one chair be ₹ 1.

Then, the selling price of 40 chairs = Cost price of 32 chairs

So, the selling price of 40 chairs = ₹ 32

$$\text{Selling price of 1 chair} = \frac{32}{40}$$

$$\text{Loss} = 1 - \frac{32}{40} = \frac{8}{40}$$

Therefore,

$$\text{Loss\%} = \frac{8/40}{1} \times 100 = 20\%$$

4. (c) Given that the cost price of piano = ₹ 30000

Total profit percentage that Rachel got = 2.5%

Hence, if the profit is x, then

$$2.5 = \frac{x}{30000} \times 100$$

$$\Rightarrow x = 2.5 \times 300 = 750$$

Therefore, the total profit = ₹ 750

Total selling price of the piano = 30000 + 750 = ₹ 30750

5. (d) Given that the selling price of Table A = ₹ 3000

Profit = 20%

So the cost price of Table A $= 3000 \times \frac{100}{120} = ₹\,2500$

Similarly, selling price of Table B = ₹ 3000

Loss = 20%

So the cost price of Table B $= 3000 \times \frac{100}{80} = ₹\,3750$

So the total cost price on both the tables = ₹ 6250

Total selling price on both the tables = ₹ 6000

Therefore,

$$\text{Loss\%} = \frac{250}{6250} \times 100 = 4\%$$

6. (b) If the cost price of the item is x, selling price of the item is

$$\left(\frac{10}{100}+1\right) = ₹\ \frac{11}{10}x$$

Now, if the cost price $= \left(x - \frac{10}{100}x\right) = \frac{9}{10}x$

and the selling price $= \frac{11}{10}x - 4$

Then, profit% = 20%

Also,

$$\frac{\text{Profit}}{\text{CP}}\times 100 = \text{Profit\%}$$

$$\Rightarrow \text{Profit} = \frac{20\times 9/10\,x}{100} = 9/50\,x$$

Comparing both equations of selling prices,

$$\frac{9}{10}x + \frac{9}{50}x = \frac{11}{10}x - 4$$

$$\Rightarrow \frac{2}{10}x - \frac{9}{50}x = 4$$

$$\Rightarrow \frac{1}{50}x = 4$$

$$\Rightarrow x = ₹\ 200$$

7. (c) Let the cost price of one bag = ₹ 1

So the cost price of 25 bags = ₹ 25

And the cost price of 20 bags = ₹ 20

Now, we know that the selling price of 25 bags = Cost price of 20 bags = ₹ 20

So the loss = CP − SP = ₹ 25 − 20 = ₹ 5

Loss% will be

$$\frac{5}{25}\times 100 = 20\,\%$$

8. (d) Given that the total cost price to make the boxes by Mark = ₹ 7850

Total number of boxes made = 310

Total selling price = 310×26.5= ₹ 8215

So the profit = ₹ 365

Therefore, the profit% will be

$$\frac{365}{7850}\times 100 = 4.65\%$$

9. (d) It is given that the marked price of computer = ₹ 32000

Discount = 8%

So,

$$\text{Discount} = \frac{8\times 32000}{100} = ₹\ 2560$$

Selling price of the computer = ₹ 29440

10. (b) It is given that the price of the book = ₹ 465

If the list price is x and discount is 7%, then

$$7 = \frac{\text{Discount}}{x}\times 100$$

$$\Rightarrow \text{Discount} = \frac{7}{100}x$$

Price = List price − Discount

$$465 = x - \frac{7}{100}x$$

$$\Rightarrow 465 = \frac{93x}{100}$$

$\Rightarrow x = 500$

Therefore, the marked or list price of the book = ₹ 500

11. (a) It is given that the marked price of bed = ₹ 32000

Selling price = ₹ 28000

Discount = ₹ 4000

Therefore,

$$\text{Discount\%} = \frac{4000}{32000}\times 100 = 12.5\%$$

12. (a) It is given that the marked price of the product = ₹ 32700

Discount = 15%

$$\text{Discount offered} = \frac{15\times 32700}{100} = ₹\ 4905$$

So the selling price of the product = ₹ 32700 − 4905 = ₹ 27795

If the cost price is x and profit% = 10%, then

$$10 = \frac{\text{Profit}}{x}\times 100$$

$$\Rightarrow \text{Profit} = x\,/\,10$$

$$\Rightarrow \frac{x}{10} = 27795 - x$$

$$\Rightarrow \frac{11x}{10} = 27795$$

$$\Rightarrow x = \frac{277950}{11} = 25268.18$$

Therefore, the cost price of the product = ₹ 25268.18

13. (c) Let the cost price be ₹ 100.

Then gain = 5% of 100 = ₹ 5

Selling price = ₹ 105

Now, if the marked price be x. Then, discount will be

$$25\% \text{ of } x = \frac{25}{100}x = \frac{1}{4}x$$

Selling price will be

$$x - \frac{1}{4}x = \frac{3}{4}x$$

Therefore,

$$\frac{3}{4}x = 105$$

$$\Rightarrow x = \frac{105 \times 4}{3} = ₹\,140$$

Marked price = ₹ 140

If the cost price is ₹ 100 and the marked price is ₹ 140, the percentage increase will be 40%.

14. (b) Let the marked price be ₹ 100

After the first discount, price = ₹ (100 – 10) = ₹ 90

After the second discount, price $= 90 - \frac{5}{100} \times 90 = ₹\,90 - 4.5 = ₹\,85.5$

Total discount = ₹ (10 + 4.5) = ₹ 14.5

Thus, the single discount% = 14.5%

15. (d) Let the marked price be ₹ 100

After the first discount, price = ₹ (100 – 5) = ₹ 95

After the second discount, price $= 95 - \frac{10}{100} \times 95 =$ ₹ 95 – 9.5 = ₹ 85.5

After third discount, price $= 85.5 - \frac{20}{100} \times 85.5 = 85.5 - 17.1 = 68.4$

So the total discount = ₹ (100 – 68.4) = ₹ 31.6

16. (d) Marked price = ₹ 5800

First discount = 12%

So the price after the first discount

$$= ₹\left(5800 - \frac{12}{100} \times 5800\right) = ₹\ 5104$$

Second discount = 8%

So the price after the second discount

$$= ₹\left(5104 - \frac{8}{100} \times 5104\right) = ₹\ (5104 - 408.32) =$$

₹ 4695.68

Thus, the net selling price = ₹ 4695.68

17. (c) Marked price of the item = ₹ 2000

Successive discount offered are of 10% and 5%.

After first discount, price $= 2000 - \frac{10}{100} \times 2000 = 1800$

After second discount, price $= 1800 - \frac{5}{100} \times 1800 = 1710$

Hence, net selling price = ₹ 1710

Now, the total profit% = 20%

If the cost price be x, profit will be (1710 – x). Therefore,

$$20 = \frac{1710 - x}{x} \times 100$$

$$\Rightarrow x = (1710 - x) \times 5$$

$$\Rightarrow 6x = 8550$$

$$\Rightarrow x = 1425$$

Hence, the cost price = ₹ 1425

18. (a) Cost price of the item = ₹ 6400

Now, profit percent = 20%

Therefore, if profit is x, then

$$20 = \frac{x}{6400} \times 100$$

$$x = 20 \times 64$$

$$x = 1280$$

Selling price of the item = ₹ 7680

Now, list price of the item = ₹ 10240

Thus, if the discount% is y, then

$$y = \frac{10240 - 7680}{10240} \times 100$$

$$\Rightarrow y = \frac{256000}{10240}$$

$$\Rightarrow y = 25\%$$

19. (b) Selling price of the watch = ₹ 5040

Profit% = 12%

If the cost price be x, then

$$12 = \frac{5040 - x}{x} \times 100$$

$$x = 4500$$

Hence, cost price of the watch will be ₹ 4500. Now, if watch is sold at ₹ 4797, then

$$\text{Profit\%} = \frac{4797 - 4500}{4500} \times 100$$

$$= \frac{297}{4500} \times 100 = 6.6\%$$

20. (d) Selling price of the phone = ₹ 1980

Loss% = 10%

If the cost price be x, then

$$10 = \frac{x - 1980}{x} \times 100$$
$$\Rightarrow x = 2200$$

Hence, cost price of the phone = ₹ 2200

Now, if the phone is sold at a loss of 32.5% and y is the new loss, then

$$32.5 = \frac{y}{2200} \times 100$$
$$\Rightarrow y = 715$$

Selling price of the phone for 32.5% loss = 2200 − 715 = ₹ 1485

21. (b) Let C.P. be x and S.P. be y.

Then,

$$3(y - x) = (2y - x) \Rightarrow y = 2x$$

Profit = $(y - x) = (2x - x) =$ ₹ x

Profit% = $\left(\frac{x}{x} \times 100\right) = 100\%$

22. (b) Let cost price = ₹ 100. Then, profit = ₹ 320 and selling price = ₹ 420.

Now, increased cost price = ₹ 125.

We are given that the selling price remains the same. Hence, selling price = ₹ 420

Profit = (420 − 125) = ₹ 295

Thus, required percentage = $\left(\frac{295}{420} \times 100\right) \simeq 70\%$

23. (c) We are given that selling price of a plot is ₹ 18700 and loss% is 15%.

We know that,

$$\text{SP} = \frac{100 - \text{Loss\%}}{100} \times \text{CP}$$
$$\Rightarrow 18700 = \frac{100 - 15}{100} \times \text{CP}$$
$$\Rightarrow \text{CP} = \frac{18700 \times 100}{85} = 22000$$

Now, we need to find the selling price such that profit is 15%. Let the selling price be x.

$$x = \frac{100 + 15}{100} \times 22000 = 115 \times 220 = 25300$$

Hence, selling price should be ₹ 25300 in order to gain 15%.

24. (a) We are given that the difference of cost price of 17 bells and selling price of 17 bells is equal to the cost price of 5 bells.

Hence, cost price of 12 bells = selling price of 17 bells = ₹ 720.

Cost price of 1 ball = $\frac{720}{12}$ = ₹ 60.

25. (d) Let the number of articles bought is 30 (= 5 × 6).

Cost price of 30 articles = ₹ $\left(\frac{5}{6} \times 30\right)$ = ₹ 25

Selling price of 30 articles = ₹ $\left(\frac{6}{5} \times 30\right)$ = ₹ 36

Thus, profit% = $\left(\frac{11}{25} \times 100\right) = 44\%$

26. (c) The ratio in which A and B are mixed is not given. Hence, both I and II together cannot give the answer.

27. (a) We are given that profit% = 20%

According to statement I, profit = ₹ 40 (i.e. the difference between selling price and cost price)

Hence, I gives the answer, but II does not.

28. (a) According to statement I, if the list price is ₹ x.

Then, selling price = $\frac{95}{100} \times x = \frac{19}{20}x$

According to statement II, gain = 20%.

If selling price is ₹ x, then

Cost price = $\frac{100}{120} \times x = \frac{5x}{6}$

Therefore, gain = $\left(\frac{19x}{20} - \frac{5x}{6}\right) = \left(\frac{57x - 50x}{60}\right) = \frac{7x}{60}$

and gain% = $\left(\frac{7x}{60} \times \frac{6}{5x} \times 100\right) = 14\%$

Hence, I and II only give the answer.

29. (d) The rate of discount on the soda = 20%

The discount on the soda = $\frac{20}{100} \times 12$ = ₹ 2.40

The selling price of the soda = 12 − 2.4 = ₹ 9.60

30. (d) Let the marked price of the vase be ₹ x. We are given that discount is ₹ 7.40.

Thus,

$$\frac{10}{100} \times x = 7.40$$
$$\Rightarrow x = 74$$

Hence, the marked price is ₹ 74.

CHAPTER 4

SIMPLE INTEREST AND COMPOUND INTEREST

INTRODUCTION

When a sum of money is loaned from party A to B, the initial amount of sum is called *principal* (P).

The total amount of time for which the money is borrowed is called *period* (N).

The extra amount of money paid back after the period is called *interest* (I).

The rate at which the interest is calculated on the principal is called *rate of interest* (R).

The total amount of money paid after the time period is called *amount* (A).

When interest is paid as it falls due, it is called *simple interest* (SI). The interest throughout the loan period is charged on the principal borrowed.

When the money is lent at *compound interest* (CI), the interest per annum or any fixed amount of time period is added to the principal amount and the interest for the next year is calculated from the new principal value. The process is repeated until the amount for the last period is found.

SOME IMPORTANT FORMULAE

Some of the common formulae used are as follows:

1. **To calculate simple interest:** If the original sum is P, rate of interest is R and time period is N, then the simple interest SI and amount A will be

$$SI = \frac{P \times N \times R}{100}$$

$$A = P + SI$$

2. **To calculate compound interest:** If the original sum is P, rate of interest is R and N is the time period multiplied by the number of times the interest is compounded per year, then the amount A and compound interest CI will be

$$A = P\left(1 + \frac{R}{100}\right)^N$$

$$CI = A - P$$

SOLVED EXAMPLES

1. What is the amount Rajesh will receive on a sum of ₹28000 after 2 years at 6.5% interest per annum?

Solution: Given that principle = ₹28000

Time period = 2 years

Rate of interest = 6.5%

So the simple interest $= \dfrac{28000 \times 6.5 \times 2}{100} = 280 \times 13$ = ₹3640

So the total amount Rajesh will receive after 2 years = ₹(28000 + 3640) = ₹31640

2. A sum of ₹12000 amounted to ₹13380 in 2 years on a certain rate of interest. What is the rate of interest?

Solution: Let the rate of interest be x

Given that amount = ₹13380

Sum = ₹12000

So the total interest on the sum = ₹(13380 − 12000) = ₹1380

Now,

$$1380 = \frac{12000 \times x \times 2}{100} = 240x$$

$$\Rightarrow x = \frac{1380}{240} = 5.75$$

Hence, the rate of interest on the amount = 5.75%

3. What will be the ratio of simple interest earned by a certain amount at the same rate of interest for 2 years and that for 6 years?

Solution: We know that the simple interest after 2 years $= \dfrac{\text{Sum} \times \text{Rate} \times 2}{100}$

Simple interest of same amount after 6 years $= \dfrac{\text{Sum} \times \text{Rate} \times 6}{100}$

Hence,

$$\text{Ratio} = \frac{\text{Sum} \times \text{Rate} \times 2}{100} \div \frac{\text{Sum} \times \text{Rate} \times 6}{100} = \frac{2}{6} = \frac{1}{3}$$

Ratio = 1 : 3

4. If the simple interest on a certain sum is ₹890 after 4 years at a rate of 5% per annum. If the interest was compounded annually for the same time period at a same rate, what would be the amount?

Solution: It is given that the simple interest on the sum = ₹890

If sum of money is x, then

$$890 = \frac{x \times 4 \times 5}{100}$$

$$\Rightarrow x = 4450$$

Thus, the sum of money = ₹4450

If the interest was compounded annually, then

$$\text{Amount} = 4450\left(1 + \frac{5}{100}\right)^4$$

$$= 4450\left(\frac{105}{100}\right)^4 = ₹5409.003$$

5. What will be the amount of ₹20000 for 3 years compounded annually, the rate of interest being 2% for the first year, 5% for the second year and 10% for the third year?

Solution: Given that sum = ₹20000

$R_1 = 2\%$ p.a.

$R_2 = 5\%$ p.a.

$R_3 = 10\%$ p.a.

$$\text{A} = 20000\left(1 + \frac{2}{100}\right) \times \left(1 + \frac{5}{100}\right) \times \left(1 + \frac{10}{100}\right)$$

$$= 20000 \times \frac{102}{100} \times \frac{105}{100} \times \frac{110}{100} = \frac{102 \times 105 \times 110 \times 2}{100}$$

$$= 23562$$

Hence, the total amount = ₹23562

6. What is the difference between the compound interests on ₹8000 for 1 year at 10% per annum compounded yearly and half-yearly?

Solution: It is given that the sum of money = ₹8000

If compounded annually then $n = 1$, so the compound interest will be

$$CI = 8000\left(1 + \frac{10}{100}\right)^1 - 8000 = ₹800$$

If compounded annually then $n = 2$, then

$$CI = 8000\left(1 + \frac{10}{100}\right)^2 - 8000$$

$$= 8000\left(\frac{11}{10}\right)^2 - 8000 = (80 \times 121) - 8000$$

$$= 9680 - 8000 = ₹1680$$

Difference between the compound interest = ₹(1680 − 800) = ₹880

7. If a sum of money becomes 5/3 of itself in 2 years at a certain rate of simple interest, then what is the rate of interest?

Solution: Let the principle sum be x, then

$$\text{Amount} = \frac{5}{3}x$$

Simple interest will be

$$\frac{5}{3}x - x = \frac{2}{3}x$$

Time period = 2 years

Now, the rate will be

$$\frac{SI \times 100}{P \times T} = \frac{(2/3)x \times 100}{x \times 2} = \frac{100}{3} = 33.33\%$$

8. A sum of money placed at a compound interest doubles itself in 4 years. In how many years will it amount to 4 times itself?

Solution: Let the sum be x, then, we have

$$2x = x\left(1 + \frac{R}{100}\right)^4$$

$$\Rightarrow \left(1 + \frac{R}{100}\right) = 2^{1/4}$$

Also,

$$4x = x\left(1 + \frac{R}{100}\right)^T$$

$$\Rightarrow 4 = (2^{1/4})^T$$

$$\Rightarrow 4 = 2^{T/4}$$

$$\Rightarrow 2^2 = 2^{T/4}$$

$$\Rightarrow \frac{T}{4} = 2$$

$$\Rightarrow T = 8$$

Hence, the amount becomes 4 times of itself in 8 years.

9. The simple interest on a sum of money is 1/8 of the sum. If the number of years is twice the rate percent per annum, then what is the time period and rate of interest?

Solution: If the sum of money is x, then simple interest is $\frac{1}{8}x$.

Let the rate of interest be y and time period be $2y$, then

$$\frac{1}{8}x = \frac{x \times y \times y/2}{100}$$

$$\Rightarrow y^2 = \frac{100}{16}$$

$$\Rightarrow y = \frac{10}{4} = \frac{5}{2}$$

Hence, the rate of interest is (5/2)% and time period is 5 years.

10. If the sum of ₹16000 is divided into two parts. Interest (simple) of one part is calculated at a rate of 10% for 2 years and interest (compound) of the other part is calculated with the same rate for same duration. If the total interest was ₹3270, then calculate the two parts of the original sum.

Solution: If the part of simple interest is x and part of compound interest is $(16000 - x)$

So the simple interest will be

$$\frac{x \times 10 \times 2}{100} = \frac{x}{5}$$

Compound interest will be

$$(16000 - x)\left(1 + \frac{10}{100}\right)^2 - (16000 - x)$$

Total interest will be

$$\frac{20x}{100} + (16000 - x)\left(1 + \frac{10}{100}\right)^2 - (16000 - x) = 3270$$

$$\Rightarrow \frac{20x}{100} + (16000 - x)\left[\left(\frac{121}{100} - 1\right)\right] = 3270$$

$$\Rightarrow \frac{20x}{100} + (16000 - x)\left[\left(\frac{21}{100}\right)\right] = 3270$$

$$\Rightarrow \frac{20x}{100} + (16000 - x)\left[\left(\frac{21}{100}\right)\right] = 3270$$

$$\Rightarrow 20x + 336000 - 21x = 327000$$

$$\Rightarrow x = 9000$$

Hence, the part on which the simple interest is calculated is ₹9000 and the part on which compound interest is calculated is ₹7000.

11. What will be the ratio of interest earned by certain amount at the same rate of interest for an year compounded yearly and half-yearly?

Solution: Let the principal be P, rate of interest be $R\%$ and time period be 1 year.

Compound interest for T when compounded yearly
$$= P\left(1+\frac{R}{100}\right)^1 - P$$

Compound interest for T when compounded half-yearly $= P\left(1+\frac{R}{100}\right)^2 - P$

Required ratio

$$= \frac{P\left(1+\frac{R}{100}\right)^1 - P}{P\left(1+\frac{R}{100}\right)^2 - P} = \frac{\left(1+\frac{R}{100}\right)^1 - 1}{\left(1+\frac{R}{100}\right)^2 - 1}$$

$$= \frac{\left(\frac{R}{100}\right)}{\left(\frac{10000 + R^2 + 200R - 10000}{10000}\right)} = \frac{100R}{R^2 + 200R}$$

$$= \frac{100}{R+200}$$

12. A person borrows ₹ 5000 for 2 years at 4% p.a. simple interest. He immediately lends to another person at $6\frac{1}{4}$ p.a. for 2 years. Find his gain in the transaction per year.

Solution: Total interest when person borrows money $= \left(5000 \times \frac{25}{4} \times \frac{2}{100}\right) =$ ₹ 625

Total interest when person lends money to another person $= \left(\frac{5000 \times 4 \times 2}{100}\right) =$ ₹ 400

Gain in 2 years = 625 – 400 = ₹ 225

Thus, gain in 1 year = ₹ $\left(\frac{225}{2}\right)$ = ₹ 112.50

13. A sum of money amounts to ₹ 9800 after 5 years and ₹ 12005 after 8 years at the same rate of simple interest. What is the rate of interest per annum?

Solution: Simple Interest for 3 years = ₹ (12005 – 9800) = ₹ 2205

Simple Interest for 5 years = ₹ $\left(\frac{2205}{3} \times 5\right) =$ ₹ 3675

Thus, principal = ₹ (9800 – 3675) = ₹ 6125

Required rate $= \left(\frac{100 \times 3675}{6125 \times 5}\right) = 12\%$

14. A sum of ₹ 12500 amounts to ₹ 15500 in 4 years at the rate of simple interest. What is the rate of interest?

Solution: Simple interest for 4 years = ₹ (15500 – 12500) = ₹ 3000

Rate of interest $= \left(\frac{100 \times 3000}{12500 \times 4}\right) = 6\%$

15. A man took loan from a bank at the rate of 12% p.a. simple interest. After 3 years he had to pay ₹ 5400 interest only for the period. What is the principal amount borrowed by him?

Solution: We are given the values of time, interest and rate. Hence, we can calculate the principal amount using the formula,

$$\text{Principal} = \left(\frac{100 \times 5400}{12 \times 3}\right) = \text{₹ } 15000$$

16. Mr. Batra invested an amount of ₹ 13900 divided in two different schemes A and B at the simple rate of 14% p.a. and 11% p.a. respectively. If the total amount of simple interest earned in 2 years be ₹ 3508, what was the amount invested in scheme B?

Solution: Let the sum invested in scheme A be ₹ x and that in scheme B be ₹ $(13900 - x)$.
Then,

$$\left(\frac{x \times 14 \times 2}{100}\right) + \left(\frac{(13900 - x) \times 11 \times 2}{100}\right) = 3508$$
$$\Rightarrow 28x - 22x = 350800 - (13900 \times 22)$$
$$\Rightarrow 6x = 45000$$
$$\Rightarrow x = 7500$$

So, sum invested in scheme B = ₹ (13900 – 7500) = ₹ 6400

17. Reena took a loan of ₹ 1200 with simple interest for as many years as the rate of interest. If she paid ₹ 432 as interest at the end of the loan period, what was the rate of interest?

Solution: Let rate of interest be x% and time be x years. Then,

$$\left(\frac{1200 \times x \times x}{100}\right) = 432 \Rightarrow x^2 = \frac{432}{12}$$
$$\Rightarrow x = \sqrt{36} = 6$$

Since, time cannot be a negative quantity we do not consider –6.

Hence, the rate of interest = 6% p.a. and time = 6 years.

18. How much time will it take for an amount of ₹ 450 to yield ₹ 81 as interest at 4.5% per annum of simple interest?

Solution: We are given the values of principal, interest and rate. Hence, we can calculate the principal amount using the formula,

$$\text{Time} = \left(\frac{100 \times 81}{450 \times 4.5}\right) = 4 \text{ years}$$

19. There is 60% increase in an amount in 6 years at simple interest. What will be the compound interest of ₹12,000 after 3 years at the same rate?

Solution: Let P be x. Then, S.I. = $0.6x$.

$$R = \left(\frac{100 \times 0.6x}{x \times 6}\right) = 10\%\text{p.a.}$$

Now, the principal amount to be compounded = ₹12000, time = 3 years and rate = 10% p.a.

$$\text{C.I.} = \left[12000 \times \left\{\left(1 + \frac{10}{100}\right)^3 - 1\right\}\right] = \left(12000 \times \frac{331}{1000}\right)$$

$$= 3972$$

Hence, the total compound interest = ₹3972

20. The compound interest on ₹30000 at 7% per annum is ₹4347. What is the total time period?

Solution: Amount = ₹(30000 + 4347) = ₹34347
Let the time be x years.

Then,

$$30000\left(1 + \frac{7}{100}\right)^x = 34347 \Rightarrow x = 2$$

$$\Rightarrow \left(\frac{107}{100}\right)^x = \frac{34347}{30000} = \frac{11449}{10000} = \left(\frac{107}{100}\right)^2$$

Thus, the time period is 2 years.

21. What is the difference between the compound interests on ₹5000 for $1\frac{1}{2}$ years at 4% per annum compounded yearly and half-yearly?

Solution: Compound interest when interest is compounded yearly $= \left[5000 \times \left(1 + \frac{4}{100}\right) \times \left(1 + \frac{\frac{1}{2} \times 4}{100}\right)\right]$

$$= \left[5000 \times \frac{26}{25} \times \frac{51}{50}\right] = ₹5304$$

Compound interest when interest is compounded half-yearly $= \left[5000 \times \left(1 + \frac{2}{100}\right)^3\right]$

$$= \left[5000 \times \frac{51}{50} \times \frac{51}{50} \times \frac{51}{50}\right] = ₹5306.04$$

Difference = ₹(5306.04 − 5304) = ₹2.04

22. The compound interest on a certain sum of 2 years at 10% per annum is ₹525. What is the simple interest on the same sum for double the time at half the rate percent per annum?

Solution: Let the sum be x.

Then,

$$\left[x\left(1 + \frac{10}{100}\right)^2 - x\right] = 525$$

$$\Rightarrow x\left[\left(\frac{11}{10}\right)^2 - 1\right] = 525 \Rightarrow x = \left(\frac{525 \times 100}{21}\right) = 2500$$

∴ Sum = ₹2500

Hence, simple interest $= \left(\frac{2500 \times 5 \times 4}{100}\right) =$ ₹500

23. The difference between compound interest and simple interest on an amount of ₹15,000 for 2 years is ₹96. What is the rate of interest per annum?

Solution: Let the rate of interest be x.

$$\left[15000 \times \left(1 + \frac{x}{100}\right)^2 - 15000\right] - \left(\frac{15000 \times x \times 2}{100}\right) = 96$$

$$\Rightarrow 15000\left[\left(1 + \frac{x}{100}\right)^2 - 1 - \frac{2x}{100}\right] = 96$$

$$\Rightarrow 15000\left[\frac{(100 + x)^2 - 10000 - (200x)}{10000}\right] = 96$$

$$\Rightarrow x^2 = \left(\frac{96 \times 2}{3}\right) = 64 \Rightarrow x = 8$$

24. What is the least number of complete years in which a sum of money put out at 20% compound interest will be more than doubled?

Solution: Say the number of years is x.

Thus, according to the data given

$$P\left(1 + \frac{20}{100}\right)^x > 2P$$

$$\Rightarrow \left(\frac{6}{5}\right)^x > 2$$

Now, $\left(\frac{6}{5} \times \frac{6}{5} \times \frac{6}{5} \times \frac{6}{5}\right) > 2$

Thus $x = 4$ years.

25. Simple interest on a certain sum of money for 3 years at 8% per annum is half the compound interest on ₹4000 for 2 years at 10% per annum. What is the sum placed on simple interest?

Solution: The compound interest can be calculated using the following formula,

$$\text{C.I.} = \left[4000 \times \left(1 + \frac{10}{100}\right)^2 - 4000\right]$$

$$= \left(4000 \times \frac{11}{10} \times \frac{11}{10} - 4000\right) = ₹\,840$$

Let the principal amount be x and simple interest $= \left(\frac{1}{2} \times 840\right) = ₹\,420$

Thus, sum $= ₹\left(\frac{420 \times 100}{3 \times 8}\right) = ₹\,1750$

PRACTICE EXERCISE

1. A sum of money at simple interest amounts to ₹815 in 3 years and to ₹854 in 4 years. What is the rate of interest?

(a) 5.59% (b) 6.35%
(c) 7.36% (d) 3.67%

2. Ted took a loan of ₹1600 with simple interest for as many years as the rate of interest. If he paid ₹400 as interest at the end of the loan period, what was the rate of interest?

(a) 3.5% (b) 6%
(c) 5% (d) 8%

3. Mr. Gupta invested an amount of ₹12000 divided in two different schemes A and B at the simple interest rate of 14% per annum and 11% per annum respectively. If the total amount of simple interest earned in 2 years be ₹3015, what was the amount invested in Scheme B?

(a) ₹5750 (b) ₹4655
(c) ₹6250 (d) ₹3650

4. A sum of ₹800 is lent in the beginning of a year at a certain rate of interest. After 8 months, a sum of ₹360 more is lent but at the rate twice the former. At the end of the year, ₹31.20 is earned as interest from both the loans. What was the original rate of interest?

(a) 7.5% (b) 3.5% (c) 6% (d) 3%

5. Ankita lent ₹4250 to Bhina for 2 years and ₹3100 to Akhil for 4 years on a simple interest and at the same rate of interest and received ₹1985.5 in all from both of them as interest. The rate of interest per annum is

(a) 7.75% (b) 9.5%
(c) 5% (d) 10%

6. What will be the ratio of simple interest earned by a certain amount at the same rate of interest for 6 years and that for 9 years?

(a) 3:2 (b) 2:3 (c) 1:2 (d) 3:5

7. A person borrows ₹5000 for 2 years at 4% per annum simple interest. He immediately lends it to another person at 6.25% per annum for 2 years. What is his gain in the transaction per year?

(a) ₹425 (b) ₹365
(c) ₹225 (d) ₹300

8. What will be the amount of ₹14700 for 3 years compounded annually, the rate of interest being 8% for the first year, 16% for the second year and 20% for the third year?

(a) ₹22099.39 (b) ₹26577
(c) ₹19504.72 (d) ₹43893.96

9. If the simple interest on a certain sum is ₹200 after 2 years at a rate of 4% per annum. If the interest was compounded annually for the same time period at a same rate, what would be the amount?

(a) ₹2704 (b) ₹4650
(c) ₹2500 (d) ₹3377

10. The difference between simple and compound interests compounded annually on a certain sum of money for 2 years at 4% per annum is ₹20. The sum (in ₹) is

(a) ₹5000 (b) ₹7500
(c) ₹10000 (d) ₹12500

11. What is the difference between the compound interests on ₹6200 for 2 years at 8% per annum compounded yearly and half-yearly?

(a) ₹1375.75 (b) ₹1544.32
(c) ₹1203.35 (d) ₹1155.65

12. At what rate of compound interest per annum will a sum of ₹1200 become ₹1348.32 in 2 years?

(a) 8% (b) 4%
(c) 6% (d) 5.5%

13. Simple interest on a certain sum of money for 3 years at 8% per annum is half the compound interest on ₹4000 for 2 years at 10% per annum. The sum placed on simple interest is

(a) ₹1750 (b) ₹1500
(c) ₹4840 (d) ₹1275

14. If a sum of money becomes 7/4 of itself in 3 years at a certain rate of simple interest, then what is the rate of interest?

(a) 12% (b) 15%
(c) 25% (d) 27%

15. A sum of money placed at compound interest doubles itself in 3 years. In how many years will it amount to 16 times itself?

(a) 8 years (b) 12 years
(c) 24 years (d) 32 years

16. A sum of ₹160000 is deposited by Arun in a bank for 3 years. If the rate of interest provided by the bank is 8% per annum, then what amount will Arun get after 3 years?

(a) ₹198400 (b) ₹180000
(c) ₹208200 (d) ₹188600

17. After how many years will a sum of ₹12500 become ₹17500 at the rate of 10% per annum?

(a) 1 years (b) 2 years
(c) 3 years (d) 4 years

18. The simple interest on a sum of money is 1/9 of the sum. If the number of years is numerically equal to the rate percent per annum, then what is the rate percent of interest?

(a) 10/3% (b) 9%
(c) 6% (d) 12%

19. Anya finds that an increase in the rate of interest from $4\frac{7}{8}\%$ to $5\frac{1}{8}\%$ per annum increases her yearly income by ₹50. What is her initial investment?

(a) ₹10000 (b) ₹20000
(c) ₹15000 (d) ₹25000

20. If a sum of money becomes 8 times itself in 3 years at compound interest, what is the rate of interest?

(a) 25% (b) 50% (c) 75% (d) 100%

Directions for Q21–Q24: Each of the questions given below consists of a question and two or three statements. You have to decide whether the data provided in the statement(s) is/are sufficient to answer the given question.

21. The simple interest on a sum of money is ₹50. What is the sum?

I. The interest rate is 10% p.a.
II. The sum earned simple interest in 10 years.

(a) I alone sufficient while II alone not sufficient to answer.
(b) II alone sufficient while I alone not sufficient to answer.
(c) Both I and II are not sufficient to answer.
(d) Both I and II are necessary to answer.

22. What is the sum which earned interest?

I. The total simple interest was ₹7000 after 7 years.
II. The total of sum and simple interest was double of the sum after 5 years.

(a) I alone sufficient while II alone not sufficient to answer.
(b) II alone sufficient while I alone not sufficient to answer.
(c) Both I and II are not sufficient to answer.
(d) Both I and II are necessary to answer.

23. What is the rate of simple interest?

I. The total interest earned was ₹4000.
II. The sum was invested for 4 years.

(a) I alone sufficient while II alone not sufficient to answer.
(b) II alone sufficient while I alone not sufficient to answer.
(c) Both I and II are not sufficient to answer.
(d) Both I and II are necessary to answer.

24. What percentage of simple interest per annum did Amy pay to Derek?

I. Amy borrowed ₹8000 from Derek for four years.
II. Amy returned ₹8800 to Derek at the end of two years and settled the loan.

(a) I alone sufficient while II alone not sufficient to answer.
(b) II alone sufficient while I alone not sufficient to answer.
(c) Both I and II are not sufficient to answer.
(d) Both I and II are necessary to answer.

Directions for Q25–Q28: Each of the questions given below consists of a question and two or three statements. You have to decide whether the data provided in the statement(s) is/are sufficient to answer the given question.

25. What will be the compound interest earned on an amount of ₹5000 in 2 years?

I. The simple interest on the same amount at the same rate of interest in 5 years is ₹2000.
II. The compound interest and the simple interest earned in one year is the same.
III. The amount became more than double on compound interest in 10 years.

(a) I only (b) I and II only
(c) II and III only (d) I or II or III

26. What is the compound interest earned at the end of 3 years?

I. Simple interest earned on that amount at the same rate and for the same period is ₹4500.
II. The rate of interest is 10% p.a.
III. Compound interest for 3 years is more than the simple interest for that period by ₹465.

(a) I and II only
(b) II and III only
(c) Either II or III only
(d) Any two of the three

27. What is the rate of compound interest?

I. The principal was invested for 4 years.
II. The earned interest was ₹1491.

(a) I alone sufficient while II alone not sufficient to answer.
(b) II alone sufficient while I alone not sufficient to answer.
(c) Either I or II alone sufficient to answer.
(d) Both I and II are not sufficient to answer.

28. What will be the compounded amount?

I. ₹200 was borrowed for 192 months at 6% compounded annually.
II. ₹200 was borrowed for 16 years at 6%.

(a) I alone sufficient while II alone not sufficient to answer.
(b) II alone sufficient while I alone not sufficient to answer.
(c) Either I or II alone sufficient to answer.
(d) Both I and II are not sufficient to answer.

29. A bank offers 5% compound interest calculated on half-yearly basis. A customer deposits ₹16000 each on 1st January and 1st July of a year. At the end of the year, what amount would he have gained by way of interest?

(a) ₹120 (b) ₹121
(c) ₹122 (d) ₹123

30. The difference between simple and compound interests compounded annually on a certain sum of money for 2 years at 4% per annum is Re. 1. What is the sum?

(a) ₹625 (b) 630
(c) ₹640 (d) ₹650

31. What will be the compound interest on a sum of ₹25000 after 3 years at the rate of 12 p.a.?

(a) ₹10123.20 (b) ₹10483.20
(c) ₹9720 (d) ₹9000.30

32. Albert invested an amount of ₹8000 in a fixed deposit scheme for 2 years at compound interest rate 5% p.a. How much amount will Albert get on maturity of the fixed deposit?

(a) ₹8600 (b) ₹8820
(c) ₹8620 (d) ₹8750

33. What is the effective annual rate of interest corresponding to nominal rate of 6% p.a. payable half-yearly?

(a) 6.06% (b) 6.07%
(c) 6.08% (d) 6.09%

34. A sum fetched a total simple interest of ₹4016.25 at the rate of 9% p.a. in 5 years. What is the sum?

(a) ₹4462.5 (b) ₹8032.5
(c) ₹8900 (d) ₹8925

35. A person claims to be lending money at simple interest, but he includes the interest every six months for calculating the principal. If he is charging an interest of 10%, then what is the rate of interest?

(a) 10% (b) 10.25%
(c) 10.5% (d) 11%

36. A lent ₹5000 to B for 2 years and ₹3000 to C for 4 years on simple interest at the same rate of interest and received ₹2200 in all from both of them as interest. What is the rate of interest per annum?

(a) 5% (b) 7%
(c) 10% (d) 12%

37. A sum of ₹725 is lent in the beginning of a year at a certain rate of interest. After 8 months, a sum of ₹362.50 more is lent but at the rate twice the former. At the end of the year, ₹33.5 is earned as interest from both the loans. What was the original rate of interest?

(a) 3.46% (b) 4.5%
(c) 5% (d) 6%

38. A sum of money at simple interest amounts to be ₹815 in 3 years and to ₹854 in 4 years. What is the sum?

(a) ₹650 (b) ₹690
(c) ₹698 (d) ₹700

Directions for Q39 to Q40: Each of the questions given below consists of a question and two or three statements. You have to decide whether the data provided in the statement(s) is/are sufficient to answer the given question.

39. A man borrowed a sum of money on compound interest. What will be the amount to be repaid if he is repaying the entire amount at the end of 2 years?

I. The rate of interest is 5% p.a.
II. Simple interest fetched on the same amount in one year is ₹600.

III. The amount borrowed is 10 times the simple interest in 2 years.

(a) I or II only
(b) II and either I or III only
(c) III only
(d) All I, II and III are required

40. What is the principal sum?

I. The sum amounts to ₹690 in 3 years at S.I.
II. The sum amounts to ₹750 in 5 years at S.I.
III. The rate of interest is 5% p.a.

(a) I and II only (b) II and III only
(c) I and III only (d) Any two of the three

ANSWERS

1. (a)	**7.** (c)	**13.** (a)	**19.** (b)	**25.** (a)	**31.** (a)	**37.** (a)
2. (c)	**8.** (a)	**14.** (c)	**20.** (d)	**26.** (c)	**32.** (b)	**38.** (c)
3. (a)	**9.** (a)	**15.** (b)	**21.** (d)	**27.** (d)	**33.** (d)	**39.** (b)
4. (d)	**10.** (d)	**16.** (a)	**22.** (d)	**28.** (c)	**34.** (d)	**40.** (d)
5. (b)	**11.** (c)	**17.** (d)	**23.** (c)	**29.** (b)	**35.** (b)	
6. (b)	**12.** (c)	**18.** (a)	**24.** (d)	**30.** (a)	**36.** (c)	

EXPLANATIONS AND HINTS

1. (a) Let the rate of interest be x%. So the simple interest after 3 years will be

$$\frac{\text{Sum} \times x \times 3}{100}$$

Amount = Sum + Simple interest

Thus, the amount after three years will be

$$\text{Sum} + \frac{\text{Sum} \times x \times 3}{100}$$

So

$$815 = \text{Sum}\left(1 + \frac{3x}{100}\right)$$

Similarly, the amount after 4 years will be

$$\text{Sum} + \frac{\text{Sum} \times x \times 4}{100}$$

$$\Rightarrow 854 = \text{Sum} + \left(1 + \frac{4x}{100}\right)$$

Dividing the two equations, we get

$$\frac{815}{854} = \frac{(1 + 3x/100)}{(1 + 4x/100)}$$

$$\Rightarrow 815\left(1 + \frac{4x}{100}\right) = 854\left(1 + \frac{3x}{100}\right)$$

$$\Rightarrow 815 + \frac{3260x}{100} = 854 + \frac{2562x}{100}$$

$$\Rightarrow \frac{698}{100}x = 39$$

$$\Rightarrow x = 5.59\%$$

Thus, the rate of interest = 5.59%

2. (c) Let the rate of interest be x.

Given that the sum taken by Ted = ₹1600

Simple interest on that sum = ₹400

Also, the rate of interest = Time period. Hence,

$$400 = \frac{1600 \times x \times x}{100}$$

$$\Rightarrow x^2 = 25$$
$$\Rightarrow x = 5$$

Thus, the rate of interest on the amount = 5%

3. (a) Total amount invested by Mr. Gupta = ₹12000

If the amount invested in scheme A be x.

Thus, the amount invested in scheme B = $12000 - x$

Simple interest from scheme A will be

$$\frac{14 \times x \times 2}{100}$$

Simple interest from scheme B will be

$$\frac{11 \times (12000 - x) \times 2}{100}$$

Thus, the total simple interest will be

$$\frac{14 \times x \times 2}{100} + \frac{11 \times (1200 - x) \times 2}{100}$$

We know that the total simple interest = ₹3015, thus,

$$\frac{14\times x\times 2}{100}+\frac{11\times(1200-x)\times 2}{100}=3015$$

$$\Rightarrow \frac{28x}{100}+2640-\frac{22x}{100}=3015$$

$$\Rightarrow \frac{6x}{100}=375$$

$$\Rightarrow x=\frac{375\times 100}{6}=6250$$

Therefore, amount invested in scheme B = ₹(12000 − 6250) = ₹5750

4. (d) If the original rate be x, therefore, the new rate of interest = $2x$

Simple interest from the original amount will be

$$\frac{800\times x\times 1}{100}$$

Simple interest from the new amount will be

$$\frac{350\times 2x\times 1}{100\times 3}$$

Total simple interest will be

$$8x+\frac{240x}{100}=10.4x$$

Also, we know that total simple interest = ₹31.2

Therefore,

$$10.4x = 31.2$$

$$\Rightarrow x = 3$$

Thus, the original rate of interest = 3%

5. (b) Given that the total money lent by Akhil to Bhina = ₹4250

Let the rate of interest be x. So the simple interest on the money lent to Bhina will be

$$\frac{4250\times x\times 2}{100}=₹85x$$

Total money lent to Chaman = ₹3100

So the simple interest on the money lent to Chaman will be

$$\frac{3100\times x\times 4}{100}=₹124x$$

Total simple interest = ₹1985.5, so

$$124x + 85x = 1985.5$$

$$\Rightarrow x=\frac{1985.5}{209}=9.5$$

Therefore, the rate of interest = 9.5%.

6. (b) Simple interest after 6 years will be

$$\frac{\text{Sum}\times\text{Rate of interest}\times 6}{100}$$

Simple interest of same amount after 9 years

$$\frac{\text{Sum}\times\text{Rate of interest}\times 9}{100}$$

So the ratio $=\frac{6}{9}=\frac{2}{3}$

7. (c) Given that the sum of money borrowed = ₹5000

So the total interest on 4% per annum $=\frac{500\times 2\times 4}{100}$ = ₹400

Total amount the person has to return = ₹(5000 + 400) = ₹5400

Now, the person lend the money at 6.25%

So the interest on the money will be

$$\frac{5000\times 6.25\times 2}{100}=₹625$$

So the total amount be required = ₹5625

And gain = 5625 − 5400 = ₹225

8. (a) Given that sum = ₹14700

$R_1 = 8\%$ p.a.

$R_2 = 16\%$ p.a.

$R_3 = 20\%$ p.a.

$$A=14700\left(1+\frac{8}{100}\right)\times\left(1+\frac{16}{100}\right)\times\left(1+\frac{20}{100}\right)$$

$$=14700\times\frac{108}{100}\times\frac{116}{100}\times\frac{120}{100}=\frac{147\times 108\times 116\times 6}{5\times 100}$$

$$= ₹22099.39$$

9. (a) Given that the simple interest on the sum = ₹200

If the sum of money is x, then

$$200=\frac{x\times 2\times 4}{100}$$

$$x = 2500$$

Thus, the sum of money = ₹2500

If the interest was compounded annually, then

$$\text{Amount}=2500\left(1+\frac{4}{100}\right)^2$$

$$=2500\left(\frac{104}{100}\right)^2=₹2704$$

10. (d) Let the sum be x. Simple interest on the sum will be

$$\frac{x\times 2\times 4}{100}=\frac{8}{100}x$$

Compound interest on the sum will be

$$x\left(1+\frac{4}{100}\right)^2-x=\left(\frac{104}{100}\right)^2x-x$$

Difference between compound interest and simple interest = ₹ 20

Thus,

$$\left(\frac{104}{100}\right)^2x-x-\frac{8}{100}x=20$$

$$\Rightarrow\left(\frac{104}{100}\right)^2x-\frac{108}{100}x=20$$

$$\Rightarrow x\left(\frac{10816-10800}{10000}\right)=20$$

$$\Rightarrow x=\frac{20\times 10000}{16}=12500$$

The sum of money = ₹ 12500

11. (c) We know that the sum of money = ₹ 6200

If compounded annually then $n = 2$, so

$$\text{Compound interest}=6200\left(1+\frac{8}{100}\right)^2-6200$$
$$=₹\,1031.68$$

If compounded annually then $n = 4$, so the compound interest will be

$$6200\left(1+\frac{8}{100}\right)^4-6200=₹\,2235.03$$

Difference between the compound interest = ₹ (2235.03 − 1031.68) = ₹ 1203.35

12. (c) Let the rate of interest be x.

Given that the sum = ₹ 1200

Amount = ₹ 1348.32

Time period = 2 years

Using the formula of compound interest, we have

$$1348.32=1200\left(1+\frac{x}{100}\right)^2$$

$$\Rightarrow\frac{1348.32}{1200}=\left(1+\frac{x}{100}\right)^2$$

$$\Rightarrow 1.1236=\left(1+\frac{x}{100}\right)^2$$

$$\Rightarrow 1.06=1+\frac{x}{100}\Rightarrow x=0.06\times 100=6$$

Thus, the rate of interest = 6%

13. (a) Let the sum of money be x and the sum which is compounded is ₹ 4000, so the compound interest will be

$$400\left(1+\frac{10}{100}\right)^2-4000$$
$$=4000\left(\frac{11}{10}\right)^2-4000=40\times 121-4000$$
$$=\ 4840-4000\ =₹\,840$$

Now, the simple interest $=\frac{1}{2}\times 840=₹\,420$

So

$$420=\frac{x\times 3\times 8}{100}\Rightarrow x=\frac{420\times 100}{24}=1750$$

So the original sum of money = ₹ 1750

14. (c) Let the principle sum be x

So the amount $=\frac{7}{4}x$

Simple interest $=\frac{7}{4}x-x=\frac{3}{4}$

Time period = 3 years

Now, the rate will be

$$\frac{SI\times 100}{P\times T}=\frac{(3/4)x\times 100}{x\times 3}=\frac{100}{4}=25\%$$

15. (b) Let the sum be x and rate be R. Then, using the formula for compound interest, we have

$$2x=x\left(1+\frac{R}{100}\right)^3$$

$$\Rightarrow\left(1+\frac{R}{100}\right)=2^{1/3}$$

Also,

$$16x=x\left(1+\frac{R}{100}\right)^T$$
$$\Rightarrow 16=(2^{1/3})^T$$
$$\Rightarrow 16=2^{T/3}$$
$$\Rightarrow 2^4=2^{T/3}$$
$$\Rightarrow\frac{T}{3}=4$$
$$\Rightarrow T=12$$

Hence, the amount becomes 16 times of itself in 12 years.

16. (a) Simple interest will be

$$\frac{160000\times 8\times 3}{100}=1600\times 24=₹38400$$

Amount = Principle + Simple interest

Amount $=₹160000+38400=₹198400$

17. (d) Given that the principle sum = ₹12500

Amount = ₹17500

Simple interest = ₹5000

Now, simple interest is given by

$$5000=\frac{12500\times 10\times T}{100}$$

$$\Rightarrow T=\frac{500}{125}=4$$

Time period in which the sum of ₹12500 will amount to ₹17500 is 4 years.

18. (a) Let the sum of money be x.

Then, the simple interest is $\frac{1}{9}x$

Let the rate of interest and time period be y. Then using the formula of simple interest, we get

$$\frac{1}{9}x=\frac{x\times y\times y}{100}$$

$$\Rightarrow y^2=\frac{100}{9}$$

$$\Rightarrow y=\frac{10}{3}$$

Hence, the rate of interest is $\frac{10}{3}\%$

19. (b) Let the sum of money be x. Then,

$$\frac{x\times(41/8)}{100}-\frac{x\times(39/8)}{100}=50$$

$$\Rightarrow \frac{2}{8}x=5000$$

$$\Rightarrow x=5000\times 4=20000$$

Hence, the initial investment is ₹20000.

20. (d) Let the sum of money be x

Then, the amount = $8x$

Hence, using the formula of compound interest we get

$$8x=x\left(1+\frac{R}{100}\right)^3$$

$$\Rightarrow \left(1+\frac{R}{100}\right)^3=2^3$$

$$\Rightarrow \left(1+\frac{R}{100}\right)=2$$

$$\Rightarrow \frac{R}{100}=1$$

$$\Rightarrow R=100$$

Hence, the rate of interest on the amount compounded annually is 100%.

21. (d) We are given that S.I. = ₹50

According to statement I, rate = 10% p.a.

According to statement II, time = 10 years.

Thus, sum $=\left(\frac{100\times \text{S.I.}}{\text{T}\times \text{R}}\right)=\left(\frac{100\times 50}{10\times 10}\right)=₹50$

Thus, I and II together give the answer.

22. (d) Let the sum be ₹ x.

According to statement I, simple interest = ₹7000 and time = 7 years

According to statement II, sum + S.I. for 5 years = 2 × Sum ⇒ Sum = S.I. for 5 years.

Now, S.I. for 7 years = ₹7000.

$\therefore$ S.I. for 1 year $=\frac{7000}{7}=₹1000$

Thus, I and II both are needed to get the answer.

23. (c) We know that, $\text{R}=\left(\frac{100\times \text{S.I.}}{\text{P}\times \text{T}}\right)$

Now, according to statement I, simple interest = ₹4000

According to statement II, time = 4 years

But principal is unknown and we cannot find the value of rate of interest.

Hence, the given data is insufficient to get r.

24. (d) Let the rate be r% p.a.

According to statement I, principal = ₹8000 and time = 4 years.

According to statement II, simple interest = (8800 − 8000) = ₹800

$$\therefore \text{R}=\left(\frac{100\times \text{S.I.}}{\text{P}\times \text{T}}\right)=\left(\frac{100\times 800}{8000\times 4}\right)=2\frac{1}{2}\%\text{ p.a.}$$

Thus, I and II both are needed to get the answer.

25. (a) We are given that principal = ₹5000 and time = 2 years.

According to statement I, simple interest on ₹5000 in 5 years is ₹2000.

$$\frac{5000\times \text{R}\times 5}{100}=2000\Rightarrow \text{R}=8$$

Thus, only I gives the answer.

26. (c) According to statement I, simple interest for 3 years = ₹ 4500

According to statement II, rate of interest = 10% p.a.

According to statement III, Compound interest – Simple interest = ₹ 465

Clearly, using I and III, we get compound interest = ₹ (465 + 4500).

Hence, II is redundant.

Also, from I and II, we get a sum = $\left(\frac{100 \times 4500}{10 \times 3}\right)$

$= 15000$

Now, Compound interest on ₹ 15000 at 10% p.a. for 3 years can be obtained. Hence, III is redundant.

Therefore, either II or III is redundant.

27. (d) Let Principal = ₹ P and Rate = R% p.a. Then,

Amount = ₹ $\left[P\left(1+\frac{R}{100}\right)^4\right]$

Thus,

$$\text{C.I.} = P\left[\left(1+\frac{R}{100}\right)^4 - 1\right] \Rightarrow P\left[\left(1+\frac{R}{100}\right)^4 - 1\right] = 1491$$

It is clear that it does not give the answer.

28. (c) According to statement I, amount = ₹ $\left[200 \times \left(1+\frac{6}{100}\right)^{16}\right]$

According to statement II, amount = ₹ $\left[200 \times \left(1+\frac{6}{100}\right)^{16}\right]$

Thus, I as well as II gives the answer.

29. (b) We are given, principal = ₹ 1600, time = 1 year compounded half yearly and rate = 5% p.a.

Amount

$$= \left[1600 \times \left(1+\frac{5}{2\times 100}\right)^2 + 1600 \times \left(1+\frac{5}{2\times 100}\right)\right]$$

$$= \left[1600 \times \frac{41}{40} \times \frac{41}{40} + 1600 \times \frac{41}{40}\right]$$

$$= \left[\frac{1600 \times 41 \times 81}{40 \times 40}\right] = ₹\,3321$$

Therefore, compound interest = ₹ (3321 – 3200) = ₹ 121

30. (a) Let the sum be ₹ x. Then,

Compound interest =

$$\left[x\left(1+\frac{4}{100}\right)^2 - x\right] = \left(\frac{676}{625}x - x\right) = \frac{51}{625}x$$

Simple interest = $\left(\frac{x \times 4 \times 2}{100}\right) = \frac{2x}{25}$

$\therefore \frac{51x}{625} - \frac{2x}{25} = 1 \Rightarrow x = 625$

Hence, sum = ₹ 625.

31. (a) We are given, principal = ₹ 25000, time = 3 years and rate = 12% p.a.

Thus, amount = $\left[25000 \times \left(1+\frac{12}{100}\right)^3\right]$

$= \left[25000 \times \frac{28}{25} \times \frac{28}{25} \times \frac{28}{25}\right] = ₹\,35123.20$

Thus, C.I. = ₹ (35123.20 – 25000) = ₹ 10123.20

32. (b) We are given, principal = ₹ 8000, time = 2 years and rate = 5% p.a.

Thus, amount = $\left[8000 \times \left(1+\frac{5}{100}\right)^2\right] = \left[8000 \times \frac{21}{20} \times \frac{21}{20}\right] = ₹\,8820$

33. (d) Let the principal amount be ₹ 100.

Amount on that principal for 1 year when compounded half-yearly = ₹ $\left[100 \times \left(1+\frac{3}{100}\right)^2\right] = ₹\,106.09$

Thus, effective rate = (106.09 – 100) = 6.09%

34. (d) We are given total simple interest = ₹ 4016.25, rate = 9% p.a. and time = 5 years.

Thus,

$$\text{Sum} = \left(\frac{100 \times 4016.25}{9 \times 5}\right) = \left(\frac{401625}{45}\right) = ₹\,8925$$

35. (b) Let the sum be ₹ 100. Then,

Simple interest for first 6 months = $\left(\frac{100 \times 10 \times 1}{100 \times 2}\right) = ₹\,5$

Simple interest for last 6 months = $\left(\frac{105 \times 10 \times 1}{100 \times 2}\right) =$ ₹ 5.25

Thus, the amount at the end of 1 year = ₹ (100 + 5 + 5.25) = ₹ 110.25

Thus, the effective rate = 110.25 – 100 = 10.25%

36. (c) Let the rate be x% p.a.

Then,

$$\left(\frac{500 \times x \times 2}{100}\right) + \left(\frac{3000 \times x \times 4}{100}\right) = 2200$$

$$\Rightarrow 100x + 120x = 2200 \Rightarrow x = \frac{2200}{220} = 10$$

Hence, the rate = 10%.

37. (a) Let the original rate be x. Thus, new rate $= 2x$.

Also, original rate is for 1 years and the new rate is for 4 months $= \frac{1}{3}$ years.

Thus,

$$\left(\frac{725 \times x \times 1}{100}\right) + \left(\frac{362.5 \times 2x \times 1}{100 \times 3}\right) = 33.5$$

$$\Rightarrow (2175 + 725)\, x = 33.5 \times 100 \times 3$$

$$\Rightarrow (2900)\, x = 10050 \Rightarrow x = \frac{10050}{2900} = 3.46$$

Hence, original rate $= 3.46\%$.

38. (c) Simple interest for 1 year $=$ ₹ $(854 - 815) =$ ₹ 39

Simple interest for 3 years $=$ ₹ $(39 \times 3) =$ ₹ 117

Thus, sum $=$ ₹ $(815 - 117) =$ ₹ 698.

39. (b) According to statement I, rate $= 5\%$ p.a.

According to statement II, S.I. for 1 year $=$ ₹ 600.

According to statement III, sum $= 10 \times$ (S.I. for 2 years).

Now, I and II give the sum. For this sum, the compound interest and hence the amount can be obtained.

Thus, III is redundant.

Also, II gives Simple interest for 2 years $=$ ₹ $(600 \times 2) =$ ₹ 1200.

Now, from III, sum $=$ ₹ $(10 \times 1200) =$ ₹ 12000.

Thus, rate $= \left(\frac{100 \times 1200}{2 \times 12000}\right) = 5\%$ p.a.

Hence, I is redundant and I or III redundant.

40. (d) According to statement I, amount $=$ ₹ 690 and time $= 3$ years.

According to statement II, amount $= 750$ and time $=$ 5 years.

According to statement III, rate of interest $= 5\%$ p.a.

From I and II, if sum is x and rate is r.

We have,

$$x + \frac{x \times r \times 3}{100} = 690 \Rightarrow x(100 + 3r) = 69000 \qquad \text{(i)}$$

Similarly,

$$x + \frac{x \times r \times 5}{100} = 750 \Rightarrow x(100 + 5r) = 75000 \qquad \text{(ii)}$$

Dividing Eq.(ii) by Eq.(i), we get

$$\frac{(100 + 5r)}{(100 + 3r)} = \frac{75000}{69000} \Rightarrow 6900 + 345r$$

$$= 7500 + 225r \Rightarrow 120r = 600$$

$$\Rightarrow r = 5$$

$$x(100 + 5 \times 5) = 75000 \Rightarrow x(125)$$

$$= 75000 \Rightarrow x = 600$$

Thus, the sum is ₹ 600.

From I and III, or II and III, we have the value of rate and we can put in the formula of the amount to calculate the sum.

Hence, we can use any of the two statements in order to calculate the sum.

CHAPTER 5

TIME AND WORK

INTRODUCTION

Work is defined as the amount of job or task completed. Problems on work are based on the application of the concept of ratio of time and speed.

In physics, a force is said to do work if, when it acts on a body, there is a displacement of the point of application in the direction of force. However, when we talk about work in time, speed and distance problems, then an analogy between time, speed, distance and work exists.

$$\text{Work} \equiv 1 \equiv \text{Distance}$$

Hence, rate at which the work is done = speed

Number of days required to do the work = time

Also, problems on pipes and cisterns are closely related to problems on work. Filling or emptying a cistern can be considered as work done. A pipe is called an *inlet* if it fills the cistern, and it is called an *outlet* if it empties the cistern.

IMPORTANT FORMULAS AND CONCEPTS

Some important formulas and concepts of work are as follows:

1. A man can do a piece of work in a number of days, then in 1 day $(1/a)$th of the work is done. Similarly, if a man does $(1/a)$th of a work in 1 day, then he can complete the work in a days.
2. If A is a times as good a workman as B, then he will take $(1/a)$th of the time taken by B to do the same work.
3. If A and B can do a piece of work in x and y days, respectively, the amount of time they take to finish the work together will be $xy/(x + y)$ days, and the total amount of work done by them in 1 day will be $[xy/(x + y)]$th of the total work.
4. If number of men to do a job is changed in the ratio $a{:}b$, then time required to do the work will be in the ratio $b{:}a$, assuming amount of work done by each of them in the given time is the same.

5. If two men A and B together can finish a job in x days, and if A working alone takes a days more than A and B working together, and B working alone takes b days more than A and B working together, then

$$x = \sqrt{ab}$$

6. Sometimes, two operations or work are being performed which are opposite in nature. A tank being filled by a pipe and being emptied by another one or a worker throwing rocks into the truck and another working picking those up and throwing them back are examples of work of opposite nature. In such situations, for the purpose of calculations one value is taken as positive and the other is taken with a sign negative.

   Usually for pipes and cisterns problems, the pipe which drains the container is said to do negative work.

7. Efficiency is basically defined by the formula $\frac{\text{Input}}{\text{Output}} \times 100$. However, in the problems encountered in this chapter, you will come across efficiency used to define the work capacity of entity with respect to another.

   Thus, if A is 50% as efficient as B, then A has half the work capacity as B. Hence, if we are given the capacity of one of the two, we can calculate the capacity of the other.

SOLVED EXAMPLES

1. A worker A can finish work in 3 days and B can finish work in 9 days. How long will it take to finish the work if they both worked together?

Solution: We know that

A's 1 day's work $= \frac{1}{3}$

B's 1 day's work $= \frac{1}{9}$

$(A + B)$'s 1 day's work $= \left(\frac{1}{3} + \frac{1}{9}\right) = \frac{3+1}{9} = \frac{4}{9}$

Thus, the time taken by A and B to complete the work together $= \frac{9}{4}$ days

2. A and B can do a piece of work in 10 and 20 days, respectively. In how many days can A do the work if he is assisted by B on every third day?

Solution: We know that

A's 2 days' work $= \left(\frac{1}{10} \times 2\right) = \frac{1}{5}$

$(A + B)$'s 1 day's work $= \left(\frac{1}{10} + \frac{1}{20}\right) = \frac{3}{20}$

Work done in 3 days $= \left(\frac{3}{20} + \frac{1}{5}\right) = \frac{7}{20}$

Thus, $\frac{7}{20}$ work is done in 3 days.

Therefore, whole work will be done in $3 \times \frac{20}{7}$ days $= \frac{60}{7}$ days

3. If 5 men and 3 women can do a piece of work in 6 days while 1 man and 3 women can do the same in 30 days, then what is the time taken by 3 men and 4 women in doing the same type of work?

Solution: Let 1 man's 1 day's work be x and 1 woman's 1 day's work be y. Then,

$$5x + 3y = \frac{1}{6} \qquad \text{(i)}$$

$$1x + 3y = \frac{1}{30} \qquad \text{(ii)}$$

Solving the two equations, we get

$$x = \frac{1}{30};\ y = 0$$

(3 men + 4 women)'s 1 day's work $= \left(\frac{3}{30} + 0\right) = \frac{1}{10}$

Thus, 3 men and 4 women can do the work in 10 days.

4. Suppose A is twice as good as workman as B. If A and B together can finish the work in 30 days, then how long will they take to finish the work alone?

Solution: Let A takes x days to finish the work alone and B takes $2x$ days to finish the work alone. So, A and B take 30 days to finish the work together.

$$(A + B)\text{'s 1 day's work} = \frac{1}{30}$$

Then,

$$\frac{1}{x} + \frac{1}{2x} = \frac{1}{30}$$

$$\Rightarrow \frac{3}{2x} = \frac{1}{30}$$

$$\Rightarrow x = 45$$

Hence, A can finish work alone in 45 days and B can finish work alone in 90 days.

5. If 100 workers can finish a work in 50 days, then how many days does it take for 30 workers to finish the work?

Solution: Given that 100 workers can finish the work in 50 days.

So the total work by 100 workers in 1 day $= \frac{1}{50}$

Total work done by 25 workers in 1 day $= \frac{1}{200}$

So the time taken by 25 workers to complete the work = 200 days

6. Worker A alone can complete a work in 16 days and B alone in 12 days. Starting with A, they work on alternate days. How many days will it take for the total work to be completed?

Solution: We know that

$(A + B)$'s 2 days' work $= \left(\frac{1}{16} + \frac{1}{12}\right) = \frac{7}{48}$

Work done in 12 days $= \left(\frac{7}{48} \times 6\right) = \frac{7}{8}$

So the remaining work $= \frac{1}{8}$

Work done by A on 13th day $= \frac{1}{16}$

So the remaining work $= \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$

On 14th day, work is done by B.

Now, B's 1 day's work $= \frac{1}{12}$

$\frac{1}{16}$ work is done by B in $\frac{3}{4}$ day

Therefore, the total time taken $= 13\frac{3}{4}$ days

7. Two pipes A and B can fill a tank in 8 and 12 min, respectively. Both the pipes are opened together, but after 3 min, pipe A is turned off. What is the total time required to fill the tank?

Solution: We know that

Part filled in 4 min $= 3\left(\frac{1}{8} + \frac{1}{12}\right) = \frac{5}{8}$

Remaining part $= \frac{3}{8}$

Part filled by B in 1 min $= \frac{1}{12}$

Total time taken to fill the remaining part $= \frac{3}{8} \times 12 = \frac{9}{2}$

The tank will be full in $\left(4 + \frac{9}{2}\right)$ minutes = 8 min 30 s

8. A large tank can be filled using two pipes A and B in 3 h and 5 h, respectively. How long will it take to fill the entire tank if both the pipes are used to fill half of the tank and the rest of the half by A alone?

Solution: We know that

A's 1 h work $= \frac{1}{3}$

B's 1 h work $= \frac{1}{5}$

Now, (A and B)'s 1 h work $= \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$

A and B together fill the tank in 15/8 h.

A and B together fill half of the tank in 15/16 h.

A can fill the tank alone in 3 h. Hence, A can fill half of the tank in 3/2 h.

Thus, the total time taken to fill the tank $= \frac{15}{16} + \frac{3}{2} = \frac{39}{16}$ h

9. There are two inlet pipes and an outlet pipe. The efficiency of one of the inlet pipes is double than that of the other inlet pipe. The efficiency of the outlet pipe is half that of the lesser-efficient inlet pipe. The empty tank gets filled in 8 h when all the pipes are opened. How many hours will it take to fill the empty tank when the most efficient inlet pipe is plugged and the rest kept opened?

Solution: Let the number of hours taken by the outlet and two inlet pipes be $4x$, $2x$ and x respectively. So, in part of tank filled in 1 h will be

$$\frac{1}{x} + \frac{1}{2x} - \frac{1}{4x} = \frac{1}{12}$$

$$\Rightarrow \frac{5}{4x} = \frac{1}{8}$$

$$\Rightarrow x = 10$$

Hence, inlet pipe with higher efficiency fills tank in 10 h. Now, the second inlet piped is plugged. Thus, the desired pipes fill

$$\frac{1}{2x} - \frac{1}{4x} = \frac{1}{4x}$$

Time taken $= 4x = 4 \times 10 = 40$ h

10. There are three kinds of workers A, B and C in the ratio of 3:2:1. There are 20 workers of type A, 30 of type B and 36 of type C in a factory. Their weekly wages amount of ₹ 780 is divided in the ratio of work done by the workers. What will be the wages of 15 workers of type A, 21 workers of type B and 30 workers of type C for 2 weeks?

Solution: We know that

Ratio of wages of A, B, C = (3×20):(2×30):(1×36) = 5:5:3

So the total wages of 20 workers of type A = $\frac{5}{13} \times 780 = ₹\ 300$

And wages of 1 worker of type A = ₹ 15

Similarly, wages of 30 workers of type B = $\frac{5}{30} \times 780 = ₹\ 300$

And wages of 1 worker of type B = ₹ 10

Similarly, wages of 36 workers of type $C = \frac{3}{13} \times 780 = ₹\ 180$

And wages of 1 worker of type C = ₹ 5

Therefore, total weekly wages of 15 workers of type A, 21 workers of type B and 30 workers of type C

$$15 \times 15 + 21 \times 10 + 30 \times 5 = ₹\ 585$$

Thus, total wages for 2 weeks = ₹ 1170

11. Two pipes A and B together can fill a cistern in 4 hours. Had they been opened separately, then B would have taken 6 hours more than A to fill the cistern. How much time will be taken by A to fill the cistern separately?

Solution: Let the cistern be filled by pipe A alone in x hours.

Then, pipe B will fill it in $(x + 6)$ hours.

$$\frac{1}{x} + \frac{1}{(x+6)} = \frac{1}{4} \Rightarrow \frac{x+6+x}{x(x+6)} = \frac{1}{4}$$

$$\Rightarrow x^2 - 2x - 24 = 0 \Rightarrow (x-6)(x+4) = 0$$

$\Rightarrow x = 6$ (Since, we ignore negative values)

12. Two pipes A and B can fill a tank in 15 minutes and 20 minutes respectively. Both the pipes are opened together but after 4 minutes, pipe A is turned off. What is the total time required to fill the tank?

Solution: Part filled in 4 minutes $= 4\left(\frac{1}{15} + \frac{1}{20}\right) = \frac{7}{15}$

Part remaining $= \left(1 - \frac{7}{15}\right) = \frac{8}{15}$

Part filled by B in 1 minute $= \frac{1}{20}$

$\therefore \frac{1}{20} : \frac{8}{15} :: 1 : x$

$\Rightarrow x = \left(\frac{8}{15} \times 1 \times 20\right) = 10$ min 40 sec

Thus, the tank will be full in (4 min + 10 min 40 sec) = 14 min 40 sec.

13. One pipe can fill a tank three times as fast as another pipe. If together the two pipes can fill the tank in 36 minutes, then how long will it take for the slower pipe to fill the tank alone?

Solution: Let the slower pipe alone fill the tank in x minutes. Then, the faster pipe will fill it in $x/3$ minutes.

$$\frac{1}{x} + \frac{3}{x} = \frac{1}{36} \Rightarrow \frac{4}{x} = \frac{1}{36} \Rightarrow x = 144 \text{ minutes.}$$

14. Kanan and Biswa are working on an assignment. Kanan takes 6 hours to type 32 pages on a computer while Biswa takes 5 hours to type 40 pages. How much time will they take working together on two different computers to type an assignment of 110 pages.

Solution: Number of pages typed by Kanan in 1 hour $= \frac{32}{6} = \frac{16}{3}$

Number of pages typed by Biswa in 1 hour $= \frac{40}{5} = 8$

Number of pages typed by both in one hour $= \left(\frac{16}{3} + 8\right) = \frac{40}{3}$

Time taken by both to type 110 pages $= \left(110 \times \frac{3}{40}\right) = 8\frac{1}{4}$ hours

15. A, B and C can complete a piece of work in 24, 6 and 12 days respectively. How long will they take to complete the same work together?

Solution: (A + B + C)'s 1 day's work $= \left(\frac{1}{24} + \frac{1}{6} + \frac{1}{12}\right) = \frac{7}{24}$

Thus, they complete the entire work in $\frac{24}{7} = 3\frac{3}{7}$ days

16. A can lay railway track between two given stations in 16 days and B can do the same job in 12 days. With help of C, they did the job in 4 days only. How long does it take for C to do the job alone?

Solution: (A + B + C)'s 1 day's work $= \frac{1}{4}$

A's 1 day's work $= \frac{1}{16}$

B's 1 day's work $= \frac{1}{12}$

Therefore, C's 1 day's work $= \frac{1}{4} - \left(\frac{1}{16} + \frac{1}{12}\right) = \frac{5}{48}$

So, C can do the work alone in $\frac{48}{5}$ days.

17. A is thrice as good as workman as B and therefore is able to finish a job in 60 days less than B. How long will it take them to finish when working together?

Solution: Ratio of times taken by A and B = 1:3
If the time difference is 2 days, then B takes 3 days.

If difference is 60 days, then B takes $\left(\frac{3}{2}\times 60\right) = 90$ days.

So, A takes 30 days to do the work.

A's 1 day's work $= \frac{1}{30}$

B's 1 day's work $= \frac{1}{90}$

(A + B)'s 1 day's work $= \left(\frac{1}{30} + \frac{1}{90}\right) = \frac{4}{90} = \frac{2}{45}$

Therefore, A and B together can do it in $\frac{45}{2} = 22\frac{1}{2}$ days.

18. A tank is fitted with 8 pipes, some of them that fill the tank and others that are waste pipes meant to empty the tank. Each of the pipes that fill the tank can fill it in 8 hours, while each of those that empty the tank can empty it in 6 hours. If all the pipes are kept open when the tank is full, it will take exactly 6 hours for the tank to empty. How many of these are fill pipes?

Solution: Let the number of fill pipes be x. Therefore, total number of waste pipes = $8 - x$.

Each of the fill pipes can fill the tank in 8 hours.

Therefore, each of the fill pipes will fill $\left(\frac{1}{8}\right)^{\text{th}}$ of the tank in an hour.

Hence, x fill pipes will fill $\left(\frac{x}{8}\right)^{\text{th}}$ of the tank in an hour. Similarly, each of the waste pipes will drain the full tank in 6 hours or $\left(\frac{1}{6}\right)^{\text{th}}$ of the tank in an hour.

Therefore, $(8 - x)$ waste pipes will drain $\left(\frac{8-x}{6}\right)^{\text{th}}$ of the tank in an hour.

Now, when all the pipes are open, it takes the tank 6 hours to drain. Hence, $\left(\frac{1}{6}\right)^{\text{th}}$ of the tank gets drained in an hour.

(Amount of water filled by fill pipes − amount of water drained by waste pipes in 1 hour) = $\left(\frac{1}{6}\right)^{\text{th}}$ of the tank gets drained in an hour.

$$\frac{x}{8} - \frac{8-x}{6} = -\frac{1}{6} \Rightarrow \frac{6x - 64 + 8x}{48} = -\frac{1}{6}$$
$$\Rightarrow 14x - 64 = -8 \Rightarrow x = 4$$

PRACTICE EXERCISE

1. A can do a work in 10 days and B in 15 days. If they work on it together for 5 days, then what fraction of the work is left?

(a) 2/3 (b) 5/6
(c) 1/2 (d) 1/6

2. X can lay a railway track between two given stations in 15 days and Y can do the same job in 10 days. With the help of Z, they did the job in 5 days only. How long does it take for Z to complete the job alone?

(a) 1/6 (b) 1/30
(c) 5/6 (d) 9/10

3. A, B and C can do a piece of work in 20, 30 and 60 days, respectively. In how many days can A do the work if he is assisted by B and C on every third day?

(a) 12 days (b) 15 days
(c) 18 days (d) 16 days

4. If 6 men and 8 boys can do a piece of work in 10 days while 26 men and 48 boys can do the same in 2 days, then what is the time taken by 15 men and 20 boys in doing the same type of work?

(a) 2 days (b) 5 days
(c) 4 days (d) 6 days

5. Peter does 75% of work in 12 days. He then calls Charlie for help and they both complete the rest of the work in 3 days. How many days would Charlie have taken to complete the work alone?

(a) 18 (b) 24
(c) 72 (d) 48

6. If A is twice as good as workman as B and therefore is able to finish a job in 40 days less than B, how many days will it take to finish the same job if A and B work together?

(a) $28\frac{1}{2}$ (b) 40
(c) $26\frac{2}{3}$ (d) 22

7. Pipe A can fill a cistern in 36 min and pipe B in 48 min. If both the pipes are opened together, when should pipe B be closed so that the cistern may be full in 24 min?

(a) 16 min (b) 10 min
(c) 12 min (d) 15 min

8. Pipes A and B can fill a tank in 4 and 8 h, respectively. Pipe C can empty it in 16 h. If all the three pipes are opened together, then how long will it take to fill the tank?

(a) 4.5 h (b) 2 h
(c) 6 h (d) 3.2 h

9. Two pipes A and B can fill a tank in 4 h. If only pipe A is open, then it would take 4 h longer to fill the tank. How much longer would it take if only pipe B is open?

(a) 7 h (b) 4 h
(c) 8 h (d) 5 h

10. A pump can fill a tank with water in 1.5 h. However, due to a leak it took 2.5 h to fill the tank. How long will the leak take to drain all the water in the tank?

(a) 15/4 h (b) 14/9 h
(c) 15/2 h (d) 12/5 h

11. There are two inlet pipes and an outlet pipe. The efficiency of one of the inlet pipes is double than that of the other inlet pipe. The efficiency of the outlet pipe is half that of the lesser-efficient inlet pipe. The empty tank gets filled in 12 h when all the pipes are opened. How many hours will it take to fill the empty tank when the lesser-efficient inlet pipe is plugged and the rest kept opened?

(a) 20 h (b) 25 h
(c) 15 h (d) 12 h

12. Two pipes A and B can fill a tank in 12 and 16 min, respectively. Both the pipes are opened together, but after 4 min, pipe A is turned off. What is the total time required to fill the tank?

(a) 8 min (b) 6 min 40 s
(c) 10 min 40 s (d) 12 min 30 s

13. Three taps A, B and C can fill a tank in 12, 15 and 20 h, respectively. If A is open all the time and B and C are open for 1 h each alternatively, how long will it take to fill the tank?

(a) 8.5 h (b) 10 h
(c) 7 h (d) 5 h

14. Three pipes A, B and C can fill a tank in 8 h. After working together for 2 h, C is closed and A and B can fill the remaining part in 10 h. What is the number of hours taken by C alone to fill the tank?

(a) 16/3 h (b) 20 h
(c) 15 h (d) 11 h

15. A large tank can be filled using two pipes A and B in 2 and 4 h, respectively. How long will it take to fill the entire tank if both the pipes are used to fill half of the tank alone?

(a) 2 h (b) 5 h
(c) 3.5 h (d) 3 h

16. Pipes A, B and C are attached to a tank and each can act as either an inlet or an outlet pipe. They take 6, 12 and 18 h to fill an empty tank or empty the full tank. In the first hour, pipes A and B work as inlet and C works as outlet. In the second hour, pipes B and C work as inlet and pipe A as outlet. In the third hour, A and C work as inlet and pipe B as outlet. What part of the tank will be filled in 3 h?

(a) 11/18 (b) 5/28
(c) 17/20 (d) 5/9

17. If Karan can do 1/4 of total work in 3 days and Arjun can do 1/6 of the same work in 4 days, how much will Karan get if both work together and are paid ₹ 300 in total?

(a) ₹ 150 (b) ₹ 200
(c) ₹ 100 (d) ₹ 225

18. If A and B can complete a work in 8 days, B and C in 12 days, C and A in 24 days, then how long will it take for A, B and C to complete the work if working together?

(a) 8 days (b) 6 days
(c) 9 days (d) $7\frac{2}{3}$ days

19. A sum of money is sufficient to pay A's wages for 20 days and B's wages for 30 days. For how many days is the same amount of money sufficient to pay both A and B together?

(a) 15 (b) 12
(c) $21\frac{2}{3}$ (d) 14

20. Worker A alone can do a piece of work in 6 days and B alone in 8 days. A and B undertook to do it for ₹ 4000. With the help of worker C, they completed the work in 3 days. How much money will be given to C?

(a) ₹ 600 (b) ₹ 350
(c) ₹ 400 (d) ₹ 500

Directions for Q21–Q24: Each of these questions is followed by multiple statements. You have to study the question and all the statements given to decide whether any information provided in the statement(s) is redundant and can be dispensed with while answering the given question.

21. In how many days can 10 women finish a job?

I. 10 men can complete the work in 6 days.
II. 10 men and 10 women together can complete the work in $\frac{24}{7}$ days
III. If 10 men work for 3 days and thereafter 10 women replace them, the remaining work is completed in 4 days.

(a) Any two of the three (b) I and II only
(c) II and III only (d) I and III only

22. How many workers are required for completing the construction work in 10 days?

I. 20% of the work can be completed by 8 workers in 8 days.
II. 20 workers can complete the work in 16 days.
III. One-eighth of the work can be completed by 8 workers in 5 days.

(a) I only (b) II and III only
(c) I and III only (d) I or II or III

23. A and B together can complete a task in 7 days. B alone can do it in 20 days. What part of the work was carried out by A?

I. A completed the job alone after A and B worked together for 5 days.
II. Part of the work done by A could have been done by B and C together in 6 days.

(a) I alone is sufficient while II alone is insufficient.
(b) II alone is sufficient while I alone is insufficient.
(c) Both I and II alone are sufficient.
(d) Both I and II are necessary to answer.

24. How long will machine Y, working alone, take to produce x candles?

I. Machine X produces x candles in 5 minutes.
II. Machine X and Machine Y working at the same time produce x candles in 2 minutes.

(a) I alone is sufficient while II alone is insufficient.
(b) II alone is sufficient while I alone is insufficient.
(c) Both I and II alone are sufficient.
(d) Both I and II are necessary to answer.

25. A and B can do a work in 8 days, B and C can do the same work in 12 days. A, B and C together can finish it in 6 days. How long will it take for A and C to do it together?

(a) 4 days (b) 6 days
(c) 8 days (d) 12 days

26. A and B together can do a piece of work in 30 days. A having worked for 16 days, B finishes the remaining work alone in 44 days. In how many days shall B finish the whole work alone?

(a) 30 days (b) 40 days
(c) 60 days (d) 70 days

27. A and B can do a job together in 7 days. A is $1\frac{3}{4}$ times as efficient as B. How long does it take for A to do it alone?

(a) $9\frac{1}{3}$ days (b) 11 days
(c) $\frac{15}{2}$ days (d) $17\frac{1}{3}$ days

28. A can finish a work in 24 days, B in 9 days and C in 12 days. B and C start the work but are forced to leave after 3 days. How long did it take for A to complete the remaining work?

(a) 10 days (b) 8 days
(c) 4 days (d) $\frac{21}{2}$ days

29. X can do a piece of work in 40 days. He works at it for 8 days and then Y finished it in 16 days. How long will they together take to complete the work?

(a) $13\frac{1}{3}$ days (b) 15 days
(c) 20 days (d) 26 days

30. Three pipes A, B and C are connected to a tank. These pipes can fill the tank separately in 5 hr, 10 hr and 15 hr respectively. When all the three pipes were opened simultaneously, it was observed that pipes A and B were supplying water at $\left(\frac{3}{4}\right)^{\text{th}}$ of their normal rates for the 1^{st} hour after which they supplied water at normal rate. Pipe C supplied water at $\left(\frac{2}{3}\right)^{\text{rd}}$ of its normal rate for first 2 hour, after which it supplied at its normal rate. In how much time the tank would be filled?

(a) 1.05 hours (b) 2.05 hours
(c) 3.05 hours (d) 4.05 hours

31. There are 12 pipes that are connected to a tank. Some of them are fill pipes and the others are drain pipes. Each of the fill pipes can fill the tank in 8 hours and each of the drain pipes can drain the tank completely in 6 hours. If all the fill pipes and drain pipes are kept open, an empty tank gets filled in 24 hours. How many of the 12 pipes are fill pipes?

(a) 8 (b) 6
(c) 7 (d) 5

32. A cylindrical overhead tank is filled by two pumps P1 and P2. P1 can fill the tank in 8 hr while P2 can fill the tank in 12 hr. There is a pipe P3 which can empty the tank in 8 hr. Both the pumps are opened simultaneously. The supervisor of the tank, before going out on a work, sets a timer to open P3 when the tank is half filled so that the tank is exactly filled up by the time he is back. Due to technical fault P3 opens when the tank is one-third filled. If the supervisor comes back as per the plan what percent of the tank is still empty?

(a) 17.5% (b) 15%
(c) 12% (d) 10%

33. Two pipes A and B can fill a cistern in $37\frac{1}{2}$ minutes and 45 minutes respectively. Both pipes are opened. How long is B turned after if the cistern will be filled in just half an hour?

(a) 5 min (b) 9 min
(c) 10 min (d) 15 min

34. Two pipes can fill a tank in 20 and 24 minutes respectively and a waste pipe can empty 3 gallons per minute. All the three pipes working together can fill the tank in 15 minutes. What is the capacity of the tank?

(a) 60 gallons (b) 100 gallons
(c) 120 gallons (d) 180 gallons

35. A tank is filled in 5 hours by three pipes A, B and C. The pipe C is twice as fast as B and B is twice as fast as A. How much time will pipe A alone take to fill the tank?

(a) 20 hours (b) 25 hours
(c) 35 hours (d) Indeterminate

36. Three pipes A, B and C can fill a tank in 6 hours. After working at it together for 2 hours, C is closed and A and B can fill the remaining part in 7 hours. How long does it take for C alone to fill the tank?

(a) 8 (b) 10
(c) 12 (d) 14

37. A can finish a work in 18 days and B can do the same work in 15 days. B worked for 10 days and left the job. In how many days, A alone can finish the remaining work?

(a) 6 (b) 5.5
(c) 5 (d) 4

38. A and B can together finish a work 30 days. They worked together for 20 days and then B left. After another 20 days, A finished the remaining work. In how many days A alone can finish the work?

(a) 50 (b) 60
(c) 65 (d) 70

39. P can complete a work in 12 days working 8 hours a day. Q can complete the same work in 8 days working 10 hours a day. If both P and Q work together, working 8 hours a day, in how many days can they complete the work?

(a) $5\frac{5}{11}$ (b) $5\frac{6}{11}$
(c) $6\frac{5}{11}$ (d) $6\frac{6}{11}$

40. A is 30% more efficient than B. How much time will they, working together, take to complete a job which A alone could have done in 23 days?

(a) 8 days (b) 11 days
(c) 12 days (d) 13 days

ANSWERS

1. (d)	**6.** (c)	**11.** (a)	**16.** (a)	**21.** (a)	**26.** (c)	**31.** (c)	**36.** (d)
2. (b)	**7.** (a)	**12.** (c)	**17.** (b)	**22.** (d)	**27.** (b)	**32.** (d)	**37.** (a)
3. (b)	**8.** (d)	**13.** (c)	**18.** (a)	**23.** (a)	**28.** (a)	**33.** (b)	**38.** (b)
4. (c)	**9.** (b)	**14.** (b)	**19.** (b)	**24.** (d)	**29.** (a)	**34.** (c)	**39.** (a)
5. (d)	**10.** (a)	**15.** (d)	**20.** (d)	**25.** (c)	**30.** (c)	**35.** (c)	**40.** (d)

EXPLANATIONS AND HINTS

1. (d) We know that A's 1 day's work $= \frac{1}{10}$

And B's 1 day's work $= \frac{1}{15}$

So $(A + B)$'s 1 day's work $= \left(\frac{1}{10} + \frac{1}{15}\right) = \frac{1}{6}$

Therefore, $(A + B)$'s 5 days' work $= 5 \times \frac{1}{6} = \frac{5}{6}$

Thus, the remaining work $= \left(1 - \frac{5}{6}\right) = \frac{1}{6}$

2. (b) We know that $(X + Y + Z)$'s 1 day's work $= \frac{1}{5}$

X's 1 day's work $= \frac{1}{15}$

Y's 1 day's work $= \frac{1}{10}$

Therefore, Z's 1 day's work $= \frac{1}{5} - \left(\frac{1}{15} + \frac{1}{10}\right) = \frac{1}{5} - \left(\frac{1}{6}\right) = \frac{1}{30}$

So Z will take 30 days to complete the job.

3. (b) We know that A's 2 days' work $= \left(\frac{1}{20} \times 2\right) = \frac{1}{10}$

$(A + B + C)$'s 1 day's work $= \left(\frac{1}{20} + \frac{1}{30} + \frac{1}{60}\right) = \frac{6}{60} = \frac{1}{10}$

Work done in 3 days $= \left(\frac{1}{10} + \frac{1}{10}\right) = \frac{1}{5}$

Thus, $\frac{1}{5}$ work is done in 3 days.

Therefore, whole work will be done in 3×5 days = 15 days

4. (c) Let 1 man's 1 day's work be x and 1 boy's 1 day's work be y. Then,

$$6x + 8y = \frac{1}{10} \tag{1}$$

$$26x + 48y = \frac{1}{2} \tag{2}$$

Solving the two equations, we get

$$x = \frac{1}{100};\ y = \frac{1}{200}$$

$(15 \text{ men} + 20 \text{ boys})$ 1 day's work $= \left(\frac{15}{100} + \frac{20}{200}\right) = \frac{1}{4}$

Thus, 15 men and 20 boys can do the work in 4 days.

5. (d) Given that Peter does 75% of work in 12 days.

So the total work will be completed by Peter in $12 \times \frac{4}{3} = 16$ days

Now, $\left(1 - \frac{3}{4}\right)$ amount of work is done by Peter and Charlie in 3 days.

Total work would be completed by Peter and Charlie together in $3 \times 4 = 12$ days

Peter and Charlie's one day's work $= \frac{1}{12}$

Charlie's 1 day's work $= \frac{1}{12} - \frac{1}{16} = \frac{1}{48}$

Charlie alone would do the work in 48 days.

6. (c) Given that the ratio of times taken by A and $B = 1{:}2$

If A takes 1 day to finish work, then B takes 2 days to finish the same work.

So the difference in time taken by A and $B = 1$ day

If difference is 1 day then B takes 2 days.

Thus, if difference is 40 days then B will take $2 \times 40 = 80$ days

So, A takes 40 days to do the work.

A's 1 day's work $= \frac{1}{40}$

B's 1 day's work $= \frac{1}{80}$

$(A + B)$'s 1 day's work $= \frac{1}{40} + \frac{1}{80} = \frac{3}{80}$

Thus, A and B together complete the work in $\frac{80}{3} = 26\frac{2}{3}$

7. (a) Let pipe B be turned off after x min.

So the part filled in x min $= x\left(\frac{1}{36} + \frac{1}{48}\right) = \frac{7x}{144}$

Part filled in $24 - x$ min $= \frac{24 - x}{36}$

Hence,

$$\frac{7x}{144} + \frac{24 - x}{36} = 1$$

$$\Rightarrow \frac{7x + 96 - 4x}{144} = 1$$

$$x = 16 \text{ min}$$

8. (d) We know that the part filled in 1 h $= \left(\frac{1}{4}+\frac{1}{8}\right)$ $= \frac{3}{8}$

Total part emptied in 1 h $= \frac{1}{16}$

So the part filled in 1 h $= \left(\frac{3}{8}-\frac{1}{16}\right) = \frac{5}{16}$

Therefore, tank will be full in $\frac{16}{5}$ h = 3.2 h

9. (b) We know that part filled if both A and B are open $= \frac{1}{4}$

Part filled if only A is open $= \frac{1}{8}$

Part filled if only B is open $= \frac{1}{4}-\frac{1}{8} = \frac{1}{8}$

Thus, B alone would take 8 h to fill the tank.

Hence, B alone will take (8 − 4) = 4 h longer to fill.

10. (a) Work done by the leak in 1 h $= \left(\frac{2}{3}-\frac{2}{5}\right) = \frac{4}{15}$

Thus, leak alone will empty tank in $\frac{15}{4}$ h.

11. (a) Let the number of hours taken by the outlet and the two inlet pipes be $4x$, $2x$ and x respectively. So, the part of tank filled in 1 h will be

$$\frac{1}{x}+\frac{1}{2x}-\frac{1}{4x} = \frac{1}{12}$$

$$\Rightarrow \frac{5}{4x} = \frac{1}{12}$$

$$\Rightarrow x = 15$$

Hence, inlet pipe with higher efficiency fills tank in 15 h.

Now, second inlet pipe is plugged.

Thus, the desired pipes fill $\frac{1}{x}-\frac{1}{4x} = \frac{3}{4x}$ part of the tank

Time taken $= \frac{4x}{3} = \frac{4\times 15}{3} = 20$ h

12. (c) We know that the part filled in 4 min $= 4\left(\frac{1}{12}+\frac{1}{16}\right) = \frac{7}{12}$

So the remaining part $= \frac{5}{12}$

Part filled by B in 1 min $= \frac{1}{16}$

So the total time taken to fill remaining part $= \frac{5}{12}\times 16 = \frac{20}{3}$

The tank will be full in $\left(4+\frac{20}{3}\right)$ min = 10 min 40 s

13. (c) We know that $(A + B)$'s 1 h work $= \left(\frac{1}{12}+\frac{1}{15}\right) = \frac{9}{60} = \frac{3}{20}$

And $(A + C)$'s 1 h work $= \left(\frac{1}{12}+\frac{1}{20}\right) = \frac{8}{60} = \frac{2}{15}$

So the part filled in 2 h $= \left(\frac{3}{20}+\frac{2}{15}\right) = \frac{17}{60}$

And the part filled in 6 h $= \frac{17}{60}\times 3 = \frac{17}{20}$

So the remaining part $= 1-\frac{17}{20} = \frac{3}{20}$

The remaining part is filled by $(A + B)$ in 1 h.

Thus, the total time taken to fill the tank = 6 + 1 = 7 h

14. (b) We know that $(A + B + C)$'s 1 h work $= \frac{1}{8}$

Total part filled in 2 h $= \frac{1}{4}$

Remaining part $= 1-\frac{1}{4} = \frac{3}{4}$

Now, C is closed.

Thus, $(A + B)$'s 10 h work $= \frac{3}{4}$

$(A + B)$'s 1 h work $= \frac{3}{40}$

Hence, C's 1 h work $= \frac{1}{8}-\frac{3}{40} = \frac{2}{40} = \frac{1}{20}$

C alone can fill the tank in 20 h.

15. (d) We know that A's 1 h work $= \frac{1}{2}$

And B's 1 h work $= \frac{1}{4}$

Now, A will fill $\frac{1}{2}$ of total tank in 1 h.

And B will fill $\frac{1}{2}$ of total tank in 1 × 2 = 2 h.

Thus, total time taken to fill the tank = 2 + 1 h = 3 h

16. (a) In the cycle of 3 h, pipes A, B and C are working as inlet pipes for 2 h each and they work as outlet pipes for an hour each.

So, part of tank filled in 3 h $= 2\times\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{18}\right)-\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{18}\right) = 2\times\left(\frac{6+3+2}{18}\right)-\left(\frac{6+3+2}{18}\right) = \frac{11}{18}$

17. (b) We know that whole work is done by Karan in = 3 × 4 = 12 days

Whole work is done by Arjun in = 4 × 6 = 24 days

Karan's wages : Arjun's wages = Karan's 1 day's work : Arjun's 1 day's work

Thus, $\frac{1}{12}:\frac{1}{24} = 2:1$

So Karan's share $= \frac{2}{3}\times 300 =$ ₹ 200

18. (a) We know that $(A + B)$'s 1 day's work $= \frac{1}{8}$

And $(B + C)$'s 1 day's work $= \frac{1}{12}$

And $(C + A)$'s 1 day's work $= \frac{1}{24}$

So $(A + B + C)$'s 1 day's work $= \frac{1}{2}\left(\frac{1}{8}+\frac{1}{12}+\frac{1}{24}\right)$

$= \frac{1}{2}\left(\frac{6}{24}\right) = \frac{1}{8}$

Thus, A, B and C together can complete work in 8 days.

19. (b) Let the total money be ₹ x.

So A's 1 day's wages = ₹ $\frac{x}{20}$

And B's 1 day's wages = ₹ $\frac{x}{30}$

Thus, $(A + B)$'s 1 day's wages = ₹ $\left(\frac{x}{20}+\frac{x}{30}\right)$ = ₹ $\frac{x}{12}$

So the money is sufficient to pay the wages of both for 12 days.

20. (d) We know that C's 1 day's work $= \frac{1}{3}-\left(\frac{1}{6}+\frac{1}{8}\right)$

$= \frac{1}{3}-\frac{7}{24} = \frac{1}{24}$

A'swages : B'swages : C'swages $= \frac{1}{6}:\frac{1}{8}:\frac{1}{24} = 4:3:1$

Thus, C's share $= \left(\frac{1}{8}\times 4000\right) =$ ₹ 500

21. (a) According to statement I,

(10 × 6) men can complete the work in 1 day.

$\Rightarrow$ 1 man's 1 day's work $= \frac{1}{60}$

According to statement II,

$\left(10\times\frac{24}{7}\right)$ men + $\left(10\times\frac{24}{7}\right)$ women can complete the work in 1 day.

$\Rightarrow \left(\frac{240}{7}\right)$ men's 1 day work + $\left(\frac{240}{7}\right)$ women's 1 day work = 1

$\Rightarrow \left(\frac{240}{7}\right)\times\frac{1}{60}$ men's 1 day work + $\left(\frac{240}{7}\right)$ women's 1 day work = 1

$\Rightarrow \left(\frac{240}{7}\right)$ women's 1 day work $= \left(1-\frac{4}{7}\right) = \frac{3}{7}$

$\Rightarrow$ 10 women's 1 day's work $= \left(\frac{3}{7}\times\frac{7}{240}\times 10\right) = \frac{1}{8}$

So, 10 women can finish the work in 8 days. 1 woman's one day work $= \frac{1}{80}$

According to statement III,

(10 men's work for 3 days) + (10 women's work for 4 days) = 1

$\Rightarrow$ 30 men's 1 day's work + 40 women's 1 day's work = 1

Hence, I and II, or I and III or II and III give us the answer.

22. (d) According to statement I,

1/5 of the work can be completed by (8 × 8) workers in 1 day.

Then, (8 × 8 × 5) workers can complete work in 1 day.

$\frac{8\times 8\times 5}{10}$ workers can complete work in 10 days = 32 workers in 10 days.

According to statement II,

20 workers can complete work in 16 days.

(20 × 16) workers can complete work in 1 day.

Then, $\frac{20\times 16}{10}$ workers can complete work in 10 days = 32 workers in 10 days.

According to statement III,

8 workers can complete 1/8 work in 5 days.

8 workers can complete the entire work in 40 days.

(8 × 4) workers can complete the entire work in 10 days = 32 workers in 10 days.

Thus, any one of the three statements gives us the answer.

23. (a) B's one day work $= \frac{1}{20}$

(A + B)'s one day work $= \frac{1}{7}$

According to statement I,

(A + B)'s 5 day's work $= \frac{5}{7}$

Work remaining $= \left(1-\frac{5}{7}\right) = \frac{2}{7}$

Hence, $\frac{2}{7}$ work was carried out by A.

Statement II is irrelevant.

24. (d) According to statement I,

Machine X produces $\frac{x}{5}$ candles in 1 min.

According to statement II,

Machine X and Y produce $\frac{x}{2}$ candles in 1 min.

From statement I and II, Y produces $\left(\frac{x}{2}-\frac{x}{5}\right) = \frac{3x}{10}$ candles in 1 min.

Hence, x candles will be produced by Y in $\left(\frac{10}{3x}\times x\right)$ min $= \frac{10}{3}$ min

Thus, statements I and II are both necessary to get the answer.

25. (c) (A + B + C)'s 1 day's work $= \frac{1}{6}$

(A + B)'s 1 day's work $= \frac{1}{8}$

(B + C)'s 1 day's work $= \frac{1}{12}$

Thus, (A + C)'s 1 day's work $= \left(2\times\frac{1}{6}\right)-\left(\frac{1}{8}+\frac{1}{12}\right)$

$= \left(\frac{1}{3}-\frac{5}{24}\right) = \frac{3}{24} = \frac{1}{8}$

Hence, A and C will together to the work in 8 days.

26. (c) Let A's 1 day's work $= x$ and B's 1 day's work $= y$

Then, according to the given data

$$x + y = \frac{1}{30} \qquad \text{(i)}$$

$$16x + 44y = 1 \qquad \text{(ii)}$$

Eq.(2) − 16 × Eq. (i),

$$y = \frac{1}{60}$$

B's 1 day's work $= \frac{1}{60}$

Hence, B alone can finish the whole work in 60 days.

27. (b) (A's 1 day's work) : (B's 1 day's work) $= \frac{7}{4} : 1 = 7 : 4$

Let A's and B's 1 day's work be $7x$ and $4x$ respectively.

Then,

$7x + 4x = \frac{1}{7} \Rightarrow x = \frac{1}{77}$

Thus, A's 1 day's work $= \left(\frac{1}{77}\times 7\right) = \frac{1}{11}$

A alone can complete the work in 11 days.

28. (a) (B + C)'s 1 day's work $= \left(\frac{1}{9}+\frac{1}{12}\right) = \frac{7}{36}$

Work done by B and C in 3 days $= \frac{7}{36}\times 3 = \frac{7}{12}$

Work remaining $= \frac{5}{12}$

A completes the entire work in 24 days. Hence, A completes $\frac{5}{12}$ work in $\left(\frac{5}{12}\times 24\right) = 10$ days.

29. (a) Work done by X in 8 days $= \left(\frac{1}{40}\times 8\right) = \frac{1}{5}$

Work remaining $= \frac{4}{5}$

In 16 days, Y does $\frac{4}{5}$ work. Hence, Y's 1 day's work $= \frac{4}{5}\times\frac{1}{16} = \frac{1}{20}$

(X + Y)'s 1 day's work $= \left(\frac{1}{40}+\frac{1}{20}\right) = \frac{3}{20}$

Total time taken to complete work by X and Y $= \frac{20}{3} = 13\frac{1}{3}$ days

30. (c) The part of the tank filled by A and B in first two hours $= \frac{3}{4}\times\left(\frac{1}{5}+\frac{1}{10}\right)+\left(\frac{1}{5}+\frac{1}{10}\right) = \frac{21}{40}$

The part of tank filled by C in first two hours =

$2\times\frac{2}{3}\times\frac{1}{15}=\frac{4}{45}$

Remaining part = $\left(\frac{139}{360}\right)$

In 1 hour, all the three pipes together will fill $=\frac{11}{30}$

Hence, the time taken to fill the remaining tank =

$\frac{139}{360}\times\frac{30}{11}=1.053\text{ hour}$

Thus, the total time taken to fill the remaining tank = 3.05 hours.

31. (c) Let there be 'n' fill pipes attached to the tank.

Therefore, there will be $(12-n)$ drain pipes attached to the tank. Each pipe fills the tank in 8 hours. Thus, each of the fill pipes will fill $\left(\frac{1}{8}\right)^{th}$ of the tank in an hour.

Hence, n fill pipes will fill $\left(\frac{n}{8}\right)$ of the tank in an hour.

Each drain pipe will drain the tank in 6 hours. Therefore, each of the drain pipes will drain $\left(\frac{1}{6}\right)^{\text{th}}$ of the tank in an hour.

Hence, $(12-n)$ drain pipes will drain $\left(\frac{12-n}{6}\right)$ of the tank in an hour.

When all these 12 pipes are kept open, it takes 24 hours for an empty tank to overflow. Therefore, in an hour $\left(\frac{1}{24}\right)^{\text{th}}$ of the tank gets filled. Hence,

$$\frac{n}{8}-\frac{12-n}{6}=\frac{1}{24}$$

$$\Rightarrow\frac{3n-4(12-n)}{24}=\frac{1}{24}\Rightarrow 7n-48=1$$

$$\Rightarrow n=7$$

32. (d) P1 and P2 can fill the tank in $\frac{24}{5}$

It takes $\frac{12}{5}$ hrs to fulfil half the tank. For remaining half of the tank P3 will open and this will take 6 hours. Supervisor has gone out for $\left(\frac{12}{5}+6\right)$ hrs.

Now, $\left(\frac{1}{3}\right)^{\text{rd}}$ tank will fill in $\left(\frac{8}{5}\right)$ hr. In remaining $\left(\frac{42}{5}\right)$ hr only $\left(\frac{33}{60}\right)^{\text{th}}$ part of the tank will fill.

Part of the tank which is empty $=1-\left(\frac{1}{3}+\frac{33}{60}\right)=\frac{1}{10}=10\%$

33. (b) Let B be turned off after x minutes. Then,

Part filled by (A + B) in x min + Part filled by A in $(30-x)$ min = 1

$$\therefore x\left(\frac{2}{75}+\frac{1}{45}\right)+(30-x)\times\frac{2}{75}=1$$

$$\Rightarrow\frac{11x}{225}+\frac{(60-2x)}{75}=1$$

$$\Rightarrow 11x+180-6x=225\Rightarrow x=9$$

34. (c) Work done by the waste pipe in 1 minute = $\frac{1}{15}-\left(\frac{1}{20}+\frac{1}{24}\right)=\left(\frac{1}{15}-\frac{11}{120}\right)=\left(-\frac{1}{40}\right)$ [–ve sign means emptying]

Volume of $\frac{1}{40}$ part = 3 gallons.

Volume of whole = (3 × 40) gallons = 120 gallons.

35. (c) Let pipe A take x hours to fill the tank.

Then, pipes B and C will take $\frac{x}{2}$ and $\frac{x}{4}$ hours, respectively, to fill the tank.

$$\therefore\frac{1}{x}+\frac{2}{x}+\frac{4}{x}=\frac{1}{5}$$

$$\Rightarrow\frac{7}{x}=\frac{1}{5}\Rightarrow x=35$$

Thus, pipe A takes 35 hours to fill the tank.

36. (d) Total part filled in 2 hours $=\frac{2}{6}=\frac{1}{3}$

Remaining part $=\left(1-\frac{1}{3}\right)=\frac{2}{3}$

(A + B)'s 7 hour's work $=\frac{2}{3}$

(A + B)'s 1 hour's work $=\frac{2}{3\times 7}=\frac{2}{21}$

C's 1 hour's work = (A + B + C)'s 1 hour's work – (A + B)'s 1 hour's work $=\left(\frac{1}{6}-\frac{2}{21}\right)=\frac{1}{14}$

Hence, C alone can fill the tank in 14 hours.

37. (a) B's 10 day's work $=\left(\frac{1}{15}\times 10\right)=\frac{2}{3}$

Work remaining $=\left(1-\frac{2}{3}\right)=\frac{1}{3}$

Now, $\frac{1}{18}$ work is done by A in 1 day.

Therefore, $\frac{1}{3}$ work is done by A in $\left(18\times\frac{1}{3}\right) = 6$ days

38. (b) (A + B)'s 20 day's work $= \left(\frac{1}{30}\times 20\right) = \frac{2}{3}$

Remaining work $= \left(1-\frac{2}{3}\right) = \frac{1}{3}$

Now, $\frac{1}{3}$ work is done by A in 20 days. Therefore, the whole work will be done by A in $(20 \times 3) =$ 60 days.

39. (a) P can complete the work in (12×8) hours = 96 hours

Q can complete the work in (8×10) hours = 80 hours

P's 1 hour's work $= \frac{1}{96}$ and Q's 1 hour's work $= \frac{1}{80}$

Hence, (P + Q)'s 1 hour's work $= \left(\frac{1}{96}+\frac{1}{80}\right) = \frac{11}{480}$

So, both P and Q will finish the work in $\left(\frac{480}{11}\right)$ hours.

$\therefore$ Number of days of 8 hours each $= \left(\frac{480}{11}\times\frac{1}{8}\right) = \frac{60}{11} = 5\frac{5}{11}$ days

40. (d) Ratio of times taken by A and B = 100 : 130 = 10 : 13.

Suppose B takes x days to do the work.

Then, $10 : 13 :: 23 : x \Rightarrow x = \left(\frac{23\times 13}{10}\right) = \frac{299}{10}$

A's 1 day's work $= \frac{1}{23}$

B's 1 day's work $= \frac{10}{299}$

(A + B)'s 1 day's work $= \left(\frac{1}{23}+\frac{10}{299}\right) = \frac{23}{299} = \frac{1}{13}$

Thus, A and B together can complete the work together in 13 days.

CHAPTER 6

AVERAGE, MIXTURE AND ALLIGATION

AVERAGE

The term *average* has a broad meaning and can be used in many ways. Average of a set of values is defined as the sum of all the values divided by the number of values. It is an effective way to represent a group of values by a single value. Mathematically,

$$\text{Average} = \frac{\text{Sum of all values}}{\text{Total number of values}}$$

Some important points when working with problems concerning *average* are as follows:

1. If each value in the group is increased/decreased by a constant k, then the average of the group also increases/decreases by the same constant k.
2. If each value in the group is multiplied/divided by a constant k, then the average of the group is also multiplied/divided by the same constant k.
3. The average value can never be smaller than the smallest value of the group and can never be larger than the largest value of the group. Hence, the average value always lies between the smallest and the largest value of the group.

Weighted Average

In some problems, individual frequencies of each value in the group are given to us. In that case, the formula given before does not apply. This average is known as *weighted average*. Mathematically,

$$\text{Average} = \frac{n_1x_1 + n_2x_2 + n_3x_3 + \cdots + n_kx_k}{n_1 + n_2 + n_3 + \cdots + n_k}$$

where x_1, x_2, x_3, ..., x_k are the corresponding values in the list and n_1, n_2, n_3, ..., n_k are the frequencies of the corresponding values in the list.

MIXTURE AND ALLIGATION

When two different substances are mixed together, it is called *simple mixture*.

When two or more simple mixtures are mixed together to form another mixture, it is called *compound mixture*.

According to the *alligation rule*, if two solutions are mixed in the ratio x:y and the concentration of the two solutions is c_x and c_y, respectively, then the ratio of their quantities is inversely proportional to the difference in their concentration from the mean concentration. Mathematically,

$$\frac{x}{y} = \frac{c_y - c}{c - c_x}$$

where

$$c = \frac{x \cdot c_x + y \cdot c_y}{x + y}$$

Figure 1 represents the pictorial representation of the concept of alligation for better understanding.

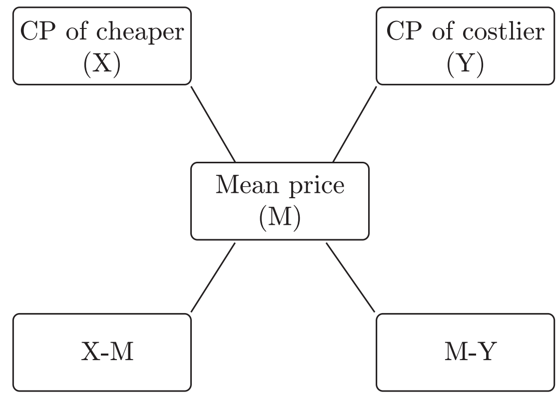


Figure 1 | Flowchart of solution using alligation rule.

SOLVED EXAMPLES

1. Marks of a student in six subjects are 97, 92, 85, 80, 92 and 79. What are the average marks obtained by the student?

Solution: Given that the total number of subjects = 6
Total marks in six subjects = 97 + 92 + 85 + 80 + 92 + 79 = 525

Average marks of the student in six subjects = $\frac{525}{6} = 87.5$

2. Two kilograms of wheat costing ₹20 per kilogram and 6 kg of wheat costing ₹25 per kilogram are mixed together. What is the net cost of the mixture?

Solution: Net cost of the mixture is

$\frac{1\times 20 + 3\times 24}{1+3}$ (because 2 and 6 are in the ratio 1:3)

$$= \frac{20+72}{4}$$
$$= \frac{92}{4}$$
$$= 23$$

Hence, net cost = ₹23 per kilogram

3. A man buys milk at ₹9 per litre and after adding water, sells it at ₹10 per litre, thereby making a profit of 25%. How much water is in the mixture?

Solution: Given that the selling price of the mixture = ₹10 per litre

Cost price of the mixture = $\frac{100}{125} \times 10$ = ₹8 per litre

Now, cost price of the mixture is only due to the cost of milk.

We know that the cost price of milk = ₹9 per litre

Thus, for ₹8, the quantity of milk = $\frac{1000}{9} \times 8 =$ 888.89 mL

Total water in the mixture = 1000 − 888.89 = 111.11 mL

4. The average marks of a class of 30 students are 82. If one student's marks from another section are added, then the average becomes 81. What are the marks of the student from another section?

Solution: Let the marks of the added student be x. Then

$$\frac{30\times 82 + x}{31} = 81$$
$$\Rightarrow 2460 + x = 2511$$
$$\Rightarrow x = 2511 - 2460 = 51$$

Hence, the marks of the added student are 51.

5. In what ratio must Tim mix two varieties of sugar worth ₹20 per kilogram and ₹32 per kilogram so that by selling the mixture at ₹36 per kilogram he may gain 20%?

Solution: It is given that the selling price of the mixture = ₹36 per kilogram
Gain = 20%

So the cost price of the mixture = $\frac{100}{120} \times 36 =$ ₹30

If the amount of sugar worth ₹20 be x

Thus, the amount of sugar worth ₹ 32 be $(1000 - x)$, thus

$$\frac{20 \times x}{1000} + \frac{32(1000 - x)}{1000} = 30$$
$$\Rightarrow 20x + 32000 - 32x = 30000$$
$$\Rightarrow 12x = 2000$$
$$\Rightarrow x = \frac{500}{3}$$

Quantities of both the sugar = 500/3 g and 2500/3 g, so the ratio will be

$$\text{Ratio} = \frac{500/3}{2500/3} = \frac{1}{5} = 1:5$$

6. In a mixture of 30 litre, the ratio of milk and water is 3:2. If this ratio is to be 1:2, then what is the quantity of water to be further added?

Solution: Given that the total quantity of the mixture = 30 litre

Ratio of milk and water = 3:2

Total quantity of milk will be

$$3 \times \frac{30}{3+2} = 18 \text{ liter}$$

Total quantity of water will be

$$2 \times \frac{30}{3+2} = 12 \text{ liter}$$

Now, if x litre of water is added.

New quantity of mixture = 30 + x

Milk = 18 litre

Now, quantity of water = 12 + x, now

$$\frac{18}{12+x} = \frac{1}{2}$$
$$\Rightarrow 36 = 12 + x$$
$$\Rightarrow x = 24$$

Thus, the quantity of water to be added = 24 litre

7. The average age of three people is 36. If their ages are in the ratio 3 : 2 : 4, what are the ages of the three persons?

Solution: Let the ages be $3x$, $2x$ and $4x$. Then,

$$\frac{3x + 2x + 4x}{3} = 36$$
$$\Rightarrow 9x = 108$$
$$\Rightarrow x = 12$$

Hence, the three ages are 39, 24 and 48.

8. The speed of a train from A to B was 40 km/h while when coming back from B to A, its speed was 60 km/h. What is the average speed during the whole journey?

Solution: Let the distance between A and B be x

So the time taken from A to B = $\dfrac{x}{40}$

And the time taken from B to A = $\dfrac{x}{60}$

Total distance covered = $2x$

Now the average speed will be

$$\frac{2x}{(x/40) + (x/60)}$$
$$= \frac{2x \times 40 \times 60}{40x + 60x}$$
$$= \frac{4800x}{100x} = 48$$

Hence, the average speed over the journey = 48 km/h

9. Average of five numbers is 40. The last number is replaced and the average becomes 39. If the last number before being replaced was 42, then what is the new last number?

Solution: Let the sum of four numbers be x and the replaced number be y. Now, initially the average of the five numbers is given by

$$\frac{x + 42}{5} = 40$$
$$\Rightarrow x = 200 - 42 = 158$$

Hence, the sum of the first four numbers = 158

Now, using the above value, after replacing the number 42 with y, we get

$$\frac{158 + y}{5} = 39$$
$$\Rightarrow y = 195 - 158 = 37$$

The last number after replacing is 37.

10. In a class, 5 students got 19 marks, 4 students got 17, 7 students got 15, 3 students got 14 and 1 student got 20. What are the average marks of the class?

Solution: We know that total students in the class = 5 + 4 + 7 + 3 + 1 = 20
Average marks of all the students in the class will be

$$\frac{5 \times 19 + 4 \times 17 + 7 \times 15 + 3 \times 14 + 1 \times 20}{20}$$
$$= \frac{95 + 68 + 105 + 42 + 20}{20} = \frac{330}{20} = 16.5$$

11. A vessel is filled with liquid, 3 parts of which are water and 5 parts syrup. How much of the mixture must be drawn off and replaced with water so that the mixture may be half water and half syrup?

Solution: Suppose the vessel initially contains 8 litres of liquid. Let x litres of this liquid be replaced with water.

Quantity of water in new mixture $= \left(3 - \frac{3x}{8} + x\right)$ litres

Quantity of syrup in new mixture $= \left(5 - \frac{5x}{8}\right)$ litres

Since the quantities of water and syrup is same,

$$\left(3 - \frac{3x}{8} + x\right) = \left(5 - \frac{5x}{8}\right)$$
$$\Rightarrow 5x + 24 = 40 - 5x$$
$$\Rightarrow 10x = 16 \Rightarrow x = \frac{8}{5}$$

Hence, part of the mixture replaced $= \left(\frac{8}{5} \times \frac{1}{8}\right) = \frac{1}{5}$

12. A 20 litre mixture of milk and water contains milk and water in the ratio 3: 2. 10 litres of the mixture is removed and replaced with pure milk and the operation is repeated once more. At the end of the two removals and replacement, what is the ratio of milk and water in the resultant mixture?

Solution: Ratio of milk and water in the mixture initially = 3:2

Say, quantity of milk is $3x$ and quantity of water is $2x$.

Hence, $3x + 2x = 20 \Rightarrow x = 4$

Thus, in the beginning we have 12 litres of milk and 8 litres of water.

After the first removal, we have 10 litres of solution containing 6 litres of milk and 4 litres of water. We then add 10 litres of pure milk in the solution. The new ratio of milk and water in the solution is 16:4 or 4:1.

After the second removal, we have 10 litres of solution containing 8 litres of milk and 2 litres of water. We then add 10 litres of pure milk in the solution again. The new ratio of milk and water in the solution is 18:2 or 9:1.

13. In what ratio must a grocer mix two varieties of pulses costing ₹15 and ₹20 per kg respectively so as to get a mixture worth ₹16.50 kg?

Solution: According to alligation rule,

$$\frac{x}{y} = \frac{c_y - c}{c - c_x}$$

Let x be the quantity of pulses costing ₹15 per kg and y be the quantity of pulses costing ₹20 per kg Thus, putting the values given to us,

$$\frac{x}{y} = \frac{20 - 16.5}{16.5 - 15} = \frac{3.5}{1.5} = \frac{7}{3}$$

14. A servant stole juice from the jar which was 40% concentration and replaced it with juice with concentration 16%. The jar only had juice of concentration 24%. How much of the juice did the servant steal?

Solution: Using alligation rule,

$$\frac{x}{y} = \frac{c_y - c}{c - c_x} = \frac{40 - 24}{24 - 16} = \frac{16}{8} = \frac{2}{1}$$

Where x is the quantity of juice with 16% concentration and y is the quantity of juice with 40% concentration.

Thus, 1/3 of the juice was left. Hence, the servant took 2/3 of the original juice.

15. In what ratio must water be mixed with milk to gain $16\frac{2}{3}\%$ on selling the mixture at cost price?

Solution: Let C.P. of 1 litre milk be ₹ x.
S.P. of 1 litre of mixture = ₹ x

Gain = $16\frac{2}{3}\%$

Thus, C.P. of 1 litre of mixture $= \left(100 \times \frac{3}{350} x\right) = \frac{6}{7}x$

Using alligation rule,

$$\frac{x}{y} = \frac{x - \frac{6}{7}x}{\frac{6}{7}x - 0} = \frac{1}{7} : \frac{6}{7} = 1 : 6 \qquad \text{[C.P. of water = 0]}$$

Hence, ratio of water and milk = 1:6.

16. The average weight of 16 boys in a class is 50.25 kg and that of the remaining 8 boys is 45.15 kg. Find the average weights of all the boys in the class.

Solution: Using the formula of weighted average, we can get the required average =

$$\left(\frac{50.25 \times 16 + 45.15 \times 8}{16 + 8}\right) = \left(\frac{804 + 361.2}{24}\right)$$
$$\left(\frac{1165.2}{24}\right) = 48.55$$

Thus the average weight of all the boys in the class is 48.55 kg.

17. A student's marks were wrongly entered as 83 instead of 63. Due to that the average marks for the class got increased by half. How many students are there in the class?

Solution: Let the total number of students in the class be x.
Total increase in marks $= \dfrac{x}{2}$
Thus,

$$\frac{x}{2} = (83 - 63) \Rightarrow \frac{x}{2} = 20 \Rightarrow x = 40$$

18. The captain of a football team of 11 members is 26 years old and the wicket keeper is 3 years older. If the ages of these two are excluded, the average age of the remaining players is one year less than the average age of the whole team. What is the average age of the team?

Solution: Let the average age of the whole team be x years.
Thus,

$$11x - (26 + 29) = 9(x - 1)$$
$$\Rightarrow 11x - 9x = 46 \Rightarrow 2x = 46$$
$$\Rightarrow x = 23$$

Hence, average age of the team is 23 years.

19. A person buys petrol at ₹7.50, ₹8 and ₹8.50 per litre for three successive years. What approximately is the average cost per litre of petrol if he spends ₹4000 each year?

Solution: Total quantity of petrol consumed in 3 years $= \left(\dfrac{4000}{7.5} + \dfrac{4000}{8} + \dfrac{4000}{8.5}\right) = \left(\dfrac{76700}{51}\right)$ litres

Total amount spent $= 3 \times 4000 =$ ₹12000

Thus, average cost spent on petrol over the three years $= \left(\dfrac{12000}{76700/51}\right) = \dfrac{6120}{767} =$ ₹7.98 per litre

20. The average monthly income of P and Q is ₹5050. The average monthly income of Q and R is ₹6250 and the average monthly income of P and R is ₹5200. What is the monthly income of P?

Solution: Let x, y and z represent the respective monthly incomes of P, Q and R. Thus, we can write

$$x + y = (5050 \times 2) = 10100 \quad (1)$$
$$y + z = (6250 \times 2) = 12500 \quad (2)$$
$$x + z = (5200 \times 2) = 10400 \quad (3)$$

Adding Eqs. (1), (2) and (3), we get

$$2(x + y + z) = 33000 \Rightarrow x + y + z = 16500 \quad (4)$$

Subtracting Eq. (2) from (4), we get

$$x = 4000.$$

Hence, P's monthly income = ₹4000.

PRACTICE EXERCISE

1. Marks of 10 students in English are 46, 48, 47, 44, 40, 49, 39, 42, 44 and 41. What are the average marks obtained in English?

(a) 44 (b) 45 (c) 43.5 (d) 46.4

2. Average weight of 5 people is 64.4. If the weight of four people is 64, 66, 58 and 72, then what is the weight of the fifth person?

(a) 65 (b) 60 (c) 62 (d) 64

3. The average marks of students of class 5A in a subject are 38 and of class 5B are 42.5. If the number of students in class 5A is 25 and 5B is 20, then what are the average marks of all the students taken together?

(a) 42 (b) 40 (c) 39.5 (d) 41.5

4. Five kilogram of flour costing ₹6 per kilogram and 10 kg of flour costing ₹9 per kilogram are mixed together. What is the net cost of the mixture?

(a) ₹7.2 per kilogram (b) ₹6.5 per kilogram
(c) ₹10 per kilogram (d) ₹8 per kilogram

5. A man buys milk at ₹5 per litre and after adding water, sells it at ₹6 per litre, thereby making a profit of $33\frac{1}{3}\%$. What is the proportion of milk to water in the mixture?

(a) 1:9 (b) 9:10 (c) 9:1 (d) 4:5

6. The average weight of a group of 20 boys is 60 kg. If the weight of another boy is added then the average weight becomes 58. What is the weight of the boy?

(a) 39 kg (b) 41 kg (c) 35.5 kg (d) 48 kg

7. The average temperature from Monday to Friday was recorded to be 12.5°C. If the average temperature of Monday and Tuesday was 14°C and the temperature on Wednesday and Thursday was 10°C and 11°C, respectively, then what was the temperature on Friday?

(a) 12°C (b) 13.5°C
(c) 11.4°C (d) 14°C

8. In what ratio must a grocer mix two varieties of salt worth ₹40 per kilogram and ₹48 per kilogram so that by selling the mixture at ₹50.6 per kilogram he may gain 10%?

(a) 1:2 (b) 1:3 (c) 2:3 (d) 4:1

9. In a mixture of 48 litre, the ratio of milk and water is 2:1. If this ratio is to be 1:3, then what is the quantity of water to be further added?

(a) 96 litre (b) 72 litre (c) 60 litre (d) 80 litre

10. The average of 10 results is 20 and that of 20 more results is 10. What is the average for all the results taken together?

(a) 12.5 (b) 14.75 (c) 13.33 (d) 12

11. Average of four numbers is 8. If the second and fourth number is same, first number is twice the second number and third number is twice the first number, then what is the smallest number?

(a) 8 (b) 6 (c) 4 (d) 2

12. The ratio of the number of boys and girls in a college is 5:4. If the percentage increase in the number of boys and girls be 20% and 25%, respectively, what will be the ratio of the total number of students before to the total number of students after increase?

(a) 9:11 (b) 5:11 (c) 4:9 (d) 8:9

13. Jack has a collection of 280 balls. If 35% of them are red, 70 of them are yellow and the rest of the balls are black, then what is the ratio of red balls to black balls?

(a) 7:8 (b) 3:4 (c) 5:6 (d) 2:5

14. The average age of three people is 52. If their ages are in the ratio 2:5:6, which of the following is not the age of any of the three persons?

(a) 60 (b) 64 (c) 24 (d) 72

15. The speed of a train from A to B is 50 km/h while when coming back from B to A, its speed is 75 km/h. What is the average speed during the whole journey?

(a) 65 km/h (b) 62.5 km/h
(c) 60 km/h (d) 67.5 km/h

16. A lab assistant mixes Zn and Cl in the ratio of 1:2 in order to produce ZnCl2. If the total amount of ZnCl2 produced is 50.7 g, what is the amount of Zn added?

(a) 21.3 g (b) 25.35 g (c) 30.8 g (d) 16.9 g

17. Average of 5 numbers is 24. The last number is replaced and the average becomes 26. If the last number before being replaced was 25, then what is the new last number?

(a) 30 (b) 35 (c) 24 (d) 28

18. A tank contains 100 litre of pure milk. After every 10 min, 10 litre of solution is replaced with water. What will be the remaining amount of milk in the solution after 40 min?

(a) 65.61 litre (b) 58.65 litre
(c) 72.25 litre (d) 62.45 litre

19. A person made a profit of 20% on 25% of the total quantity sold and on the rest he makes a profit of 10%. What was the net profit percentage in the whole transaction?

(a) 15% (b) 12.5% (c) 25% (d) 17.5%

20. A man bought 5 apples worth ₹5 each, 10 bananas worth ₹2 each, 3 oranges worth ₹4 each and 7 kiwis worth ₹12 each. What is the average of the total amount of fruits bought?

(a) ₹7 (b) ₹6.54 (c) ₹4.35 (d) ₹5.36

21. If the average marks of three batches of 55, 60 and 45 students, respectively, are 50, 55, 60, then what are the average marks of all the students?

(a) 53.33 (b) 55
(c) 54.68 (d) None of these

22. The average of 20 numbers is zero. Of them, at the most, how many may be greater than zero?

(a) 0 (b) 1 (c) 10 (d) 19

23. The average age of husband, wife and their child 3 years ago was 27 years and that of wife and the child 5 years ago was 20 years. What is the present age of the husband?

(a) 35 years (b) 40 years
(c) 50 years (d) 60 years

Directions for Q24 to Q25: Each of the questions given below consists of a question and two or three statements. You have to decide whether the data provided in the statement(s) is/are sufficient to answer the given question.

24. The average age of P, Q, R and S is 30 years. How old is R?

I. The sum of ages of P and R is 60 years.
II. S is 10 years younger than R.

(a) I alone sufficient while II alone not sufficient to answer.
(b) II alone sufficient while I alone not sufficient to answer.
(c) Both I and II are not sufficient to answer.
(d) Both I and II are necessary to answer.

25. In a cricket team, the average age of eleven players in 28 years. What is the age of the captain?

I. The captain is eleven years older than the youngest player.

II. The average age of 10 players, other than the captain is 27.3 years.

III. Leaving aside the captain and the youngest player, the average ages of three groups of three players each are 25 years, 28 years and 30 years respectively.

(a) All I, II and III
(b) II and III only
(c) II only or I and III only
(d) Any two of the three

26. A can contains a mixture of two liquids A and B is the ratio 7:5. When 9 litres of mixture are drawn off and the can is filled with B, the ratio of A and B becomes 7:9. How many litres of liquid A was contained by the can initially?

(a) 10 (b) 20 (c) 21 (d) 25

27. Tea packs worth ₹126 per kg and ₹135 per kg are mixed with a third variety in the ratio 1:1:2. If the mixture is worth ₹153 per kg, what will the price of the third variety per kg?

(a) ₹175.50 (b) ₹170
(c) ₹169.50 (d) ₹180

28. Find the ratio in which sugar at ₹7.20 per kg be mixed with rice at ₹5.70 a kg to produce a mixture worth ₹6.30 a kg.

(a) 1:3 (b) 2:3 (c) 3:4 (d) 4:5

29. How many kilogram of sugar costing ₹9 per kg must be mixed with 27 kg of sugar costing ₹7 per kg so that there may be a gain of 10% by selling the mixture at ₹9.24 per kg?

(a) 63 kg (b) 54 kg (c) 42 kg (d) 36 kg

30. A container contains 40 litres of milk. From this container 4 litres of milk was taken out and replaced by water. This process was repeated further two times. How much milk is now contained by the container?

(a) 26.34 litres (b) 27.36 litres
(c) 28 litres (d) 29.16 litres

ANSWERS

1. (a)	**5.** (c)	**9.** (d)	**13.** (a)	**17.** (b)	**21.** (c)	**25.** (c)	**29.** (a)
2. (c)	**6.** (a)	**10.** (c)	**14.** (b)	**18.** (a)	**22.** (d)	**26.** (c)	**30.** (d)
3. (b)	**7.** (b)	**11.** (c)	**15.** (c)	**19.** (b)	**23.** (b)	**27.** (a)	
4. (d)	**8.** (b)	**12.** (a)	**16.** (d)	**20.** (d)	**24.** (c)	**28.** (b)	

EXPLANATIONS AND HINTS

1. (a) Given that the total number of students = 10

Total marks of the students in English = 46 + 48 + 47 + 44 + 40 + 49 + 39 + 42 + 44 + 41

= 440

Average marks of 10 students in English $= \frac{440}{10} = 44$

2. (c) Given that the average weight of 5 people = 64.4

Let the weight of the fifth person be x. Then,

$$\frac{64+66+58+72+x}{5} = 64.4$$
$$\Rightarrow 260 + x = 322$$
$$\Rightarrow x = 62$$

3. (b) Given that the total students in class 5A = 25 and total students in class 5B = 20

The students are in the ratio of 25:20 = 5:4

Hence, average marks of all students taken together will be

$$\frac{5\times 38 + 4\times 42.5}{5+4} = \frac{360}{9} = 40$$

4. (d) Net cost of the mixture is

$$\frac{1\times 6 + 2\times 9}{1+2} = \frac{24}{3} = 8$$

Hence, the net cost = ₹8 per kilogram

5. (c) Given that the selling price of the mixture = ₹6 per litre.

So the cost price of the mixture will be

$$\frac{100}{133.33}\times 6 = ₹4.5 \text{ per liter}$$

Now, the cost price of the mixture is only due to the cost of milk

We know that the cost price of milk = ₹5 per litre

Thus, for ₹4.5, we would get milk $= \frac{1000}{5}\times 4.5 =$ 900 mL

The mixture contains 900 mL milk and 100 mL water

Thus, the ratio of milk to water = 900:100 = 9:1

6. (a) Let the weight of the boy be x. Then,

$$\frac{20\times 60+x}{21}=58$$
$$\Rightarrow 1200+x=59\times 21$$
$$\Rightarrow x=1239-1200=39$$

Hence, the weight of the boy = 39 kg

7. (b) Let the temperature on Friday be x.

Now, the average temperature of Monday and Tuesday = 14°C

Hence, the total temperature of Monday and Tuesday = 28°C

Temperature recorded on Wednesday = 10°C

Temperature recorded on Thursday = 11°C

Now, average of temperature from Monday to Friday can be given as

$$\frac{28+10+11+x}{5}=12.5$$
$$\Rightarrow 49+x=12.5\times 5$$
$$\Rightarrow x=62.5-49=13.5$$

Hence, the temperature on Friday = 13.5°C

8. (b) Given that the selling price of the mixture = ₹50.6 per kilogram

Gain = 10%

Cost price of the mixture $=\frac{100}{110}\times 50.6=$ ₹46

If the amount of salt worth ₹40 be x

The amount of salt worth ₹48 be $(1000-x)$

Thus,

$$\frac{40\times x}{1000}+\frac{48(1000-x)}{1000}=46$$
$$\Rightarrow 40x+48000-48x=46000$$
$$\Rightarrow 8x=2000$$
$$\Rightarrow x=250$$

Quantities of both the salts = 250 g and 750 g

$$\text{Ratio}=\frac{250}{750}=\frac{1}{3}=1:3$$

9. (d) Given that the total quantity of mixture = 48 litre

Ratio of milk and water = 2:1

Total quantity of the milk $=2\times\frac{48}{2+1}=32$ liter

Total quantity of water $=1\times\frac{48}{2+1}=16$ liter

Now, if x litre of water is added. Then the new quantity of mixture = 48 + x

Milk = 32 litre

Now quantity of water = 16 + x

Now,

$$\frac{32}{16+x}=\frac{1}{3}$$
$$\Rightarrow 3\times 32=16+x$$
$$\Rightarrow x=96-16=80$$

Thus, the quantity of water to be added = 80 litre

10. (c) The average is given by

$$\frac{10\times 20+20\times 10}{10+20}$$
$$=\frac{400}{30}=13.33$$

11. (c) If the second number is x.

The first number = $2x$

Third number = $4x$

Fourth number = x

Average of four numbers = 8

Average is given by

$$\frac{2x+x+4x+x}{4}=8$$
$$\Rightarrow 8x=32$$
$$\Rightarrow x=4$$

Hence, the smallest number = 4

12. (a) Given that the ratio of boys to girls = 5:4

Increase in the number of boys = 20%

Increase in the number of girls= 25%

New ratio of boys to girls will be

$$5+\frac{20}{100}\times 5:4+4\times\frac{25}{100}=6:5$$

If x is a constant such that total students before = $5x$ + $4x$ = $9x$

Similarly, total students after increment = $6x$ + $5x$ = $11x$

Ratio of students before and after the increase = $9x:11x$ = 9:11

13. (a) Given that the total balls that Jack has = 280 and the total yellow balls = 70

Total percentage of yellow balls $=\frac{70}{280}\times 100=25\%$

Total red balls = 35%

Total black balls = 100 – (25 + 35)% = 40%

Ratio of red to black balls = 35/40 = 7/8 = 7:8

14. (b) Let the ages be $2x$, $5x$ and $6x$. Then,

$$\frac{2x+5x+6x}{3}=52$$
$$\Rightarrow 13x=156$$
$$\Rightarrow x=12$$

Hence, the three ages are 24, 60 and 72.

So the answer is 64.

15. (c) Let the distance between A and B be x

Time taken from A to B = $\frac{x}{50}$

Time taken from B to A = $\frac{x}{75}$

Total distance covered = $2x$

Now, average speed will be

$$\frac{2x}{(x/50)+(x/75)}=\frac{2x\times 50\times 75}{75x+50x}$$
$$=\frac{7500x}{125x}=60$$

Hence, the average speed over the journey will be 60 km/h.

16. (d) Given that the ratio of zinc and chloride = 1:2

Total amount of $ZnCl_2$ produced = 50.7 g

If Zn added be x and Cl added be $2x$

Thus, the total amount $=x+2x=50.7$

$$\Rightarrow 3x=50.7$$
$$\Rightarrow x=\frac{50.7}{3}=16.9$$

Total amount of Zn mixed = 16.9 g

17. (b) Let the sum of four numbers be x and the replaced number be y. Now, initially the average of the five numbers is given by

$$\frac{x+25}{5}=24$$
$$\Rightarrow x=120-25=95$$

Hence, the sum of the first four numbers = 95

Now, using the above value, after replacing the number 25 with y, we get

$$\frac{95+y}{5}=26$$
$$\Rightarrow y=130-95=35$$

The last number after replacing is 35.

18. (a) Given that after 10 min, 10 litre of milk is replaced with water.

Hence, in the solution, milk = 90 litre and water = 10 litre

Now, after 20 min, 10 litre of the solution is replaced with water.

The solution has milk and water in the ratio of 9:1.

Hence, the remaining amount of milk = 81 litre

Remaining amount of water = 19 litre

Now, solution has milk and water in the ratio of 81:19.

Now, after 30 min, 10 litre of the solution is replaced with water.

The solution has milk and water in the ratio of 81:19.

Hence, milk taken out from the solution = 8.1 litre

Water taken out from the solution = 1.9 litre

Remaining amount of milk = 72.9 litre

Remaining amount of water = 27.1 litre

After 40 min, 10 litre of solution will have 7.29 litre milk and 2.71 litre water and is replaced with water.

Remaining amount of milk = 72.9 − 7.29 = 65.61 litre

19. (b) If the total quantity is x, then

Profit made on 25% of the total quantity = $\frac{20}{100}\times\frac{x}{4}=\frac{x}{20}$

Profit made on 75% of the total quantity = $\frac{10}{100}\times\frac{3x}{4}=\frac{3x}{40}$

So the total profit

$$\frac{x}{20}+\frac{3x}{40}=\frac{5x}{40}$$

Total profit percentage = 12.5%

20. (d) Total cost of apples = ₹5 × 5 = ₹25

Total cost of bananas = ₹2 × 10 = ₹20

Total cost of oranges = ₹4 × 3 = ₹12

Total cost of kiwis = ₹11 × 7 = ₹77

Total cost of all the fruits = ₹77 + 12 + 20 + 25 = ₹134

Total fruits = 5 + 3 + 10 + 7 = 25

Average cost of all the items = $\frac{134}{25}$ = ₹5.36

21. (c) According to formula of weighted average,

$$\text{required average}=\left(\frac{55\times 50+60\times 55+45\times 60}{55+60+45}\right)$$
$$=\left(\frac{2750+3300+2700}{160}\right)=\frac{8750}{160}=54.68$$

22. (d) Average of 20 numbers = 0.

Therefore, sum of 20 numbers (0 × 20) = 0.

It is possible that 19 of these numbers are positive with sum = x and the 20^{th} number be $-x$.

23. (b) Sum of the present ages of husband, wife and child = $(27 \times 3 + 3 \times 3)$ years = 90 years

Sum of the present ages of wife and child = $(20 \times 2 + 5 \times 2)$ years = 50 years

Thus, husband's present age = (90 – 50) = 40 years.

24. (c) P + Q + R + S = (30 × 4)

$\Rightarrow$ P + Q + R + S = 120 (i)
I. P + R = 60 (ii)
II. S = (R – 10) (iii)

From Eqs. (i), (ii) and (iii), we cannot find R. Thus the data is insufficient.

25. (c) Total age of 11 players = (28 × 11) years = 308 years.

I. C = Y + 11 $\Rightarrow$ C – Y = 11 (i)
II. Total age of 10 players (excluding captain) = (27.3 × 10) years = 273 years

Thus, age of captain = (308 – 273) years = 35 years

Thus, C = 35 (ii)

Putting value from Eq. (ii) in (i), we get

Y = 35 – 11 = 24

III. Total age of 9 players = $[(25 \times 3) + (28 \times 3) + (30 \times 3)]$ = 249 years

Thus, C + Y = (308 – 249) years = 59 years (iii)

From Eq. (i) and (iii), we get

C = 35

Thus, II alone gives the answer. Also, I and III together give the answer.

26. (c) Suppose the can initially contains $7x$ and $5x$ of mixtures A and B, respectively.

Quantity of A in mixture left = $\left(7x - \frac{7}{12} \times 9\right) = \left(7x - \frac{21}{4}\right)$ litres.

Quantity of B in mixture left = $\left(5x - \frac{5}{12} \times 9\right) = \left(5x - \frac{15}{4}\right)$ litres.

$$\therefore \frac{\left(7x - \frac{21}{4}\right)}{\left(5x - \frac{15}{4}\right) + 9} = \frac{7}{9}$$

$$\Rightarrow \frac{28x - 21}{20x + 21} = \frac{7}{9} \Rightarrow 252x - 189 = 140x + 147$$

$\Rightarrow 112x = 336$

$\Rightarrow x = 3$

Hence, the can contained 21 litres of A.

27. (a) We know that first and second varieties are mixed in equal proportions.

Thus, average price = $\left(\frac{126 + 135}{2}\right) = ₹130.5$

So, their mixture is formed by mixing two varieties, on at ₹130.50 per kg and the other at ₹X in the ratio 1:1.

Hence, by using alligation formula,

$$\frac{x - 153}{22.5} = 1 \Rightarrow x = 175.5$$

28. (b) Using alligation rule,

$$\frac{x}{y} = \frac{6.3 - 5.7}{7.2 - 6.3} = \frac{0.6}{0.9} = \frac{2}{3}$$

29. (a) S.P. of 1 kg of mixture = ₹9.24

Gain = 10%

$\therefore$ C.P. of 1 kg of mixture = ₹$\left(\frac{100}{110} \times 9.24\right)$ = ₹8.40

Using the formula for alligation,

$$\frac{x}{y} = \frac{8.4 - 7}{9 - 8.4} = \frac{1.4}{0.6} = \frac{7}{3}$$

Let z kg of sugar of 1st kind be mixed with 27 kg of 2^{nd} kind.

Then,

$$7 : 3 = z : 27$$

$$\Rightarrow z = \left(\frac{7 \times 27}{3}\right) = 63 \text{ kg}$$

30. (d) Container contains 40 litres of milk. After removing 4 litres of milk by water, we have a mixture of 36 litres milk and 4 litres of water. The ratio of milk and water is 9:1. Hence, when 4 litres of mixture is removed again, 3.6 litres of milk and 0.4 litres of water is removed. Now, remaining mixture will have $(36 - 3.6) = 32.4$ litres of milk and 7.6 litres of water.

When we remove 4 litres of mixture again, 3.24 litres of milk and 0.76 litres of water is replaced by 4 litres of water.

Hence, the final mixture contains $(32.4 - 3.24) = 29.16$ litres.

CHAPTER 7

RATIO, PROPORTION AND VARIATION

RATIO

A *ratio* is a fraction of two numbers of the same kind. It is used to tell how many times a given number is in comparison to the other number or what part a given number is in comparison to the other number.

Ratio of two numbers, a and b, is denoted by $a{:}b$ and is calculated by $\frac{a}{b}$.

In the above expression, numerator "a" is called *antecedent* and denominator "b" is called *consequent*.

If

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$$

Then, each ratio will be

$$\frac{a+c+e}{b+d+f}$$

The same principle can also be applied after multiplying numerator and denominator of any fraction by the same number.

If

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$$

Then each ratio will be

$$\left(\frac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{1/n}$$

where p, q, r and n may have any values except that they must not all be zeros.

The *compound ratio* of r_1 and r_2, where $r_1=a:b$ and $r_2=x:y$ is defined as $r_1r_2=ax:by$. Hence, compound ratios are simply a multiplication of two ratios.

While comparing two numbers in terms of ratio:

1. Both the numbers or quantities should be of the same kind. For example, you cannot calculate the ratio between rupees and apples.
2. Units of measurements of numbers or quantities must be same. For example, you cannot calculate the ratio between ₹20 and 200 paise. You can either convert ₹ 20 into 2000 paise or convert 200 paise into ₹ 2 to calculate the correct ratio.

3. Ratio of any quantity is always dimensionless and unitless. It is a pure number.
4. The ratio remains unchanged if both the antecedent and consequent are multiplied and divided by the same number.

Till now we have just focused on ratios when they are equal. However, ratios sometimes do not follow equality. For unequal ratios, we have

$$\frac{a}{b} \neq \frac{c}{d}$$

For unequal ratios, the two important points to be considered are as follows:

1. The sense of an inequality remains unchained if both members of the ratio are multiplied (or divided) by the same positive number.

   If $a > b$ and $k > 0$, then $ka > kb$ and $\frac{a}{k} > \frac{b}{k}$.

2. The sense of an inequality is reversed if both members of the ratio are multiplied (or divided) by the same negative number.

   If $a > b$ and $k < 0$, then $ka < kb$ and $\frac{a}{k} < \frac{b}{k}$.

PROPORTION

When two ratios are equal to each other, then the four terms constituting the ratios are said to be in *proportion*. If $a{:}b = c{:}d$, then a, b, c and d are in proportion.

The terms a and d are called *extremes*, while b and c are called *means*.

When $c = b / c$, then a, b and c are said to be in *continued proportion* and b is called the *geometric mean* or *mean proportional* between a and c.

Also,

$$b^2 = a \times c$$

Then,

$$b = \sqrt{a \times c}$$

Some properties of ratios and proportions are as follows:

1. **Invertendo:** Inverse ratios of two equal ratios are equal. Thus, if

   $$\frac{a}{b} = \frac{c}{d}$$

   Then,

   $$\frac{b}{a} = \frac{d}{c}$$

2. **Alternendo:** Ratios of antecedents and consequents of two equal ratios are equal. Thus, if

   $$\frac{a}{b} = \frac{c}{d}$$

   Then,

   $$\frac{a}{c} = \frac{b}{d}$$

3. **Componendo:** It is the result we get after adding 1 to both sides of the equality. Thus, if

   $$\frac{a}{b} = \frac{c}{d}$$

   Then,

   $$\frac{a+b}{b} = \frac{c+d}{d}$$

4. **Dividendo:** It is the result we get after subtracting 1 from both sides of the equality. Thus, if

   $$\frac{a}{b} = \frac{c}{d}$$

   Then

   $$\frac{a-b}{b} = \frac{c-d}{d}$$

5. **Componendo–Dividendo:** It is the result we get by dividing results of componendo by dividendo. Thus, if

   $$\frac{a}{b} = \frac{c}{d}$$

   Then

   $$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

VARIATION

Variation means change of a quantity with respect to another quantity. It is the relationship between the change of two or more quantities. Now, quantities can vary broadly in the following ways:

1. If a is directly proportional to b then, the nature of change is similar in the values of a and b. Hence, if a becomes twice its value then b also becomes twice its value.

   $$a \propto b$$

2. If a is indirectly proportional to b, then the nature of change is inverse in the value of a and b. Hence, if a becomes twice its value, then b becomes half its value.

   $$a \propto \frac{1}{b}$$

SOLVED EXAMPLES

1. Two numbers x and y are 20% and 40% more than a number z. What is the ratio of the two numbers?

Solution: We have three numbers x, y and z. Also,

$$x = z + 20\% \text{ of } z = \frac{6}{5}z$$

$$y = z + 40\% \text{ of } z = \frac{7}{5}z$$

$$\therefore x : y = \frac{6/5z}{7/5z}$$

$$\Rightarrow x : y = \frac{6}{7} = 6 : 7$$

2. A sum of money is to be distributed among A, B, C and D in the proportion of 3:2:1:4. If A gets ₹ 1500, what is D's share?

Solution: Given that A:B:C:D = 3:2:1:4

Now, we know that A's share = ₹ 1500

So A:D = 3:4

Now, if A = ₹ 1500, then

$$D = \frac{4}{3} \times 1500 = 2000$$

Hence, D's share = ₹ 2000

3. If a sum of money is to be distributed among A, B, C and D in the proportion of 1:7:4:2. If B gets ₹ 900 more than C, what is D's share?

Solution: Given that A:B:C:D = 1:7:4:2

Let B get x amount of the total. Then,

$$\text{C} = \frac{4x}{7}$$

Also,

$$\text{B} - \text{C} = 900$$

Hence,

$$x - \frac{4x}{7} = 900$$

$$\Rightarrow \frac{3x}{7} = 900$$

$$\Rightarrow x = 300 \times 7 = 2100$$

Thus, B's share = ₹ 2100

$$\text{Total share of D} = \frac{2}{7} \times 2100 = ₹\ 600$$

4. The ratio of the number of boys and girls in a college is 7:5. If the percentage increase in the number of boys and girls be 20% and 40%, respectively, what will be the new ratio?

Solution: Given that the ratio of boys to girls = 7:5

Increase in the number of boys = 20%

Increase in the number of girls = 40%

So the new ratio of boys to girls will be

$$7 + \frac{20}{100} \times 7 : 5 + 5 \times \frac{25}{100} = 6 : 5 = 9 : 11$$

5. If x:13::121:143, then what is the value of x?

Solution: It is given that

$$\frac{x}{13} = \frac{121}{143}$$

$$\Rightarrow x = \frac{121 \times 13}{143}$$

$$\Rightarrow x = 11$$

6. A sum of three numbers is 84. The ratio of the first and the second number is 5:3 and the ratio of the second and the third number is 3:4. What are the three numbers?

Solution: If x is a constant such that

$$5x + 3x + 4x = 84$$

$$\Rightarrow 12x = 84$$

$$\Rightarrow x = 7$$

First number = 5 × 7 = 35

Second number = 3 × 7 = 21

Third number = 4 × 7 = 28

So, the three numbers are 35, 21 and 28.

7. The ratio of sale of shirts to trousers of a store is 4:3 and the ratio of sale of tie to trouser is 4:5. What is the ratio of sale of shirts to tie?

Solution: Given that the ratio of sold shirts to trousers = 4:3

Ratio of sold tie to trousers = 4:5

Ratio of sold trousers to tie = 5:4

Ratio of sold shirts to tie = $\frac{4}{3} \times \frac{5}{4} = \frac{5}{3} = 5:3$

8. What is the third proportional to 8 and 12?

Solution: Let the third proportional be x. Then, 8:12::12:x

$$\Rightarrow \frac{8}{12} = \frac{12}{x}$$

$$x = \frac{12 \times 12}{8} = 3 \times 6 = 18$$

Thus, the third proportional is 18.

9. What ratio is equal to the ratio $8^3{:}16^{2.5}$?

Solution: Given that the ratio is $8^3{:}16^{2.5}$.

Now, the ratio can be rewritten as $[(2)^3]^3 : [(2)^4]^{2.5}$

$$\Rightarrow 2^9 : 2^{10} = 1:2$$

Thus, the ratio equal to $8^3{:}16^{1.5}$ = 1:2

10. A car running at a certain speed increases its speed to 4/3 times and reaches its destination 15 min earlier. How long will it take to reach the destination with the original speed?

Solution: Let the original speed be x km/h.
So the new speed = $4/3x$

If the time taken originally be t_1 and with the new speed be t_2, then

$$t_1 \times x = t_2 \times \frac{4}{3}x$$
$$\Rightarrow t_1 = t_2 \times \frac{4}{3}$$
$$\Rightarrow t_2 = t_1 \times \frac{3}{4}$$

Also, $t_1 - t_2 = 15$ min. Hence

$$t_1 - t_1 \times \frac{3}{4} = 15$$
$$\Rightarrow \frac{1}{4}t_1 = 15$$
$$\Rightarrow t_1 = 60$$

Hence, the time taken originally = 60 min

11. A metal bar ten feet long weighs 128 kg. What is the weight of a similar bar that is two feet four inches long?

Solution: Length of the second bar = 2 feet 4 inches = $\frac{28}{12} = \frac{7}{3}$ feet

Say, weight of the bar is x kg. Thus,

$$\frac{10}{128} = \frac{7/3}{x} \Rightarrow x = \frac{7}{3} \times \frac{128}{10} = \frac{448}{15} = 29.87 \text{ kg}$$

12. What is the fourth proportional to 5, 8 and 15?

Solution: Let the fourth proportional to 5, 8, 15 be x.

Then, 5:8:15:x

$$\Rightarrow 5x = (8 \times 15)$$
$$\Rightarrow x = 24$$

13. Two numbers are in the ratio 3:5. If 9 is subtracted from each, the new numbers are in the ratio 12:23. What is the larger number?

Solution: Let the two numbers be $3x$ and $5x$. Then,

$$\frac{3x-9}{5x-9} = \frac{12}{23}$$
$$\Rightarrow 23(3x-9) = 12(5x-9)$$
$$\Rightarrow 9x = 99 \Rightarrow x = 11$$

Thus, the smaller number is $(5 \times 11) = 55$.

14. The salaries A, B, C are in the ratio 2:3:5. If the increments of 15%, 10% and 20% are allowed, respectively, in their salaries, then what will be new ratio of their salaries?

Solution: Let A = $2x$, B = $3x$ and C = $5x$.

$$\text{A's new salary} = \left(\frac{115}{100} \times 2x\right) = \frac{23x}{10}$$

$$\text{B's new salary} = \left(\frac{110}{100} \times 3x\right) = \frac{33x}{10}$$

$$\text{C's new salary} = \left(\frac{120}{100} \times 5x\right) = 6x$$

$$\text{Thus, new ratio} = \left(\frac{23x}{10} : \frac{33x}{10} : 6x\right) = 23:33:6$$

15. The ratio of the number of boys and girls in a college is 7:8. If the percentage increase in the number of boys and girls be 20% and 10%, respectively, what will be the new ratio?

Solution: Originally, let the number of boys and girls in the college be $7x$ and $8x$, respectively.

Number of boys and girls after increase is $\left(\frac{120}{100} \times 7x\right)$ and $\left(\frac{110}{100} \times 8x\right)$

Hence, the required ratio = $\left(\frac{120}{100} \times 7x\right) : \left(\frac{110}{100} \times 8x\right)$ = 21 : 22

16. If ₹ 736 was divided into three parts, proportional to $\frac{1}{2} : \frac{2}{3} : \frac{3}{4}$, then what is the first part?

Solution: Given ratio = $\frac{1}{2} : \frac{2}{3} : \frac{3}{4} = 6:8:9$.

$$\text{The first part} = 736 \times \frac{6}{6+8+9} = 736 \times \frac{6}{23} = ₹\ 192$$

17. In a mixture 60 litres, the ratio of milk and water 2:1. If this ratio is to be 1:2, then what quantity of water is to be further added?

Solution: Quantity of milk = $\left(60 \times \frac{2}{3}\right)$ = 40 litres

Quantity of water in it = $(60 - 40)$ = 20 litres

New ratio = 1:2

Let quantity of water to be added further be x litres.

Then, ratio of milk and water $= (40 : 20 + x)$

Now,

$$\left(\frac{40}{20+x}\right) = \frac{1}{2}$$

$$\Rightarrow 20 + x = 80 \Rightarrow x = 60$$

Therefore, quantity of water to be added = 60 litres.

18. A sum money is to be distributed among A, B, C, D in the proportion of 5:2:4:3. If C gets ₹ 1000 more than D, what is B's share?

Solution: Let the shares of A, B, C and D be ₹ $5x$, ₹ $2x$, ₹ $4x$ and ₹ $3x$, respectively.
Then,

$$4x - 3x = 1000 \Rightarrow x = 1000$$

Thus, B's share = ₹ $2x = 2 \times 1000 =$ ₹ 2000

19. Two numbers are, respectively, 20% and 50% more than a third number. What is the ratio of the two numbers?

Solution: Let the third number be x.

Then, first number $= \dfrac{120x}{100} = \dfrac{6x}{5}$

Second number $= \dfrac{150x}{100} = \dfrac{3x}{2}$

Therefore, ratio of first two numbers $= \left(\dfrac{6x}{5} : \dfrac{3x}{2}\right)$
$= 12x : 15x = 4 : 5$

19. The sum of three numbers is 98. If the ratio of the first to second is 2:3 and that of the second to the third is 5:8, then what is the second number?

Solution: Let the three parts be x, y and z. Then,

$x : y = 2 : 3$ and $y : z = 5 : 8 = \left(5 \times \dfrac{3}{5}\right) : \left(8 \times \dfrac{3}{5}\right) = 3 : \dfrac{24}{5}$

$$\Rightarrow x : y : z = 2 : 3 : \frac{24}{5} = 10 : 15 : 24$$

$$\Rightarrow y = \left(98 \times \frac{15}{49}\right) = 30$$

20. If 0.75:x::5:8, then what is the value of x?

Solution: We are given that $0.75 : x :: 5 : 8$. Hence,

$$(x \times 5) = (0.75 \times 8)$$

$$\Rightarrow x = \left(\frac{6}{5}\right) = 1.2$$

21. If 40% of a number is equal to two-third of another number, what is the ratio of first number to the second number?

Solution: Let 40% of A $= \dfrac{2}{3}$B

Then,

$$\frac{40\text{A}}{100} = \frac{2\text{B}}{3}$$

$$\Rightarrow \frac{2\text{A}}{5} = \frac{2\text{B}}{3} \Rightarrow \frac{\text{A}}{\text{B}} = \left(\frac{2}{3} \times \frac{5}{2}\right) = \frac{5}{3}$$

$$\therefore \text{A} : \text{B} = 5 : 3$$

PRACTICE EXERCISE

1. Two numbers x and y are 25% and 50% more than a number z. What is the ratio of the two numbers?

(a) 1/2 (b) 4/5
(c) 2/3 (d) 5/6

2. A sum of money is to be distributed among A, B, C and D in the proportion of 2:3:5:4. If C gets ₹ 3000, what is A's share?

(a) ₹ 600 (b) ₹ 1200
(c) ₹ 1500 (d) ₹ 750

3. If a sum of money is to be distributed among A, B, C and D in the proportion of 3:1:4:6. If A gets ₹ 1500, what is the total sum of money?

(a) ₹ 6500 (b) ₹ 7000
(c) ₹ 5000 (d) ₹ 8000

4. If a sum of money is to be distributed among A, B, C and D in the proportion of 1:3:4:2. If C gets ₹ 1000 more than D, what is B's share?

(a) ₹ 1500 (b) ₹ 2000
(c) ₹ 1000 (d) ₹ 3000

5. A car running at a certain speed increases its speed to 5/4 times and reaches its destination 20 min earlier. How long will it take to reach the destination with the original speed?

(a) 120 min (b) 80 min
(c) 100 min (d) 60 min

6. A car running at a certain speed increases its speed to 2/3 times and reaches its destination 45 min earlier. How long will it take to reach the destination with the original speed?

(a) 120 min (b) 90 min
(c) 60 min (d) 75 min

7. Two quantities A and B vary directly. If initially $A = 20$ and $B = 50$, then what is the value of B if $A = 24$?

(a) 96 (b) 72
(c) 60 (d) 80

8. The ratio of the number of boys and girls in a college is 5:4. If the percentage increase in the number of boys and girls be 20% and 25%, respectively, what will be the new ratio?

(a) 6:5 (b) 5:3
(c) 1:1 (d) 6:7

9. The ratio of the two numbers is 1:3 and the LCM of the numbers is 39. What are the two numbers?

(a) 4 and 12 (b) 11 and 33
(c) 13 and 39 (d) 5 and 15

10. If x:42::11:6, then what is the value of x?

(a) 7 (b) 66
(c) 77 (d) 11/6

11. A sum of three numbers is 88. The ratio of the first and second number is 1:3 and the ratio of the second and third number is 3:4. What are the three numbers?

(a) 10, 30, 40 (b) 12, 36, 48
(c) 24, 28, 32 (d) 11, 33, 44

12. If the ratio of the amount of money with A, B and C is 3:2:8. If they get an increment of $33\frac{1}{3}$%, 50% and 25%, respectively, what is the new ratio of the money A, B and C have after increment?

(a) 3:2:8 (b) 4:3:10
(c) 4:6:7 (d) 2:3:5

13. A fruit shop sold apples and oranges in the ratio of 5:2, respectively. The ratio of oranges to bananas sold is 5:3. What is the ratio of apples to bananas sold?

(a) 25:3 (b) 25:6
(c) 5:3 (d) 2:3

14. A and B started a shop with initial investments in the ratio 5:7. If after one year their profits were in the ratio 1:2 and the period for A's investment was 7 months, then what was the time period for which B invested the money?

(a) 12 months (b) 9 months
(c) 14 months (d) 11 months

15. The salaries of three co-workers are in the ratio 3:4:5. If the increments of 15%, 20% and 25% are allowed, respectively, in their salaries, then what will be the new ratio of their salaries?

(a) 4:7:11 (b) 10:13:17
(c) 3:4:5 (d) 69:96:125

16. A, B and C invested money in the ratio 2:3:5 and the total time of their investments is 2:3:1. What is the ratio of their profits?

(a) 2:6:11 (b) 2:5:7
(c) 3:6:14 (d) 4:12:15

17. What is the third proportional to 0.12 and 0.24?

(a) 0.48 (b) 0.42
(c) 0.36 (d) 0.60

18. What ratio is equal to the ratio $4^{4.5}:8^2$?

(a) 4:1 (b) 10:1
(c) 8:1 (d) 6:1

19. What fraction bears the same ratio to 13/27 that 11/13 does to 5/27?

(a) 11/27 (b) 11/5
(c) 13/11 (d) 5/13

20. If ratio of two natural numbers a and b is x and that of b and a is y, then what is the value of $x + y$?

(a) $x + y \leq 2$ (b) $x + y \geq 0$
(c) $x + y \geq 1$ (d) $x + y \geq 2$

21. A and B together have ₹ 1210. If $\frac{4}{15}$ of A's amount is equal to $\frac{2}{5}$ of B's amount, how much amount does B have?

(a) ₹ 460 (b) ₹ 484
(c) ₹ 550 (d) ₹ 664

22. Seats for Electronics, Computers and Mechanical in an engineering college are in the ratio 5:7:8. There is a proposal to increase these seats by 40%, 50% and 75%, respectively. What will be the ratio of increased seats?

(a) 2:3:4 (b) 6:7:8
(c) 6:8:9 (d) None of these

23. Salaries of Ravi and Sumit are in the ratio 2:3. If the salary of each is increased by ₹ 4000, the new ratio becomes 40:57. What is Sumit's salary?

(a) 17000 (b) 20000
(c) 25500 (d) 38000

24. In a bag, there are coins of 25 p, 10 p and 5 p in the ratio of 1:2:3. If there is ₹ 30 in all, how many 5 p coins are there?

(a) 50 (b) 100
(c) 150 (d) 200

25. The marks scored by a student in three subjects are in the ratio of 4:5:6. If the candidate scored an overall aggregate of 60% of the sum of the maximum marks and the maximum marks in all three subjects is the same, in how many subjects did he score more than 60%?

(a) 1 (b) 2
(c) 3 (d) None of the subject

26. The ratio of boys to girls in a class is 5:3. The class has 16 more boys than girls. How many girls are there in the class?

(a) 6 (b) 16
(c) 24 (d) 34

27. The ratio of marks obtained by Akshay and Shikhar is 6:5. If the combined average of their percentage is 68.75 and their sum of the marks is 275, find the total marks for which exam was conducted.

(a) 150 (b) 200
(c) 400 (d) 600

28. The proportion of milk and water in 3 samples is 2:1, 3:2 and 5:3. A mixture comprising of equal quantities of all 3 samples is made. What is the proportion of milk and water in the mixture?

(a) 2:1 (b) 5:1
(c) 99.61 (d) 227:133

29. A picture measuring 3.5″ high by 5″ wide is to be enlarged so that the width is now 9 inches. How tall will the picture be?

(a) 6.3 inches (b) 5.7 inches
(c) 5.5 inches (d) 7 inches

30. The monthly salaries of Tejasvini and Sahay are in the ratio of 4:7. If each receives an increase of ₹ 25 in the salary, the ratio is altered to 3:5. What is the value of their respective salaries?

(a) 120 and 210 (b) 80 and 140
(c) 180 and 300 (d) 200 and 350

ANSWERS

1. (d)	**6.** (b)	**11.** (d)	**16.** (d)	**21.** (b)	**26.** (c)
2. (b)	**7.** (c)	**12.** (b)	**17.** (a)	**22.** (a)	**27.** (b)
3. (b)	**8.** (a)	**13.** (b)	**18.** (c)	**23.** (d)	**28.** (d)
4. (a)	**9.** (c)	**14.** (a)	**19.** (b)	**24.** (c)	**29.** (a)
5. (c)	**10.** (c)	**15.** (d)	**20.** (d)	**25.** (a)	**30.** (d)

EXPLANATIONS AND HINTS

1. (d) We have three numbers x, y and z. Also,

$$x = z + 25\% \text{ of } z = \frac{5}{4}z$$

$$y = z + 50\% \text{ of } z = \frac{6}{4}z$$

$$x:y = \frac{5/4\,z}{6/4\,z}$$

$$\Rightarrow x:y = \frac{5}{6}$$

2. (b) Given that A:B:C:D = 2:3:5:4. Now, we know that C = ₹ 3000

Therefore,

$$\text{A:C} = 2:5$$

Now, if

$$\text{C} = ₹\ 3000$$

Then

$$\text{A} = \frac{2}{5}\times 3000 = 1200$$

Thus, A's share = ₹ 1200

3. (b) Given that A:B:C:D = 3:1:4:6

Now, we know that A = ₹ 1500. So

$$\text{B} = \frac{1}{3}\times 1500 = 500$$

$$\text{C} = \frac{4}{3}\times 1500 = 2000$$

$$\text{D} = \frac{6}{3}\times 1500 = 3000$$

$$\text{Total sum of money} = ₹\,(1500 + 500 + 2000 + 3000) = ₹\ 7000$$

4. (a) Given that A:B:C:D = 1:3:4:2

Let C get x amount of the total, then,

$$D = \frac{x}{2}$$

Also,

$$C - D = 1000$$
$$x - \frac{x}{2} = 1000$$
$$x = 2 \times 1000 = 2000$$

Thus,

$$C = ₹\ 2000$$
$$B = \frac{3}{4} \times 2000 = 1500$$

So the total share of B = ₹ 1500

5. (c) Let the original speed be x km/h

Them the new speed = $5/4x$

If the time taken originally be t_1 and with the new speed be t_2

$$t_1 \times x = t_2 \times \frac{5}{4}x$$
$$t_1 = t_2 \times \frac{5}{4}$$
$$t_2 = t_1 \times \frac{4}{5}$$

Also, $t_1 - t_2 = 20$ min

$$t_1 - t_1 \times \frac{4}{5} = 20$$
$$\frac{1}{5}t_1 = 20$$
$$t_1 = 100$$

Hence, the time taken originally = 100 min

6. (b) Let the original speed be x km/h

New speed = $2/3x$

If the time taken originally be t_1 and with the new speed be t_2

$$t_1 \times x = t_2 \times \frac{2}{3}x$$
$$t_1 = t_2 \times \frac{2}{3}$$
$$t_2 = t_1 \times \frac{3}{2}$$

Also, $t_2 - t_1 = 45$ min

$$t_1 \times \frac{3}{2} - t_1 = 45$$
$$\Rightarrow \frac{1}{2}t_1 = 45$$
$$\Rightarrow t_1 = 90$$

Hence, time taken originally = 90 min

7. (c) Since the values of A_1 and B_1 vary directly with respect to A_2 and B_2, respectively. Then,

$$\frac{A_1}{B_1} = \frac{A_2}{B_2}$$
$$\Rightarrow \frac{20}{50} = \frac{24}{B_2}$$
$$\Rightarrow B_2 = \frac{24 \times 50}{20} = 60$$

8. (a) Given that the ratio of boys and girls = 5:4

Increase in the number of boys = 20%

Increase in the number of girls = 25%

So the new ratio of boys to girls

$$5 + \frac{20}{100} \times 5 : 4 + 4 \times \frac{25}{100}$$
$$= 5 + 1 : 4 + 1 = 6:5$$

9. (c) Ratio of the two numbers = 1:3

If the two numbers are x and $3x$. Hence,

$$x \times 3x = \text{H.C.F} \times 39$$

Now, H.C.F of the two numbers is x. So

$$x \times 3x = x \times 39$$
$$\Rightarrow 3x^2 = 39x$$
$$\Rightarrow x = 13$$

Hence, the two numbers are 13 and 39

10. (c) Given that

$$\frac{x}{42} = \frac{11}{6}$$
$$\Rightarrow x = \frac{11 \times 42}{6}$$
$$\Rightarrow x = 77$$

11. (d) If x is a constant such that

$$1x + 3x + 4x = 88$$
$$\Rightarrow 8x = 88$$
$$\Rightarrow x = 11$$

First number = 11

Second number = 3 × 11 = 33

Third number = 4 × 11 = 44

So the three numbers are 11, 33 and 44

12. (b) Given that the ratio of A, B and C = 3:2:8

Increment of A, B, C = $33\frac{1}{3}$%, 50% and 25%, respectively

New ratio after the increment will be

$$3+3\times\frac{33\frac{1}{3}}{100}:2+2\times\frac{50}{100}:8+8\times\frac{25}{100}$$

$$=3+1:2+1:8+2=4{:}3{:}10$$

13. (b) Given that the ratio of sold apples to oranges $= 5{:}2$

Ratio of sold oranges to bananas $= 5{:}3$

Ratio of sold apples to bananas $=\frac{5}{2}\times\frac{5}{3}=\frac{25}{6}=25:6$

14. (a) Let the investments of A and B, respectively, be $3x$ and $5x$.

If the time period for B's investments is y months

$$\frac{3x\times10}{5x\times y}=\frac{1}{2}$$

$$\Rightarrow y=3\times2\times2$$

$$\Rightarrow y=12$$

Hence, the time period for B's investments is 12 months

15. (d) Given that the ratio of salaries of co-workers $=$ 3:4:5

Ratio of salaries after increment will be

$$3+3\times\frac{15}{100}:4+4\times\frac{20}{100}:5+5\times\frac{25}{100}$$

$$=3+\frac{9}{20}:4+\frac{16}{20}:5+\frac{25}{20}=69{:}96{:}125$$

16. (d) Given that the ratio of money invested by A, B and C be 2:3:5.

Hence, let the money invested by A, B and C be $2x$, $3x$ and $5x$, respectively.

Ratio of time of investment $= 2{:}4{:}3$

And let the time of investment by A, B and C be $2y$, $4y$ and $3y$, respectively.

Ratio of profit of A: profit of B: profit of C

$$=(2x)(2y){:}(3x)(4y){:}(5x)(3y)$$

$$=4{:}12{:}15$$

17. (a) Let the third proportional be x, then,

$$\mathbf{0.12{:}0.24{::}0.24{:}}x$$

$$\Rightarrow\frac{0.12}{0.24}=\frac{0.24}{x}$$

$$\Rightarrow x=\frac{0.24\times0.24}{0.12}$$

$$=0.24\times2=0.48$$

Thus, the third proportional is 0.48.

18. (c) Given that the ratio is $4^{4.5}{:}8^2$

Now, the ratio can be rewritten as $[(2)^2]^{4.5}:[(2)^3]^2$

$$\Rightarrow 2^9:2^6$$

$$=2^3=8$$

Thus, the ratio equal to $4^{4.5}{:}8^2 = 8{:}1$

19. (b) If the required fraction is x. Then,

$$x:\frac{13}{27}::\frac{11}{13}:\frac{5}{27}$$

$$\Rightarrow\frac{(x/13)}{27}=\frac{(11/13)}{(5/27)}\Rightarrow x=\frac{11}{13}\times\frac{13}{27}\times\frac{27}{5}$$

$$\Rightarrow x=\frac{11}{5}$$

Thus, the required fraction $= 11/5$

20. (d) We have, $\quad x=\frac{a}{b}$

$$y=\frac{b}{a}$$

$$\Rightarrow x+y=\frac{a}{b}+\frac{b}{a}$$

$$=\frac{a^2+b^2}{ab}$$

Since the two numbers are natural numbers, the lowest value can be if $a = b = 1$. In that case,

$$x+y=\frac{(1)^2+(1)^2}{1\times1}=2$$

Thus, the value of $x+y\geq2$.

21. (b) We are given that

$$\frac{4}{15}\text{A}=\frac{2}{5}\text{B}$$

$$\Rightarrow\text{A}=\left(\frac{2}{5}\times\frac{15}{4}\right)\text{B}=\frac{3}{2}\text{B}$$

$$\Rightarrow\frac{\text{A}}{\text{B}}=\frac{3}{2}\Rightarrow\text{A:B}=3:2$$

Thus, B's share $=$ ₹ $\left(1210\times\frac{2}{5}\right)=$ ₹ 484

22. (a) Let the number of seats of Electronics, Computers and Mechanical be $5x$, $7x$ and $8x$, respectively.

Number of increased seats $=$ (140% of $5x$), (150% of $7x$) and (175% of $8x$).

$$\Rightarrow\left(\frac{140}{100}\times5x\right),\left(\frac{150}{100}\times7x\right),\left(\frac{175}{100}\times8x\right)$$

$$\Rightarrow 7x,\frac{21x}{2},14x$$

Therefore, the required ratio = $7x : \frac{21x}{2} : 14x$

$$\Rightarrow 14x : 21x : 28x$$
$$\Rightarrow 2 : 3 : 4$$

23. (d) Let the original salaries of Ravi and Sumit be ₹ $2x$ and ₹ $3x$, respectively.

Then,

$$\frac{2x+4000}{3x+4000} = \frac{40}{57}$$
$$\Rightarrow 57(2x+4000) = 40(3x+4000)$$
$$\Rightarrow 6x = 68000 \Rightarrow 3x = 34000$$

Sumit's present salary = $(3x + 4000)$ = ₹ (34000 + 4000) = ₹ 38000

24. (c) Let the number of 25 p, 10 p and 5 p coins be x, $2x$ and $3x$, respectively.

Then sum of their values = ₹ $\left(\frac{25x}{100} + \frac{20x}{100} + \frac{15x}{100}\right)$ = ₹ $\frac{60x}{100}$

Thus,

$$\frac{60x}{100} = 30 \Rightarrow x = \frac{30 \times 100}{60} = 50$$

Hence, the number of 5 p coins = $(3 \times 50) = 150$.

25. (a) Let the maximum marks in each of the three subjects be 100. Maximum marks attainable = 300

Therefore, the candidate scored 60% of 300 marks = 180 marks.

Let the marks scored in the three subjects be $4x$, $5x$ and $6x$.

Then, $4x + 5x + 6x = 180$

$$\Rightarrow 15x = 180 \text{ or } x = 12.$$

Therefore, marks scored by the candidate in the three subjects are

$$4 \times 12 = 48$$
$$5 \times 12 = 60$$
$$6 \times 12 = 72$$

Hence, the candidate has scored more than 60% in one subject.

26. (c) Let the number of boys in the class be $5x$ and the number of girls in the class be $3x$.

The class has 16 more boys, i.e., the difference between the number of boys and girls is 16

i.e., $5x - 3x = 16$

$\Rightarrow x = 8$

Number of girls = $3x = 3 \times 8 = 24$

27. (b) Let Akshay's marks be $6x$ and Shikhar's marks be $5x$. Therefore, the sum of the marks = $11x$

Total marks = 275

Thus, $11x = 275 \Rightarrow x = 25$

Akshay's marks = $6x = 6 \times 25 = 150$

Shikhar's marks = $5x = 5 \times 25 = 125$

Therefore, the combined average of their marks = $\frac{(150+125)}{2} = 137.5$

If total marks is 100, then their combined average of their percentage is 68.75.

Hence, if their combined average of their percentage is 137.5 then the total marks = $\frac{137.5}{68.75} \times 100 = 200$

28. (d) Proportion of milk in 3 samples is 2/3, 3/5, 5/8.

Proportion of water in the 3 samples is 1/3, 2/5, 3/8.

We are given that equal quantities are taken.

Total proportion of milk = $\frac{2}{3} + \frac{3}{5} + \frac{5}{8} = \frac{227}{120}$

Total proportion of water = $\frac{1}{3} + \frac{2}{5} + \frac{3}{8} = \frac{133}{120}$

Proportion of milk and water in the solution is 227:133.

29. (a) The initial ratio of height to width is 3.5 to 5 inches.

Let the height to be calculated be x inches.

Hence,

$$\frac{3.5}{5} = \frac{x}{9} \Rightarrow x = 6.3$$

30. (d) Let the salaries of Tejasvini and Sahay be $4x$ and $7x$.

Therefore,

$$\frac{4x+25}{7x+25} = \frac{3}{5}$$
$$\Rightarrow 5(4x+25) = 3(7x+25)$$
$$\Rightarrow x = 125 - 75 = 50$$

Therefore, their salaries are $4 \times 50 = 200$ and $7 \times 50 = 350$.

CHAPTER 8

SPEED, DISTANCE AND TIME

INTRODUCTION

Speed of an object is defined as the distance covered by that object per unit time. It is a scalar quantity. The unit of speed is m/s.

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

The average speed of an object in an interval of time is the distance traveled by the object divided by the duration of the interval.

$$\text{Average speed} = \frac{\text{Total distance traveled}}{\text{Total time taken}}$$

Relative speed of an object A, with a reference to object B, is equal to $(s_A - s_B)$ m/s if A and B are traveling in the same direction, and $(s_A + s_B)$ m/s if A and B are traveling in the opposite direction. Here, s_A is the speed of object A and s_B is the speed of object B.

SOME IMPORTANT FORMULAS

List of some of the common formulas used for speed, distance and time are as follows:

1. If a body covers the distance d_1 and d_2 meters at a speed of s_1 and s_2 m/s, respectively, in time t_1 and t_2, then the total time taken T will be

$$T = t_1 + t_2 = \frac{d_1}{s_1} + \frac{d_2}{s_2}$$

   Total distance covered D will be

$$D = d_1 + d_2 = s_1 t + s_2 t_2$$

2. While traveling a certain distance d, if a man changes his speed in the ratio $x{:}y$, then ratio of time taken becomes $y{:}x$.
3. If a certain distance d is covered at a speed of a m/s from A to B, and the same distance is covered again from B to A at b m/s, then average speed s_{av} during the whole journey is given by

$$S_{av} = \frac{2ab}{a+b} \text{ m/s}$$

Also, if t_1 and t_2 are the time taken to travel from A to B, and B to A, respectively, then distance d from A to B is given by

$$d = (t_1 + t_2)\frac{ab}{a+b}$$

$$d = (t_1 - t_2)\frac{ab}{a-b}$$

$$d = (a-b)\frac{t_1 t_2}{t_1 - t_2}$$

4. If a body travels a distance d from A to B with speed a m/s in time t_1, and travels back from B to A with x/y of the usual speed a m/s, then change in time taken to cover the same distance is given by

$$\text{Change in time} = \begin{cases} \left(\frac{y}{x} - 1\right) \times t_1; & \text{for } y > x \\ \left(1 - \frac{y}{x}\right) \times t_1; & \text{for } x > y \end{cases}$$

5. Time taken by a moving object a meters long in passing a stationary object of negligible length from the time they meet is same as the time taken by the moving object to cover "a" meters with its own speed.

6. Time taken by a moving object a meters long in passing a stationary object b meters long from the time they meet is same as the time taken by the moving object to cover "$a + b$" meters with its own speed.

7. If two objects (such as a train or car) of length a and b meters move in the same direction at s_A and s_B m/s, then time taken t to cross each other from the time they meet is

$$t = \begin{cases} \frac{a+b}{s_A - s_B} & \text{if } a > b \\ \frac{a+b}{s_B - s_A} & \text{if } b > a \end{cases}$$

8. If two objects (such as a train or car) of length a and b meters move in the opposite direction at s_A and s_B m/s, respectively, then time t to cross each other from the time they meet is

$$t = \frac{a+b}{s_A + s_B}$$

9. If speed of a boat in still water is x m/s and speed of the stream is y m/s, then
Speed of boat while traveling upstream $s_u = (x + y)$ m/s

10. Speed of boat while traveling downstream $s_d = (x - y)$ m/s. In a circular track, the speed of boat in still water, s $= \frac{1}{2} \times (s_u + s_d)$ m/s
length of the track is given by $2\pi r$ where r is the radius of the circular track.

11. In a circular race, the total length of the race is given by $2\pi rn$ where n is the number of times the racer covers the track.

SOLVED EXAMPLES

1. A policeman goes after a thief who has a 1000 m start. The policeman runs 1 km in 5 min and the thief 1 km in 8 min. How far did the thief go before he was overtaken?

Solution: We know that

Speed of the policeman $= \frac{1}{5}$ km/min

Speed of the thief $= \frac{1}{8}$ km/min

Policeman gains $= \frac{1}{5} - \frac{1}{8} = \frac{3}{40}$ km/min $= \frac{3000}{18}$ m/min $= \frac{500}{3}$ m/min

To gain 1000 m, time required $= \frac{1000}{500/3}$ min $= 6$ min

Hence, the thief has gone ahead by $6 \times \frac{1}{8} = \frac{3}{4}$ km or 750 m

2. A man rows upstream 18 km and downstream 30 km taking 3 h each time. What is the velocity of the current?

Solution: We know that,

$$\frac{18}{\text{Speed upstream}} = 3$$

and $\frac{30}{\text{Speed downstream}} = 3$

Thus, speed upstream $= (18/3)$ km/h $= 6$ km/h

and speed downstream $= (30/3)$ km/h $= 10$ km/h

Speed of the river $= \frac{1}{2}(10 - 6) = \frac{1}{2} \times 4 = 2$ km/h

3. A train leaves from Delhi to Jalandhar at 6 am and another train leaves from Jalandhar at 7 am. The speed of the train from Jalandhar is 1/3 times

faster than the train leaving from Delhi. If the distance between Delhi and Jalandhar is roughly 360 km and both trains meet at 8:30 am, then what is the speed of the two trains?

Solution: Let the speed of the train from Delhi is x km/h. Hence, speed of the train from Jalandhar $= \dfrac{4x}{3}$ km/h

Distance traveled by the train from Delhi till 8:30 am $= x \times \dfrac{5}{2}$

Distance traveled by the train from Jalandhar till 8:30 am $= \dfrac{4x}{3} \times \dfrac{3}{2} = 2x$

Now, total distance = 360 km

Hence,

$$\frac{5}{2}x + 2x = 360$$
$$\Rightarrow \frac{9}{2}x = 360$$
$$\Rightarrow x = 80$$

Speed of train from Delhi = 80 km/h

Speed of train from Jalandhar $= \dfrac{4 \times 80}{3}$ km/h $= 106.67$ km/h

4. A train traveling at 60 km/h passes a cyclist going in the same direction in 8 s. If the cyclist would have been in the opposite direction, the train would have passed him in 4 s. What is the length of the train?

Solution: Let the speed of the cyclist is x km/h. We know that

$$\text{Length} = \text{Relative speed} \times \text{Time}$$
$$= (60 - x) \times 8$$

Similarly, length $= (60 + x) \times 4$

Solving the above equations, we get

$$(60 - x) \times 8 = (60 + x) \times 4$$
$$\Rightarrow 60 = 3x$$
$$\Rightarrow x = 20$$

Thus, the speed of the cyclist = 20 km/h

Length of the train $= (60 - 20) \times \dfrac{5}{18} \times 8$. Thus, the length of the train = 88.89 m.

5. A train running with the speed of 90 km/h crosses a platform in 15 s. If the length of the train is 200 m, what is the length of the platform?

Solution: Let the length of the platform be x meters.
We know that the length of the train = 200 m

Time taken = 15 s

Speed of the train = 90 km/h = 90 km/h $= 90 \times \dfrac{5}{18}$ m/s = 25 m/s

Relative speed = 25 m/s (since platform is stationary)

Hence,

$$\frac{200 + x}{15} = 25$$
$$\Rightarrow x = 375 - 200 = 175\,\text{m}$$

6. A sailor covers a distance in 5 h downstream and covers the same distance in 3 h upstream. If the speed of the river is 4 m/s, then what is the speed of the boat?

Solution: Let the speed of the boat in still water is x m/s.

Speed of the boat while going upstream $= (x - 4)$ m/s

Speed of boat while going downstream $= (x + 4)$ m/s

Since distance remains the same in both cases,

$$(x + 4) \times 3 = (x - 4) \times 5$$
$$\Rightarrow 3x + 12 = 5x - 20$$
$$\Rightarrow x = 16$$

The speed of the boat in still water = 16 m/s.

7. A train increases its speed by 10% and reaches 4 min early. What was the original time the train took to reach the destination?

Solution: Let the initial speed be x.

So the new speed $= \dfrac{11x}{10}$

Now, if the initial time was y, time becomes $\dfrac{10y}{11}$. Hence,

$$y - \frac{10y}{11} = 4$$
$$\Rightarrow y = 44$$

Thus, the initial time taken to reach the destination is 44 min.

8. Rachyita goes to office at a speed of 7 km/h and returns to her home at a speed of 4 km/h. If she takes 22 h in total, what is the distance between her office and her home?

Solution: Let the distance between office and home be x km. Then

$$\frac{x}{7}+\frac{x}{4}=22$$
$$\Rightarrow \frac{11x}{28}=22$$
$$\Rightarrow x=56\,\text{km}$$

Total distance = 56 km

9. Without stoppage, a train travels at an average speed of 60 km/h and with stoppages it covers the same distance at an average speed of 48 km/h. How many stoppages are there if each stoppage is of exactly 3 min?

Solution: The average speeds of the trains are 60 km/h and 48 km/h.

To cover a distance of 48 km, the slower train will take 60 min.

The faster train will cover that distance in 48 min.

Hence, for the faster train to cover the same distance as the slower train, total time of stoppage = 12 min.

Now, since each stoppage is of 3 min, total number of stoppages = 4.

10. A train requires 3 s to pass a pole while it requires 8 s to cross a stationary train which is 150 meters long. What is the speed of the train?

Solution: Let the length of moving train be x, and speed of train be y. To cross a pole

$$\text{Time}=\frac{\text{Length of train}}{\text{Speed of train}}$$
$$3=\frac{x}{y}$$

Also, to cross a stationary train, $8=\dfrac{x+250}{y}$

$$8=3+\frac{150}{y}$$
$$y=30\text{ km/h}$$

Hence, speed of the moving train = 30 km/h.

11. Aishling drives from Plymouth to Southampton, a distance of 160 miles, in 4 hours. She then drives from Southampton to London, a distance of 90 miles, in 1 hour and 30 minutes. Determine her average speed of her each journey.

Solution: From Plymouth to Southampton,

$$\text{Average speed}=\frac{160}{4}=40\ \text{mph}$$

From Southampton to London,

Time taken = 1 hour and 30 minutes = $\dfrac{3}{2}$ hours

$$\text{Average speed}=\frac{90}{3/2}=60\ \text{mph}$$

12. Jane drives at an average speed of 45 mph on a journey of 175 km. How long does the journey take?

Solution: We are given that,

Distance = 175 km

Average speed = 45 mph = $45\times 1.61=72.45$ kmph

$$\text{Time}=\frac{\text{Distance}}{\text{Speed}}=\frac{175}{72.45}=2.415\text{ hours}$$

13. A man covered a certain distance at some speed. Had he moved 3 kmph faster, he would have taken 40 minute less. If he had moved 2 kmph slower, he would have taken 40 minutes more. What is the distance?

Solution: Let the distance be x km and speed = y kmph.

Then,

$$\frac{x}{y}-\frac{x}{y+3}=\frac{40}{60}\Rightarrow 2y(y+3)=9x \qquad \text{(i)}$$
$$\frac{x}{y-2}-\frac{x}{y}=\frac{40}{60}\Rightarrow y(y-2)=3x \qquad \text{(ii)}$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{2y(y+3)}{y(y-2)}=\frac{9x}{3x}\Rightarrow 2(y+3)=3(y-2)\Rightarrow y=12$$

Putting this value in Eq. (ii), we get

$$12(12-2)=3x\Rightarrow x=40$$

14. A farmer travelled a distance of 61 km in 9 hours. He travelled partly on foot at the rate of 4 km/hr and partly on bicycle at 9 km/hr. What is the distance travelled on foot?

Solution: Let the distance travelled on foot be x km.

Then, distance travelled on bicycle = $(61-x)$ km

So,

$$\frac{x}{4}+\frac{(61-x)}{9}=9\Rightarrow 9x+4(61-x)=9\times 36$$
$$\Leftrightarrow 5x=80\Rightarrow x=16\text{ km}$$

15. It takes eight hours for a 600 km journey, if 120 km is done by train and the rest by car. It takes 20 minutes more, if 200 km is done by train and the rest by car. What is the ratio of the speed of the train to that of the cars?

Solution: Let the speed of the train be x km/hr and that of the car be y km/hr.

Then,

$$\frac{120}{x}+\frac{480}{y}=8 \Rightarrow \frac{1}{x}+\frac{4}{y}=\frac{1}{15} \qquad \text{(i)}$$

$$\frac{200}{x}+\frac{400}{y}=\frac{25}{3} \Rightarrow \frac{1}{x}+\frac{2}{y}=\frac{1}{24} \qquad \text{(ii)}$$

Eq. (i) − (ii), we get

$$\frac{4}{y}-\frac{2}{y}=\frac{1}{15}-\frac{1}{24}$$

$$\Rightarrow \frac{2}{y}=\frac{8-5}{120} \Rightarrow \frac{2}{y}=\frac{3}{120} \Rightarrow y=80$$

Substituting value of y in Eq. (i), we get

$$\frac{1}{x}+\frac{4}{80}=\frac{1}{15} \Rightarrow \frac{1}{x}=\frac{1}{15}-\frac{1}{20}=\frac{4-3}{60} \Rightarrow \frac{1}{x}=\frac{1}{60}$$

$$\Rightarrow x=60$$

Therefore, $x = 60$ and $y = 80$.

Thus, ratio of speeds = 60:80 = 3:4.

16. Hatem is travelling on his cycle and has calculated to reach point A at 2 P.M. if he travels at 10 kmph, he will reach there at 12 noon if he travels at 15 kmph. At what speed must he travel to reach A at 1 P.M?

Solution: Let the distance travelled by x km.

Then,

$$\frac{x}{10}-\frac{x}{15}=2 \Rightarrow 3x-2x=60 \Rightarrow x=60\,\text{km}$$

Time taken to travel 60 km at 10 km/hr $=\left(\frac{60}{10}\right)$ hrs $= 6$ hrs.

So, Hatem started 6 hours before 2 p.m., i.e. at 8 a.m.

Therefore, required speed $=\left(\frac{60}{5}\right)=12$ kmph

17. In covering a distance of 30 km, Abhay takes 2 hours more than Sameer. If Abhay doubles his speed, then he would take 1 hour less than Sameer. What is Abhay's speed?

Solution: Let Abhay's speed be x kmph.

Then,

$$\frac{30}{x}-\frac{30}{2x}=3 \Rightarrow 6x=30 \Rightarrow x=5\,\text{km/hr}.$$

18. In a flight of 600 km, an aircraft was slowed down due to bad weather. Its average speed for the trip was reduced by 200 km/hr and the time of flight increased by 30 minutes. What is the duration of the flight?

Solution: Let the duration of the flight be x hrs. Then,

$$\frac{600}{x}-\frac{600}{x+(1/2)}=200$$

$$\Rightarrow \frac{600}{x}-\frac{1200}{2x+1}=200$$

$$\Rightarrow x(2x+1)=3$$

$$\Rightarrow 2x^2+x-3=0 \Rightarrow (2x+3)(x-1)=0$$

$x = 1$ hr [neglecting the negative value of x]

Thus, the duration of the flight = 1 hr.

19. A man completes a journey in 10 hours. He travels first half of the journey at the rate of 21 km/hr and second half at the rate of 24 km/hr. What is the total length of the journey?

Solution: We are given that total time of the journey = 10 hours. Let the distance be x km. Then,

$$\frac{(1/2)x}{21}+\frac{(1/2)x}{24}=10 \Rightarrow \frac{x}{21}+\frac{x}{24}=20$$

$$\Rightarrow 15x=168\times 20 \Rightarrow x=\left(\frac{168\times 20}{15}\right)$$

$$x = 224\text{ km}$$

Thus, the total length of the journey = 224 km.

20. The ratio between the speeds of two trains is 7:8. If the second train runs 400 km in 4 hours, then what is the speed of the first train?

Solution: Let the speed of the two trains be $7x$ and $8x$ km/hr.

Then,

$$8x=\left(\frac{400}{4}\right)=100$$

$$\Rightarrow x=\left(\frac{100}{8}\right)=12.5$$

Therefore, speed of first train $=(7\times 12.5)=87.5$ km/hr.

21. A man on tour travels first 160 km at 64 km/hr and the next 160 km at 80 km/hr. What is the average speed for the first 320 km of the tour?

Solution: Total time taken for the entire journey $=\left(\frac{160}{64}+\frac{160}{80}\right)=\frac{9}{2}$ hrs

Therefore, average speed $=\left(320\times\frac{2}{9}\right)=71.11$ km/hr.

22. A car travelling with $\frac{5}{7}$ of its actual speed covers 42 km in 1 hr 40 min 48 sec. What is the actual speed of the car?

Solution: We are given that time taken = 1 hr 40 min 48 sec = 1 hr $40\frac{4}{5}$ min = $\frac{126}{75}$ hr.

Let the actual speed be x km/hr.

Then,

$$\frac{5}{7}x \times \frac{126}{75} = 42 \Rightarrow x = \left(\frac{42 \times 7 \times 75}{5 \times 126}\right) = 35 \text{ km/hr}$$

23. Excluding stoppages, the speed of a bus is 54 kmph and including stoppages, it is 45 kmph. For how many minutes does the bus stop per hour?

Solution: Due to stoppages, it covers (54 − 45) km less in one hour.

Hence, the bus covers 9 km less.

Time taken to cover 9 km = $\left(\frac{9}{54}\right) = \frac{1}{6}$ hr = $\frac{1}{6} \times 60 = 10$ min.

24. A train can travel 50% faster than a car. Both start from point A at the same time and reach point B 75 km away from A at the same time. On the way, however, the train lost about 12.5 minutes while stopping at the stations. What is the speed of the car?

Solution: Let speed of the car be x kmph.

Then, the speed of the train = $\frac{150x}{100} = \left(\frac{3}{2}x\right)$ kmph.

Thus,

$$\frac{75}{x} - \frac{75}{(3/2)x} = \frac{125}{10 \times 60} \Rightarrow \frac{75}{x} - \frac{50}{x} = \frac{5}{24}$$

$$\Rightarrow x = \left(\frac{25 \times 24}{5}\right) = 120$$

Hence, the speed of the car is 120 kmph.

25. If a person walks at 14 km/hr instead of 10 km/hr, he would have walked 20 km more. What is the actual distance travelled by him?

Solution: Let the actual distance travelled by x km.

Then,

$$\frac{x}{10} = \frac{x+20}{14} \Rightarrow 14x = 10x + 200$$

$$\Rightarrow 4x = 200$$

$$\Rightarrow x = 50 \text{ km}$$

Hence, the actual distance travelled is 50 km.

PRACTICE EXERCISE

1. A car covers first 300 m at a speed of 54 km/h and the next 500 m at a speed of 72 km/h. What is the average speed of the car?

(a) 50 km/h (b) 47 km/h
(c) 60 km/h (d) 64 km/h

2. A policeman goes after a thief who has a 500 m start. The policeman runs 1 km in 6 min and the thief 1 km in 9 min. How far did the thief go before he was overtaken?

(a) 1 km (b) $1\frac{1}{2}$ m
(c) $1\frac{1}{4}$ km (d) 2 km

3. A man rows upstream 13 km and downstream 28 km, taking 5 h each time. What is the velocity of the current?

(a) 1.5 km/h (b) 1.2 km/h
(c) 3.6 km/h (d) 2 km/h

4. A train overtakes two villagers walking at a rate of 3 km/h and 4 km/h, respectively, in the same direction and completely passes them in 8 and 10 s. Assuming that the train is traveling at a uniform speed, what is the speed of the train in km/h?

(a) 8 (b) 12.5
(c) 15 (d) 10

5. In the previous question, what is the length of the train?

(a) 50 m (b) 11.11 m
(c) 20.50 m (d) 12 m

6. A train leaves from Delhi to Jammu at 7 pm and another train leaves from Jammu at 8 pm. The speed of the train from Delhi is 60 km/h and the speed of the train from Jammu is 1/4 times faster. If the distance between Delhi and Jammu is roughly 600 km, then at what distance from Delhi will the two trains meet?

(a) 350 km (b) 276 km
(c) 324 km (d) 288 km

7. Jack runs $1\frac{3}{4}$ times faster than Jeremy. If Jack gives Jeremy a start of 90 m, how far must the winning post be so that it may be a dead heat?

(a) 250 m (b) 140 m
(c) 175 m (d) 210 m

8. A train traveling at 40 km/h passes a runner going in the same direction in 6 s. If the runner would have been in the opposite direction, the train would have passed him in 4 s. What is the length of the train?

(a) 72.5 m (b) 55 m
(c) 60 m (d) 67 m

9. Two men A and B run a 4-km race on a circular course of 1/2 km. If their speeds are in the ratio of 5:4, how often does the winner pass the other?

(a) Once (b) Twice
(c) Thrice (d) Four times

10. A train running with the speed of 72 km/h crosses a platform in 20 s. If the length of the train is 100 m, what is the length of the platform?

(a) 200 m (b) 300 m
(c) 400 m (d) 500 m

11. A sailor covers a distance in 6 h downstream and covers the same distance in 3 h upstream. If the speed of the river is 5 m/s, then what is the speed of the boat?

(a) 10 m/s (b) 12 m/s
(c) 15 m/s (d) 18 m/s

12. A train increases its speed by 25% and reaches 5 min early. What was the original time the train took to reach the destination?

(a) 30 min (b) 25 min
(c) 40 min (d) 15 min

13. A car travels from A to B at x_1 km/h, travels back from B to A at x_2 km/h and again goes back from A to B at x_2 km/h. What is the average speed of the car?

(a) $\dfrac{2x_1x_2}{x_1+2x_2}$ (b) $\dfrac{2x_1x_2}{2x_1+x_2}$

(c) $\dfrac{3x_1x_2}{x_1+2x_2}$ (d) $\dfrac{3x_1x_2}{2x_1+x_2}$

14. Charlie reaches his college 6 min late if he walks at a speed of 10 km/h. Next time he doubles his speed but finds that he is still 2 min late. What is the distance between Charlie's college and house?

(a) 2.5 km (b) 5 km
(c) 7.5 km (d) 10 km

15. Rachyita goes to office at a speed of 5 km/h and returns to her home at a speed of 3 km/h. If she takes 8 h in total, what is the distance between her office and her home?

(a) 15 km (b) 20 km
(c) 7.5 km (d) 10 km

16. A boat rows 16 km upstream and 30 km downstream taking 5 h each time. What was the speed of the boat in still water?

(a) 6 km/h (b) 8 km/h
(c) 5 km/h (d) 2 km/h

17. Two trains are traveling in the same direction at 60 km/h and 30 km/h, respectively. The faster train crosses a man in the slower train in 9 s. What is the length of the faster train?

(a) 50 km (b) 60 km
(c) 75 km (d) 100 km

18. Without stoppage, a train travels at an average speed of 60 km/h, and with stoppages it covers the same distance at an average speed of 40 km/h. How many stoppages are there if each stoppage is of exactly 2 min?

(a) 8 (b) 10
(c) 5 (d) 6

19. A train requires 5 s to pass a pole while it requires 12 s to cross a stationary train which is 350 m long. What is the speed of the train?

(a) 50 km/h (b) 85 km/h
(c) 35 km/h (d) 65 km/h

20. A train decreases its speed by 20% and reaches 4 min late. What was the original time the train took to reach the destination?

(a) 30 min (b) 24 min
(c) 12 min (d) 16 min

21. A person crosses a 600 m long street in 5 minutes. What is his speed in kmph?

(a) 3.6 (b) 7.2
(c) 8.4 (d) 10

22. An aeroplane covers a certain distance at a speed of 240 kmph in 5 hours. To cover the same distance in $1\frac{2}{3}$ hours, what must be its speed?

(a) 300 kmph (b) 360 kmph
(c) 600 kmph (d) 720 kmph

Directions for Q23 to Q26: Each of the questions given below consists of a question and two or three statements. You have to decide whether the data provided in the statement(s) is/are sufficient to answer the given question.

23. Two towns are connected by railway. What is the distance between them?

I. The speed of the mail train is 12 km/h more than that of an express train.
II. A mail train takes 40 minutes less than an express train to cover the distance.

(a) I alone is sufficient while II alone is not sufficient to answer.
(b) II alone is sufficient while I alone is not sufficient to answer.
(c) Either I or II alone sufficient to answer.
(d) Both I and II are not sufficient to answer.

24. The towns A, B and C are on a straight line. Town C is between A and B. The distance from A to B is 100 km, How far is A from C?

I. The distance from A to B is 25% more than the distance from C to B.
II. The distance from A to C is 1/4 of the distance C to B.

(a) I alone is sufficient while II alone is not sufficient to answer.
(b) II alone is sufficient while I alone is not sufficient to answer.
(c) Either I or II alone sufficient to answer.
(d) Both I and II are not sufficient to answer.

25. Two cars pass each other in opposite direction. How long would they take to be 500 km apart?

I. The sum of their speeds is 135 kmph.
II. The difference of their speed is 25 kmph.

(a) I alone is sufficient while II alone is not sufficient to answer.
(b) II alone is sufficient while I alone is not sufficient to answer.
(c) Either I or II alone sufficient to answer.
(d) Both I and II are necessary to answer.

26. How much time did X take to reach the destination?

I. The ratio between the speed of X and Y is 3:4.
II. Y takes 36 minutes to reach the same destination.

(a) I alone is sufficient while II alone is not sufficient to answer.
(b) II alone is sufficient while I alone is not sufficient to answer.
(c) Either I or II alone sufficient to answer.
(d) Both I and II are necessary to answer.

27. In an industry, the raw materials and the finished goods are transported on the conveyor belt. There are two conveyor belt, one for carrying parts from P to point Q and another for carrying parts from R to point Q. P, Q and R in that order are in a straight line. Sometimes, the belt serves to transport cart, which can themselves move with respect to the belts. The two belts move at a speed of 0.5 m/s and the cart move at a uniform speed of 2m/s with respect to the belts. A cart goes from point P to R and back to P taking a total of 64 s. Find the distance PR in meters. Assume that the time taken by the cart to turn back at R is negligible?

(a) 64 (b) 60
(c) 54 (d) 48

28. A, B and C start simultaneously from X to Y. A reaches Y, turns back and meet B at a distance of 11 km from Y. B reached Y, turns back and meet C at a distance of 9 km from Y. If the ratio of the speeds of A and C is 3:2, what is the distance between X and Y?

(a) 99 m (b) 100 m
(c) 120 m (d) 142 m

29. Two friends A and B run around a circular track of length 510 metres, starting from the same point, simultaneously and in the same direction. A who runs faster laps B in the middle of the 5th round. If A and B were to run a 3 km race long race, how much start, in terms of distance, should A give B so that they finish the race in a dead heat?

(a) 545.45 m (b) 600 m
(c) 666.67 m (d) 857.14 m

30. Three athletes A, B and C run a race, B finished 24 meters ahead of C and 36 m ahead of A, while C finished 16 m ahead of A. If each athlete runs the entire distance at their respective constant speeds, what is the length of the race?

(a) 108 m (b) 90 m
(c) 80 m (d) 96 m

31. From a point P, on the surface of radius 3cm, two cockroaches A and B started moving along two different circular paths, each having the maximum possible radius, on the surface of the sphere, that lie in the two different planes which are inclined at an angle of 45 degree to each other. If A and B takes 18 sec and 6 sec, respectively, to complete one revolution along their respective circular paths, then after how much time will they meet again, after they start from P?

(a) 27 sec (b) 24 sec
(c) 18 sec (d) 9 sec

32. Arpit and Gaurav leave points x and y towards y and x, respectively, simultaneously and travel in the same route. After meeting each other on the way, Arpit takes 4 hours to reach her destination, while Gaurav takes 9 hours to reach his destination. If the speed of Arpit is 48 km/hr, what is the speed of Gaurav?

(a) 72 mph (b) 32 mph
(c) 20 mph (d) None of these

33. A man driving his bike at 24 kmph reaches his office 5 minutes late. Had he driven 25% faster on an average he would have reached 4 minutes earlier than the scheduled time. How far is his office?

(a) 24 km (b) 18 km
(c) 72 km (d) 36 km

34. Two motorists Anil and Sunil are practicing with two different sports cars on the circular racing track, for the car racing tournament to be held next month. Both Anil and Sunil start from the same point on the circular track. Anil completes one round of the track in 1 min and Sunil takes 2 min to complete a round. While Anil maintains speed for all the rounds, Sunil halves his speed after the completion of each round. How many times Anil and Sunil will meet between 6th round and 9th round of Sunil (6th and 9th round is excluded)? Assume that the speed of Sunil remains steady throughout each round and changes only after the completion of that round.

(a) 382 (b) 347
(c) 260 (d) 189

35. A river flows at 12 km/h. A boy who can row at $\frac{25}{18}$ m/s in still water had to cross it in the least possible time. The distance covered by the boy is how many times the width of the river?

(a) 2.1 (b) 2.3
(c) 2.6 (d) 2.9

36. In a 3600 m race around a circular track of length 400 m, the faster runner and the slowest runner meet at the end of the fourth minute, for the first time after the start of the race. All the runners maintain uniform speed throughout the race. If the faster runner runs at thrice the speed of the slowest runner. Find the time taken by the faster runner to finish the race.

(a) 36 minutes (b) 24 minutes
(c) 16 minutes (d) 12 minutes

37. If the wheel of a bicycle makes 560 revolutions in travelling 1.1 km, what is its radius?

(a) 31.25 cm (b) 37.75 cm
(c) 35.15 cm (d) 11.25 cm

38. A ship develops a leak 12 km from the shore. Despite the leak, the ship is able to move towards the shore at a speed of 8 km/hr. However, the ship can stay afloat only for 20 minutes. If a rescue vessel were to leave from the shore towards the ship and it takes 4 minutes to evacuate the crew and passengers of the ship, what should be the minimum speed of the rescue vessel in order to be able to successfully rescue the people aboard the ship?

(a) 53 km/hr (b) 37 km/hr
(c) 28 km/hr (d) 44 km/hr

39. When an object is dropped, the number of feet N that it falls is given by the formula $N = \frac{1}{2}gt^2$, where t is the time in seconds from the time it was dropped and g is 32.2. If it takes 5 seconds for the object to reach the ground, how many feet does it fall during the last 2 seconds?

(a) 64.4 (b) 96.6
(c) 161.0 (d) 257.6

40. By walking at $3/4^{\text{th}}$ of his usual speed, a man reaches office 20 minutes later than usual. What is his usual time?

(a) 30 minutes (b) 60 minutes
(c) 70 minutes (d) 50 minutes

41. In a 500m race Dishu beats Abhishek by 100 m or 5 seconds. In another race on the same track at the same speeds. Abhishek and Prashant start at one end while Dishu starts at the opposite end. How many metres would Abhishek have covered, by the time Dishu meets Prashant given that Dishu's speed is 10 m/sec more than that of Prashant.

(a) 200 m (b) 225 m
(c) 250 m (d) 275 m

42. The speed of a motor boat itself is 20 km/h and the rate of flow of the river is 4 km/h. Moving with the stream the boat went 120 km. What distance will the boat cover during the same time going against the stream?

(a) 80 km (b) 180 km
(c) 60 km (d) 100 km

43. Akhil covers a part of the journey at 20 kmph and the balance at 70 kmph taking total of 8 hours to cover the distance of 400 km. How many hours has he been driving at 20 kmph?

(a) 2 hours (b) 3 hours 20 minutes
(c) 4 hours 40 minutes (d) 3 hours 12 minutes

44. I travel the first part of my journey at 40 kmph and the second part at 60 kmph and cover the total distance of 240 km to my destination in 5 hours. How long did the first part of my journey last?

(a) 4 hours (b) 2 hours
(c) 3 hours (d) 2 hours 30 minutes

45. Two men are walking towards each other alongside a railway track. A freight train overtakes one of them in 20 seconds and exactly 10 minutes later meets the other man coming from the opposite direction. The train passes this man is 18 seconds. Assume the velocities are constant throughout. How long after the train has passed the second man will the men meet?

(a) 89.7 minutes (b) 90 minutes
(c) 90.3 seconds (d) 91 seconds

46. P and Q walk from A to B, a distance of 27 km at 5 km/hr and 7 km/hr, respectively. Q reaches B and immediately turns back meeting P at T. What is the distance from A to T?

(a) 25 km (b) 22.5 km
(c) 24 km (d) 20 km

47. A bus without stopping travels at an average speed of 60 km/hr and with stoppages at an average speed of 40 km/hr. What is the total time taken by the bus for stoppages on a route of length 300 km?

(a) 4 hours (b) 3 hours
(c) 2.5 hours (d) 3.5 hours

48. Distance between the points is P. A man with the speed of s and reaches the other point late by 2 hours. Calculated the speed with which the man has to travel to reach in time, where t is the actual time.

(a) $\frac{s\times(t+1)}{(t+2)}$ (b) $\frac{s\times t}{(t+2)}$

(c) $\frac{s\times(t+2)}{t}$ (d) $s\times\frac{t}{(t+1)}$

49. The speed of a bus during the second hour of its journey is twice that in the first hour. Also, its speed during the third hour is two-third the sum of its speeds in the first two hours. Had the bus travelled for three hours at the speed of the first hour, it would have travelled 120 km less. Find the average speed of the bus for the first three hours.

(a) 100 kmph (b) 80 kmph
(c) 70 kmph (d) 60 kmph

50. P beats Q by 5 seconds in a race of 1000 m and Q beats R by 5 meters in a race of 100 m. By how many seconds does P beat R in a race of 1000 m?

(a) 5 sec (b) 7 sec
(c) 10 sec (d) Cannot be determined

ANSWERS

1. (d)	**6.** (c)	**11.** (c)	**16.** (c)	**21.** (b)	**26.** (d)	**31.** (d)	**36.** (b)	**41.** (c)	**46.** (b)
2. (a)	**7.** (d)	**12.** (b)	**17.** (c)	**22.** (d)	**27.** (b)	**32.** (c)	**37.** (a)	**42.** (a)	**47.** (c)
3. (a)	**8.** (c)	**13.** (d)	**18.** (b)	**23.** (d)	**28.** (a)	**33.** (b)	**38.** (b)	**43.** (d)	**48.** (c)
4. (a)	**9.** (a)	**14.** (d)	**19.** (a)	**24.** (c)	**29.** (c)	**34.** (a)	**39.** (d)	**44.** (c)	**49.** (a)
5. (b)	**10.** (b)	**15.** (a)	**20.** (d)	**25.** (a)	**30.** (d)	**35.** (c)	**40.** (b)	**45.** (b)	**50.** (d)

EXPLANATIONS AND HINTS

1. (d) Time taken to cover first 300 m $= \frac{300}{54\times(5/18)} = \frac{300}{15} = 20$ s

Time taken to cover next 500 m $= \frac{500}{72\times(5/18)}$

$= \frac{500}{20} = 25$ s

So the total time = 45 s

Total distance = 800 m

So the Average speed $= \frac{800}{45}\times\frac{18}{5} = \frac{1600}{25} = 64$ km/h

2. (a) We know that the speed of the policeman $= \frac{1}{6}$ km/min

Speed of the thief $= \frac{1}{9}$ km/min

Policeman gains $= \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$ km/min $= \frac{1000}{18}$ m/min $= \frac{500}{9}$ m/min

To gain 500 m, time required $= \frac{500}{500/9}$ min $= 9$ min

Hence, the thief has gone ahead by $9 \times \frac{1}{9} = 1$ km

3. (a)

We know that,

$$\frac{13}{\text{Speed upstream}} = 5$$

and
$$\frac{28}{\text{Speed downstream}} = 5$$

Thus, speed upstream $= \frac{13}{5}$ km/h and speed downstream $= \frac{28}{5}$ km/h

Speed of the river $= \frac{1}{2}\left(\frac{28}{5} - \frac{13}{5}\right) = \frac{1}{2} \times \frac{15}{5} = 1.5$ km/h

4. (a) In each case, train has to travel its own length to pass each man.

Let the speed of the train is x km/h, then

$$\text{Length} = \text{Relative speed} \times \text{Time}$$
$$= (x-3) \times \frac{8}{60 \times 60}$$

Also, length $= (x-4) \times \frac{10}{60 \times 60}$

Now, length in both cases remains the same. So

$$(x-3) \times \frac{8}{60 \times 60} = (x-4) \times \frac{10}{60 \times 60}$$
$$\Rightarrow 4x - 12 = 5x - 20$$
$$\Rightarrow x = 8 \text{ km/h}$$

Thus, the speed of the train is 8 km/h.

5. (b) As already calculated, the speed of the train is 8 km/h.

Also,

$$\text{Length} = \text{Relative speed} \times \text{Time}$$
$$= (8-3) \times \frac{8}{60 \times 60} \text{ km}$$
$$= \frac{5 \times 8}{60 \times 60} \times 1000 \text{ m} = 11.11 \text{ m}$$

So the length of the train = 11.11 m

6. (c) Let the two trains meet after x h.

Speed of the train from Delhi = 60 km/h

Speed of the train from Jammu = 75 km/h

Then,

$$60 \times x + 75 \times (x-1) = 600$$
$$\Rightarrow 60x + 75x = 675$$
$$\Rightarrow 125x = 675$$
$$\Rightarrow x = \frac{27}{5} \textbf{ h}$$

Hence, they meet after $\frac{27}{5}$ h.

So the total distance from Delhi when they meet $= \frac{27}{5} \times 60 = 27 \times 12 = 324$ km

7. (d) Given that Jack is $1\frac{3}{4}$ times faster than Jeremy.

Speeds of Jack and Jeremy are in the ratio $1\frac{3}{4} : 1 = 7 : 4$

In a race of 7 m, A gains 3 m over B.

So to gain 90 m, race must be of $90 \times \frac{7}{3} = 210$ m

8. (c) Let the speed of the runner be x km/h.

$$\text{Length} = \text{Relative speed} \times \text{Time}$$
$$= (40 - x) \times 6$$

Similarly,

$$\text{Length} = (40 + x) \times 4$$

Solving the above equations, we get

$$(40 - x) \times 6 = (40 + x) \times 4$$
$$\Rightarrow 80 = 10x$$
$$\Rightarrow x = 8$$

Thus, the speed of the runner = 8 km/h

$$\text{Length} = \frac{36 \times 6 \times 1000}{3600} = \frac{6000}{100} \text{ m} = 60 \text{ m}$$

Thus, the length of the train = 60 m.

9. (a) Given that A runs 5 rounds and B runs 4 rounds.

A passes B every time in $5 \times \frac{1}{2} = \frac{5}{2}$ km

Hence, A passes B only once.

10. (b) Let the length of the platform is x m.

We know that the length of the train = 100 m

Time taken = 20 s

Speed of the train = 72 km/h $= 72 \times \frac{5}{18}$ m/s = 20 m/s

Relative speed = 20 m/s (since platform is stationary)

Hence,

$$\frac{100+x}{20} = 20$$
$$\Rightarrow x = 400 - 100 = 300 \text{ m}$$

11. (c) Let the speed of the boat in still water is x m/s.

Speed of the boat while going upstream = $(x-5)$ m/s

Speed of the boat while going downstream = $(x+5)$ m/s

Since distance remains same in both the cases, we have

$$(x+5)\times 6 = (x-5)\times 12$$
$$\Rightarrow x+5 = 2x-10$$
$$\Rightarrow x = 15$$

So the speed of the boat in still water = 15 m/s

12. (b) Let the initial speed be x.

New speed $= \dfrac{5x}{4}$

Now, if the initial time was y, time becomes $\dfrac{4y}{5}$.

Hence,

$$y - \frac{4y}{5} = 5$$
$$\Rightarrow y = 25$$

Thus, the initial time taken to reach the destination = 25 min

13. (d) We know that

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

Let the distance from A to B = y

Total distance = $3y$

Total time $= \dfrac{y}{x_1} + \dfrac{y}{x_2} + \dfrac{y}{x_2}$

$$\text{So the average speed} = \frac{3y}{(y/x_1)+(y/x_2)+(y/x_2)}$$
$$= \frac{3}{(1/x_1)+(2/x_2)} = \frac{3x_1x_2}{2x_1+x_2}$$

14. (d) Let distance from house to college be x km. Hence

$$\frac{x}{10} = t + \frac{6}{60} \quad \text{(i)}$$

Also,
$$\frac{x}{20} = t + \frac{2}{60} \quad \text{(ii)}$$

Subtracting Eq. (2) from Eq. (1), we get

$$\frac{x}{10} - \frac{x}{20} = \frac{6}{60} - \frac{2}{60}$$
$$\Rightarrow x = \frac{4}{60}\times 20 = \frac{4}{3} \text{ km}$$

Total distance = 10 km

15. (a) Let the distance between office and home is x km. Then

$$\frac{x}{5} + \frac{x}{3} = 8$$
$$\Rightarrow \frac{8x}{15} = 8$$
$$\Rightarrow x = 15 \text{ km}$$

Therefore, total distance = 15 km

16. (c) Let the speed of the boat in still water be x km/h and speed of the stream be y km/h.

$$\frac{12}{x-y} = 4 \Rightarrow x - y = 3 \quad \text{(i)}$$
$$\frac{28}{x+y} = 4 \Rightarrow x + y = 7 \quad \text{(ii)}$$

Adding the above two equations, we get

$$2x = 10$$
$$\Rightarrow x = 5$$

Hence, the speed of boat in still water = 5 km/h

17. (c) Let x is the length of the faster train.

Now, relative speed = 60 km/h − 30 km/h = 30 km/h

$= \dfrac{25}{3}$ m/s

Time = 9 s

So,

$$\text{Time} = \frac{\text{Distance}}{\text{Relative speed}}$$
$$\Rightarrow 9 = \frac{x}{25/3}$$
$$\Rightarrow x = \frac{25}{3}\times 9 = 75 \text{ m}$$

So the length of the faster train = 75 m

18. (b) Given that the average speeds of the trains are 60 km/h and 40 km/h.

To cover a distance of 40 km, the slower train will take 60 min.

The faster train will cover that distance in 40 min.

Hence, for the faster train to cover the same distance as the slower train, total time of stoppage = 20 min.

Now, since each stoppage is of 2 min, total number of stoppages = 10.

19. (a) Let the length of the moving train be x and speed be y. To cross a pole,

$$\text{Time} = \frac{\text{Length of train}}{\text{Speed of train}}$$

$$\Rightarrow 5 = \frac{x}{y}$$

Also, to cross a stationary train,

$$12 = \frac{x+350}{y}$$

$$\Rightarrow 12 = 5 + \frac{350}{y}$$

$$\Rightarrow y = 50 \text{ km/h}$$

Hence, the speed of the moving train is 50 km/h.

20. (d) Let the initial speed be x.

So the new speed $= \dfrac{4x}{5}$

Now, if the initial time was y, time becomes $\dfrac{5y}{4}$

Hence,

$$\frac{5y}{4} - y = 4$$

$$\Rightarrow \frac{y}{4} = 4$$

$$\Rightarrow y = 16$$

Thus, the initial time taken to reach the destination is 16 min.

21. (b) Time taken by a person = 5 minutes = 300 seconds

Speed of the person $= \left(\dfrac{600}{300}\right) = 2\,\text{m/s}$

Converting speed from m/s into km/h, we get

$$= \left(2 \times \frac{18}{5}\right) = 7.2 \text{ km/h}$$

22. (d) Distance $= (240 \times 5) = 1200$ km

We know that, speed = distance/time

Speed $= \dfrac{1200}{5/3} = 720$ km/h

23. (b) Let the distance between the two stations be x km.

I. Then, speed of the mail train = $(y + 12)$ km/hr

II. $\dfrac{x}{y} - \dfrac{x}{(y+12)} = \dfrac{40}{60}$

Thus, even I and II together do not give x.

24. (c) Let distance from A to C be x. Then distance from C to B = $(100 - x)$.

I. Distance from A to B = 125% of C to B.

$$\Rightarrow 100 = \frac{125}{100} \times (100 - x)$$

$$\Rightarrow (100 - x) = \frac{100 \times 100}{125} = 80$$

$$\Rightarrow x = 20$$

Thus, distance from A to C is 20 km. Hence, I alone can give the answer.

II. Distance from A to C $= \dfrac{1}{4}$(Distance from C to B)

$$x = \frac{1}{4}(100 - x) \Rightarrow 5x = 100 \Rightarrow x = 20$$

Thus, distance between A to C = 20 km.

Hence, II alone can give the answer.

25. (a) I. Relative speed = 135 km/hr

Thus, time taken $= \dfrac{500}{135}$ hrs.

II. does not give the relative speed.

Thus, I alone gives the answer and II is irrelevant.

26. (d) Since ratio of speed of X:Y is 3:4, then ratio of time will be 4:3.

I. If Y takes 3 min, then X takes 4 min.

II. If Y takes 36 min, then X takes $\left(\dfrac{4}{3} \times 36\right)$ min = 48 min.

Thus, I and II together give the answer.

27. (b) Let the speed of the belts be a and that of the trolley be b.

Let PQ = x and QR = y.

Time taken for cart to cover PR $= \dfrac{x}{a+b} + \dfrac{y}{b-a}$

Time taken for trolley to cover RP $= \dfrac{y}{a+b} + \dfrac{x}{b-a}$

Total time $= (x+y)\left[\dfrac{1}{a+b} + \dfrac{1}{b-a}\right] = 64\,\text{s}$

Hence, $x + y = \left[\dfrac{2^2 - 0.5^2}{2 \times 2}\right] \times 64 = 60$

28. (a) Let the distance between X and Y be d km

In the first instance distance travelled by A = $d + 11$

In the first instance distance travelled by B = $d - 11$

Also, the time taken by both is same.

$$\Rightarrow \frac{d+11}{A} = \frac{d-11}{B}$$

$$\Rightarrow \frac{A}{B} = \frac{d+11}{d-11} \quad \text{(i)}$$

In the second instance, distance travelled by B = d + 9 while distance travelled by C = d – 9

$$\Rightarrow \frac{B}{C} = \frac{d+9}{d-9} \quad \text{(ii)}$$

From Eq. (1) and (2), we get

$$\frac{A}{C} = \frac{A}{B} \times \frac{B}{C} = \frac{3}{2} \Rightarrow \frac{d+11}{d-11} \times \frac{d+9}{d-9} = \frac{3}{2}$$

$$\Rightarrow \frac{\left(d^2 + 20d + 99\right)}{\left(d^2 - 20d + 99\right)} = \frac{3}{2}$$

$$\Rightarrow 2\left(d^2 + 20d + 99\right) = 3\left(d^2 - 20d + 99\right)$$

$$\Rightarrow d^2 - 100d + 99 = 0$$

$$\Rightarrow (d-1)(d-99) = 0$$

Hence, d = 1 or 99. Since, d = 1 is not possible since distance travelled by B cannot be negative, we get d = 99.

29. (c) A and B run around a circular track. A laps B in the middle of the 5^{th} lap, i.e. when A has run four and a half laps, he has covered a distance which is 1 lap greater than that covered by B's.

Therefore, when A runs $\frac{9}{2}$ laps, B runs $\frac{7}{2}$ laps.

This is same as saying when A runs 9 laps, B runs 7 laps, i.e. in a race that is 9 laps long, A can give B a start of 2 laps.

So, if the race is of 3000 m long, then A can give B a start of $\left(\frac{2}{9}\right) \times 3000 = 666.67$ m

30. (d) Let the length of the race be d.

When B finished the race, A and C would have run (d – 36) and (d – 24) m, respectively.

When C finishes the race, A would have run (d – 16) m.

The ratio of speeds of C and A is given by

$$\frac{d-24}{d-36} = \frac{d}{d-16} \Rightarrow d = 96 \text{ m}$$

31. (d) Both the circular paths have the maximum possible radius, hence both have a radius of 3 cm each. Irrespective of the angle between the planes of their circular paths, the two cockroaches will meet again, at the point Q only, which is at a diametrically opposite end of P. A reaches point Q after 9 seconds.

On the other hand, B would have completed 3/2 of his revolution and it will also reach point Q after 9 seconds.

32. (c) Arpit and Gaurav travel for the same amount of time till the time they meet between x and y.

So, the distance covered by them will be the same as the ratio of their speeds. Let the time that they have taken to meet each other be x hours from the time they have started.

Therefore, to cover the entire distance, Arpit would take x + 4 hours and Gaurav would take x + 9 hours.

Ratio of time taken (Arpit:Gaurav) = $x + 4 : x + 9$

Hence, ratio of speeds of Arpit and Gaurav = $x + 9 : x + 4 = 1 : \frac{x+4}{x+9}$

By the time Arpit and Gaurav meet, Arpit would have travelled = $48x$ km.

After meeting, Gaurav takes 9 hours to cover this distance.

Hence, Gaurav's speed = $\frac{48x}{9}$ km/h.

But we know that the ratio of Arpit and Gaurav speeds are $1 : \frac{x+4}{x+9}$

Therefore,

$$48 : \frac{48x}{9} :: 1 : \frac{x+4}{x+9} \Rightarrow \frac{x}{9} = \frac{x+4}{x+9}$$

$$\Rightarrow x^2 + 9x = 9x + 36 \Rightarrow x^2 = 36$$

$$\Rightarrow x = 6$$

Hence, speed of Gaurav = $\frac{48x}{9} = \frac{48 \times 6}{9} = 32$ kmph $\simeq$ 20 mph.

33. (b) Let x km be the distance between his house and office.

Thus, while travelling at 24 kmph, he would take $\frac{x}{24}$ hours.

While travelling at 25% faster speed, i.e. 24 + 25% pf 24 = 24 $\times\left(\frac{1}{4}\right)$ = 30 kmph, time taken would be $\frac{x}{30}$ hours.

We are given that, time difference = 5 min late + 4 min early = 9 min.

$$\Rightarrow \frac{x}{24}-\frac{x}{30}=9 \Rightarrow \frac{6x}{24\times 30}=\frac{9}{60} \Rightarrow x=9\times 2=18$$

34. (a) Time taken by Sunil for 1st round = 2min

2nd round = 4min
3rd round = 8 min
4th round = 16 min
5th round = 32 min
6th round = 64 min
7th round = 128 min
8th round = 256 min

We are given that, Anil tales one minute for every round.

He meets 127 times in 7th and 255 times in 8th round. Thus, total time Anil and Sunil meet = 127 + 255 = 382.

35. (c) Speed of the river = 12 kmph.

Speed of the boy = 5 kmph

Let the time taken by the boy to cross the river in still water = t.

Width of the river = $5t$

The boy takes the least time when he is travelling directly across the river. But the river current pushes him in a direction perpendicular to the flow.

Distance travelling along the river = $12t$

Effective distance travelled by the boy = $\sqrt{(12t)^2+(5t)^2}=13t$

$$\Rightarrow \frac{\text{Distance covered by the boy}}{\text{Width of the river}}=\frac{13t}{5t}=2.6$$

36. (b) We are given that the faster runner is thrice as fast as the slowest runner. Hence, the faster runner would have completed three rounds by the time the slowest runner completes one round.

Also, their first meeting takes place after the fastest runner takes 4 min to complete one and the half round. Thus,

$$400\times\left(\frac{3}{2}\right)=600\,\text{m}$$

Hence, he takes $\left(\frac{3600}{600}\right)\times 4=24$ minutes to finish the race.

37. (a) The distance covered by the wheel in 560 revolutions = 1100 m

Hence, the distance covered per revolution = $\frac{1100}{560}$ $=\frac{55}{28}$ meters

The distance covered in one revolution = circumference of the wheel.

Circumference$=2\pi r=2\times\frac{22}{7}\times r \Rightarrow \frac{55}{28}=2\times\frac{22}{7}\times r$

$$r = 31.25 \text{ cm}$$

38. (b) Distance between the rescue vessel and the ship, which is 12 km has to be covered in 16 minutes. (The ship can stay afloat only 20 minutes and it takes 4 minutes to evacuate the people aboard the ship).

Therefore, the two vessels should move towards each other at a speed of

$$\frac{12}{\left(\frac{16}{60}\right)}=\frac{12\times 60}{16}=45\,\text{km/h}$$

The ship is moving at a speed of 8 km/h. Therefore, the rescue vessel should move at a speed = $45-8=37$ km/h.

39. (d) In 5 seconds, the object travels = $\left(\frac{1}{2}\right)\times 32.2\times 52$ $=16.1\times 25=402.5$

In first 3 seconds, the object travels = $\left(\frac{1}{2}\right)\times 32.2\times 32$ $=16.1\times 9=144.9$

Hence, in the last 2 seconds it travelled = $402.5-144.9=257.6$ feet.

40. (b) If a man is travelling with 3/4 of his usual speed, then he takes 4/3 of his usual time to cover the same distance.

Hence, extra time taken = $\frac{4}{3}-1=\frac{1}{3}$

We are given that, 1/3 of time is 20 minutes.

Usual time = $20\times 3=60$ minutes

41. (c) Abhishek's speed = $\frac{100}{5}=20$ m/s

Time taken by Abhishek to cover 500 m = $\frac{500}{20}=25$ seconds

Dishu's speed = $\frac{500}{20}=25$ m/s

Prashant's speed = 15 m/s

Time taken by Dishu to meet Prashant in 500 m race in opposite direction = $\frac{500}{(15+25)}=12.5$ s

Distance covered by Abhishek = $12.5\times(20)=250$ m

42. (a) Let the distance to be covered by the boat when it is travelling against the stream be x.

The boat goes down the river at a speed = 20 + 4 = 24 km/h

and up the river at a speed = $20 - 4 = 16$ km/h.

Since the time taken is same,

$$\frac{120}{24} = \frac{x}{16}$$

Therefore, $x = 80$ km.

43. (d) Let x be the number of hours he travels at 20 kmph.

Distance covered in this x hours = $20x$ km.

He would have therefore, travelled the rest $(8 - x)$ hours at 70 kmph.

Distance covered in this time = $(8-x)\times 70 = (560 - 70x)$ km

The total distance travelled = $20x + 560 - 70x = 400$

$$\Rightarrow 160 = 50x$$

$$\Rightarrow x = \left(\frac{160}{50}\right)$$

Thus, $x = 3.2$ hours = 3 hours and 12 minutes.

44. (c) We know that the total time of journey = 5 hours.

Let x hours be the time that I travelled at 40 kmph. Therefore, $5 - x$ hours would be time that I travelled at 60 kmph.

Hence, I would have covered in 5 hours = $(x\times 40) + (5 - x)60 = 240$

$$\Rightarrow 300 - 20x = 240 \Rightarrow x = 3 \text{ hours}$$

45. (b) Let the length of the train be L, x be the speed of the first man, y be the speed of the second man and z be the speed of the train. Thus, we can write

$$20 = \frac{1}{(z-x)} \text{ and } 18 = \frac{1}{(z+y)}$$

$$\Rightarrow z = 10x + 9y$$

Distance between the two men = $600(z + y)$ m

$$\text{Time} = \frac{600(z+y) - 600(z+y)}{(x+y)} = \frac{600(9x+9y)}{(x+y)}$$

= 90 minutes

46. (b) Let the distance be x km from A.

So total distance travelled by P = x at a speed of 5 km/h.

Total distance travelled by Q = 27 + $(27 - x)$ = $(54 - x)$ at a speed of 7 km/h.

Total time taken by P = $\frac{x}{5}$ and by Q = $\frac{(54-x)}{7}$.

Since, they have met at the same time, they would have travelled for the same time.

Hence,

$$\frac{x}{5} = \frac{(54-x)}{7} \Rightarrow x = 22.5 \text{ km}$$

47. (c) Let r = running time of the train, s = stopping time of the train and d = total distance travelled by train.

We have,

$$\frac{d}{r} = 60,\ \frac{d}{r+s} = 40$$

$$\Rightarrow \frac{(r+s)}{r} = \frac{3}{2} \Rightarrow \frac{s}{r} = \frac{1}{2}$$

As, $d = 300$ km

$$\frac{300}{r} = 60 \Rightarrow r = 5 \text{ hrs and } s = 2.5 \text{ hours.}$$

48. (c) Assuming x as the actual speed.

Then,

$$\frac{p}{x} = t \text{ and } \frac{p}{s} = t + 2$$

So,

$$x \times t = s \times (t+2)$$

$$\Rightarrow x = \frac{s\times(t+2)}{t}$$

49. (a) Let the speed for the first hour be s kmph.

Speed during the second hour = $2s$ kmph

Speed in the third hour = $\frac{2}{3}\times(3s) = 2s$ kmph

Distance travelled = $5s$ km

Had it travelled at s kmph, distance travelled would have been equal to $3s$ kmph.

We are given that,

$$5s - 3s = 120$$

$$\Rightarrow 2s = 120 \Rightarrow s = 60$$

Thus, average speed for the first three hours = $\frac{5s}{3} = \frac{5\times 60}{3} = 100$ kmph

50. (d) If P takes t seconds to run 1000 m, then Q takes $(t + 5)$ seconds.

In $(t + 5)$ s, Q runs 100 m whereas R runs only 950 m.

R runs 1000 m in $\frac{1000}{950}\times(t+5)$ sec

P takes t seconds to cover 1000 m, while R takes $\left(\frac{20t}{19} + \frac{100}{95}\right)$ sec.

Thus, from the given data we cannot determine t.

UNIT 2: ALGEBRA

CHAPTER 1

PERMUTATION AND COMBINATION

PERMUTATION

The notion of *permutation* is used with several slightly different meanings, all related to the act of permuting (rearranging) objects or values. Informally, a permutation of a set of objects is an arrangement of those objects in a particular order. When doing permutation, the order of the items is important. For example, permutations of three items a, b, c are ab, ba, ac, ca, bc and cb.

The number of permutations of n things taking r at a time is denoted by ${}^nP_r$, and the expression is

$${}^nP_r = \frac{n!}{(n+r)!}$$

Some other formulae on permutations are as follows:

1. The number of permutations of n different things taken r at a time is given by
$${}^nP_r = n^r$$
2. The number of permutations of n things taken all together, when x of the things are alike of one kind, y of the things are alike of one kind and z are rest of the things that are alike, is given by
$${}^nP_r = \frac{n!}{x!y!z!}$$
3. The number of ways of arranging n distinct objects along a round table is given by
$$(n-1)!$$
4. The number of ways of arranging n persons along a round table so that no person has the same two neighbors is given by
$$\frac{1}{2}(n-1)!$$

COMBINATION

A *combination* is a way of selecting several things out of a larger group, where (unlike permutations) order does not matter. For example, combinations of three items a, b, c are ab, ac and bc.

Number of combinations of n things taking r at a time is denoted by ${}^nC_r$, and the expression is

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

Some other formulae on combinations are as follows:

1. The number of ways in which $a + b$ things can be divided into two groups containing a and b things, respectively, is

$${}^{a+b}C_b = \frac{(a+b)!}{a!b!}$$

2. The number of ways in which a selection can be made out of $a + b + c$ things of which a are alike of one kind, b are alike of another kind and remaining c are different is given by

$$(a+1)(b+1)2^c - 1$$

3. The number of ways of arranging n distinct objects along a round table is given by

$$(n-1)!$$

4. The number of ways of arranging n persons along a round table so that no person has the same two neighbors is given by

$$\frac{1}{2}(n-1)!$$

Partitions

A combination is nothing but portioning a set into two subsets, one containing k objects and the other containing the remaining $n - k$ objects. Hence, in general the set $S = \{1, 2, \ldots, n\}$ can be partitioned into r subsets.

Consider the following iterative situation leading to partition of S. We first select a subset of n_1 elements from S. Having chosen the first subset, we select second subset containing n_2 elements from the remaining $n - n_1$ elements and so on until no elements remain. This procedure yields a partition of S into r subsets, with the p^{th} subset containing exactly n_p elements.

Now we wish to count the number of such partitions. We know that there are $\dfrac{n!}{n!(n-n_1)!}$ ways to form the first subset.

Hence, we can formulate that there are $\dfrac{\left(n-n_1-\cdots-n_{p-1}\right)!}{n_p!\left(n-n_1-\cdots-n_p\right)!}$ ways to form the p^{th} subset. Using the counting principle, the total number of partitions is then given by $\dfrac{n!}{n_1!n_2!\ldots n_r!}$.

This expression is called a multinomial coefficient.

COUNTING

There are two fundamental principles of counting, which are *principle of addition* and *principle of multiplication*. The two principles form the base of permutations and combinations.

Fundamental Principle of Addition

If there are two tasks such that they can be performed independently in m and n ways, respectively, then either of the two jobs can be performed in $(m + n)$ ways.

Fundamental Principle of Multiplication

If there are two tasks such that one of them can be completed in m ways and when it has been completed the second task can be completed in n ways, then the two tasks in succession can be completed in $(m \times n)$ ways.

SOLVED EXAMPLES

1. How many words can be formed out of the letters of the word "UNITED" taking all the letters at a time considering no letter is to be repeated?

Solution: The word "UNITED" consists of 6 different letters. Hence the required number of permutations $= {}^6P_6 = 6! = 6\times5\times4\times3\times2\times1 = 720$

2. From a group of 8 men and 4 women, 4 persons are to be selected to form a committee so that at least 2 women are there on the committee. In how many ways can it be done?

Solution: We may have (2 men and 2 women) or (1 man and 3 women) or (0 man and 4 women).

Therefore, the required number of ways is

$$({}^8C_2 \times {}^4C_2) + ({}^8C_1 \times {}^4C_3) + ({}^4C_4)$$
$$= \left(\frac{8\times 7}{2} \times \frac{4\times 3!}{4}\right) + (8\times 4) + (1)$$
$$= 168 + 33 + 1 = 201$$

3. How many words can be formed from the letters of the word "COMMUTE" taking all the letters at a time considering no letter is to be repeated and the first and last letter of the words are "M"?

Solution: The word "COMMUTE" has two M's. The first and the last letter should be M. The remaining letters are C, O, U, T, E. So, the number of permutations is

$${}^5P_5 = 5\times 4\times 3\times 2\times 1 = 120$$

4. In how many ways 6 prizes can be given to 10 students when a student may receive at most two prizes?

Solution: The first prize can be given in 10 ways as any student can receive it. Also, since a student may receive at most two prizes, the second prize can be given in 10 ways. Now, the third prize can be given in 9 ways since a student cannot get more than two prizes. Similarly, the fourth prize can be given in 9 ways. The fifth and sixth prize can be given in 8 ways each.

Hence, the total number of ways in which the prizes can be given to 10 students when no student can get more than two prizes $= 10\times 10\times 9\times 9\times 8\times 8 = 518400$

5. How many natural numbers less than 1000 can be formed with the digits 1, 2, 3, 4 and 5, considering repetition of digits is not allowed?

Solution: The numbers can be of 1, 2 or 3 digits. The numbers can be formed using the digits 1, 2, 3, 4, 5. So the total number of natural numbers less than 1000 using 1, 2, 3, 4, 5 is

$$= 5 + 5\times 4 + 5\times 4\times 3$$
$$= 5 + 20 + 60 = 85$$

6. In how many ways can 5 beads of different colors form a necklace?

Solution: Here, we are to form circular permutations of 5 things taken all at a time. This can be done in $(5-1)!$ ways. But in case of a necklace, clockwise and anti-clockwise, permutations are same. Hence, the required number of ways is

$$\frac{1}{2}(5-1)! = \frac{1}{2}4! = \frac{1}{2}\times 24$$
$$= 12$$

7. Mary has 4 bananas, 6 oranges and 5 apples. How many selections of fruits can be made by selecting at least one of them?

Solution: Five apples can be selected in 6 different ways, that is, we can choose 1, 2, 3, 4, 5 or none of the apple. Similarly, 4 bananas can be selected in 5 different ways and 6 oranges can be selected in 7 different ways. Therefore,

$$\text{Number of ways} = 6\times 5\times 7 = 210$$

This also includes the case in which none of the fruits is selected. Rejecting this case, we get required number of ways = 210 − 1 = 209.

8. In how many ways can the word "APOCALYPSE" be arranged such that all the vowels come together?

Solution: The word "APOCALYPSE" has 10 letters. Now, all the vowels are supposed to come together, we consider all the vowels "A, O, A, E" as one.

Hence, number of ways this can be arranged in $(7!/2!)$ (since 4 vowels are treated as 1, the number of arrangements become 7!, and because "P" is repeated twice).

Also, the vowels can be arranged in 4! number of ways within themselves but the vowel "A" is repeated twice. Hence, the total number of arrangements of the word "APOCALYPSE" such that all the vowels come together is

$$\frac{7!}{2!}\times\frac{4!}{2!} = 7!\times 3!$$
$$= 5040\times 6 = 30240$$

9. If ${}^{n-1}P_3 : {}^nP_4 = 1:6$, then what is the value of n?

Solution: We have

$${}^{n-1}P_3 : {}^nP_4 = 1:6$$
$$\Rightarrow \frac{(n-1)!}{(n-1-3)!}\times\frac{(n-4)!}{(n)!} = \frac{1}{6}$$
$$\Rightarrow \frac{(n-1)!}{(n-4)!}\times\frac{(n-4)!}{(n)!} = \frac{1}{6}$$
$$\Rightarrow \frac{(n-1)!}{n!} = \frac{1}{6}$$
$$\Rightarrow \frac{(n-1)!}{n\cdot(n-1)!} = \frac{1}{6}$$
$$\Rightarrow n = 6$$

10. How many triangles can be formed by joining the vertices of an octagon?

Solution: Total number of vertices in an octagon = 8.

We know that a triangle is formed by joining three vertices. Hence, this can be done in $^8C_3$ number of ways. Thus, the total number of triangles that can be formed is

$$^8C_3 = \frac{8!}{3!\times 5!} = \frac{8\times 7\times 6}{3\times 2} = 56$$

PRACTICE EXERCISE

1. How many numbers are there between 100 and 1000 in which all the digits are distinct?

(a) 1000 (b) 810
(c) 729 (d) 648

2. How many words can be formed out of the letters of the word "UNITED" taking all the letters at a time considering no letter is to be repeated?

(a) 120 (b) 720
(c) 216 (d) 600

3. From a group of 6 men and 4 women, four persons are to be selected to form a committee so that at least 2 men are there on the committee. In how many ways can it be done?

(a) 172 (b) 190
(c) 185 (d) 216

4. How many words can be formed from the letters of the word "APPLE" taking all the letters at a time considering no letter is to be repeated and the first and last letter of the words are "P"?

(a) 6 (b) 120
(c) 36 (d) 12

5. Out of 6 consonants and 3 vowels, how many words can be formed considering exactly 3 consonants and exactly 2 vowels?

(a) 7200 (b) 3600
(c) 60 (d) 120

6. Six candidates are to be examined, 2 in Electronics and the remaining in different disciplines. In how many ways can they be seated in a row so that the two examinees in Electronics may not sit together?

(a) 240 (b) 480
(c) 360 (d) 720

7. How many words can be formed with the letters of the word "SQUARE" considering all the words start with "S" and end with "E"?

(a) 720 (b) 15
(c) 72 (d) 24

8. In how many ways 4 prizes can be given to 5 students when any student may receive any number of prizes?

(a) 320 (b) 120
(c) 625 (d) 5

9. How many natural numbers less than 1000 can be formed with the digits 1, 2, 3, 4 and 5, considering repetition of digits is allowed?

(a) 200 (b) 125
(c) 155 (d) 120

10. Simon has 5 apples, 3 bananas and 6 oranges. In how many ways can they be arranged in a row?

(a) 90 (b) 168168
(c) 172880 (d) 272800

11. In how many ways can 6 beads of different colors form a necklace?

(a) 60 (b) 100
(c) 120 (d) 720

12. If $^nC_3 = {^nC_5}$, then what is the value of n?

(a) 10 (b) 12
(c) 6 (d) 8

13. If $^nC_r : {^nC_{r+1}} = 1:2$ and $^nC_{r+1} : {^nC_{r+2}} = 2:3$, then what is the value of n and r?

(a) 9, 2 (b) 11, 3
(c) 14, 4 (d) 17, 5

14. Roger has 4 dark chocolates, 3 milk chocolates and 5 caramel chocolates. How many selections of chocolates can be made by selecting at least one of them?

(a) 119 (b) 149
(c) 150 (d) 120

15. A man has 7 relatives, 3 men and 4 women and his wife also has 7 relatives, 4 men and 3 women. In how many ways can they invite a dinner party of 4 women and 4 men so that there are 3 of man's relatives and 3 of the wife's relatives?

(a) 365 (b) 465
(c) 625 (d) 485

16. In how many ways can the word "CONSTANTINE" be arranged such that all the vowels come together?

(a) 64260 (b) 967680
(c) 80640 (d) 72000

17. In how many ways can the word "CONSTANTINE" be arranged such that all the consonants come together?

(a) 25200 (b) 42500
(c) 50400 (d) 604800

18. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$, then what is the value of n?

(a) 4 (b) 3
(c) 2 (d) 1

19. If ${}^{10}P_5 + 4 \cdot {}^{10}P_6 = k \cdot {}^{10}P_5$, then what is the value of k?

(a) 10 (b) 21
(c) 5 (d) 3

20. How many triangles can be formed by joining the vertices of a pentagon?

(a) 20 (b) 15 (c) 12 (d) 10

ANSWERS

1. (d)	**4.** (a)	**7.** (d)	**10.** (b)	**13.** (c)	**16.** (c)	**19.** (b)
2. (b)	**5.** (a)	**8.** (c)	**11.** (a)	**14.** (a)	**17.** (c)	**20.** (d)
3. (c)	**6.** (b)	**9.** (c)	**12.** (d)	**15.** (d)	**18.** (a)	

EXPLANATIONS AND HINTS

1. (d) Total numbers between 100 and 1000 considering digits are distinct

Total distinct hundredth digit = 9 (Since we cannot use 0)

Total distinct tenth digit = 9 (since we cannot use the digit used at hundredth digit but can use 0)

Similarly, total distinct ones digit = 8.

$$9 \times 9 \times 8 = 648$$

2. (b) The word "UNITED" consists of 6 different letters. Hence the required number of permutations is

$${}^6P_6 = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

3. (c) We may have (2 men and 2 women) or (3 men and 1 women) or (4 men). Therefore, the required number of ways is

$$({}^6C_2 \times {}^4C_2) + ({}^6C_3 \times {}^4C_1) + ({}^6C_4)$$
$$= \left(\frac{6\times 5}{2} \times \frac{4\times 3}{2}\right) + \left(\frac{6\times 5\times 4}{3\times 2} \times 4\right) + \left(\frac{6\times 5}{2}\right)$$
$$= 90 + 80 + 15 = 185$$

4. (a) The word "APPLE" has two P's. The first and the last letter should be P. The remaining letters are A, L, E. So the number of permutations = $3\times 2\times 1 = 6$.

5. (a) Number of ways of selecting 3 consonants out of 6 and 2 vowels out of 3 are

$$^6C_3 \times ^3C_2 = \frac{6\times 5\times 4}{3\times 2}\times 3 = 60$$

Number of groups, each having 3 consonants and 2 vowels = 60. Each group contains 5 letters.

Number of ways of arranging 5 letters among themselves = $5! = 5\times 4\times 3\times 2\times 1 = 120$

Therefore, required number of ways = 60 × 120 = 7200

6. (b) The total number of ways in which 6 candidates can sit = $^6P_6 = 720$.

When two candidates of Electronics sit together, we can consider them as one candidate. Now, total candidates become 5. They can be seated in $^5P_5$ ways. Two Electronic students can arrange themselves in 2! ways. Therefore, number of ways in which Electronics students sit together = 120 × 2 = 240.

Hence, number of ways in which two candidates in Electronics do not sit together = 720 − 240 = 480.

7. (d) The word "SQUARE" has 6 different letters. Since, the first letter of every word is "S" and last letter of every word is "E." So the total numbers of ways the rest of the four words can be arranged = $4\times 3\times 2\times 1 = 24$.

8. (c) First prize can be given in 5 ways. Since, a student can receive any number of prizes.

The number of ways, the 4 prizes can be distributed = $5\times 5\times 5\times 5 = 625$.

9. (c) The numbers can be of 1, 2 or 3 digits. The numbers can be formed using the digits 1, 2, 3, 4, 5. The total number of natural numbers less than 1000 using 1, 2, 3, 4, 5 is

$$\begin{aligned} &5 + 5\times 5 + 5\times 5\times 5 \\ &= 5 + 25 + 125 = 155 \end{aligned}$$

10. (b) Total number of fruits = 14.

Five are of one kind (apples), three are of second kind (bananas) and six are of third kind (oranges). Hence

$$\frac{14!}{5!\times 3!\times 6!} = \frac{14\times 13\times 12\times 11\times 10\times 9\times 8\times 7\times 6!}{(5\times 4\times 3\times 2)\times(3\times 2)\times 6!}$$
$$= 168168$$

11. (a) Here, we are to form circular permutations of 6 things taken all at a time. This can be done in $(6-1)!$ ways. But in the case of a necklace, clockwise and anti-clockwise permutations are same. Hence, the required number of ways is

$$\frac{1}{2}(6-1)! = \frac{1}{2}5! = \frac{1}{2}\times 120 = 60$$

12. (d) We have

$$\begin{aligned} &^nC_3 = {}^nC_5 \\ &\Rightarrow C(n, n-3) = C(n,5) \\ &\text{Since, } C(n,r) = C(n,\ n-r) \\ &\Rightarrow n-3 = 5 \\ &\Rightarrow n = 8 \end{aligned}$$

13. (c) We have

$$\begin{aligned} &^nC_r : {}^nC_{r+1} = 1:2 \\ &\Rightarrow \frac{^nC_r}{^nC_{r+1}} = \frac{1}{2} \\ &\Rightarrow \frac{n!}{r!(n-r)!}\times\frac{(r+1)!(n-r-1)!}{n!} = \frac{1}{2} \\ &\Rightarrow \frac{(r+1)}{n-r} = \frac{1}{2} \\ &\Rightarrow 2r+2 = n-r \\ &\quad n-3r = 2 \qquad \text{(i)} \\ &^nC_{r+1} : {}^nC_{r+2} = 2:3 \\ &\Rightarrow \frac{^nC_{r+1}}{^nC_{r+2}} = \frac{2}{3} \\ &\frac{n!}{(r+1)!(n-r-1)!}\times\frac{(r+2)!(n-r-2)!}{n!} \\ &\Rightarrow \frac{(r+2)}{(n-r-1)} = \frac{2}{3} \\ &\Rightarrow 3r+6 = 2n-2r-2 \\ &\quad 2n-5r = 8 \qquad \text{(ii)} \end{aligned}$$

Solving Eqs. (1) and (2), we get

$$\begin{aligned} &2(2+3r)-5r = 8 \\ &\Rightarrow 4+6r-5r = 8 \\ &\Rightarrow r = 4 \\ &n = 2+3(4) = 14 \end{aligned}$$

Therefore, value of n and r = 14, 4.

14. (a) Four dark chocolates can be selected in 5 different ways, that is, we can choose 1, 2, 3, 4 or none of the dark chocolate. Similarly 3 milk chocolates can be selected in 4 different ways and 5 caramel chocolates can be selected in 6 different ways. Therefore, the number of ways

$$6 \times 5 \times 4 = 120$$

This also includes the case in which none of the chocolates is selected. Rejecting this case, we get required number of ways = 120 − 1 = 119.

15. (d) Let M and W denote men and women, respectively. Under the given restrictions of the problem, the different possibilities are

(a) 3W from man's relatives and 3M from wife's relatives.
(b) 2W, 1M from man's relatives and 1W, 2M from wife's relatives.
(c) 1W, 2 M from man's relatives and 2W, 1M from wife's relatives.
(d) 3M from man's relatives and 3 W from wife's relatives.

Therefore, the required number of ways of inviting a dinner party

$$\begin{aligned}&{}^4C_3 \times {}^4C_3 + {}^4C_2 \times {}^3C_1 \times {}^3C_1 \times {}^4C_2 + {}^4C_1 \\ &\qquad \times {}^3C_2 \times {}^3C_2 \times {}^4C_1 + {}^3C_3 \times {}^3C_3 \\ &= 4 \times 4 + 6 \times 3 \times 3 \times 6 + 4 \times 3 \times 3 \times 4 + 1 \times 1 \\ &= 16 + 324 + 144 + 1 = 485\end{aligned}$$

16. (c) The word "CONSTANTINE" has 11 letters.

Now, since all the vowels are supposed to come together, we consider all the vowels "O, A, I, E" as one.

Hence, the number of ways this can be arranged is

$$\frac{8!}{2! \times 3!}$$

(since 4 vowels are treated as 1, the number of arrangements become 8!, and because "N" is repeated thrice and "T" is repeated twice.)

Also, the vowels can be arranged in 4! number of ways within themselves. Hence, the total number of arrangements of the word "CONSTANTINE" such that all the vowels come together will be

$$\begin{aligned}&\frac{8!}{2! \times 3!} \times 4! \\ &= 8! \times 2 = 80640\end{aligned}$$

17. (c) The word "CONSTANTINE" has 11 letters. Now, since all the vowels are supposed to come together, we consider all the vowels "C, N, S, T, N, T, N" as one letter. Hence, number of ways this can be arranged is 5!. Also, number of ways in which the consonants can be arranged within themselves will be

$$\frac{7!}{2! \times 3!}$$

Hence, the total number of arrangements of the word "CONSTANTINE" such that all the consonants come together will be

$$\begin{aligned}&\frac{7!}{2! \times 3!} \times 5! \\ &= 7! \times 10 = 50400\end{aligned}$$

18. (a) We have

$$\begin{aligned}&{}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5 \\ &\frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5} \\ &\Rightarrow \frac{(2n+1)(2n)(2n-1)!}{(n+2)(n+1)(n)(n-1)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5} \\ &\Rightarrow \frac{2(2n+1)!}{(n+2)(n+1)} = \frac{3}{5} \\ &\Rightarrow 10(2n+1) = 3(n+2)(n+1) \\ &\Rightarrow 20n + 10 = 3n^2 + 9n + 6 \\ &\Rightarrow 3n^2 - 11n - 4 = 0 \\ &\Rightarrow (n-4)(3n+1) = 0 \\ &\Rightarrow n = 4\end{aligned}$$

19. (b) We have

$${}^{10}P_5 + 4 \cdot {}^{10}P_6 = k \cdot {}^{10}P_5$$

Now,

$$\begin{aligned}&{}^{10}P_5 + 4 \cdot {}^{10}P_6 \\ &= \frac{10!}{5!} + 4 \cdot \frac{10!}{4!} \\ &= 10!\left[\frac{1}{5!} + \frac{1}{3!}\right] = 10!\left[\frac{21}{120}\right] \\ &= 10!\left[\frac{21}{5!}\right] \\ &= 21 \times \frac{10!}{5!}\end{aligned}$$

Equating the above expression with right hand side, we get

$$21 \times \frac{10!}{5!} = k \times \frac{10!}{5!}$$

Therefore, $k = 21$

20. (d) Total number of vertices in a pentagon $= 5$

We know that a triangle is formed by joining three vertices. Hence, this can be done in ${}^5C_3$ number of ways. Thus, the total number of triangles that can be formed is ${}^5C_3$

$$= \frac{5!}{3! \times 2!}$$

$$= 5 \times 2 = 10$$

CHAPTER 2

PROGRESSION

ARITHMETIC PROGRESSION

An *arithmetic progression* (AP) is a sequence of numbers such that the difference between the consecutive terms is constant. This constant value is called common difference. In other words, any term of an arithmetic progression can be obtained by adding the common difference to the preceding term.

Let a be the first term, n be the number of terms, d be the common difference, a_n be the nth term, S_n be the sum of first n terms and S be the sum of the entire sequence. Then,

$$a_n = a + (n-1) \times d$$

$$S_n = \frac{n}{2} \times [2a + (n-1) \times d]$$

$$S = \frac{n}{2} \times (\text{First term} + \text{Last term})$$

Arithmetic progression can be represented as a, $a + d$, $a + 2d$, …, $[a + (n - 1)d]$. Here, quantity d is to be added to any chosen term to get the next term of the progression.

GEOMETRIC PROGRESSION

A *geometric progression* (GP) is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio. In other words, any term of a geometric progression can be obtained by multiplying preceding number by the common ratio.

Let a be the first term, n be the number of terms, r be the common ratio, a_n be nth term, S_n be the sum of first n terms. Then,

$$a_n = ar^{(n-1)}$$

$$S_n = \frac{a(1-r^n)}{1-r} \text{ if } r < 1$$

$$S_n = \frac{a(r^n-1)}{r-1} \text{ if } r > 1$$

Geometric progression can be represented as a, ar, ar^2, …, ar^n. Here, a is the first term and quantity r is the common ratio of the geometric progression.

Infinite Geometric Progression

If $-1 < r < +1$ or $|r| < 1$, then the sum of a geometric progression does not increase infinitely; it converges to a particular value. Such a GP is called *infinite geometric progression*. Mathematically, the sum of an infinite geometric progression is

$$S_\infty = \frac{a}{1-r}$$

HARMONIC SERIES

1. **Arithmetic mean (AM):** The most commonly used average is the arithmetic mean or simply the average. Arithmetic mean of n numbers $x_1, x_2, x_3, \ldots, x_n$ is denoted by $\bar{X}$ and calculated as

$$\bar{X} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

$$\Rightarrow \bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

If values $x_1, x_2, x_3, \ldots, x_n$ are the assigned weights $w_1, w_2, w_3, \ldots, w_n$, respectively, then the *weighted arithmetic mean* is given as

$$x_w = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

2. **Geometric mean (GM):** Geometric mean of n numbers $x_1, x_2, x_3, \ldots, x_n$ is nth root of their products. It is a type of mean or average, which indicates the central tendency or typical value of a set of numbers by using the product of their values. Geometric mean of n numbers $x_1, x_2, x_3, \ldots, x_n$ is denoted by $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$ and calculated as

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} = \sqrt[n]{x_1 x_2 x_3 \cdots x_4}$$

3. **Harmonic mean (HM):** Harmonic mean is the special case of the power mean. As it tends strongly toward the least elements of the list, it may (compared to the arithmetic mean) mitigate the influence of large outliers and increase the influence of small values. Harmonic mean of n numbers $x_1, x_2, x_3, \ldots, x_n$ is calculated as

$$\text{HM} = \frac{n}{\sum_{i=1}^{n} (1/x_i)}$$

RELATION BETWEEN AM, GM AND HM

Consider two numbers a and b.

Arithmetic mean $= (a+b)/2$; Geometric mean $= \sqrt{ab}$; Harmonic mean $= 2ab/(a+b)$

Therefore,

$$(\text{GM})^2 = \text{AM} \times \text{HM}$$

Also,

Arithmetic mean > Geometric mean > Harmonic mean

SOLVED EXAMPLES

1. What is the tenth term of an arithmetic progression whose fifth term is 19 and common difference d is 4?

Solution: We know that

$$a_n = a + (n-1) \times d$$
$$\Rightarrow 19 = a + (5-1) \times 4$$
$$\Rightarrow a = 3$$

Now,

$$a_{10} = 3 + (10-1) \times 4 = 3 + (9) \times 4 = 39$$

2. If in an AP, tenth term is 86 and 86th term is 10, then what will be the 50th term?

Solution: We know that

$$86 = a + (10-1) \times d$$
$$10 = a + (86-1) \times d$$

Therefore, subtracting the above equations, we get

$$-76d = 76$$
$$\Rightarrow d = -1$$

Substituting this value of d, we get

$$86 = a + (10-1) \times (-1)$$
$$\Rightarrow a = 86 + 9 = 95$$

Now, 50th term of the AP is given as

$$a_{50} = 95 + (50-1)(-1) = 95 - 49 = 46$$

3. Sum of the first 21 terms of an AP is 1071. What is the first term if the common difference is 6?

Solution: We know that

$$S_n = \frac{n}{2} \times [2a + (n-1) \times d]$$

Now,

$$1071 = \frac{21}{2} \times [2a + (21-1) \times 6]$$
$$\Rightarrow 2a = \frac{1071 \times 2}{21} - 120$$
$$\Rightarrow a = -18/2 = -9$$

4. Find out the 10^{th} term of a geometric progression and the sum if $a_1 = 35$ and the common ratio, $r = 2$.

Solution: Use the formula,

$$a_n = a_1 \times r^{n-1}$$

that gives the n^{th} term to find a_{10} as follows:

$$a_6 = 35 \times (2)^{10-1} = 35 \times (2)^9 = 35 \times 512 = 17920$$

Hence, the 10^{th} term of a geometric sequence is 17920.

Series of the sequence,

$$s_n = \frac{a_1\left(1 - r^n\right)}{1 - r}$$
$$s_{10} = \frac{35\left(1 - 2^{10}\right)}{1 - 2} = \frac{35(-1023)}{-1} = 35805$$

5. The sum of the first two terms of a G.P is 2 and the sum of the first four terms is 20. Find the GP

Solution: Consider the GP a, ar, ar^2, …
We are given,

$$a + ar = 2 \Rightarrow a(1 + r) = 2 \qquad (1)$$
$$a + ar + ar^2 + ar^3 = 20a(1 + r)\left(1 + r^2\right) = 20 \qquad (2)$$

Using Eq. (1) and (2), we get

$$2\left(1 + r^2\right) = 20 \Rightarrow 1 + r^2 = 10$$
$$\Rightarrow r^2 = 9 \Rightarrow r = \pm 3$$

Substituting $r = \pm 3$ in Eq. (1),

If $r = 3$, then $a = 1/2$

If $r = -3$, then $a = -1$.

The geometric progression is $-1, 3, -9, 27, \ldots$

6. If a, b, c and d are in GP then prove that $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$.

Solution: We are given that a, b, c and d are in GP Thus,

$$b = ar, c = ar^2, d = ar^3.$$

Considering L.H.S.

$$(b-c)^2 + (c-a)^2 + (d-b)^2 = \left(ar - ar^2\right)^2 + \left(ar^2 - a\right)^2 + \left(ar^3 - ar\right)^2$$
$$= a^2\left[\left(r - r^2\right)^2 + \left(r^2 - 1\right)^2 + \left(r^3 - r\right)^2\right]$$
$$= a^2\left(r^6 - 2r^3 + 1\right) = \left(ar^3 - a\right)^2$$
$$= \left(a - ar^3\right)^2$$
$$= (a - d)^2 = \text{R.H.S.}$$

7. The sixth and the tenth term of a GP are 70 and 17920, respectively. Find the GP

Solution: We are given that the sixth and tenth terms are 70 and 17920. Thus,

$$\frac{ar^9}{ar^5} = \frac{17920}{70} \Rightarrow r^4 = 256 \Rightarrow r = \pm 4$$

Substituting $r = 4$, we get

$$a(4)^5 = 70 \Rightarrow a = \frac{70}{1024} = \frac{35}{512}$$

If $r = -4$, then we get $a = -\dfrac{35}{512}$

Therefore, the GP is $\dfrac{35}{512}, \dfrac{140}{512}, \dfrac{560}{512}, \ldots$ or $-\dfrac{35}{512}, -\dfrac{140}{512}, -\dfrac{560}{512}, \ldots$

8. Find three numbers in GP whose sum is 14 and product is 64.

Solution: Let the numbers be $\dfrac{a}{r}$, a and ar. Product of the numbers $= \dfrac{a}{r} \times a \times ar = 64$

$$a^3 = 64 \Rightarrow a = 4$$

Sum of then numbers =

$$\frac{a}{r} + a + ar = a\left(\frac{1}{r} + 1 + r\right) = 14$$
$$2\left(1 + r + r^2\right) = 7r \Rightarrow 2r^2 - 5r + 2 = 0$$
$$\Rightarrow r = \frac{1}{2}, 2$$

If $r = 2$, the numbers are 2, 4, 8.

If $r = \dfrac{1}{2}$, the numbers are 8, 4, 2.

9. Find out the 7^{th} term of a geometric progression and the sum if $a_1 = 79$ and the common ratio $r = 2$.

Solution: Using the formula,

$$a_n = a_1 \times r^{n-1}$$

$$a_7 = a_1 \times (2)^{7-1} = 79 \times (2)^6 = 79 \times 64 = 5056$$

The 7^{th} term of the GP sequence is 5056.

Series of the sequence,

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_7 = \frac{79(1-2^7)}{1-2} = \frac{79(-127)}{-1} = 10033$$

10. Find out the 5^{th} term of a geometric progression and the sum if $a_1 = 46$ and the common ratio $r = 2$.

Solution: Using the formula,

$$a_n = a_1 \times r^{n-1}$$

$$a_5 = 46 \times (2)^{5-1} = 46 \times (2)^4 = 46 \times 16 = 736$$

The 5^{th} term of a geometric progression is 736.

Sum of the sequence,

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_5 = \frac{46(1-2^5)}{1-2} = \frac{46(-31)}{-1} = 1426$$

11. What is the geometric mean of the following set of data

5, 6, 8, 12, 2

Solution: Given that, $a_1 = 5$, $a_2 = 6$, $a_3 = 8$, $a_4 = 12$, $a_5 = 2$. Here $n = 5$.
Geometric mean formula is

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} = \sqrt[n]{a_1 a_2 a_3 \cdots a_n} = \sqrt[5]{5 \times 6 \times 8 \times 12 \times 2}$$

$$= \sqrt[5]{5760} = 5.65$$

12. Find the harmonic mean of the following data:

8, 9, 6, 11, 10, 5

Solution: Given data: {8, 9, 6, 11, 10, 5}

$$\text{So harmonic mean} = \frac{6}{\frac{1}{8}+\frac{1}{9}+\frac{1}{6}+\frac{1}{11}+\frac{1}{10}+\frac{1}{5}}$$

$$\text{H} = \frac{6}{0.7936} = 7.560$$

13. Which term of the AP 3, 15, 27, 39, … will be 132 more than its 54^{th} term?

Solution: We are given an arithmetic series as 3, 15, 27, 39, …

Here, $a = 3$, $d = 15 - 3 = 12$

Since, $a_n = a_k = (n-k)d$

$$a_n - a_{54} = (n-54)d \Rightarrow 132 = 12n - (54 \times 12)$$
$$\Rightarrow 12n = 132 + 648 = 780$$
$$\Rightarrow n = 65$$

Thus, the 65^{th} term of the AP will be 132 more than its 54^{th} term.

14. Find the 31^{st} term of an AP whose 11^{th} term is 38 and the 16^{th} term is 73.

Solution: Let a be the 1^{st} term and d be the common difference. We are given that,

$$a_{11} = a + 10d = 38 \qquad \text{(i)}$$
$$a_{16} = a + 15d = 73 \qquad \text{(ii)}$$

Subtracting Eq. (2) from (1), we get

$$a + 10d - 1 - 15d = 38 - 73$$
$$\Rightarrow -5d = -35$$
$$\Rightarrow d = 7$$

Now, substituting $d = 7$ in Eq. (1), we get

$$a + (10 \times 7) = 38$$
$$\Rightarrow a = 38 - 70 = -32$$
$$\Rightarrow a_{31} = a + 30d = -32 + (30 \times 7)$$
$$= -32 + 210 = 178$$

Hence, 31^{st} term is 178.

15. Is -150 a term of the series 11, 8, 5, 2, …?

Solution: Here, $a = 11$, $d = 8 - 11 = -3$. Let $a_n = -150$
Therefore,

$$a + (n-1)d = -150$$
$$\Rightarrow 11 + (n-1)(-3) = -150$$
$$\Rightarrow -3(n-1) = -161 \Rightarrow n - 1 = \frac{161}{3}$$

Since, $\frac{161}{3}$ is not an integral number, -150 is not a term of the given series.

PRACTICE EXERCISE

1. What is the sum of all the odd numbers up to 100?

(a) 1250 (b) 2500
(c) 3200 (d) 2700

2. Sum of the first four and twelve terms of an AP is 26 and 222, respectively. What is the sum of the first 15 terms of the AP?

(a) 345 (b) 295
(c) 400 (d) 370

3. An AP consists of 50 terms of which third term is 12 and the last term is 106. Find the 29th term.

(a) 58 (b) 76
(c) 64 (d) 52

4. Find the sum of the following infinite series 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$, …

(a) 8 (b) 6
(c) 4 (d) 2

5. What is the maximum sum of the series 60, 56, 52, 48, …?

(a) 540 (b) 620
(c) 720 (d) 480

6. Three numbers whose sum is 15 are in AP If 8, 6 and 4 be added to them, respectively, then these are in GP. What are the three numbers?

(a) 3, 5, 7 (b) 0, 5, 10
(c) 2, 5, 8 (d) 1, 5, 9

7. The second term of a GP is 25/4 and the eighth term is 16/625, find the first term of the GP?

(a) $\frac{16}{5}$ (b) $\frac{125}{8}$
(c) $\frac{25}{4}$ (d) $\frac{16}{9}$

8. Insert three terms between 1 and 16 if they are in GP.

(a) 3, 6, 9 (b) 2, 4, 8
(c) 5, 9, 13 (d) 2, 8, 12

9. Insert three terms between 2 and 18 if they are in AP.

(a) 5, 10, 15 (b) 6, 9, 12
(c) 4, 8, 12 (d) 6, 10, 14

10. What is the ratio of common difference d_1 and d_2 of two arithmetic progressions if respective nth terms are in the ratio of $2n+3 : n-1$?

(a) 4:1 (b) 2:1
(c) 1:2 (d) 1:4

11. How many terms are there in an arithmetic sequence whose first and fifth terms are −12 and 0, respectively, and the sum of terms is 135?

(a) 8 (b) 15
(c) 20 (d) 10

12. What is the sum up to 10 terms of the series 2/3, 8/9, 26/27, 80/81, …?

(a) $\frac{3^{10}+1}{2\cdot 3^{10}}$ (b) $\frac{21\cdot 3^{10}+1}{2\cdot 3^{10}}$
(c) $\frac{20\cdot 3^{10}+1}{2\cdot 3^{10}}$ (d) $\frac{19\cdot 3^{10}+1}{2\cdot 3^{10}}$

13. If the ratio of the sum of the first eight terms of a GP to the sum of the first four terms of the same GP is 9, what is the common ratio of that GP?

(a) 1 (b) 2 (c) 3 (d) 4

14. What is the sum to infinity of the series, $3+6x^2+9x^4+12x^6+\cdots$ given that $|x|<1$.

(a) $\frac{3}{(1+x^2)^2}$ (b) $\frac{3}{(1-x^2)}$
(c) $\frac{3}{(1+x^2)}$ (d) $\frac{3}{(1-x^2)^2}$

15. What is the harmonic mean of two numbers whose geometric mean and arithmetic mean are 6 and 12, respectively?

(a) 5 (b) 2 (c) 9 (d) 3

16. If GM = 12 and HM = 7.2, then what will be the value of AM?

(a) 20 (b) 15 (c) 9 (d) 12.5

17. Arithmetic mean of the two numbers is 25 and geometric mean is 7. What are the two numbers?

(a) 2 and 10 (b) 25 and 25
(c) 1 and 49 (d) 2 and 49/2

18. For what value of x, the following numbers are in geometric progression?

$$\frac{2}{3},\ x,\ \frac{8}{27}$$

(a) $\pm\frac{4}{9}$ (b) $\pm\frac{8}{9}$ (c) $\pm\frac{16}{9}$ (d) $\pm\frac{2}{9}$

19. What is the sum of the following infinite series?

$$\frac{3}{4}-\frac{5}{4^2}+\frac{3}{4^3}+\frac{5}{4^4}+\cdots$$

(a) 8/15 (b) 7/15
(c) 7/13 (d) 8/17

20. The third term of a geometric progression is 2. What is the product of the first five terms?

(a) 4 (b) 8
(c) 16 (d) 32

ANSWERS

1. (b)	**4.** (b)	**7.** (b)	**10.** (b)	**13.** (c)	**16.** (a)	**19.** (b)
2. (a)	**5.** (d)	**8.** (b)	**11.** (b)	**14.** (d)	**17.** (c)	**20.** (d)
3. (c)	**6.** (a)	**9.** (d)	**12.** (d)	**15.** (d)	**18.** (a)	

EXPLANATIONS AND HINTS

1. (b) Given numbers are 1, 3, 5, …, 99. This sequence is an AP with

$$a = 1,\ d = 2$$

If the number of terms be n, then

$$1 + (n - 1)2 = 99$$

$$\Rightarrow n = 50$$

$$\text{Sum} = \frac{n}{2}(\text{first term} + \text{last term})$$

$$= \frac{50}{2}(1+99) = 2500$$

2. (a) Sum of the first four terms = 26

Sum of the first 12 terms = 222. Therefore,

$$\frac{4}{2}(2a+3d) = 26$$

$$\Rightarrow 2a + 3d = 13$$

Similarly,

$$\frac{12}{2}(2a+11d) = 222$$

$$\Rightarrow 2a + 11d = 37$$

Solving the above equations, we get

$$a = 2,\ d = 3$$

Sum of the first 15 terms will be

$$\frac{15}{2}[2\times 2+(14)\times 3]$$

$$= 15(2+7\times 3) = 15(23) = 345$$

3. (c) If a is the first term and d the common difference, then

$$a + 2d = 12$$

$$a + 49d = 106$$

Solving the above two equations, we get

$$a = 8,\ d = 2$$

Now, the 29th term of the sequence will be

$$a_{29} = 8 + 28 \times 2 = 8 + 56 = 64$$

4. (b) We know that $a = 2$ and $r = (2/3) < 1$, then

$$S_\infty = \frac{2}{(1-2)/3} = \frac{2}{(1-2)/3} = 6$$

5. (d) First term of the series, $a = 60$ and the common difference, $d = -4$.

Now, maximum sum will be till the time we have positive numbers in the series. Hence,

$$0 = 60 + (n - 1)(-4)$$

$$\Rightarrow 60 - 4n + 4 = 0$$

$$\Rightarrow n = 16$$

Thus, to obtain the maximum sum, we need to calculate the sum till the 16th term. Hence,

$$S = \frac{16}{2}[2\times 60+(16-1)(-4)]$$

$$= 16(60-30) = 16(30)$$

$$= 480$$

6. (a) Let the three numbers in AP be $a - d$, a, $a + d$. Hence

$$\text{Sum} = (a - d) + a + (a + d) = 15$$

$$3a = 15 \Rightarrow a = 5.$$

Also,

$$(a - d + 8), (a + 6), (a + d + 4) \text{ are in GP}$$
$$5 - d + 8, 5 + 6, 5 + d + 4 \text{ are in GP}$$
$$13 - d, 11, 9 + d \text{ are in GP}$$
$$(13 - d)(9 + d) \text{ are in GP}$$
$$(13 - d)(9 + d) = (11)^2$$
$$d^2 - 4d + 4 = 0$$
$$(d - 2)^2 = 0 \qquad d = 2$$

Hence, the numbers are

$$5 - 2 = 3, 5 \text{ and } 5 + 2 = 7$$

7. (b) Let a be the first term and r be the common ratio of a given GP, then

$$ar^{2-1} = \frac{25}{4}$$
$$ar^{8-1} = \frac{16}{625}$$

Dividing the above two equations, we get

$$\frac{ar^7}{ar} = \frac{16}{625} \times \frac{4}{25}$$
$$\Rightarrow r = \frac{2^4 \times 2^2}{5^4 \times 5^2} = \left(\frac{2}{5}\right)^6$$
$$\Rightarrow r = \frac{2}{5}$$

Substituting value of r in any of the above equations

$$a \cdot \frac{2}{5} = \frac{25}{4}$$
$$\Rightarrow a = \frac{125}{8}$$

8. (b) It is given that first term, $a = 1$ and fifth term $= 16$. Hence

$$ar^4 = 16$$
$$\Rightarrow 1 \times r^4 = 16$$
$$\Rightarrow r = 2$$

Thus, three terms between 1 and 16 are 2, 4 and 8.

9. (d) We know that $a = 2$ and $a_5 = 18$. Hence

$$18 = 2 + 4d \Rightarrow d = 4$$

Thus, the three terms between 2 and 18 are 6, 10 and 14.

10. (b) The ratio of nth terms is

$$\frac{2n+3}{n-11}$$

If $n = 1$ in $2n + 3$ series, then the first term is $2(1) + 3 = 5$.

If $n = 2$, then the second term is $2(2) + 3 = 7$.

Hence,

$$d_1 = 7 - 5 = 2$$

Now

If $n = 1$ in $n - 11$ series, first term is $1 - 11 = -10$

If $n = 2$ in $n - 11$ series, second term is $2 - 11 = -9$

Hence,

$$d_2 = -9 - (-10) = 1 \qquad \Rightarrow d_1 : d_2 = 2 : 1$$

11. (b) We know that the first term $= -12$ and the fifth term $= 0$, so

$$0 = -12 + (4)d$$
$$d = 3$$

Now, the sum of sequence $= 135$, therefore,

$$135 = \frac{n}{2}[2 \times (-12) + (n-1)3]$$
$$\Rightarrow 270 = 3n^2 - 27n$$
$$\Rightarrow 3n^2 - 27n - 270 = 0$$

Solving the above quadratic equation for n, we get

$$n = \frac{-(-9) \pm \sqrt{81+360}}{2}$$
$$= \frac{9 \pm 21}{2}$$

Since, the number of terms cannot be negative, hence,

$$n = \frac{9+21}{2} = 15$$

12. (d) Given series is $\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} + \cdots$ up to 10 terms. Series can be expressed as

$$\left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{9}\right) + \left(1 - \frac{1}{27}\right) + \left(1 - \frac{1}{81}\right) + \cdots$$
$$= 10 - \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots + \frac{1}{3^{10}}\right)$$
$$= 10 - \frac{1}{3}\left[\frac{1-(1/3)^{10}}{1-(1/3)}\right]$$
$$= 10 - \frac{3^{10} - 1}{2 \times 3^{10}}$$
$$= \frac{20 \times 3^{10} - 3^{10} + 1}{2 \times 3^{10}}$$
$$= \frac{19 \times 3^{10} + 1}{2 \times 3^{10}}$$

13. (c) The sum of first n terms is given as

$$\frac{ar^n - a}{r-1}$$

The sum of first eight terms is

$$\frac{a(r^8-1)}{r-1}$$

Similarly, the sum of first four terms is

$$\frac{a(r^4-1)}{r-1}$$

$$\frac{[a(r^8-1)]/(r-1)}{[a(r^4-1)]/(r-1)} = 82 \Rightarrow \frac{r^8-1}{r^4-1} = 82$$

$$\Rightarrow r^4 + 1 = 82$$

$$\Rightarrow r^4 = 81 \qquad \Rightarrow r = 3$$

14. (d) Given series is

$$f(x) = 3 + 6x^2 + 9x^4 + 12x^6 + \cdots$$

$$\Rightarrow x^2 f(x) = 3x^2 + 6x^4 + 9x^6 + 12x^8 + \cdots$$

$$\Rightarrow f(x) = x' f(x) = (3 + 3x^2 + 3x^4 + 3x^6 + \cdots)$$

$$= 3(1 + x^2 + x^4 + x^6 + \cdots)$$

$$f(x)(1-x^2) = 3\left(\frac{1}{1-x^2}\right) \qquad [\text{since } \quad |x| < 1]$$

$$\Rightarrow f(x) = 3\left(\frac{1}{1-x^2}\right)^2 = \frac{3}{(1-x^2)^2}$$

15. (d) We know that

$$\text{GM}^2 = \text{AM} \times \text{HM}$$

and it is given that

$$\text{GM} = 6 \text{ and AM} = 12$$

Hence,

$$\text{HM} = \frac{6 \times 6}{12} = 3$$

16. (a) We know that

$$\text{GM}^2 = \text{AM} \times \text{HM}$$

and it is given that

$$\text{GM} = 12 \text{ and HM} = 7.2$$

Hence,

$$\text{AM} = \frac{12 \times 12}{7.2} = 20$$

17. (c) If the two numbers are x and y, then

$$25 = \frac{x+y}{2}$$

$$\Rightarrow x + y = 50$$

and

$$\sqrt{xy} = 7$$

$$\Rightarrow xy = 49$$

Now,

$$y(50-y) = 49$$

$$\Rightarrow y^2 - 50y + 49 = 0$$

$$\Rightarrow (y-49)(y-1) = 0$$

$$\Rightarrow y = 1, 49$$

Hence, the two numbers are 1 and 49.

18. (a) It is given that

$$\frac{2}{3}, x, \frac{8}{27}$$

are in GP, then

$$\frac{x}{2/3} = \frac{8/27}{x}$$

$$\Rightarrow x^2 = \frac{16}{81}$$

$$\Rightarrow x = \pm\frac{4}{9}$$

19. (b) Sum of the series is given by

$$\left(\frac{3}{4} + \frac{3}{4^3} + \frac{3}{4^5} + \cdots + \infty\right) - \left(\frac{5}{4^2} + \frac{5}{4^4} + \frac{5}{4^6} + \cdots + \infty\right)$$

$$= \frac{3/4}{1-(1/4)^2} - \frac{5/4^2}{1-(1/4)^2} = \frac{3}{4} \times \frac{16}{15} - \frac{5}{16} \times \frac{16}{15}$$

$$= \frac{4}{5} - \frac{1}{3} = \frac{7}{15}$$

20. (d) If a is the first term of the GP and r is the common ratio. Then, the first five terms of the GP are given by a, ar, ar^2, ar^3 and ar^4. So the product of five terms will be

$$a \times ar \times ar^2 \times ar^3 \times ar^4 = (ar^2)^5$$

Now, $ar^2 = 2$, thus the product of first five terms $= (2)^5 = 32$.

CHAPTER 3

PROBABILITY

INTRODUCTION

Probability is used to measure the degree of certainty or uncertainty of the occurrence of events. Probability (P) can be mathematically given as

$$P = \frac{\text{Number of favourable outcomes}}{\text{Total outcomes}}$$

Hence, probability can also be defined as the ratio of number of favorable outcomes of an event to the total possible outcomes of the same event.

SOME BASIC CONCEPTS OF PROBABILITY

1. An experiment which, when repeated under identical conditions, does not produce the same outcome every time is called a *random experiment*. Example: Tossing a die.
2. An experiment which, when repeated under identical conditions, produces the same outcome every time is called a *deterministic experiment*. Example: If a stone is dropped from a window, then the stone will go down.
3. All the possible outcomes of a random experiment are called *events* and the set representing all the events of a random experiment is called *sample space*. Example: Tossing a die is an event and $\{1, 2, 3, 4, 5, 6\}$ is its sample space.
4. An event that can never occur or the probability of its occurrence is zero, is called an *impossible event*. Example: The probability of getting a number greater than 6 when a die is tossed once.
5. If the occurrence of one event means the non-occurrence of another event, then the set of those events is called *mutually exclusive events*. Example: If a coin is tossed one, it can either result in heads or tails but not both.
6. A set of events that includes all the possible outcomes of the sample space is said to be an *exhaustive set of events*. Example: If we have the event of tossing a die, then A = $\{1, 2, 3\}$ and B = $\{4, 5, 6\}$ are said to be exhaustive set of events.
7. If two events have equal probability of occurrence, then the two events are said to be *equally likely events*. Example: The probability of getting heads

when a coin is tossed once is the same as the probability of getting tails.

8. If the occurrence of one event has no effect on the occurrence or probability of another event, then the two events are called *independent events*. Example: We can flip a coin and gets a head and flip a second coin and get a tails. Thus, the two coins do not influence each other.
9. The occurrence of an event A such that event B has already occurred is denoted by $P(A|B)$ and is called *conditional probability*. Example: Event of drawing two kings from a deck of cards.

SOME IMPORTANT THEOREMS

To solve the algebraic problems of probability, following are some important and commonly used theorems:

1. If A, B and C are events of a random experiment, then

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$
$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B)$$
$$-P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$$

2. If A and B are two events of a random experiment, then

$$P(\overline{A}\cap B) = P(B) - P(A\cap B)$$
$$P(A\cap \overline{B}) = P(A) - P(A\cap B)$$

3. For any two events A and B,

$$P(A\cap B) \le P(A) \le P(A\cup B) \le P(A) + P(B)$$

4. If A and B are two events of a random experiment, then

$$P(A) + P(B) - 2P(A\cup B) = P(A\cup B) - P(A\cap B)$$

SOLVED EXAMPLES

1. A book contains 50 pages. A page is chosen at random. What is the chance that the sum of the digits on the page is equal to 6?

Solution: The pages whose sum of the digits equal to 6 are 6, 15, 24, 33, 42. Thus, there are five favorable outcomes out of a possible 50 outcomes. So

$$\text{Probability} = \frac{5}{50} = 0.1$$

2. A bag contains 4 red balls and 3 black balls and a man draws 2 balls at random. What is the probability of both the balls being red?

Solution: Total number of balls is 4 + 3 = 7
So the number of ways in which 2 balls can be drawn = ${}^7C_2$

And the number of ways in which 2 red balls can be drawn = ${}^4C_2$

So the probability of both the balls being red is

$$\frac{{}^4C_2}{{}^7C_2}$$

$$= \frac{4!\times 5!}{2!\times 7!} = \frac{4!}{7\times 6\times 2} = \frac{2}{7}$$

3. Three digit numbers are formed by using the digits 1, 2, 3, 4, 5 and 6 without repeating any digit. What is the probability that a chosen number is an odd number?

Solution: Three digits numbers using 1, 2, 3, 4, 5 and 6 can be formed in $6\times 5\times 4$ ways. So the total number of ways will be 120.
If the number is even, then the last digit can be selected in two ways; therefore, the number of ways in which even number can be selected is $3\times 5\times 4 = 60$.

So the probability of number being even $= \dfrac{60}{120} = \dfrac{1}{2}$.

4. A fair dice is rolled twice. What is the probability that the sum of the results is at least 9?

Solution: The total number of outcomes when one fair dice is rolled is 36. So the number of favorable outcomes will be

$$\{(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),$$
$$(6,3),(6,4),(6,5),(6,6)\}$$

So the probability of sum of the results to be at least 9 $= \dfrac{10}{36} = \dfrac{5}{18}$.

5. In a biased coin, head occurs four times as frequently as tail occurs. If the coin is tossed 4 times, what is the probability of getting three heads?

Solution: We know that

$$P(H) = 4P(T)$$

$$P(\mathrm{H}) + P(\mathrm{T}) = 1 \Rightarrow 4P(\mathrm{T}) + P(\mathrm{T}) = 1$$

$$\Rightarrow 5P(\mathrm{T}) = 1$$

$$\Rightarrow P(\mathrm{T}) = \frac{1}{5}$$

Therefore,

$$P(\mathrm{H}) = \frac{4}{5}$$

Since the coin is tossed 4 times, three heads may occur if {(H H H T), (H H T H), (H T H H), (T H H H)}.

Therefore, required probability is

$$\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{1}{5}+\frac{4}{5}\times\frac{4}{5}\times\frac{1}{5}\times\frac{4}{5}+\frac{4}{5}\times\frac{1}{5}\times\frac{4}{5}\times\frac{4}{5}+\frac{1}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}$$

$$= 4\times\left(\frac{4}{5}\times\frac{4}{5}\times\frac{4}{5}\times\frac{1}{5}\right) = \frac{256}{625}$$

6. A positive integer is chosen at random from 1 to 100. What is the probability that the integer chosen is either a multiple of 3 or a multiple of 5?

Solution: Say E_1 is the event when the chosen integer is multiple of 3 and E_2 is the event when the chosen integer is multiple of 5. Hence

$$P(E_1) = \frac{33}{100} \text{ and } P(E_2) = \frac{20}{100} = \frac{1}{5}$$

Also, $E_1 \cap E_2$ is an event such that the numbers are divisible by 3 and 5. Hence

$$P(E_1 \cap E_2) = \frac{6}{100} = \frac{3}{50}$$

Thus, the required probability will be

$$P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{33}{100} + \frac{1}{5} - \frac{2}{50}$$

$$= \frac{33+20-4}{100} = \frac{49}{100}$$

7. From a well-shuffled deck of 52 cards, three cards are drawn at random without replacing them. What is the probability that the first card is of hearts, second card is of spades and third is of clubs?

Solution: We know that total cards = 52
So the total cards of hearts = 13

Hence, the probability that the first card is hearts $= \frac{13}{52}$

After 1 card is drawn, cards left = 51

And the total cards of spades = 13

So the probability that the second card is spades $= \frac{13}{51}$

After 2 cards are drawn, cards left = 50

and the total cards of clubs = 13.

So the probability that third card is clubs $= \frac{13}{50}$

So the required probability $= \frac{1}{4}\times\frac{13}{51}\times\frac{13}{50} = \frac{169}{10200}$

8. From a bag containing 6 vegetables and 4 fruits, an item is chosen at random. If out of 6 vegetables in the bag 2 are cabbages, 4 are ladyfingers, and out of 4 fruits, 3 are apples and 1 is orange, then what is the probability that the item chosen at random will be an apple?

Solution: We know that total items = 10. Say $P(A)$ be the probability of choosing a fruit, then

$$P(A) = \frac{4}{10} = \frac{2}{5}$$

Now, out of 4 fruits, total number of apples = 3

Say $P(B)$ be the probability of choosing an apple if the bag contains only fruits and no vegetables

$$P(B) = \frac{3}{4}$$

Now, if $P(C)$ is the probability of choosing an apple if bag contains all the items, then

$$P(C) = P(A)\times P(B)$$

$$P(C) = \frac{2}{5}\times\frac{3}{4} = \frac{3}{10}$$

9. The letters of the word "MESSAGE" are placed in a row at random. What is the probability that the two "S" come together?

Solution: The word "MESSAGE" contains 7 letters and "E" and "S" are repeated twice. Hence,

Total arrangements $= \frac{7!}{2!\times 2!}$

Now, taking both the "S" as one, total number of arrangements $= \frac{6!}{2!}$

Probability that the two "S" come together is

$$\frac{6!/2!}{7!/2!\times 2!} = \frac{6!\times 2!}{7!} = \frac{2}{7}$$

10. Calculate the value of $P(A\cup B)$, if $P(A) = 1/3$, $P(B) = 1/4$ and $P(A\cap B) = 1/36$.

Solution: We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{2}{5} + \frac{1}{4} - \frac{1}{20} = \frac{8+5-1}{20}$$
$$= \frac{12}{20} = \frac{3}{5}$$

11. A bag contains 6 blacks and 8 white balls. One ball is drawn at random. What is the probability that the ball drawn is white?

Solution: Let number of balls = (6 + 8) = 14
Number of white balls = 8

P (drawing a white ball) = $\frac{8}{14} = \frac{4}{7}$

12. One card is drawn at random from a pack of 52 cards. What is the probability that the card drawn is a face card (Jack, Queen and King only) or an ace?

Solution: We know that there are 52 cards, out of which there are 12 face and 4 ace cards.

Required probability = $\frac{16}{52} = \frac{4}{13}$

13. Two cards are drawn together from a pack of 52 cards. What is the probability that one is a spade and one is a heart?

Solution: Total cards in a deck = 52

Total spades = 13

Probability of drawing a spades = $\frac{13}{52}$

Total hearts = 13

Probability of drawing a heart after a spade = $\frac{13}{52} \times \frac{13}{51} = \frac{169}{2652}$

Also, if we draw a heart first and then a spade is the same = $\frac{13}{52} \times \frac{13}{51} = \frac{169}{2652}$

Thus, the required probability that one is a spade and one is a heart = $\frac{169}{2652} + \frac{169}{2652} = \frac{13}{102}$

14. A card is drawn from a pack of 52 cards. What is the probability of getting a queen of club or a king of heart?

Solution: Total cards in a deck = 52.

Probability of getting a queen of club or a king of heart = $\frac{2}{52} = \frac{1}{26}$.

15. From a pack of 52 cards, two cards are drawn together at random. What is the probability of both the cards being kings?

Solution: Total cards in the deck = 52

If sample space is S, then $n(S) = {}^{52}C_2 = \frac{(52 \times 51)}{2 \times 1}$ $= 1326$

Say E = event of getting 2 kings out of 4.

Thus,

$$n(E) = {}^{4}C_2 = \frac{(4 \times 3)}{2 \times 1} = 6$$

The required probability = $P(E) = \frac{n(E)}{n(S)} = \frac{6}{1326}$ $= \frac{1}{221}$.

16. Two dice are tossed. What is the probability that the total is a prime number?

Solution: Clearly, $n(S) = (6 \times 6) = 36$

Say E = Event that the sum is a prime number

Then, $E = \{(1, 1), (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (4, 1), (4, 3), (5, 2), (5, 6), (6, 1), (6, 5)\}$

Therefore,

$$n(E) = 15$$

Hence, required probability = $P(E) = \frac{n(E)}{n(S)} = \frac{15}{36}$ $= \frac{5}{12}$

17. In a lottery, there are 10 prizes and 25 blanks. A lottery is drawn at random. What is the probability of getting a prize?

Solution: Sample space = 10 + 25 = 35

Favorable outcome = 10

Hence, probability of getting a prize = $\frac{10}{35} = \frac{2}{7}$

18. In a class, there are 15 boys and 10 girls. Three students are selected at random. What is the probability that 1 girl and 2 boys are selected?

Solution: Let S be the sample space and E be the even of selecting 1 girl and 2 boys.

Then,

$n(S)$ = number of ways of selecting 3 students out of 25 = ${}^{25}C_3 = \frac{(25 \times 24 \times 23)}{3 \times 2 \times 1} = 2300$

$$n(E) = \left({}^{10}C_1 \times {}^{15}C_2\right) = \left[10 \times \frac{(15 \times 14)}{2 \times 1}\right] = 1050$$

Therefore, $P(E) = \frac{n(E)}{n(S)} = \frac{1050}{2300} = \frac{21}{46}$

19. Two dice are thrown simultaneously. What is the probability of getting two numbers whose product is even?

Solution: When two dices are thrown simultaneously then the simultaneous sample space can be defined as

$$n(S) = (6 \times 6) = 36$$

Let event E can be defined as getting two numbers who product is even.

Then, $E = \{(1, 2), (1, 4), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 5) (2, 6), (3, 2), (3, 4), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5) (4, 6), (5, 2), (5, 4), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5) (6, 6)\}$

Thus, $n(E) = 27$.

Hence, required probability $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{27}{36} = \dfrac{3}{4}$

20. Three unbiased coins are tossed. What is the probability of getting at most two heads?

Solution: The sample space when three unbiased coins are tossed = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

Thus, $n(S) = 8$

Let E be the event of getting at most two heads.

Then E = {TTT, TTH, THT, HTT, THH, HTH, HHT}

$n(E) = 7$

Therefore, required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{7}{8}$

21. What is the probability of getting a sum of 9 from two throws of a dice?

Solution: When two dices are thrown simultaneously then the simultaneous sample space can be defined as

$$n(s) = (6 \times 6) = 36$$

Let E be the even of getting a sum of 9 = $\{(3, 6), (4, 5), (5, 4), (6, 3)\}$

Thus, $n(E) = 4$.

Therefore, required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{4}{36} = \dfrac{1}{9}$

22. Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5?

Solution: Here, sample space $S = \{1, 2, 3 \ldots, 19, 20\}$

Let E be the event of getting a multiple of 3 or 5 = $\{3, 6, 9, 12, 15, 18, 5, 10, 20\}$

Thus, $n(E) = 9$

Therefore, required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{9}{20}$

23. Debra was born between October 6^{th} and 10^{th} (6^{th} and 10^{th} excluding). Her year of birth is also known. What is the probability of Debra being born on a Saturday?

Solution: Since the year of birth is known, the birthday being on Saturday can have a zero probability.
Also, since between 6^{th} and 10^{th}, there are three days, i.e. 7^{th}, 8^{th} and 9^{th}.

Therefore, probability of birthday falling on Saturday can be 1/3.

24. Amit, Sumit and Pramit go to a seaside town to spend a vacation there and on the first day everybody decides to visit different tourist locations. After breakfast, each of them boards a different tourist vehicle from the nearest bus-depot. After three hours, Sumit who had gone to a famous beach, calls on the mobile of Pramit and claims that he has observed a shark in the waters. Pramit learns from the local guide that at that time of the year, only eight sea-creatures (including a shark) are observable and the probability of observing any creature is equal. However, Amit and Pramit later recall during their discussion that Sumit has a reputation for not telling the truth five out of six times. What is the probability that Sumit actually observed a shark in the waters?

Solution: The probability that Sumit actually sees a shark, given that he claimed to have seen one.

$$\Rightarrow \frac{P(\text{ He actually sees the shark and reports truth})}{P(\text{ He claims of seeing a shark})}$$

$$\Rightarrow \frac{P(\text{Sees the shark}) \times P(\text{Reports truth})}{P(\text{Sees the shark}) \times P(\text{reports truth}) + P(\text{Doesn't see}) \times P(\text{reports false})}$$

$$= \frac{\left(\frac{1}{8} \times \frac{1}{6}\right)}{\left(\frac{1}{8} \times 1.6\right) + \left(\frac{7}{8} \times \frac{5}{6}\right)} = \frac{1}{36}$$

25. A bag contains 10 balls numbered from 0 to 9. The balls are such that the person picking a ball out of the bag is equally likely to pick anyone of them.

A person picked a ball and replaced it in the bag after noting its number. He repeated this process 2 more times. What is the probability that the ball picked first is numbered higher than the ball picked second and the ball picked second is numbered higher than the ball picked third?

Solution: Let the number of the ball picked first = x, second = y and third = z.

The three numbers x, y and z are distinct.

Three distinct balls can be picked in $(10 \times 9 \times 8)$ ways.

The order of x, y and z can be as follows:

$$x > y > z;$$
$$x > z > y;$$
$$y > z > x;$$
$$y > x > z;$$
$$z > x > y;$$
$$z > y > x;$$

They will occur equal number of times.

Thus, the number of ways in which $(x > y > z)$ = $\left(\frac{1}{6}\right) \times (10 \times 9 \times 8) = 120$

Hence, required probability $= \frac{120}{10 \times 10 \times 10} = \frac{3}{25}$

26. Rachyita thought of a two-digit number and divided the number by the sum of the digits of the number. She found that the remainder is 3. Mehek also thought of a two-digit number and divided the number by the sum of the digits of the number. She also found that the remainder is 3. Find the probability that the two digit number thought by Rachyita and Mehek is same?

Solution: Let the two digit number that Rachyita thought be xy, where x and y are single digit numbers.
Therefore,

$$10x + y = k(x + y) + 3$$

where k is a natural number.

$$k = \left(\frac{10x + y - 3}{x + y}\right)$$

Also $y + x > 3$

Possible values of x and y for which k is a natural number are:

$(x = 1,\ y = 5)$, $(x = 2,\ y = 3)$, $(x = 3,\ y = \{1, 3, 5, 9\})$, $(x = 4,\ y = 7)$, $(x = 5,\ y = \{1, 2, 9\})$, $(x = 6,\ y = 0)$,

$(x = 7,\ y = \{3, 5, 8\})$, $(x = 8,\ y = 0)$, $(x = 9,\ y = 4)$

There are 14 such two-digit numbers that give a remainder of 3 when divided by the sum of the digits.

Probability that Mehek thought of the same number as Rachyita $= \frac{1}{14}$.

27. An eight-digit telephone number consists of exactly two zeroes. One of the digits is repeated thrice. Remaining three digits are all distinct. If the first three digits (from left to right) are 987, then what is the probability of having only one 9, one 8 and one 7 in the telephone number?

Solution: Let us consider two possible cases.

Case 1: There is only one 9, one 8 and one 7 in the number. Hence, there has to be one digit from $\{1, 2, 3, 4, 5, 6\}$ repeated thrice.

Total number of ways in which such a number can exist $= {}^6C_1 \times \frac{5!}{3! \times 2!} = 60$

Case 2: One of the three digits $\{9, 8, 7\}$ is repeated thrice. Hence, there will be one digit from $\{1, 2, 3, 4, 5, 6\}$.

Total number of ways in which such a number can exist $= {}^3C_1 \times {}^6C_1 \times \frac{5!}{3! \times 2!} = 540$

Total possible telephone numbers = 60 + 540 = 600

Probability $= \frac{60}{600} = \frac{1}{10}$

28. If Raman picks a number from 1 to 6 and the operator rolls three dice. If the number Raman picked comes up on all three dice, the operator pays Raman ₹ 3. If it comes up on two dice, Raman is paid ₹ 2 and if it comes up on just one dice, then Raman is paid ₹ 1.

Only if the number Raman picked does not come up at all, then Raman pays the operator ₹ 1. What is the probability that Raman will win money playing in this game is?

Solution: If Raman picks up a number out of six numbers, then the only case in which he will lose money is if none of the three dice shows the picked number on the top surface.

Required probability of losing the game $= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$

Probability of winning the game $= 1 - \frac{125}{216} = \frac{91}{216}$ = 0.42

29. An Insurance company issues standard, preferred and ultra-preferred policies. Among the company's policy holders of a certain age, 50% are standard with the probability of 0.01 dying in the next year, 30% are preferred with a probability of 0.008 of dying in the next year and 20% are ultra-preferred with a probability of 0.007 of dying in the next year. If a policy holder of that age dies in the next year, what is the probability of the decreased being a preferred policy holder?

Solution: The probability of holders dying = 0.01% for standard, 0.008% for preferred and 0.007% for ultra-preferred.

The expected number of deaths among all the policy holders of the given age, say X, during the next year $= T \times \left(\frac{50 \times 0.01}{100} + \frac{30 \times 0.008}{100} + \frac{20 \times 0.007}{100}\right)$

$= T \times \left(\frac{0.88}{100}\right)$

where $X =$ Total number of policy holder of age X.

If any of these policy holders (who die during the next year) is picked at random, the probability that he is a preferred policy holder =

$$\left(\frac{\frac{30 \times 0.008 \times T}{100}}{\frac{0.88 \times T}{100}}\right) = \frac{24}{88} = \frac{3}{11} = 0.2727$$

30. Sum of the digits of a 5 digit number is 41. What is the probability that such a number is divisible by 11?

Solution: In order to get the sum equal to 41, the following 5-digit combination exists:

99995 – number of combinations $= \frac{5!}{4!} = 5$

99986 – number of combinations $= \frac{5!}{3!} = 20$

99977 – number of combinations $= \frac{5!}{3! \times 2!} = 10$

99887 – number of combinations $= \frac{5!}{2! \times 2!} = 30$

98888 – number of combinations $= \frac{5!}{4!} = 5$

Total number of combinations = 70

Now for a 5 digit number of form (abcde) to be divisible by 11

$$(a+c+e)-(b+d) = 11$$
$$(a+c+e)+(b+d) = 41$$

Thus,

$$(a+c+e) = 26, \quad (b+d) = 15$$

Hence, $(a,c,e) = (9,9,8)$ and $(b,d) = (8,7)$ (i)

Or $(a,c,e) = (9,9,8)$ and $(b,d) = (9,6)$ (ii)

Using Eq. (i), we can construct $\frac{3!}{2!} \times 2! = 6$ numbers

Using Eq. (ii), we can construct $\frac{3!}{2!} \times 2! = 6$ numbers

Number of favorable cases = 12.

Hence, required probability $= \frac{12}{70} = \frac{6}{35}$

PRACTICE EXERCISE

1. A book contains 100 pages. A page is chosen at random. What is the chance that the sum of the digits on the page is equal to 8?

(a) 0.08 (b) 0.09 (c) 0.90 (d) 0.10

2. From a class of 11 boys and 9 girls, a group of 5 students is selected in such a way that every group of 5 students is equally likely to be selected. What is the probability that there are exactly three girls in the selected group?

(a) 472/555 (b) 195/644
(c) 308/323 (d) 924/949

3. A bag contains 5 red balls and 6 black balls and a man draws 4 balls at random. What is the probability of these being all black?

(a) 1/22 (b) 6/125
(c) 15/254 (d) 4/11

4. Three digit numbers are formed by using the digits 1, 2, 3, 4 and 5 without repeating any digit. What is the probability that a chosen number is an even number?

(a) 2/5 (b) 3/5 (c) 3/16 (d) 3/4

5. Three people aim at a target and their respective probabilities of hitting the target are 2/3, 3/8 and 5/7. What is the probability that at least two of them hit the target?

(a) 107/168 (b) 5/28
(c) 17/56 (d) 5/84

6. Two fair dice are rolled once. What is the probability that the sum of the results is at least 8?

(a) 1/9 (b) 5/12 (c) 7/12 (d) 17/36

7. A pair of coins is tossed once. What is the probability of having exactly one head?

(a) 1/4 (b) 2/3 (c) 3/4 (d) 1/2

8. A positive integer is chosen at random from 1 to 100. What is the probability that the integer chosen is either a multiple of 5 or a multiple of 8?

(a) 8/25 (b) 6/11 (c) 3/10 (d) 1/50

9. In a biased coin, head occurs three times as frequently as tail occurs. If the coin is tossed 3 times, what is the probability of getting two heads?

(a) 9/64 (b) 1/16 (c) 3/64 (d) 27/64

10. A fair dice is rolled twice. What is the probability that at least one of the two results is divisible by 2?

(a) 1/2 (b) 3/4 (c) 2/5 (d) 1/4

11. From a well-shuffled deck of 52 cards, a card is drawn at random. What is the chance that it is a face card (i.e., Jack, Queen and King)?

(a) 10/13 (b) 6/13 (c) 3/13 (d) 5/26

12. From a well-shuffled deck of 52 cards, three cards are drawn at random without replacing them. What is the probability that all the three cards are of hearts?

(a) 11/850 (b) 9/13 (c) 1/64 (d) 3/64

13. From a well-shuffled deck of 52 cards, three cards are drawn at random and the cards are replaced as they are drawn, what is the probability that all the three cards are of spades?

(a) 3/64 (b) 3/4 (c) 1/4 (d) 1/64

14. If A and B are two possible events such that $P(A \cap B) = 0.6$ and $P(A) = 0.3$, then what is the value of $P(B)$ such that A and B are mutually exclusive events?

(a) 0.4 (b) 0.3 (c) 0.6 (d) 0.5

15. The probability that at least one of A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2. What is the value of $P(\overline{A}) + P(\overline{B})$?

(a) 1 (b) 0.8 (c) 0.6 (d) 1.2

16. From a bag containing 6 balls and 4 squares, an item is chosen at random. If the bag has 3 red balls, 3 black balls, 2 red squares and 2 black squares, then what is the probability that the item chosen at random will be a red ball?

(a) 3/5 (b) 2/5 (c) 3/10 (d) 1/20

17. Calculate the value of $P(A \cup B)$, if $P(A) = 1/3$, $P(B) = 1/4$ and $P(A \cap B) = 1/36$.

(a) 1/9 (b) 5/9 (c) 11/10 (d) 2/3

18. An integer is chosen at random from 1 to 50. What is the probability that the integer selected is a multiple of 2, 3 and 5?

(a) 3/5 (b) 31/50 (c) 7/10 (d) 19/50

19. From a full deck of cards, the face cards are removed and four cards are drawn at random. What is the probability that all the four cards are of different suit?

(a) 1000/9139 (b) 1/64
(c) 2197/124950 (d) 1995/124950

20. The letters of the word "ADDRESS" are placed in a row at random. What is the probability that the two vowels come together?

(a) 1/6 (b) 2/3 (c) 1/7 (d) 2/7

21. In a game there are 70 people in which 40 are boys and 30 are girls, out of which 10 people are selected at random. One from the total group, thus selected is selected as a leader at random. What is the probability that the person, chosen as the leader is a boy?

(a) 4/7 (b) 4/9 (c) 5/7 (d) 2/7

22. A number is selected at random from first thirty natural numbers. What is the chance that it is multiple of either 3 or 13?

(a) 17/30 (b) 2/5 (c) 11/30 (d) 4/15

23. A bag contains 2 red, 3 green and 2 blue balls are to be drawn randomly. What is the probability that the balls drawn contain no blue ball?

(a) 5/7 (b) 10/21 (c) 2/7 (d) 11/21

24. Four boys and three girls stand in queue for an interview. What is the probability that they stand in alternate positions?

(a) 1/35 (b) 1/34 (c) 1/17 (d) 1/68

25. If the probability that X will live 15 years is 7/8 and that Y will live 15 years is 9/10, then what is the probability that both will live for 15 years?

(a) 1/20 (b) 1/5 (c) 63/80 (d) 5/16

26. There are three events A, B and C, one of which must and only can happen. If the odds are 8:3 against A, 5:2 against B, then what are the odds against C?

(a) 13:7 (b) 3:2 (c) 43:34 (d) 43:77

27. A 5-digit number is formed by the digits 1, 2, 3, 4 and 5 without repetition. What is the probability that the number formed is a multiple of 4?

(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/5

28. A and B play a game where each is asked to select a number from 1 to 5. If the two numbers match, both of them win a prize. What is the probability that they will not win a prize in a single trial?

(a) 1/25 (b) 24/25 (c) 2/25 (d) 23/25

29. The probability that an arrow fired from a point will hit the target is 1/4. Three such arrows are fired simultaneously towards the target from that very point. What is the probability that the target will be hit?

(a) 19/64 (b) 23/64 (c) 23/67 (d) 37/64

30. Abhi has 9 pairs of dark blue socks and 9 pairs of black socks. He keeps them all in the same bag. If he picks out three socks at random, then what is the probability that he will get a matching pair?

(a) 1 (b) $\dfrac{2 \times {}^9C_2 \times {}^9C_1}{{}^{18}C_3}$

(c) $\dfrac{{}^9C_3 \times {}^9C_1}{{}^{18}C_3}$ (d) None of these

31. There are 2 positive integers x and y. What is the probability that $x + y$ is odd?

(a) 1/4 (b) 1/3 (c) 1/2 (d) 1/5

32. What is the probability that a two digit number selected at random will be a multiple of 3 and not a multiple of 5?

(a) 4/45 (b) 2/45 (c) 1/15 (d) 4/15

33. A bag contains 4 blue, 5 white and 6 green balls. Two balls are drawn at random. What is the probability that both the balls are blue?

(a) 2/35 (b) 1/17 (c) 1/15 (d) 2/21

34. Six dice are tossed together. What is the probability of getting different faces in all of the dice?

(a) $\dfrac{1}{6^5}$ (b) $\dfrac{1}{6^6}$ (c) $\dfrac{6!}{6^5}$ (d) $\dfrac{6!}{6^6}$

35. Six dice are tossed together. What is the probability of getting the same face in all the dice?

(a) $\dfrac{1}{6^5}$ (b) $\dfrac{1}{6^6}$ (c) $\dfrac{6!}{6^5}$ (d) $\dfrac{6!}{6^6}$

36. There are 12 prizes and 24 blanks in a lottery. If John has taken a lottery, what is the probability for him to get a prize?

(a) 4/5 (b) 1/3 (c) 3/4 (d) 1/2

37. Four different objects 1,2,3,4 are distributed at random in four places marked 1,2,3,4. What is the probability that none of the objects occupy the place corresponding to its number?

(a) 17/24 (b) 3/8 (c) 1/2 (d) 5/8

38. An anti-aircraft gun can fire four shots at a time. If the probabilities of the first, second, third and the last shot hitting the enemy aircraft are 0.7, 0.6, 0.5 and 0.4, what is the probability that four shots aimed at an enemy aircraft will bring the aircraft down?

(a) 0.084 (b) 0.916 (c) 0.036 (d) 0.964

39. The letters B, G, I, N and R are rearranged to form a word. Then what is the probability that the word is "BRING"?

(a) $\dfrac{1}{5^4}$ (b) $\dfrac{1}{24}$ (c) $\dfrac{1}{120}$ (d) $\dfrac{1}{76}$

40. I forgot the last digit of a 7-digit telephone number. If 1 randomly dials the final 3 digits after correctly dialing the first four, then what is the chance of dialing the correct number?

(a) $\dfrac{1}{1001}$ (b) $\dfrac{1}{990}$ (c) $\dfrac{1}{999}$ (d) $\dfrac{1}{1000}$

ANSWERS

1. (b)	**8.** (c)	**15.** (d)	**22.** (b)	**29.** (d)	**36.** (b)
2. (c)	**9.** (d)	**16.** (c)	**23.** (b)	**30.** (a)	**37.** (c)
3. (a)	**10.** (b)	**17.** (b)	**24.** (a)	**31.** (c)	**38.** (d)
4. (a)	**11.** (c)	**18.** (b)	**25.** (c)	**32.** (d)	**39.** (c)
5. (a)	**12.** (a)	**19.** (a)	**26.** (c)	**33.** (a)	**40.** (d)
6. (b)	**13.** (d)	**20.** (d)	**27.** (d)	**34.** (d)	
7. (d)	**14.** (b)	**21.** (a)	**28.** (b)	**35.** (a)	

EXPLANATIONS AND HINTS

1. (b) The pages whose sum of the digits equal to 8 are 8, 17, 26, 35, 53, 62, 71 and 80.

Thus, there are 9 favorable outcomes. So,

$$\text{Probability} = \frac{9}{100} = 0.09$$

2. (c) The number of ways in which five students can be selected from 20 students is

$$^{20}C_4 = \frac{20!}{16! \times 4!} = 4845$$

Number of ways in which three girls can be selected from nine girls

$$^{9}C_3 = \frac{9!}{3!6!} = 84$$

Number of ways in which 2 boys can be selected from 11 boys

$$^{11}C_2 = \frac{11!}{2! \times 9!} = 55$$

Favorable outcome = 55 × 84. So

$$\text{Probability} = \frac{55 \times 84}{4845} = \frac{924}{969} = \frac{308}{323}$$

3. (a) We know the total number of balls = 5 + 6 = 11

So the number of ways in which 4 balls can be drawn = $^{11}C_4$

And the number of ways in which 4 black balls can be drawn = $^{6}C_4$

Probability of all 4 balls being black will be

$$\frac{^{6}C_4}{^{11}C_4}$$

$$= \frac{6!}{2! \times 4!} \times \frac{4! \times 7!}{11!}$$

$$= \frac{6!}{2! \times 11 \times 10 \times 9 \times 8} = \frac{6 \times 5 \times 4 \times 3 \times 2}{11 \times 10 \times 9 \times 8 \times 2} = \frac{1}{22}$$

4. (a) Three digits numbers using 1, 2, 3, 4 and 5 can be formed in $5 \times 4 \times 3$ ways.

So the total number of ways = 60

If the number is even, then the last digit can be selected in two ways. Therefore the number of ways in which even number can be selected $= 2 \times 4 \times 3 = 24$

So the probability of number being even $= \frac{24}{60} = \frac{2}{5}$

5. (a) Let A, B and C be the events that the targets are hit, therefore,

$$P(A) = 2/3;\ P(\overline{A}) = 1/3$$
$$P(B) = 3/8;\ P(\overline{B}) = 5/8$$
$$P(C) = 5/7;\ P(\overline{C}) = 2/7$$

Event $(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$ denotes that exactly two persons hit the target, and $A \cap B \cap C$ is the event that all the three persons hit the target. Hence

$$P(A \cap B \cap \overline{C}) = P(A) \cdot P(B) \cdot P(\overline{C})$$
$$= \frac{2}{3} \times \frac{3}{8} \times \frac{2}{7} = \frac{1}{14} P(A \cap \overline{B} \cap C)$$
$$= P(A) \cdot P(\overline{B}) \cdot P(C)$$
$$= \frac{2}{3} \times \frac{5}{8} \times \frac{5}{7} = \frac{25}{84}$$
$$P(\overline{A} \cap B \cap C) = P(\overline{A}) \cdot P(B) \cdot P(C)$$
$$= \frac{1}{3} \times \frac{3}{8} \times \frac{5}{7} = \frac{5}{56}$$
$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$
$$= \frac{2}{3} \times \frac{3}{8} \times \frac{5}{7} = \frac{5}{28}$$

Required probability will be

$$P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C)$$
$$= \frac{1}{14} \times \frac{25}{84} \times \frac{5}{56} \times \frac{5}{28} = \frac{107}{168}$$

6. (b) We know that the total number of outcomes when two fair dice are rolled = 36

Number of favorable outcomes will be

$$\{(2,6),(3,5),(3,6),(4,4),(4,5),(4,6),(5,3),(5,4),$$
$$(5,5),(5,6),(6,2),(6,3_3),(6,4),(6,5),(6,6)\}$$

So the probability of the sum of results to be at least $8 = \frac{15}{36} = \frac{5}{12}$

7. (d) Total number of outcomes when two coins are tossed = {(H, H), (H, T), (T, T), (T, H)}

And favorable outcomes = {(H, T), (T, H)}

So the probability of having exactly one head $= \frac{2}{4} = \frac{1}{2}$

8. (c) Say E_1 is the event that chosen integer is multiple of 5 and E_2 is the event that chosen integer is multiple of 8. Hence

$$P(E_1) = \frac{20}{100} = \frac{1}{5} \text{ and } P(E_2) = \frac{12}{100} = \frac{3}{25}$$

Also, $E_1 \cap E_2$ is an event such that the numbers are divisible by 5 and 8. So

$$P(E_1 \cap E_2) = \frac{2}{100} = \frac{1}{50}$$

Thus, the required probability will be

$$\begin{aligned} &P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ &= \frac{1}{5} + \frac{3}{25} - \frac{1}{50} \\ &= \frac{10+6-1}{50} = \frac{15}{50} = \frac{3}{10} \end{aligned}$$

9. (d) We know that

$$\begin{aligned} P(\text{H}) &= 3P(\text{T}) \\ P(\text{H}) + P(\text{T}) &= 1 \quad \text{and} \quad 3P(\text{T}) + P(\text{T}) = 1 \\ 4P(\text{T}) &= 1 \\ P(\text{T}) &= \frac{1}{4} \end{aligned}$$

Therefore,

$$P(\text{H}) = \frac{3}{4}$$

Since the coin is tossed three times, two heads may occur if {(H H T), (H T H), (T H H)}. Therefore, the required probability will be

$$\begin{aligned} &\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4} + \frac{3}{4}\times\frac{1}{4}\times\frac{3}{4} + \frac{1}{4}\times\frac{3}{4}\times\frac{3}{4} \\ &= \frac{9}{64} + \frac{9}{64} + \frac{9}{64} = \frac{27}{64} \end{aligned}$$

10. (b) When a fair dice is rolled twice, the total outcomes are 36. If one of the two results is divisible by 2, then total outcomes are {(1, 2), (1, 4), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 2), (3, 4), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 2), (5, 4), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

So the total favorable outcomes = 27

And the required probability $= \frac{27}{36} = \frac{3}{4}$

11. (c) We know that the total number of cards = 52

Total face cards in a deck = 12

So the probability that the card drawn is a face card $= \frac{12}{52} = \frac{3}{13}$

12. (a) We know that total cards = 52

And total cards of hearts = 13

So the probability that the first card is hearts $= \frac{13}{52}$

After 1 card is drawn, cards left = 51

And total cards of hearts = 12

So the probability that the second card is hearts $= \frac{12}{51}$

After 2 cards are drawn, cards left = 50

And total cards of hearts = 11

So the probability that the third card is hearts $= \frac{11}{50}$

So the required probability will be

$$\frac{13}{52}\times\frac{12}{51}\times\frac{11}{50} = \frac{33}{2550} = \frac{11}{850}$$

13. (d) We know that total number of cards = 52

And total number of cards that are spades = 13

So the probability of drawn card to be of spades $= \frac{13}{52}$

Since, the cards are replaced, the probability of drawing a spade everytime a card is drawn remains the same. Hence the required probability will be

$$\frac{13}{52}\times\frac{13}{52}\times\frac{13}{52} = \frac{1}{64}$$

14. (b) If A and B are mutually exclusive events, then

$$\begin{aligned} &P(A \cap B) = 0 \\ &\Rightarrow P(A \cup B) = P(A) + P(B) \\ &\Rightarrow 0.6 = 0.3 + P(B) \\ &\Rightarrow P(B) = 0.3 \end{aligned}$$

15. (d) We know that

$$P(A \cup B) = 0.6 \text{ and } P(A \cap B) = 0.2$$

Now

$$\begin{aligned} P(\bar{A}) + P(\bar{B}) &= 1 - P(A) + 1 - P(B) \\ &= 2 - [P(A) + P(B)] \\ &= 2 - [P(A) + P(B) - P(A \cap B)] \\ &\qquad - P(A \cap B) \\ &= 2 - P(A \cup B) - P(A \cap B) \\ &= 2 - 0.6 - 0.2 = 1.2 \end{aligned}$$

16. (c) We know that total items = 10

Say $P(A)$ be the probability of choosing a ball, then

$$P(A) = \frac{6}{10} = \frac{3}{5}$$

Now, out of 6 balls, total number of red balls = 3

Say $P(B)$ be the probability of choosing a red ball if the bag contains only balls and no squares

$$P(B) = \frac{3}{6} = \frac{1}{2}$$

Now, if $P(C)$ is the probability of choosing a red ball if bag contains all the items, then

$$P(C) = P(A) \times P(B)$$
$$P(C) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$$

17. (b) We know that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{3} + \frac{1}{4} - \frac{1}{36} = \frac{7}{12} - \frac{1}{36} \\ &= \frac{20}{36} = \frac{5}{9} \end{aligned}$$

18. (b) We know that the total numbers = 50

Say, A = Event of getting multiple of 2, B = Event of getting multiple of 3 and C = Event of getting multiple of 5. Then,

$$\begin{aligned} A &= \{2, 4, 6,\ldots, 48, 50\} \\ B &= \{3, 6, 9,\ldots, 45, 48\} \\ C &= \{5, 10, 15,\ldots, 45, 50\} \\ A \cap B &= \{6, 12, 18,\ldots, 42, 48\} \\ B \cap C &= \{15, 30, 45\} \\ A \cap C &= \{10, 20, 30, 40, 50\} \\ A \cap B \cap C &= \{30\} \end{aligned}$$

$$P(A) = \frac{25}{50}$$
$$P(B) = \frac{16}{50}$$
$$P(C) = \frac{10}{50}$$

$$P(A \cap B) = \frac{8}{50},\ P(A \cap C) = \frac{5}{50},\ P(B \cap C) = \frac{3}{50}$$
$$P(A \cap B \cap C) = \frac{1}{50}$$

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(B) &- P(A \cap B) \\ - P(A \cap C) &- P(B \cap C) \\ + P(A \cap B \cap C) & \end{aligned}$$

$$= \frac{25}{50} + \frac{11}{50} + \frac{10}{50} - \frac{8}{50} - \frac{5}{50} - \frac{3}{50} + \frac{1}{50} = \frac{31}{50}$$

19. (a) We know that there are 3 face cards in every suit. Hence, 12 cards are removed from the deck. Remaining cards in the deck = 52 − 12 = 40.

Let, $P(A)$ = Probability of drawing a card from any of the four suit

$P(B)$ = Probability of drawing a card from any of the remaining three suit

$P(C)$ = Probability of drawing a card from any of the remaining two suit

$P(D)$ = Probability of drawing a card from the remaining suit

$P(E)$ = Probability that all the four cards are of the different suit

Therefore,

$$P(A) = \frac{40}{40}$$
$$P(B) = \frac{30}{39} = \frac{10}{13}$$
$$P(C) = \frac{20}{38} = \frac{10}{19}$$
$$P(D) = \frac{10}{37}$$

Now,

$$P(E) = P(A) \times P(B) \times P(C) \times P(D)$$
$$\Rightarrow P(E) = \frac{40}{40} \times \frac{10}{13} \times \frac{10}{19} \times \frac{10}{37} = \frac{1000}{9139}$$

20. (d) The word "ADDRESS" has seven letters and can be arranged in 7! number of ways.

Hence, the total outcome = 7!

Now, "A" and "E" can be arranged in 6! ways in the row and 2! ways within themselves.

Thus, the probability of vowels coming together will be

$$\frac{6! \times 2!}{7!} = \frac{2}{7}$$

21. (a) The total groups contains boys and girls in the ratio 4:3.

If the leader is chosen at random from the selection, the probability of him being a boy = 4/(4 + 3)= 4/7

22. (b) The probability that the number is a multiple of 3 is $\frac{10}{30}$.

Similarly, the probability that the number is a multiple of 13 is $\frac{2}{30}$.

Neither 3 nor 13 has common multiple from 1 to 30. Hence, these events are mutually exclusive events.

Thus, the chance that the selected number is a multiple of 3 or 13 = $\frac{10+2}{30} = \frac{12}{30} = \frac{2}{5}$.

23. (b) 2 balls can be drawn in the following ways:

1 red, 1 green or 2 red or 2 green.

Required probability $= \left(\frac{{}^2C_1 \times {}^3C_1}{{}^7C_2}\right) + \frac{{}^2C_2}{{}^7C_2} + \frac{{}^3C_2}{{}^7C_2}$

$= \frac{6}{21} + \frac{1}{21} + \frac{3}{21} = \frac{10}{21}$

24. (a) Total number of possible arrangements for 4 boys and 3 girls in a queue = 7!

When they occupy alternate position the arrangement would be like: B G B G B G B

Thus, total number of possible arrangements for boys $= (4 \times 3 \times 2)$

Total number of possible arrangements for girls $= (3 \times 2)$

Required probability $= \frac{(4 \times 3 \times 2 \times 3 \times 2)}{7!} = \frac{1}{35}$

25. (c) We are given that probability that X will live 15 years = 7/8

Also, we know that probability that Y will lie 15 years = 9/10

Probability that X and Y will live 15 years $= \frac{7}{8} \times \frac{9}{10} = \frac{63}{80}$

26. (c) According to the question,

$$\frac{P(A')}{P(A)} = \frac{8}{3}, P(A) = \frac{3}{11} \text{ and } P(A') = \frac{8}{11}$$

Also, $\frac{P(B')}{P(B)} = \frac{5}{2}$

$$\Rightarrow P(B) = \frac{2}{7} \text{ and } P(B') = \frac{5}{7}$$

Now, out of A, B and C, one and only one can happen.

$$P(A) + P(B) + P(C) = 1$$

$$P(C) = \frac{34}{77}$$

$$P(C') = 1 - P(C) = \frac{43}{77}$$

Thus, odds against $C = \frac{P(C)}{P(C')} = \frac{43}{34}$

27. (d) Total number of 5 digits number = 5! = 120.

Now to be a multiple of 4, the last two digits of the number have to be divisible by 4, i.e. they must be 12, 24, 32 or 52.

Corresponding to each of these ways there are 3! = 6, i.e. 6 ways of filling the remaining 3 places.

The required probability $= \frac{(4 \times 6)}{120} = \frac{1}{5}$

28. (b) Total number of ways in which both of them can select a number each $= 5 \times 5 = 25$

Probability that they win the prize $= 1 \times \frac{1}{25} = \frac{1}{25}$

Probability that they do not win a prize $= 1 - \frac{1}{25} = \frac{24}{25}$

29. (d) Probability of an arrow not hitting the target = 3/4

Probability that none of the 3 arrow will hit the target $= \left(\frac{3}{4}\right)^3 = \frac{27}{64}$

Probability that the target will hit at least once $= 1 - \frac{27}{64} = \frac{37}{64}$

30. (a) If he draws any combination of 3 socks, he will definitely have the matching pair of either colour. Hence, the probability is 1.

31. (c) Here we have four cases:

Case 1: x is even, y is even.

Case 2: x is odd, y is even.

Case 3: x is even, y is odd.

Case 4: x is odd, y is odd.

Out of these four cases, in cases 2 and 3. Thus, required probability $= \frac{2}{4} = \frac{1}{2}$

32. (d) There are a total of 90 two digit numbers. Every third number will be divisible by 3. Therefore, there are 30 of those numbers that are divisible by 3.

Of these 30 numbers, the numbers that are divisible by 5 are those that are multiples of 15, i.e. numbers that are divisible by both 3 and 5.

There are 6 such numbers 15, 30, 45, 60 and 90.

Hence, the numbers that are divisible by 3 and not by 5 = 30 − 6 = 24

24 out of the 90 numbers are divisible by 3 and not by 5. The required probability $= \frac{24}{90} = \frac{4}{15}$

33. (a) Let E be the event that both the balls are blue.

$n(E)$ = number of ways in which 2 balls can be drawn from 4 blue balls $= {}^4C_2$

$n(S)$ = number of ways in which 2 balls can be drawn from 15 balls $= {}^{15}C_2$

$$P(E) = \frac{n(E)}{n(S)} = \frac{{}^4C_2}{{}^{15}C_2} = \frac{\left(\frac{4}{2} \times \frac{3}{1}\right)}{\left(\frac{15}{2} \times \frac{14}{1}\right)} = \frac{12}{210} = \frac{2}{35}$$

34. (d) Total number of outcomes possible when a die is tossed = 6 (since, any one face out of the 6 faces is possible)

Hence, total number of outcomes possible when 6 dice are thrown, $n(S) = 6^6$

$n(E)$ = number of ways of getting different faces in all the dice = the number of arrangements of 6 numbers 1, 2, 3, 4, 5, 6 by taking all at a time = 6!

$$P(E) = \frac{n(E)}{n(S)} = \frac{6!}{6^6}$$

35. (a) Total number of outcomes possible when a die is tossed = 6 (since, any one face out of the 6 faces is possible)

Hence, total number of outcomes possible when 6 dice are thrown, $n(S) = 6^6$

$n(E)$ = number of ways of getting same face in all the dice = ${}^6C_1 = 6$

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{6^6} = \frac{1}{6^5}$$

36. (b) Total number of prizes, $n(E) = 12$

Total number of possible outcomes, $n(S) = 12 + 24 = 36$

Thus, required probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{12}{36} = \dfrac{1}{3}$

37. (c) Let a particular number, say 2, occupies position 1.

Then all possible arrangements are given as:

(2, 1, 3, 4), (2, 1, 4, 3), (2, 3, 4, 1), (2, 4, 1, 4), (2, 4, 1, 3), (2, 4, 3, 1).

Out of these, three are not acceptable to us [(2, 1, 3, 4), (2, 3, 1, 4), (2, 4, 3, 1)] are not acceptable because 3 and 4 occupy the correct positions.

Hence required probability = $\dfrac{3}{6} = \dfrac{1}{2}$

38. (d) The enemy aircraft will be brought down even if one of the four shots hits the aircraft.

The opposite of this situation is that none of the four shots hit the aircraft.

The probability that none of the four shots hit the aircraft is given by

$$= (1-0.7)(1-0.6)(1-0.5)$$
$$= 0.3 \times 0.4 \times 0.5 \times 0.6 = 0.036$$

Thus, the probability that at least one of the four hits the aircraft = $1 - 0.036 = 0.964$

39. (c) Total number of words that can be made out of the letters B, G, I, N, R are 5! = 120

Favorable outcome = 1

Hence, the required probability = $\dfrac{1}{120}$

40. (d) It is given that last three digits are randomly dialled. Then, each of the digits can be selected out of 10 digits in 10 ways.

Hence, the required probability = $\left(\dfrac{1}{10}\right)^3 = \dfrac{1}{1000}$

CHAPTER 4

SET THEORY

INTRODUCTION

A set is a collection of distinct objects, considered as single entity of size equal to the number of distinct objects.

A set is described by listing elements, separated by commas, within braces {}. For example, a set of even numbers can be described as {2, 4, 6, …}.

A set is said to be empty or null or void set, if it has no elements. Two sets A and B are said to be equal sets if every element of A is a member of B, and vice versa.

If every element of A is an element of B, then A is called a subset of B.

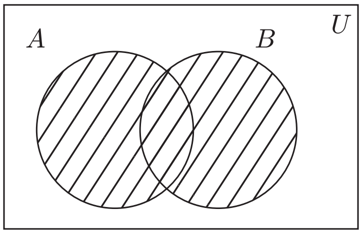
A set that contains all sets under consideration is called the universal set and is denoted by U.

Venn Diagrams

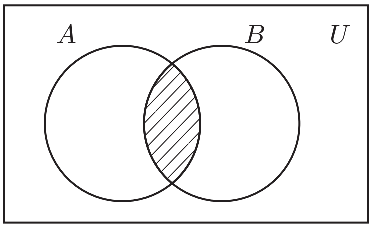
The diagrams drawn to represent sets are called Venn–Euler diagrams or simply Venn diagrams. In Venn diagrams, the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle. If a set A is a subset of B, then the circle representing A is drawn inside the circle representing B. If A and B are not equal but they have some common elements, then to represent A and B we draw two intersecting circles. Two disjoint sets are represented by two non-intersecting circles.

OPERATION ON SETS

1. **Union of sets:** $A \cup B$

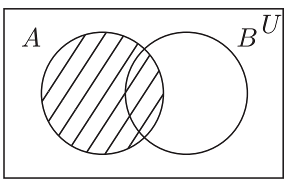


2. Intersection of sets: $A \cap B$

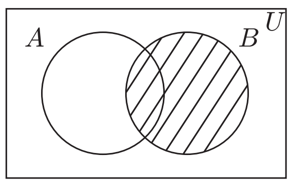


3. Difference of sets: $A - B$ or $B - A$

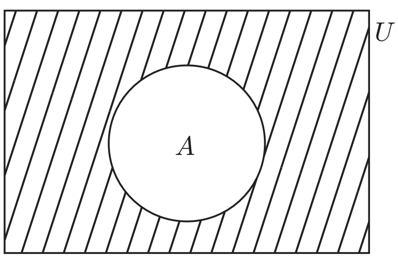
$A - B$



$B - A$



4. Complement of a set: A'



VENN DIAGRAM WITH TWO ATTRIBUTES

In this section, we discuss the Venn diagram with two attributes and give the generalized formulae. Figure 1 shows Venn diagram representation of two sets.

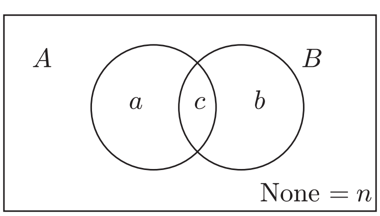


Figure 1 Venn diagram representation of two sets A and B.

1. $n(A) = a + c$
2. $n(B) = b + c$
3. Things belonging to exactly one attribute $= a + b$
4. $n(A \cap B) = c$
5. Things belonging to none of the attributes $= n$
6. $n(A \cup B) = a + b + c$

VENN DIAGRAM WITH THREE ATTRIBUTES

In this section, we discuss the Venn diagram with two attributes and give the generalized formulae. Figure 2 shows Venn diagram representation of three sets.

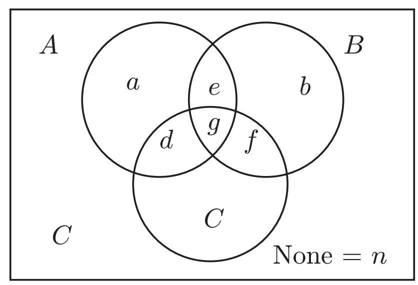
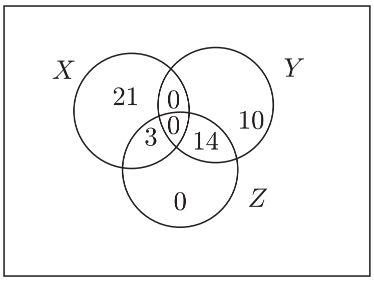


Figure 2 Venn diagram representation of three sets A, B and C.

1. $n(A) = a + e + g + d$
2. $n(B) = b + f + e + g$
3. $n(C) = c + d + g + f$
4. Things belonging to exactly one attribute $= a + b + c$
5. $n(A \cap B) = e + g$
6. $n(B \cap C) = g + f$
7. $n(A \cap C) = d + g$
8. $n(A \cap B \cap C) = g$
9. Things belonging to none of the attributes $= n$

SOLVED EXAMPLES

Direction (Q1–Q5): Solve the examples on the basis of the following diagram:



Note: X represents even numbers, Y represents odd numbers and Z represents prime numbers in the sample.

1. How many even numbers are there which are not prime?

Solution: Total even numbers which are not prime are the numbers belonging to X only $= 21$.

2. How many odd numbers are there which are prime?

Solution: Total odd numbers, which are prime, are the numbers belonging to $Y \cap Z = 14$.

3. How many even numbers are prime?

Solution: Total even prime numbers are numbers belonging to $X \cap Z = 3$.

4. How many numbers are present which are not prime?

Solution: Total numbers which are not prime $=$ (X only) $+$ (Y only)

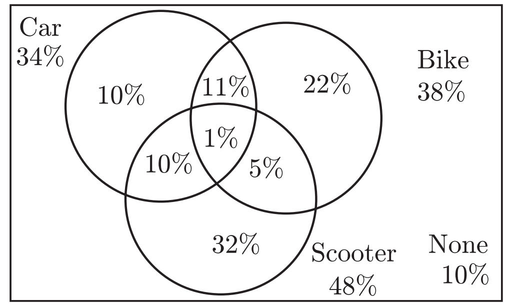
$$= 21 + 10 = 31.$$

5. How many total odd numbers are present?

Solution: Total odd numbers $= (Y \text{ only}) + (Y \cap Z)$

$$= 10 + 14 = 24.$$

Direction (Q6–Q10): In a survey conducted for a society, 10% of the houses have just car, 22% have just bike, 32% have just scooter, 1% have a car, bike and scooter, and 10% have none of the three things. So, 40 houses do not have any vehicle.



Say x is (car $\cap$ bike), y is (car $\cap$ scooter) and z is (bike $\cap$ scooter). Therefore,

$$31 = 10 + x + 1 + y$$
$$38 = 22 + x + 1 + z$$
$$48 = 32 + y + 1 + z$$

Solving the above equations for x, y and z, we get

$$x = 10\%$$
$$y = 10\%$$
$$z = 5\%$$

Also, total houses surveyed $= \dfrac{100 \times 40}{10} = 400.$

6. How many houses have bikes only?

Solution: Total houses with bikes only $= 22\%$
Total houses $= 400$

Houses with bikes only $= \dfrac{22}{100} \times 400 = 88$

7. How many houses have exactly two vehicles?

Solution: Total houses with more than one vehicle $= 25\%$

Total houses with three vehicles $= 1\%$

Total houses with exactly two vehicles $= 24\%$

$$= \frac{24}{100} \times 400 = 96$$

8. How many houses have only cars?

Solution: Total houses with cars only $= 10\%$

$$= \frac{10}{100} \times 400 = 40$$

9. How many houses have only one vehicle?

Solution: Total houses with only cars, bikes or scooters $= 10\% + 32\% + 22\%$

$$= 64\%$$

$$= \frac{64}{100} \times 400 = 256$$

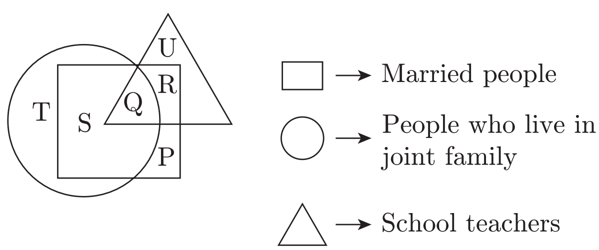
10. How many houses do not have a scooter?

Solution: Total houses which do not have a scooter $= 10\% + 10\% + 22\%$

$$= 42\%$$

$$= \frac{42}{100} \times 400 = 168$$

Direction (Q11–Q13): The following Venn diagram depicts people living in a building who are married, who live in joint family and who are school teacher. Study the chard carefully and answer the following.



11. By which letter, the married teachers who live in joint family are represented?

Solution: In this question, we basically need to find the intersection of all the three sets. Hence, it is represented by Q.

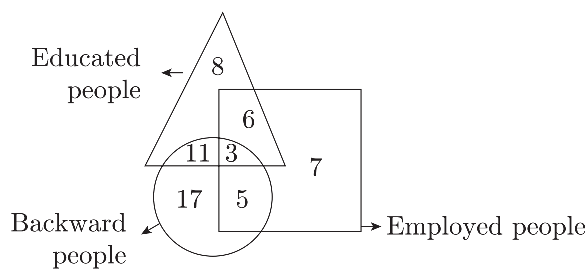
12. By which letter, the married people who live in joint family but not are school teachers are represented?

Solution: We have to find the intersection of just married people and who live in joint family. This is represented by S.

13. The people who live in joint family but are neither married nor teachers are represented by which letter?

Solution: We have the find the letter which represents only the people who live in joint family, represented by T.

Direction (Q14–Q15): The following Venn diagram depicts people who are educated, backward and employed. Study the diagram carefully and answer the following.



14. How many educated people are employed?

Solution: Number of educated people who are employed = 3 + 6 = 9.

15. How many backward people are educated?

Solution: Number of backward people who are educated = 11 + 3 = 14.

PRACTICE EXERCISE

Direction (Q1–Q5): During the cultural day of a college, 225 students participated. Out of them, 100 participated in art, 105 participated in music, 95 participated in dance, 20 participated in none of these three, 45 participated in exactly two of the above three events and 20 students participated in all three events.

1. If 45 students participated in only art, then how many students participated in music and dance?

(a) 10 (b) 15
(c) 20 (d) 25

2. If 20 students participated in both art and dance but not music, then how many students participated in music only?

(a) 55 (b) 60
(c) 65 (d) 70

3. If total students participated in dance only are 40, then how many students participated in both art and music?

(a) 5 (b) 10
(c) 15 (d) 20

4. If 50 students participated in music but not dance, find the number of students who participated in art or music?

(a) 110 (b) 130
(c) 140 (d) 120

5. How many students participated in art or music or dance only?

(a) 110 (b) 130
(c) 150 (d) 170

Direction (Q6–Q10): In a party there were 60 people who took tea and 40 people who took coffee. If there were 80 people who attended the party then answer the following questions.

6. What is the maximum number of people who took none of these two?

(a) 20 (b) 40
(c) 10 (d) 30

7. What is the maximum possible number of people who took at least one drink?

(a) 10 (b) 60
(c) 80 (d) 70

8. What is the minimum possible number who took both the drinks?

(a) 40 (b) 10
(c) 20 (d) 30

9. What is the minimum possible number of people who took none of these two drinks?

(a) 10 (b) 20
(c) 30 (d) 0

10. What is the maximum possible number of people who took exactly one of the two drinks?

(a) 20 (b) 40
(c) 60 (d) 70

Direction (Q11–Q15): Say we have three quantities A, B and C. Total value of A is 40, B is 32, C is 50, $A \cap B = 4$, $A \cap C = 5$, $B \cap C = 7$ and $A \cap B \cap C = 2$. Use the given data to answer the following questions.

11. What is the values of only A?

(a) 29 (b) 31
(c) 35 (d) 27

12. What is the value of only B?

(a) 19 (b) 21
(c) 29 (d) 17

13. What is the value of only C?

(a) 18 (b) 22
(c) 30 (d) 36

14. How many values are present in only one quantity?

(a) 122 (b) 84
(c) 96 (d) 78

15. How many values are present in at least two quantities?

(a) 22 (b) 15
(c) 18 (d) 19

Direction (Q16–Q20): A survey was conducted among 300 people to find readership of three newspapers: Times of India, Hindustan Times and Delhi Times. It is known that 100 people read at least two of these newspapers, 230 people read Times of India or Delhi Times, 180 people read exactly one, 80 read neither Times of India nor Hindustan Times, 130 read the Times of India or Hindustan Times but not Delhi Times.

16. How many people read at least one of the other two newspapers along the Delhi Times?

(a) 50 (b) 80
(c) 60 (d) 70

17. How many read only Hindustan Times or only Delhi Times?

(a) 80 (b) 110
(c) 135 (d) 150

18. How many people read Times of India only?

(a) 35 (b) 50
(c) 65 (d) 70

19. How many people read exactly one newspaper among the three?

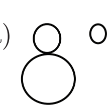
(a) 180 (b) 220
(c) 200 (d) 160

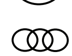
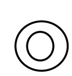
20. How many people read none of these three newspapers?

(a) 10 (b) 30
(c) 40 (d) 20

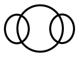
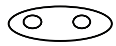
Direction (Q21–Q25): Each of these questions given below contains three elements. These elements may or may not have some inter linkage. Each group of elements may fit into one of these diagrams at (A), (B), (C) or (D). You have to indicate the group of elements which correctly fits into the diagrams.

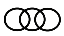
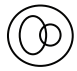
21. Which of the following diagrams indicates the best relation between travelers, train and bus?

(a)  (b) 

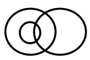
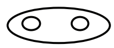
(c)  (d) 

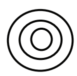
22. Which of the following diagrams indicates the best relation between profit, dividend and bonus?

(a)  (b) 

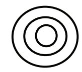
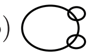
(c)  (d) 

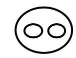
23. Which of the following diagrams indicates the best relation between women, mothers and engineers?

(a)  (b) 

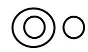
(c)  (d) 

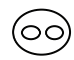
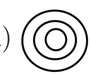
24. Which of the following diagrams indicates the best relation between factory, product and machinery?

(a)  (b) 

(c)  (d) 

25. Which of the following diagrams indicates the best relation between author, lawyer and singer?

(a)  (b) 

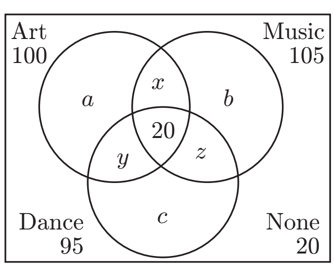
(c)  (d) 

ANSWERS

1. (a)	**6.** (a)	**11.** (a)	**16.** (d)	**21.** (c)
2. (b)	**7.** (c)	**12.** (a)	**17.** (c)	**22.** (b)
3. (b)	**8.** (c)	**13.** (d)	**18.** (b)	**23.** (a)
4. (d)	**9.** (d)	**14.** (b)	**19.** (a)	**24.** (d)
5. (c)	**10.** (c)	**15.** (c)	**20.** (d)	**25.** (b)

EXPLANATIONS AND HINTS

Solution to Q1–Q5:



We know that, $x + y + z = 45$

$$a + x + y + 20 = 100$$
$$b + x + z + 20 = 105$$
$$y + z + c + 20 = 95$$

1. (a) Total students who participated in art only $= 45$. Thus, $a = 45$

$$45 + x + y + 20 = 100$$
$$\Rightarrow x + y = 35$$
$$\Rightarrow z = 45 - 35 = 10.$$

Students who participated in music and dance together $= 10$.

2. (b) Total students who participated in art and dance $= 20$. Thus, $y = 20$

$$x + 20 + z = 45 \quad \Rightarrow x + z = 25$$
$$b + x + z + 20 = 105 \quad \Rightarrow b + 25 + 20 = 105$$
$$\Rightarrow b = 60$$

Students who participated in music only $= 60$.

3. (b) Total students who participated in dance only $= 40$. Thus, $C = 40$

$$y + z + 40 + 20 = 95$$
$$\Rightarrow y + z = 35$$
$$5 \Rightarrow x + y + z = 45$$
$$\Rightarrow x + 35 = 45$$
$$\Rightarrow x = 10$$

Students who participated in both art and music $= 10$.

4. (d) We have $x + b = 50$. Hence,

$$x + b + z + 20 = 105 \quad \Rightarrow z + 50 + 20 = 105$$
$$\Rightarrow z = 35$$
$$x + y + z = 45$$
$$\Rightarrow x + y = 10$$
$$a + x + y + 20 = 100$$
$$\Rightarrow a = 70$$
$$a + x + b = 70 + 50 = 120$$

Thus, total students who participated in art or music = 120.

5. (c) We have $x + y + z = 45$. Hence,

$$2(x + y + z) = 90$$
$$a + b + c + 2(x + y + z) + 60 = 300$$
$$\Rightarrow a + b + c + 90 + 60 = 300$$
$$\Rightarrow a + b + c = 150$$

Thus, total students who participated in only art, music or dance = 150.

Solution to Q6–Q10:

6. (a) Total people who took tea = 60.

That is, at least 60 people took tea or coffee.

Assuming that whomsoever had coffee, had tea also, we get maximum number of people who did not have anything = 80 − 60 = 20.

7. (c) For number of people who took at least one drink to be maximum, the number of people who did not take even a single drink have to be minimum = 0.

Thus, the number of people who took at least one drink = 80.

8. (c) People who took coffee = 40.

People who took tea = 60.

Total people who attended the party = 100.

Thus, minimum number of people who took both = 120 − 100 = 20.

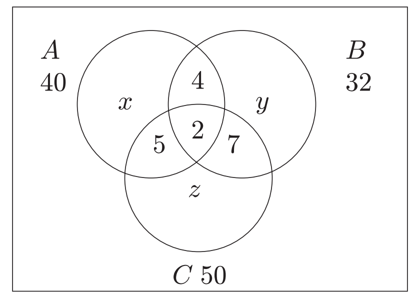
9. (d) Minimum possible number of people who did not take any drink means maximum possible number of people who took at least one drink = 80.

Hence, minimum possible number of people who did not take any drinks = 0.

10. (c) To maximize the number of only one drink taken try to minimize both and none.

Maximum number of only one drink taken = (60 − 20) + (40 − 20) = 60.

Solution to Q11–Q15:



11. (a) Value of only $A \Rightarrow 40 = (x + 4 + 2 + 5)$
$\Rightarrow x = 40 - 11 = 29$.

12. (a) Value of only $B \Rightarrow 32 = (y + 4 + 2 + 7)$
$\Rightarrow y = 32 - 13 = 19$.

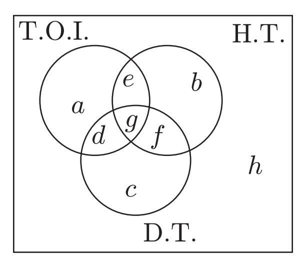
13. (d) Value of $C \Rightarrow 50 = (z + 5 + 2 + 7)$
$\Rightarrow z = 50 - 14 = 36$.

14. (d) Values present only in one quantity = 29 + 19 + 39 = 84.

15. (c) Values present in at least two quantities
$= A \cap B + A \cap C + B \cap C + A \cap B \cap C$

$= 4 + 5 + 7 + 2 = 18$

Solution to Q16–Q25:



$$a + b + c + d + e + f + g + h = 300$$
$$e + d + g + f = 100$$
$$a + b + c = 180 \Rightarrow h = 20$$
$$a + b + c + e + f + g = 230$$
$$b + h + 230 = 300$$
$$b = 50$$
$$c = 80$$
$$a = 50$$
$$a + e + b = 130 \Rightarrow e = 30$$
$$e + f + g = 230 - (a + b + c)$$
$$= 230 - 180 = 50$$
$$d = 100 - 50 = 50$$

16. (d) As already calculated, $e = 30$. Hence,

$$d + g + e + f = 100$$

Total people who read at least one newspaper along with Delhi Times $= d + g + f$

$$= 100 - 30 = 70$$

17. (c) Total people who read only Hindustan Times or Delhi Times $= b + c$

$$= 50 + 80 = 135$$

18. (b) Total people who read Times of India only $= a = 50$.

19. (a) Total people who read exactly one newspaper $= a + b + c = 180$.

20. (d) Total people who read none of the three newspaper $= h = 20$.

21. (c) Bus and train are different from each other but some travelers travel by bus and some travel by train.

22. (b) Bonus and dividend are different from each other but both these are parts of profit.

23. (a) All mothers are women and some mothers and some women may be engineers.

24. (d) Product and machinery are different from each other but both are found in factory.

25. (b) All the three professions are different.

CHAPTER 5

SURDS, INDICES AND LOGARITHMS

SURDS

Irrational numbers that are of the form $\sqrt[n]{x}$ are called surds of order 'n'. Some important rules of surds are as follows:

1. $\left(\sqrt[n]{x}\right)^n = (x^{1/n})^n = x$

2. $\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}$

3. $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$

4. $\left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$

5. $\sqrt[a]{\sqrt[b]{\sqrt[c]{x}}} = \sqrt[abc]{x} = x^{(1/abc)}$

INDICES

A surd expressed in the form x^n is called fractional index. Some important rules of indices are as follows:

1. $x^m \times x^n = x^{m+n}$

2. $\dfrac{x^m}{x^n} = x^{m-n}$

3. $(x^m)^n = x^{mn}$

4. $(x \times y)^n = x^n \times y^n$

5. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$

6. $x^m = x^n \quad \text{if} \quad m = n$

7. $x^{-n} = \dfrac{1}{x^n}$

8. $x^0 = 1, \quad x$ is any number

LOGARITHM

The logarithm of a number is the exponent to which the base must be raised to produce that number.

If $a^x = b$, then x is called logarithm of a number 'b' to the base 'a'. It is represented as $\log_a b = x$. Some of the important rules of logarithms are as follows:

1. $\log_x(a \times b) = \log_x a + \log_x b$
2. $\log_x\left(\dfrac{a}{b}\right) = \log_x(a) - \log_x(b)$
3. $\log_x(a^b) = b\log_x a$
4. $\log_{x^y}(a) = \dfrac{1}{y}\log_x(a)$
5. $\log_x a = \dfrac{\log_n a}{\log_n x}$
6. $\log_x a = \dfrac{1}{\log_a x}$ or $\log_x a \times \log_a x = 1$
7. $\log_x 1 = 0$ where x is any natural number
8. $x^{\log_x a} = a$

SOLVED EXAMPLES

1. Simplify $4^{2x-1} \cdot 16 = 1$.

Solution: We have

$$\begin{aligned} &4^{2x-1} \cdot 16 = 1 \\ \Rightarrow\ &2^{(2x-1)} \cdot 2^4 = 1 \\ \Rightarrow\ &2^{(8x-4)} = 2^0 \\ \Rightarrow\ &8x - 4 = 0 \\ \Rightarrow\ &x = \frac{1}{2} \end{aligned}$$

2. Simplify $3^{(3x+1)} = 81$.

Solution: We have

$$\begin{aligned} &3^{(3x+1)} = 81 \\ \Rightarrow\ &3^{(3x+1)} = 3^4 \\ \Rightarrow\ &3x+1 = 4 \\ \Rightarrow\ &3x = 3 \\ \Rightarrow\ &x = 1 \end{aligned}$$

3. Which is greater $\sqrt[3]{3}$ or $\sqrt[5]{5}$?

Solution: We have

$$\sqrt[3]{3} = 3^{1/3} \quad \text{and} \quad \sqrt[5]{5} = 5^{1/5}$$

L.C.M. of 3 and 5 = 15. Thus,

$$(3^{1/3})^{15} = 3^5 = 243$$
$$(5^{1/5})^{15} = 5^3 = 125$$

Hence, $\sqrt[3]{3} > \sqrt[5]{5}$.

4. Which is greater $\sqrt[2]{3}$ or $\sqrt[3]{4}$?

Solution: We have

$$\sqrt[2]{3} = 3^{1/2} \quad \text{and} \quad \sqrt[3]{4} = 4^{1/3}$$

L.C.M. of 2 and 3 = 6. Thus,

$$(3^{1/2})^6 = 3^3 = 27$$
$$(4^{1/3})^6 = 4^2 = 16$$

Hence, $\sqrt[2]{3} > \sqrt[3]{4}$.

5. Simplify $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$.

Solution: We have

$$\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$$

After multiplying numerator and denominator with $(\sqrt{5}-\sqrt{2})$, we get

$$\frac{(\sqrt{5}-\sqrt{2})}{(\sqrt{5}+\sqrt{2})} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$= \frac{(\sqrt{5}-\sqrt{2})^2}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{5+2-2\sqrt{10}}{5-2} = \frac{7-2\sqrt{10}}{3}$$

6. Simplify $\dfrac{\sqrt{a^2+1}+\sqrt{a^2-1}}{\sqrt{a^2+1}-\sqrt{a^2-1}}$.

Solution: We have

$$\frac{\sqrt{a^2+1}+\sqrt{a^2-1}}{\sqrt{a^2+1}-\sqrt{a^2-1}}$$

After multiplying numerator and denominator with $\sqrt{a^2+1}+\sqrt{a^2-1}$, we get

$$\frac{\left(\sqrt{a^2+1}+\sqrt{a^2-1}\right)^2}{\left(\sqrt{a^2+1}\right)^2-\left(\sqrt{a^2-1}\right)^2}$$

$$= \frac{(a^2+1)+(a^2-1)+2\sqrt{a^4-1}}{2}$$

$$= \frac{2a^2+2\sqrt{a^4-1}}{2} = a^2+\sqrt{a^4-1}$$

7. If $2^{x-4} + \frac{1}{16^{x-2}}$, then what is the value of x?

Solution: We have

$$(2)^{x-4} = \frac{1}{(16)^{x-2}}$$

$$\Rightarrow (2)^{x-4} = (16)^{-(x-2)}$$

$$\Rightarrow (2)^{x-4} = (2)^{-4(x-2)}$$

Since, the base is same, we can equate powers

$$\Rightarrow x - 4 = -4x + 8$$
$$\Rightarrow 5x = 12$$
$$\Rightarrow x = \frac{12}{5}$$

8. What is the value of x, if $\sqrt[3]{16} = 4^{2x}$?

Solution: We have

$$\sqrt[3]{16} = 4^{2x}$$

$$\Rightarrow 16^{1/3} = 4^{2x}$$

$$\Rightarrow 4^{2/3} = 4^{2x}$$

After equating the powers of bases, we have

$$2x = \frac{2}{3}$$

$$\Rightarrow x = 1/3$$

9. Simplify $\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$.

Solution: We have

$$\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$$

After multiplying numerator and denominator by $(\sqrt{6}+\sqrt{2})$, we get

$$\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$

$$= \frac{(\sqrt{6}+\sqrt{2})^2}{(\sqrt{6})^2-(\sqrt{2})^2} = \frac{6+2+2\sqrt{12}}{6-2}$$

$$= \frac{8+4\sqrt{3}}{4} = 2+\sqrt{3}$$

10. If $2^{-x} = 3^{-y} = 6^{z}$, then what is the value $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$?

Solution: Let $2^{-x} = 3^{-y} = 6^{z} = k$

Thus, $2 = k^{-1/x}$, $3 = k^{-1/y}$, $6 = k^{1/z}$

Now, $2 \times 3 = 6$. Hence

$$k^{-1/x} \cdot k^{-1/y} = k^{1/z}$$

After comparing the powers, we get

$$-\frac{1}{x} - \frac{1}{y} = +\frac{1}{z}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$$

11. What is the logarithm of 64 to the base 4?

Solution: $\log_4 64 = \log_4 (4)^3$

$$= 3\log_4 4 = 3 \times 1 = 3$$

12. If $\log_3 x = 6$, then what is the value of x?

Solution: We have

$$\log_3 x = 6$$

$$\Rightarrow x = 3^6 = 729$$

13. What is the value of $4^{3+\log_4 2}$?

Solution: We have

$$4^{3+\log_4 2} = 4^3 \times 4^{\log_4 2}$$
$$= 64 \times 2$$
$$= 128$$

14. What is the value of $\log_4 256 - \log_3 9$?

Solution: Now,

$$\log_4 256 - \log_3 9$$

$$= \log_4 (4)^4 - \log_3 (3)^2$$

$$= 4 - 2 = 2$$

15. What is the value of x, if $\log_{27} x = \frac{1}{3}$?

Solution: We have

$$\log_{27} x = \frac{1}{3}$$

$$\Rightarrow x = 27^{1/3}$$
$$\Rightarrow x = 3$$

16. What is the value of x, if $\log_x \sqrt{5} = \frac{1}{4}$?

Solution: We have

$$\log_x \sqrt{5} = \frac{1}{4} \Rightarrow \log_x (5)^{1/2} = \frac{1}{4}$$

$$\Rightarrow x^{1/4} = 5^{1/2}$$

$$\Rightarrow x = 5^2 = 25$$

17. What is the value of $\log_{16} x + \log_8 x + \log_4 x - \log_2 x = 1$?

Solution: Now,

$$\log_{16} x + \log_8 x + \log_4 x - \log_2 x = 1$$

$$\Rightarrow \frac{1}{4}\log_2 x + \frac{1}{3}\log_2 x + \frac{1}{2}\log_2 x - \log_2 x = 1$$

$$\Rightarrow \frac{3\log_2 x + 4\log_2 x + 6\log_2 x - 12\log_2 x}{12} = 1$$

$$\Rightarrow \frac{1}{12}\log_2 x = 1$$

$$\Rightarrow \log_2 x = 12$$

$$\Rightarrow x = 12^2 = 144$$

18. What is the value of x if $\log_2 16 + \log_2 64 - \log_3 9 + \log_3 x = 9$?

Solution: We have

$$\log_2 16 + \log_2 64 - \log_3 9 + \log_3 x = 9$$

$$\Rightarrow \log_2 (2)^4 + \log_2 (2)^6 - [\log_3 (9/x)] = 9$$

$$\Rightarrow 4 + 6 - 9 = \log_3 (9/x)$$

$$\Rightarrow 1 = \log_3 (9/x) \Rightarrow \frac{9}{x} = 1^3$$

$$\Rightarrow x = 9$$

19. What is the value of $\log \frac{x^2}{yz} + \log \frac{y^2}{zx} + \log \frac{z^2}{xy}$?

Solution:

$$\log \frac{x^2}{yz} + \log \frac{y^2}{zx} + \log \frac{z^2}{xy} = \log\left(\frac{x^2}{yz} \times \frac{y^2}{zx} \times \frac{z^2}{xy}\right)$$

$$= \log\left(\frac{x^2y^2z^2}{x^2y^2z^2}\right) = \log(1) = 0$$

20. What is the value of x, if $\log \sqrt[3]{9} x = 7\frac{1}{2}$?

Solution: We have

$$\log_{\sqrt[3]{9}} x = 7\frac{1}{2} = \frac{15}{2}$$

$$\Rightarrow x = (3^{2/3})^{15/2}$$

$$\Rightarrow x = 3^5 = 243$$

21. If $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$, then find the value of $x^2 + x^{-2}$.

Solution: We are given that,

$$x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3 \qquad (1)$$

Squaring Eq. (1), we get

$$\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2 = 3^2 \Rightarrow x + x^{-1} + 2 = 9$$

$$\Rightarrow x + x^{-1} = 7 \qquad (2)$$

Squaring Eq. (2), we get

$$\left(x + x^{-1}\right)^2 = 7^2 \Rightarrow x^2 + x^{-2} + 2 = 49$$

$$\Rightarrow x^2 + x^{-2} = 47$$

22. If $\log_{12} 3 = x$ then find the value of $\log_{\sqrt{3}} 8$in terms of x.

Solution: We can rewrite $\log_{\sqrt{3}} 8$ as

$$\log_{\sqrt{3}} 8 = \frac{\log_{12} 8}{\log_{12} \sqrt{3}} = \frac{\log_{12} 2^3}{\log_{12} 3^{\frac{1}{2}}}$$

$$= \frac{3}{1/2}\left(\frac{\log_{12} 2}{\log_{12} 3}\right) = 6\left(\frac{\log_{12} 2}{\log_{12} 3}\right)$$

$$= 3\left(\frac{\log_{12} 2^2}{\log_{12} 3}\right) = 3\left(\frac{\log_{12}(12/3)}{\log_{12} 3}\right)$$

$$= 3\left(\frac{\log_{12} 12 - \log_{12} 3}{\log_{12} 3}\right) = 3\left(\frac{1-x}{x}\right)$$

23. Find the value of $\frac{x^3 + 2x^2y + y^3}{xy(x+3y)}$ given that $\frac{x}{y} = \sqrt{2} - 1$.

Solution: We need to find the value of $\frac{x^3 + 2x^2y + y^3}{xy(x+3y)}$.

$$\frac{x^3 + 2x^2y + y^3}{xy(x+3y)} = \frac{\left(\frac{x}{y}\right)^3 + 2\left(\frac{x}{y}\right)^2 + 1}{\left(\frac{x}{y}\right)\left[\left(\frac{x}{y}\right) + 3\right]}$$

$$= \frac{\left(\sqrt{2}-1\right)^3 + 2\left(\sqrt{2}-1\right)^2 + 1}{\left(\sqrt{2}-1\right)\left[\left(\sqrt{2}-1\right) + 3\right]}$$

$$= \frac{\left(\sqrt{2}-1\right)\left(3-2\sqrt{2}\right) + 2\left(3-2\sqrt{2}\right)}{\left(3-2\sqrt{2}\right) + 3\left(\sqrt{2}-1\right)}$$

$$= \frac{5\sqrt{2} - 7 + 6 - 4\sqrt{2} + 1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

24. If $x = \dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$, find the value of $8x - x^2$.

Solution: We are given that,

$$x = \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$= \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{\left(\sqrt{5}-\sqrt{3}\right)^2}{\left(\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2}$$

$$= \frac{5+3-2\sqrt{15}}{2} = 4 - \sqrt{15}$$

Putting value of x in $8x - x^2$, we get

$$8\left(4-\sqrt{15}\right) - \left(4-\sqrt{15}\right)^2 = 32 - 8\sqrt{15} - 16$$
$$-15 + 8\sqrt{15} = 32 - 31 = 1$$

25. Solve the equation $\dfrac{x\sqrt{x}}{x^{\log x}} = 10^{-1}$.

Solution: We are given,

$$\frac{x\sqrt{x}}{x^{\log x}} = 10^{-1} \Rightarrow \log x^{\frac{3}{2}-\log x}$$
$$= -1 \Rightarrow \left(\frac{3}{2} - \log x\right)\log x$$
$$= -1 \Rightarrow (\log x)^2 - \frac{3}{2}\log x - 1 = 0$$
$$\Rightarrow 2(\log x)^2 - 3\log x - 2 = 0$$
$$\Rightarrow (2\log x + 1)(\log x - 2) = 0$$
$$\Rightarrow \log x = -\frac{1}{2} \text{ or } \log x = 2$$
$$\Rightarrow x = \frac{1}{\sqrt{10}} \text{ or } 100$$

PRACTICE EXERCISE

1. What is the value of $(\sqrt[3]{27})^2$?

(a) 3 (b) 9
(c) 16 (d) 27

2. What is the value of x, if $\sqrt{6^x} = 36$?

(a) 1 (b) 2
(c) 4 (d) 0

3. What is the value of $\sqrt{16} \cdot \sqrt[3]{8x} = 8$?

(a) 2 (b) 4
(c) 6 (d) 8

4. What is the value of x, if $\dfrac{4^2 \times 8^3 \times 2^x}{16^2 \times 4^3} = 16$?

(a) 2 (b) 3
(c) 5 (d) 6

5. What is the value of x, if $(\sqrt{3})^7 \times 3^4 = 3^x \times 3^{3/2}$?

(a) 6 (b) 5
(c) 4 (d) 2

6. What is the value of $\sqrt[3]{x^{-1}y} \cdot \sqrt[3]{y^{-1}z} \cdot \sqrt[3]{z^{-1}x}$, if x, y and z are real numbers?

(a) xyz (b) $x^{-1}y^{-1}z$
(c) 0 (d) 1

7. What is the value of x, if $2^{x+4} - 2^{x+2} = 3$?

(a) -2 (b) $+2$
(c) -4 (d) $+4$

8. What is the value of x, if $3^{x+3} = \dfrac{1}{9^{x-3}}$?

(a) -1 (b) 1
(c) -3 (d) 3

9. What is the value of x if $a^x = b^y = c^z$ and $a^2 = bc$?

(a) $yz/(y+z)$ (b) $yz/(y-z)$
(c) $2yz/(y-z)$ (d) $\dfrac{2yz}{y+z}$

10. Simplify $\dfrac{\sqrt{a^2+1} - \sqrt{a^2-1}}{\sqrt{a^2+1} + \sqrt{a^2-1}}$.

(a) $a^2 + \sqrt{a^4-1}$ (b) $a^2 + a^4$
(c) $a^2 - a^4$ (d) $a^2 - \sqrt{a^4-1}$

11. What is the value of $\log_{216} 6$?

(a) 3 (b) -3
(c) 1/3 (d) $-1/3$

12. What is the value of x, if $\log_4 x = 5/2$?

(a) 8 (b) 32
(c) 64 (d) 128

13. What is the value of x, if $\log_8 x + \log_4 x = 5$?

(a) 64 (b) 125
(c) 25 (d) 16

14. What is the value of $\log_2(\log_3 6561)$?

(a) 3 (b) 4
(c) 5 (d) 2

15. What is the value of $\left[\frac{1}{(\log_a bc)+1}+\frac{1}{(\log_b ca)+1}+\frac{1}{(\log_c ab)+1}\right]$?

(a) 3 (b) 2
(c) 3/2 (d) 1

16. What is the value of x, if $\log_2[\log_4(\log_3 x)] = 0$?

(a) 81 (b) 9
(c) 64 (d) 16

17. What is the value of $\log_{16} 64 + \log_8 16$?

(a) 5/6 (b) 10/3
(c) 17/6 (d) 12

18. If $\log x + \log y = \log(x - y)$, then what is the value of y?

(a) $x/x - 1$ (b) $\frac{x+1}{x}$
(c) $x/x + 1$ (d) $\frac{x-1}{x}$

19. If $\log_4(x^2 - 1) - \log_4(x + 1) = 3$, then what is the value of x?

(a) 63 (b) 65
(c) 125 (d) 81

20. What is the value of $16^{\log_4 5}$?

(a) 32 (b) 125
(c) 64 (d) 25

21. If $\log_5\left(5^{2x+1} - 20\sqrt{5}\right) = 2x$, then what is the value of x?

(a) 3/4 (b) 1/4
(c) 1/2 (d) 3/2

22. Find the value of the expression $\frac{1}{\log_a(ab)} + \frac{1}{\log_b(ab)}$.

(a) ab (b) a/b
(c) 10 (d) 1

23. Solve the equation $\log_x 8 + \log_8 x = \frac{13}{6}$.

(a) 2, 4 (b) 16, 2
(c) $16\sqrt{2}, 4$ (d) $16\sqrt{2}, 4\sqrt{2}$

24. Solve the simultaneous equations for x and y:

$$\log_2 xy = 5 \quad \text{and} \quad \log_8 x = \log_4 y$$

(a) $x = 4,\ y = 8$ (b) $x = 2,\ y = 3$
(c) $x = 8,\ y = 4$ (d) $x = 4,\ y = 3$

25. If $\log_{10} 2 = 0.3010$, then what is the value of $\log_{10} 80$?

(a) 1.6020 (b) 1.9030
(c) 3.9030 (d) 4.2210

26. Find the value of $\frac{1}{1+a^{(x-y)}} + \frac{1}{1+a^{(y-x)}}$.

(a) 0 (b) 1/2
(c) 1 (d) a^{x+y}

27. Find the value of x in the equation $\frac{(25)^{7.5} \times (5)^{2.5}}{(125)^{1.5}} = 5^x$.

(a) 13 (b) 16
(c) 17.5 (d) 20

28. Given that $x = 10^{0.48}$, $y = 10^{0.70}$ and $x^z = y^2$, then what is the approximate value of z?

(a) 1.45 (b) 1.88
(c) 3.7 (d) 2.9

29. What is the value of $\left(\frac{1}{\log_3 60} + \frac{1}{\log_4 60} + \frac{1}{\log_5 60}\right)$?

(a) 0 (b) 1
(c) 5 (d) 60

30. If a and b are whole numbers such that $m^n = 121$, then what is the value of $(m-1)^{n+1}$?

(a) 1 (b) 10
(c) 121 (d) 1000

ANSWERS

1. (b)	**6.** (d)	**11.** (c)	**16.** (a)	**21.** (a)	**26.** (c)
2. (c)	**7.** (a)	**12.** (b)	**17.** (c)	**22.** (d)	**27.** (a)
3. (d)	**8.** (b)	**13.** (a)	**18.** (c)	**23.** (c)	**28.** (d)
4. (c)	**9.** (d)	**14.** (a)	**19.** (b)	**24.** (c)	**29.** (b)
5. (a)	**10.** (d)	**15.** (d)	**20.** (d)	**25.** (b)	**30.** (d)

EXPLANATIONS AND HINTS

1. (b) We have

$$(\sqrt[3]{27})^2 = (3)^2 = 9$$

2. (c) We have

$$\sqrt{6^x} = 36$$
$$\Rightarrow 6^{x/2} = 6^2$$
$$\Rightarrow \frac{x}{2} = 2 \quad \Rightarrow \quad x = 4$$

3. (d) We have

$$\sqrt{16} \cdot \sqrt[3]{x} = 8$$
$$\Rightarrow 4 \cdot \sqrt[3]{x} = 8$$
$$\Rightarrow \sqrt[3]{x} = 2$$
$$\Rightarrow x = 2^3 = 8$$

4. (c) We have

$$\frac{4^2 \times 8^3 \times 2^x}{16^2 \times 4^3} = 16$$
$$\Rightarrow \frac{2^4 \times 2^9 \times 2^x}{2^8 \times 2^6} = 2^4$$
$$\Rightarrow 2^{(4+9+x-8-6)} = 2^4$$

After equating the powers, we get

$$4 + 9 + x - 8 - 6 = 4$$
$$\Rightarrow x = 14 - 9 = 5$$

5. (a) We have

$$(\sqrt{3})^7 \times 3^4 = 3^x \times 3^{3/2}$$
$$\Rightarrow (3)^{7/2} \times 3^4 = 3^x \times 3^{3/2}$$
$$\Rightarrow (3)^{7/2+4} = 3^{x+3/2}$$

After equating higher powers, we get

$$7/2 + 4 = x + 3/2$$
$$\Rightarrow x = \frac{4}{2} + 4$$
$$\Rightarrow x = 6$$

6. (d) We have

$$\sqrt[3]{x^{-1}y} \cdot \sqrt[3]{y^{-1}z} \cdot \sqrt[3]{z^{-1}x}$$
$$= (x^{-1}y)^{1/3} \cdot (y^{-1}z)^{1/3} \cdot (z^{-1}x)^{1/3}$$
$$= [(x^{-1}y)(y^{-1}z)(z^{-1}x)]^{1/3}$$
$$= (1)^{1/3} \quad = \quad 1$$

7. (a) We have

$$2^{x+4} - 2^{x+2} = 3$$
$$\Rightarrow 2^{x+2}(2^2 - 1) = 3$$
$$\Rightarrow 2^{x+2} = 1 = 2^0$$

After equating higher powers, we get

$$x + 2 = 0$$
$$\Rightarrow x = -2$$

8. (b) We have

$$3^{x+3} = \frac{1}{9^{x-3}}$$
$$\Rightarrow 3^{x+3} = \frac{1}{3^{2x-6}} \Rightarrow 3^{x+3} = 3^{-(2x-6)}$$

After equating higher powers, we get

$$x + 3 = -2x + 6 \quad \Rightarrow \quad 3x = 3 \quad \Rightarrow x = 1$$

9. (d) Let $a^x = b^y = c^z = k$

Then,

$$a = k^{1/x}$$
$$b = k^{1/y}$$
$$c = k^{1/z}$$

Also, $a^2 = bc$. Thus,

$$k^{2/x} = k^{1/y} \cdot k^{1/z}$$
$$\Rightarrow k^{2/x} = k^{(1/y)+(1/z)}$$

After equating higher powers, we get

$$\frac{2}{x} = \frac{1}{y} + \frac{1}{z}$$
$$\Rightarrow \frac{2}{x} = \frac{y+z}{yz} \Rightarrow x = \frac{2yz}{y+z}$$

10. (d) We have

$$\frac{\sqrt{a^2+1} - \sqrt{a^2-1}}{\sqrt{a^2+1} + \sqrt{a^2-1}}$$

After multiplying numerator and denominator with $\dfrac{\sqrt{a^2+1} - \sqrt{a^2-1}}{\sqrt{a^2+1} - \sqrt{a^2-1}}$, we get

$$\frac{\sqrt{a^2+1} - \sqrt{a^2-1}}{\sqrt{a^2+1} + \sqrt{a^2-1}} \times \frac{\sqrt{a^2+1} - \sqrt{a^2-1}}{\sqrt{a^2+1} - \sqrt{a^2-1}}$$

$$= \frac{\left(\sqrt{a^2+1}-\sqrt{a^2-1}\right)^2}{\left(\sqrt{a^2+1}\right)^2-\left(\sqrt{a^2-1}\right)^2}$$

$$= \frac{(a^2+1)+(a^2-1)-2\sqrt{a^4-1}}{a^2+1-a^2+1}$$

$$= \frac{2a^2-2\sqrt{a^4-1}}{2} = a^2-\sqrt{a^4-1}$$

11. (c) We have

$$\log_{216} 6 = x$$

Then, $(216)^x = 6$, hence,

$$\Rightarrow (6^3)^x = 6$$

$$\Rightarrow 6^{3x} = 6^1$$

After equating higher powers, we get

$$3x = 1$$

$$\Rightarrow x = 1/3$$

12. (b) We have

$$\log_4 x = \frac{5}{2}$$

$$\Rightarrow x = 4^{5/2} = \sqrt{4^5} = 2^5$$

$$\Rightarrow x = 32$$

13. (a) We have

$$\log_8 x + \log_4 x = 5$$

$$\Rightarrow \frac{\log x}{\log 8} + \frac{\log x}{\log 4} = 5$$

$$\Rightarrow \frac{\log x}{3\log 2} + \frac{\log x}{2\log 2} = 5$$

$$\Rightarrow \frac{2\log x + 3\log x}{6\log 2} = 5 \Rightarrow 5\log x = 30\log 2$$

$$\Rightarrow \log x = 6\log 2$$

$$\Rightarrow \log x = \log 2^6$$

$$\Rightarrow x = 64$$

14. (a) We have

$$\log_2(\log_3 6561)$$

$$= \log_2[\log_3(3)^8]$$

$$= \log_2(8)$$

$$= \log_2(2)^3 = 3$$

15. (d) We have

$$\frac{1}{\log_a bc + \log_a a} + \frac{1}{\log_b ca + \log_b b} + \frac{1}{\log_c ab + \log_c c}$$

$$= \frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)}$$

$$= \log_{abc} a + \log_{abc} b + \log_{abc} c$$

$$= \log_{abc}(abc) = 1$$

16. (a) We have

$$\log_2[\log_4(\log_3 x)] = 0$$

$$\Rightarrow \log_4(\log_3 x) = 1$$

$$\Rightarrow \log_3 x = 4^1$$

$$\Rightarrow x = 3^4 = 81$$

17. (c) We have

$$\log_{16} 64 + \log_8 16$$

$$= \log_{4^2} 4^3 + \log_{2^3} 2^4$$

$$= \frac{3}{2}\log_4 4 + \frac{4}{3}\log_2 2$$

$$= \frac{3}{2} + \frac{4}{3}$$

$$= \frac{9+8}{6} = \frac{17}{6}$$

18. (c) We have

$$\log x + \log y = \log(x-y)$$

$$\log(xy) = \log(x-y)$$

$$xy = x - y$$

$$y(x+1) = x$$

$$y = \frac{x}{x+1}$$

19. (b) We have

$$\log_4(x^2-1) - \log_4(x+1) = 3$$

$$\Rightarrow \log_4\frac{(x^2-1)}{(x+1)} = 3$$

$$\Rightarrow \log_4\frac{x^2-1}{x+1} = \log_4 64$$

$$\Rightarrow \frac{x^2-1}{x+1} = 64 \quad \Rightarrow \frac{(x+1)(x-1)}{x+1} = 64$$

$$\Rightarrow x - 1 = 64$$

$$\Rightarrow x = 65$$

20. (d) We have

$$16^{\log_4 5}$$

Now, we know that

$$a^{\log_a x} = x$$

Hence,

$$16^{\log_4 5} = 4^{2\log_4 5}$$
$$= 4^{\log_4 25}$$
$$= 25$$

21. (a) We have,

$$\log_5\left(5^{2x+1} - 20\sqrt{5}\right) = 2x$$
$$\Rightarrow 5^{2x+1} - 20\sqrt{5} = 5^{2x}$$
$$\Rightarrow 5\left(5^{2x}\right) - 5^{2x} = 20\sqrt{5}$$
$$\Rightarrow 4\left(5^{2x}\right) = 20\sqrt{5} \Rightarrow\Rightarrow \left(5^{2x}\right) = 5\sqrt{5}$$
$$\Rightarrow 5^{2x} = 5^{\frac{3}{2}} \Rightarrow 2x = \frac{3}{2} \Rightarrow x = \frac{3}{4}$$

22. (d) We have to find the value of $\dfrac{1}{\log_a(ab)} + \dfrac{1}{\log_b(ab)}$

Now, let $x = \log_a(ab)$ and $y = \log_b(ab)$. Therefore, $a^x = ab$ and $b^y = ab$.

Hence, $a = (ab)^{\frac{1}{x}}$ and $b = (ab)^{\frac{1}{y}}$.

$$\Rightarrow ab = (ab)^{\frac{1}{x}}(ab)^{\frac{1}{y}}$$
$$= (ab)^{\frac{1}{x}+\frac{1}{y}}$$

When the bases are same, the powers can be equated.

Thus, $\dfrac{1}{x} + \dfrac{1}{y} = 1 \Rightarrow \dfrac{1}{\log_a(ab)} + \dfrac{1}{\log_b(ab)} = 1$

23. (c) We are given,

$$\log_x 8 + \log_8 x = \frac{13}{6}$$
$$\Rightarrow \frac{1}{\log_8 x} + \log_8 x = \frac{13}{6}$$
$$\Rightarrow 6\left(\log_8 x\right)^2 - 13\left(\log_8 x\right) + 6 = 0$$
$$\Rightarrow \left(2\log_8 x - 3\right)\left(3\log_8 x - 2\right) = 0 \Rightarrow \log_8 x = \frac{3}{2}, \frac{2}{3}$$
$$\Rightarrow x = 8^{\frac{3}{2}}, 8^{\frac{2}{3}} \Rightarrow x = \left(2^3\right)^{\frac{3}{2}}, \left(2^3\right)^{\frac{2}{3}}$$
$$\Rightarrow x = 16\sqrt{2}, 4$$

24. (c) We are given two equations:

$$\log_2 xy = 5 \text{ and } \log_8 x = \log_4 y$$

$$\log_2 xy = 5 \Rightarrow xy = 2^5 = 32 \qquad \text{(i)}$$

$$\log_8 x = \log_4 y$$
$$\Rightarrow \frac{\log_2 x}{\log_2 8} = \frac{\log_2 y}{\log_2 4} \Rightarrow \frac{\log_2 x}{\log_2 2^3} = \frac{\log_2 y}{\log_2 2^2}$$
$$\Rightarrow \frac{1}{3}\log_2 x = \frac{1}{2}\log_2 y$$
$$\Rightarrow \log_2 x^{\frac{1}{3}} = \log_2 y^{\frac{1}{2}} \Rightarrow x^{\frac{1}{3}} = y^{\frac{1}{2}}$$
$$\Rightarrow x = y^{\frac{3}{2}} \qquad \text{(ii)}$$

Substituting Eq. (2) in Eq. (1), we get

$$\left(y^{\frac{3}{2}}\right)(y) = 32 \Rightarrow y^{\frac{5}{2}} = 32 \Rightarrow y = (32)^{\frac{2}{5}} = 4$$

$$x = y^{\frac{3}{2}} = (4)^{\frac{3}{2}} = 8$$

25. (b) We can rewrite $\log_{10} 80$ as follows:

$$\log_{10} 80 = \log_{10}(8 \times 10) = \log_{10} 8 + \log_{10} 10$$
$$= \log_{10}\left(2^3\right) + 1 = 3\log_{10} 2 + 1 = (3 \times 0.3010) + 1$$
$$= 1.9030$$

26. (c) We can rewrite $\dfrac{1}{1+a^{(x-y)}} + \dfrac{1}{1+a^{(y-x)}}$ as follows:

$$\frac{1}{1+a^{(x-y)}} + \frac{1}{1+a^{(y-x)}} = \frac{1}{\left(1+\dfrac{a^x}{a^y}\right)} + \frac{1}{\left(1+\dfrac{a^y}{a^x}\right)}$$
$$= \frac{a^y}{\left(a^y + a^x\right)} + \frac{a^x}{\left(a^y + a^x\right)} = \frac{\left(a^y + a^x\right)}{\left(a^y + a^x\right)} = 1$$

27. (a) We have $\dfrac{(25)^{7.5} \times (5)^{2.5}}{(125)^{1.5}} = 5^x$

Then,

$$\frac{\left(5^2\right)^{7.5} \times (5)^{2.5}}{\left(5^3\right)^{1.5}} = 5^x \Rightarrow \frac{5^{15} \times 5^{2.5}}{5^{4.5}} = 5^x$$

Since, the base is same, we can equate the powers.

$$\frac{\left(5^2\right)^{7.5} \times (5)^{2.5}}{\left(5^3\right)^{1.5}} = 5^x \Rightarrow x = 15 + 2.5 - 4.5 = 13$$

28. (d) We are given that,

$$\Rightarrow 10^{(0.48z)} = 10^{(2 \times 0.70)} = 10^{1.4}$$

$$\Rightarrow 0.48z = 1.4$$

$$\Rightarrow z = \frac{140}{48} = \frac{35}{12} = 2.9$$

29. (b) We are given that

$$\log_{60} 3 + \log_{60} 4 + \log_{60} 5 = \log_{60}(3 \times 4 \times 5)$$

$$= \log_{60} 60$$

$$= 1$$

30. (d) We know that,

$$11^2 = 121$$

Putting $a = 11$ and $b = 2$, we get

$$(11-1)^{2+1} = (10)^{(3)} = 10^3 = 1000$$

UNIT 3: REASONING AND DATA INTERPRETATION

CHAPTER 1

CUBES AND DICES

CUBES

A cuboid is any three-dimensional figure with length, breadth and height. If all the three sides of a cuboid are same, then the figure is called a cube. Figure 1 shows a Rubik's cube which is a cube made of 27 smaller cubes.

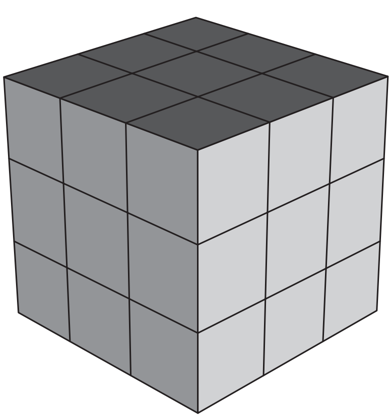


Figure 1 | Rubik's cube.

In a cube:

1. There are 8 corners/vertices.
2. There are 6 faces, all equal in area.
3. There are 12 edges, all equal in area.
4. It should be noted that

$$\text{Number of edges} = \text{Number of vertices} + \text{Number of face} - 2$$

Some important tips to solve questions based on cubes are as follows:

1. If we paint a cube of the dimension $n \times n \times n$ in any one colour and cut it to get n^3 cubelets, then the number of cubes with only one face painted = $(n-2)^2 \times 6$.
2. If we paint a cube of the dimension $n \times n \times n$ in any one colour and cut it to get n^3 cubelets, then the number of cubes with two faces painted = $(n-2) \times 12$.
3. If we paint a cube of the dimensions $n \times n \times n$ in any one colour and cut it to get n^3 cubelets, then the number of cubes with three faces painted = $(n-2)^3$.

4. If we cut a bigger cube into n^3 identical cubelets, using minimum number of cuts, we need a total of $3(n-1)$ cuts, such that $(n-1)$ cuts parallel to each of these faces which are joining to corner.
5. If number of cuts are not multiple of three then cube can never be cut into identical cubes but still it can be cut into maximum number of identical cuboidal pieces. To maximize such number of pieces we need to split the number of cuts into three parts which are closest.

DICES

Dices are small cubical structures with numbers 1 to 6 (or dots) marked on each face.

When sum of number of opposite pair of faces is same, then the dice is called a symmetric dice. Hence, for a symmetrical dice, the faces opposite to each other will be 1→6, 2→5, 3→4. Figure 2 shows symmetrical dices.

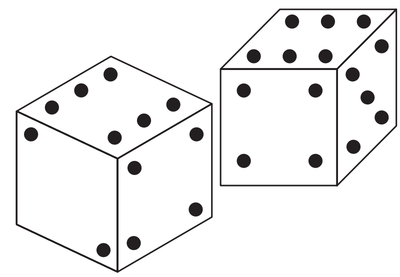


Figure 2 | Symmetric dices.

When sum of numbers on opposite pair of faces is different, then the dice is called an asymmetric dice. Figure 3 shows asymmetrical dices.

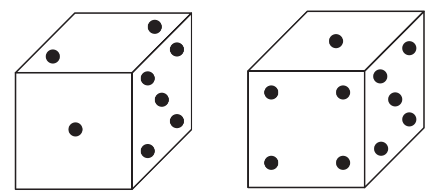


Figure 3 | Asymmetric dices.

SOLVED EXAMPLES

Direction for Q1–Q5: If a cube is cut into n^3 identical cubelets using minimum number of cuts, after painting all faces of cubes with white colour, then answer the following questions.

1. What is the maximum number of cuts required?

   Solution: There are a minimum of $(n-1)$ equidistant cuts parallel to each of three faces which are joining the corner.
   Hence, total number of cuts required is $3(n-1)$.

2. How many cubelets will have exactly two faces painted?

   Solution: Cubes with exactly two faces painted = $12(n-2)$, where n is the number of parts into which side is divided.

3. How many cubelets will have exactly one face painted?

   Solution: Cubes with exactly one face pained = $6(n-2)^2$, where n is the number of parts into which side is divided.

4. How many cubelets will have at most two faces painted?

   Solution: Total number of cubelets with at most two faces painted = Total number of cubes – Number of cubes with three faces painted.
   Total number of cubes = n^3

   Number of cubes with three faces painted = 8

   Hence, total number of cubelets with at most two faces painted = $n^3 - 8$

5. How many cubelets will have at least one face painted?

   Solution: Total number of cubelets with at least one face painted = Total number of cubes – Number of cubes with no-face painted.
   Number of cubes with no face painted = $(n-2)^3$

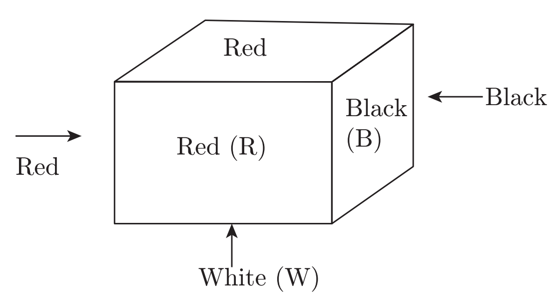
   Hence, total number of cubelets with at least one face painted = $n^3 - (n-2)^3$

   Three adjacent faces of one cube are painted in red, one adjacent pair of faces is painted in black and remaining faces are painted in white.

   The cube is then cut into 216 identical cubelets.

   Direction for Q6–Q10: A cube is cut into 216 cubelets = $6 \times 6 \times 6$

   We can draw the pattern of painting as shown in the following figure:



Now, let us analyze the colour combination.

Corners:

RRR = 1, BBR = 1

RRB = 2, BBW = 1

RRW = 1, RWB = 2

Edges:

RR = 3, RW = 2

RB = 3, BW = 2

BB = 1

Faces: R = 3

B = 2

W = 1

6. How many cubelets have all the three colours on them?

Solution: The small pieces will be formed from the corners. The cubelets having all three colours (red, black and white) are 2.

7. How many cubelets have exactly two colours on them?

Solution: The cubelets with exactly two colours found at corners = 5
Total edges having different colours on either side = 8

Therefore, total cubelets $= 8(6-2)+5 = 37$

8. How many cubelets have exactly one colour on them?

Solution: The cubelets having exactly one colour at corner (RRR) = 1

At the three edges each of RR categories = $3(6-2) = 12$

At one edge of BB category $= 1(6-2) = 4$

At middle of each of the six faces $= 6(6-2)^2 = 96$

Hence, total cubelets with exactly one colour = $1 + 12 + 4 + 96 = 113$

9. How many cubelets do not have red colour on them?

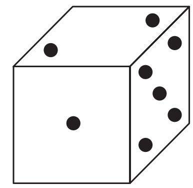
Solution: The number of cubelets with no red colour = Total cubes − (Total cubelets from one red surface + Total cubelets from two red surface + Total cubelets from three red surfaces) = 216 − (36 + 30 + 25) = 125

10. How many cubelets have black or white colour on them but not red colour on them?

Solution: The number of cubelets with no red colour = 125

The number of cubelets with black or white but not red $= 125 - (6-2)^3 = 125 - 64 = 61$

11. Observe the dots on a dice (1–6 dots) in the following figures. How many dots are contained on the face opposite to that containing 4 dots?



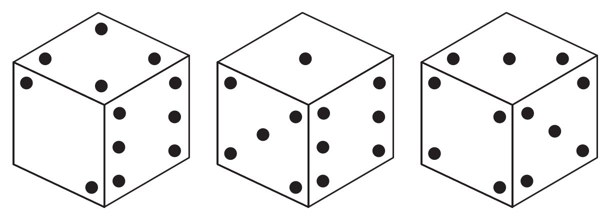
(a) (b)

Solution: We can see from Figs. (a) and (b) that the side of the dice with 5 dots remains stationary.

Hence, when we shift the dice such that we move the side with 1 dot, we observe that one side has 2 dots and the other side has 4 dots.

Since these two faces are opposite to each other, 2 dots are contained on the face opposite to that containing 4 dots.

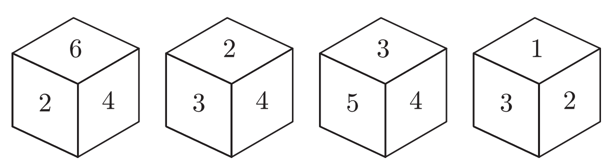
12. Three different positions of a dice are shown in the following figures. How many dots lie opposite 2 dots?



(a) (b) (c)

Solution: From Figs. (b) and (c), we conclude that 1, 6, 3 and 4 dots lie adjacent to 5 dots. Therefore, 2 dots must lie opposite 5 dots. Conversely, 5 dots must lie opposite 2 dots.

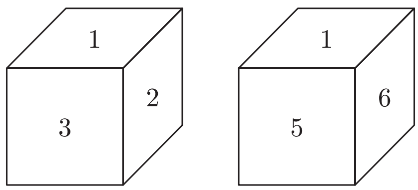
13. A dice is thrown four times and its four different positions are shown in the following figures. What is the number on the face opposite the face showing 2?



(a) (b) (c) (d)

Solution: From Figs. (a), (b) and (d), we conclude that 6, 4, 3 and 1 lie adjacent to 2. Hence, 5 must lie opposite 2.

14. Two positions of a cube with its surfaces numbered are shown in the following figures. When the surface 4 touches the bottom, which surface will be on the top?



(a) (b)

Solution: In these 2 positions, one common face with number 1 is in the same position. Also, we can see that 2, 3, 5 and 6 are adjacent to 1. Thus, 1 and 4 are opposite.

Therefore, when surface 4 touches the bottom, 1 will be on the top.

15. A dice is numbered from 1 to 6 in different ways. If 2 is opposite to 3 and adjacent to 4 and 6, then the statement "1 is adjacent to 2 and 3" is true or false?

Solution: If 2 is opposite to 3, then 1 cannot lie opposite to either of the two numbers, i.e. 2 or 3.

Hence, 1 is necessarily adjacent to both 2 and 3. Thus, the statement is true.

Direction for Q16–Q20: There are 128 cubes which are coloured according to two schemes, viz.

1. 64 cubes each having two adjacent red faces and one yellow and other blue on their opposite faces while green on the rest.
2. 64 cubes each having two adjacent blue faces and one red and other green on their opposite faces, while red on the rest. They are then mixed up.

Using this information, answer the following questions.

16. How many cubes have at least two-coloured red faces each?

Solution: According to two schemes, both types of cubes are such who have at least two red-coloured faces each. Therefore, the total number of the required cubes is 128.

17. What is the total number of red faces?

Solution: Number of red faces among first 64 cubes = 128
Number of red faces among second 64 cubes = 192
Thus, the total number of red faces = 128 + 192 = 320

18. How many cubes have two adjacent blue faces each?

Solution: According to the second scheme, 64 cubes have two adjacent blue faces.

19. How many cubes have only one red face each?

Solution: None of the schemes have only one face red.

20. Which two colours have the same number of faces?

Solution: First 64 cubes are such that each of whose two faces are green and second 64 cubes are such that each of whose two faces are blue.

Therefore, green and blue colours have the same number of faces.

PRACTICE EXERCISE

Direction for Q1–Q5: A cuboid is divided into 192 identical cubelets. This is done by making minimum number of cuts possible. All cuts are parallel to some of the faces. But before doing so the cube is painted with green colour on one set of opposite faces. But on the other set of opposite faces and red on remaining their part of opposite faces.

1. What is the maximum number of cubelets possible which are coloured with green colour only?

(a) 48 (b) 64 (c) 72 (d) 96

2. What is the minimum number of cuts being made?

(a) 15 (b) 18 (c) 21 (d) 12

3. What is the minimum number of cubelets possible which are painted with green and blue colours?

(a) 8 (b) 16 (c) 24 (d) 32

4. What is the number of cubelets which are painted with exactly three colours?

(a) 2 (b) 4 (c) 8 (d) 16

5. What is the number of cubelets possible which are painted with no colour?

(a) 16 (b) 24
(c) 36 (d) 48

Direction for Q6–Q10: A cube is painted and then divided cut into 336 smaller but identical pieces by making the minimum number of cuts possible. All cuts are parallel to some faces.

6. What is the minimum number of cuts required?

(a) 21 (b) 18
(c) 24 (d) None of the above

7. How many smaller pieces have exactly three painted faces?

(a) 8 (b) 12 (c) 20 (d) 24

8. How many smaller pieces have at least two faces painted?

(a) 60 (b) 64 (c) 66 (d) 68

9. How many smaller pieces have at most one face painted?

(a) 234 (b) 248 (c) 264 (d) 268

10. How many smaller pieces have no face painted?

(a) 108 (b) 120 (c) 124 (d) 144

Direction for Q11–Q15: A wooden cube of side 4 cm has been painted with different colours. The two opposite surfaces are painted red, the other two with green colour. Out of the remaining two surfaces, one is painted white and the other is painted black. Now the cube is cut into 64 equal cubes.

11. How many cubes have only green colour?

(a) 8 (b) 12 (c) 16 (d) 20

12. How many cubes have only two colours, red and black?

(a) 2 (b) 4 (c) 6 (d) 12

13. How many cubes have only black colour?

(a) 16 (b) 8 (c) 4 (d) 2

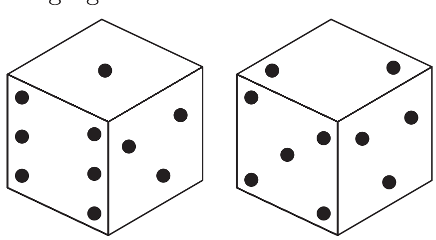
14. How many cubes do not have any coloured face?

(a) 8 (b) 4 (c) 16 (d) 12

15. How many cubes have three colours red, green and white?

(a) 12 (b) 2 (c) 8 (d) 4

16. Two positions (a and b) of a dice are shown in the following figures:



(a) (b)

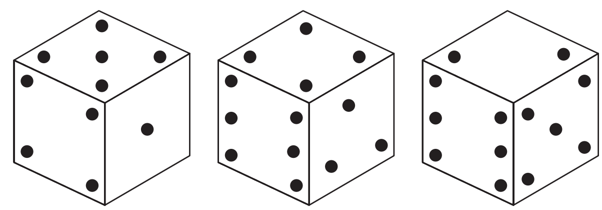
When 2 is at the bottom, what number will be at the top?

(a) 4 (b) 6 (c) 1 (d) 5

17. A dice is numbered from 1 to 6 in different ways. If 1 is opposite to 5 and 2 is opposite to 3, then

(a) 4 is adjacent to 3 and 6
(b) 2 is adjacent to 4 and 6
(c) 4 is adjacent to 5 and 6
(d) 6 is adjacent to 3 and 4

18. Three positions (a, b and c) of a dice are shown in the following figures:



(a) (b) (c)

Which number lies opposite to 6?

(a) 2 (b) 4 (c) 1 (d) 5

Direction for Q19–Q20: A dice is prepared in the following manner:

1. 1 should be between 2 and 3.
2. 2 should be opposite to 3.
3. 4 should lie between 5 and 6.
4. 5 and 6 should lie opposite to each other.
5. 4 should lie face down.

19. The face opposite to 1 is

(a) 2 (b) 4 (c) 5 (d) 6

20. The faces adjacent to 3 are

(a) 1, 2, 5 and 6 (b) 2, 4, 5 and 6
(c) 1, 2, 4 and 5 (d) 1, 4, 5 and 6

ANSWERS

1. (a)	**3.** (b)	**5.** (d)	**7.** (a)	**9.** (d)	**11.** (a)	**13.** (c)	**15.** (d)	**17.** (b)	**19.** (b)
2. (a)	**4.** (c)	**6.** (b)	**8.** (d)	**10.** (b)	**12.** (b)	**14.** (a)	**16.** (c)	**18.** (c)	**20.** (d)

EXPLANATIONS AND HINTS

1. (a) Cuboid has to be painted with green colour on set of opposite faces of 6×8.

Hence, number of green coloured cubelets $= (6-2) \times (8-2) = 24$

Now there are two such faces hence maximum total number of only green coloured cubelets $= 24 \times 2 = 48$

2. (a) The number of cuts when 192 is factorized into three closest factors.

Hence, $192 = 4 \times 6 \times 8$

Therefore, we have two opposite faces of 4×6, 6×8 and 4×8.

Now, the minimum number of cuts $= (4-1) + (6-1) + (8-1) = 3 + 5 + 7 = 15$

3. (b) To minimize the number of cubelets with blue and green, blue should be selected on (4×6) or (6×8) and similarly green should be coloured on (6×8) or (4×6) cubelets, respectively, from the edges where green and blue faces will be meeting $=$ $4+4+4+4=16$

4. (c) Only the cubelets from corners of cuboid will be painted with 3 colours.

Hence, 8 cubelets will be coloured in 3 colours.

5. (d) Number of unpainted cubelets $=(4-2)\times(6-2)\times(8-2)=2\times 4\times 6=48$

Solution to Q6–Q10: Number of identical pieces, $336=8\times 7\times 6$

Hence, we need 7 cuts in Z-direction, 6 cuts in Y-direction and 5 cuts in X-direction.

Hence, cube is cut into 8 parts in Z-direction, 7 parts in Y-direction and 6 parts in X-direction.

6. (b) Minimum number of cuts $=(8-1)+(7-1)+(6-1)=18$

7. (a) All pieces from the corners of cube $=2^3=8$

8. (d) Total cubelets with three faces painted are corner faces $=8$.

Total cubelets with two faces painted $=4[(8-2)+(7-2)+(6-2)]=4(6+5+4)=60$

Hence, required cubelets $=60+8=68$

9. (d) Total cubelets with no face painted $=(8-2)(7-2)(6-2)=6\times 5\times 4=120$

Total cubelets with only one face painted

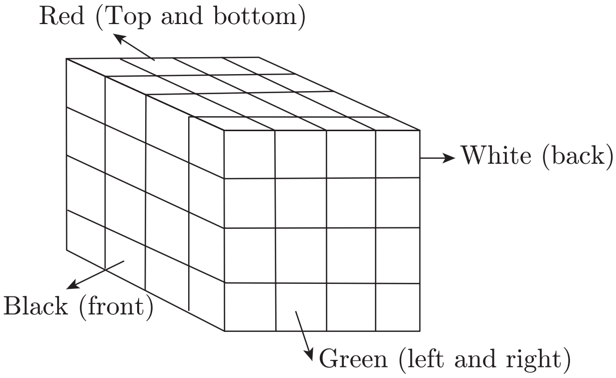
$$2[(8-2)(7-2)+(6-2)(7-2)+(6-2)(8-2)]$$
$$=2(6\times 5+5\times 4+4\times 6)=148$$

Hence, total cubelets with no face painted $=120+48=268$

10. (b) Total cubelets with no face painted

$$(8-2)(7-2)(6-2)=6\times 5\times 4=120$$

Solution to Q11–Q15: Consider the following figure:



Total cubes $=4^3=64$.

Number of cubes with no face painted $=(n-2)^3=(4-2)^3=8$

Number of cubes with only black colour $=(n-2)^2=(4-2)^2=4$

Number of cubes with only green colour $=2(n-2)^2$

$=2\times 4=8$

Number of cubes with red and black colour $=2(n-2)=4$

Number of cubes with red, green and white colour $=4$

11. (a) Total cubes with only green colour $=8$

12. (b) Total cubes with only two colours, red and black $=4$

13. (c) Total cubes with only black colour $=4$

14. (a) Total cubes with no colour $=8$

15. (d) Total cubes with three colours (red, green and white) $=4$

16. (c) Number 3 is common to both Figs. (a) and (b). Now, let us rotate the dice in Fig. (b) in a way such that 3 is unmoved.

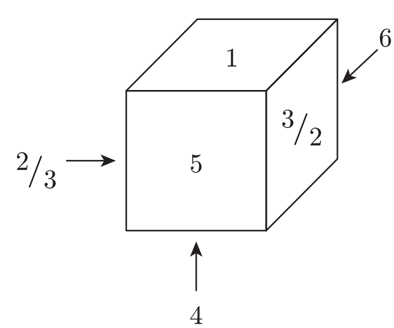
Moreover, the numbers 5 and 2 come on opposite sides of 6 and 1, respectively. Hence, when 2 is at the bottom, 1 is at the top.

17. (b) If 1 is opposite to 5 and 2 is opposite to 3, then 4 is opposite to 6. Therefore, 2 cannot lie opposite to any of the two numbers, 4 or 6. Hence, 2 lies adjacent to 4 and 6.

18. (c) From Figs. (a) and (b), we conclude that 1, 5, 6 and 3 lie adjacent to 4.

Therefore, 2 must lie adjacent to 4. From Figs. (b) and (c), we conclude that 4, 3, 2 and 5 lie adjacent to 6. Thus, 1 must lie opposite to 6.

Solution to Q19 and Q20: Consider the following figure:



19. (b) The face opposite to 1 will be 4.

20. (d) We can conclude that 3 and 2 will be opposite. Hence, 1, 4, 5, 6 are adjacent to 3.

CHAPTER 2

LINE GRAPH

INTRODUCTION

A line graph (or line chart) is a graph which displays information as a series of data points connected by straight line segments. A line graph shows a particular data that change at equal intervals of time.

Line graph is the simplest way to represent data. Single set or multiple sets of data can be shown in a graph. Figure 1 shows a line graph representing average daily temperature for two states, A and B.

Usually, we can analyze the following things using line graphs:

1. Increase in profit in absolute terms or in percentage terms.
2. Average annual growth rate.
3. Average profit.
4. Capability utilization.

$$\text{Capacity utilization} = \frac{\text{Total production}}{\text{Total capacity}} \times 100$$

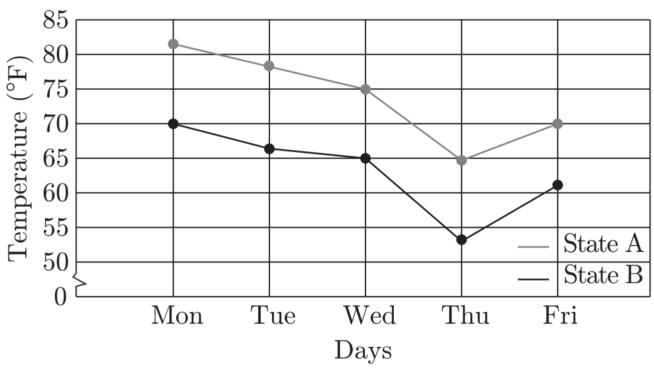


Figure 1 | Line graph representing average daily temperature for two states.

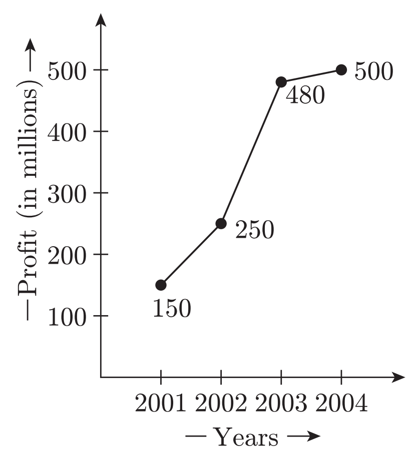
SOLVED EXAMPLES

1. The total capacity of a Hyundai i10 plant is 150 cars per day. In the month of April 2009, the plant manufactured at the rate of 120 cars per day. Find the capacity utilization in the month of April.

Solution: We know that capacity utilization is given by

$$\text{Capacity utilization} = \frac{\text{Total production}}{\text{Total capacity}} \times 100$$
$$= \frac{120}{150} \times 100$$
$$= 80\%$$

Direction for Q2 and Q3: Consider the following graph:



Balance sheet of XYZ Corporation

2. What was the percent increase in profit in 2001–02 and 2003–04?

Solution: Percent increase in profit in 2001–2002

$$= \frac{250 - 150}{150} \times 100 = 66.67\%$$

Present increase in profit in 2003–04

$$= \frac{500 - 480}{480} \times 100 = 4.16\%$$

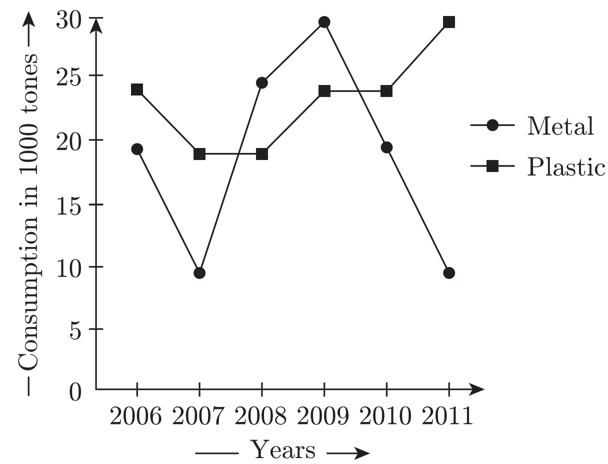
3. What was the average annual growth rate for XYZ Corporation?

Solution:

Average annual growth rate

$$= \frac{\text{Increase in profit for duration}}{\text{Base years profit}} \times \frac{100}{\text{Number of years}}$$
$$= \frac{500 - 150}{150} \times \frac{100}{3}$$
$$= \frac{350}{150} \times \frac{100}{3} = 66.67\%$$

Direction for Q4 and Q5: Consider the following graph:



The above graph is showing consumption of metals versus plastics in the given years (2006–11) for car manufacturing.

4. Which item (and for which year) shows the highest percentage change in consumption over the previous year?

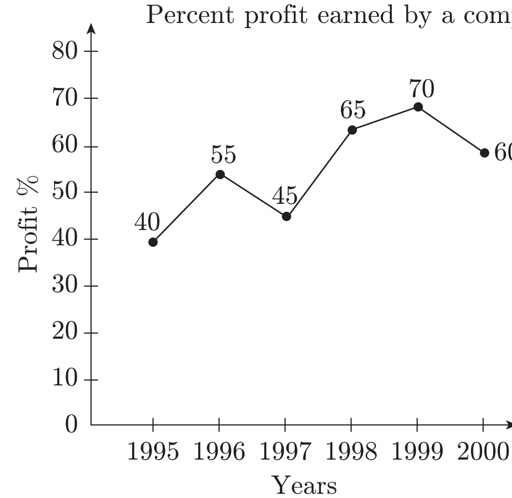
Solution: Percentage change in consumption is highest for metals in 2008.

5. Find out the number of years for which the consumption of metals was less than the consumption of plastic over the given time period.

Solution: For 2006, 2007, 2010 and 2011, the consumption of metals was less than plastics. Hence, for four years the consumption of metals was less than plastics.

Direction for Q6–Q8: Consider the following graph:

Percent profit earned by a company



6. If the expenditures in 1996 and 1999 are equal, then what is the approximate ratio of the income in 1996 and 1999, respectively?

Solution: Let the expenditure in 1996 be x.

Also, let the incomes in 1996 and 1999 be a and b, respectively.

Then, for the year 1996, we have

$$\frac{a-x}{x}\times 100 = 55 \Rightarrow a = \frac{155x}{100} \quad (1)$$

$$\frac{b-x}{x}\times 100 = 70 \Rightarrow b = \frac{170x}{100} \quad (2)$$

From Eqs. (1) and (2), we get

$$\frac{a}{b} = \frac{(155x/100)}{(170x/100)} = \frac{155}{170} \approx 9:10$$

7. What is the average profit earned for the given years?

Solution: Average profit for the given years

$$= \frac{40+55+45+65+70+60}{6} = \frac{335}{6} = 55.83$$

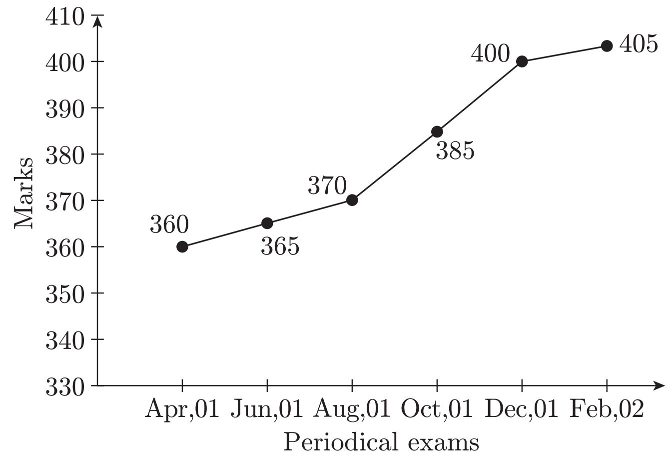
8. If the income in 1998 was ₹264 crores, what was the expenditure in 1998?

Solution: Let the expenditure is 1998 be ₹ x crores.

$$\frac{264-x}{x}\times 100 = 65 \Rightarrow x = \frac{264\times 100}{165} = 160$$

Expenditure in 1998 = ₹160 crores.

Direction for Q9 and Q10: Consider the following graph:



The above graph shows marks obtained by a student in six periodicals held in every two months during the year in the session 2001–2002. In addition, the maximum total marks in each periodical exam are 500.

9. What is the percentage of marks obtained by the student in the periodical exams of August, 01 and October, 01 taken together?

Solution: Required percentage $= \left(\frac{370+385}{500+500}\right)\times 100$

$$= \frac{755}{1000}\times 100 = 75.5\%$$

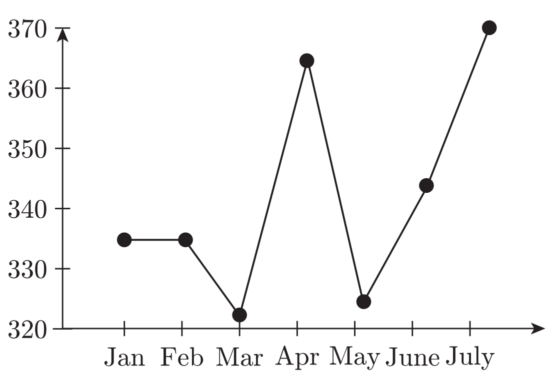
10. What are the average marks obtained by the student in all the periodical exams during the last session?

Solution: Average marks obtained in all the periodical exams $= \left(\frac{360+365+370+385+400+405}{6}\right) = \frac{2285}{6}$

$$= 380.83$$

PRACTICE EXERCISE

Direction for Q1–Q5: The following graph depicts the consumer price index (CPI) of 2000–2001.



Consider the graph and answer the following questions.

1. Which month showed the highest absolute difference in the consumer price index over the previous month?

(a) March (b) April
(c) May (d) July

2. For how many months was the CPI greater than 350?

(a) 1 (b) 2
(c) 3 (d) 4

3. Which month showed the highest percentage differences in the CPI over the previous month?

(a) March (b) April
(c) May (d) July

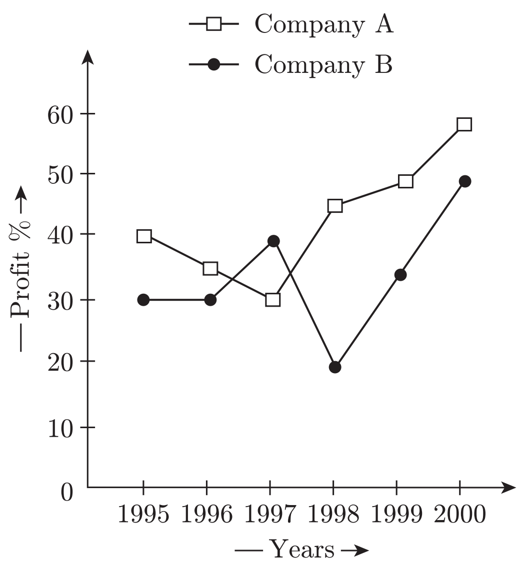
4. The difference in the number of months in which there was an increase in the CPI and the number of months in which there was a decrease was

(a) 1 (b) 2 (c) 3 (d) 4

5. In how many months was there a decrease in the CPI over the previous month?

(a) 1 (b) 2
(c) 3 (d) 4

Direction for Q6–Q10: The following graph shows the annual profit of two companies for six years.



Consider the graph and answer the following questions.

6. If the expenditure of company B in year 1998 was ₹200 crore, what was its income?

(a) ₹360 crore (b) ₹300 crore
(c) ₹240 crore (d) ₹120 crore

7. If the income of company A in year 2000 was ₹600 crore, what was its expenditure?

(a) ₹210 crore (b) ₹275 crore
(c) ₹340 crore (d) ₹375 crore

8. If the income of the two companies were equal in 1996, what was the ratio of their expenditures?

(a) 26:27 (b) 100:67
(c) 1:2 (d) 5:27

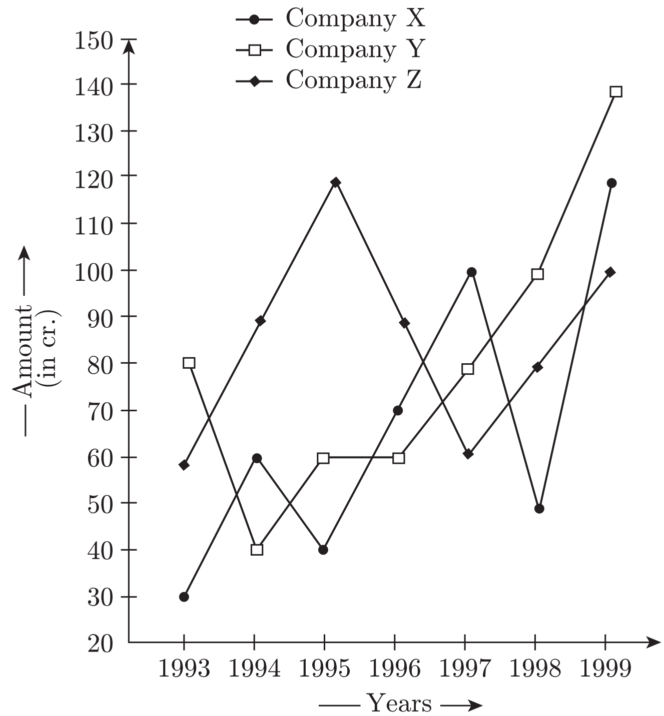
9. If the income of company B in 1996 was ₹200 crore, what was its profit in 1997?

(a) ₹21.5 crore (b) ₹46.15 crore
(c) ₹153 crore (d) Cannot be determined

10. What is the percent increase in the present profit for company B from year 1998 to 1999?

(a) 25% (b) 50%
(c) 75% (d) 90%

Direction for Q11–Q15: The following graph depicts export from three companies over the years.



Consider the graph and answer the following questions.

11. For which of the following pairs of years, the total exports from the three companies together are equal?

(a) 1995 and 1998 (b) 1996 and 1998
(c) 1997 and 1998 (d) 1995 and 1996

12. Average annual exports during the given period for company Y is approximately what percent of the average annual export for company Z?

(a) 93.33% (b) 91.21%
(c) 89.64% (d) 87.12%

13. In which year was the difference between the exports from companies X and Y the minimum?

(a) 1995 (b) 1996
(c) 1997 (d) 1998

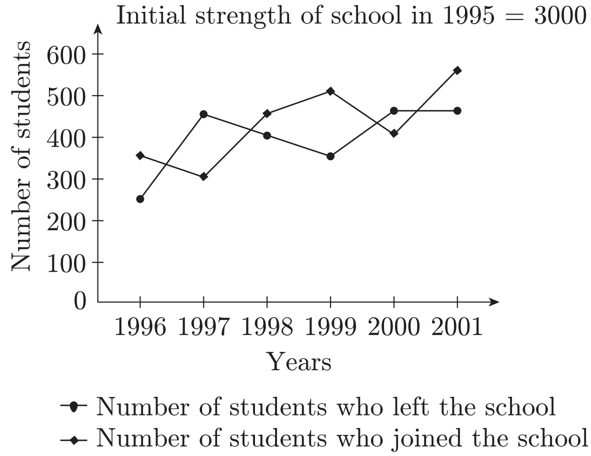
14. What was the difference between the average exports of the three companies in 1993 and average exports in 1998?

(a) ₹20 crore (b) ₹22.17 crore
(c) ₹25.55 crore (d) ₹17.5 crore

15. In how many given years were the exports from company Z more than the average annual exports over the given years?

(a) 1 (b) 2
(c) 3 (d) 4

Direction for Q16–Q20: The following graph depicts the number of students who joined and left the school in the beginning of year for six years, from 1996 to 2001.



Consider the graph and answer the following questions.

16. By approximately what percent, the strength of school increased/decreased from 1997 to 1998?

(a) 1.2% (b) 1.7% (c) 2.1% (d) 2.4%

17. For which year, the percentage rise/fall in the number of students who left the school compared to the previous year is maximum?

(a) 1997 (b) 1998
(c) 1999 (d) 2000

18. How many students were studying in the school during 1999?

(a) 2950 (b) 3000
(c) 3100 (d) 3150

19. The number of students studying in the school in 1998 was what percent of the number of students studying in the school in 2001?

(a) 92.13% (b) 93.75%
(c) 96.88% (d) 97.25%

20. What is the ratio of the least number of students who joined the school to the maximum number of students who left the school in any of the years during the given period?

(a) 7:9 (b) 4:5 (c) 3:4 (d) 2:3

ANSWERS

1. (b)	**4.** (a)	**7.** (d)	**10.** (c)	**13.** (b)	**16.** (b)	**19.** (b)
2. (b)	**5.** (b)	**8.** (a)	**11.** (d)	**14.** (a)	**17.** (a)	**20.** (d)
3. (b)	**6.** (c)	**9.** (d)	**12.** (a)	**15.** (d)	**18.** (d)	

EXPLANATIONS AND HINTS

1. (b) It is visible from the graph given that April showed the highest absolute difference in CPI over the previous month.

2. (b) For April and July, the CPI was greater than 350.

3. (b) It is visible from the graph given that April showed the highest percentage difference in the CPI over the previous month.

4. (a) The CPI increased in three months: April, June and July

The CPI decreased in two months: March and May

Hence, the difference $= 3 - 2 = 1$

5. (b) The CPI decreased in March and May. Hence, there was a decrease in 2 months.

6. (c) Income of company B in $1998 = 200 \times \dfrac{120}{100} =$ ₹240 crore

7. (d) Expenditure of company A in $2000 = 600 \times \dfrac{100}{160} =$ ₹ 375 crore

8. (a) Ratio of the expenditures of companies A and B in $1996 = \dfrac{100}{135} \times \dfrac{130}{100} = 26:27$

9. (d) We can calculate the profit for 1996. However, we do not know the income for 1997 and hence the data are insufficient to calculate the profit in 1997.

10. (c) Total profit percent in 1998 = 20%

Total profit percent in 1999 = 35%

Total percent increase for company B from 1998 to $1999 = \dfrac{35 - 20}{20} \times 100 = 75\%$

11. (d) Total export of companies X, Y, Z in 1995 = ₹$(40 + 60 + 120)$ crore = ₹220 crore

Total export of companies X, Y, Z in 1996 = ₹$(70 + 60 + 90)$crore = ₹220 crore

Total export of companies X, Y, Z in 1997 = ₹$(100 + 80 + 60)$crore = ₹240 crore

Total export of companies X, Y, Z in 1998 = ₹(50 + 100 + 80) crore = ₹230 crore

Hence, export of companies X, Y, Z was same in 1995 and 1996.

12. (a) Average annual exports of company $Y = \frac{1}{7} \times (80 + 40 + 60 + 60 + 80 + 100 + 140) = \frac{560}{7} = 80$

Average annual exports of company $Z = \frac{1}{7} \times (60 + 90 + 120 + 90 + 60 + 80 + 100) = \frac{600}{7}$

Therefore, required percentage $= \left[\frac{80}{(600/7)} \times 100\right]$
$= 93.33\%$

13. (b) Difference between the exports of company X and Y in 1995 = ₹(60 – 40) crore = ₹20 crore

Difference between the exports of company X and Y in 1996 = ₹(70 – 60) crore = ₹10 crore

Difference between the exports of company X and Y in 1997 = ₹(100 – 80) crore = ₹20 crore

Difference between the exports of company X and Y in 1998 = ₹(100 – 50) crore = ₹50 crore

Hence, the difference in exports of company X and company Y is minimum in 1996.

14. (a) Average exports of three companies X, Y, Z in 1993 $= \frac{1}{3} \times (30 + 80 + 60) =$ ₹ $\frac{170}{3}$ crore

Average exports of three companies X, Y, Z in 1998 $= \frac{1}{3} \times (50 + 100 + 80) =$ ₹ $\frac{230}{3}$ crore

Difference $= \left[\left(\frac{230}{3}\right) - \left(\frac{170}{3}\right)\right] = \frac{60}{3} =$ ₹20 crore

15. (d) Average annual exports of company Z during the given period is $\frac{1}{7} \times (60 + 90 + 120 + 90 + 60 + 80 + 100) = \frac{600}{7} =$ ₹85.71 crore

It is visible from the graph given that the exports of company Z are more than the average annual exports of company Z during 1994, 1995, 1996 and 1999, that is, 4 years.

16. (b) From the graph, it can be noted that

- In 1996: Number of students left = 250 and number of students joined = 350.
- In 1997: Number of students left = 450 and number of students joined = 300.
- In 1998: Number of students left = 400 and number of students joined = 450.
- In 1999: Number of students left = 350 and number of students joined = 500.
- In 2000: Number of students left = 450 and number of students joined = 400.
- In 2001: Number of students left = 450 and number of students joined = 550.

Therefore, the number of students studying in the school in various years is as follows:

- In 1995 = 3000 (given).
- In 1996 = 3000 − 250 + 350 = 3100.
- In 1997 = 3100 − 450 + 300 = 2950.
- In 1998 = 2950 − 400 + 450 = 3000.
- In 1999 = 3000 − 350 + 500 = 3150.
- In 2000 = 3150 − 450 + 400 = 3100.
- In 2001 = 3100 − 450 + 550 = 3200.

Therefore, percentage increase in the strength of the school from 1997 to 1998 $= \left[\frac{(3000 - 2950)}{2950} \times 100\right] = 1.69\% \simeq 1.7\%$

17. (a) The percentage rise/fall in the number of students who left the school (compared to the previous year) during various years is:

For 1997 $= \left[\frac{(450 - 250)}{250} \times 100\right]\% = 80\%$ rise

For 1998 $= \left[\frac{(450 - 400)}{450} \times 100\right]\% = 11.11\%$ fall

For 1999 $= \left[\frac{(400 - 350)}{400} \times 100\right]\% = 12.5\%$ fall

For 2000 $= \left[\frac{(450 - 350)}{350} \times 100\right]\% = 28.57\%$ rise

For 2001 $= \left[\frac{(450 - 450)}{450} \times 100\right]\% = 0\%$ rise

Clearly, the maximum percentage rise/fall is for 1997.

18. (d) The number of students studying in the school in 1999 = 3000 − 350 + 500 = 3150

19. (b) Total students studying in the school in 1998 = 3000

Total students studying in the school in 2001 = 3200

Required percentage $= \frac{3000}{3200} \times 100 = 93.75\%$

20. (d) Least number of students who joined in any year = 300

Maximum number of students who left in any year = 450

Required ratio $= \frac{300}{450} = \frac{2}{3}$

CHAPTER 3

TABLES

INTRODUCTION

A table is used to represent data in rows and columns. Tables are arguably the most common way of arranging a given data. Evidently, tables are used almost everywhere, in print media, books, presentation slides, etc. Tables differ significantly in variety, structure, flexibility, notation, representation and use.

Tables are advantageous to us as they can arrange a large amount of data easily and without any complexity. However, locating pattern or visual observations are difficult.

SOLVED EXAMPLES

Direction for Q1–Q5: The following table shows sales of cars of five companies over the years (in thousand).

Company	**Years**				
	2000	**2001**	**2002**	**2003**	**2004**
Honda	20	21	50	35	75
Hyundai	29	31	23	46	42
Maruti	31	29	27	22	16
Tata	33	14	33	37	48
Skoda	15	17	32	39	47
Total	128	112	165	179	228

Consider the table and answer the following questions.

1. For which company did the amount of sales of cars increase continuously over the years?

Solution: For Skoda, we can see that the sales of each year are always greater than the previous year.

2. For which company did the amount of sales of cars decrease continuously over the years?

Solution: For Maruti, we can see that the sales of each year are always lesser than the previous year.

3. In 2001, which company had the biggest increase in sales from 2000?

Solution: Three companies had more sales in 2001 than 2000.

Percentage increase for Honda $= \frac{21-20}{20} \times 100 = 5\%$

Percentage increase for Hyundai $= \frac{31-29}{29} \times 100 =$ 6.9%

Percentage increase for Skoda $= \frac{17-15}{15} \times 100 =$ 13.33%

Hence, Skoda had the biggest percentage increase in sales of 2001.

4. In which year did Maruti have the largest percentage decrease in sales?

Solution: Percentage decrease in sales in 2001 $= \frac{31-29}{31} \times 100 = 6.45\%$

Percentage decrease in sales in 2002 $= \frac{29-27}{29} \times 100 = 6.9\%$

Percentage decrease in sales in 2003 $= \frac{27-22}{27} \times 100 = 18.52\%$

Percentage decrease in sales in 2004 $= \frac{22-16}{22} \times 100 = 27.27\%$

Hence, in 2004, the decrease in sales for Maruti was the highest.

5. Which company sold the most number of cars in the five years?

Solution: Total sales of Honda = 20 + 21 + 50 + 35 + 75 = 201000
Total sales of Hyundai = 29 + 31 +23 + 46 + 42 = 171000
Total sales of Maruti = 31 + 29 + 27 + 22 + 16 = 125000
Total sales of Tata = 33 + 14 + 33 + 37 + 48 = 165000
Total sales of Skoda = 15 + 17 + 32 + 39 + 47 = 150000

Hence, Honda had the highest sales from 2000 to 2004.

Direction for Q6–Q10: The following table shows total revenue of five companies over the years (in \$ millions).

Company	**Years**			
	2000	**2001**	**2002**	**2003**
A	20	17	8	18
B	10	12	16	20
C	17	15	4	2
D	6	16	3	9
E	18	30	10	11

Consider the table and answer the following questions.

6. What was the total revenue of company D from 2000 to 2003?

Solution: Total revenue of D = 6 + 16 + 3 + 9 = \$ 34 million

7. What was the revenue of all the companies in the year 2002?

Solution: Total revenue is 2002 = 8 + 16 + 4 + 3 + 10 = \$ 41 million

8. From 2000 to 2003, in which company was there a continuous increase in revenue?

Solution: Company B had a continuous increase in revenue from 2000 to 2003.

9. In which two years did company B and D have the same amount of revenue?

Solution: In 2000 and 2001, revenue of company B = 10 + 12 = \$ 22 million
In 2000 and 2001, revenue of company D = 6 + 16 = \$ 22 million
Hence, in 2000 and 2001, the sum of revenue of B and D is same.

10. What was the percentage decrease in total revenue from 2001 to 2003?

Solution: Total revenue in 2001 = 17 + 12 + 15 + 16 + 30 = \$ 90 million
Total revenue in 2003 = 18 + 20 + 2 + 9 + 11 = \$ 60 million
Percentage decrease in revenue from 2001 to 2003 = $\frac{90-60}{90} \times 100 = 33.33\%$

PRACTICE EXERCISE

Direction for Q1–Q5: The following table depicts countries foreign trade (in crores) from 1990–1991 to 1995–1996.

Year	Export	Import	Deficit
1990–1991	4661	8248	3587
1991–1992	5790	11606	5816
1992–1993	8019	13750	5731
1993–1994	6756	14110	7354
1994–1995	10420	18300	7880
1995–1996	12550	20010	7460

Consider the table and answer the following questions.

1. Which of the following showed an increase every year?

(a) Exports (b) Imports
(c) Deficit (d) All of the above

2. In which year the ratio of imports to exports was maximum?

(a) 1993–94 (b) 1992–93
(c) 1991–92 (d) 1990–91

3. The percentage increase in exports over previous year was maximum in which year?

(a) 1991–1992 (b) 1992–1993
(c) 1994–1995 (d) 1995–1996

4. What was the total trade deficit for the last three years?

(a) ₹22694 crore (b) ₹21745 crore
(c) ₹18980 crore (d) ₹24407 crore

5. In which year was the difference between imports and exports maximum?

(a) 1990–1991 (b) 1991–1992
(c) 1992–1993 (d) 1994–1995

Direction for Q6–Q10: The following table depicts productions of main crops in India (in million tons).

Crops	Years				
	2000–01	2001–02	2002–03	2003–04	2004–05
Pulses	20.5	24.6	28.2	23.5	22.4
Oats	32.4	40.8	52.4	42.4	34.6
Rice	80.4	88.2	90.8	92.6	86.4
Sugarcane	140.8	152.2	172.5	160.3	150.2
Wheat	130.2	146.8	158.4	141.6	138.4

Consider the table and answer the following questions.

6. What was the total amount of crops produced in 2002–03?

(a) 479.5 million tons
(b) 510.7 million tons
(c) 502.3 million tons
(d) 468.4 million tons

7. What was the total production of rice?

(a) 410.2 million tons
(b) 422.5 million tons
(c) 438.4 million tons
(d) 472. 6 million tons

8. Production of which type of crop was always increasing in the given years?

(a) Pulses (b) Oats
(c) Sugarcane (d) None

9. What was the average production of oats in the given years?

(a) 40.52 million tons
(b) 39.65 million tons
(c) 42.25 million tons
(d) 37.28 million tons

10. Production of wheat was what percentage of total production in 2002–03?

(a) 29.71% (b) 32.06%
(c) 34.75% (d) 31.53%

Direction for Q11–Q15: The following table consists of average number of people receiving certain assistance and the cost required for that assistance.

Category of assistance	Average number receiving help		Total cost of help (in crores, ₹)		Cost paid by Centre (in crores, ₹)	
	1995	1996	1995	1996	1995	1996
A	36097	38263	38.4	34.8	18.4	17.4
B	6632	5972	5.0	3.2	2.6	1.6
C	32545	31804	76.4	59.4	13.0	10.0
D	13992	11782	26.4	42.6	6.6	10.6
E	21275	228795	216.6	242.8	55.0	62.6

Consider the table and answer the following questions.

11. The category receiving the least percentage help from the centre (in the entire data) is

(a) Category B in 1995
(b) Category C in 1996
(c) Category B in 1996
(d) Category D in 1995

12. The difference between the average costs paid by the centre during 1995 and 1996 is

(a) ₹66 lakh (b) ₹13.2 crore
(c) ₹132 lakh (d) ₹13.2 lakh

13. Monthly cost of the city receiving E category assistance in 1996 is nearly

(a) ₹1.8 crore less than that in 1995
(b) ₹2.1 crore more than that in 1995
(c) ₹2.1 crore less than that in 1995
(d) ₹1.8 crore more than that in 1995

14. Assuming that 50% of the persons receiving category B help in 1995 were adults caring for minor children, but the city's contribution towards maintaining these adults was 40% of the total contribution to B program in 1995, average amount paid by the city for each adult per year in 1995 is nearly

(a) ₹5900 (b) ₹6000
(c) ₹7500 (d) ₹3000

15. Monthly costs to the city of category D during 1995 and 1996 bear a ratio

(a) 2:3 (b) 5:3
(c) 3:2 (d) 3:5

Direction for Q16–Q20: The following table depicts the number of candidates that appeared and qualified in a competitive examination from different states over the years.

State	Year									
	1997		1998		1999		2000		2001	
	Appeared	Qualified	Appeared	Qualified	Appeared	Qualified	Appeared	Qualified	Appeared	Qualified
M	5200	720	8500	980	7400	850	6800	775	9500	1125
N	7500	840	9200	1050	8450	920	9200	980	8800	1020
P	6400	780	8800	1020	7800	890	8750	1010	9750	1250
Q	8100	950	9500	1240	8700	980	9700	1200	8950	995
R	7800	870	7600	940	9800	1350	7600	945	7990	885

Consider the table and answer the following questions.

16. Total number of candidates qualified from all stages together in 1997 is approximately what percentage of the total number of candidates qualified from all the states together in 1998?

(a) 80% (b) 77%
(c) 75% (d) 73%

17. What is the average number of candidates who appeared from state Q during the given years?

(a) 8700 (b) 8760
(c) 8990 (d) 8920

18. In which of the given years, the number of candidates that appeared from state P has maximum percentage of qualified candidates?

(a) 1997 (b) 1998
(c) 1999 (d) 2001

19. Combining the states P and Q together in 1998, what is the percentage of the qualified candidates to that of the appeared candidates?

(a) 10.87% (b) 11.49%
(c) 12.35% (d) 12.54%

20. What is the percentage of candidates that qualified over the candidates that appeared from state N during all the years together?

(a) 12.36% (b) 12.16%
(c) 11.47% (d) 11.15%

ANSWERS

1. (b)	**5.** (d)	**9.** (a)	**13.** (b)	**17.** (c)
2. (a)	**6.** (c)	**10.** (d)	**14.** (b)	**18.** (d)
3. (c)	**7.** (c)	**11.** (b)	**15.** (d)	**19.** (c)
4. (a)	**8.** (d)	**12.** (c)	**16.** (a)	**20.** (d)

EXPLANATIONS AND HINTS

1. (b) As it is visible from the table, imports showed an increase every year.

2. (a) Ratio of import to export in 1990–1991 =
$\frac{8248}{4661} = 1.77$
Ratio of import to export in 1991–1992 =
$\frac{11606}{5790} = 2.00$
Ratio of import to export in 1992–1993 =
$\frac{13750}{8019} = 1.71$
Ratio of import to export in 1993–1994 =
$\frac{14110}{6756} = 2.09$
Hence, the ratio was maximum for 1993–94.

3. (c) Percentage increase in 1991–1992 =
$\frac{5790-4661}{4661} \times 100 = 24.22\%$
Percentage increase in 1992–1993 =
$\frac{8019-5790}{5790} \times 100 = 38.497\%$
Percentage increase in 1994–1995 =
$\frac{10420-6756}{6756} \times 100 = 54.23\%$
Percentage increase in 1995–1996 =
$\frac{12550-10420}{10420} \times 100 = 20.44\%$

Hence, the maximum percentage increase was in 1994–1995.

4. (a) Total trade deficit for the last three years = $(7354 + 7880 + 7460)$ = ₹22694 crore

5. (d) As it is visible from the table, the maximum trade deficit was in 1994–1995.

6. (c) Total crops production in 2002–03 = 28.2 + 52.4 + 90.8 + 172.5 + 158.4 = 502.3 million tons

7. (c) Total production of rice = 80.4 + 88.2 + 90.8 + 92.6 + 86.4 = 438.4 million tons

8. (d) As it is visible from the table, production of all caps decreases at least once in five years.

9. (a) The average production of oats is the given years $= \frac{32.4 + 40.8 + 52.4 + 42.4 + 34.6}{5}$

$= \frac{202.6}{5} = 40.52$ million tons

10. (d) Total production of wheat in 2002–03 = 158.4 million tons
Total production of all crops in 2002–03 = 502.3 million tons
Total percentage $= \frac{158}{502.3} \times 100 = 31.53\%$

11. (b) Percentage for category B in 1995 $= \left(\frac{2.6}{5.0}\right) \times 100 = 52\%$

Percentage for category C in 1996 $=\left(\dfrac{10.0}{59.4}\right)\times 100=$ 16.8%

Percentage for category B in 1996 $=\left(\dfrac{1.6}{3.2}\right)\times 100=$ 50%

Percentage for category B in 1995 $=\left(\dfrac{55.0}{216.6}\right)\times 100=$ 25.4%

Thus, the least percentage of help from Center is for category C in 1996.

12. (c) The average cost in 1995 $= \dfrac{18.4+2.6+13.0+6.6+55.0}{5}=\dfrac{95.6}{5}=$ ₹19.12 crore

The average cost in 1996 $= \dfrac{17.4+1.6+10.0+10.6+62.6}{5}=\dfrac{102.2}{5}=$ ₹20.44 crore

Difference $= 20.44-19.12=$ ₹1.32 crore = ₹132 lakh

13. (b) Monthly cost of city receiving E category assistance in 1995 $=\dfrac{216.6}{12}=\ 18.05$

Monthly cost of city receiving E category assistance in 1996 $=\dfrac{242.8}{12}=20.23$

Difference $=20.23-18.05=2.183$ crore $\simeq$ ₹2.1 crore

14. (b) 50% of persons receiving B category help during 1995 = 3316

City's contribution to maintenance $=\dfrac{40}{100}\times 5=$ ₹2 crore

Therefore, average amount paid $=\dfrac{20000000}{3316}=$ 6031.36 $\simeq$ ₹6000

15. (d) Monthly costs of category D in 1995 $=\dfrac{26.4}{12}$

Monthly costs of category D in 1996 $=\dfrac{42.6}{12}$

Therefore, required ratio = 26.4:42.6 = 3:5.

16. (a) Total candidates who qualified from all states in 1997 = 720 + 840 + 780 + 950 + 870 = 4160

Total candidates who qualified from all states in 1998 = 980 + 1050 + 1020 + 1240 + 940 = 5230

Required percentage $=\left(\dfrac{4160}{5230}\times 100\right)=79.54\% \simeq 80\%$

17. (c) Total students who appeared from state Q in the given five years = 8100 + 9500 + 8700 + 9700 + 8950 = 44950

Therefore, required average $=\dfrac{44950}{5}=8990$

18. (d) The percentage of qualified candidates to appeared candidates from state P during different years is:

For 1997: $\left(\dfrac{780}{6400}\right)\times 100=12.19\%$

For 1998: $\left(\dfrac{1020}{6400}\times 100\right)=11.59\%$

For 1999: $\left(\dfrac{890}{7800}\times 100\right)=11.41\%$

For 2000: $\left(\dfrac{1010}{8750}\times 100\right)=11.54\%$

For 2001: $\left(\dfrac{1250}{9750}\times 100\right)=12.82\%$

Therefore, the maximum percentage is for the year 2001.

19. (c) Total candidates who qualified in 1998 for P and Q = 1020 + 1240 = 2260

Total candidates who applied in 1998 for P and Q = 8800 + 9500 = 18300

Therefore, required percentage $=\left(\dfrac{2260}{18300}\times 100\right)\%=$ 12.35%

20. (d) Total candidates who qualified from state N = 840 + 1050 + 920 + 980 + 1020 = 4810

Total candidates who appeared from state N = 7500 + 9200 + 8450 + 9200 + 8800 = 43150

Therefore, required percentage $=\left(\dfrac{4810}{43150}\times 100\right)=$ 11.15%

CHAPTER 4

BLOOD RELATIONSHIP

INTRODUCTION

GATE comprises of various questions on blood relations. The easiest and non-confusing way to solve these questions is to draw a family tree diagram and increase the levels in the hierarchy as follows:

1. **First stage:** *Grandparents* – Grandfather, grandmother, grand uncle and grand aunt.
2. **Second stage:** *Parents and in-laws* – Father, mother, uncle, aunt, father-in-law and mother-in-law.
3. **Third stage:** *Siblings, spouse and in-laws* – Brother, sister, cousin, wife, husband, brother-in-law and sister-in-law.
4. **Fourth stage:** *Children and in-laws* – Son, daughter, niece, nephew, son-in-law and daughter-in-law.
5. **Fifth stage:** *Grandchildren* – Grandson and granddaughter.

Some of the common relationships are given in Table 1.

Table 1 | Description of relation

Description	Relation
Mother's/Father's brother	Uncle
Mother's/Father's sister	Aunt
Mother's/Father's mother	Grandmother
Mother's/Father's father	Grandfather
Grandmother's/Grandfather's brother	Grand uncle
Grandmother's/Grandfather's sister	Grand aunt
Children's children	Grandchildren
Uncle or Aunt's son/daughter	Cousins
Daughter's husband	Son-in-law
Son's wife	Daughter-in-law
Sister's/Brother's daughter	Niece
Sister's/Brother's son	Nephew
Wife's/Husband's brother or sister's husband	Brother-in-law
Wife's/Husband's sister or brother's wife	Sister-in-law

STANDARD CODING TECHNIQUE

1. Proper nouns should always be denoted by capital letters.
   Example: Mark and Rachel can be coded as Mark – M, Rachel – R.
2. In questions where the gender is of importance to solving it, you can denote females by underlining the notation used.
   Example: Eve is the only girl student in her course. Eve – $\underline{\text{E}}$.
3. When defining a relationship, always put the elder generation on top of the younger generation. Hence, grandparents come above parents and parents come above children.
   Example: A is grandfather of B. B is mother of two boys C and D. This can be coded as follows:

   A

   $\underline{\text{B}}$

   C D
4. When defining spousal relationship, use →. Conventionally, write the male on the left and the female on the right.
   Example: Aman is Prerna's husband. A → $\underline{\text{P}}$.
5. When defining a relationship between siblings, use '..'.
   Example: Sana and Shreya are sisters. $\underline{\text{S}}$..$\underline{\text{SH}}$
   Anupam is Kittu's brother. A..$\underline{\text{K}}$
6. When the gender of a person is not confirmed, write it in a box.
   Example: A is the only son of B.

   $\boxed{\text{B}}$

   A

SOLVED EXAMPLES

1. Leena and Meena are Rahul's wives. Shalini is Meena's stepdaughter. How is Leena related to Shalini?

   Solution: Shalini is Meena's stepdaughter. Hence, Shalini is the daughter of the other wife of Rahul, so Shalini is the daughter of Leena or Leena is Shalini's mother.

2. Pointing to a man in a photograph a man said to a woman "His mother is the only daughter of your father." How is the woman related to the man in the photograph?

   Solution: The only daughter of the woman's father is the woman herself, and hence the man in the photograph is her son. Therefore, the woman is the mother of the man in the photograph.

3. A party consists of grandmother, father, mother, four sons and their wives, and one son and two daughters of each of the sons. How many females are there?

   Solution: Grandmother is one female, mother is another, wives of four sons are another 4 and two daughters of each son make it 8 more females. Hence, total females = 1 + 1 + 4 + 8 = 14

4. Introducing Marcelle to guests, Aman said, "Her father is the only son of my father." How is Marcelle related to Aman?

   Solution: According to Aman, Marcelle's father is the only son of Aman's father. Hence, Aman is Marcelle's father. Therefore, Marcelle is the daughter of Aman.

Direction for Q5 and Q6: A + B means A is the daughter of B. $\text{A} \times \text{B}$ means A is the son of B. A – B means A is the wife of B.

5. What is the meaning of $\text{P} \times \text{Q} - \text{S}$?

   Solution: $\text{P} \times \text{Q} - \text{S}$ means P is the son of Q who is the wife of S.

6. If $\text{Z} \times \text{T} - \text{S} \times \text{U} + \text{P}$, then what is the relationship between U and Z?

   Solution: $\text{Z} \times \text{T} - \text{S} \times \text{U}$ means Z is the son of T who is the wife of S who is the son of U, that is Z is the son of S who is the son of U, that is Z is the grandson of U or U is the grandmother of Z.

Direction for Q7–Q10: A is married to B and L is A's brother-in-law. A has two daughters. I is the cousin brother of J and is the brother of K. E and F are B's son-in-laws. E has two daughters and one son, F has one son and one daughter. G and H are two daughters of C. K and A share a granddaughter and a grandfather relationship. D is also the member of this family.

7. How is C related to I?

8. How is L related to D?

9. How is E related to C?

10. How is G related to B?

Solution: Now, A – B..L

A has two daughters, D_1 and D_2.

I is K's brother and cousin of J. Hence, I..K, J

E and F are husbands of D_1 and D_2. Therefore,

$$E - D_1..D_2 - F$$

Now, E has two daughters and one son,

$$E - D_1$$

$$D_1'..D_2'..s_1'$$

and D_2 – F has one son and one daughter

$$F - D_2$$

$$D_3'..s_2'$$

G and H are daughters of C, so C is wife of E because only E has two daughters.

Now, the complete tree can be formed as follows:

$$A - B..L$$

$$E - \boxed{C}..\boxed{D} - F$$

$$\downarrow \quad \downarrow$$

$$\boxed{H}..\boxed{G}..J, I..\boxed{K}$$

7. C is the sister of D who is I's mother. Hence, C is the aunt of I.

8. D is daughter of B, L is B's brother. Thus, L is D's uncle.

9. E and C have two daughters and E is the son-in-law of A and B. C is daughter of A and B. Thus, E is husband of C.

10. We know that G is daughter of C. Also, C is daughter of B.

Hence, G is B's granddaughter.

Direction for Q11–Q12: A + B means A is the daughter of B.

A + B means A is the daughter of B.

A × B means A is the brother of B.

11. What is the meaning of $P + Q \times R$?

Solution: $P + Q \times R$ means that P is the daughter of Q who is the brother of R.

12. If $P \times Q + R$, then what is the relation between P and R?

Solution: $P \times Q + R$ means P is the brother of Q who is the daughter of R. Hence, P is the son of R.

Direction for Q13–Q15: Read the information carefully and answer the following questions.

1. A family consists of 6 members P, Q, R, X, Y, Z.

2. Q is the son of R but R is not the mother of Q.

3. P and R is a married couple.

4. Y is the brother of R, X is the daughter of P.

5. Z is the brother of P.

13. How is Q related to X?

Solution: From the information given above, we can draw the following tree:

$$Y..R \rightarrow \underline{P}..Z$$

$$Q..\underline{X}$$

Thus, Q is the brother of X.

14. How is Y related to P?

Solution: Y's brother is P's husband. Hence, Y is P's brother in law.

15. How many females are there in the family?

Solution: There are two females in the family, P and X.

PRACTICE EXERCISE

Direction for Q1–Q5: Read the following information carefully and answer the questions. All the six members of a family P, Q, R, S, T and U are travelling together. Q is the son of R but R is not the mother of Q, and P and R are a married couple. T is the brother of R. S is the daughter of P. U is the brother of Q.

1. Who is the mother of Q?

(a) S (b) U
(c) T (d) P

2. How many children does P have?

(a) One (b) Two
(c) Three (d) Four

3. How is U related to S?

(a) Father (b) Brother
(c) Uncle (d) Indeterminate

4. Who is the wife of T?

(a) P (b) U
(c) Q (d) Indeterminate

5. How many male members are there in the family?

(a) 1 (b) 2
(c) 3 (d) 4

6. Looking at the portrait of a man, Tinku said, "His mother is the wife of my father's son. I have no brothers or sisters." At whose portrait was Tinku looking?

(a) His child (b) His father
(c) His grandson (d) His brother

7. If $A + B \rightarrow A$ is the mother of B

$A - B \rightarrow A$ is the brother of B

$A\%B \rightarrow A$ is the father of B

$A \times B \rightarrow A$ is the sister of B

then which of the following shows P is the maternal uncle of Q?

(a) $P - M + N \times Q$ (b) $Q - N + M \times P$
(c) $Q - S\%P$ (d) $P + S \times N - Q$

8. If A is the brother of B, B is the sister of C and C is the father of D, how is D related to A?

(a) Brother (b) Sister
(c) Uncle (d) Indeterminate

9. Pointing to a gentleman, Deepak said "His only brother is the father of my daughter's father." How is the gentleman related to Deepak?

(a) Brother (b) Father
(c) Uncle (d) Grandfather

10. If $A + B$ means A is the brother of B

$A - B$ means A is the sister of B

and $A \times B$ means A is the father of B,

then which of the following means that C is the son of M?

(a) $M - N \times C + F$
(b) $M \times N - C + F$
(c) $F - C + N \times M$
(d) $N + M - F \times C$

Direction for Q11–Q13: P is the son of Q. Q's sister has a son S and daughter T. U is the maternal uncle of S.

11. How many nephew(s) does U have?

(a) Zero (b) One
(c) Two (d) Three

12. How is P related to S?

(a) Cousin (b) Nephew
(c) Uncle (d) Brother

13. How is T related to U?

(a) Sister
(b) Daughter
(c) Niece
(d) Wife

Direction for Q14–Q16: $A + B$ means A is the daughter of B. $A - B$ means A is the husband of B. $A \times B$ means A is the brother of B.

14. For $P + Q - R$, which of the following is true?

(a) R is the mother of P
(b) R is the sister-in-law of P
(c) R is the aunt of P
(d) R is the mother-in-law of P

15. For $P \times Q + R$, which of the following is true?

(a) P is the brother of R
(b) P is the uncle of R
(c) P is the son of R
(d) P is the father of R

16. For $P + Q \times R$, which of the following is true?

(a) P is the niece of R
(b) P is the daughter of R
(c) P is cousin of R
(d) P is the daughter-in-law of R

Direction for Q17–Q20: A family consists of six members P, Q, R, X, Y and Z. Q is the son of R but R is not the mother of Q. P and R are married couple. Y is the brother of R, X is the daughter of P. Z is the brother of P.

17. Who is the brother-in-law of R?

(a) P (b) Z
(c) Y (d) X

18. How many children does P have?

(a) One (b) Two
(c) Three (d) Four

19. How many female members are there in the family?

(a) One (b) Two
(c) Three (d) Four

20. Which is a pair of brothers?

(a) P and X (b) P and Z
(c) Q and X (d) R and Y

ANSWERS

1. (d)	**5.** (d)	**9.** (c)	**13.** (c)	**17.** (b)
2. (c)	**6.** (a)	**10.** (b)	**14.** (a)	**18.** (b)
3. (b)	**7.** (a)	**11.** (c)	**15.** (c)	**19.** (b)
4. (d)	**8.** (d)	**12.** (a)	**16.** (a)	**20.** (d)

EXPLANATIONS AND HINTS

Solution to Q1–Q5:

Q is son of R

R is not the mother, R is the father of Q.

P and R are married couple, hence, $\underline{P} - R$

T is R's brother, hence $T..R - \underline{P}$

S is P's daughter.

U is brother of Q, hence $Q..U$

The tree can be formed as follows:

$$\begin{array}{l} T..R - \underline{P} \\ Q..U..\underline{S} \end{array}$$

1. (d) The mother of Q is the wife of R which is P.

2. (c) P has three children, Q, U and S.

3. (b) U is the brother of S.

4. (d) There is no mention of T's wife. Hence, it is indeterminate.

5. (d) There are four male members in the family, T, R, Q and U.

6. (a) Tinku has no brothers or sisters, so Tinku is single child. Wife of father's son is Tinku's wife.

Tinku's wife is the mother of the person whose portrait it is. So portrait is of Tinku's child.

7. (a) P is the brother of $M \to P - M$

M is the mother of $N \to M + N$

N is the sister of $Q \to N \times Q$

Hence, if P is the maternal uncle of Q, then $P - M + N \times Q$.

8. (d) If D is male, the answer is nephew.

And if D is female, then the answer is niece.

However, the sex of D is not known, the relation of D and A cannot be determined.

9. (c) Father of my daughter's father is Deepak's father.

Brother of Deepak's father is Deepak's uncle.

10. (b) M is the father of $N \to M \times N$

N is the sister of $C \to N - C$

C is the brother of $F \to C + F$

Thus, M is the father of C if $M \times N - C + F$.

Solution to Q11–Q13: From the data given in the question, we can form the tree as follows:

$$\begin{array}{lll} Q.. & \underline{R} & ..U \\ \mid & \mid & \\ P & S.. & \underline{T} \end{array}$$

11. (c) U has two nephews, a son of R and a son of Q.

12. (a) P is cousin brother of S.

13. (c) T is the daughter of the sister of U. Hence, T is the niece of U.

14. (a) $P + Q - R$ means P is the daughter of Q who is the husband of R, i.e. R is P's mother.

15. (c) $P \times Q + R$ means P is the brother of Q who is the daughter of R, i.e. P is the son of R.

16. (a) $P + Q \times R$, means P is the daughter of Q who is the brother of R, i.e. P is the niece of R.

Solution to Q17–Q20: From the data given we can form the family tree,

$$\mathrm{Y..R - \underline{P}..Z}$$
$$\mathrm{Q..\underline{X}}$$

17. (b) Brother-in-law of R is brother of P, i.e. Z.

18. (b) P has two children Q and X.

19. (b) There are two female members, P and X.

20. (d) R and Y are the only pair of brothers.

CHAPTER 5

BAR DIAGRAM

INTRODUCTION

A bar diagram or bar graph is a chart with rectangular bars in which the lengths of bars are in proportion to the values of the entities they represent.

A vertical bar chart is called a column bar chart (Fig. 1a) and a horizontal bar chart is called a row bar chart (Fig. 1b).

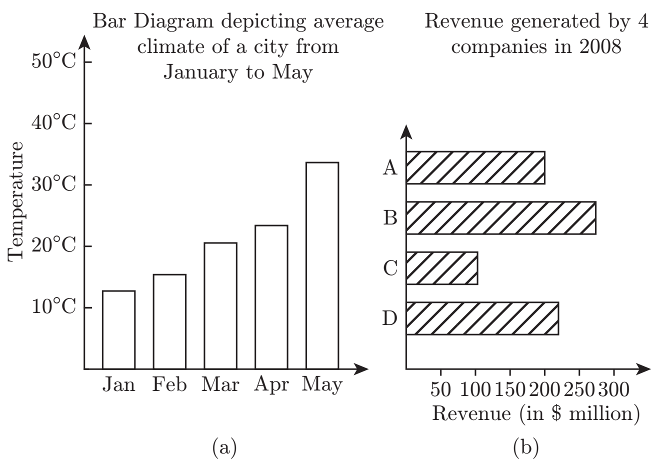


Figure 1 Bar diagram: (a) Column bar chart and (b) row bar chart.

Bar diagrams are often used to provide visual presentation of categorical data and are considered very reliable. Bar graphs can also be used for comparison of complex data. Also, a bar chart is very useful for displaying discrete data values.

Multiple sets of interrelated data can be represented using a multiple bar diagram. It is used to compare more than one quantity. The technique of a simple bar diagram is used to draw the diagram. However, different quantities are represented using different shapes, sizes, colours, patterns, etc.

Figure 2 shows a regular multiple bar diagram.

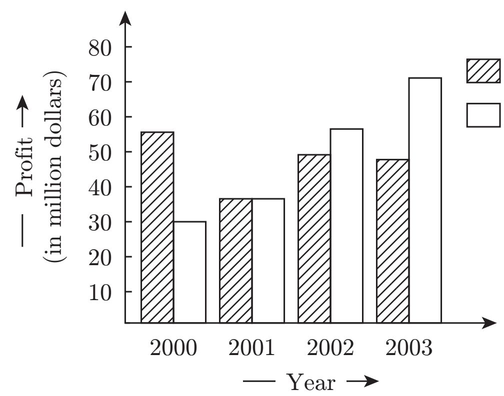


Figure 2 Multiple bar diagram.

A compound bar diagram is used to combine or compare two or more types of information in one chart. The bars are stacked on top of one another. However, different quantities are represented using different shapes, sizes, colours, patterns, etc. Figure 3 shows a regular compound bar diagram.

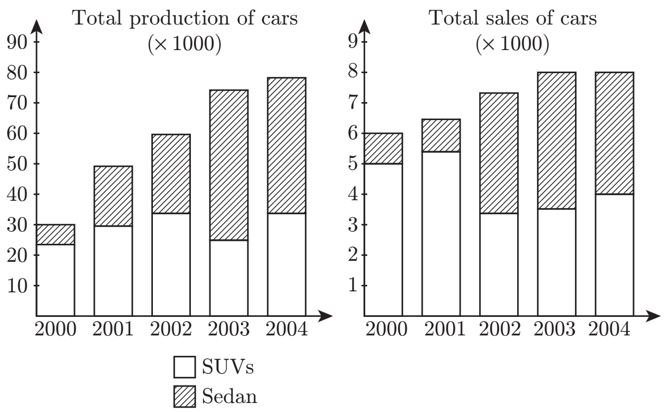


Figure 3 | Compound bar diagram.

A bar chart may be drawn on a percentage basis. To obtain a percentage bar diagram, bars of length equal to 100 for each class are drawn and subdivided in the proportion of the percentage of their component. Figure 4 shows a regular percentage bar diagram.

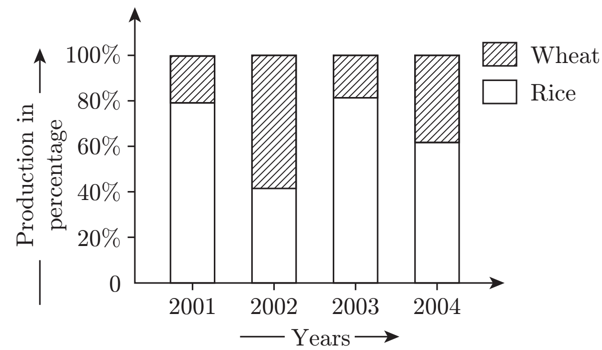
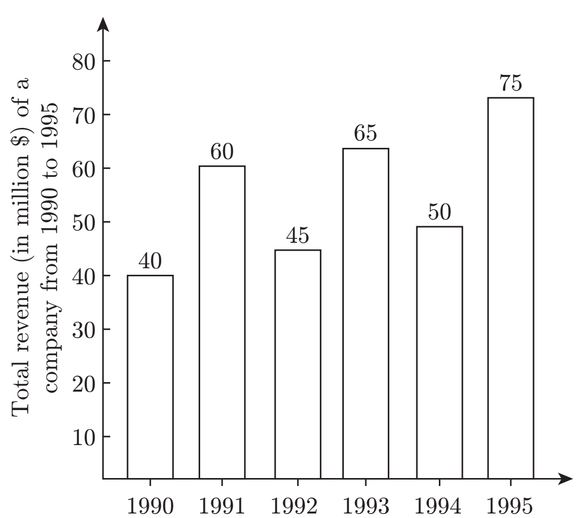


Figure 4 | Percentage bar diagram.

SOLVED EXAMPLES

Direction for Q1–Q4: Study the following bar diagram carefully and answer the questions.



1. Which pair of years had the same average revenue as that of 1990 and 1995?

Solution: Total revenue in 1990 = \$ 40 million
Total revenue in 1995 = \$ 75 million

Average revenue in 1990 and 1995 = $\dfrac{40+75}{2}$ = \$ 57.5 million

Total revenue in 1993 = \$ 65 million

Total revenue in 1994 = \$ 50 million

Average revenue for 1993 and 1994 = $\dfrac{65+50}{2}$ = \$ 57.5 million

Hence, average revenue of 1993 and 1994 was same as 1990 and 1995.

2. In which year(s), the percentage increase in production was maximum from the previous year?

Solution: Percentage increase from 1990 to 1991 is

$$\frac{60-40}{40}\times 100$$
$$=\frac{20}{40}\times 100 = 50\%$$

Also, percentage increase from 1994 to 1995 is

$$\frac{75-50}{50}\times 100 = 50\%$$

Thus, percentage increase was maximum in 1991 and 1995.

3. What was the percentage drop in the revenue from 1991 to 1992?

Solution: Total revenue in 1991 = \$ 60 million
Total revenue in 1992 = \$ 45 million

Percentage drop was

$$\frac{60-45}{60}\times 100$$

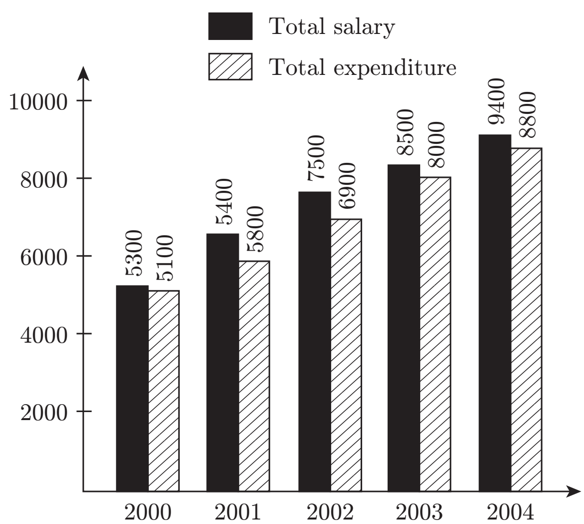
$$=\frac{15}{60}\times 100 = 25\%$$

4. What was the difference between revenue in 1995 and 1990?

Solution: Total revenue in 1995 = \$ 75 million
Total revenue in 1990 = \$ 40 million

Difference in revenue = 75 − 40 = \$ 30 million

Direction for Q5–Q7: Study the following bar diagram carefully and answer the questions.



5. What was the percentage increase in the "Total Salary" in 2002 as compared to 2000?

Solution: Total salary in 2002 = 7500
Total salary in 2000 = 5300

Percentage increase was

$$\frac{7500-5300}{7500}\times 100 = 29.33\%$$

6. If profit = total salary − expenditure, then in 2003 what percentage of total salary was the profit made?

Solution: Total salary in 2003 = 8500
Total expenditure in 2003 = 8000

Total profit = 500

$$\text{Profit }\% = \frac{500}{8500}\times 100 = 5.88\%$$

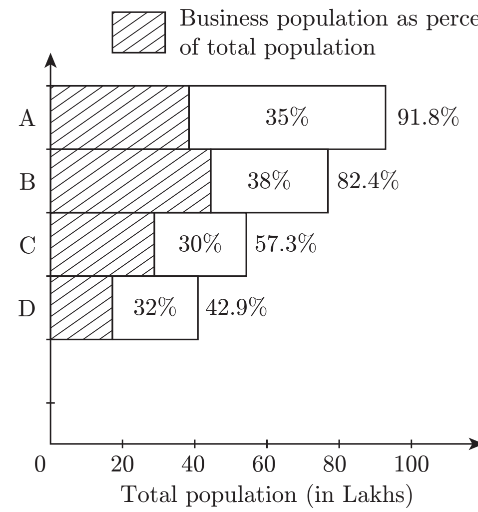
7. By what amount had the expenditure increased over the period 2000–2004?

Solution: Total expenditure in 2000 = ₹5100
Total expenditure in 2004 = ₹8800

Total increase in expenditure from 2000–2004 was

8800 − 5100 = ₹3700

Direction for Q8–Q10: Study the following bar diagram carefully and answer the questions.



8. What is total business population of A?

Solution: Total population of A = 91.8 lakh
Total business population = 35%

$$=\frac{35}{100}\times 91.8\text{ lakh} = 32.13\text{ lakh}$$

9. What place has the highest business population?

Solution: Business population of A $=\frac{35}{100}\times 91.8$

$$= 32.13\text{ lakh}$$

Business population of B $=\frac{38}{100}\times 82.4 = 31.31$ lakh

Business population of C $=\frac{30}{100}\times 57.3 = 17.19$ lakh

Business population of D $=\frac{32}{100}\times 42.9 = 13.728$ lakh

Hence, A has the highest business population.

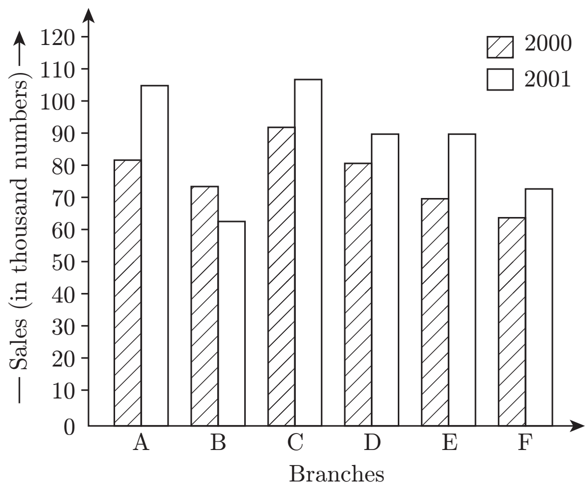
10. What is the difference between business populations of C and D?

Solution: Business population of C $=\frac{30}{100}\times 57.3$
$= 17.19$ lakh

Business population of D $= \frac{32}{100} \times 49.9 = 13.728$ lakh

Difference between business population of C and D $= 17.19 - 13.728 = 3.462$ lakh

Direction for Q11–Q15: Study the following bar diagram carefully and answer the questions.



11. What is the ratio of the total sales of branch B for both years to the total sales of branch D for both years?

Solution: Total sales of branch B = (75 + 65) × 1000 = 140000
Total sales of branch D = (85 + 95) × 1000 = 180000

Required ratio $= \frac{140}{180} = \frac{7}{9}$

12. What is the total sale of branches A, C and E together for both the years (in thousands)?

Solution: Total sales of branches A, C and E for both the years (in thousands)

$$= (80 + 105) + (95 + 110) + (75 + 95)$$
$$= 560$$

13. What percent of the average sales of branches A, B and C in 2001 is the average sales of branches A, C and F in 2000?

Solution: Average sales (in thousands) of branches A, C and F in 2000 $= \frac{1}{3} \times (80 + 95 + 70) = \frac{245}{3}$

Average sales (in thousands) of branches A, B and C in 2001

$$= \frac{1}{3} \times (105 + 65 + 110) = \frac{280}{3}$$

$$\text{Required percentage} = \left[\frac{245/3}{280/3} \times 100\right] = 87.5\%$$

14. Total sale of branch F for both the years is what percent of the total sales of branch C for both the years?

Solution: Total sales of branch F = (70 + 80) ×1000
Total sales of branch F = (95 + 110) ×1000

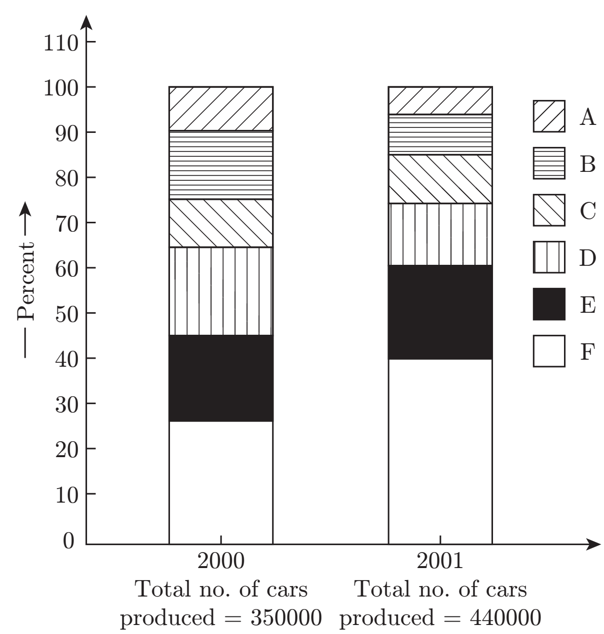
$$\text{Required percentage} = \left[\frac{70+80}{95+110} \times 100\right] = 73.17\%$$

15. What is the average sale of all the branches (in thousands) for the year 2000?

Solution: Average sales of all the six branches (in thousands) for the year 2000

$$= \frac{1}{6} \times (80 + 75 + 95 + 85 + 75 + 70) = 80$$

Direction for Q16–Q20: Study the following bar diagram carefully and answer the questions.



16. What is the difference in the number of E-type cars produced in 2000 and that produced in 2001?

Solution: Total number of E-type cars produced in 2001 $= \frac{(60 - 40) \times 440000}{100} = 88000$

Total number of E-type cars produced in 2000 = $\dfrac{(45-25)\times 350000}{100} = 70000$

Required difference = 88000 − 77000 = 11000

17. If the percentage of production of F-type cars in 2001 were the same as that in 2000, then the number of F-type cars produced in 2001 would have been?

Solution: We are given that the percentage production of F-type cars in 2001 = percentage production of F-type cars in 2000 = 25%

Then, number of F-type cars produced in 2001 = 25% of 4,40,000 = 110000

18. Total number of cars of model B, E and F manufactured in 2000 is?

Solution: In 2000, total number of cars produced = 350000

$$F = (25-0)\%\text{ of }350000 = \frac{25}{100}\times 350000 = 87500$$

$$E = (45-25)\%\text{ of }350000 = \frac{20}{100}\times 350000 = 70000$$

$$D = (65-45)\%\text{ of }350000 = \frac{20}{100}\times 350000 = 70000$$

$$C = (75-65)\%\text{ of }350000 = \frac{10}{100}\times 350000 = 35000$$

$$B = (90-75)\%\text{ of }350000 = \frac{15}{100}\times 350000 = 52500$$

$$A = (100-90)\%\text{ of }350000 = \frac{10}{100}\times 350000 = 35000$$

In 2001, total number of cars produced = 440000

$$F = (40-0)\%\text{ of }440000 = \frac{40}{100}\times 440000 = 176000$$

$$E = (60-40)\%\text{ of }440000 = \frac{20}{100}\times 440000 = 88000$$

$$D = (75-60)\%\text{ of }440000 = \frac{15}{100}\times 440000 = 66000$$

$$C = (85-75)\%\text{ of }440000 = \frac{10}{100}\times 440000 = 44000$$

$$B = (95-85)\%\text{ of }440000 = \frac{10}{100}\times 440000 = 44000$$

$$A = (100-95)\%\text{ of }440000 = \frac{5}{100}\times 440000 = 22000$$

Total number of cars of model B, E and F manufactured in 2000 = (52500 + 70000 + 87500) = 210000

19. If 85% of the C-type cars produced in each year were sold by the company, how many C-type cars remain unsold?

Solution: Number of C-type cars which remain unsold in 2000 = $\dfrac{15}{100}\times 35000 = 5250$

Number of C-type cars which remained unsold in 2001 = $\dfrac{15}{100}\times 44000 = 6600$

Therefore, total number of C-type cars which remained unsold = 5250 + 6600 = 11850

20. For which model was the percentage rise/fall in production from 2000 to 2001 minimum?

Solution: The percentage change in production from 2000 to 2001 for various models is:

$$F = \left[\frac{(176000-87500)}{105000}\times 100\right] = 101.14\%\text{ rise}$$

$$E = \left[\frac{(88000-70000)}{70000}\times 100\right] = 25.71\%\text{ rise}$$

$$D = \left[\frac{(70000-66000)}{70000}\times 100\right] = 5.71\%\text{ fall}$$

$$C = \left[\frac{(44000-35000)}{35000}\times 100\right] = 25.71\%\text{ rise}$$

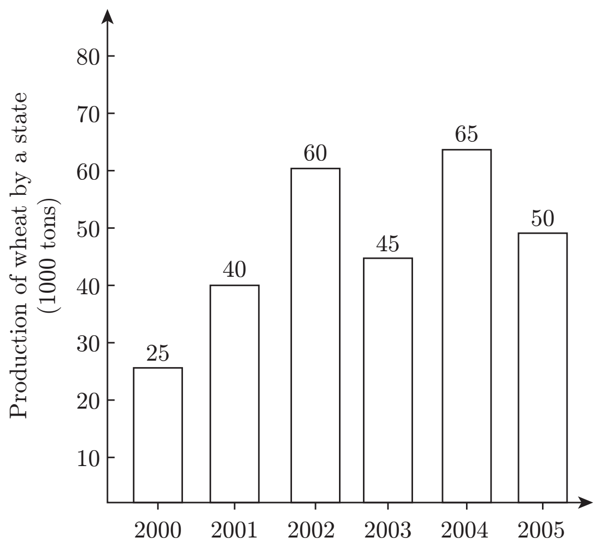
$$B = \left[\frac{(52500-44000)}{52500}\times 100\right] = 16.19\%\text{ fall}$$

$$A = \left[\frac{(35000-22000)}{35000}\times 100\right] = 37.14\%\text{ fall}$$

Therefore, minimum change is observed in model D.

PRACTICE EXERCISE

Direction for Q1–Q5: Study the following graph carefully and answer the questions.



1. What was the difference in the production of wheat between 2002 and 2004?

(a) 10000 tons (b) 15000 tons
(c) 20000 tons (d) 5000 tons

2. What was the percentage increase in production of wheat from 2001 to 2002?

(a) 25% (b) 50%
(c) 75% (d) 100%

3. What was the percentage decrease in production of wheat from 2002 to 2003?

(a) 10% (b) 20%
(c) 25% (d) 33.3%

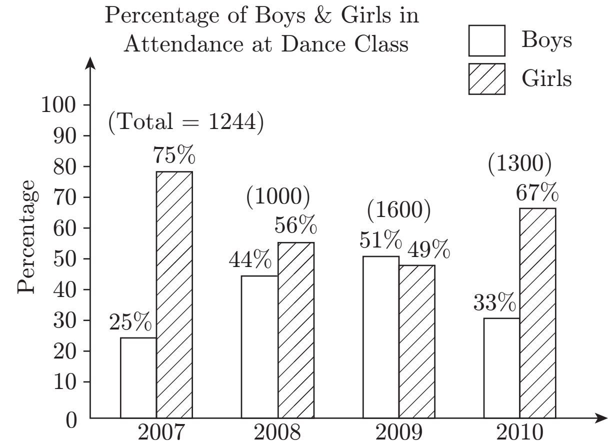
4. The average production of wheat in 2001 and 2004 was equal to the average of which two years?

(a) 2000 and 2001 (b) 2001 and 2003
(c) 2002 and 2003 (d) 2003 and 2005

5. In how many given years was the production of wheat more than average production of all years?

(a) 3 (b) 2
(c) 1 (d) 4

Direction for Q6–Q10: Study the following graph carefully and answer the questions:



6. How many boys attended dance class in 2007?

(a) 311 (b) 404 (c) 300 (d) 364

7. Which year had the largest number of boys attending dance school?

(a) 2007 (b) 2008 (c) 2009 (d) 2010

8. Which year had the lowest number of girls attending dance school?

(a) 2007 (b) 2008 (c) 2009 (d) 2010

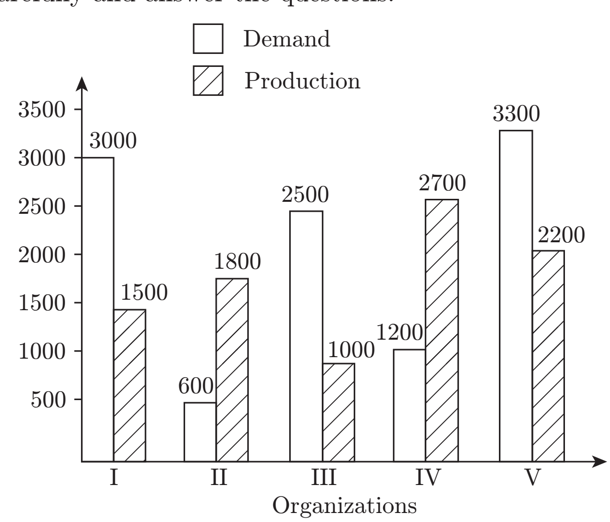
9. In which year(s) shown did approximately the same numbers of girls and boys attend the camp?

(a) 2008 (b) 2009
(c) 2007 and 2008 (d) 2008 and 2010

10. In how many years was the number of girls more than the average number of girls?

(a) 1 year (b) 2 year
(c) 3 year (d) Indeterminate

Direction for Q11–Q15: Study the following diagram carefully and answer the questions.



11. What is the difference between average demand and average production of organizations I, II and IV taken together?

(a) 400 (b) 200 (c) 500 (d) 800

12. Demand of organization II is what percent of organization I?

(a) 10% (b) 15% (c) 25% (d) 20%

13. The production of organization II is how many times that of production of organization IV?

(a) 2/3 (b) 2 (c) 3 (d) 3/2

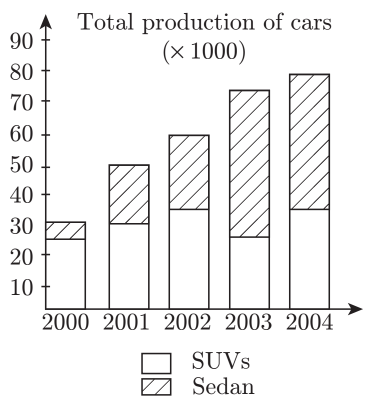
14. What is the ratio of demand and production of organization V?

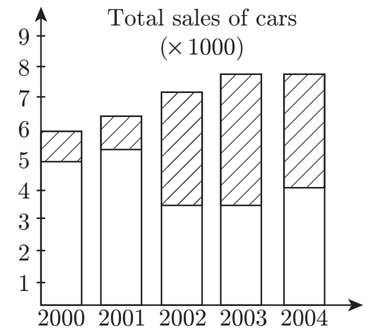
(a) 2:3 (b) 4:1 (c) 3:2 (d) 1:4

15. Production of organization III is what percent of organization I?

(a) 15% (b) 33.33% (c) 20% (d) 30%

Direction for Q16–Q20: Study the following bar diagram carefully and answer the questions.





16. Sales of session in 2002 were what percentage of production of sedans in 2002?

(a) 16% (b) 10%
(c) 25% (d) 8%

17. How many sedans were sold from 2000 to 2004?

(a) 13500 (b) 17500
(c) 15000 (d) 14500

18. How many SUVs were produced from 2000 to 2004?

(a) 175000 (b) 145000
(c) 120000 (d) 155000

19. What was the ratio of total SUVs produced in 2001 to the total cars produced?

(a) 2:5 (b) 5:2
(c) 3:5 (d) 4:7

20. What percent of total cars produced from 2000 to 2004 were sold?

(a) 15.5% (b) 17.3%
(c) 10.1% (d) 12.2%

ANSWERS

1. (d)	**4.** (c)	**7.** (c)	**10.** (b)	**13.** (a)	**16.** (a)	**19.** (c)
2. (b)	**5.** (a)	**8.** (c)	**11.** (a)	**14.** (c)	**17.** (d)	**20.** (d)
3. (c)	**6.** (a)	**9.** (b)	**12.** (d)	**15.** (b)	**18.** (b)	

EXPLANATIONS AND HINTS

1. (d) Production of wheat in 2002 = 60000 tons

Production of wheat in 2004 = 65000 tons

Difference between production of wheat = 5000 tons

2. (b) Total production of wheat in 2001 = 40000 tons

Total increase = (60000 − 40000) tons = 20000 tons

Total percentage increase $= \frac{20000}{40000} \times 100 = 50\%$

3. (c) Productions in 2002 = 60000 tons

Productions in 2003 = 45000 tons

Total decrease = 15000 tons

Total percentage decrease $= \frac{15000}{60000} \times 100 = 25\%$

4. (c) Total productions of wheat in 2001 and 2004 = 40 + 65 = 105000 tons

Total productions of wheat in 2002 and 2003 = 40 + 65 = 105000 tons

Hence, average production in same as 2001 and 2004 in 2002 and 2003.

5. (a) Total production of wheat from 2000 to 2005 = 25 + 40 + 60 + 45 + 65 + 50 = 285000 tons

Average production per year $= \dfrac{285000}{6} = 47500$ tons

Therefore, the production is less than the average for 3 years.

6. (a) Total students in 2007 = 1244

Total boys $= \dfrac{25}{100} \times 1244 = 311$

7. (c) Total boys who attended dance school in 2007 were

$$\frac{25}{100} \times 1244 = 311$$

Total boys who attended dance school in 2008 were

$$\frac{44}{100} \times 1000 = 440$$

Total boys who attended dance school in 2009 were

$$\frac{57}{100} \times 1600 = 816$$

Total boys who attended dance school in 2010 =

$$\frac{33}{100} \times 1300 = 429$$

Hence, maximum number of boys who attended dance school were 816 in 2009.

8. (c) Total girls who attended dance school in 2007 were

$$\frac{75}{100} \times 1244 = 933$$

Total girls who attended dance school in 2008 were

$$\frac{56}{100} \times 1000 = 560$$

Total girls who attended dance school in 2009 =

$$\frac{49}{100} \times 1600 = 784$$

Total girls who attended dance school in 2010 =

$$\frac{67}{100} \times 1300 = 871$$

Hence, the lowest number of girls who attended dance school was 784 in 2009.

9. (b) In 2009, the difference between percentage of boys and girls was approximately same.

10. (b) Total number of girls = 933 + 560 + 784 + 871 = 3148

Average number of girls = 787

Hence, girls are more than the average for 2 years.

11. (a) Average demand of organizations I, II and IV is

$$\frac{3000 + 600 + 1200}{3} = 1600$$

Average production of organizations I, II and IV is

$$\frac{1500 + 1800 + 2700}{3} = 2000$$

Difference between average production and average demand = 400

12. (d) Demand of organization I = 3000

Demand of organization II = 600

Total percentage $= \dfrac{600}{3000} \times 100 = 20\%$

13. (a) Production of organization II = 1800

Production of organization IV = 2700

Now, $\dfrac{1800}{2700} = \dfrac{2}{3}$

Hence, production of organization II is 2/3 times organization IV.

14. (c) Demands of organization V = 3300

Productions of organization V = 2200

Ratio of demands to production $= 3300 : 2200 = 3 : 2$

15. (b) Production of organization III = 1000

Production of organization I = 1500

Total percentage $= \dfrac{500}{1500} \times 100 = 33.33\%$

16. (a) Sales of sedans in 2002 = 4000

Production of sedans in 2002 = 25000

Now, total percentage $= \dfrac{4000}{25000} \times 100 = 16\%$

17. (d) Total sales of sedans from 2000 to 2004 = (1 + 1 + 4 + 4.5 +4) × 1000 = 14.5 × 1000 = 14500

18. (b) Total production of SUVs from 2000 to 2004 = (25 + 30 + 35 + 25 + 30) × 1000 = 145 × 1000 = 145000

19. (c) Total cars produced in 2002 = 50000

Total SUVs produced in 2002 = 30000

Ratio of SUVs produced to the cars produced in 2002 $= \dfrac{30000}{50000} = 3:5$

20. (d) Total cars produced = (30 + 50 + 60 + 75 + 80) × 1000 = 295000

Total cars sold $= (6 + 6.5 + 7.5 + 8 + 8) \times 1000 = 36000$

Total percent of cars sold $= \dfrac{36000}{295000} \times 100 = 12.2\%$

CHAPTER 6

PIE CHART

INTRODUCTION

A pie chart is a circular chart divided into sectors which represent numerical proportions. In a pie chart, the arc length of each sector is proportional to the quantity it represents. The pie chart is also called a circle graph.

Pie charts are widely used to represent numbers and figures in the corporate world meetings and presentations.

In a pie chart,

$$\begin{aligned} 100\% &= 360^\circ \\ 1\% &= 3.6^\circ \\ &= \left(\frac{18}{5}\right)^\circ \end{aligned}$$

Conversely, $1^\circ = \frac{5}{15}\%$

TYPES OF PIE CHARTS

3-D Pie Chart

A 3-D pie cake, or perspective pie cake, is used to give the chart a three-dimensional look. The third dimension is often used for aesthetic reasons and does not improve the reading of the chart in any way. Figure 1 represents a basic 3-D pie chart.

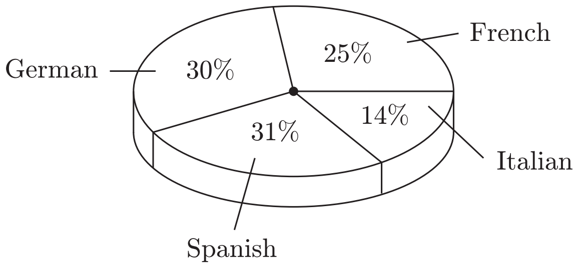


Figure 1 | Pie chart representing the percentage of students taking a certain foreign language.

Doughnut Chart

A doughnut chart is like an ordinary pie chart with the exception of a blank center. Moreover, we can use it to display multiple statistics at once. The blank center can be used to display additional data. Figure 2 represents a basic doughnut chart.

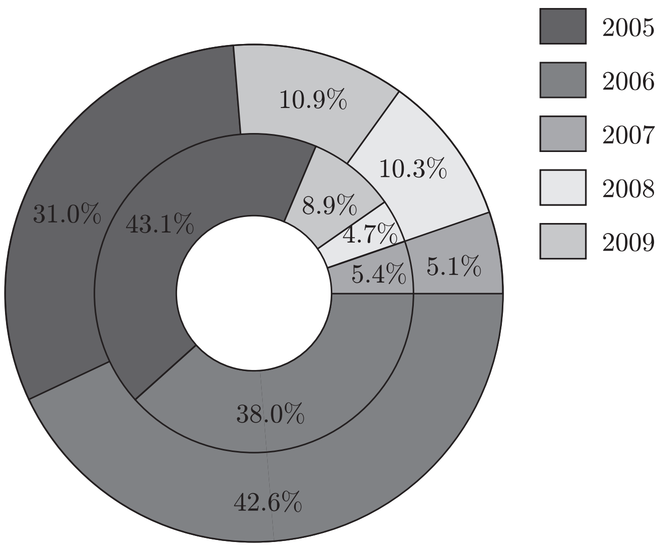


Figure 2 | Pie chart representing sale of two products over 5 years.

Exploded Pie Chart

A chart with one or more sectors separated from the rest of the disk is known as an exploded pie chart. This effect is used to highlight either a sector or segments of the chart with small proportions. Figure 3 represents a basic exploded pie chart.

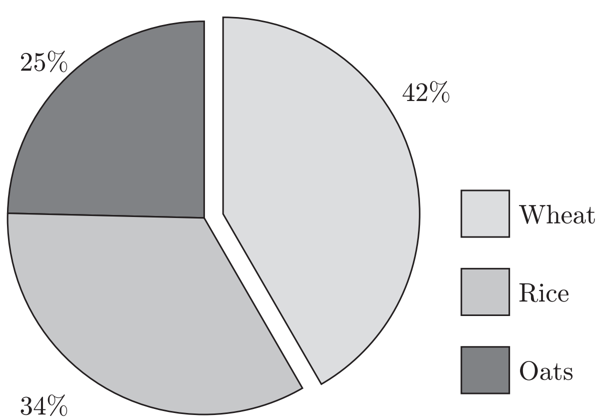


Figure 3 | Pie chart representing the percentage of food items consumed for a certain year.

Polar Area Chart

The only difference between a normal pie chart and a polar area chart is that for the former, the sectors have the same distance from the center of the circle but have different angles, whereas for the latter the sectors have equal angles and differ rather in how far each sector extends from the center of the circle. The polar area diagram is used to plot cyclic phenomena. Figure 4 represents a basic polar area chart.

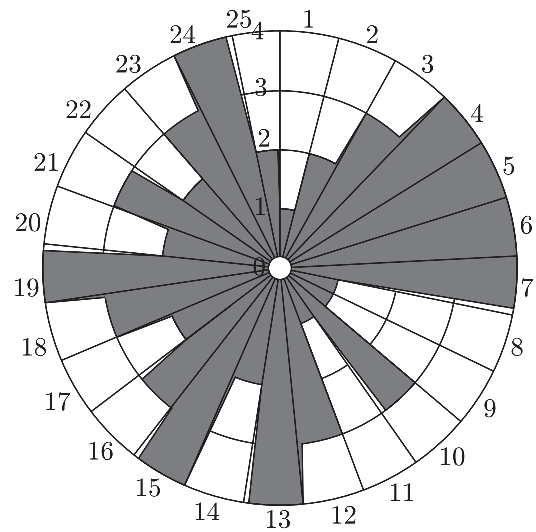


Figure 4 | Polar area chart.

Ring Chart/Multilevel Pie Chart

A ring chart is used to visualize hierarchical data depicted by concentric circles. The circle in the center represents the root node, with the hierarchy moving outward from the center. A segment of the inner circle has a hierarchical relationship to those segments of the outer circle which lie within the angular sweep of the inner segment. Figure 5 represents a basic ring chart.

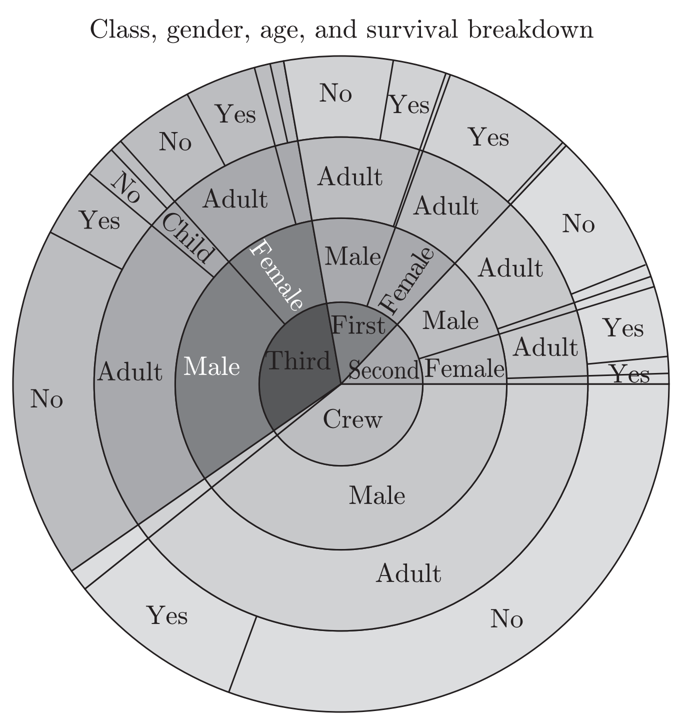
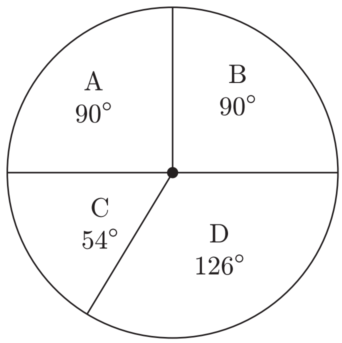


Figure 5 | Ring chart.

SOLVED EXAMPLES

Direction for Q1–Q3: The data of section-wise distribution of 1080 student is shown in the following figure.



Consider the figure and answer the following questions.

1. How many percent of students are more in section D than that in A?

Solution: Section D = 126°
Section A = 90°

$$\Rightarrow \frac{126-90}{90}\times 100$$

$$\Rightarrow \frac{36}{90}\times 100 = 40\%$$

2. What is the percentage composition of students in various sections?

Solution: We know

$$360° = 100\%$$

$$\Rightarrow 1° = \frac{5}{18}\%$$

Section A = $90\times\frac{5}{18} = 25\%$

Section B = $90\times\frac{5}{18} = 25\%$

Section C = $54\times\frac{5}{18} = 15\%$

Section D = $126\times\frac{5}{18} = 35\%$

3. What is the number of students in each section?

Solution: We know
Total students = 1080

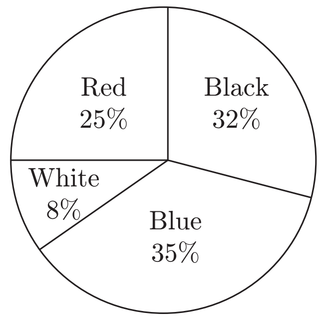
So

$$100\% \text{ or } 360° = 1080$$

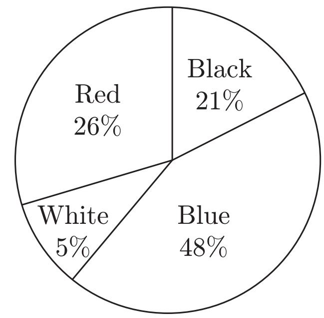
$$\Rightarrow 1° = 3$$

Section A has 3 × 90 = 270 students
Section B has 3 × 90 = 270 students
Section C has 3 × 54 = 162 students
Section D has 3 × 126 = 378 students

Direction for Q4 and Q5: The data of colours of shirts (total shirts = 5000) sold in a showroom in the years 2000 and 2010 are shown in the following figures.



2000



2010

Consider the figures and answer the following questions.

4. If there are 1302 black shirts sold in 2010, then what was the total number of white shirts sold in 2000 and 2010 combined?

Solution: Let the total shirts sold in 2010 are x.

So

$$\frac{21}{100}\times x = 1302$$

$$\Rightarrow x = 6200$$

Therefore, total number of shirts sold in 2010 = 6200

Total number of white shirts sold in 2010 = $\frac{5}{100}\times 6200 = 310$

Total number of shirts sold in 2000 = 5000

Total number of white shirts sold in 2000 = $\frac{8}{100}\times 5000 = 400$

So the total number of white shirts sold in 2000 and 2010 = 710

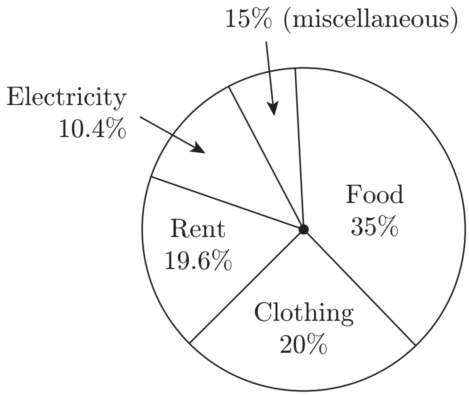
5. If there were 1302 black shirts sold in 2010, and 25% of blue shirts were light blue, then how many dark blue shirts were sold?

Solution: Total number of shirts sold in 2010 = $\frac{1302}{21}\times 100 = 6200$

Total number if blue shirts sold = 48% of 6200 = $\frac{48}{100}\times 6200 = 2976$

Total number of dark blue shirts sold = 75% of 2976 = $\frac{75}{100}\times 2976 = 2232$

Direction for Q6–Q8: The data of percentage distribution of household expenditure of a family per month is shown in the following figure.



Consider the figure and answer the following questions.

6. What is the ratio between the money spent on clothing and food?

Solution: Money spent on clothing = 20%
Money spent on food = 35%

$$\text{Ratio} = \frac{20}{35} = \frac{4}{7} \quad \text{or} \quad 4:7$$

7. If total expenditure of family is ₹45000 per month, then what is the total expenditure on food and rent?

Solution: Total expenditure = ₹45000 per month

Total expenditure on food = 35% of 45000 = $\frac{35}{100} \times 45000$ = ₹15750

Total expenditure on rent = 19.6% of 45000 = $\frac{19.6}{100} \times 45000$ = ₹8820

Total expenditure on food and rent = ₹(15750 + 8820) = ₹24570

8. If the total expenditure of family is ₹45000 per month, then what is the difference between the expenditure of clothing and electricity?

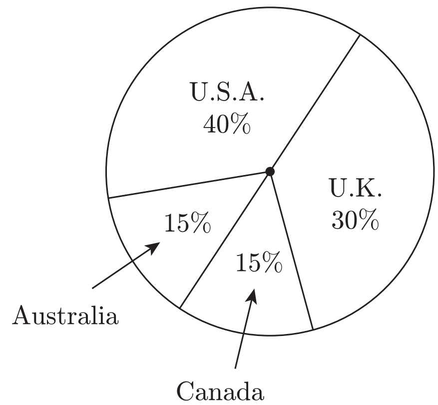
Solution: Total expenditure = ₹45000

Expenditure on clothing = $\frac{20}{100} \times 45000$ = ₹9000

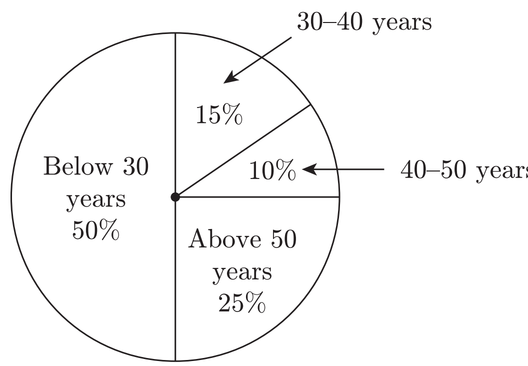
Expenditure on electricity = $\frac{10.4}{100} \times 45000$ = ₹4680

Difference between expenditure on clothing and electricity = ₹9000 − 4680 = ₹4320

Direction for Q9 and Q10: Figure A shows the number of tourists in the following countries and Figure B shows age-wise traffic of the tourists.



(a)



(b)

9. What is the ratio of number of tourists in USA to the number of tourists above 50 years?

Solution: Total tourists in USA = 40%
Total tourists above 50 years = 25%

$$\text{Ratio} = \frac{40}{25} = \frac{8}{5} = 8:5$$

10. If the number of tourists that went to UK were 360000, then how many of the total tourists are between 40 and 50 years?

Solution: Total tourists in UK = 30%
If total number of tourists are x, then,

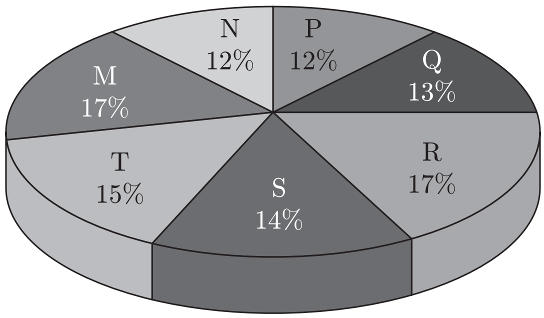
$$\frac{30}{100} \times x = 360000 \Rightarrow x = 1200000$$

Now, total tourists between 40 and 50 years

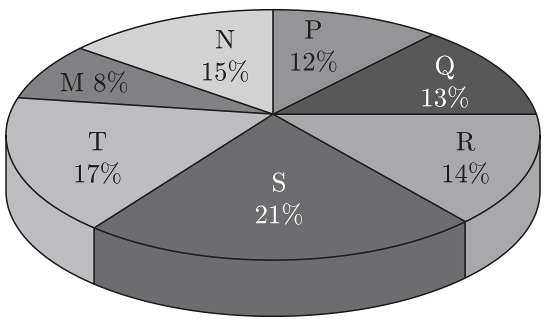
$$\frac{10}{100} \times 1200000 = 120000$$

Direction for Q11–Q15: Distribution of students at graduate and post graduate levels in seven institutes (M, N, P, Q, R, S and T).

Total number of students of graduate level = 27300



Total number of students of post graduate level = 24700



11. What is the total number of graduate and post-graduate level students in institute R?

Solution: Required number = (17% of 27300) + (14% of 24700) $= \left(\frac{17}{100} \times 27300\right) + \left(\frac{14}{100} \times 24700\right)$

= 4641 + 3458 = 8099

12. How many students of institutes of M and S are studying at graduate level?

Solution: Students of institute M at graduate level $= \frac{17}{100} \times 27300 = 4641$

Students of institute S at graduate level $= \frac{14}{100} \times 27300 = 3822$

Total number of students at graduate level in institutes M and S = 4641 + 3822 = 8463

13. What is the total number of students studying at postgraduate level from institutes N and P?

Solution: Students studying at postgraduate level from $N = \frac{15}{100} \times 24700 = 3705$

Students studying at postgraduate level from $P = \frac{12}{100} \times 24700 = 2964$

Required number = 3705 + 2964 = 6669

14. What is the ratio between the number of students studying at postgraduate level from institutes S and the number of students studying at graduate level from institute Q?

Solution: Number of students studying at post-graduate level from $S = \frac{21}{100} \times 24700 = 5187$

Number of students studying at graduate level from $Q = \frac{13}{100} \times 27300 = 3549$

Required ratio $= \frac{5187}{3549} = \frac{19}{13}$

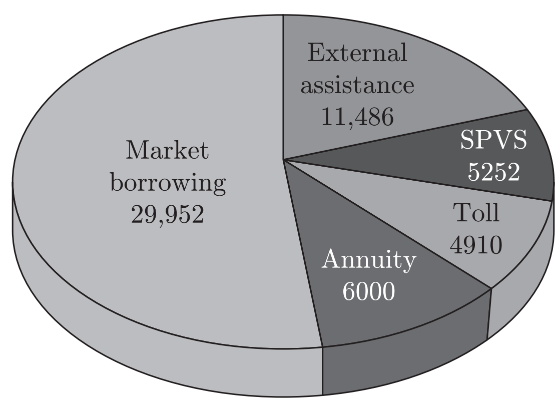
15. What is the ratio between the number of students studying at postgraduate and graduate levels from institute S?

Solution: Number of postgraduate students from institute $S = \frac{21}{100} \times 24700 = 5187$

Number of graduate students from institute S = $\frac{14}{100} \times 27300 = 3822$

Required ratio $= \frac{5187}{3822} = \frac{19}{14}$

Direction for Q16–Q20: Sources of funds to be arranged by NHAI for Phase II projects (in crores).



16. 20% of the funds are to be arranged through which source?

Solution: Total funds = 29952 + 11486 + 5252 + 4910 + 6000 = 57600

20% of the total funds to be arranged = ₹ (20% of 57600) = ₹ 11520 crores ≈ ₹ 11486 crores

Hence, 20% of the total funds are arranged through External Assistance.

17. If the Toll is to be collected through an outsourced agency by allowing a maximum 10% commission, how much amount should be permitted to be collected by the outsourced agency so that the project is supported with ₹4910 crores?

Solution: Amount permitted = (Funds required from Toll for projects of Phase II) + (10% of these funds)

$$= 4910 + \left(\frac{10}{100} \times 4910\right) = 4910 + 491$$

$$= ₹5401 \text{ crores}$$

18. What is the approximate ratio of funds to be arranged through Toll and that through Market Borrowing?

Solution: Total funds from Toll = ₹4910 crores
Total funds from Market Borrowing = ₹29952 crores

$$\text{Required ratio} = \frac{4910}{29952} \simeq \frac{1}{6}$$

19. If NHAI could receive a total of ₹9695 crores as External Assistance, by what percent (approximately) should it increase the Market Borrowing to arrange for the shortage of funds?

Solution: Shortage of funds arranged through External Assistance = 11486 − 9695 = ₹1791 crores
Thus, increase required in Market Borrowing = ₹1791 crores

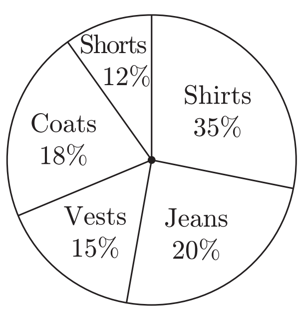
$$\text{Percentage increase required} = \left(\frac{1791}{29952} \times 100\right)\% \simeq 6\%$$

20. What is the central angle corresponding to Market Borrowing?

Solution: Central angle corresponding to Market Borrowing $= \left(\frac{29952}{57600} \times 360°\right) = 187.2°$

PRACTICE EXERCISE

Direction for Q1–Q4: The following pie chart shows the items sold by a store.



Consider the figure and answer the following questions.

1. What is the central angle of the sector for the number of jeans sold?

(a) 56° (b) 72° (c) 64° (d) 92°

2. If the total number of vests sold is 4500, then how many coats were sold?

(a) 5400 (b) 5600 (c) 3000 (d) 6000

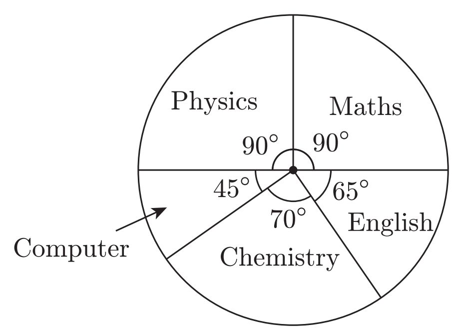
3. What is the ratio of number of jeans sold to number of shorts sold?

(a) 5 : 3 (b) 3 : 4 (c) 10 : 3 (d) 5 : 4

4. How many more number of jeans is sold compared to vests if the total number of sales of all items is 30000?

(a) 4500 (b) 6000 (c) 1000 (d) 1500

Direction for Q5–Q8: The following pie chart shows number of marks scored by a student in an examination. Here the total number of marks is 540.



Consider the figure and answer the following questions.

5. By how many marks, the marks in Maths and Chemistry exceed the marks scored in Physics and Computer?

(a) 42 (b) 55 (c) 37.5 (d) 30

6. What percentage of marks was scored in English?

(a) 22% (b) 18.06% (c) 24.25% (d) 16.55%

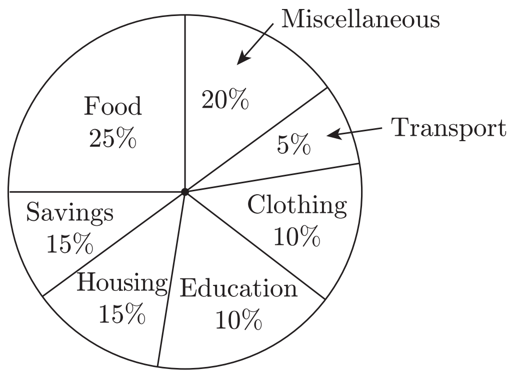
7. In which subject did the student get 105 marks?

(a) Maths (b) Physics
(c) Computer (d) Chemistry

8. What is percentage of the marks obtained in Physics, Computer and Maths of the total marks?

(a) 50% (b) 42.5% (c) 62.5% (d) 80%

Direction for Q9–Q12: The following pie chart shows the percentage distribution of money spent by a family in August, 2013.



Consider the figure and answer the following questions.

9. If the total amount spent was ₹120000, then what was the total amount spent on "housing"?

(a) ₹18000 (b) ₹21000 (c) ₹15000 (d) ₹24500

10. What is the ratio of the total amount of money spent on clothing to that on food?

(a) 5 : 2 (b) 2 : 5 (c) 3 : 4 (d) 4 : 3

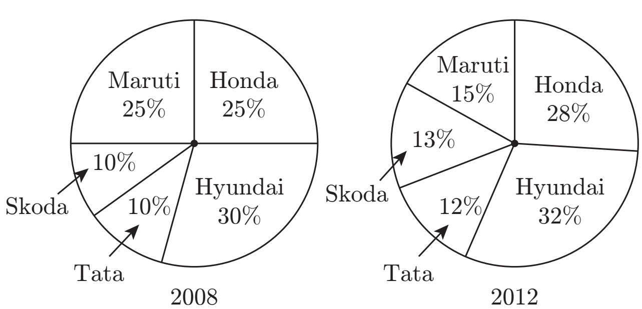
11. How much of the total money was used for savings?

(a) 36° (b) 48° (c) 52° (d) 54°

12. If total money spent on transportation was ₹13000, then how much money was spent on education?

(a) ₹13000 (b) ₹26000 (c) ₹20000 (d) ₹40000

Direction for Q13–Q20: The following pie chart depicts sales of cars in a city. Here the total number of cars sold is 8600.



Consider the figure and answer the following questions.

13. What was the total number of Honda or Hyundai cars sold in 2008?

(a) 4220 (b) 5100
(c) 4730 (d) 4400

14. If total 2150 Maruti cars were sold in 2008, then how many Skoda cars were sold?

(a) 860 (b) 1240
(c) 640 (d) 720

15. In 2012, if 800 Hyundai cars were sold, then what is the difference between the sale of Hyundai in 2008 and 2012?

(a) 2100 (b) 1780
(c) 1600 (d) 1850

16. If 1600 Hyundai cars were sold in 2012, then how many cars were sold in total?

(a) 2000 (b) 3000
(c) 4000 (d) 5000

17. If 1600 Hyundai cars were sold in 2012, then how many Honda cars were sold?

(a) 1500 (b) 1450
(c) 1400 (d) 1300

18. If 1600 Hyundai cars were sold in 2012, then how many cars were sold in 2010 and 2012 combined?

(a) 11800 (b) 12200
(c) 13600 (d) 14500

19. If sales of Tata in 2008 were 4/3 times the sales of Tata in 2012, then how many Tata cars were sold in 2008 and 2012?

(a) 1320 (b) 1410
(c) 1505 (d) 2010

20. If sales of Tata in 2008 were 4/3 times the sales of Tata in 2012, then what were the total numbers of cars sold in 2012?

(a) 4700 (b) 5500
(c) 5000 (d) 5375

ANSWERS

1. (b)	**3.** (a)	**5.** (c)	**7.** (d)	**9.** (a)	**11.** (d)	**13.** (c)	**15.** (b)	**17.** (c)	**19.** (c)
2. (a)	**4.** (d)	**6.** (b)	**8.** (c)	**10.** (b)	**12.** (b)	**14.** (a)	**16.** (d)	**18.** (c)	**20.** (d)

EXPLANATIONS AND HINTS

1. (b) Central angle of sector for jeans

$$\left(\frac{20}{100}\times 360\right) = 72°$$

2. (a) Total number of vests = 4500
Let the total items sold are x. so

$$\frac{15}{100}x = 4500$$
$$x = 30000$$

Total number of coats sold

$$\frac{18}{100}\times 30000 = 5400$$

3. (a) Total percent of jeans sold = 20%
Total percent of shorts sold = 12%

$$\text{Ratio} = \frac{20}{12} = \frac{5}{3} = 5 : 3$$

4. (d) Total number of jeans sold = 20%
Total number vests sold = 15%
Thus, 5% of more number of jeans are sold.
So the total items sold = 30000
Hence, total difference between sales of jeans and vests is

$$\frac{5}{100}\times 30000 = 1500.$$

5. (c) Marks scored in Maths and Chemistry $= \frac{160}{360}\times 540 = 240$
Marks scored in Physics and Computer $= \frac{135}{360}\times 540 = 202.5$
Excess marks in Maths and Chemistry than Physics and Computer = 37.5

6. (b) We have

$$360° = 100\%$$
$$\Rightarrow 1° = \frac{100}{360}\%$$

Now, the marks in English $= 65° = 65\times\frac{100}{360} = 18.06\%$

7. (d) Let the degree is x. So

$$\frac{x}{360}\times 540 = 105$$
$$\Rightarrow x = 105\times\frac{2}{3} = 70°$$

Hence, the subject is Chemistry.

8. (c) Total marks in Physics, Computer and Maths = $225° = 225\times\frac{100}{360} = 62.5\%$

9. (a) Total money spent = ₹120000
Money spent on housing $= 15\% = \frac{15}{100}\times 120000$ = ₹18000

10. (b) Total money spent on clothing = 10%
Total money spent on food = 25%
Ratio of money spent clothing to food $= \frac{10}{25} = \frac{2}{5} = 2:5$

11. (b) Total money spent on saving $= 15\% = 15\times\left(\frac{18}{5}\right)^{\circ} = 54°$

12. (b) Total money spent on transportation = ₹13000
If total money spent is ₹x. Then,

$$\frac{5}{100}\times x = 13000$$
$$\Rightarrow x = ₹260000$$

Total money spent on education is $\frac{10}{100}\times 260000 =$ ₹26000

13. (c) Total cars sold in 2008 = 8600
Total Honda or Hyundai sold $= 55\% = \frac{55}{100}\times 8600 = 4730$

14. (a) Total Maruti cars sold = 25% or 2150
Total Skoda cars sold = 10% or $2150\times\frac{10}{25} = 860$

15. (b) Total Hyundai cars sold in 2008 $= \frac{30}{100}\times 8600 = 2580$
Total Hyundai cars sold in 2012 = 800
Difference between sales in 2008 and 2012 = 1780

16. (b) Total Hyundai cars sold in 2012 = 1600
Let the total cars sold be x. Thus,

$$x = 1600\times\frac{100}{32} = 5000$$

17. (c) Total Hyundai cars sold in 2012 = 32% or 1600
Total Honda cars sold in 2012 $= 28\% = 1600\times\frac{28}{32} = 1400$

18. (c) Total Hyundai cars sold in 2012 = 1600
If total cars sold are x. Then

$$\frac{32}{100}\times x = 1600$$
$$\Rightarrow x = 1600\times\frac{100}{32} = 5000$$

Total cars in 2008 and 2012 = 5000 + 8600 = 13600

19. (c) Total Tata cars sold in 2008 $= \frac{10}{100}\times 8600 = 860$
Total Tata cars sold in 2012 $= \frac{3}{4}\times 8600 = 645$
Total Tata cars sold = 1505

20. (d) Total Tata cars sold in 2008 $= \frac{10}{100}\times 8600 = 860$
Total Tata cars sold in 2012 $= \frac{3}{4}\times 860 = 645$
If total cars sold in 2012 be x. Then,

$$\frac{12}{100}\times x = 645$$
$$\Rightarrow x = 645\times\frac{100}{12} = 5375$$

CHAPTER 7

PUZZLES

INTRODUCTION

A puzzle is a problem to test the reasoning ability of the student. In a puzzle, one is required to put pieces together, in a logical way, in order to arrive at the desired correct solution.

Puzzles are not only used for games and entertainment but in some cases, their solutions may be a significant contribution to mathematical research.

TYPES OF PUZZLES

Puzzles not only come in different shapes and sizes, there are different types of puzzles too. Some common categories of puzzles are:

1. Number puzzles
2. Logical puzzles
3. Picture puzzles
4. Connect the dots
5. Spot the difference

SOLVED EXAMPLES

1. In a ten-digit number, first represents number of "1" present in the number, second digit represents number of "2" in the number. Similarly, third digit represents number of "3" present in the number and so on till the ninth digit, which represents number of "9" present in the number. However, the last digit represents number of "0" present in the number. What is the ten-digit number?

Solution: We start with the smallest possible ten-digit number is 1000000009. Now, since last digit represents number of zeros, it should be 9.

Hence, the number becomes 1000000009.

Since, we have only 8 zeros so the last digit should be 8. Hence, the number becomes 1000000008.

Since, number of "8" present in the above case is one, next modification will be 1000000108.

Now, the number of "0" is only seven. Hence, the number is modified to 1000000107.

Since, the number of "7" present is one so next modification will be 10000001007.

Now, since the number of "1" present is two, the next modification will be 2000001007.

Number of "2" present is to be one so the next modification will be 2100001007.

Here, since number of zeros present is six, the next modification will be 2100001006.

Now, since number of "6" present is 1, so the next modification will be 2100010006.

The number 2100010006 satisfies all the condition so this is the desired result.

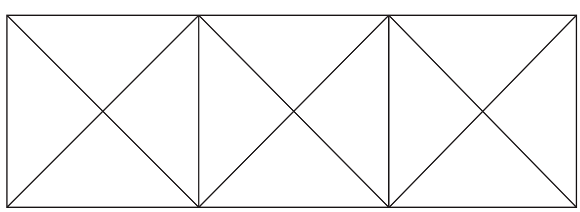
2. Fill in the number that completes the following sequence:

$$3, 5, 7, 11, 13, 17, 19, x$$

Solution: It is evident that the sequence contains prime numbers.

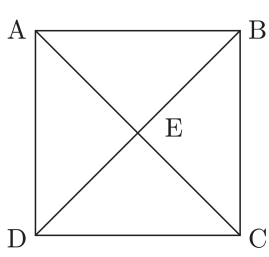
Hence, the number x is 23.

3. Find the total number of triangles in the given figure.



Solution: We can see that the figure contains three identical squares.

Hence, let us calculate the triangles in one of these squares.

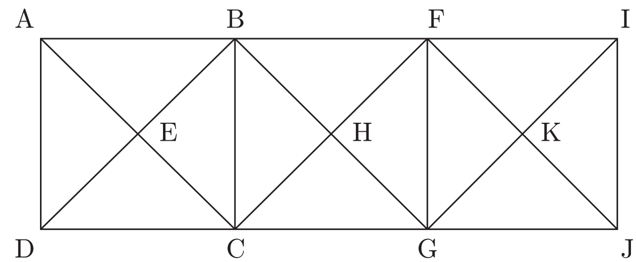


Total triangles in the square are

ΔABE, ΔBCE, ΔDEC, ΔAED,
ΔABC, ΔBCD, ΔACD, ΔDAB

Hence, there are 8 triangles in 1 square. Therefore, there will be 24 triangles in 3 squares.

Also, consider the diagram again,

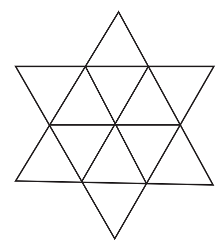


Total triangles not covered in the diagram are

ΔACF, ΔBGI, ΔDBG, ΔCFJ

Hence, total number of triangles in the figure = 24 + 4 = 28

4. Find the total number of triangles in the following figure.



Solution: Let us assume that the smallest side of a triangle to be 1 unit.

Number of triangles with side 1 unit = 1 + 5 + 5 + 1 = 12

Number of triangles with side 2 units = 1 + 2 (facing upward) + 1 + 2 (facing downward) = 6

Number of triangles with side 3 units = 1(facing upward) + 1 (facing downward) = 2

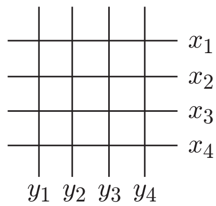
Thus, total number of triangles = 12 + 6 + 2 = 20

5. Three people checked into a hotel. They pay \$30 to the manager and go to their room. The manager suddenly remembers that the room rates are for \$25 for that night and gives \$5 to the bellboy to return to the people. However, the bellboy keeps \$2 and gives \$1 to each person. Now, each person paid \$9, totaling \$27 and the bellboy kept \$2, totaling \$29. Where is the missing \$1?

Solution: Originally, they paid \$30, of which they got \$1 and hence they have only paid \$27. Of this \$27, \$25 went to the manager for the room and \$2 went to the bellboy.

Hence, the figures add up.

6. Four parallel horizontal lines are interesting perpendicularly to four vertical parallel lines. Find the number of rectangular formed after such intersection.



Solution: Let x_1–x_4 represent four parallel vertical lines and y_1–y_4 represent four parallel horizontal lines.

To form a rectangle we need to select any two horizontal parallel lines which interest any two vertical parallel lines.

Thus,

$$^4C_2 \times {}^4C_2 = 6 \times 6 = 36$$

Hence, 36 rectangles can be formed.

7. There are four holes numbered 1, 2, 3 and 4. The unique properties of these holes are that number of mice become double, triple and four times after entering into hole number 2, 3 and 4, respectively, while it remains the same if they enter hole 1. One cat is running to eat some mice. To save themselves, the mice enter in first hole and come out, 24 of them are eaten by the cat. They, respectively, entered into second, third and fourth holes and come with double, triple and four times in number. After exit from each hole, 24 mice are eaten up by the cat each time. At the end, there is no mouse left. Find the initial number of mouse before entering into the first hole.

Solution: We know that 24 mice exit from fourth hole which were all eaten up by the cat and hence no mouse was left.

Hence, 6 mice entered the fourth hole.

Now, 24 were eaten after they exited from third hole by the cat and 6 remained. Thus, 30 mice came out of the third hole.

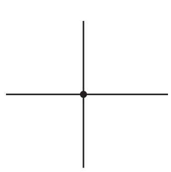
Thus, before entering into third hole there were 10 mice.

34 mice came out of the second hole, since 24 mice were again eaten by the cat. Therefore, 17 mice entered the second hole.

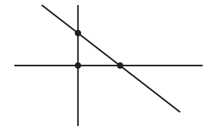
Again, 41 mice came out of first hole and hence, 41 mice entered into first hole.

8. Four lines are intersecting each other. Find how many intersecting points are possible.

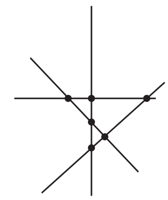
Solution: Two lines have almost one intersecting point.



Three lines can have three intersecting points.

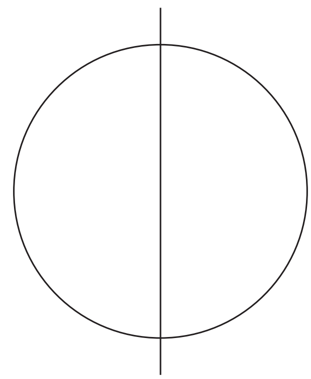


Four lines can have six intersecting points.

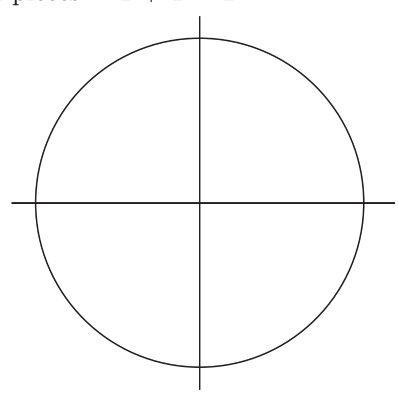


9. A cake is to be cut into different pieces. How many maximum pieces are possible if four vertical cuts are made given that no horizontal cut is allowed?

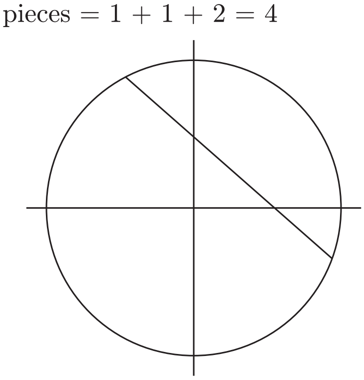
Solution:



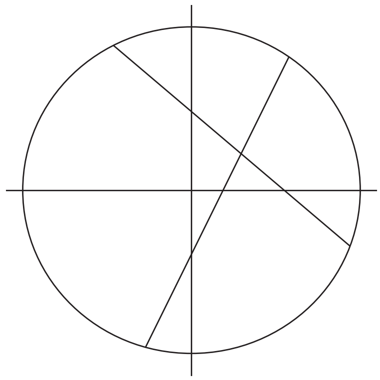
Total pieces = 1 + 1 = 2



Total pieces = 1 + 1 + 2 = 4



Total pieces = 1 + 1 + 2 + 3 = 7



Total pieces = 1 + 1 + 2 + 3 + 4 = 11

10. Flag of a nation consists of six different vertical strips. How many such flags can be possible if we use six colours to fill the strips such that no two consecutive strips will have same colours?

Solution: First strips can be filled with any six colours. Second strip can be filled with any five colours except the colour filled in strip one.

Similarly, successive strips can also be filled in the same manner, that is, 5 ways.

Hence, number of possible flags = $6 \times 5 \times 5 \times 5 \times 5 \times 5 = 18750$

PRACTICE EXERCISE

Direction for Q1–Q3: Disha brought some sweets on her 22nd birthday. She offered one less than the half of total number sweets in the gurdwara near her house. She also gave one sweet each to 3 beggars sitting on the stairs of gurdwara on the way back to home, she stopped a big group of poor children and gave them half of what was left with her. After reaching home she shared the remaining two pieces of sweets with her younger brother.

1. How many sweets did she originally have?

(a) 8 (b) 10 (c) 12 (d) 14

2. How many sweets did she offer in the gurdwara?

(a) 3 (b) 4 (c) 5 (d) 6

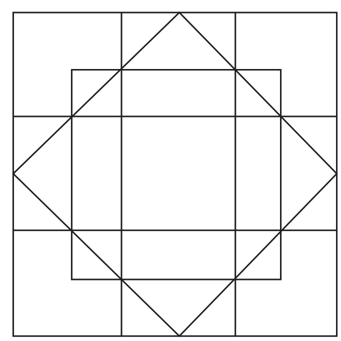
3. How many sweets did she give to poor children?

(a) 1 (b) 2 (c) 30 (d) 42

4. Mary's mom has four children. The second child is called April, third child is called May and the fourth child is called June. What is the name of the first child?

(a) March (b) Mary
(c) July (d) None of these

5. How many triangles are there in the following picture?



(a) 20 (b) 24 (c) 26 (d) 28

6. Which day comes three days after the day which comes two days after the day which comes immediately after the day which comes two days after Monday?

(a) Monday (b) Tuesday
(c) Wednesday (d) Thursday

7. Suppose you are in a dark room with a candle, a wood stove and a gas lamp. You only have one match, so what do you light first?

(a) Candle (b) Wood stove
(c) Gas lamp (d) Match

Direction for Q8–Q10: Study the pyramid of the letters given below and answer the following questions.

				B				
			M	T	R			
		G	C	S	N	P		
	Y	Q	H	K	E	A	I	
J	F	U	W	X	O	V	Z	D

8. Which letter is missing in the pyramid?

(a) L (b) F (c) I (d) P

9. If letters were to be studied vertically, which two letters happen to be neighbors that occur in alphabetical order?

(a) WX (b) PR (c) UV (d) ST

10. If all the horizontal lines were to be studied separately which neighbors in the alphabetical order are the farthest?

(a) F and U (b) Q and Y
(c) C and S (d) H and Q

11. Find the value of x.

		4	3	2		
	5	3	5	1	1	
6	1	2	8	3	3	1
	7	2	8	4	3	
		9	x	3		

(a) 4 (b) 6 (c) 7 (d) 9

12. How many squares are there in a chessboard?

(a) 204 (b) 128 (c) 64 (d) 32

13. Which number fits the empty circle?

9	1	5
4	8	3
2		7

(a) 5 (b) 6 (c) 9 (d) 2

14. Read the following phrase and answer the question:

As I was going to St. Ives,
I met a man with seven wives.
Each wife had seven sacks.
Each sack had seven cats.
Each cat has seven kits.
Kits, cats, sacks and wives,
How many people were going to St. Ives?

(a) 345 (b) 9 (c) 2 (d) 1

15. What is the value of x?

			x			
		9		25		
	4				36	
1		81		64		49

(a) 15 (b) 16 (c) 4 (d) 17

Direction for Q16–Q18: Akul, Kshitij, Nakul and Dhruv are four brothers playing a game where the loser doubles the money of each of the other players by giving them from his share at that point of time. They played four games and each brother lost one game in alphabetical order. At the end of the fourth game, each brother had ₹ 64.

16. Who started with the lowest amount?

(a) Akul (b) Kshitij (c) Nakul (d) Dhruv

17. How many rupees did Kshitij start with?

(a) 64 (b) 136 (c) 68 (d) 72

18. What was the amount left with Nakul at the end of the second round?

(a) 72 (b) 144 (c) 132 (d) 136

Direction for Q19–Q20: When zero was not invented, people could not multiply numbers. Thus, mathematics had only 9 digits (1–9) and after that came 11, 12 and so on. Find the answer to following operations in mathematics used then.

19. What is the value of $7 + 11 + 3$?

(a) 22 (b) 23 (c) 24 (d) 25

20. What is the value of $21 + 29 + 1$?

(a) 49 (b) 50 (c) 51 (d) 52

ANSWERS

1. (c)	**5.** (d)	**9.** (d)	**13.** (b)	**17.** (c)
2. (c)	**6.** (b)	**10.** (c)	**14.** (d)	**18.** (b)
3. (b)	**7.** (d)	**11.** (b)	**15.** (b)	**19.** (a)
4. (b)	**8.** (a)	**12.** (a)	**16.** (d)	**20.** (d)

EXPLANATIONS AND HINTS

1. (c) Total sweets given to the beggars = 3

If x is the total number of sweets that Disha originally had, then total sweets offered in the gurdwara $= x - \left(\frac{x}{2} - 1\right) = \frac{x}{2} + 1$

Total sweets left with her after giving to the beggars $= \frac{x}{2} + 1 - 3 = \frac{x}{2} - 2$

Total sweets left with her after giving to poor children $= \frac{1}{2}\left(\frac{x}{2} - 2\right) = \frac{x}{4} - 1$

Total number of sweets left with her after giving her brother = 2

Hence,

$$\frac{x}{4} - 1 = 2$$
$$\Rightarrow x = 3 \times 4 = 12$$

Thus, Disha originally had 12 sweets.

2. (c) Total sweets offered in gurdwara $= \frac{x}{2} - 1$

We calculated that $x = 12$

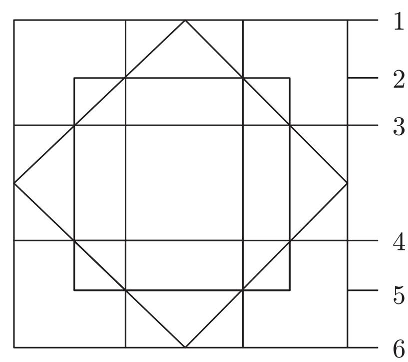
Hence, total sweets offered in gurdwara $= \frac{12}{2} - 1 = 5$

3. (b) Total sweets given to poor children $= \frac{1}{2}\left(\frac{x}{2} - 2\right)$

Since, $x = 12$, total sweets given to poor children $= \frac{1}{2}\left(\frac{12}{2} - 2\right) = 2$

4. (b) Mary's mother's first child was Mary herself.

5. (d)



Number of non-overlapping triangles between 1 and 2 = 3

Number of non-overlapping triangles between 2 and 3 = 4

Number of non-overlapping triangles between 3 and 4 = 6

Number of non-overlapping triangles between 4 and 5 = 4

Number of non-overlapping triangles between 5 and 6 = 3

Number of overlapping triangles = 8

Total number of triangles $= 3 + 4 + 6 + 4 + 3 + 8 = 28$

6. (b) Let the day be x.

Now, x comes 3 days after the day which comes 2 days after the day which comes immediately after the day which comes 2 days after Monday.

Hence, x comes after $3+2+1+2=8$ days from Monday.

Hence, $x=$ Tuesday

7. (d) You will first light the match since you cannot light anything without the match.

8. (a) L is missing in the pyramid.

9. (d) PR does not occur together in an alphabetical order. WX and UV are not vertical neighbors in a given pyramid. ST occurs together in an alphabetical order, and also S and T are vertical neighbors in a given pyramid.

10. (c) F and U are separated by 14 letters in an alphabetical order, while Q and Y are separated by 7 letters. C and S are separated by 15 letters and H and Q are separated by 8 letters.

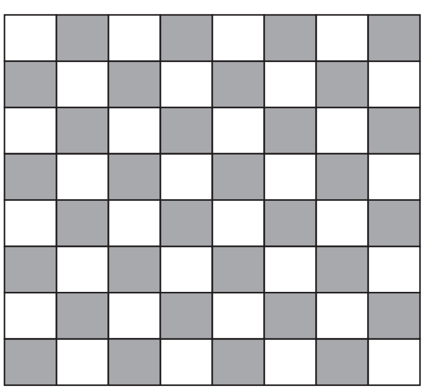
Thus, C and S are the farthest from each other among given pairs.

11. (b) Looking at the diagram in rows, the central number equals half the sum of the numbers in the other circles to the left and right of the center.

Hence,

$$x=\frac{9+3}{2}=6$$

12. (a)



Say the minimum value of the side of square is 1 unit.

Then, the total number of squares with side 1 cm $= 8\times 8=64$

Total number of squares with side 2 cm $=7\times 7=49$

Total number of squares with side 3 cm $=6\times 6=36$

Total number of squares with side 4 cm $=5\times 5=25$

Total number of squares with side 5 cm $=4\times 4=16$

Total number of squares with side 6 cm $=3\times 3=9$

Total number of squares with side 7 cm $=2\times 2=4$

Total number of squares with side 8 cm $=1\times 1=1$

Hence, total number of squares on the chessboard $= 64+49+36+25+16+9+4+1=204$

13. (b) The numbers in each row and column add up 15. Hence, the second column should also add up to 15.

Therefore,

$$1+8+x=15 \Rightarrow x=6$$

Thus, the circle should be filled with 6.

14. (d) The phrase clearly mentions, "As I was going to St. Ives."

Hence, only "I" was going. The correct answer is 1.

15. (b) Starting from Eq. (1) and moving clockwise around the triangle, numbers follow the sequence of square numbers. Hence, between 9 and 25 (i.e., square of 3 and 5, respectively), we will add 16.

Thus, $x=16$.

16. (a) The four brothers play in such a way that the loser doubles the money of the other players and they lose in an alphabetical order. Thus, working backwards we get

	Akul	Kshitij	Nakul	Dhruv
Round 4	64	64	64	64
Round 3	32	32	32	160
Round 2	16	16	144	80
Round 1	8	136	72	40
Initially	132	68	36	20

Thus, Dhruv started with the lowest amount.

17. (c) Kshitij started with ₹68.

18. (b) At the end of the second round, Nakul was left with ₹144.

19. (a) We have to calculate $7+11+3$

Now, $7+3=10$.

However, since 0 was not invented, so

$$7+3=11$$
$$11+11=22$$

20. (d) We have to calculate $21+29+1$

Now, $29+1=30$

Since, 0 is not invented, $29+1=30$

$$\Rightarrow 31+21=52$$

CHAPTER 8

ANALYTICAL REASONING

INTRODUCTION

Analytical reasoning represents judgments made on statements that are based on the virtue of the statement's own content. Beyond an understanding of the meanings of the words used, no particular experience or knowledge is required for analytical reasoning.

SOLVED EXAMPLES

Direction for Q1—Q5: Read the passage and answer the following questions:

A, B, C and D are four friends living together in a flat and they have an agreement that whatever edible comes they will share equally among themselves. One day A's uncle came to him and gave a box of chocolates. Since nobody was around, A divided that chocolates in four equal parts and ate his share after which he put the rest in the box. As he was closing the box, B walked in and took the box. He again divided remaining chocolates in four equal parts. A and B ate one part each and kept the remaining chocolates in the box. Suddenly C appeared and snatched the box. He again divided the chocolates in four equal parts; the three of them ate one part each and kept the remaining chocolates in the box. Later when D came, he again divided the chocolates in four equal parts and all four ate their share, respectively. In total D ate three chocolates.

1. How many chocolates did C eat in total?
2. How many chocolates did B eat in total?
3. How many chocolates did A eat in total?
4. How many chocolates were given to A by his uncle?
5. How many chocolates did A eat the first time?

Solution: We know that D ate three chocolates.

Before D ate the chocolates, there should be $3 \times 4 = 12$ chocolates.

Before C ate the chocolates, there should be $12 \times 3 + 12 = 48$ chocolates.

Before B ate the chocolates, there should be $24 \times 2 + 48 = 96$ chocolates.

Before A ate the chocolates, there should be $\dfrac{96 \times 4}{3} = 128$ chocolates.

At the beginning, we have 128 chocolates.

	A	B	C	D	Remaining
A's share	32				96
B's share	24	24			48
C's share	12	12	12		12
D's share	3	3	3	3	0
Total	71	39	15	3	

1. C ate 15 chocolates in total.

2. B ate 39 chocolates in total.

3. A ate 71 chocolates in total.

4. 128 chocolates were given to A by his uncle.

5. A ate 32 chocolates the first time.

Direction for Q6–Q10: Read the passage and answer the following questions:

Five courses: A, B, C, D and E each of one month duration are to be taught from January to May one after the other though not necessarily in the same order by lecturers P, Q, R, S and T. P teaches course B but not in the month of April or May. Q teaches course A in the month of March. R teaches in the month of January but does not teach course C or D.

6. What course is taught by S?

7. Which lecturer's course immediately follows course B?

8. Which course is taught in the month of January?

9. When was Q's course taught?

10. Which lecturer taught in February?

Solution: Let us consider the conditions given in the passage.

1. P teaches course B but not in the month of April or May.

2. Q teaches course A in the month of March.

3. R teaches in the month of January but does not teach courses C or D. Now, we make a tabular chart.

Months → Course ↓	January	February	March	April	May
A	×	×	✓	×	×
B	×	✓	×	×	×
C	×	×	×		
D	×	×	×		
E	✓	×	×	×	×

Lecturer → Course ↓	P	Q	R	S	T
A	×	✓	×	×	×
B	✓	×	×	×	×
C	×	×	×		
D	×	×	×		
E	×	×	✓	×	×

6. Either C or D.

7. Q's course A immediately follows course B.

8. Course E is taught in the month of January.

9. Q taught course A in the month of March.

10. In February, course B was taught. Course B is taught by lecture P.

Direction for Q11–Q14: Read the passage and answer the following questions:

Each of the five friends A, B, C, D and E have different likings, viz., reading, playing, travelling, singing and writing.

B has liking for singing whereas C does not like reading or writing. E likes to play, while A has liking for writing.

11. What does D like?

12. What does C like?

13. If reading, singing and writing are categorized as indoor likings whereas travelling and playing are categorized as outdoor likings, then which friends have indoor likings?

14. How many friends have outdoor likings?

Solution: From the passage, we can tabulate the following data:

Liking → Friends ↓	Read	Play	Travel	Sing	Write
A	×	×	×	×	✓
B	×	×	×	✓	×
C	×	×	✓	×	×
D	✓	×	×	×	×
E	×	✓	×	×	×

11. D likes to read.

12. C likes to travel.

13. A, B and D have indoor likings.

14. C and E have outdoor likings.

Direction for Q15–Q20: Read the passage and answer the following questions:

In a family of six, there are three men L, M and N and three women R, S and T. The six persons are Architect, Lawyer, CA, Professor, Doctor and Engineer by profession but not in the same order.

- There are two married couples and two unmarried persons.
- N is not R's husband.
- The Doctor is married to a Lawyer. R's grandfather is a Professor.
- M is neither L's son, nor an Architect or a Professor.
- The Lawyer is T's daughter-in-law.
- N is T's son and the Engineer's father.
- L is married to the CA.

15. How is T related to M?

16. Which are the correct pairs of married couples?

17. What is the profession of L's wife?

18. Who is an architect?

19. What is the profession of M's father?

20. Who are the two unmarried persons?

Solution: Professor is the grandfather of R therefore, R will be in third generation and professor will be in the first generation. Now it is given that Doctor is married to lawyer and lawyer is T's daughter-in-law. From here it is clear that one of the married couples is Doctor-Lawyer and is in the second generation. T is the wife of the professor in the first generation. From the information (2) and (4), we conclude that married couples are L – T and N – S.

Hence, we can tabulate the result as follows:

Grand-parents	1st generation	L-Professor	T-C.A.
Parents	2nd generation	N-Doctor	S-Lawyer
Children	3rd generation	R-Architect	M-Engineer

15. We know that T is a female and T is a grandparent of M. Hence, T is M's grandmother.
16. L-T and N-S are the two married couples.
17. L's wife T is a C.A.
18. R is the architect.
19. M's father is N and he is a doctor.
20. R and M are the two unmarried couples.

PRACTICE EXERCISE

Direction for Q1–Q4: Read the information given below and answer the following questions.

There is a family of six persons A, B, C, D, E and F. Their professions are Engineer, Doctor, Teacher, Salesman, Manager and Lawyer. There are two married couples in the family. The Manager is the grandfather of F, who is an Engineer. C, the Salesman, is married to the lady Teacher. B is the mother of F and E. The doctor, D is married to the Manager.

1. How many male members are there in the family?

(a) Two (b) Three
(c) Four (d) Data inadequate

2. What is the profession of A?

(a) Lawyer (b) Salesman
(c) Manager (d) Teacher

3. Who are the two married couples in the family?

(a) AB and DC (b) CF and DE
(c) AE and CD (d) AD and BC

4. How is A related to E?

(a) Father (b) Grandfather
(c) Mother (d) Wife

Direction for Q5–Q9: Read the information given below and answer the following questions.

A, B, C, D, E, E, G and H are eight friends. Three of them play cricket and table tennis each and two of them play football. Each one of them has a different height. The tallest does not play football and the shortest does not play cricket. F is taller than A and D but shorter than H and B. E, who does not play cricket, is taller than B and is second to the tallest. G is shorter than D but taller than A. H, who is fourth from the top, plays table tennis with D. G does not play either cricket or football, B does not play football.

5. Which of the following pairs of friends play football?

(a) EF (b) EA
(c) HF (d) Data inadequate

6. What is F's position from the top when they are arranged in descending order of their height?

(a) Fifth (b) Sixth
(c) Seventh (d) None of these

7. Who is the tallest?

(a) A (b) B
(c) C (d) H

8. Who is the shortest?

(a) A (b) G
(c) D (d) F

9. Which of the following group of friends play cricket?

(a) C-A-E (b) C-B-F
(c) C-B-A (d) None of these

Direction for Q10–Q14: Read the information given below and answer the following questions.

There is a family of six members A, B, C, D, E and F. There are two married couples in the family and the family members represent three generations. Each member has a distinct choice of a colour among Green, Yellow, Black, Red, White and Pink.

No lady member like either Green or White.

C, who likes Black colour, is the daughter-in-law of E.

B is the brother of F and son of D and like Pink.

A is the grandmother of F and F does not like Red.

The husband has a choice for Green colour, but his wife likes Yellow.

10. Which of the following is the colour preference of A?

(a) Red (b) Yellow
(c) Either Yellow or Red (d) None of these

11. Which of the following is the colour combination of one of the couples?

(a) Yellow–Red (b) Green–Black
(c) Red–Yellow (d) Yellow–Green

12. Which of the following is one of the married couples?

(a) CD (b) DA
(c) AC (d) Cannot be determined

13. Which of the following is true about F?

(a) Sister of B (b) Brother of B
(c) Daughter of C (d) None of the above

14. How many male members are there in the family?

(a) Two (b) Three
(c) Four (d) None of these

Direction for Q15–Q17: Read the information given below and answer the following questions.

Aman planted some plants in his lawn but in a certain fixed pattern:

In most of the rows there is neither Rose nor Marigold.

There are two more rows of Orchids than Tulips and two more rows of Rose than Orchids.

There are four more rows of Rose than Tulips.

There are not as many rows of Lily as Fireball.

There is one less Marigold row than Rose.

There is just one row of Tulips.

The maximum number of rows he planted is six.

15. How many rows of Rose he planted?

(a) Two (b) Five
(c) Four (d) Cannot be determined

16. What is the sum of the rows of Orchids and Marigold he planted?

(a) 7 (b) 5
(c) 9 (d) Cannot be determined

17. How many rows of Fireball did he plant?

(a) 2 (b) 5
(c) 6 (d) Cannot be determined

Direction for Q18–Q20: Read the following passage carefully and answer the following questions.

Three kids, Sam, Mona and Tanya went on a picnic with their dog Skipper and carried with them few chocolates, without counting. They rested under a tree and slept for a while. After sometime, Sam woke up, gave one chocolate from the total to the Skipper and distributed the remaining into three equal parts, ate his share and slept. After some time, Mona woke up, gave one chocolate to the Skipper and distributed the remaining into three equal parts, ate her share and slept. After sometime Tanya woke up and repeated the same. A little later, all of them woke up together, gave one chocolate from the total to Skipper and divided the remaining chocolates among them and each ate their share. However, we know that no chocolate was broken and there were less than 150 chocolates.

18. How many chocolates were there in the beginning?

(a) 66 (b) 84
(c) 118 (d) 79

19. In what ratio did Sam and Tanya eat the chocolates?

(a) 11:6 (b) 6:11
(c) 26:11 (d) 7:26

20. How many more chocolates did Tanya eat from Skipper?

(a) 18 (b) 12
(c) 22 (d) 14

ANSWERS

1. (d)	**3.** (d)	**5.** (b)	**7.** (c)	**9.** (b)	**11.** (d)	**13.** (b)	**15.** (b)	**17.** (c)	**19.** (a)
2. (c)	**4.** (b)	**6.** (a)	**8.** (a)	**10.** (b)	**12.** (a)	**14.** (b)	**16.** (a)	**18.** (d)	**20.** (d)

EXPLANATIONS AND HINTS

Solutions to Q1–Q4: From the information given in the question it is clear that there is a family tree of three generations.

1st Generation	A (Manager)	D (Doctor)
2nd Generation	C (Salesman)	B (Teacher)
3rd Generation	F (Engineer)	E (Lawyer)

Also, from the information, it is clear that A-D and B-C are married couples whereas sex of E and F cannot be determined from the information given.

1. (d) Since the sex of E and F cannot be determined. Therefore, we cannot find the total number of male members from the given data.

2. (c) A is the Manager.

3. (d) The two married couples are A-D and B-C.

4. (b) A is the grandfather of E.

Solutions to Q5–Q9: We know that three play Cricket, three play TT and two play Football.

Also, each of them has a different height, say (1) being the tallest and (8) being the shortest.

Also, tallest does not play Football and shortest does not play Cricket.

Therefore, (1) Fb (8) Cr

Now, F is taller than A and D but shorter than H and B. Therefore,

↑ H, B
F
A, D

Now, the heights of H and B, and A and D are not given.

E does not play Cricket, hence

$$E \to \text{Cr}$$

and is taller than B and also second to the tallest. So E's position is (2). G is shorter than D but taller than A. Hence,

↑ D
G
A

H is fourth from top so its position is H → (4)

H plays TT with D so,

$$\text{H} \to \text{TT}$$
$$\text{D} \to \text{TT}$$

G does not play Cricket or Football, hence

$$\text{G} \to \text{TT}$$

Now, we can conclude that

Order of Height	Friend	Cricket	Football	Table tennis
(1)	C	✓	×	
(2)	E	×	✓	
(3)	B	✓		
(4)	H			✓
(5)	F	✓		
(6)	D			✓
(7)	G			✓
(8)	A	×	✓	

5. (b) E and A play Football.

6. (a) F is fifth from the top.

7. (c) C is the tallest.

8. (a) A is the shortest.

9. (b) C-B-F play cricket.

Solutions to Q10–Q14: From the information given to us, we can make the following table:

Grandparents	1st Generation	Green	E – Male	A – Female	Yellow
Parents	2nd Generation	Red	D – Male	C – Female	Black
Children	3rd Generation	White	F – Male	B – Female	Pink

10. (b) A prefers Yellow.

11. (d) Green–Yellow is the colour combination of one of the couples.

12. (a) CD is one of the married couples.

13. (b) Since no lady member likes either Green or White. Hence F will be a male and brother of B.

14. (b) There are three male members, E, D and F.

Solutions to Q15–Q17: Say number of rows of Orchids, Tulips, Roses and Marigold is O, T, R and M.

Then,

$$\mathrm{O} = \mathrm{T} + 2 \quad (1)$$

$$\mathrm{R} = \mathrm{O} + 2 \quad (2)$$

$$\mathrm{R} = \mathrm{M} + 1 \quad (3)$$

and $\mathrm{T} = 1$

Solving Eqs. (1)–(3), we get

$$\mathrm{O} = 3,\ \mathrm{R} = 5 \text{ and } \mathrm{M} = 4$$

Also, we know that there are not as many rows of Lily as Fireball.

Lily < Fireball

So,

Lily = 2, Fireball = 6

15. (b) Total rows of Roses planted = 5

16. (a) Total rows of Orchids = 3

Total rows of Marigold = 4

Therefore, sum of rows of Orchids and Marigold planted = 7

17. (c) Total rows of Fireball planted = 6

Solutions to Q18–Q20: Let x be the number of chocolates in the beginning.

When Sam wakes up, total chocolates left for others $= \dfrac{2}{3}(x-1)$

When Mona wakes up, total chocolates left for others $= \dfrac{2}{3}\left[\dfrac{2}{3}(x-1)-1\right] = \dfrac{4x-10}{9}$

Similarly, when Tanya wakes up, total chocolates left $= \dfrac{2}{3}\left[\left(\dfrac{4x-10}{9}\right)-1\right] = \dfrac{8x-38}{27}$

Then, $\dfrac{8x-38}{27} - 1$ was divided into three equal parts.

$\Rightarrow 8x - 65$ is divisible by 81.

Thus, $8x - 65 = 81n$ where n is an integer.

So, $81n + 65$ should be divisible by 8. Thus, $81n + 65$ is an even number and $81n$ is odd.

Now, putting values $n = 1, 3, 5\ldots$ we get $n = 7$ and 15.

However, $n = 15$ is not possible since number of chocolates is less than 150.

Thus, $n = 7$ is the accepted value.

Number of chocolates initially $= \dfrac{(81\times 7)+65}{8} = 79$

Now, we can tabulate our results as follows:

	Sam	**Mona**	**Tanya**	**Skipper**
First	26	26	26	1
Second	17	17	17	1
Third	11	11	11	1
Fourth	7	7	7	1
Total = 79	33	24	18	4

18. (d) There were 79 chocolates initially.

19. (a) Sam ate a total of 33 chocolates.

Tanya ate a total of 18 chocolates.

Ratio = 33:18 = 11:6

20. (d) Tanya ate a total of 18 chocolates.

Skipper ate a total of 4 chocolates.

Hence, total chocolates that Tanya ate more than Skipper = 14

GENERAL APTITUDE

SOLVED GATE PREVIOUS YEARS' QUESTIONS

1. Which of the following options is the closest in meaning to the word below?

Circuitous

(a) Cyclic (b) Indirect
(c) Confusing (d) Crooked

(GATE 2010, 1 Mark)

Solution: Circuitous means deviating from a straight course ⇒ indirect

(a) Cyclic means recurring in cycle
(b) Indirect means not leading by straight line
(c) Confusing means lacking clarity
(d) Crooked means for shapes (irregular in shape)

Ans. (a)

2. The question below consists of a pair of related words followed by four pairs of words. Select the pair that best expresses the relation in the original pair.

Unemployed: Worker

(a) Fallow : Land (b) Unaware : Sleeper
(c) Wit : Jester (d) Renovated : House

(GATE 2010, 1 Mark)

Solution: Unemployed: Worker. Here one is opposite to other.

(a) Fallow: Land ⇒ Fallow means undeveloped land
(b) Unaware: Sleeper ⇒ Both are same
(c) Wit: Jester ⇒ Wit means ability to make jokes and jester means a joker
(d) Renovated: House ⇒ Renovate means to make better and house can be renovated

Ans. (a)

3. Choose the most appropriate word from the options given below to complete the following sentence:

If we manage to ___________ our natural resources, we would leave a better planet for our children.

(a) uphold (b) restrain
(c) cherish (d) conserve

(GATE 2010, 1 Mark)

Solution:

(a) Uphold means cause to remain ⇒ not appropriate
(b) Restrain means keep under control ⇒ not appropriate
(c) Cherish means be fond of ⇒ not related
(d) Conserve means keep in safety and protect from harm, decay, loss, or destruction ⇒ most appropriate

Ans. (d)

4. Choose the most appropriate word from the options given below to complete the following sentence:

His rather casual remarks on politics ___________ his lack of seriousness about the subject.

(a) masked (b) belied
(c) betrayed (d) suppressed

(GATE 2010, 1 Mark)

Solution:

(a) Masked means hide under a false appearance ⇒ opposite
(b) Belied means be in contradiction with ⇒ not appropriate
(c) Betrayed means reveal unintentionally ⇒ most appropriate
(d) Suppressed means to put down by force or authority ⇒ irrelevant

Ans. (c)

5. 25 persons are in a room. 15 of them play hockey, 17 of them play football and 10 of them play both hockey and football. Then the number of persons playing neither hockey nor football is

(a) 2 (b) 17
(c) 13 (d) 3

(GATE 2010, 1 Mark)

Solution: Using the set theory formula, we have

$n(A)$: Number of people who play hockey = 15

$n(B)$: Number of people who play football =17

$n(A \cap B)$: Persons who play both hockey and football = 10

$n(A \cup B)$: Persons who play either hockey or football or both

Using the formula, we have

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \cup B) = 15 + 17 - 10 = 22$$

Thus people who play neither hockey not football = 25 − 22 = 3

Ans. (d)

6. Modern warfare has changed from large scale clashes of armies to suppression of civilian populations. Chemical agents that do their work silently appear to be suited to such warfare; and regretfully, there exist people in military establishments who think that chemical agents are useful tools for their cause.

Which of the following statements best sums up the meaning of the above passage?

(a) Modern warfare has resulted in civil strife.
(b) Chemical agents are useful in modern warfare.
(c) Use of chemical agents in warfare would be undesirable.
(d) People in military establishments like to use chemical agents in war.

(GATE 2010, 2 Marks)

Solution:

(a) Modern warfare has resulted in civil strife: There is no direct consequence of warfare given, so it is not appropriate.

(b) Chemical agents are useful in modern warfare: Passage does not say whether chemical agents are useful or not, so it is not appropriate.

(c) Use of chemical agents in warfare would be undesirable: Given that people in military think these are useful, undesirable is wrong.

(d) People in military establishments like to use chemical agents in war. Correct choice as last statement tells that military people think that chemical agents are useful tools for their cause (work silently in warfare).

Ans. (d)

7. If 137 + 276 = 435 how much is 731 + 672?

(a) 534 (b) 1403
(c) 1623 (d) 1531

(GATE 2010, 2 Marks)

Solution: 7 and 6 added is becoming 5 means the given two numbers are added on base 8.

$$\begin{array}{r} (137)_8 \\ +(276)_8 \\ \hline (435)_8 \end{array}$$

Hence, we have to add the another two given set of numbers also on base 8.

$$\begin{array}{r} (731)_8 \\ +(672)_8 \\ \hline (1623)_8 \end{array}$$

Hence, the overall problem was based on identifying base, which was 8, and adding number on base 8.

Ans. (c)

8. 5 skilled workers can build a wall in 20 days; 8 semiskilled workers can build a wall in 25 days; 10 unskilled workers can build a wall in 30 days. If a team has 2 skilled, 6 semiskilled and 5 unskilled workers, how long will it take to build the wall?

(a) 20 days (b) 18 days
(c) 16 days (d) 15 days

(GATE 2010, 2 Marks)

Solution: Let one day work of skilled semi-skilled and unskilled worker be a, b, c units, respectively.
$5a \times 20 = 8b + 25 = 10c \times 30$ = Total units of work

$$100a = 200b = 300c$$

$$a = 2b = 3c$$

$$\Rightarrow b = \frac{a}{2} \text{ and } c = \frac{a}{3}$$

Given that 2 skilled, 6 semi-skilled and 5 unskilled workers are working. Consider that they finish the work in 'x' days.

$(2a + 6b + 5c)x = 5a + 20$ = Total units of work

$$\left(2a + 3a + \frac{5}{3}a\right)x = 5a \times 20$$

$$\frac{20a}{3}x = 5a \times 20$$

$$x = 15 \text{ days}$$

Ans. (d)

9. Given digits 2, 2, 3, 3, 3, 4, 4, 4, 4 how many distinct 4 digit numbers greater than 3000 can be formed?

(a) 50 (b) 51
(c) 52 (d) 54

(GATE 2010, 2 Marks)

We have to make 4 digit numbers, so the number should start with 3 or 4, two cases possible;

Case 1: Thousands digit is 3. Now other three digits may be any from 2, 2, 3, 3, 4, 4, 4, 4.

(a) Using 2, 2, 3

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(b) Using 2, 2, 4

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(c) Using 2, 3, 3

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(d) Using 2, 3, 4

$$3! = 6 \text{ numbers are possible}$$

(e) Using 2, 4, 4

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(f) Using 3, 3, 4

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(g) Using 3, 4, 4

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(h) Using 4, 4, 4

$$\frac{3!}{3!} = 1 \text{ number is possible}$$

Total 4-digit numbers in Case 1 = 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 1 = 25

Case 2: Thousands digit is 4. Now other three digits may be any from 2, 2, 3, 3, 3, 4, 4, 4.

(a) Using 2, 2, 3

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(b) Using 2, 2, 4

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(c) Using 2, 3, 3

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(d) Using 2, 3, 4

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(e) Using 2, 4, 4

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(f) Using 3, 3, 3

$$\frac{3!}{3!} = 1 \text{ number is possible}$$

(g) Using 3, 3, 4

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(h) Using 3, 4, 4

$$\frac{3!}{2!} = 3 \text{ numbers are possible}$$

(i) Using 4, 4, 4

$$\frac{3!}{3!} = 1 \text{ number is possible}$$

Total 4-digit numbers in Case 2 = 3 + 3 + 3 + 3 + 3 + 3 + 1 + 3 + 3 + 1 = 26

Thus total 4 digits numbers using Case 1 and Case 2 = 25 + 26 = 51

Ans. (b)

10. Hair (H), Gita (G), Irfan (I) and Saira (S) are siblings (i.e. brothers and sisters). All were born on 1st January. The age difference between any two successive siblings (that is born one after another) is less than 3 years. Given the following facts:

1. Hari's age + Gita's age > Irfan's age + Saira's age.
2. The age difference between Gita and Saira is 1 year. However, Gita is not the oldest and Saira is not the youngest.
3. There are no twins.

In what order were they born (oldest first)?

(a) HSIG (b) SGHI (c) IGSH (d) IHSG

(GATE 2010, 2 Marks)

Solution: Suppose: Hari's age: H, Gita's age: G, Saira's age: S, Irfan's age: 1

- H + G > I + S
- Using statement (2) both G − S = I or S − G = I; G can't be oldest and S can't be youngest.
- There are no twins; thus using statement (2) either GS or SG is possible.

(a) HSIG: Not possible as there is between S and G which is not possible using statement (3).

(b) SGHI: SG order is possible, S > G > H; > I and G + H > S + I (possible) because if {S = G + I; and G = H + I and H = I + 2 then G + (I + 2) > (G + I) + 1}

(c) IGSH: According to this I > G and S > H; thus adding these both inequalities we get I + S > G + H which is possible of statement (2). Thus not possible.

(d) IHSG: According to this I > H and S > G; Thus adding both inequalities I + S > H which is opposite of statement (2). Thus not possible.

Ans. (b)

11. Choose the word from the options given below that is most nearly opposite in meaning to the given word:

Amalgamate

(a) merge (b) split
(c) collect (d) separable

(GATE 2011, 1 Mark)

Solution: The answer is *split*.

Ans. (b)

12. Which of the following options is the closest, in the meaning to the word below:

Inexplicable

(a) Incomprehensible (b) Indelible
(c) Inextricable (d) Infallible

(GATE 2011, 1 Mark)

Solution: The answer is *incomprehensible*.

Ans. (a)

13. If $\log(P) = (1/2)\log(Q) = (1/3)\log(R)$, then which of the following options is TRUE?

(a) $P^2 = Q^3R^2$ (b) $Q^2 = PR$
(c) $Q^2 = R^3P$ (d) $R = P^2Q^2$

(GATE 2011, 1 Mark)

Solution: $\log(P) = \dfrac{1}{2}\log(Q) = \dfrac{1}{3}\log(R)$

$$\Rightarrow \log(P) = \log(Q)^{1/2} = \log(R)^{1/3} = K$$

$$\Rightarrow P = (Q)^{1/2} = (R)^{1/3} = K$$

$$\Rightarrow P = K,\ Q = K^2,\ R = K^3 \qquad \text{(i)}$$

Only option (b), $Q^2 = PR$ satisfies

From Eq. (i), we have

$$Q^2 = (K^2)^2 = K^4$$
$$PR = K * K^3 = K^4$$

Here, $Q^2 = PR$ holds true.

Ans. (b)

14. Choose the most appropriate word(s) from the options given below to complete the following sentence.

In contemplated __________ Singapore for my vacation but decided against it.

(a) to visit (b) having to visit
(c) visiting (d) for a visit

(GATE 2011, 1 Mark)

Solution: The answer is 'visiting'.

Ans. (c)

15. Choose the most appropriate word from the options given below to complete the following sentence.

If you are trying to make a strong impression on your audience, you cannot do so by being understated, tentative or

(a) hyperbolic (b) restrained
(c) argumentative (d) indifferent

(GATE 2011, 1 Mark)

Solution: The answer is 'restrained'.

Ans. (b)

16. The question below consists of a pair of related words followed by four pairs of words. Select the pair that best expresses the relation in the original pair:

Gladiator : Arena

(a) dancer : stage (b) commuter : train
(c) teacher : classroom (d) lawyer : courtroom

(GATE 2011, 1 Mark)

Solution: Like gladiator fights in an arena as people watch, a dancer performs on a stage as audience watch his performance. Hence, dancer : stage best expresses the relation gladiator : arena.

Ans. (a)

17. There are two candidates P and Q in an election. During the campaign 40% of the voters promised to vote for P, and rest for Q. However, on the day of election 15% of the voters went back on their promise to vote for P and instead voted for Q. 25% of the voters went back on their promise to vote for Q and instead voted for P. Suppose, P lost by 2 votes, then what was the total number of voters?

(a) 100 (b) 110
(c) 90 (d) 95

(GATE 2011, 1 Mark)

Solution: Let there be overall x candidates. Initially, those who decided to vote for P and Q are $0.4x$ and $0.6x$, respectively.

On the day of election 15% of $0.4x = 0.06x$ went from P's side to Q's side and 25% of $0.6x = 0.15x$ went from Q's side to P's side.

Now, after transfer P has

$$0.4x - 0.06x + 0.15x = 0.49x$$

and after transfer Q has

$$0.6x - 0.15x + 0.06x = 0.51x$$

Given, in the question that P lost by 2 votes

$$\text{Q} - \text{P} = 2 \text{ votes}$$
$$0.51x - 0.49x = 0.02x = 2 \text{ votes}$$

Therefore, total number of votes are 100.

Ans. (a)

18. Choose the most appropriate word from the options given below to complete the following sentence:

It was her view that the country's problems had been ________ by foreign technocrats, so that to invite them to come back would be counter-productive.

(a) identified (b) ascertained
(c) exacerbated (d) analysed

(GATE 2011, 1 Mark)

Solution: The given sentence says that inviting foreign technocrats back would be counter-productive. Hence, we can conclude that the foreign technocrats are the reason for the country's problems. Therefore, the best answer from the options is *exacerbated.*

Ans. (c)

19. Choose the word from the options given below that is most nearly opposite in meaning to the given word:

Frequency

(a) periodicity (b) rarity
(c) gradualness (d) persistency

(GATE 2011, 1 Mark)

Solution: The term *frequency* is associated with how quickly the change is occurring. The word opposite to it is *gradualness* which refers to slow development of a process.

Ans. (c)

20. Choose the most appropriate word from the options given below to complete the following sentence:

Under ethical guidelines recently adopted by the Indian Medical Association, human genes are to be manipulated only to correct diseases for which ________ treatments are unsatisfactory.

(a) similar (b) most
(c) uncommon (d) available

(GATE 2011, 1 Mark)

Solution: Uncommon

Ans. (c)

21. A container originally contains 10 litres of pure spirit. From this container 1 litre of spirit is replaced with 1 litre of water subsequently, 1 litre of the mixture is again replaced with 1 litre of water and this processes is repeated one more time. How much spirit is now left in the container?

(a) 7.58 litres (b) 7.84 litres
(c) 7 litres (d) 7.29 litres

(GATE 2011, 2 Marks)

Solution: We have 10 liters of pure spirit. Now 1 liter is replaced with water.

The mixture now has 9 liters of spirit and 1 liter of water. Thus, when 1 liter of mixture is removed, we remove 0.9 liter of spirit and 0.1 liter of water.

Now, the mixture has 8.1 liters of spirit and 1.9 liters of water. When 1 liter of mixture is removed, we remove 0.81 liter of spirit and 0.19 liter of water and replace it with 1 liter of water.

Therefore, the mixture now contains 7.29 liters.

Ans. (d)

22. Few school curricula include a unit on how to deal with bereavement and grief, and yet all students at some point in their lives suffer from losses through death and parting.

Based on the above passage which topic would not be included in a unit on bereavement?

(a) how to write a letter of condolence
(b) what emotional stages are passed through in the healing process
(c) what the leading causes of death are
(d) how to give support to a grieving friend

(GATE 2011, 2 Marks)

Solution: Option (c) would not be included.

Ans. (c)

23. The variable cost (V) of manufacturing a product varies according to the equation V = $4q$, where q is the quantity produced. The fixed cost (F) of production of same product reduces with q according to the equation F = $100/q$. How many units should be produced to minimize the total cost (V + F)?

(a) 5 (b) 4
(c) 7 (d) 6

(GATE 2011, 2 Marks)

Solution: Total cost can be given as

$$\mathrm{TC} = 4q + \frac{100}{q}$$

For cost to be minimum, we have

$$\frac{d}{dq}\left(4q + \frac{100}{q}\right) = 0$$

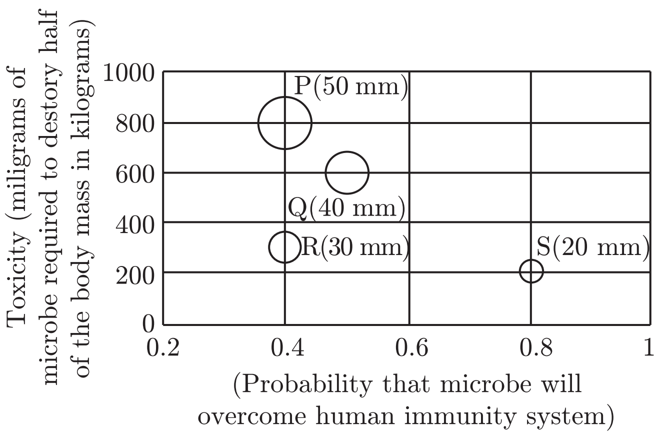
$$\Rightarrow q^2 = 25 \text{ or } q = 5$$

$$\frac{d}{dq}\left(4q + \frac{100}{q}\right)_{q=5} > 0$$

Hence, total cost is minimum at $q = 5$

Ans. (a)

24. P, Q, R and S are four types of dangerous microbes recently found in a human habitat. The area of each circle with its diameter printed in brackets represents the growth of a single microbe surviving human immunity system within 24 hours of entering the body. The danger to human beings varies proportionately with the toxicity, potency and growth attributed to a microbe shown in figure below:



A pharmaceutical company is contemplating the development of a vaccine against the most dangerous microbe. Which microbe should the company target in its first attempt?

(a) P (b) Q
(c) R (d) S

(GATE 2011, 2 Marks)

Solution: We can understand that the danger of a microbe to human being will be directly proportional to potency and growth. At the same time, it will be inversely proportional to toxicity defined (more dangerous will a microbe be if lesser of its milligram is required). So level of danger is proportional to $\frac{P\uparrow \times G\uparrow}{T\downarrow}$ where, P, G and T are the potency, growth and toxicity as defined in question.

So, $$D_i = \frac{KPG}{T} \quad \text{(i)}$$

where K is constant of proportionality. So level of danger of S will be maximum and is given by

$$D_P = \frac{0.4 \times \pi(50\text{mm})^2}{800} = 3.927$$

$$D_Q = \frac{0.5 \times \pi(40\text{mm})^2}{600} = 4.189$$

$$D_R = \frac{0.4 \times \pi(30\text{mm})^2}{300} = 3.77$$

$$D_S = \frac{0.8 \times \pi(20\text{mm})^2}{200} = 5.026$$

Hence, D_S is maximum and most dangerous among them.

Ans. (d)

25. A transporter receives the same number of orders each day. Currently, he has some pending orders (backlog) to be shipped. If he uses 7 trucks, then at the end of the 4^{th} day he can clear all the orders. Alternately, if he uses only 3 trucks, then all the orders are cleared at the end of the 10^{th} day. What is the minimum number of trucks required so that there will be no pending order at the end of the 5^{th} day?

(a) 4 (b) 5 (c) 6 (d) 7

(GATE 2011, 2 Marks)

Solution: Let y be the backlog with transporter and x be the number of orders each day. So, as per conditions given in question

$$4x + y = 28 \quad \text{(i)}$$

$$10x + y = 30 \quad \text{(ii)}$$

Solving Eqs. (i) and (ii),

$$x = \frac{1}{3} \text{ and } y = \frac{80}{3}$$

Since, we need to find out number of trucks so that no pending order will be there at the end of 5^{th} day.

$$5x + y = n \times 5$$

So we need to find n, where n is the number of trucks required.

$$n = \frac{5x+y}{5} = \frac{5 \times \frac{1}{3} + \frac{80}{3}}{5} = \frac{\frac{85}{3}}{5}$$

Hence, 5.66 truck will be required. As number of trucks have to be natural number, 6 trucks will be required.

Ans. (c)

26. The horse has played a little known but very important role in the field of medicine. Horses were injected with toxins of diseases until their blood built up immunities. Then a serum was made from their blood. Serums to fight with diphtheria and tetanus were developed this way.

It can be inferred from the passage that horses were

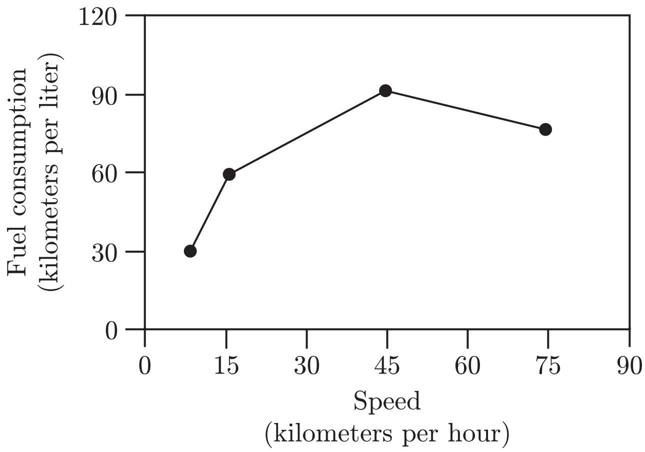
(a) given immunity to disease
(b) generally quite immune to disease
(c) given medicines to fight toxins
(d) given diphtheria and tetanus serums

(GATE 2011, 2 Marks)

Solution: It can be inferred that the horses were given medicines to fight toxins.

Ans. (c)

27. The fuel consumed by a motorcycle during a journey while travelling at various speeds is indicated in the graph below



The distances covered during four laps of the journey are listed in the table below:

Lap	Distance (kilometer)	Average speed (kilometer per hour)
P	15	15
Q	75	45
R	40	75
S	10	10

From the given data, we can conclude that the fuel consumed per kilometer was least during the lap

(a) P (b) Q (c) R (d) S

(GATE 2011, 2 Marks)

Solution: Fuel consumed per kilometer will we least, when mileage (kilometers per litres) mentioned on y axis of graph will be maximum irrespective of number of kilometers travelled.
From the graph we can observe that mileage (kilometers per litres) is maximum when vehicle is driven at 45 kilometers per hour. Thus, fuel consumption will be lowest for lap Q.

Ans. (b)

28. Three friends, R, S and T shared toffee from a bowl. R took $1/3^{\text{rd}}$ of the toffees, but returned four to the bowl. S took $1/4^{\text{th}}$ of what was left but returned three toffees to the bowl. T took half of the remainder but returned two back into the bowl. If the bowl had 17 toffees left, how many toffees were originally there in the bowl?

(a) 38 (b) 31 (c) 48 (d) 41

(GATE 2011, 2 Marks)

Solution: We are given that R took $1/3^{\text{rd}}$ of toffees initially. Therefore, total number of toffees have to be multiple of 3. There is only one option (c) as 48 which is multiple of 3.

Ans. (c)

29. Given that $f(y) = |y|/y$, and q is any non-zero real numerator, the value of $|f(q) - f(-1)|$ is

(a) 0 (b) −1 (c) 1 (d) 2

(GATE 2011, 2 Marks)

Solution: $f(y) = \dfrac{|y|}{y}$

As we know,

$$|y| = \left.\begin{matrix} y & y \geq 0 \\ -y & y < 0 \end{matrix}\right\} \qquad (1)$$

Therefore, from Eq. (1), we have

$$f(y) = \frac{y}{y} = 1 \qquad \text{if } y \geq 0$$

$$f(y) = \frac{-y}{y} = -1 \quad \text{if } y < 0$$

Hence, $|f(q) - f(-q) = |1 - (-1)| = 2$

Ans. (d)

30. The sum of n terms of the series 4 + 44 + 444 + … is

(a) $(4/81)\ [10^{n+1} - 9n - 1]$
(b) $(4/81)\ [10^{n-1} - 9n - 1]$
(c) $(4/81)\ [10^{n+1} - 9n - 10]$
(d) $(4/81)\ [10^{n} - 9n - 10]$

(GATE 2011, 2 Marks)

Solution: 4 + 44 + 444 +…
If we see first term of series = 4

Hence, sum up to 1st term is also = 4

Put $n = 1$ (first term), only option (c) satisfies.

$$\frac{4}{81}\left[10^{n+1} - 9n - 10\right] = \frac{4}{81}[10^{1+1} - 9 \times 1 - 10]$$

$$= \frac{4}{81} \times 81 = 4$$

Hence, option (c) is correct answer. (Note: Always solve these questions by putting values.)

Alternative Solution

$$4 + 44 + 444 + \ldots$$

$$= \frac{4}{9}[9 + 99 + 999 + \ldots]$$

$$= \frac{4}{9}\left[(10-1)+(10^2-1)+(10^3-1)+\ldots\right]$$

$$= \frac{4}{9}\left[(10+10^2+10^3+\ldots+10^n)-(1+1+1)\ldots\right]$$

$$= \frac{4}{81}\left[10^{n+1}-9n-10\right]$$

Ans. (c)

31. The cost function for a product in a firm is given by $5q^2$, where q is the amount of production. The firm can sell the product at a market price of 50 per unit. The number of units to be produced by the firm such that the profit is maximized is

(a) 5 (b) 10 (c) 15 (d) 25

(GATE 2012, 1 Mark)

Solution: $P = 50q - 5q^2$

For max/min, $\frac{dP}{dq} = 0$

$\Rightarrow 50 - 10q = 0$

$q = 5$

$$\left.\frac{d^2P}{dq^2}\right|_{q=5} = -10 \text{ which is negative}$$

Therefore, maximum profit will happen at $q = 5$.

Ans. (a)

32. Choose the most appropriate alternative from the options given below to complete the following sentence:

Despite several ________ the mission succeeded in its attempt to resolve the conflict.

(a) attempts (b) setbacks
(c) meetings (d) delegations

(GATE 2012, 1 Mark)

Solution: *Setbacks* is the correct answer.
Despite several setbacks the mission succeeded in its attempt to resolve the conflict.

The word *despite* indicates that there has to be a contrast in the sentence, use of the word *setbacks* in the blank indicates that despite many problems the mission was successful.

Ans. (b)

33. Which one of the following options is the closest in meaning to the word given below?

Mitigate

(a) Diminish (b) Divulge
(c) Dedicate (d) Denote

(GATE 2012, 1 Mark)

Solution: *Diminish* is the correct answer.
Mitigate means to reduce, to lessen etc. So, only the word *diminish* is close. Rest all choices have no link with the given word.

Ans. (a)

34. Choose the grammatically INCORRECT sentence:

(a) They gave us the money back less than service charges to Three Hundred rupees.
(b) This country's expenditure is not less than that of Bangladesh.
(c) The committee initially asked for a funding of Fifty Lakh rupees, but later settled for a lesser sum.
(d) This country's expenditure on educational reforms is very less.

(GATE 2012, 1 Mark)

Solution: The grammatically incorrect sentence is "They gave us the money back less than service charges to Three Hundred rupees."

Ans. (a)

35. Choose the most appropriate alternative from the options given below to complete the following sentence:

Suresh's dog is the one ________ was hurt in the stampede.

(a) that (b) which (c) who (d) whom

(GATE 2012, 1 Mark)

Solution: Suresh's dog is the one that was hurt in the stampede.
That is used with restrictive clauses.

Ans. (a)

36. Choose the most appropriate alternative from the options given below to complete the following sentence:

If the tired soldier wanted to lie down, he ________ the mattress out on the balcony.

(a) should take (b) shall take
(c) should have taken (d) will have taken

(GATE 2012, 1 Mark)

Solution: If the tired soldier wanted to lie down, he *should have taken* the mattress out on the balcony.
As the sentence is in past tense, options (a), (b) and (d) are incorrect.

Ans. (c)

37. If $(1.001)^{1259} = 3.52$, $(1.001)^{2052} = 7.85$, then $(1.001)^{3321}$

(a) 2.23 (b) 4.33 (c) 11.37 (d) 27.64

(GATE 2012, 1 Mark)

Solution: $(1.001)^{1259} = 3.52$ and $(1.001)^{2062} = 7.85$
$(1.001)^{1259} \times (1.001)^{2062} = (1.001)^{3321}$

$$3.52 \times 7.85 = (1.001)^{3321} \text{ (as } a^m \times a^n = a^{m+n})$$

Hence $(1.001)^{3321} = 27.632 \approx 27.64$

Ans. (d)

38. One of the parts (A, B, C, D) in the sentence given below contains an ERROR. Which one of the following is INCORRECT?

I requested that he should be given the driving test today instead of tomorrow.

(a) requested that (b) should be given
(c) the driving test (d) instead of tomorrow

(GATE 2012, 1 Mark)

Solution: The incorrect part of the sentence is should be given.

Ans. (b)

39. Which one of the following options is the closest in meaning to the word given below?

Latitude

(a) Eligibility (b) Freedom
(c) Coercion (d) Meticulousness

(GATE 2012, 1 Mark)

Solution: *Latitude* refers to freedom of action, freedom of expression from restrictions etc. For example, he allowed his children a fair amount of latitude.

Coercion refers to force which is an opposite of the word *latitude*. *Meticulousness* refers to being extremely careful and conscientious.

Ans. (b)

40. Choose the most appropriate word from the options given below to complete the following sentence:

Given the seriousness of the situation that he had to face, his ____________ was impressive.

(a) beggary (b) nomenclature
(c) jealously (d) nonchalance

(GATE 2012, 1 Mark)

Solution: Given the seriousness of the situation that he had to face, his nonchalance was impressive.
Nonchalance refers to casual/non-serious/indifferent attitude. The author wants to convey that in spite of the seriousness of the situation, he was very casual.

No other word fits in the blank. *Beggary* is word that is related to extreme poverty (Beggar). *Nomenclature* refers to the art of naming.

Ans. (d)

41. Wanted Temporary, Part-time persons for the post of Field Interviewer to conduct personal interviews to collect and collate economic data. Requirements; High School-pass, must be available for Day, Evening and Saturday work, Transportation paid, expenses reimbursed.

Which one of the following is the best inference from the above advertisement?

(a) Gender-discriminatory
(b) Xenophobic
(c) Not designed to make the post attractive
(d) Not gender-discriminatory

(GATE 2012, 2 Marks)

Solution: Option (a) cannot be considered since there is no gender discrimination mentioned in the argument.

Option (b), xenophobic is one who has fear of foreigners, no link with the given argument.

Option (c), it is wrong to say that the profile has not been designed to make the post attractive, since there are certain features which have been added to make the profile lucrative (which are given towards the end of the advertisement, like Transportation paid, expenses reimbursed).

Hence, the best inference is *Not gender-discriminatory*.

Ans. (d)

42. A political party orders an arch for the entrance to the ground in which the annual conventions is being held. The profile of the arch follows the equation $y = 2x - 0.1x^2$ where y is the height of the arch in meters. The maximum possible height of the arch is

(a) 8 meters (b) 10 meters
(c) 12 meters (d) 14 meters

(GATE 2012, 2 Marks)

Solution: $y = 2x - 0.1x^2$

For y (height) to be maximum, $\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = 2x - 2x = 0$$

$$\Rightarrow x = 10$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=10} = -0.2(-\text{ve}),\ \frac{d^2y}{dx^2} < 0$$

Thus, maximum height will be at $x = 10$.

$$y = 2 \times 10\ 0\ 0.1 \times (10)^2 = 10$$

Thus, the maximum height = 10 meters.

Ans. (b)

43. An automobile plant contracted to buy shock absorbers from two supplies X and Y. X supplies 60% and Y supplies 40% of the shock absorbers. All shock absorbers are subjected to a quality test. The ones that pass the quality test are considered reliable. Of X's shock absorbers, 96% are reliable. OF Y's shock absorbers, 72% are reliable.

The probability that a randomly chosen shock absorber, which is found to be reliable, is made by Y is

(a) 0.288 (b) 0.334 (c) 0.667 (d) 0.720

(GATE 2012, 2 Marks)

Solution: Let total 100 shock absorbs are supplied. Then,
X supplies = 60 (60% of 100)

Y supplies = 40 (40% of 100)

Reliable supply by X = 96% of 60 = 57.6

Reliable supply by Y = 72% of 40 = 28.8

Therefore, the required probability that random chosen reliable shock absorber is made by Y is

$$\text{Required probability} = \frac{28.8}{57.6 + 28.8} = 0.333$$

= 0.334

Ans. (b)

44. Which of the following assertions are CORRECT?

P: Adding 7 to each entry in a list adds 7 to the mean of the list

Q: Adding 7 to each entry in a list adds 7 to the standard deviation of the list

R: Doubling each entry in a list doubles the mean of the list

S: Doubling each entry in a list leaves the standard deviation of the list unchanged

(a) P, Q (b) Q, R (c) P, R (d) R, S

(GATE 2012, 2 Marks)

Solution: $$\frac{x_1 + x_2 + x_3 \ldots x_N}{N} = \overline{x}$$

$$\frac{(x_1 + 7) + (x_2 + 7) + \cdots + (x_N + 7)}{N} = \overline{x} + 7$$

and $$\frac{2(x_1 + x_2 \ldots x_N)}{N} = 2\overline{x}$$

Therefore, only P, R are correct.

Ans. (c)

45. Given the sequence of terms, AD, CG, FK, JP, the next time is

(a) OV (b) OW (c) PV (d) PW

(GATE 2012, 2 Marks)

Solution: AD, CG, FK, JP are given terms.

We can observe that the first letter of the terms are differentiated by 1, 2 , 3 and so on. (A and C have 1 letter in between, C and F have 2 letters in between etc.). Thus, J and the next term will have 4 letters in between. The first letter of the next term is O.

Also in the series AD, CG, FK, JP, two letters in each term are separated by 2, 3, 4, 5 letters in between them, respectively. Hence, the next term should be OV as there should be a gap of 6 alphabets between O and V. Therefore, answer is OV.

Ans. (a)

46. Raju has 14 currency notes in his pocket consisting of only 20 notes and 10 notes. The total money value of the notes is 230. The number of 10 notes that Raju has is

(a) 5 (b) 6 (c) 9 (d) 10

(GATE 2012, 2 Marks)

Solution: Let the number of 20 notes be a and 10 notes be b.

So,

$$a + b = 14 \tag{1}$$

$$20a + 10b = 230 \tag{2}$$

Multiplying Eq. (1) by 10 and subtracting from Eq. (2), we get

$$a = 9$$

Putting a in Eq. (1), we get

$$b = 5$$

Hence, number of 10 notes are 5.

Ans. (a)

47. One of the legacies of the Roman legions was discipline. In the legions, military law prevailed and discipline was brutal. Discipline on the battlefield kept units obedient, intact and fighting, even when the odds and conditions were against them.

Which one of the following statements best sums up the meaning of the above passage?

(a) Thorough regimentation was the main reason for the efficiency of the Roman legions even in adverse circumstances.
(b) The legions were treated inhumanly as if the men were animals.

(c) Discipline was the armies' inheritance from their seniors.
(d) The harsh discipline to which the legions were subjected to led to the odds and conditions being against them.

(GATE 2012, 2 Marks)

Solution: Through regimentation was the main reason for the efficiency of the Roman legions even in adverse circumstances. As the author says, 'Discipline on the battlefield kept units obedient, intact and fighting, even when the odds and conditions were against them. So option (a) emerges as the best answer option.

We have been asked to summarize the paragraph in the best possible manner. Option (c), presents one-sided picture of the paragraph. Summary is the essence or the gist of the paragraph.

Option (b) and option (d) are wrong as per the information mentioned in the paragraph.

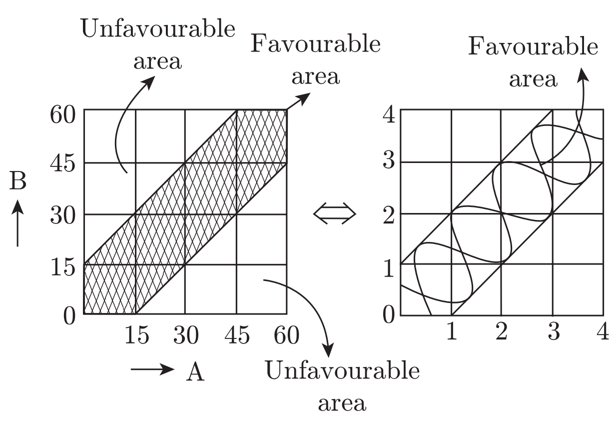
Ans. (a)

48. A and B are friends. They decide to meet between 1 PM and 2 PM on a given day. There is a condition that whoever arrives first will not wait for the other for more than 15 minutes. The probability that they will meet on that day is

(a) 1/4 (b) 1/16 (c) 7/16 (d) 9/16

(GATE 2012, 2 Marks)

Solution: Probability that A and B will meet will be given by the graphical representation where shaded region represents favourable area.



$$\text{Required probability} = \frac{\text{Favourable area}}{\text{Total area}} = \frac{7}{16}$$

Ans. (c)

49. The data given in the following table summarizes the monthly budget of an average household.

Category	Amount
Food	4000
Clothing	1200
Rent	2000
Savings	1500
Others	1800

The approximate percentage of the monthly budget NOT spent on savings is

(a) 10% (b) 14% (c) 81% (d) 86%

(GATE 2012, 2 Marks)

Solution:

Category	Amount
Food	4000
Clothing	1200
Rent	2000
Savings	1500
Others	1800
Total	10500

So total expenses are 10,500.

Expenses excluding savings are 9,000.

Therefore, required percentage

$$= \frac{9,000}{10,500} \times 100$$

$= 85.714\% \approx 86\%$

Ans. (d)

50. There are eight bags of rice looking alike, seven of which have equal and one is slightly heavier. The weighing balance is of unlimited capacity. Using this balance the minimum number of weighting required to identify the heavier bag is

(a) 2 (b) 3 (c) 4 (d) 8

(GATE 2012, 2 Marks)

Solution: In case of a weighing balance (i.e. beam balance), the following model can be observed:

$1 \rightarrow 3 \rightarrow$ weighing required

$4 - 9 - 2$ weighing required

$10 - 27 \rightarrow 3$ weighing required

$28 - 81 \rightarrow 4$ weighing required

and the process will go so on.

As there are eight objects. Minimum number of weighing required will be only 2.

Ans. (a)

51. Friendship, no matter how ________ it is, has its limitation.

(a) cordial (b) intimate
(c) secret (d) pleasant

(GATE 2013, 1 Mark)

Solution: Friendship, no matter how intimate it is, has its limitation.
Intimate refers to close personal relations, an intimate friend. It is also characterized by or involving warm friendship or a personally close or familiar association or feeling.

Ans. (b)

52. A number is as much greater than 75 as it is smaller than 117. The number is

(a) 91 (b) 93 (c) 89 (d) 96

(GATE 2013, 1 Mark)

Solution: We have to choose the largest option between 75 and 117 which is 96.

Ans. (d)

53. The pair that best express a relationship similar to that expression in the pair:

Medicine : Health

(a) Science : Experiment
(b) Wealth : Peace
(c) Education : Knowledge
(d) Money : Happiness

(GATE 2013, 1 Mark)

Solution: Medicine leads to good health. Similarly education leads to knowledge. Science does not lead to experiment. Wealth may not necessarily lead to peace. Also money may not also lead to happiness all the time.

Ans. (c)

54. Which of the following options is closest in meaning to the word given below: "Primeval"?

(a) Modern (b) Historic
(c) Primitive (d) Antique

(GATE 2013, 1 Mark)

Solution: Primeval pertains to the first age or ages, especially of the word. Example: primeval form of life.

Ans. (c)

55. The Professor (I) ordered to (II) the student to go (III) out of the class (IV)

The incorrect one is

(a) (I) (b) II (c) (III) (d) (IV)

(GATE 2013, 1 Mark)

Solution: The correct sentence is,

'The Professor ordered the students to go out of the class.'

Hence, the incorrect one is *ordered* to.

Ans. (b)

56. Complete the sentence:

Universalism is to particularism as diffuseness is to ________

(a) specificity (b) neutrality
(c) generality (d) adaptation

(GATE 2013, 1 Mark)

Solution: Specificity.
Universalism and particularism are opposite words. Similarly, we require a word that indicates an opposite to diffuseness.

Diffuseness refers to spreading or causing to spread in all directions. Specificity is an appropriate opposite.

Ans. (a)

57. Were you a bird, you ________ in the sky.

(a) would fly (b) shall fly
(c) should fly (d) shall have flown

(GATE 2013, 1 Mark)

Solution: Would fly.

This is a sentence based on conditional sentences. In the first clause imagination (were) has been used, so the main clause should comprise of 'would + I form of verb' usage.

Ans. (a)

58. Which one of the following options is the closest in meaning to the word given below?

Nadir

(a) Highest (b) Lowest
(c) Medium (d) Integration

(GATE 2013, 1 Mark)

Solution: *Lowest.*

Nadir which is opposite to the word Zenith (the highest point) means the lowest point.

Ans. (b)

59. Choose the grammatically, INCORRECT sentence:

(a) He is of Asian origin.
(b) They belonged to Africa

(c) She is an European
(d) They migrated from India to Australia

(GATE 2013, 1 Mark)

Solution: 'She is an European' is grammatically incorrect.

The usage of articles *a* and *an* is not decided by the first alphabet of the word but the sound/pronunciation of the word. European though starts with 'E' which is a vowel but it does not yield a vowel sound. When read aloud, it leaves the sound of 'Y' which is a consonant sound.

So the correct sentence should be, 'She is a European'.

Ans. (c)

60. What will be the maximum sum of 44, 42, 40, …?

(a) 502 (b) 504 (c) 506 (d) 5000

(GATE 2013, 1 Mark)

Solution: For sum to be maximum, we will consider only positive terms

$$44 + 42 + \cdots + 2$$

or
$$2 + 4 + 6 + \cdots + 2 + 44$$

$$2[1 + 2 + 3 + \cdots + 22]$$

$$= \frac{2[22 \times 23]}{2} = 506$$

Ans. (c)

61. Choose the grammatically CORRECT sentence:

(a) Two and two add four.
(b) Two and two become four.
(c) Two and two are four.
(d) Two and two make four

(GATE 2013, 1 Mark)

Solution: This is a famous idiom in English.

Ans. (d)

62. Statement: You can always give me a ring whenever you need.

Which one of the following is the best inference from the above statement?

(a) Because I have a nice caller tune.
(b) Because I have a better telephone facility.
(c) Because a friend in need is a friend indeed.
(d) Because you need not pay towards the telephone bills when you give me a ring.

(GATE 2013, 1 Mark)

Solution: 'You can always give me a ring whenever you need' implies that I am available even at a single call (whenever you need me). *Ring* has been used as a metaphor here and not literally.

Ans. (c)

63. Complete the sentence:

Dare ________ mistakes.

(a) commit (b) to commit
(c) committed (d) committing

(GATE 2013, 1 Mark)

Solution: Commit.

Ans. (a)

64. They were requested not to quarrel with others.

Which one of the following options is the closest in meaning to the word quarrel?

(a) make out (b) call out
(c) dig out (d) fall out

(GATE 2013, 1 Mark)

Solution: The word *quarrel* means to pick up a fight or have a heated argument. Thus, the phrase closest to it is to fall out.

Ans. (d)

65. In the summer of 2012, in New Delhi, the mean temperature of Monday to Wednesday was 41°C and of Tuesday to Thursday was 43°C. If the temperature on Thursday was 15% higher than that of Monday, then the temperature in °C on Thursday was

(a) 40 (b) 43 (c) 46 (d) 49.

(GATE 2013, 1 Mark)

Solution:

$$\frac{\text{Mon} + \text{Tue} + \text{Wed}}{3} = 41$$

$$\Rightarrow \text{Mon} + \text{Tue} + \text{Wed} = 123 \quad (1)$$

$$\frac{\text{Tue} + \text{Wed} + \text{Thu}}{3} = 43$$

$$\Rightarrow \text{Tue} + \text{Wed} + \text{Thu} = 129 \quad (2)$$

From Eq. (2) − Eq. (1), we get

$$\text{Thu} - \text{Mon} = 6$$

Also
$$\text{Thu} = 1.15 \text{ Mon}$$

$$1.15 \text{ Mon} - \text{Mon} = 6$$

$$\Rightarrow 15 \text{ Mon} = 6$$

$$\Rightarrow \text{Mon} = 40$$

$$\text{Thu} = 1.15 \text{ Mon}$$

$$\Rightarrow \text{Thu} = 1.15 \times 40$$

$$\Rightarrow \text{Thu} = 46$$

Ans. (c)

66. A firm is selling its product at 60/unit. The total cost of production is 100 and firm is earning total profit of 500. Later the total cost increased by 30%. By what percentage the price should be increased to maintain the same profit level.

(a) 5 (b) 15 (c) 10 (d) 30

(GATE 2013, 2 Marks)

Solution: Total cost of production = 100

Profit = 500

Total selling price = 500 + 100 = 600

Now, increased total cost of production = $100 + \frac{30}{100} \times 100 = 130$

Profit = 500

Increased total selling price = 500 + 130 = 630

Therefore, Percentage increase in selling price = $\frac{630 - 600}{600} \times 100 = 5\%$

Ans. (a)

67. Abhishek is elder to Savan, Savan is younger to Anshul. The correct relations is

(a) Abhishek is elder to Anshul
(b) Anshul is elder to Abhishek
(c) Abhishek and Anshul are of same age
(d) No conclusion can be drawn

(GATE 2013, 2 Marks)

Solution: It is given that both Abhishek and Anshul are elder than Savan. However, any statement to establish a relation between the two is not given. Hence, no conclusion can be drawn.

Ans. (d)

68. Following table provides figures (in rupees) on annual expenditure of a firm for two years 2010 and 2011

Category	2010	2011
Raw material	5200	6240
Power & fuel	7000	9450
Salary & wages	9000	12600
Plant & machinery	20000	25000
Advertising	15000	19500
Research & Development	22000	26400

In 2011, which of the following two categories have registered increase by same percentage?

(a) Raw material and salary and wages
(b) Salary and wages and advertising
(c) Power and fuel and advertising
(d) Raw material and research and development

(GATE 2013, 2 Marks)

Solution: % of raw material and salary and wages

$$= \frac{6240 - 5200}{5200} \times 100 = 20\%$$

% of raw material and research and development

$$= \frac{26400 - 22000}{22000} \times 100 = 20\%$$

Ans. (d)

69. If $|4x - 7| = 5$ then the value of $2|x| - |-x|$ is

(a) $2, \frac{1}{3}$ (b) $\frac{1}{2}, 3$ (c) $\frac{2}{3}, \frac{1}{3}$ (d) $\frac{2}{9}, 3$

(GATE 2013, 2 Marks)

Solution:
$$|4x - 7| = 5$$
$$4x - 7 = 5$$

and $4x - 7 = -5$

$\Rightarrow$ $x = 3$

and $x = 0.5$

$$2|x| = - |-x| = 2 \times 3 - 3 = 3$$

and $2|x| - |-x| = 2 \times \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$

Hence, the values are $\frac{1}{2}$ and 3.

Ans. (b)

70. x and y are two positive real numbers, such that equation

$$2x + y \leq 6; \quad x + 2y \leq 8$$

For which values of (x, y), the function $f(x, y) = 3x + 6y$ will give maximum value

(a) 4/3, 10/3 (b) 8/3, 20/3
(c) 8/3, 10/3 (d) 4/3, 20/3

(GATE 2013, 2 Marks)

Solution:
$$2x + y \leq 6$$
$$x + 2y \leq 8$$
$$x \leq 1.33$$
$$y \leq 3.33$$

Ans. (a)

71. Out of the 2-digit integers between 1 and 100, a 2-digit number has to be selected at random. What is the probability that the selected number is not divisible by 7?

(a) 13/90 (b) 12/90 (c) 78/90 (d) 77/90

(GATE 2013, 2 Marks)

Solution: There are ninety (90) two-digit numbers between 1 and 100 (typically 10 to 99). Among these numbers which are divisible by 7 are (14, 21, …, 98) only 13 numbers.

$$\left[\because \left(\frac{100}{7}\right) - 1 = 13\right]$$

So, the required probability that the number is not divisible by $1 - \frac{13}{90} = \frac{77}{90}$

Ans. (d)

72. A tourist covers half of this journey by train at 60 km/h, half of the remainder by bus at 30 km/h and the rest by cycle at 10 km/h. The average speed of the tourist in km/h during his entire journey is

(a) 36 (b) 30 (c) 24 (d) 18

(GATE 2013, 2 Marks)

Solution: Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$\frac{D}{\frac{D/2}{60} + \frac{D/2}{30} + \frac{D/4}{10}} = 24 \text{ kmph}$$

Ans. (c)

73. Find the sum of the expression

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + ... + \frac{1}{\sqrt{80}+\sqrt{81}}$$

(a) 7 (b) 8 (c) 9 (d) 10

(GATE 2013, 2 Marks)

Solution: The equation can be written as

$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + ... + \frac{1}{\sqrt{8}+\sqrt{80}}$$

Rationalize each term by multiplying it with its conjugate in both numerator and denominator.

$$\left[\frac{1}{\sqrt{2}+\sqrt{1}} \times \frac{\sqrt{2}-\sqrt{1}}{\sqrt{2}-\sqrt{1}}\right] \times \left[\frac{1}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right]$$
$$\times \left[\frac{1}{\sqrt{8}\times\sqrt{80}} \times \frac{\sqrt{81}-\sqrt{80}}{\sqrt{81}-\sqrt{80}}\right]$$

which can be resolved as

$$\left[\sqrt{2}-\sqrt{1}\right] + \left[\sqrt{3}-\sqrt{2}\right] + ... + \left[\sqrt{81}-\sqrt{80}\right]$$

(as denominator in each term will become 1)

Now cancelling like terms we will be left with

$$\sqrt{81} - \sqrt{1} = (9-1) = 8$$

Ans. (b)

74. The current erection cost of a structure is ₹. 13,200. If the labour wages per day increases by 1/5 of the current wages and the working hours decreases by 1/24 of the current period, then the new cost of erection in ₹., is

(a) 16,500 (b) 15,180 (c) 11,000 (d) 10,120

(GATE 2013, 2 Marks)

Solution: New cost $= 13200\left[1+\frac{1}{5}\right]\left[1-\frac{1}{24}\right]$

$$= 13200\left[\frac{6}{5}\right] \times \left[\frac{23}{24}\right]$$
$$= 15180$$

Ans. (b)

75. After several defeats in wars, Robert Bruce went in exile and wanted to commit suicide. Just before committing suicide, they came across a spider attempting tirelessly to have its net. Time and again, the spider failed but that did not deter it to refrain from making attempts. Such attempts by the spider made Bruce curious. Thus, Bruce started observing the near-impossible goal of the spider to have the net. Ultimately, the spider succeeded in having its net despite several failures. Such act of the spider encouraged. Bruce not to commit suicide. And then, Bruce went back again and won many a battle, and the rest is history.

Which one of the following assertions is best supported by the above information?

(a) Failure is the pillar of success
(b) Honesty is the best policy
(c) Life begins and ends with adventures
(d) No adversity justifies giving up hope

(GATE 2013, 2 Marks)

Solution: No adversity justifies giving up hope. Options (b) and (c) can be clearly discarded since honesty and adventure making have no relationship with the given argument.
Option (a) may appear close. However, remember, every failure may not result into success as it happens in the case of spider.

Ans. (d)

76. A car travels 8 km in the first quarter of an hour, 6 km in the second quarter and 16 km in the third quarter. The average speed of the car in km per hour over the entire journey is

(a) 30 (b) 36 (c) 40 (d) 24

(GATE 2013, 2 Marks)

Solution:

$$\text{Average speed of the car} = \frac{\text{Total distance}}{\text{Total time}}$$

$$\frac{\text{Total distance}}{\text{Total time}} = \frac{(8+6+16)}{\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)}$$

$$\frac{30}{\frac{3}{4}} = 40 \text{ km/hr}$$

Ans. (c)

77. Find the sum to n terms of the series 10 + 84 + 734 + ...

(a) $\frac{9(9^n+1)}{10}+1$ (b) $\frac{9(9^n-1)}{8}+1$

(c) $\frac{9(9^n-1)}{8}+n$ (d) $\frac{9(9^n-1)}{8}+n^2$

(GATE 2013, 2 Marks)

Solution: $S = 10 + 84 + 734 + \ldots$

$S = (9 + 1) + (9^2 + 3) + (9^3 + 5) + \ldots$

$S = (9 + 9^2 + 9^3 + \ldots 9^n) + (1 + 3 + 5 + \ldots + (2n - 1)$

$$S = \frac{9(9^n-1)}{(9-1)}+n^2$$

$$\left(\begin{array}{l}\text{As } a + ar + ar^2 + \ldots + a^{n-1} \\ = \dfrac{a(r^n-1)}{(r-1)} \quad \text{for} \quad r > 1\end{array}\right)$$

$$\left(\begin{array}{l}\text{and } 1+3+5+\ldots+(2n-1) = n^2 \\ \text{Sum of odd natural numbers} = n^2\end{array}\right)$$

So, $S = \frac{9(9^n-1)}{8}+n^2$

Ans. (d)

78. The set of values of p for which the roots of the equation $3x^2 + 2x + p(p - 1) = 0$ are of opposite sign is

(a) $(-\infty, 0)$ (b) $(0, 1)$
(c) $(1, \infty)$ (d) $(0, \infty)$

(GATE 2013, 2 Marks)

Solution: $3x^2 + 2x + p(p - 1) = 0$
If roots are of opposite signs product of the roots will be negative.

$$\frac{p(p-1)}{3} < 0$$

$$p(p - 1) < 0$$
$$0 < p < 1$$
$$P \in (0, 1)$$

Ans. (b)

79. What is the chance that a leap year, selected at random, will contain 53 Saturdays?

(a) $\frac{2}{7}$ (b) $\frac{3}{7}$ (c) $\frac{1}{7}$ (d) $\frac{5}{7}$

(GATE 2013, 2 Marks)

Solution: Every leap year will have 366 days.
1 leap year = 366 days

$= 52 \times 7 + 2$ days

Every leap year will have 52 complete weeks. For 53rd Saturday, there are 2 favourable cases from 7 possible cases.

Thus, required probability $= \frac{2}{7}$

Ans. (a)

80. Statement: There were different streams of freedom movements in colonial India carried out by the moderates, liberals, radicals, socialists, and so on.

Which one of the following is the best inference from the above statement?

(a) The emergence of nationalism in colonial India led to our Independence.
(b) Nationalism in India emerged in the context of colonialism.
(c) Nationalism in India is homogeneous.
(d) Nationalism in India is heterogeneous.

(GATE 2013, 2 Marks)

Solution: As the given argument states that there were different streams of freedom movement during colonialism, it clearly implies that nationalism in India is heterogeneous.
Option (c) is ruled out as it states the contrary, and options (a) and (b) fail to establish a direct link with the argument.

Ans. (d)

81. Which of the following options is the closest in meaning to the phrase underlined in the sentence below?

It is fascinating to see life forms <u>cope with</u> varied environmental conditions.

(a) adopt to (b) adapt to
(c) adept in (d) accept with

(GATE 2014, 1 Mark)

Solution: 'Adapt to' is closest in meaning.

Ans. (b)

82. Choose the most appropriate word from the options given below to complete the following sentence.

He could not understand the judges awarding her the first prize, because he thought that her performance was quite ________.

(a) superb (b) medium
(c) mediocre (d) exhilarating

(GATE 2014, 1 Mark)

Solution: 'Mediocre' is the appropriate word.

Ans. (c)

83. In a press meet on the recent scam, the minister said, 'The buck stops here'. What did the minister convey by the statement?

(a) He wants all the money.
(b) He will return the money.
(c) He will assume final responsibility.
(d) He will resist all enquiries.

(GATE 2014, 1 Mark)

Solution: The minister conveys 'He will assume final responsibility'.

Ans. (c)

84. If $(z + 1/z)^2 = 98$, compute $(z^2 + 1/z^2)$.

(GATE 2014, 1 Mark)

Solution: The given equation is

$$\left(z + \frac{1}{z}\right)^2 = 98$$

Expanding the equation, we get

$$z^2 + \frac{1}{z^2} + 2z \cdot \frac{1}{z} = 98 \Rightarrow z^2 + \frac{1}{z^2} = 96$$

Ans. 96

85. The roots of $ax^2 + bx + c = 0$ are real and positive. a, b and c are real. Then $ax^2 + b|x| + c = 0$ has

(a) no roots (b) 2 real roots
(c) 3 real roots (d) 4 real roots

(GATE 2014, 1 Mark)

Solution: We are given

$$ax^2 + bx + cx = 0$$

For roots to be real and positive,

$$b^2 - 4ac > 0$$

This will have 2 real positive roots. Now consider

$$ax^2 + b|x| + c = 0$$

This can be written as

$$ax^2 + bx + c = 0;\ ax^2 - bx + c = 0$$

For $ax^2 + bx + c = 0$,

Discriminant $= b^2 - 4ac > 0$

For $ax^2 - bx + c = 0$,

$$(-b)^2 - 4ac \Rightarrow b^2 - 4ac > 0$$

Thus, it will have real roots.

Thus, in total the equation has 4 real roots.

Ans. (d)

86. Choose the most appropriate phrase from the options given below to complete the following sentence.

India is a post-colonial country because ______ .

(a) it was a former British colony
(b) Indian Information Technology professionals have colonized the world
(c) India does not follow any colonial practices
(d) India has helped other countries gain freedom

(GATE 2014, 1 Mark)

Solution: Option (a) is the most appropriate phrase.

Ans. (a)

87. Who ________ was coming to see us this evening?

(a) you said (b) did you say
(c) did you say that (d) had you said

(GATE 2014, 1 Mark)

Solution: Option (b) fits the blank appropriately.

Ans. (b)

88. Match the columns.

Column 1	Column 2
1) eradicate	P) misrepresent
2) distort	Q) soak completely
3) saturate	R) use
4) utilize	S) destroy utterly

(a) 1:S, 2:P, 3:Q, 4:R (b) I:P, 2:Q, 3:R, 4:S
(c) 1:Q, 2:R, 3:S, 4:P (d) 1:S, 2:P, 3:R, 4:Q

(GATE 2014, 1 Mark)

Solution: Option (a) is correctly matched.

Ans. (a)

89. What is the average of all multiples of 10 from 2 to 198?

(a) 90 (b) 100 (c) 110 (d) 120

(GATE 2014, 1 Mark)

Solution: Total multiples between 2 and 198 = {10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190}

Hence, total number of multiples = 19

$$\text{Average} = \frac{1900}{19} = 100$$

Ans. (b)

90. The value of $\sqrt{12+\sqrt{12+\sqrt{12+\cdots}}}$ is

(a) 3.464 (b) 3.932
(c) 4.000 (d) 4.444

(GATE 2014, 1 Mark)

Solution: Let

$$\sqrt{12+\sqrt{12+\sqrt{12+\cdots}}} = y$$

$$\Rightarrow \sqrt{12+y} = y$$

$$\Rightarrow 12+y = y^2$$

$$\Rightarrow (y-4)(y+3) = 0$$

$$\Rightarrow y = 4, -3$$

Ans. (c)

91. <u>While trying to collect</u> (I) an envelope <u>from under the table,</u> (II) <u>Mr. X fell down</u> (III) and <u>was losing consciousness.</u> (IV)

Which one of the above underlined parts of the sentence is NOT appropriate?

(a) I (b) II (c) III (d) IV

(GATE 2014, 1 Mark)

Solution: IV part is not appropriate.

Ans. (d)

92. If she ________ how to calibrate the instrument, she ________ done the experiment.

(a) knows, will have
(b) knew, had
(c) had known, could have
(d) should have known, would have

(GATE 2014, 1 Mark)

Solution: Option (c) completes the sentence appropriately.

Ans. (c)

93. Choose the word that is opposite in meaning to the word 'coherent'.

(a) sticky (b) well-connected
(c) rambling (d) friendly

(GATE 2014, 1 Mark)

Solution: 'Rambling' is opposite of 'coherent'

Ans. (c)

94. Which number does not belong to the series below?
2, 5, 10, 17, 26, 37, 50, 64

(a) 17 (b) 37
(c) 64 (d) 26

(GATE 2014, 1 Mark)

Solution: The difference between the second and first terms is 3. The difference between the third and second terms is 5. Similarly, the difference between the fourth and third terms is 7 and so on.

Hence, the number which does not belong to the series is 64 since the difference between 64 and 50 is even.

Ans. (c)

95. The table below has question-wise data to the performance of students in an examination. The marks for each question are also listed. There is no negative or partial marking in the examination.

Q. No.	Marks	Answered Correctly	Answered Wrongly	Not Attempted
1	2	21	17	6
2	3	15	27	2
3	2	23	18	3

What is the average of the marks obtained by the class in the examination?

(a) 1.34 (b) 1.74
(c) 3.02 (d) 3.91

(GATE 2014, 1 Mark)

Solution: Total marks obtained = $(21 \times 2) + (15 \times 3) + (23 \times 2) = 133$
Total number of students = 44

$$\text{Average} = \frac{133}{44} = 3.02$$

Ans. (c)

96. Choose the most appropriate phrase from the options given below to complete the following sentence.

The aircraft ________ take off as soon as its flight plan was filed.

(a) is allowed to (b) will be allowed to
(c) was allowed to (d) has been allowed to

(GATE 2014, 1 Mark)

Solution: The correct answer is option (c).

Ans. (c)

97. Read the statements:
All women are entrepreneurs.
Some women are doctors.
Which of the following conclusions can be logically inferred from the above statements?

(a) All women are doctors
(b) All doctors are entrepreneurs
(c) All entrepreneurs are women
(d) Some entrepreneurs are doctors

(GATE 2014, 1 Mark)

Solution: The correct answer is option (d).

Ans. (d)

98. Choose the most appropriate word from the options given below sentence.

Many ancient cultures attributed disease to supernatural causes. However, modern science has largely helped ________ such notions.

(a) impel (b) dispel
(c) propel (d) repel

(GATE 2014, 1 Mark)

Solution: The correct answer is option (b).

Ans. (b)

99. The statistics of runs scored in a series by four batsmen are provided in the following table. Who is the most consistent batsman of these four?

Batsman	Average	Standard deviation
K	31.2	5.21
L	46.0	6.35
M	54.4	6.22
N	17.9	5.90

(a) K (b) L (c) M (d) N

(GATE 2014, 1 Mark)

Solution: 'K' is the most consistent batsman.

Ans. (a)

100. What is the next number in the series?

12 35 81 173 357

(GATE 2014, 1 Mark)

Solution: We can observe from the sequence,

$$35 - 12 = 23$$
$$81 - 35 = 46 = (23 \times 2) = (23 \times 2^1)$$
$$173 - 81 = 92 = (23 \times 4) = (23 \times 2^2)$$
$$357 - 173 = 184 = (23 \times 8) = (23 \times 2^3)$$

Thus,

$$x - 357 = (23 \times 2^4) = 23 \times 16 \Rightarrow x = 725$$

101. Choose the most appropriate phrase from the options given below to complete the following sentence.
Communication and interpersonal skills are ________ important in their own ways.

(a) each (b) both (c) all (d) either

(GATE 2014, 1 Mark)

Solution: The correct answer is 'both'.

Ans. (b)

102. Which of the options given below best completes the following sentence?
She will feel much better if she ________.

(a) will get some rest
(b) gets some rest
(c) will be getting some rest
(d) is getting some rest

(GATE 2014, 1 Mark)

Solution: The correct answer is option (b).

Ans. (b)

103. Choose the most appropriate pair of words from the options given below to complete the following sentence.
She could not ________ the thought of ________ the election to her bitter rival.

(a) bear, loosing (b) bare, loosing
(c) bear, losing (d) bare, losing

(GATE 2014, 1 Mark)

Solution: The correct answer is option (c).

Ans. (c)

104. A regular die has six sides with numbers 1 to 6 marked on its sides. If a very large number of throws show the following frequencies of occurrence: $1 \to 0.167$; $2 \to 0.167$; $3 \to 0.152$; $4 \to 0.166$; $5 \to 0.168$; $6 \to 0.180$. We call this die

(a) irregular (b) biased
(c) Gaussian (d) insufficient

(GATE 2014, 1 Mark)

Solution: For a very large number of throws, the frequency should be same for unbiased throw. As it is not same, the die is biased.

Ans. (b)

105. Fill in the missing number in the series

2 3 6 15 ________ 157.5 630

(GATE 2014, 1 Mark)

Solution: In the series given above, $\frac{2^{\text{nd}}\text{ number}}{1^{\text{st}}\text{ number}}$ is in increasing order.

$$3/2 = 1.5$$
$$6/3 = 2$$
$$15/6 = 2.5$$

Hence, the next number is $15 \times 3 = 45$.

Ans. 45

106. 'India is a country of rich heritage and cultural diversity.'

Which one of the following facts best supports the claim made in the above sentence?

(a) India is a union of 28 states and 7 union territories.
(b) India has a population of over 1.1 billion.
(c) India is home to 22 official languages and thousands of dialects.
(d) The Indian cricket team draws players from over ten states.

(GATE 2014, 1 Mark)

Solution: Diversity is shown in terms of difference language.

Ans. (c)

107. The value of one U.S. dollar is 65 Indian Rupees today, compared to 60 last year. The Indian rupee has ________.

(a) depressed (b) depreciated
(c) appreciated (d) stabilized

(GATE 2014, 1 Mark)

Solution: The correct answer is 'depreciated'.

Ans. (b)

108. 'Advice' is ________.

(a) a verb
(b) a noun
(c) an adjective
(d) both a verb and a noun

(GATE 2014, 1 Mark)

Solution: Advice is a 'noun'.

Ans. (b)

109. The next term in the series 81, 54, 36, 24, … is ________.

(GATE 2014, 1 Mark)

Solution: $81 - 54 = 27;\ 27 \times \frac{2}{3} = 18$

$$54 - 36 = 18;\ 18 \times \frac{2}{3} = 12$$

$$36 - 24 = 12;\ 12 \times \frac{2}{3} = 8$$

Therefore

$$24 - 8 = 16$$

Ans. 16

110. In which of the following options will the expression P < M be definitely true?

(a) M < R > P > S (b) M > S < P < F
(c) Q < M < F = P (d) P = A < R < M

(GATE 2014, 1 Mark)

Solution: The expression P < M will be definitely true in option 'P = A < R < M'.

Ans. (d)

111. Which one of the following options is the closest in meaning to the word underlined in the sentence below?
In a democracy, everybody has the freedom to <u>disagree</u> with the government.

(a) dissent (b) descent
(c) decent (d) decadent

(GATE 2014, 1 Mark)

Solution: The correct answer is 'dissent'.

Ans. (a)

112. After the discussion, Tom said to me, 'Please revert!'. He expects me to

(a) retract (b) get back to him
(c) move in reverse (d) retreat

(GATE 2014, 1 Mark)

Solution: The correct answer is 'get back to him'.

Ans. (b)

113. While receiving the award, the scientist said, "I feel vindicated". Which of the following is closest in meaning to the word 'vindicated'?

(a) punished (b) substantiated
(c) appreciated (d) chastened

(GATE 2014, 1 Mark)

Solution: The correct answer is 'substantiated'.

Ans. (b)

114. Let $f(x, y) = x^n y^m = P$. If x is doubled and y is halved, the new value of f is

(a) $2^{n-m}P$ (b) $2^{m-n}P$
(c) $2(n-m)P$ (d) $2(m-n)P$

(GATE 2014, 1 Mark)

Solution: We have

$$f(x, y) = x^n y^m = P$$

$$P' = 2^n x^n \left(\frac{1}{2}\right)^m y^m = 2^{n-m} x^n y^m = 2^{n-m} P$$

Ans. (a)

115. In a sequence of 12 consecutive odd numbers, the sum of the first 5 numbers is 425. What is the sum of the last 5 numbers in the sequence?

(GATE 2014, 1 Mark)

Solution: Let the first number is a. Thus, the sequence is

$$a,\ a+2,\ a+4,\ a+6,\ a+8,\ a+10,\ a+12,\ a+14,\ a+16,\ a+18,\ a+20,\ a+22$$

We are given that

$$a+(a+2)+(a+4)+(a+6)+(a+8) = 425$$
$$\Rightarrow 5a+20 = 425 \Rightarrow 5a = 405 \Rightarrow a = 81$$

Sum of the last 5 numbers in the sequence =

$$(a+14)+(a+16)+(a+18)+(a+20)+(a+22)$$
$$= 5a+30+40+20+405+90 = 495$$

Ans. 495

116. A student is required to demonstrate a high level of comprehension of the subject, especially in the social sciences.

The word closest in meaning to comprehension is

(a) understanding (b) meaning
(c) concentration (d) stability

(GATE 2014, 1 Mark)

Solution: The word closest in meaning is 'understanding'.

Ans. (a)

117. Choose the most appropriate word from the options given below to complete the following sentence.

One of his biggest ________ was his ability to forgive.

(a) vice (b) virtues (c) choices (d) strength

(GATE 2014, 1 Mark)

Solution: 'Virtues' is the most appropriate word.

Ans. (b)

118. Rajan was not happy that Sajan decided to do the project on his own. On observing his unhappiness, Sajan explained to Rajan that he preferred to work independently.

Which one of the statements below is logically valid and can be inferred from the above sentences?

(a) Rajan has decided to work only in a group.
(b) Rajan and Sajan were formed into a group against their wishes.
(c) Sajan had decided to give in to Rajan's request to work with him.
(d) Rajan had believed that Sajan and he would be working together.

(GATE 2014, 1 Mark)

Solution: Option (d) is logically valid.

Ans. (d)

119. If $y = 5x^2 + 3$, then the tangent at $x = 0$, $y = 3$

(a) passes though $x = 0$, $y = 0$
(b) has a slope of $+1$
(c) is parallel to the x-axis
(d) has a slope of -1

(GATE 2014, 1 Mark)

Solution: We have the equation,

$$y = 5x^2 + 3 \quad (1)$$

Differentiating Eq. (1), we get

$$\frac{dy}{dx} = 10x$$

$$\text{Slope of tangent} = \left(\frac{dy}{dx}\right)_{x=0,y=3} = 10\times 0 = 0$$

Slope = 0 ⇒ tangent is parallel to x-axis.

Ans. (c)

120. A foundry has a fixed daily cost of ₹ 50,000 whenever it operates and a variable cost of ₹ 800Q, where Q is the daily production in tonnes. What is the cost of production in ₹ per tonne for a daily production of 100 tonnes?

(GATE 2014, 1 Mark)

Solution: Fixed daily cost = ₹ 50,000
Variable cost = ₹ $800Q$
Q = daily production in tonnes
For 100 tonnes of production daily, total cost of production $= 50000 + 800 \times 100 = 130000$
So, cost of production per tonne of daily production $= \frac{130000}{100} = ₹1300$

Ans. ₹1300

121. Choose the most appropriate word from the options given below to complete the following sentence.
A person suffering from Alzheimer's disease _________ short-term memory loss.

(a) experienced (b) has experienced
(c) is experiencing (d) experiences

(GATE 2014, 1 Mark)

Solution: 'Experiences' is the most appropriate word.

Ans. (d)

122. Choose the most appropriate word from the options given below to complete the following sentence.
_________ is the key to their happiness; they are satisfied with what they have.

(a) Contentment (b) Ambition
(c) Perseverance (d) Hunger

(GATE 2014, 1 Mark)

Solution: 'Contentment' is the most appropriate word.

Ans. (a)

123. Which of the flowing options is the closest in meaning to the sentence below?

'As a woman, I have no country.'

(a) Women have no country.
(b) Women are not citizens of any country.
(c) Women's solidarity knows no national boundaries.
(d) Women of all countries have equal legal rights.

(GATE 2014, 1 Mark)

Solution: Option (c) is the closest in meaning.

Ans. (c)

124. In any given year, the probability of an earthquake greater than magnitude 6 occurring in the Garhwal himalayas is 0.04. The average time between successive occurrences of such earthquakes is _________ years.

(GATE 2014, 1 Mark)

Solution: $P = 0.04 = \frac{4}{100}$

For 1 earthquake

$$\frac{100}{4}P = 1 \text{ earthquake}$$

Thus, $P = 25$ years.

Ans. 25 years

125. The population of a new city is 5 million and is growing at 20% annually. How many years would it take to double at this growth rate?

(a) 3–4 years (b) 4–5 years
(c) 5–6 years (d) 6–7 years

(GATE 2014, 1 Mark)

Solution: After 1 year, population $= \frac{20}{100} \times 5 + 5 =$ 6 million

After 2 years, population $= \frac{20}{100} \times 6 + 6 = 7.2$ million

After 3 years, population $= \frac{20}{100} \times 7.2 + 7.2 = 8.65$ million

After 4 years, population $= \frac{20}{100} \times 8.65 + 8.65 \simeq 10$ million

Time will be in between 3 and 4 years.

Ans. (a)

126. The Palghat Gap (or Palakkad Gap), a region about 30 km wide in the southern part of the Western Ghats in India, is lower than the hilly terrain to its north and south. The exact reasons for the formation of this gap are not clear. It results in the neighbouring regions of Tamil Nadu getting more rainfall from the South West monsoon and the neighbouring regions of Kerala having higher summer temperatures.
What can be inferred from this passage?

(a) The Palghat gap is caused by high rainfall and high temperatures in southern Tamil Nadu and Kerala.
(b) The regions in Tamil Nadu and Kerala that are near the Palghat Gap are low-lying.
(c) The low terrain of the Palghat Gap has a significant impact on weather patterns in neighbouring parts of Tamil Nadu and Kerala.
(d) Higher summer temperatures result in higher rainfall near the Palghat Gap area.

(GATE 2014, 2 Marks)

Solution: Option (c) can be inferred from the passage.

Ans. (c)

127. Geneticists say that they are very close to confirming the genetic roots of psychiatric illnesses such as depression and schizophrenia, and consequently, that doctors will be able to eradicate these diseases through early identification and gene therapy.

On which of the following assumptions does the statement above rely?

(a) Strategies are now available for eliminating psychiatric illnesses.
(b) Certain psychiatric illnesses have a genetic basis.
(c) All human diseases can be traced back to genes and how they are expressed.
(d) In the future, genetics will become the only relevant field for identifying psychiatric illnesses.

(GATE 2014, 2 Marks)

Solution: The given statement relies on assumption (b).

Ans. (b)

128. Round-trip tickets to a tourist destination are eligible for a discount of 10% on the total fare. In addition, groups of 4 or more get a discount of 5% on the total fare. If the one-way single person fare is ₹100, a group of 5 tourists purchasing round-trip tickets will be charged ₹________.

(GATE 2014, 2 Marks)

Solution:

One-way fare = ₹100

Two-way fare per person = ₹200

For 5 persons = ₹1000

Total discount applicable = 10 + 5 = 15%

Discount amount = $\frac{15}{100} \times 1000 = 150$

Amount to be paid = $1000 - 150 =$ ₹850

Ans. ₹850

129. In a survey, 300 respondents were asked whether they own a vehicle or not. If yes, they were further asked to mention whether they own a car or scooter or both. Their responses are tabulated below. What percent of respondents do not own a scooter?

		Men	Women
Own vehicle	Car	40	34
	Scooter	30	20
	Both	60	46
Do not own vehicle		20	50

(GATE 2014, 2 Marks)

Solution: Total respondents = 300

People who do not have scooter:

$$\text{Men} = 40 + 20 = 60$$

$$\text{Women} = 34 + 50 = 84$$

Total respondents who do not have a scooter = 60 + 84 = 144

Percentage of respondents who do not have a scooter = $\frac{144}{300} \times 100 = 48\%$

Ans. 48%

130. When a point inside of a tetrahedron (a solid with four triangular surfaces) is connected by straight lines to its corners, how many (new) internal planes are created with these lines?

(GATE 2014, 2 Marks)

Solution: Six internal planes are created with these lines.

Ans. 6

131. The old city of Koenigsberg, which had a German majority population before World War 2, is now called Kaliningrad. After the events of the war, Kaliningrad is now a Russian territory and has a predominantly Russian population. It is bordered by the Baltic Sea on the north and the countries of Poland to the south and west and Lithuania to the east respectively. Which of the statements below can be inferred from this passage?

(a) Kaliningrad was historically Russian in its ethnic make up
(b) Kaliningrad is a part of Russia despite it not being contiguous with the rest of Russia
(c) Koenigsberg was renamed Kaliningrad, as that was its original Russian name
(d) Poland and Lithuania are on the route from Kaliningrad to the rest of Russia

(GATE 2014, 2 Marks)

Solution: Option (b) can be inferred.

Ans. (b)

132. The number of people diagnosed with dengue fever (contracted from the bite of a mosquito) in north India is twice the number diagnosed last year. Municipal authorities have concluded that measures to control the mosquito population have failed in this region.

Which one of the following statements, if true, does not contradict this conclusion?

(a) A high proportion of the affected population has returned from neighbouring countries where dengue is prevalent

(b) More cases of dengue are now reported because of an increase in the Municipal Office's administrative efficiency

(c) Many more cases of dengue are being diagnosed this year since the introduction of a new and effective diagnostic test

(d) The number of people with malarial fever (also contracted from mosquito bites) has increased this year

(GATE 2014, 2 Marks)

Solution: Option (d) does not contradict the conclusion.

Ans. (d)

133. If x is real and $|x^2 - 2x + 3| = 11$, then the possible value of $|-x^3 + x^2 - x|$ include

(a) 2, 4 (b) 2, 14
(c) 4, 52 (d) 14, 52

(GATE 2014, 2 Marks)

Solution: We have

$$x^2 - 2x + 3 = 11$$

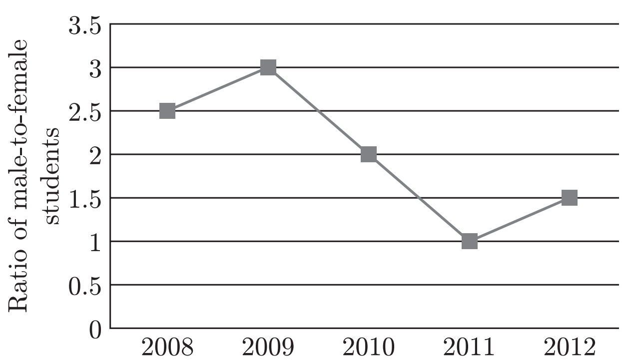
$$\Rightarrow (x-4)(x+2) = 0 \Rightarrow x = 4, -2$$

For $x = 4$, value of $\left|-x^3 + x^2 - x\right| = 52$

For $x = -2$, value of $\left|-x^3 + x^2 - x\right| = 14$

Ans. (d)

134. The ratio of male-to-female students in a college for five years is plotted in the following line graph. If the number of female students doubled in 2009, by what percent did the number of male students increase in 2009?



(GATE 2014, 2 Marks)

Solution: In 2008,

$$\frac{m}{f} = 2.5 \Rightarrow m = 2.5f$$

In 2009,

$$\frac{m'}{2f} = 3 \Rightarrow m' = 6f$$

Percentage increase $= \dfrac{3.5f}{2.5f} \times 100 = 140\%$

Ans. 140%

135. At what time between 6 a.m. and 7 a.m. will the minute hand and hour hand of a clock make an angle closest to 60°?

(a) 6:22 a.m. (b) 6:27 a.m.
(c) 6:38 a.m. (d) 6:45 a.m.

(GATE 2014, 2 Marks)

Solution: Angle by minute's hand

60 min $\rightarrow 360°$

1 min $\rightarrow \dfrac{360°}{60°} = 6°$

8 min $\rightarrow 48°$

Angle $\rightarrow 48°$ with number '6'

Angle by hours hand,

60 min $= 30°$

22 min $\rightarrow \dfrac{30}{60} \times 22 = 11$

Total angle $= 48 + 11 = 59°$

Hence, the time is 6.22 a.m.

Ans. (a)

136. A dance programme is scheduled for 10.00 a.m. Some students are participating in the programme and they need to come an hour earlier than the start of the event. These students should be accompanied by a parent. Other students and parents should come in time for the programme. The instruction you think that is appropriate for this is

(a) Students should come at 9.00 a.m. and parents should come at 10.00 a.m.

(b) Participating students should come at 9.00 a.m. accompanied by a parent, and other parents and students should come by 10.00 a.m.

(c) Students who are not participating should come by 10.00 a.m. and they should not bring their parents. Participating students should come at 9.00 a.m.

(d) Participating students should come before 9.00 a.m. Parents who accompany them should come at 9.00 a.m. All others should come at 10.00 a.m.

(GATE 2014, 2 Marks)

Solution: Option (b) is the appropriate instruction.

Ans. (b)

137. By the beginning of the 20^{th} century, several hypotheses were being proposed, suggesting a paradigm shift in our understanding of the universe.

However, the clinching evidence was provided by experimental measurements of the position of a star which was directly behind our sun.

Which of the following inference(s) may be drawn from the above passage?

(i) Our understanding of the universe changes based on the positions of stars
(ii) Paradigm shifts usually occur at the beginning of centuries
(iii) Stars are important objects in the universe
(iv) Experimental evidence was important in confirming this paradigm shift

(a) (i), (ii) and (iv)
(b) (iii) only
(c) (i) and (iv)
(d) (iv) only

(GATE 2014, 2 Marks)

Solution: Only option (iv) can be drawn from the above passage.

Ans. (d)

138. The Gross Domestic Product (GDP) in Rupees grew at 7% during 2012–2013. For international comparison, the GDP is compared in US Dollars (USD) after conversion based on the market exchange rate. During the period 2012–2013, the exchange rate for the USD increased from ₹50/USD to ₹60/USD. India's GDP in USD during the period 2012–2013

(a) increased by 5%
(b) decreased by 13%
(c) decreased by 20%
(d) decreased by 11%

(GATE 2014, 2 Marks)

Solution: Final value per ₹100 = ₹107

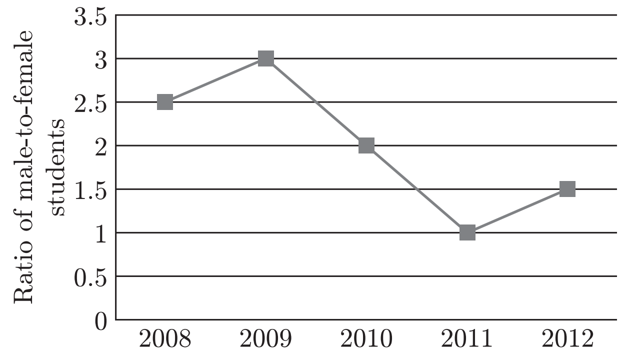
$$\Rightarrow \text{Per}\,\frac{100}{50}\,\text{dollars, final value} = \frac{107}{60}$$

$$\text{For 100 dollars, final value} = \frac{107}{60} \times 50 = 89.16$$

Hence, India's GDP in USD during the period 2012–2013 decreased by 11%.

Ans. (d)

139. The ratio of male-to-female students in a college for five years is plotted in the following line graph. If the number of female students in 2011 and 2012 is equal, what is the ratio of male students in 2012 to male students in 2011?



(a) 1:1
(b) 2:1
(c) 1.5:1
(d) 2.5:1

(GATE 2014, 2 Marks)

Solution: Take number of female students in 2011 = 100

Therefore, number of males in 2011 = 100

Number of females in 2012 = 100

Number of males in 2012 = 150

$$\text{Thus, ratio} = \frac{150}{100} = 1.5:1$$

Ans. (c)

140. Consider the equation: $(7526)_8 - (Y)_8 = (4364)_8$, where $(X)_N$ stands for X to the base N. Find Y.

(a) 1634
(b) 1737
(c) 3142
(d) 3162

(GATE 2014, 2 Marks)

Solution: We are given

$$(7526)_8 - (Y)_8 = (4364)_8$$
$$\Rightarrow (Y)_8 = (7526)_8 - (4364)_8$$

$$\begin{array}{cccc} 7 & 5 & 2 & 6 \\ 4 & 3 & 6 & 4 \\ \hline 3 & 1 & 4 & 2 \end{array}$$

When we have base 8, we borrow 8 instead of 10 as done in normal subtraction.

Ans. (c)

141. Find the odd one from the following group:

W,E,K,O I,Q,W,A F,N,T,X N,V,B,D

(a) W,E,K,O
(b) I,Q,W,A
(c) F,N,T,X
(d) N,V,B,D

(GATE 2014, 2 Marks)

Solution: Considering WEKO, the difference between the two consecutive letters is 8, 6, 4.
Considering IQWA, the difference between the two consecutive letters is 8, 6, 4.
Considering FNTX, the difference between the two consecutive letters is 8, 6, 4.
Considering NVBD, the difference between the two consecutive letters is 8, 6, 2. Thus, the odd one out is NVBD.

Ans. (d)

142. For submitting tax returns, all resident males with annual income below ₹10 lakh should fill up Form P and all resident females with income below ₹8 lakh should fill up Form Q. All people with incomes above ₹10 lakh should fill up Form R, except non-residents with income above ₹15 lakhs, who should fill up Form S. All others should fill Form T. An example of a person who should fill Form T is

(a) a resident male with annual income ₹9 lakh

(b) a resident female with annual income ₹9 lakh

(c) a non-resident male with annual income ₹16 lakh

(d) a non-resident female with annual income ₹16 lakh

(GATE 2014, 2 Marks)

Solution: Resident females in between 8 and 10 lakhs haven't been mentioned.

Ans. (b)

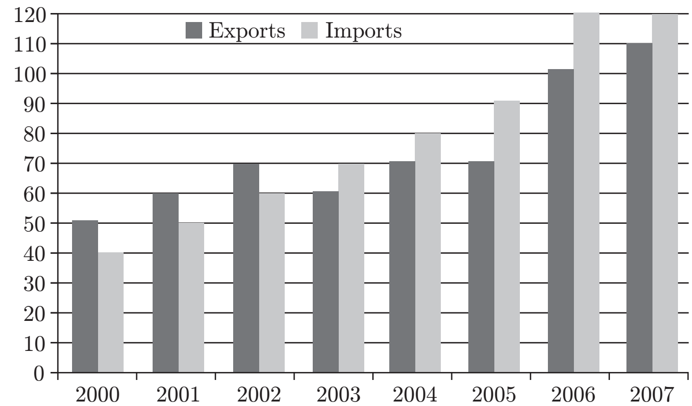
143. A train that is 280 metre long, travelling at a uniform speed, crosses a platform in 60 seconds and passes a man standing on the platform in 20 seconds. What is the length of the platform in metre?

(GATE 2014, 2 Marks)

Solution: For a train, which is 280 m long, to cross a person, it takes 20 seconds.
Also, it takes 60 seconds to cross the platform. Total distance travelled should be $280 \times 3 = 840$ m.
Including 280 m train length, so length of plates = $840 - 280 = 560$ m.

Ans. 560 m

144. The exports and imports (in crores of ₹) of a country from 2000 to 2007 are given in the following bar chart. If the trade deficit is defined as excess of imports over exports, in which year is the trade deficit 1/5th of the exports?



(a) 2005 (b) 2004 (c) 2007 (d) 2006

(GATE 2014, 2 Marks)

Solution: In the year 2004,

$$\frac{\text{Imports} - \text{Exports}}{\text{Exports}} = \frac{10}{70} = \frac{1}{7}$$

In the year 2005, $\frac{26}{76} = \frac{2}{7}$

In the year 2006, $\frac{20}{100} = \frac{1}{5}$

In the year 2007, $\frac{10}{110} = \frac{1}{11}$

Ans. (d)

145. You are given three coins: one has heads on both faces, the second has tails on both faces, and the third has a head on one face and a tail on the other. You choose a coin at random and toss it, and it comes up heads. The probability that the other face is tails is

(a) 1/4 (b) 1/3 (c) 1/2 (d) 2/3

(GATE 2014, 2 Marks)

Solution: Basically, we need to find out the probability of choosing a coin that has one head and one tail. Thus, the probability is given by

$$P = \frac{1}{3}$$

Ans. (b)

146. Find the odd one in the following group:

Q,W,Z,B B,H,K,M W,C,G,J M,S,V,X

(a) Q,W,Z,B (b) B,H,K,M

(c) W,C,G,J (d) M,S,V,X

(GATE 2014, 2 Marks)

Solution: The difference between consecutive alphabets in options (a), (b) and (d) is 6, 3 and 2. Whereas in (c), the difference between consecutive alphabets is 6, 4 and 3. Hence the odd one out is WCGJ.

Ans. (c)

147. Lights of four colours (red, blue, green, yellow) are hung on a ladder. On every step of the ladder, there are two lights. If one of the lights is red, the other light on that step will always be blue. If one of the lights on a step is green, the other light on that step will always be yellow. Which of the following statements is not necessarily correct?

(a) The number of red lights is equal to the number of blue lights

(b) The number of green lights is equal to the number of yellow lights

(c) The sum of the red and green lights is equal to the sum of the yellow and blue lights

(d) The sum of the red and blue lights is equal to the sum of the green and yellow lights

(GATE 2014, 2 Marks)

Solution: The correct answer is option (d).

Ans. (d)

148. The sum of eight consecutive odd numbers is 656. The average of four consecutive even numbers is 87. What is the sum of the smallest odd number and second largest even number?

(GATE 2014, 2 Marks)

Solution: Say the eight consecutive odd numbers are $2a-7, 2a-5, 2a-3, 2a-1, 2a+1, 2a+3, 2a+5$ and $2a+7$. Thus,

$$(2a-7)+(2a-5)+(2a-3)+(2a-1)+$$
$$(2a+1)+(2a+3)+(2a+5)+(2a+7)=656$$
$$\Rightarrow 16a = 656 \Rightarrow a = 41$$

Say the four consecutive even numbers are $2a-2$, $2a$, $2a+2$ and $2a+4$.

Thus,

$$\frac{(2a-2)+(2a)+(2a+2)+(2a+4)}{4} = 87$$
$$\Rightarrow 8a+4 = 348 \Rightarrow a = 43$$

The smallest odd number is $2\times 41 - 7 = 75$

The second largest even number is $2\times 43 + 2 = 88$

$$\text{Sum} = 75 + 88 = 163$$

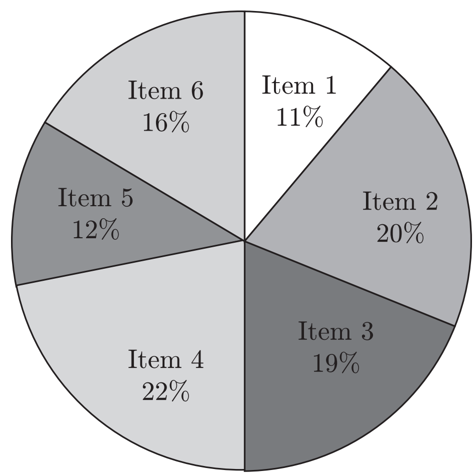
Ans. 163

149. The total exports and revenues from the exports of a country are given in the two charts shown as follows. The pie chart for exports shows the quantity of each item exported as a percentage of the total quantity of exports. The pie chart for the revenues shows the percentage of the total revenue generated through export of each item. The total quantity of exports of all the items is 500 thousand tonnes and the total revenues are 250 crore rupees. Which item among the following has generated the maximum revenue per kg?

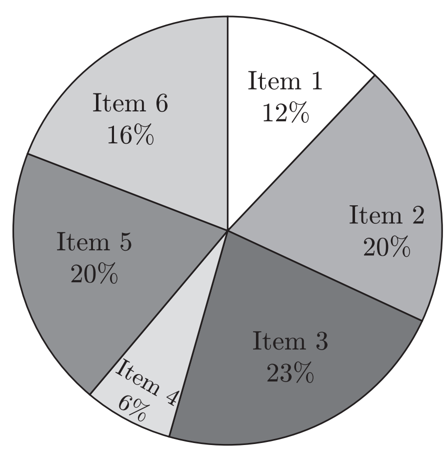
(a) Item 2 (b) Item 3

(c) Item 6 (d) Item 5

Exports



Revenues



(GATE 2014, 2 Marks)

Solution:

$$\text{Item 2 generated revenue} = \frac{\frac{20}{100}\times 250\times 10^7}{\frac{20}{100}\times 500\times 10^3}$$

$$= 5\times 10^3$$

$$\text{Item 3 generated revenue} = \frac{\frac{23}{100}\times 250\times 10^7}{\frac{19}{100}\times 500\times 10^3}$$

$$= 6.05\times 10^3$$

$$\text{Item 6 generated revenue} = \frac{\frac{19}{100}\times 250\times 10^7}{\frac{16}{100}\times 500\times 10^3}$$

$$= 5.9375\times 10^3$$

$$\text{Item 5 generated revenue} = \frac{\frac{20}{100}\times 250\times 10^7}{\frac{12}{100}\times 500\times 10^3}$$

$$= 8.33\times 10^3$$

Ans. (d)

150. It takes 30 minutes to empty a half-full tank by draining it at a constant rate. It is decided to simultaneously pump water into the half-full tank while draining it. What is the rate at which water has to be pumped in so that it gets fully filled in 10 minutes?

(a) 4 times the draining rate

(b) 3 times the draining rate

(c) 2.5 times the draining rate

(d) 2 times the draining rate

(GATE 2014, 2 Marks)

Solution:

$V_{\text{half}} = 30(s)$, where s is the drawing rate.

Total volume $= 60s$

$$(s')(10) - (s)10 = 30s$$
$$\Rightarrow s'(s) - s = 3s$$
$$\Rightarrow s' = 4s$$

Hence, $s' = 4$ times the draining rate.

Ans. (a)

151. Find the next term in the sequence: 7G, 11K, 13M, ________.

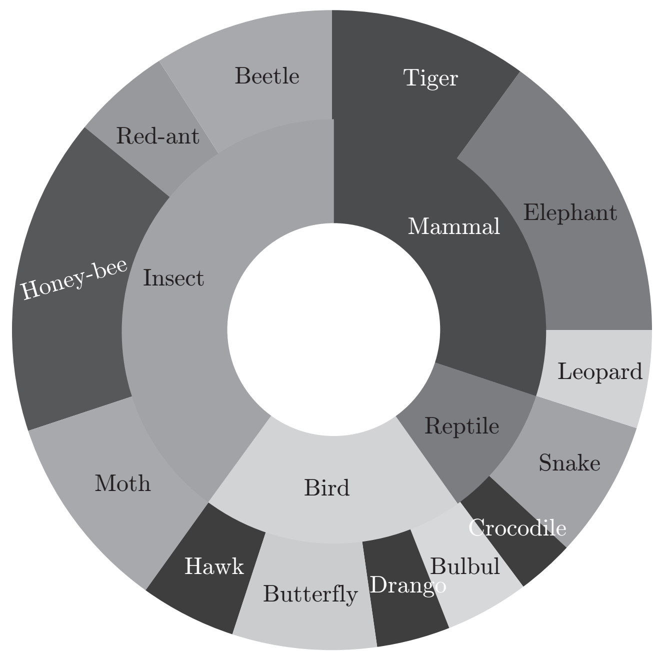
(a) 15Q (b) 17Q (c) 15P (d) 17P

(GATE 2014, 2 Marks)

Solution: The difference between the number and the alphabets is same. Hence, the difference between the next number and 13 will be the same as the difference in the places of the next alphabet and M. From the options, we can see that the correct answer is 17Q.

Ans. (b)

152. The multi-level hierarchical pie chart shows the population of animals in a reserve forest. The correct conclusions from this information are:



(i) Butterflies are birds
(ii) There are more tigers in this forest than red ants
(iii) All reptiles in this forest are either snakes or crocodiles
(iv) Elephants are the largest mammals in this forest

(a) (i) and (ii) only

(b) (i), (ii), (iii) and (iv)

(c) (i), (iii) and (iv) only

(d) (i), (ii) and (iii) only

(GATE 2014, 2 Marks)

Solution: It is not mentioned that elephant is the largest animal.

Ans. (d)

153. A man can row at 8 km per hour in still water. If it takes him thrice as long to row upstream, as to row downstream, then find the stream velocity in km per hour.

(GATE 2014, 2 Marks)

Solution: Speed of man in still water = 8 km/hr

Left distance = d

Time taken = $\frac{d}{8}$

Upstream:

Speed of stream = s

$$\text{Speed upstream} = s'' = (8-s)$$

$$t' = \left(\frac{d}{8-s}\right)$$

Downstream:

Given speed downstream = $t'' = \frac{d}{8+s}$

Now

$$3t' = t''$$

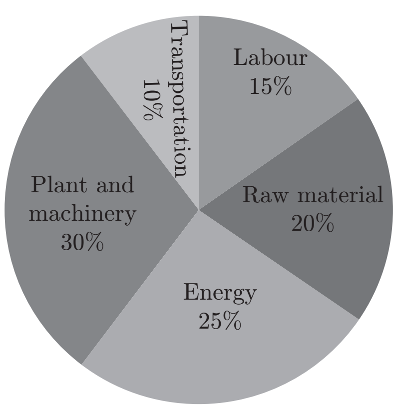
$$\Rightarrow \frac{3d}{8-s} = \frac{d}{8+s}$$

$$\Rightarrow \frac{3d}{8-s} = \frac{d}{8+s}$$

$$\Rightarrow s = 4 \text{ km/hr}$$

Ans. 4 km/hr

154. A firm producing air purifiers sold 200 units in 2012. The following pie chart presents the share of raw material, labour, energy, plant & machinery, and transportation costs in the total manufacturing cost of the firm in 2012. The expenditure on labour in 2012 is ₹4,50,000. In 2013, the raw material expenses increased by 30% and all other expenses increased by 20%. If the company registered a profit of ₹10 lakhs in 2012, at what price (in ₹) was each air purifier sold?



(GATE 2014, 2 Marks)

Solution: Total expenditure =

$$\frac{15}{100}x = 450000 \Rightarrow x = 3\times 10^6$$

$$\text{Profit} = 10 \text{ lakhs}$$

$$\text{So, total selling price} = 4000000 \quad (1)$$

$$\text{Total purifier} = 200 \quad (2)$$

Selling price of each purifier = $\frac{4000000}{200} = 20000$

Ans. 20000

155. A batch of one hundred bulbs is inspected by testing four randomly chosen bulbs. The batch is rejected if even one of the bulbs is defective. A batch typically has five defective bulbs. The probability that the current batch is accepted is ________.

(GATE 2014, 2 Marks)

Solution: Probability for one bulb to be non-defective is $\frac{95}{100}$.

Therefore, probabilities that none of the bulbs is defective is $\left(\frac{95}{100}\right)^4 = 0.8145$

Ans. 0.8145

156. Find the next term in the sequence: 13M, 17Q, 19S, ________.

(a) 21W (b) 21V (c) 23W (d) 23V

(GATE 2014, 2 Marks)

Solution: The difference between the number and the alphabets is same. Hence, the difference between the next number and 19 will be the same as the difference in the places of the next alphabet and S. From the options, we can see that the correct answer is 23W.

Ans. (c)

157. If 'KCLFTSB' stands for 'best of luck' and 'SHSWDG' stands for 'good wishes', which of the following indicates 'ace the exam'?

(a) MCHTX (b) MXHTC
(c) XMHCT (d) XMHTC

(GATE 2014, 2 Marks)

Solution: KCLFTSB in reverse order is BEST OF LUCK without the vowels.
SHSWDG in reverse order is GOOD WISHES without the vowels.
Thus, ACE THE EXAM in reverse order without the vowels will be MXHTC.

Ans. (b)

158. Industrial consumption of power doubled from 2000−2001 to 2010−2011. Find the annual rate of increase in percent assuming it to be uniform over the years.

(a) 5.6 (b) 7.2 (c) 10.0 (d) 12.2

(GATE 2014, 2 Marks)

Solution: We will use the formula

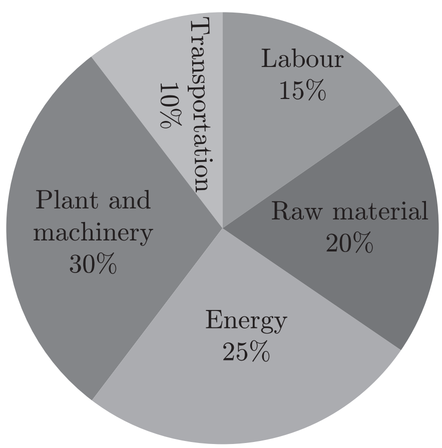
$$A = P\left(1+\frac{r}{100}\right)^n$$

where $n = 10$ years and $A = 2P$. So

$$2P = P\left(1+\frac{r}{100}\right)^{10} \Rightarrow 2 = \left(1+\frac{r}{100}\right)^{10} \Rightarrow r = 7.2$$

Ans. (b)

159. A firm producing air purifiers sold 200 units in 2012. The following pie chart presents the share of raw material, labour, energy, plant & machinery, and transportation costs in the total manufacturing cost of the firm in 2012. The expenditure on labour in 2012 is ₹4,50,000. In 2013, the raw material expenses increased by 30% and all other expenses increased by 20%. What is the percentage increase in total cost for the company in 2013?



(GATE 2014, 2 Marks)

Solution: In 2012, we are given that the cost of labour = ₹450000

Also, the cost of labour = 15% of the total cost

Hence, total cost $= 450000 \times \frac{100}{15}$ = ₹ 3000000

Cost of transportation $= 3000000 \times \frac{10}{100}$

= ₹ 300000

Cost of plant and machinery $= 3000000 \times \frac{30}{100}$

= ₹ 900000

Cost of energy $= 3000000 \times \frac{25}{100}$ = ₹ 750000

Cost of raw material $= 3000000 \times \frac{20}{100}$ = ₹ 600000

In 2013,

Cost of raw material $= 600000\left(\frac{30}{100}+1\right)$

= ₹ 780000

Cost of transportation $= 300000\left(\frac{20}{100}+1\right)$

= ₹ 360000

Cost of plant and machinery $= 900000\left(\frac{20}{100}+1\right)$

= ₹ 1080000

Cost of energy $= 750000\left(\frac{20}{100}+1\right)$ = ₹ 900000

Cost of labour $= 450000\left(\frac{20}{100}+1\right)$ = ₹ 540000

Total cost in 2013 = 3660000

Percentage increase =

$$\frac{3660000-3000000}{3000000}\times 100 = \frac{660000}{3000000}\times 100 = 22\%$$

Ans. 22%

160. A five digit number is formed using the digits 1, 3, 5, 7 and 9 without repeating any of them. What is the sum of all such possible five digit numbers?

(a) 6666660 (b) 6666600

(c) 66666666 (d) 6666606

(GATE 2014, 2 Marks)

Solution: Total possible five digit numbers = 5! = 120

For each decimal place, 24 of these are of same digit.

Thus, for a decimal place the total value = $24(1+3+5+7+9) = 600$

To get the total value of all these numbers, multiply the first result by 11111.

Thus, the total value is 6666600.

Ans. (b)

161. Find the odd one in the following group: ALRVX, EPVZB, ITZDF, OYEIK

(a) ALRVX (b) EPVZB (c) ITZDF (d) OYEIK

(GATE 2014, 2 Marks)

Solution: ALRVX → only one vowel

EPVZB → only one vowel

ITZDF → only one vowel

OYEIK → three vowels

Ans. (d)

162. Anuj, Bhola, Chandan, Dilip, Eswar and Faisal live on different floors in a six-storeyed building (the ground floor is numbered 1, the floor above it 2, and so on). Anuj lives on an even-numbered floor. Bhola does not live on an odd numbered floor. Chandan does not live on any of the floors below Faisal's floor. Dilip does not live on floor number 2. Eswar does not live on a floor immediately above or immediately below Bhola. Faisal lives three floors above Dilip. Which of the following floor-person combinations is correct?

	Anuj	Bhola	Chandan	Dilip	Eswar	Faisal
(a)	6	2	5	1	3	4
(b)	2	6	5	1	3	4
(c)	4	2	6	3	1	5
(d)	2	4	6	1	3	5

(GATE 2014, 2 Marks)

Solution:
(a) Anuj lives on even-numbered floor (2, 4, 6).
(b) Bhola lives on even-numbered floor (2, 4, 6).
(c) Chandar lives on the floor above that of Faisal.
(d) Dilip is not on 2^{nd} floor.
(e) Eswar does not live immediately above or immediately below Bhola.

From the options, it is clear that only option (b) satisfies condition (e). Hence, the correct answer is (b).

Ans. (b)

163. The smallest angle of a triangle is equal to two thirds of the smallest angle of a quadrilateral. The ratio between the angles of the quadrilateral is 3:4:5:6. The largest angle of the triangle is twice its smallest angle. What is the sum, in degrees, of the second largest angle of the triangle and the largest angle of the quadrilateral?

(GATE 2014, 2 Marks)

Solution: Let the angles of quadrilateral be $3x$, $4x$, $5x$ and $6x$.

So,

$$3x + 4x + 5x + 6x = 360 \Rightarrow 18x = 360 \Rightarrow x = 20$$

Smallest angle of quadrilateral $= 3 \times 20 = 60°$

Smallest angles of triangle are $40°, 60°, 80°$.

Largest angle of quadrilateral is $120°$.

Sum (2^{nd} largest angle of triangle + largest angle of quadrilateral) $= 60° + 120° = 180°$.

Ans. 180°

164. One percent of the people of country X are taller than 6 ft. Two percent of the people of country Y are taller than 6 ft. There are thrice as many people in country X as in country Y. Taking both countries together, what is the percentage of people taller than 6 ft?

(a) 3.0 (b) 2.5 (c) 1.5 (d) 1.25

(GATE 2014, 2 Marks)

Solution: Let number of people in country Y = 100

Hence, number of people in country X = 300

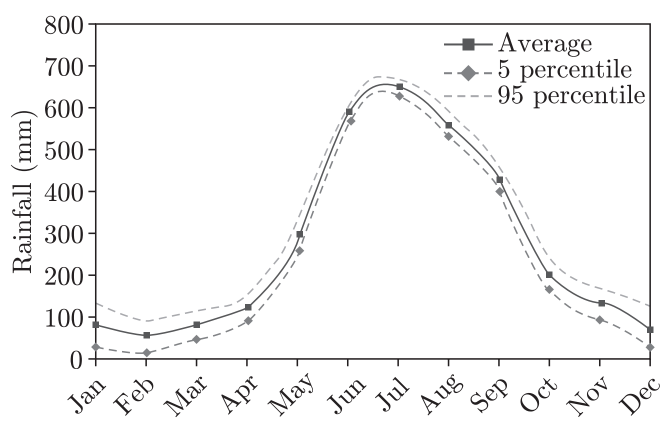
Total number of people taller than 6 ft in both the countries $= 300 \times \frac{1}{100} + 100 \times \frac{2}{100} = 5$

Percentage of people taller than 6 ft in both the countries $= \frac{5}{400} \times 100 = 1.25\%$

Ans. (d)

165. The monthly rainfall chart based on 50 years of rainfall in Agra is shown in the following figure.

Which of the following are true? (k percentile is the value such that k percent of the data fall below that value)



(i) On average, it rains more in July than in December
(ii) Every year, the amount of rainfall in August is more than that in January
(iii) July rainfall can be estimated with better confidence than February rainfall
(iv) In August, there is at least 500 mm of rainfall

(a) (i) and (ii) (b) (i) and (iii)
(c) (ii) and (iii) (d) (iii) and (iv)

(GATE 2014, 2 Marks)

Solution: In the question, the monthly average rainfall chart for 50 years has been given. Considering the given options,

(i) On average, it rains more in July than in December $\Rightarrow$ correct.
(ii) Every year, the amount of rainfall in August is more than that in January.
$\Rightarrow$ may not be correct because average rainfall is given in the question.
(iii) July rainfall can be estimated with better confidence than February rainfall.
$\Rightarrow$ From chart, it is clear that the gap between 5 percentile and 95 percentile from average is higher in February than that in July $\Rightarrow$ correct.
(iv) In August, there is at least 500 mm rainfall
$\Rightarrow$ may not be correct, because its 50 year average.
So correct option is (b) as (i) and (iii) are correct.

Ans. (b)

166. In a group of four children, Som is younger to Riaz. Shiv is elder to Ansu. Ansu is youngest in the group. Which of the following statements is/are required to find the eldest child in the group?

Statements
1. Shiv is younger to Riaz.
2. Shiv is elder to Som.

(a) Statement 1 by itself determines the eldest child.
(b) Statement 2 by itself determines the eldest child.
(c) Statements 1 and 2 are both required to determine the eldest child.
(d) Statements 1 and 2 are not sufficient to determine the eldest child.

(GATE 2014, 2 Marks)

Solution: Option (a) is required to fine the eldest child in the group.

Ans. (a)

167. Moving into a world of big data will require us to change our thinking about the merits of exactitude. To apply the conventional mindset of measurement to the digital, connected world of the twenty-first century is to miss a crucial point. As mentioned earlier, the obsession with exactness is an artefact of the information-deprived analog era. When data was sparse, every data point was critical, and thus great care was taken to avoid letting any point bias the analysis.
From 'BIG DATA' Viktor Mayer-Schonberger and Kenneth Cukier

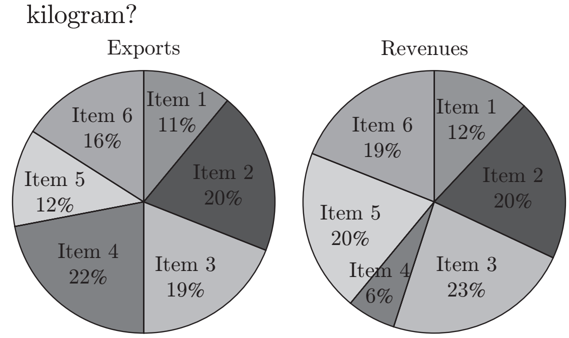
The main point of the paragraph is
(a) The twenty-first century is a digital world
(b) Big data is obsessed with exactness
(c) Exactitude is not critical in dealing with big data
(d) Sparse data leads to a bias in the analysis

(GATE 2014, 2 Marks)

Solution: Option (c) represents the main point of the paragraph.

Ans. (c)

168. The total exports and revenues from the exports of a country are given in the two pie charts below. The pie chart for exports shows the quantity of each item as a percentage of the total quantity of exports. The pie chart for the revenues show the percentage of the total revenue generated through export of each item. The total quantity of exports of all the items is 5 lakh tonnes and the total revenues are 250 crore rupees. What is the ratio of the revenue generated through export of Item 1 per kilogram to the revenue generated through export of Item 4 per kilogram?



(a) 1:2 (b) 2:1 (c) 1:4 (d) 4:1

(GATE 2014, 2 Marks)

Solution: Considering Item 1, total quantity $= \frac{11}{100}\times 5 = \frac{11}{20}$(lakh tonnes)

$$\text{Total revenue} = \frac{12}{100}\times 250 \times \frac{20}{11}(\text{crores per lakh ton})$$

$$= \frac{30}{11}\times 20 \quad (1)$$

Considering Item 4,

$$\text{Total quantity} = \frac{22}{100}\times 5 = \frac{22}{20}(\text{lakh tonnes})$$

$$\text{Total revenue} = \frac{6}{100} \times 250 \times \frac{20}{22} (\text{crores per lakh ton})$$

$$= \frac{15}{22} \times 20 \qquad (2)$$

Taking the ratio of Eq. (1) and Eq. (2), we get

$$\left(\frac{30}{11} \times 20\right) : \left(\frac{15}{22} \times 20\right) \Rightarrow \frac{30 \times 20 \times 22}{11 \times 15 \times 20} : 1$$

$$= 2 \times 2 : 1 = 4 : 1$$

Ans. (d)

169. X is 1 km northeast of Y. Y is 1 km southeast of Z. W is 1 km west of Z. P is 1 km south of W. Q is 1 km east of P. What is the distance between X and Q in km?

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

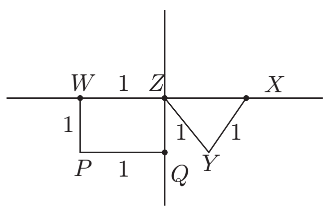
(GATE 2014, 2 Marks)

Solution: From the figure given below,

$$ZX = \sqrt{2}$$

$\Rightarrow$ Considering angle ZQX, which is a right angle

$\Rightarrow QX^2 = ZQ^2 + ZX^2 = \sqrt{1+2} = \sqrt{3}$



Ans. (c)

170. 10% of the population in a town is HIV^+. A new diagnostic kit for HIV detection is available; this kit correctly identifies HIV^+ individuals 95% of the time, and HIV^- individuals 89% of the time. A particular patient is tested using this kit and is found to be positive. The probability that the individual is actually positive is ________ .

(GATE 2014, 2 Marks)

Solution: Let total population = 100

HIV^+ patients = 10

For the patient to be +ve, he should be either +ve and test is showing the same or the patient should be −ve but the test is showing +ve.

$$\frac{0.1 \times 0.95}{0.1 \times 0.95 + 0.9 \times 0.11} = 0.489$$

Ans. 0.489

171. Given Set $A = \{2, 3, 4, 5\}$ and Set $\{11, 12, 13, 14, 15\}$, two numbers are randomly selected, one from each set. What is the probability that the sum of the two numbers equals 16?

(a) 0.20 (b) 0.25
(c) 0.30 (d) 0.33

(GATE 2015, 1 Mark)

Solution: Total possible outcomes $= 4 \times 5 = 20$

Favourable outcomes = $\{\{5,11\}, \{4, 12\}, \{3, 13\}, \{2, 14\}\} = 4$

Probability that the sum of the two numbers selected equals 16 $= \frac{4}{20} = \frac{1}{5} = 0.2$

Ans. (a)

172. Which of the following options is the closest in meaning to the sentence below?
She enjoyed herself immensely at the party.

(a) She had a terrible time at the party.
(b) She had a horrible time at the party.
(c) She had a terrific time at the party.
(d) She had a terrifying time at the party.

(GATE 2015, 1 Mark)

Solution: Option (c) is closest in meaning.

Ans. (c)

173. Which one of the following combinations is incorrect?

(a) Acquiescence – Submission
(b) Wheedle – Roundabout
(c) Flippancy – Lightness
(d) Profligate – Extravagant

(GATE 2015, 1 Mark)

Solution: 'Wheedle – Roundabout' is incorrect.

Ans. (b)

174. Based on the given statements, select the most appropriate option to solve the given question.
If two floors in a certain building are 9 ft apart, how many steps are there in a set of stairs that extends from the first floor to the second floor of the building?

Statements:

(i) Each step is 3/4 ft high.
(ii) Each step is 1 ft wide.

(a) Statement I alone is sufficient, but statement II alone is not sufficient.
(b) Statement II alone is sufficient, but statement I alone is not sufficient.
(c) Both statements together are sufficient, but neither statement alone is sufficient.
(d) Statements I and II together are not sufficient.

(GATE 2015, 1 Mark)

Solution: Both statements together are not sufficient.

Ans. (d)

175. Didn't you buy ________ when you went shopping?

(a) any paper
(b) much paper
(c) no paper
(d) a few paper

(GATE 2015, 1 Mark)

Solution: 'Any paper' fits the sentence well.

Ans. (a)

176. Choose the statement where underlined word is used correctly.

(a) The industrialist had a personnel jet.
(b) I write my experience in my personnel dairy.
(c) All personnel are being given the day off.
(d) Being religious is a personnel aspect.

(GATE 2015, 1 Mark)

Solution: Personnel – people employed in an organization or engaged in an organized undertaking such as military service.

Ans. (c)

177. Consider a function $f(x) = 1|x|$ on $-1 \le x \le 1$. The value of x at which the function attains a maximum, and the maximum value of the function are:

(a) 0, −1 (b) −1, 0 (c) 0, 1 (d) −1, 2

(GATE 2015, 1 Mark)

Solution: For $-1 \le x \le 1$, $0 \le |x| \le 1$. Hence, $f(x)$ will be maximum when $|x|$ is minimum, that is $|x| = 0 \Rightarrow x = 0$.

Therefore, maximum value of $f(x) = 1 - 0 = 1$.

Ans. (c)

178. We ______ our friend's birthday and we ______ how to make it up to him.

(a) completely forget, don't just know
(b) forgot completely, don't just know
(c) completely forgot, just don't know
(d) forgot completely, just don't know

(GATE 2015, 1 Mark)

Solution: Option (c) completes the sentence in the most appropriate way.

Ans. (c)

179. A generic term that includes various items of clothing such as a skirt, a pair of trousers and a shirt is

(a) Fabric
(b) Textile
(c) Fibre
(d) Apparel

(GATE 2015, 1 Mark)

Solution: Apparel – clothing, especially outerwear; garments; attire; raiment.

Ans. (d)

180. Based on the given statements, select the most appropriate option to solve the given question.
What will be the total weight of 10 poles each of same weight?

Statements:
(i) One fourth of the weight of a pole is 5 kg.
(ii) The total weight of these poles is 160 kg more than the total weight of two poles.

(a) Statement (i) alone is not sufficient.
(b) Statement (ii) alone is not sufficient.
(c) Either (i) or (ii) is sufficient.
(d) Both statements (i) and (ii) together are not sufficient.

(GATE 2015, 1 Mark)

Solution: Let weight of each pole be x.

(i) Given, $\frac{x}{4} = 15 \Rightarrow x = 60$ kg

All poles are of same weight. Thus, weight of 10 poles $= 10x = 600$ kg

(ii) Given,

$$10x = 2x + 160$$
$$\Rightarrow 8x = 160 \text{ kg} \Rightarrow x = 20\,\text{kg}$$

Thus, weight of 10 poles $= 10x = 200$ kg.

Ans. (c)

181. The Tamil version of ________ John Abraham-starrer Madras Café ________ cleared by the Censor Board with no cuts last week, but the film's distributors ________ no takers among the exhibitors for a release in Tamil Nadu ________ this Friday.

(a) Mr., was, found, on
(b) a, was, found, at
(c) the, was, found, on
(d) a, being, find, at

(GATE 2015, 1 Mark)

Solution: John Abraham-starrer Madras Café talks about the movie not the person, so Mr. is ruled out. 'Find no takers' is not the correct phrase. 'At' this Friday is incorrect. So, option (c) is correct.

Ans. (c)

182. If ROAD is written as URDG, then SWAN should be written as

(a) VXDQ (b) VZDQ
(c) VZDP (d) UXDQ

(GATE 2015, 1 Mark)

Solution:
R + 3 = U, O + 3 = R, A + 3 = D, D + 3 = G;
S + 3 = V, W + 3 = Z, A + 3 = D, N + 3 = Q.

Ans. (b)

183. A function $f(x)$ is linear and has a value of 29 at $x = -2$ and 39 at $x = 3$. Find its value at $x = 5$.

(a) 59 (b) 45 (c) 43 (d) 35

(GATE 2015, 1 Mark)

Solution:

$$f(x) = 2x + 33.$$

For $x = 5$,

$$f(x) = 10 + 33 = 43$$

Ans. (c)

184. Select the pair that best expresses a relationship similar to that expressed in the pair
Children: Paediatrician

(a) Adult: Orthopaedist
(b) Females: Gynaecologist
(c) Kidney: Nephrologist
(d) Skin: Dermatologist

(GATE 2015, 1 Mark)

Solution: Community of people: Doctor.

Ans. (b)

185. Extreme focus on syllabus and studying for tests has become such a dominant concern of Indian students that they close their minds to anything ________ to the requirements of the exam.

(a) related (b) extraneous
(c) outside (d) useful

(GATE 2015, 1 Mark)

Solution: Extraneous – irrelevant or unrelated to the subject being dealt with.

Ans. (b)

186. Operators Ω, $\Diamond$ and $\rightarrow$ are defined by $a \,\Omega\, b = \dfrac{a-b}{a+b}$; $a \Diamond b = \dfrac{a+b}{a-b}$; $a \rightarrow b = ab$.

Find the value of $(66\,\Omega\,6) \rightarrow (66 \Diamond 6)$.

(a) −2 (b) −1 (c) 1 (d) 2

(GATE 2015, 1 Mark)

Solution:

$$66\,\Omega\,6 = \frac{66-6}{66+6} = \frac{60}{72} = \frac{5}{6}$$

$$66 \Diamond 6 = \frac{66+6}{66-6} = \frac{72}{60} = \frac{6}{5}$$

$$(66\,\Omega\,6) \rightarrow (66 \Diamond 6) = \frac{5}{6} \times \frac{6}{5} = 1$$

Ans. (c)

187. Choose the most appropriate word from the options given below to complete the following sentence:

The principal presented the chief guest with a ________, as token of appreciation.

(a) momento (b) memento
(c) momentum (d) moment

(GATE 2015, 1 Mark)

Solution: The correct answer is 'memento'.

Ans. (b)

188. Choose the appropriate word/phrase, out of the four options given below, to complete the following sentence:

Frogs ________.

(a) croak (b) roar (c) hiss (d) patter

(GATE 2015, 1 Mark)

Solution: Frogs make 'croak' sound.

Ans. (a)

189. Choose the word most similar in meaning to the given word: Educe

(a) Exert (b) Educate
(c) Extract (d) Extend

(GATE 2015, 1 Mark)

Solution: Educe means to 'infer' or 'extract'.

Ans. (c)

190. If log$x$ (5/7) = −1/3, then the value of x is

(a) 343/125 (b) 125/343
(c) −25/49 (d) −49/25

(GATE 2015, 1 Mark)

Solution:

$$\frac{5}{7} = x^{-1/3} \Rightarrow \frac{7}{5} = x^{1/3} \Rightarrow \left(\frac{7}{5}\right)^3 \Rightarrow x = 2.74$$

Ans. (a)

191. Choose the word most similar in meaning to the given word?

Awkward

(a) Inept (b) Graceful
(c) Suitable (d) Dreadful

(GATE 2015, 1 Mark)

Solution: 'Inept' has the similar meaning to 'awkward'.

Ans. (a)

192. Choose the appropriate word/phrase, out of the four options given below, to complete the following sentence:

Dhoni, as well as the other team members of Indian team, __________ present on the occasion.

(a) were (b) was (c) has (d) have

(GATE 2015, 1 Mark)

Solution: The correct answer is 'was'.

Ans. (b)

193. Ram and Ramesh appeared in an interview for two vacancies in the same department. The probability of Ram's selection is 1/6 and that of Ramesh is 1/8. What is the probability that only one of them will be selected?

(a) 47/48 (b) 1/4 (c) 13/48 (d) 35/48

(GATE 2015, 1 Mark)

Solution: $P(\text{Ram}) = \frac{1}{6}$; $P(\text{Ramesh}) = \frac{1}{8}$

Hence, $P(\text{Ram is not selected})$

$$= \frac{5}{6},\ P(\text{Ramesh is not selected}) = \frac{7}{8}$$

$$P(\text{only one is selected}) = \frac{1}{6} \times \frac{7}{8} + \frac{5}{6} \times \frac{1}{8} = \frac{12}{48} = \frac{1}{4}$$

Ans. (b)

194. What is the adverb for the given word below?

MISOGYNOUS

(a) Misogynousness (b) Mysogynity
(c) Misogynously (d) Misogynous

(GATE 2015, 1 Mark)

Solution: The adverb is 'misogynously'.

Ans. (c)

195. An electric bus has onboard instruments that report the total electricity consumed since the start of the trip as well as the total distance covered. During a single day of operation, the bus travels on stretches M, N, O and P, in that order. The cumulative distances travelled and the corresponding electricity consumption is shown in the table below:

Stretch	Cumulative Distance (km)	Electricity Used (kWh)
M	20	12
N	45	25
O	75	45
P	100	57

The stretch where the electricity consumption per km is minimum is

(a) M (b) N
(c) O (d) P

(GATE 2015, 1 Mark)

Solution: For M, consumption $= \frac{12}{20} = 0.6$

For N, consumption $= \frac{25}{45} = 0.555$

For O, consumption $= \frac{45}{75} = 0.6$

For P, consumption $= \frac{57}{100} = 0.57$

Ans. (b)

196. Choose the most suitable one word substitute for the following expression:

Connotation of a road or way

(a) Pertinacious (b) Viaticum
(c) Clandestine (d) Ravenous

(GATE 2015, 1 Mark)

Solution: No word is relevant. Least irrelevant word is pertinacious.

Ans. (a)

197. Choose the most appropriate word from the options given below to complete the following sentence.

If the athlete had wanted to come first in the race, he __________ several hours every day.

(a) should practise
(b) should have practised
(c) practised
(d) should be practising

(GATE 2015, 1 Mark)

Solution: For a condition regarding something which has already happened, 'should have practiced' is the correct choice.

Ans. (b)

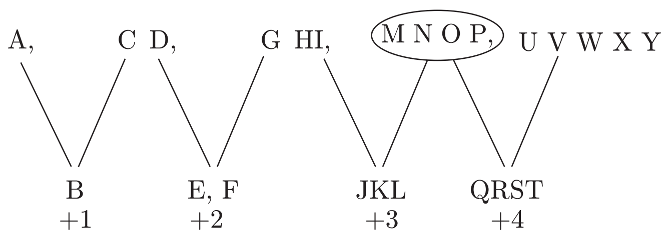
198. Find the missing sequence in the letter series below:

A, CD, GHI,?, UVWXY

(a) LMN (b) MNO
(c) MNOP (d) NOPQ

(GATE 2015, 1 Mark)

Solution:



Ans. (c)

199. Choose the correct verb to fill in below:

Let us __________.

(a) introvert (b) alternate
(c) atheist (d) altruist

(GATE 2015, 1 Mark)

Solution: 'Alternate' is the correct verb.

Ans. (b)

200. If $x > y > 1$, which of the following must be true?

(i) $\ln x > \ln y$
(ii) $e^x > e^y$
(iii) $y^x > x^y$
(iv) $\cos x > \cos y$

(a) (i) and (ii)
(b) (i) and (iii)
(c) (iii) and (iv)
(d) (ii) and (iv)

(GATE 2015, 1 Mark)

Solution: For whole numbers, greater the value, greater will be its logarithmic. Same logic applies for power of e.

Ans. (a)

201. Which of the following options is the closest in meaning to the sentence below?
She enjoyed herself immensely at the party.

(a) She had a terrible time at the party
(b) She had a horrible time at the party
(c) She had a terrific time at the party
(d) She had a terrifying time at the party

(GATE 2015, 1 Mark)

Solution: The correct answer is option (c).

Ans. (c)

202. Five teams have to compete in a league, with every team playing every other team exactly once, before going to next round. How many matches will have to be held to complete the league round of matches?

(a) 20 (b) 10 (c) 8 (d) 5

(GATE 2015, 1 Mark)

Solution: For a match to be played, we need 2 teams.
Number of matches = Number of selection 2 teams out of 5 = ${}^5C_2 = 10$

Ans. (b)

203. Choose the appropriate word/phrase, out of the four options given below, to complete the following sentence:

Apparent lifelessness __________ dormant life.

(a) harbours (b) leads to
(c) supports (d) affects

(GATE 2015, 1 Mark)

Solution: Apparent: looks like

Dormant: hidden

Harbour: give shelter

Effect (verb): results in

Ans. (a)

204. Choose the statement where underlined word is used correctly.

(a) When the teacher <u>eludes</u> to different authors, he is being elusive.
(b) When the thief keeps <u>eluding</u> the police, he is being elusive.
(c) Matters that are difficult to understand, identify or remember are <u>allusive</u>.
(d) Mirages can be <u>allusive</u>, but a better way to express them is illusory.

(GATE 2015, 1 Mark)

Solution: Elusive: difficult to answer.

Ans. (b)

205. Fill in the blank with the correct idiom/phrase.
That boy from the town was a __________ in the sleepy village.

(a) dog out of herd
(b) sheep from the heap
(c) fish out of water
(d) bird from the flock

(GATE 2015, 1 Mark)

Solution: From the statement, it appears that boy found it tough to adapt to a very different situation.

Ans. (c)

206. Tanya is older than Eric.

Cliff is older than Tanya.

Eric is older than Cliff.

If the first two statements are true, then the third statement is

(a) True
(b) False
(c) Uncertain
(d) Data insufficient

(GATE 2015, 1 Mark)

Solution: The correct answer is option (b).

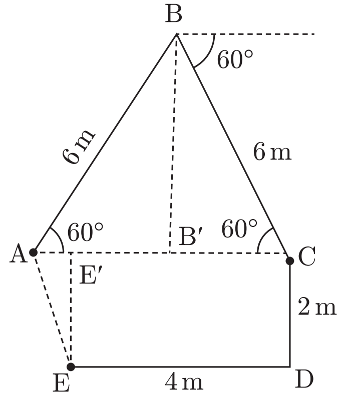
Ans. (b)

207. Mr. Vivek walks 6 m North-east, then turns and walks 6 m South-east, both at 60° to East. He further moves 2 m South and 4 m West. What is the straight distance in meters between the point he started from and the point he finally reached?

(a) $2\sqrt{2}$ (b) 2
(c) $\sqrt{2}$ (d) $1/\sqrt{2}$

(GATE 2015, 1 Mark)

Solution: Consider the figure formed from the data given in the question, where A is the starting point and E is the destination.



AB = BC = 6 m
Thus, AB′ = BC′ = 6 cos 60° = 3 m
Hence, AC = 3 + 3 = 6 m
AE′ = 6 − 4 = 2 m
EE′ = 2 m
Hence, using Pythagoras theorem, we get

$$AE = \sqrt{(EE')^2 + (AE')^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}\text{ m}$$

Ans. (a)

208. Choose the statement where underlined word is used correctly.

(a) The minister <u>insured</u> the victims that everything would be all right.
(b) He <u>ensured</u> that the company will not have to bear any loss.
(c) The actor got himself <u>ensured</u> against any accident.
(d) The teacher <u>insured</u> students of good results.

(GATE 2015, 1 Mark)

Solution: Insured – the person, group, or organization whose life or property is covered by an <u>insurance</u> policy.
Ensured – to secure or guarantee.

Ans. (b)

209. Which word is <u>not</u> a synonym for the word vernacular?

(a) Regional
(b) Indigenous
(c) Indigent
(d) Colloquial

(GATE 2015, 1 Mark)

Solution: Vernacular − expressed or written in the native language of a place.
Indigent – deficient in what is requisite.

Ans. (c)

210. Four cards are randomly selected from a pack of 52 cards. If the first two cards are kings, what is the probability that the third card is a king?

(a) 4/52
(b) 2/50
(c) (1/52) × (1/52)
(d) (1/52) × (1/51) × (1/50)

(GATE 2015, 1 Mark)

Solution: There are 4 kings in a pack of 52 cards. If 2 cards are selected and both are kings, remaining cards will be 50 out of which 2 will be kings.

Thus, the probability $= \dfrac{2}{50}$

Ans. (b)

211. Choose the most appropriate word from the options given below to complete the following sentence:

The official answered ________ that the complaints of the citizen would be looked into.

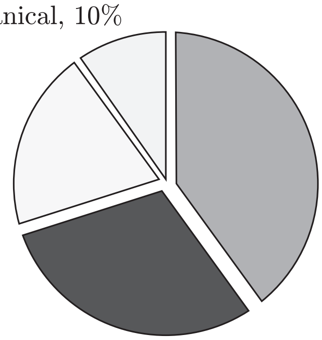
(a) respectably
(b) respectfully
(c) reputably
(d) respectively

(GATE 2015, 1 Mark)

Solution: The correct answer is 'respectfully'.

Ans. (b)

212. The pie chart below has the breakup of the number of students from different departments in an engineering college for the year 2012. The proportion of male to female students in each department is 5:4. There are 40 males in Electrical Engineering. What is the difference between the numbers of female students in the Civil department and the female students in the Mechanical department?



(GATE 2015, 2 Marks)

Solution:
Electrical male students = 40

Thus, electrical female students $= \dfrac{4}{5} \times 40 = 32$

Total female students in Mechanical department $= \dfrac{32}{2} = 16$

Total female students in Civil department $= 32 \times \dfrac{3}{2} = 48$

Difference $= 48 - 16 = 32$

Ans. 32

213. The given statement is followed by some course of action. Assuming the statement to be true, decide the correct option.

Statement:

There has been a significant drop in the water level in the lakes supplying water to the city.

Course of action:

(i) The water supply authority should impose a partial cut in supply to tackle the situation.
(ii) The government should appeal to all the residents through mass media for minimal use of water.
(iii) The government should ban the water supply in lower areas.

(a) Statements (i) and (ii) follow
(b) Statements (i) and (iii) follow
(c) Statements (ii) and (iii) follow
(d) All statements follow

(GATE 2015, 2 Marks)

Solution: Both statements (i) and (ii) follow.

Ans. (a)

214. Select the alternative meaning of the underlined part of the sentence.
The chain snatchers <u>took to their heels</u> when the police party arrived.

(a) took shelter in a thick jungle
(b) open indiscriminate fire
(c) took to flight
(d) unconditionally surrendered

(GATE 2015, 2 Marks)

Solution: 'Took to flight' has the same meaning.

Ans. (c)

215. The number of students in a class, who have answered correctly, wrongly, or not attempted each question in exam, are listed in the table below. The marks from each question are also listed. There is no negative or partial marking.

Q. No.	Marks	Answered Correctly	Answered Wrongly	Not Attempted
1	2	21	14	6
2	3	15	27	2
3	1	11	29	4
4	2	23	18	3
5	5	31	12	1

What is the average of the marks obtained by the class in the examination?

(a) 2.290 (b) 2.970
(c) 6.795 (d) 8.795

(GATE 2015, 2 Marks)

Solution: Total students = 21 + 17 + 6 = 44
Total marks = $2\times 21+3\times 15+1\times 11+2\times 23+5\times 31=299$

$$\text{Average marks} = \frac{299}{44} = 6.795$$

Ans. (c)

216. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c, respectively. Of these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Following relations are drawn in m, p and c.

(i) $p + m + c = 27/20$
(ii) $p + m + c = 13/20$
(iii) $(p) \times (m) \times (c) = 1/10$

(a) Only relation (i) is true
(b) Only relation (ii) is true
(c) Relations (ii) and (iii) are true
(d) Relations (i) and (iii) are true

(GATE 2015, 2 Marks)

Solution: Probability of passing in at least one subject

$$1-(1-m)(1-p)(1-c)=0.75 \qquad \text{(i)}$$

Probability of passing in at least two subjects

$$(1-m)pc+(1-p)mc+(1-c)mp+mpc=0.5 \quad \text{(ii)}$$

Probability of passing in exactly two subjects

$$(1-m)pc+(1-p)mc+(1-c)mp=0.4 \qquad \text{(iii)}$$

Subtracting Eq. (iii) from Eq. (ii), we get

$$mpc=0.1$$

Simplifying Eq. (i), we get

$$p+c+m-(mp+mc+pc)+mpc=0.75$$

$$p+c+m-(mp+mc+pc)=0.65 \qquad \text{(iv)}$$

After simplifying Eq. (iii), we get

$$pc+mc+mp-3mpc=0.4$$

Putting value of mpc, we get

$$pc+mc+mp=0.7$$

Putting above value in Eq. (iv), we get

$$p+c+m-0.7=0.65$$

$$p+c+m=1.35=\frac{27}{20}$$

Ans. (d)

217. If the list of letters, P, R, S, T, U, is an arithmetic sequence, which of the following are also in arithmetic sequence?

I. 2P, 2R, 2S, 2T, 2U
II. P−3, R−3, S−3, T−3, U−3
III. P^2, R^2, S^2, T^2, U^2

(a) (I) only
(b) (I) and (II)
(c) (II) and (III)
(d) (I) and (III)

(GATE 2015, 2 Marks)

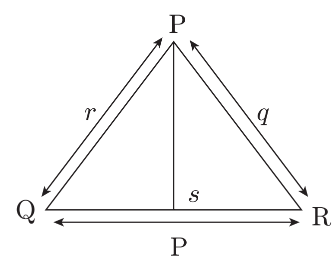
Solution: If P, R, S, T, U are in arithmetic sequence with a common difference 'd'. Then, 2P, 2R, 2S, 2T, 2U will be an arithmetic sequence with common difference '$2d$'.

P − 3, R − 3, S − 3, T − 3, U − 3 will be an arithmetic sequence with common difference 'd'.

P^2, R^2, S^2, T^2, U^2 will not be an arithmetic sequence.

Ans. (b)

218. In a triangle PQR, PS is the angle bisector of ∠QPR and ∠QPS = 60°. What is the length of PS?



(a) $\dfrac{(q+r)}{qr}$ (c) $\sqrt{(q^2+r^2)}$

(b) $\dfrac{qr}{(q+r)}$ (d) $\dfrac{(q+r)^2}{qr}$

(GATE 2015, 2 Marks)

Solution:

$$PS=\frac{2qr}{q+r}\cos\left(\frac{\angle QPR}{2}\right)$$

$$\angle QPR = 2\times\angle QPS = 120$$

$$PS=\frac{2qr}{q+r}\cos\left(\frac{120}{2}\right)=\left(\frac{qr}{q+r}\right)$$

Ans. (b)

219. Out of the following four sentences, select the most suitable sentence with respect to grammar and usage:

(a) Since the report lacked needed information, it was of no use to them.
(b) The report was useless to them because there were no needed information in it.
(c) Since the report did not contain the needed information, it was not real useful to them.
(d) Since the report lacked needed information, it would not have been useful to them.

(GATE 2015, 2 Marks)

Solution: Option (a) is the most suitable sentence.

Ans. (a)

220. If p, q, r, s are distinct integers such that:

$f(p, q, r, s) = \max(p, q, r, s)$

$g(p, q, r, s) = \min(p, q, r, s)$

$h(p, q, r, s) =$ remainder of $(p \times q)/(r \times s)$ if $(p \times q) > (r \times s)$ or remainder of $(r \times s)/(p \times q)$ of $(r \times s) > (p \times q)$

Also a function $fgh(p, q, r, s) = f(p, q, r, s) \times g(p, q, r, s) \times h(p, q, r, s)$

Also the same operations are valid with two variable functions of the form $f(p, q)$.

What is the value of $fg(h(2, 5, 7, 3), 4, 6, 8)$?

(GATE 2015, 2 Marks)

Solution:

$$fg(h(2,5,7,3),4,6,8)$$
$$= fg(1,4,6,8)$$
$$= f(1,4,6,8)\times g(1,4,6,8) = 8\times 1 = 8$$

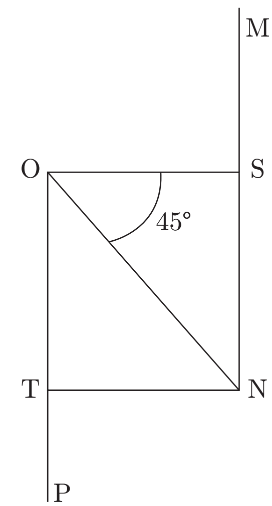
Ans. 8

221. Four branches of company are located at M, N, O and P. M is north of N at a distance of 4 km; P is south of O at a distance of 2 km; N is southeast of O by 1 km. What is the distance between M and P in km?

(a) 5.34 (b) 6.74
(c) 28.5 (d) 45.49

(GATE 2015, 2 Marks)

Solution: Given the data in the question, we can form the following figure:



$$SN = 1\sin 45^\circ = \frac{1}{\sqrt{2}} = OT$$

Vertical distance between M and P $= \left(4 + 2 - \frac{1}{\sqrt{2}}\right)$

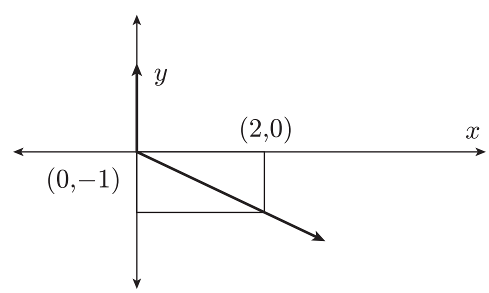
Horizontal distance $= \frac{1}{\sqrt{2}}$

Distance between M and P $= \sqrt{\left(6 - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$

$= 5.34$ km

Ans. (a)

222. Choose the most appropriate equation for the function drawn as a thick line, in the plot below.



(a) $x = y - |y|$
(b) $x = -(y - |y|)$
(c) $x = y + |y|$
(d) $x = -(y + |y|)$

(GATE 2015, 2 Marks)

Solution:

When $y = -1$, $x = 2$.

Also, when y is positive $x = 0$.

Thus, $x = -\left(y - |y|\right)$

Ans. (b)

223. Alexander turned his attention toward India, since he had conquered Persia.

Which one of the statements below is logically valid and can be inferred from the above sentence?

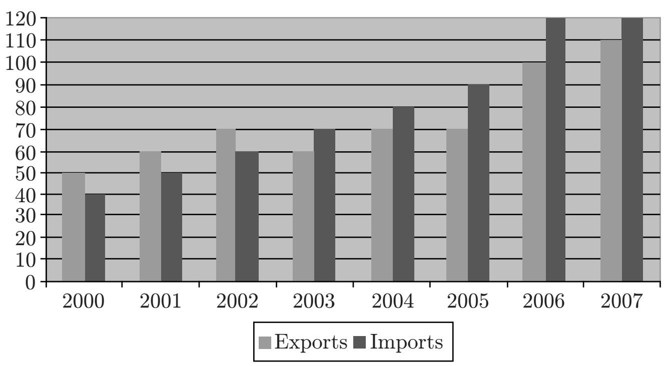
(a) Alexander would not have turned his attention towards India had he not conquered Persia.
(b) Alexander was not ready to rest on his laurels and wanted to march to India.
(c) Alexander was completely in control of his army and could command it to move toward India.
(d) Since Alexander's kingdom extended to Indian borders after the conquest of Persia, he was keen to move further.

(GATE 2015, 2 Marks)

Solution: Option (a) is logically valid.

Ans. (a)

224. The exports and imports (in crores of Rs) of a country from the year 2000 to 2007 are given in the following bar chart. In which year is the combined percentage increase in imports and exports the highest?



(GATE 2015, 2 Marks)

Solution:

$$\text{Increase in exports in 2006} = \frac{100-70}{70} = 42.8\%$$

$$\text{Increase in imports in 2006} = \frac{120-90}{90} = 33.3\%$$

which is more than any other year.

Ans. 2006

225. Most experts feel that in spite of possessing all the technical skills required to be a batsman of the highest order, he is unlikely to be so due to the lack of requisite temperament. He was guilty of throwing away his wicket several times after working hard to lay a strong foundation. His critics pointed out that until he addresses this problem, success at the highest level will continue to elude him.

Which of the statement(s) below is/are logically valid and can be inferred from the above passage?

(i) He was already a successful batsman at the highest level.
(ii) He has to improve his temperament in order to become a great batsman.
(iii) He failed to make many of his good starts count.
(iv) Improving his technical skills will guarantee success.

(a) (iii) and (iv)
(b) (ii) and (iii)
(c) (i), (ii), and (iii)
(d) (ii) only

(GATE 2015, 2 Marks)

Solution: Statements (ii) and (iii) are logically valid.

Ans. (b)

226. The head of a newly formed government desires to appoint five of the six selected members P, Q, R, S, T and U to portfolios of Home, Power, Defense, Telecom, and Finance. U does not want any portfolio if S gets one of the five. R wants either Home or Finance or no portfolio. Q says that if S gets either Power or Telecom, then she must get the other one. T insists on a portfolio if P gets one.

Which is the valid distribution of portfolios?

(a) P – Home, Q – Power, R – Defense, S – Telecom, T – Finance
(b) R – Home, S – Power, P – Defense, Q – Telecom, T = Finance
(c) P – Home, Q – Power, T – Defense, S – Telecom, U – Finance
(d) Q – Home, U – Power, T – Defense, R – Telecom, P – Finance

(GATE 2015, 2 Marks)

Solution: Since U does not want any portfolio, (c) and (d) are ruled out. R wants Home, or Finance or no portfolio, therefore (a) is not valid. Thus, option (b) is correct.

Ans. (b)

227. A cube of side 3 units is formed using a set of smaller cubes of side 1 unit. Find the proportion of the number of faces of the smaller cubes visible to those which are NOT visible.

(a) 1:4
(b) 1:3
(c) 1:2
(d) 2:3

(GATE 2015, 2 Marks)

Solution: Number of faces per cube = 6

Total number of cubes = $9 \times 3 = 27$

Therefore, total number of faces = $27 \times 6 = 162$

Hence, total number of non-visible faces = $162 - 54 = 108$

Therefore,

$$\frac{\text{Number of visible faces}}{\text{Number of non-visible faces}} = \frac{54}{108} = \frac{1}{2}$$

Ans. (c)

228. The following question presents a sentence, part of which is underlined. Beneath the sentence you find four ways of phrasing the underlined part. Following the requirements of the standard written English, select the answer that produces the most effective sentence.

Tuberculosis, together with its effects, ranks one of the leading causes of death in India.

(a) ranks as one of the leading causes of death

(b) rank as one of the leading causes of death

(c) has the rank of the leading causes of death

(d) are one of the leading causes of death

(GATE 2015, 2 Marks)

Solution: 'Ranks as one of the leading causes of death' produces most the effective sentence.

Ans. (a)

229. Humpty Dumpty sits on a wall every day while having lunch. The wall sometimes breaks. A person sitting on the wall falls if the wall breaks. Which one of the statements below is logically valid and can be inferred from the above sentences?

(a) Humpty Dumpty always falls while having lunch.

(b) Humpty Dumpty does not fall sometimes while having lunch.

(c) Humpty Dumpty never falls during dinner.

(d) When Humpty Dumpty does not sit on the wall, the wall does not break.

(GATE 2015, 2 Marks)

Solution: Option (b) is logically valid.

Ans. (b)

230. Find the missing value:

6 5 4
7 4 7 2 1
1 9 2 8 1 2 1
4 1 5 2 3
3 ? 3

(GATE 2015, 2 Marks)

Solution: Middle number is the average of the numbers on both sides.

Average of 6 and 4 is 5

Average of (7 + 4) and (2 + 1) is 7.

Average of (1 + 9 + 2) and (1 + 2 + 1) is 8.

Average of (4 + 1) and (2 + 3) is 5.

Therefore, average of 3 and 3 is 3.

Ans. 3

231. Read the following paragraph and choose the correct statement.

Climate change has reduced human security and threatened human well-being. An ignored reality of human progress is that human security largely depends upon environmental security. But on the contrary, human progress seems contradictory to environment security. To keep up both at the required level is a challenge to be addressed by one and all. One of the ways to curb the climate change may be suitable scientific innovations, while the other may be the Gandhian perspective on small scale progress with focus on sustainability.

(a) Human progress and security are positively associated with environmental security.

(b) Human progress is contradictory to environmental security.

(c) Human security is contradictory to environmental security.

(d) Human progress depends upon environmental security.

(GATE 2015, 2 Marks)

Solution: The correct answer is option (b).

Ans. (b)

232. Lamenting the gradual sidelining of the arts in school curricula, a group of prominent artists wrote to the Chief Minister last year, asking him to allocate more funds to support arts education in schools. However, no such increase has been announced in this year's budget. The artists expressed their deep anguish at their request not being approved, but many of them remain optimistic about funding in the future.

Which of the statement(s) below is/are logically valid and can be inferred from the above statements?

(i) The artists expected funding for the arts to increase this year.

(ii) The Chief Minister was receptive to the idea of increasing funding for the arts.

(iii) The Chief Minister is a prominent artist.

(iv) Schools are giving less importance to arts education nowadays.

(a) (iii) and (iv)
(b) (i) and (iv)
(c) (i), (ii) and (iv)
(d) (i) and (iii)

(GATE 2015, 2 Marks)

Solution: (i), (ii) and (iv) are logically valid.

Ans. (c)

233. If $a^2 + b^2 + c^2 = 1$, then $ab + bc + ac$ lies in the interval

(a) [1, 2/3] (b) $[-1/2, 1]$
(c) [−1, 1/2] (d) [2, 4]

(GATE 2015, 2 Marks)

Solution: It lies in the interval $[-1/2, 1]$.

Ans. (b)

234. In the following sentence, certain parts are underlined and marked P, Q and R. One of the parts may contain certain error or may not be acceptable in standard written communication. Select the part containing an error. Choose D as your answer if there is no error.

The student corrected $\underset{P}{\underline{\textbf{all the errors}}}$ that $\underset{Q}{\underline{\textbf{the instructor marked}}}$ on the $\underset{R}{\underline{\textbf{answer book.}}}$

(a) P (b) Q
(c) R (d) No Error

(GATE 2015, 2 Marks)

Solution: Part Q contains error.

Ans. (b)

235. Given below are two statements followed by two conclusions. Assuming these statements to be true, decide which one logically follows.

Statements:

(i) All film stars are playback singers.

(ii) All film directors are film stars.

Conclusions:

(i) All film directors are playback singers.

(ii) Some film stars are film directors.

(a) Only conclusion (i) follows.
(b) Only conclusion (ii) follows.
(c) Neither conclusion (i) or (ii) follows.
(d) Both conclusions (i) and (ii) follow.

(GATE 2015, 2 Marks)

Solution: Both conclusions (i) and (ii) follow.

Ans. (d)

236. A tiger is 50 leaps of its own behind a deer. The tiger takes 5 leaps per minute to the deer's 4. If the tiger and the deer cover 8 m and 5 m per leap, respectively, what distance in metres will the tiger have to run to catch the deer?

(GATE 2015, 2 Marks)

Solution: We are given

1 leap of tiger = 8 m

Speed = $5 \times 8 = 40\,\text{m/min}$

1 leap of deer = 5 m

Speed = $5 \times 4 = 20\,\text{m/min}$

Assume that at time 't' the tiger catches the deer.

Thus, distance travelled by deer + initial distance between them

Distance covered by tiger = $50 \times 8 = 400$ m

$$\Rightarrow 40 \times t = 400 + 20t \Rightarrow t = \frac{400}{20} = 20 \text{ min}$$

Total distance = 400 + 20t = 400 + 400 = 800 m

Ans. 800 m

237. Ms. X will be in Bagdogra from 01/05/2014 to 20/05/2014 and from 22/05/2014 to 31/05/2014. On the morning of 21/05/2014, she will reach Kochi via Mumbai.

Which one of the statements below is logically valid and can be inferred from the above sentences?

(a) Ms. X will be in Kochi for one day, only in May.
(b) Ms. X will be in Kochi for only one day in May.

(c) Ms. X will be only in Kochi for one day in May.
(d) Only Ms. X will be in Kochi for one day in May.

(GATE 2015, 2 Marks)

Solution: Second sentence says that Ms. X reaches Kochi on 21/05/2014. Also she has to be in Bagdogra on 22/05/2014.
Therefore, she stays in Kochi for only one day in May.

Ans. (b)

238. log tan 1° + log tan 2° + ________ + log tan 89° is ________ .

(a) 1 (b) $1/\sqrt{2}$
(c) 0 (d) −1

(GATE 2015, 2 Marks)

Solution: $\log\tan 1^\circ + \log\tan 89^\circ$

$$= \log(\tan 1^\circ \times \tan 89^\circ) = \log(\tan 1^\circ \times \cot 1^\circ)$$
$$= \log 1 = 0$$

Using the same logic total sum is '0'.

Ans. (c)

239. From a circular sheet of paper of radius 30 cm, a sector of 10% area is removed. If the remaining part is used to make a conical surface, then the ratio of the radius and height of the cone is ______.

(GATE 2015, 2 Marks)

Solution: 90% of area of sheet = Cross-sectional area of cone

$$\Rightarrow 0.9\pi \times 30 \times 30 = \pi \times r_1 \times 30$$
$$\Rightarrow r_1 = 27$$

Therefore, height of the cone $= \sqrt{30^2 - 27^2}$

$= 13.08$ cm

Ans. 13.08 cm

240. Ram and Shyam shared a secret and promised to each other it would remain between them. Ram expressed himself in one of the following ways as given in the choices below. Identify the correct way as per standard English.

(a) It would remain between you and me.
(b) It would remain between I and you.
(c) It would remain between you and I.
(d) It would remain with me.

(GATE 2015, 2 Marks)

Solution: The correct way to express is 'It would remain between you and me'.

Ans. (a)

241. In the following question, the first and the last sentence of the passage are in order and numbered 1 and 6. The rest of the passage is split into four parts and numbered as 2, 3, 4 and 5. These four parts are not arranged in proper order. Read the sentences and arrange them in a logical sequence to make a passage and choose the correct sequence from the given options.

1. On Diwali, the family rises early in the morning.
2. The whole family, including the young and the old enjoy doing this.
3. Children let off fireworks later in the night with their friends.
4. At sunset, the lamps are lit and the family performs various rituals.
5. Father, mother, and children visit relatives and exchange gifts and sweets.
6. Houses look so pretty with lighted lamps all around.

(a) 2, 5, 3, 4 (b) 5, 2, 4, 3
(c) 3, 5, 4, 2 (d) 4, 5, 2, 3

(GATE 2015, 2 Marks)

Solution: The correct sequence is '5, 2, 4, 3'.

Ans. (b)

242. Right triangle PQR is to be constructed in the *xy*-plane so that the right angle is at P and line PR is parallel to the *x*-axis. The *x* and *y* coordinates of P, Q, and R are to be integers that satisfy the inequalities: $4 \le x \le 5$ and $6 \le y \le 16$. How many different triangles could be constructed with these properties?

(a) 110 (b) 1100
(c) 9900 (d) 10,000

(GATE 2015, 2 Marks)

Solution: P can take a total of 10 × 11 = 110 coordinates.

For a given P, R can take a total of 9 coordinates and Q can take a total of 10 coordinates.

Hence, total number of different triangles = $110 \times 9 \times 10 = 9900$.

Ans. (c)

243. A coin is tossed thrice. Let X be the event that head occurs in each of the first two tosses. Let Y be the event that a tail occurs on the third toss. Let Z be the event that two tails occur in three tosses. Based on the above information, which one of the following is TRUE?

(a) X and Y are not independent
(b) Y and Z are dependent
(c) Y and Z are independent
(d) X and Z are independent

(GATE 2015, 2 Marks)

Solution: Let Y be the event that tail occurred in third toss. And Z be two tails in third toss which can be {TTH, THT, HTT}.

$$Y = \{\text{TTH, TTT}\}$$

Thus, both Y and Z are dependent.

Ans. (b)

244. Given below are two statements followed by two conclusions. Assuming these statements to be true, decide which one logically follows.

Statements:
(i) No manager is a leader.
(ii) All leaders are executives.

Conclusions:
(i) No manager is an executive.
(ii) No executive is a manager.

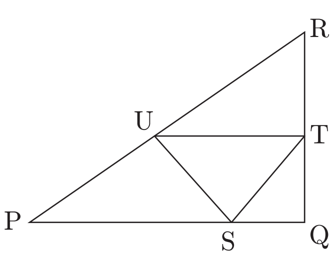
(a) Only conclusion (i) follows.
(b) Only conclusion (ii) follows.
(c) Neither conclusion (i) nor (ii) follows.
(d) Both conclusions (i) and (ii) follow.

(GATE 2015, 2 Marks)

Solution: The correct answer is option (c).

Ans. (c)

245. In the given figure, angle Q is a right angle, PS:QS = 3:1, RT:QT = 5:2 and PU:UR = 1:1. If area of triangle QTS is 20 cm^2, then the area of triangle PQR in cm^2 is ________.



(GATE 2015, 2 Marks)

Solution: Let area of triangle PQR be 'A'

$$\frac{SQ}{PQ} = \frac{1}{1+3} = \frac{1}{4}$$

$$\frac{QT}{QR} = \frac{2}{2+5} = \frac{2}{7}$$

Therefore, area of triangle QTS $= \frac{1}{2}\times SQ \times QT$

$$= \frac{1}{2}\times\left(\frac{1}{4}PQ\right)\times\left(\frac{2}{7}QR\right) = \frac{1}{4}\times\frac{2}{7}\times\left(\frac{1}{2}\times PQ\times QR\right)$$

$$= \frac{1}{14}\times \text{Area of } \Delta PQR$$

Given,

$$20 = \frac{1}{14}\times A \Rightarrow A = 14\times 20 = 280 \text{ cm}^2$$

Ans. 280 cm^2

246. Select the appropriate option in place of underlined part of the sentence.

<u>Increased productivity necessary</u> reflects greater efforts made by the employees.

(a) Increase in productivity necessary
(b) Increase productivity is necessary
(c) Increase in productivity necessarily
(d) No improvement required.

(GATE 2015, 2 Marks)

Solution: The correct answer is option (c).

Ans. (c)

247. Read the following table giving sales data of five types of batteries for years 2006–2012.

Year	Type I	Type II	Type III	Type IV	Type V
2006	75	144	114	102	108
2007	90	126	102	84	126
2008	96	114	75	105	135
2009	105	90	150	90	75
2010	90	75	135	75	90
2011	105	60	165	45	120
2012	115	85	160	100	145

Out of the following, which type of battery achieved highest growth between the years 2006 and 2012?

(a) Type V
(b) Type III
(c) Type II
(d) Type I

(GATE 2015, 2 Marks)

Solution: Type I achieved a growth of 53% in the period which is higher than any other type of battery.

Ans. (d)

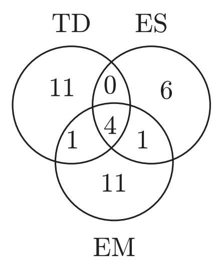
248. There are 16 teachers who can teach Thermodynamics (TD), 11 who can teach Electrical Sciences (ES), and 5 who can teach both TD and Engineering Mechanics (EM). There are a total of 40 teachers, 6 cannot teach any of the three subjects, that is EM, ES or TD. Six can teach only ES. Four teach all three subjects, that is EM, ES and TD. Four can teach ES and TD. How many can teach both ES and EM but not TD?

(a) 1 (b) 2

(c) 3 (d) 4

(GATE 2015, 2 Marks)

Solution: Using the data given in the question, we can form the Venn diagram as follows:



Ans. (a)

249. How many four digit numbers can be formed with the 10 digits, 0, 1, 2, ..., 9, if no number can start with 0 and if repetitions are not allowed?

(GATE 2015, 2 Marks)

Solution: In thousands place, 9 digits except 0 can be placed.

In hundreds place, 9 digits can be placed (including 0, excluding the one used in thousands place).

In tens place, 8 digits can be placed (excluding the ones used in thousands and hundreds place).

In ones place, 7 digits can be placed (excluding the one used in thousands, hundreds and tens place).

Total number of combinations $= 9 \times 9 \times 8 \times 7 = 4536$

Ans. 4536

250. The word similar in meaning to 'dreary' is

(a) Cheerful

(b) Dreamy

(c) Hard

(d) Dismal

(GATE 2015, 2 Marks)

Solution: Dreary – depressingly dull and bleak or repetitive.

Ans. (d)

251. The given question is followed by two statements; select the most appropriate option that solves the question.

Capacity of a solution tank A is 70% of the capacity of tank B. How many gallons of solution are in tank A and tank B?

Statements:

(i) Tank A is 80% full and tank B is 40% full

(ii) Tank A is full and contains 14,000 gallons of solution.

(a) Statement (i) alone is sufficient

(b) Statement (ii) alone is sufficient

(c) Either statement (i) or (ii) alone is sufficient.

(d) Both the statements (i) and (ii) together are sufficient.

(GATE 2015, 2 Marks)

Solution: Statement I can be used to solve the question if capacity of both tanks is already known. Statement II can be used if it is known what quantity of each tank is full/empty.

Now, let capacity of tank B be x.

$$\frac{70}{100}x = 14000$$

$$\Rightarrow x = 2000 \text{ gallons}$$

$$\text{Solution in tank A} = \frac{80}{100} \times 14000 = 11200 \text{ gallons}$$

$$\text{Solution in tank B} = \frac{40}{100} \times 20000 = 8000 \text{ gallons}$$

Thus, total solution $= 11200 + 8000 = 19200$ gallons

Ans. (d)

252. Out of the following four sentences, select the most suitable sentence with respect to grammar and usage.

(a) I will not leave the place until the minister does not meet me.

(b) I will not leave the place until the minister doesn't meet me.

(c) I will not leave the place until the minister meet me.

(d) I will not leave the place until the minister meets me.

(GATE 2016, 1 Mark)

Solution: 'Until' itself is negative word hence the sentence after 'until' must be affirmative. Minister is singular and hence 's' is added with the verb.

Ans. (d)

253. A rewording of something written or spoken is a __________.

(a) paraphrase
(b) paradox
(c) paradigm
(d) paraffin

(GATE 2016, 1 Mark)

Solution: If any written or spoken sentence is reworded, it is called paraphrasing.

Ans. (a)

254. Archimedes said, "Give me a lever long enough and a fulcrum on which to place it, and I will move the world."

The sentence above is an example of a __________ statement.

(a) figurative
(b) collateral
(c) literal
(d) figurine

(GATE 2016, 1 Mark)

Solution: The sentence under quotes is denoting figures and hence this sentence is figurative.

Ans. (a)

255. If 'relftaga' means carefree, 'otaga' means careful and 'fertaga' means careless, which of the following could mean 'aftercare'?

(a) zentaga
(b) tagafer
(c) tagazen
(d) relffer

(GATE 2016, 1 Mark)

Solution: refltaga=care free
otaga=care full

Hence, 'taga'=care

Now, fertaga=care less

Hence, 'fer'=less

Therefore, tagazen=aftercare

Ans. (c)

256. A cube is built using 64 cubic blocks of side one unit. After it is built, one cubic block is removed from every corner of the cube. The resulting surface area of the body (in square units) after the removal is __________.

(a) 56
(b) 64
(c) 72
(d) 96

(GATE 2016, 1 Mark)

Solution: Side of a cube=1 unit.
Assume that side of larger cube is made of X units.

Then volume of larger cube=Volume of all 64 cubes.

$$X^3 = 64 \times (1)^3$$
$$X = \sqrt[3]{64} = 4 \text{ units}$$

Surface area, $A = 6 \times 4^2 = 96$ units

If one cube is removed from each corner, then three surfaces one on each side will be removed and three surfaces will also be added. Hence, no change in surface area will be there.

Ans. (d)

257. The man who is now Municipal Commissioner worked as __________.

(a) the security guard at a university
(b) a security guard at the university
(c) a security guard at university
(d) the security guard at the university

(GATE 2016, 1 Mark)

Solution: The man who is now Municipal Commissioner worked as the security guard at a university.

Ans. (a)

258. Nobody knows how the Indian cricket team is going to <u>cope with</u> the difficult and seamer-friendly wickets in Australia.

Choose the option which is closest in meaning to the underlined phrase in the above sentence.

(a) put up with
(b) put in with
(c) put down to
(d) put up against

(GATE 2016, 1 Mark)

Solution: Cope with=Put up with

Ans. (a)

259. Find the odd one in the following group of words:

mock, deride, praise, jeer

(a) mock
(b) deride

(c) praise
(d) jeer

(GATE 2016, 1 Mark)

Solution: *Praise* is the odd word. Its meaning is opposite to that of the other words.

Ans. (c)

260. Pick the odd one from the following options:

(a) CADBE
(b) JHKIL
(c) XVYWZ
(d) ONPMQ

(GATE 2016, 1 Mark)

Solution: Number the alphabet as shown in the following figure:

	1	2	3	4	5		
	A	B	C	D	E	F	G
	H	I	J	K	L		
	M	N	O	P	Q	R	
U	V	W		Y	Z		

Options (a), (b) and (c) are in the order 31425.

Only option (d) is in the order 32415.

Ans. (d)

261. In a quadratic function, the value of the product of the roots (α, β) is 4. Find the value of

$$\frac{\alpha^n + \beta^n}{\alpha^{-n} + \beta^{-n}}$$

(a) n^4
(b) 4^n
(c) 2^{2n-1}
(d) 4^{n-1}

(GATE 2016, 1 Mark)

Solution: $\alpha\beta = 4$ (Given)
Now,

$$\frac{\alpha^n + \beta^n}{\alpha^{-n} + \beta^{-n}}$$

$$= \frac{\alpha^n + \beta^n}{\dfrac{1}{\alpha^n} + \dfrac{1}{\beta^n}} = \frac{\alpha^n + \beta^n}{\dfrac{\beta^n + \alpha^n}{\alpha^n \beta^n}}$$

$$= \alpha^n \cdot \beta^n \frac{(\alpha^n + \beta^n)}{\alpha^n + \beta^n} = (\alpha\beta)^n = 4^n$$

Ans. (b)

262. Which of the following is **CORRECT** with respect to grammar and usage?

Mount Everest is __________.

(a) the highest peak in the world
(b) highest peak in the world
(c) one of highest peak in the world
(d) one of the highest peak in the world

(GATE 2016, 1 Mark)

Solution: The sentence is stating the highest peak in the world. Since it is a specific thing, we need to use the definite article 'the' before it. Also the sentence is using the superlative degree and so we say 'the highest peak in the world' making option (a) the correct answer. There cannot be many highest peaks in the world and so options (c) and (d) are incorrect.

Ans. (a)

263. The policeman asked the victim of a theft, "What did you __________?"

(a) loose (b) lose
(c) loss (c) louse

(GATE 2016, 1 Mark)

Solution: The context of the sentence is asking a person who has been deprived of something because of a theft. The word to be used to fill the blank is 'lose' which means to be deprived of something. 'Loose' means something that is not fitted. 'Louse' is the singular form of the word 'lice' that is a parasite that lives in the skin of mammals and birds. 'Loss' is a noun that means the process of losing someone or something. For example, he suffered tremendous loss in his business.

Ans. (b)

264. Despite the new medicines __________ in treating diabetes, it is not __________ widely.

(a) effectiveness – prescribed
(b) availability – used

(c) prescription – available
(d) acceptance – proscribed

(GATE 2016, 1 Mark)

Solution: The sentence is looking for a contrast as it is joined by the conjunction 'despite'. The best pair of words that can fit the context of the sentence is 'effectiveness – prescribed'. Though the medicine is 'effective' in treating diabetes, it is not being 'prescribed' widely. A new medicine cannot have a 'prescription' or 'availability' for treating a disease. 'Proscribed' means forbidden by law. In case we use 'acceptance – proscribed' the sentence will not make any sense because it will mean that though the medicine is accepted widely, it is not forbidden by law.

Ans. (a)

265. In a huge pile of apples and oranges, both ripe and unripe mixed together, 15% are unripe fruits. Of the unripe fruits, 45% are apples. Of the ripe ones, 66% are oranges. If the pile contains a total of 5,692,000 fruits, how many of them are apples?

(a) 2,029,198 (b) 2,467,482
(c) 2,789,080 (d) 3,577,422

(GATE 2016, 1 Mark)

Solution: Let

T = Total number of fruits = 5,692,000

R = Ripe fruits

U = Unripe fruits

A = Apple

O = Oranges

Given U = 15% of T:

$$(15/100)\times 5{,}692{,}000 = 853{,}800$$
$$R = T - U = 4{,}838{,}200$$

$A(U)$ = 45% of U:

$$(45/100)\times 853{,}800 = 384{,}210$$

$A(R)$ = (100 − 66)% of R:

$$(34/100)\times 4{,}838{,}200 = 1{,}644{,}988$$

Therefore,

$$A(U) + A(R) = 2{,}029{,}198$$

Ans. (a)

266. Michael lives 10 km away from where I live. Ahmed lives 5 km away and Susan lives 7 km away from where I live. Arun is farther away than Ahmed but closer than Susan from where I live. From the information provided here, what is one possible distance (in km) at which I live from Arun's place?

(a) 3.00 (b) 4.99
(c) 6.02 (d) 7.01

(GATE 2016, 1 Mark)

Solution: In the question, it is given that Ahmed is 5 km away and Susan is 7 km away from where I live. Further, it is given that Arun is farther away than Ahmed from where I live and not as far as Susan. That means Arun must be living at distance more than 5 km but less than 7 km from my house which is according to given options can be 6.02 km.

Ans. (c)

267. Based on the given statements, select the appropriate option with respect to grammar and usage.

Statements:

(i) The height of Mr. X is 6 feet.
(ii) The height of Mr. Y is 5 feet.

(a) Mr. X is longer than Mr. Y.
(b) Mr. X is more elongated than Mr. Y.
(c) Mr. X is taller than Mr. Y.
(d) Mr. X is lengthier than Mr. Y.

(GATE 2016, 1 Mark)

Solution: 'Mr. X is taller than Mr. Y' is correct.

Ans. (c)

268. The students __________ the teacher on teachers' day for twenty years of dedicated teaching.

(a) facilitated
(b) felicitated
(c) fantasized
(d) facilitated

(GATE 2016, 1 Mark)

Solution: *Felicitated* is correct.

Ans. (b)

269. After India's cricket world cup victory in 1985, Shrotria who was playing both tennis and cricket till then, decided to concentrate only on cricket. And the rest is history.

What does the underlined phrase mean in this context?

(a) History will rest in peace.
(b) Rest is recorded in history books.

(c) Rest is well known.
(d) Rest is archaic.

(GATE 2016, 1 Mark)

Solution: 'Rest is well known' is correct.

Ans. (c)

270. Given (9 inches)$^{1/2}$=(0.25 yards)$^{1/2}$, which one of the following statements is TRUE?

(a) 3 inches=0.5 yards
(b) 9 inches=1.5 yards
(c) 9 inches=0.25 yards
(d) 81 inches=0.0625 yards

(GATE 2016, 1 Mark)

Solution: (9 inches)$^{1/2}$=(0.25 yards)$^{1/2}$

$$\Rightarrow \sqrt{9 \text{ inches}} = \sqrt{0.25 \text{ yards}}$$

Squaring on both sides, we get

9 inches=0.25 yards

Ans. (c)

271. S, M, E and F are working in shifts in a team to finish a project. M works with twice the efficiency of others but for half as many days as E worked. S and M have 6 hour shifts in a day, whereas E and F have 12 hours shifts. What is the ratio of contribution of M to contribution of E in the project?

(a) 1:1
(b) 1:2
(c) 1:4
(d) 2:1

(GATE 2016, 1 Mark)

Solution: Ratio of contribution of M to the contribution of E in the project is given as

$$\frac{\text{Contribution of } M}{\text{Contribution of } E} = \frac{\left[6 \times \frac{d}{2}\right] \times 2}{[12 \times d]} = \frac{1}{2}$$

Note that $\frac{d}{2}$ days have been used for M as it is given that M works for half as many days as E works. An extra factor of 2 is used for M because M has twice the efficiency of others.

Ans. (b)

272. An apple costs Rs. 10. An onion costs Rs. 8.

Select the most suitable sentence with respect to grammar and usage.

(a) The price of an apple is greater than an onion.
(b) The price of an apple is more than onion.
(c) The price of an apple is greater than that of an onion.
(d) Apples are costlier than onions.

(GATE 2016, 1 Mark)

Solution: The information given to us about both apples and onions is its price. We see that an apple costs more than an onion. The best way to frame the information with respect to grammar and usage in the given options is option (c). 'More costlier' is wrong grammatically because 'costlier' is the comparative degree of comparison and hence does not require 'more' before it. Since the prices are being compared, we cannot say the price of an apple is greater than an onion. Thus option (a) is also wrong. Option (b) is incorrect without any article before 'onion'. Thus, option (c) that correctly compares the price of both the items is the correct answer.

Ans. (c)

273. The Buddha said, 'Holding on to anger is like grasping a hot coal with the intent of throwing it at someone else; you are the one who gets burnt'.

Select the word below which is closest in meaning to the word underlined above.

(a) burning
(b) igniting
(c) clutching
(d) flinging

(GATE 2016, 1 Mark)

Solution: 'Burning' or igniting' means to give fire to something. 'Clutching' means to hold on to something. 'Flinging' means to hurl something forcefully. From the options, we see that the word closest in meaning to the word 'grasping' is clutching'.

Ans. (c)

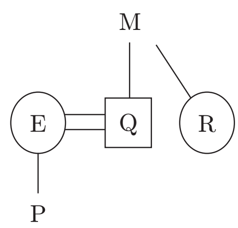
274. M has a son Q and a daughter R. He has no other children. E is the mother of P and daughter-in-law of M. How is P related to M?

(a) P is the son-in-law of M
(b) P is the grandchild of M
(c) P is the daughter-in law of M
(d) P is the grandfather of M

(GATE 2016, 1 Mark)

Solution: Given information can be represented as

Symbol in Diagram	Meaning
○	Female
□	Male
═	Married Couple
—	Siblings
│	Difference of A Generation



Hence, we can say that, P is grandchild of M.

Ans. (b)

275. The number that least fits this set: (324, 441, 97 and 64) is __________.

(a) 324
(b) 441
(c) 97
(d) 64

(GATE 2016, 1 Mark)

Solution: Analyzing given numbers given, we get

$$324 = 18^2$$
$$441 = 21^2$$
$$64 = 8^2$$

But 97 is not square of any numbers, hence, least fits the given set.

Ans. (c)

276. It takes 10 s and 15 s, respectively, for two trains travelling at different constant speeds to completely pass a telegraph post. The length of the first train is 120 m and that of the second train is 150 m. The magnitude of the difference in the speeds of the two trains (in m/s) is __________.

(a) 2.0
(b) 10.0
(c) 12.0
(d) 22.0

(GATE 2016, 1 Mark)

Solution: If a train crosses a pole, it has to travel distance equal to its own length.
So,

Speed of first train $= 120/10 = 12$ m/s

Speed of second train $= 150/15 = 10$ m/s

Therefore,

Difference in the speed $= 12 - 10 = 2$ m/s

Ans. (a)

277. The chairman requested the aggrieved shareholders to ________ him.

(a) bare with (b) bore with
(c) bear with (d) bare

(GATE 2016, 1 Mark)

Solution: The chairman requested the aggrieved shareholders to bear with him.

Ans. (c)

278. Identify the correct spelling out of the given options:

(a) Managable (b) Manageable
(c) Mangaeble (d) Managible

(GATE 2016, 1 Mark)

Solution: *Manageable* has the correct spelling.

Ans. (b)

279. Pick the odd one out in the following:

13, 23, 33, 43, 53

(a) 23 (b) 33
(c) 43 (d) 53

(GATE 2016, 1 Mark)

Solution: 33 is not a prime number. 13, 23, 43, 53 are primes.

Ans. (b)

280. R2D2 is a robot, R2D2 can repair aeroplanes. No other robot can repair aeroplanes.

Which of the following can be logically inferred from the above statements?

(a) R2D2 is a robot which can only repair aeroplanes.
(b) R2D2 is the only robot which can repair aeroplanes.
(c) R2D2 is a robot which can repair only aeroplanes.
(d) Only R2D2 is a robot.

(GATE 2016, 1 Mark)

Solution: Since no other robot can repair airplanes, R2D2 is the only robot that can repair aeroplanes.

Ans. (b)

281. If $|9y - 6| = 3$, then $y^3 - 4y/3$ is ________.

(a) 0 (b) $+\frac{1}{3}$
(c) $-\frac{1}{3}$ (d) Undefined

(GATE 2016, 1 Mark)

Solution: $9y - 6 = \pm 3$

$$\Rightarrow 3y - 2 = \pm 1$$
$$\Rightarrow 9y^2 + 4 - 12y = -1$$
$$\Rightarrow 9y^2 - 12y = -3$$
$$\Rightarrow y^2 - \frac{12}{9}y = \frac{-3}{9}$$
$$\Rightarrow y^2 - \frac{4}{3}y = \frac{-1}{3}$$

Ans. (c)

282. The volume of a sphere of diameter 1 unit is ______ than the volume of a cube of side 1 unit.

(a) least (b) less
(c) lesser (d) low

(GATE 2016, 1 Mark)

Solution:

$$\text{Volume of sphere} = \frac{\pi d^3}{6} = \frac{\pi}{6}(1)^3 = \frac{\pi}{6} = 0.5236$$

$$\text{Volume of cube} = a^3 = (1)^3 = 1$$

Clearly, volume of sphere is less than volume of cube.

Ans. (b)

283. The unruly crowd demanded that the accused be ______ without trial.

(a) hanged (b) hanging
(c) hankering (d) hung

(GATE 2016, 1 Mark)

Solution: The correct entry for the blank is hanged.

Ans. (a)

284. Choose the statement(s) where the underlined word is used correctly:

(i) A <u>prone</u> is a dried plum.
(ii) He was lying <u>prone</u> on the floor.
(iii) People who eat a lot of fat are <u>prone</u> to heart disease.

(a) (i) and (iii) only (b) (iii) only
(c) (i) and (ii) only (d) (ii) and (iii) only

(GATE 2016, 1 Mark)

Solution: The word 'prone' has been correctly used only in statements (ii) and (iii).

Ans. (d)

285. Fact: If it rains, then the field is wet.

Read the following statements:

(i) It rains. (ii) The field is not wet.
(iii) The field is wet. (iv) It did not rain.

Which one of the options given below is **NOT** logically possible, based on the given fact?

(a) If (iii), then (iv). (b) If (i), then (iii).
(c) If (i), then (ii). (d) If (ii), then (iv).

(GATE 2016, 1 Mark)

Solution: (a) If (iii), then (iv)

$\Rightarrow$ If the field is wet, then it did not rain.

(b) If (i), then (iii)

$\Rightarrow$ If it rains, then the field is wet.

(c) If (i), then (ii)

$\Rightarrow$ If it rains, then the field is not wet.

(d) If (ii), then (iv).

$\Rightarrow$ If the field is not wet, then it did not rain.

According to the given fact, if it rains, then the field is wet. Opposite of this may not be true. Therefore, option (c) is not logically possible.

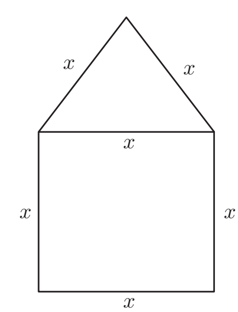
Ans. (c)

286. A window is made up of a square portion and an equilateral triangle portion above it. The base of the triangular portion coincides with the upper side of the square. If the perimeter of the window is 6 m, the area of the window in m^2 is ______.

(a) 1.43 (b) 2.06
(c) 2.68 (d) 2.88

(GATE 2016, 1 Mark)

Solution:



Perimeter of the window = 6 m

$$\Rightarrow (x + x + x) + (x + x) = 6 \text{ m}$$
$$\Rightarrow 5x = 6 \Rightarrow x = 6/5$$

Now, area of the window is,

$$= x^2 + \frac{\sqrt{3}}{4}x^2 = x^2\left(1 + \frac{\sqrt{3}}{4}\right) = \left(\frac{6}{5}\right)^2\left(1 + \frac{\sqrt{3}}{4}\right)$$
$$= 2.06 \text{ m}^2$$

Ans. (b)

287. If I were you, I ______ that laptop. It's much too expensive.

(a) won't buy (b) shan't buy
(c) wouldn't buy (d) would buy

(GATE 2016, 1 Mark)

Solution: First sentence is influencing the second form of helping verb in the blank. Hence 'would not buy' is correct

Ans. (c)

288. He turned a deaf ear to my request.

What does the underlined phrasal verb mean?

(a) ignored (b) appreciated
(c) twisted (d) returned

(GATE 2016, 1 Mark)

Solution: 'Turned a deaf ear' is a phrasal verb which means disregard something. Hence, 'ignored' is correct.

Ans. (a)

289. Choose the most appropriate set of words from the options given below to complete the following sentence. ______ ______ is a will, ______ is a way.

(a) Wear, there, their (b) Were, their, there
(c) Where, there, there (d) Where, their, their

(GATE 2016, 1 Mark)

Solution: Where there is a will, there is a way.

Ans. (c)

290. (x% of y) + (y% of x) is equivalent to ______.

(a) 2% of xy (b) 2% of $(xy/100)$
(c) xy% of 100 (d) 100% of xy

(GATE 2016, 1 Mark)

Solution: $x\%$ of $y + y\%$ of $x = \frac{x}{100}\times y + \frac{y}{100}\times x$

$$= \frac{xy}{100} + \frac{yx}{100}$$

$$= \frac{2xy}{100} = \frac{2}{100}\times xy = 2\% \text{ of } xy$$

Ans. (a)

291. The sum of the digits of a two digit number is 12. If the new number formed by reversing the digits is greater than the original number by 54, find the original number.

(a) 39 (b) 57
(c) 66 (d) 93

(GATE 2016, 1 Mark)

Solution: Let the number be ab. So it can be represented as $(10a + b)$.

Then, $a + b = 12$ (1)

Reversing the digits means the new number is ba. This number can be represented as $(10b + a)$.

Now ba is greater than ab by 54.

Therefore,

$$(10b + a) - (10a + b) = 54$$

$$-9a + 9b = 54$$

$$a - b = -6 \quad (2)$$

Adding Eqs. (1) and (2), we get

$$a + b = 12$$

$$\Rightarrow a - b = -6$$

$$\Rightarrow 2a = 6$$

$$\Rightarrow a = 3 \text{ and } b = 9$$

Hence, the number is 39.

Ans. (a)

292. A shaving set company sells four different types of razors, Elegance, Smooth, Soft and Executive. Elegance sells at Rs. 48, Smooth at Rs. 63, Soft at Rs. 78 and Executive at Rs. 173 per piece. The table below shows the numbers of each razor sold in each quarter of a year.

Quarter/ Product	Elegance	Smooth	Soft	Executive
Q1	27300	20009	17602	9999
Q2	25222	19392	18445	8942
Q3	28976	22429	19544	10234
Q4	21012	18229	16595	10109

Which product contributes the greatest fraction to the revenue of the company in that year?

(a) Elegance
(b) Executive
(c) Smooth
(d) Soft

(GATE 2016, 2 Marks)

Solution:

(i) Elegance $= \sum Q \times 48$ Rs.
$= 1{,}02{,}510 \times 48$
$=$ Rs. 49,20,480

(ii) Smooth $= \sum Q \times 63$ Rs.
$= 80{,}059 \times 63$
$=$ Rs. 50,43,717

(iii) Soft $= \sum Q \times 78$ Rs.
$= 72{,}186 \times 78$
$=$ Rs. 56,30,508

(iv) Executive $= \sum Q \times 173$ Rs.

$= 39{,}284 \times 173$

$=$ Rs. 67,96,132

Hence, maximum contribution is made by Executive.

Ans. (b)

293. Indian currency notes show the denomination indicated in at least seventeen languages. If this is not an indication of the nation's diversity, nothing else is.

Which of the following can be logically inferred from the above sentences?

(a) India is a country of exactly seventeen languages.
(b) Linguistic pluralism is the only indicator of a nation's diversity.
(c) Indian currency notes have sufficient space for all the Indian languages.
(d) Linguistic pluralism is strong evidence of India's diversity.

(GATE 2016, 2 Marks)

Solution: Linguistic pluralism means existence of different languages, and this is a strong evidence of nation's diversity. Hence, the best inference can be made is explained in last line.

Ans. (d)

294. Consider the following statements relating to the level of poker play of four players P, Q, R and S.

I. P always beats Q
II. R always beats S
III. S loses to P only sometimes
IV. R always loses to Q

Which of the following can be logically inferred from the above statements?

(i) P is likely to beat all the three other players
(ii) S is the absolute worst player in the set

(a) (i) only
(b) (ii) only
(c) (i) and (ii)
(d) Neither (i) nor (ii)

(GATE 2016, 2 Marks)

Solution: (a) S loses to P only sometimes means S beats P most of the times. Hence, statement (i) is not logical.

(b) R always beats S, R always loses to Q and P always beats Q means P is better than Q and R but S sometimes loses to P hence S is not the absolute worst player. Hence, statement (ii) also is not logical.

Ans. (d)

295. If $f(x) = 2x^7 + 3x - 5$, which of the following is a factor of $f(x)$?

(a) $(x^3 + 8)$
(b) $(x - 1)$
(c) $(2x - 5)$
(d) $(x + 1)$

(GATE 2016, 2 Marks)

Solution: $f(x) = 2x^7 + 3x - 5$
Put $x = 1$ in the above equation, we get

$$f(1) = 2 \times 1^7 + 3 \times 1 - 5 = 0$$

Hence, $(x - 1)$ is a factor of $f(x)$.

Ans. (b)

296. In a process, the number of cycles to failure decreases exponentially with an increase in load. At a load of 80 units, it takes 100 cycles for failure. When the load is halved, it takes 10000 cycles for failure. The load for which the failure will happen in 5000 cycles is ________.

(a) 40.00
(b) 46.02
(c) 60.01
(d) 92.02

(GATE 2016, 2 Marks)

Solution: Let the number of cycles $= N$ and load $= P$.
Exponential decrease can be represented as

$$N = C_1 e^{-C_2 P} \quad (1)$$

where C_1 and C_2 are arbitrary constants.

As per the given conditions:

$$100 = C_1 e^{-80C_2} \quad (2)$$

$$10000 = C_1 e^{-40C_2} \quad (3)$$

Using Eqs. (2) and (3) to find the values of the constant, we have

$$C_2 = 0.1151,\ C_1 = 10^6$$

Substituting the values of C_1 and C_2 in Eq. (1), we have

$$N = (10^6)e^{-0.1151P} \tag{4}$$

Now, for $N = 5000$ cycles find the value of P, using Eq. (4).

So $$P = 46.02$$

Ans. (b)

297. Among 150 faculty members in an institute. 55 are connected with each other through Facebook® and 85 are connected through WhatsApp®. 30 faculty members do not have Facebook® or WhatsApp® accounts. The member of faculty members connected only through Facebook® accounts is ________.

(a) 35
(b) 45
(c) 65
(d) 90

(GATE 2016, 2 Marks)

Solution: Let faculties on Facebook® $= v = 55$
Let faculties on WhatsApp® $= w = 85$

Let faculties on both $= x$

Let faculties on none $= y = 30$

Therefore,

$$v + w - x + y = 150$$

$$\Rightarrow 55 + 85 - x + 30 = 150$$

$$\Rightarrow x = 20$$

Out of 55 on Facebook®, 20 are also on WhatsApp®. That is, $55 - 20 = 35$ are only on Facebook®.

Ans. (a)

298. Computers were invented for performing only high-end useful computations. However, it is no understatement that they have taken over our world today. The internet, for example, is ubiquitous. Many believe that the internet itself is an unintended consequence of the original invention. With the advent of mobile computing on our phones, a whole new dimension is now enabled. One is left wondering if all these developments are good or, more importantly, required.

Which of the statement(s) below is/are logically valid and can be inferred from the above paragraph?

(i) The author believes that computers are not good for us.
(ii) Mobile computers and the internet are both intended inventions.

(a) (i) only
(b) (ii) only
(c) Both (i) and (ii)
(d) Neither (i) nor (ii)

(GATE 2016, 2 Marks)

Solution: The author calls the internet an "unintended consequence". Hence, (ii) cannot be true. The author is questioning if internet and mobile communications are good or not. Therefore, (i) is not an inference.

Ans. (d)

299. All hill-stations have a lake. Ooty has two lakes.

Which of the statement(s) below is/are logically valid and can be inferred from the above sentences?

(i) Ooty is not a hill-station.
(ii) No hill-station can have more than one lake.

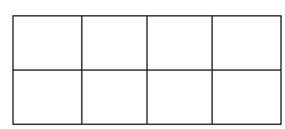
(a) (i) only
(b) (ii) only
(c) Both (i) and (ii)
(d) Neither (i) nor (ii)

(GATE 2016, 2 Marks)

Solution: (i) can be correct when "All hill-stations should have at most one lake. Ooty has 2 lakes. Then (i) can be inferred. Hence, false. Option (ii) also has similar explanation.

Ans. (d)

300. In a 2×4 rectangle grid shown below, each cell is a rectangle. How many rectangles can be observed in the grid?



(a) 21
(c) 27
(c) 30
(d) 36

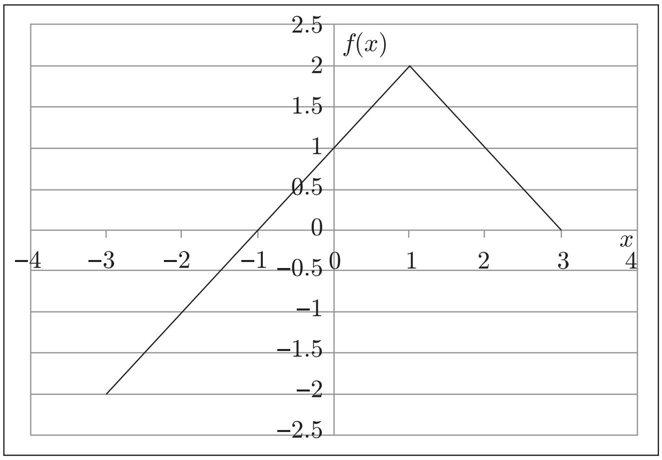
(GATE 2016, 2 Marks)

Solution: In a 4×2 grid, we have 5×3 grid points. Any rectangle will need 2 points on a column and 2 points on a row.
Therefore,

$$\left[\frac{5!}{(5-2)!2!}\right] \times \left[\frac{3!}{(3-2)!2!}\right] = 30$$

Ans. (c)

301.



Choose the correct expression for $f(x)$ given in the above graph.

(a) $f(x)=1-|x-1|$
(b) $f(x)=1+|x-1|$
(c) $f(x)=2-|x-1|$
(d) $f(x)=2+|x-1|$

(GATE 2016, 2 Marks)

Solution: Given graph has two parts.
For $x=-3$ to $x=1$, equation of line is

$$f(x)=x+1$$

For $x=1$ to $x=3$, equation of line is

$$f(x)=-x+3$$

Hence, option (c) fits these cases.

Ans. (c)

302. A person moving through a tuberculosis prone zone has a 50% probability of becoming infected. However, only 30% of infected people develop the disease. What percentage of people moving through a tuberculosis prone zone remains infected but does not show symptoms of disease?

(a) 15 (b) 33
(c) 35 (d) 37

(GATE 2016, 2 Marks)

Solution: Percentage probability of being infected $=P(A)=50\%$

Percentage probability of infected person developing disease is having system $=P(B)=30\%$

Therefore, percentage probability of infected person not showing symptoms $=P(\bar{B})=70\%$

Therefore, percentage probability of person moving though a TB prone zone remaining infected but not showing symptoms

$$=P(A)\cdot P(\bar{B})=(50/100)\times(70/100)=35\%$$

Ans. (c)

303. In a world filled with uncertainty, he was glad to have many good friends. He had always assisted them in times of need and was confident that they would reciprocate. However, the events of the last week proved him wrong.

Which of the following inference(s) is/are logically valid and can be inferred from the above passage?

(i) His friends were always asking him to help them.
(ii) He felt that when in need of help, his friends would let him down.
(iii) He was sure that his friends would help him when in need.
(iv) His friends did not help him last week.

(a) (i) and (ii) (b) (iii) and (iv)
(c) (iii) only (d) (iv) only

(GATE 2016, 2 Marks)

Solution: The paragraph states that the subject was very confident about his good friends helping him in his times of need because he had always helped them before in their time. Thus, inference (iii) follows. Since the events of the last week proved him wrong, this means that his confidence was broken and his friends had not helped him. Thus, inference (iv) also follows.

Ans. (b)

304. Leela is older than her cousin Pavithra. Pavithra's brother Shiva is older than Leela. When Pavithra and Shiva are visiting Leela, all three like to play chess. Pavithra wins more often than Leela does.

Which one of the following statements must be **TRUE** based on the above?

(a) When Shiva plays chess with Leela and Pavithra, he often loses.
(b) Leela is the oldest of the three.
(c) Shiva is a better chess player than Pavithra.
(d) Pavithra is the youngest of the three.

(GATE 2016, 2 Marks)

Solution: According to given information, the points we got are as follows:

(i) Shiva is the brother of Pavithra.
(ii) Shiva and Pavithra are cousins of Leela.
(iii) According to their ages, Shiva > Leela > Pavithra.
(iv) They all like to play chess.
(v) Pavithra wins more often than Leela but information about winning cases of Shiva is not given.

So from the given option statements, which one is clearly true is that Pavithra is the youngest of all.

Ans. (d)

305. If $q^{-a}=\frac{1}{r}$ and $r^{-b}=\frac{1}{s}$ and $s^{-c}=\frac{1}{q}$, the value of abc is __________.

(a) $(rqs)^{-1}$ (b) 0
(c) 1 (d) $r+q+s$

(GATE 2016, 2 Marks)

Solution: $q^{-a}=1/r$; $r^{-b}=1/s$ and $s^{-c}=1/q$
Therefore,

$$q^a=r;\ r^b=s \text{ and } s^c=q$$

Therefore,

$$a\log q=\log r \quad (1)$$

and

$$b\log r=\log s \quad (2)$$

and

$$c\log s=\log q \quad (3)$$

Multiplying Eqs. (1)–(3), we get

$$abc(\log q)(\log r)(\log s)=(\log r)(\log s)(\log q)$$

Therefore,

$$abc=1$$

Ans. (c)

306. P, Q, R and S are working on a project. Q can finish the task in 25 days, working alone for 12 hours a day. R can finish the task in 50 days, working alone for 12 hours per day. Q worked 12 hours a day but took sick leave in the beginning for two days. R worked 18 hours a day on all days. What is the ratio of work done by Q and R after 7 days from the start of the project?

(a) 10:11 (b) 11:10
(c) 20:21 (d) 21:20

(GATE 2016, 2 Marks)

Solution: After 7 days from start of project, Q took sick leave on first 2 days.
Therefore, man hours by Q = 5 × 12

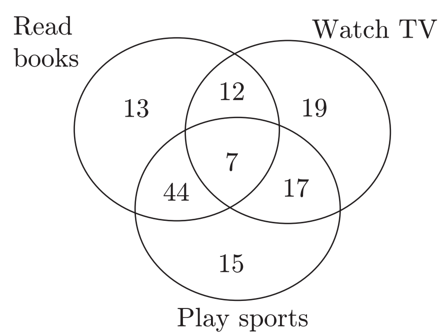
Therefore, work done by Q = 5 × 12 × [1/(25 × 12)] = 15 man hours by R = 7 × 18

Therefore, work done by R = [1/(50 × 12)] × 7 × 18 = 21,100

Therefore, ratio of work done by Q to work done by R = (1/5):(21/100) = 100/(5 × 21) = 20:21

Ans. (c)

307. The Venn diagram shows the preference of the student population for leisure activities.



From the data given, the number of students who like to read books or play sports is __________.

(a) 44
(b) 51
(c) 79
(d) 108

(GATE 2016, 2 Marks)

Solution: From the given Venn diagram:
Total students who play sports = 15 + 44 + 17 + 7 = 83

Total students who read books = 13 + 44 + 7 + 12 = 76

Total students who either play sports or read books = 83 + 76 − (44 + 7) = 108

Ans. (d)

308. Social science disciplines were in existence in an amorphous form until the colonial period when they were institutionalized. In varying degrees, they were intended to further the colonial interest. In the time of globalization and the economic rise of postcolonial countries like India, conventional ways of knowledge production have become obsolete.

Which of the following can be logically inferred from the above statements?

(i) Social science disciplines have become obsolete.

(ii) Social science discipline had a pre-colonial origin.

(iii) Social science disciplines always promote colonialism.

(iv) Social science must maintain disciplinary boundaries.

(a) (ii) only
(b) (i) and (ii) only
(c) (ii) and (iv) only
(d) (iii) and (iv) only

(GATE 2016, 2 Marks)

Solution: On the basis of given passage, it can be concluded that statement (ii) 'social science disciplines had a pre-colonial origin' is correct. Therefore, option (a) is only correct.

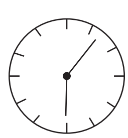
Ans. (a)

309. Two and a quarter hours back, when seen in a mirror, the reflection of a wall clock without number markings seemed to show 1:30. What is the actual current time shown by the clock?

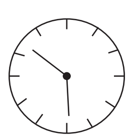
(a) 8:15
(b) 11:15
(c) 12:15
(d) 12:45

(GATE 2016, 2 Marks)

Solution: Two and a quarter hours back, the mirror image of the wall clock was as shown below:



Therefore, actual time was as shown below:



The actual time, two and a quarter hours back, was 10:30.

Actual current time shown by the clock=10:30+2 Hours 15 minutes

=12:45

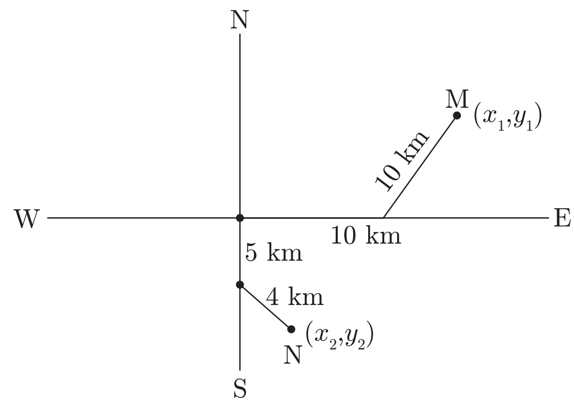
Ans. (d)

310. M and N start from the same location. M travels 10 km East and then 10 km North-East. N travels 5 km South and then 4 km South-East. What is the shortest distance (in km) between M and N at the end of their travel?

(a) 18.60
(b) 22.50
(c) 20.61
(d) 25.00

(GATE 2016, 2 Marks)

Solution:



Position of M$\equiv$(17.07, 7.07)

Position of N$\equiv$(2.03, $-$7.03)

The shortest distance between their final positions is

$$\sqrt{(x_1 - x_2)^2 + (y_1 + y_2)^2} = 20.61$$

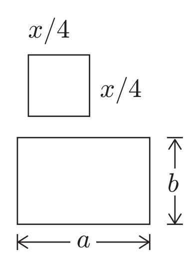
Ans. (c)

311. A wire of length 340 mm is to be cut into two parts. One of the parts is to be made into a square and the other into a rectangle where sides are in the ratio of 1:2. What is the length of the side of the square (in mm) such that the combined area of the square and the rectangle is a MINIMUM?

(a) 30
(b) 40
(c) 120
(d) 180

(GATE 2016, 2 Marks)

Solution:



Let the two parts of wire are of x min and $(340-x)$ mm, respectively.

Side of the square $= \dfrac{x}{4}$ mm

Perimeter of the rectangle $= 2(a+b) = 340 - x$

But it is given that $a = 2b$. Therefore,

$$6b = 340 - x$$
$$\Rightarrow b = \frac{340}{6} - \frac{x}{6}$$

Therefore,

$$a = 2b = \frac{340}{3} - \frac{x}{3}$$

Combined area of square and rectangle

$$A = \frac{x^2}{16} + 2\left(\frac{340}{6} - \frac{x}{6}\right)^2$$

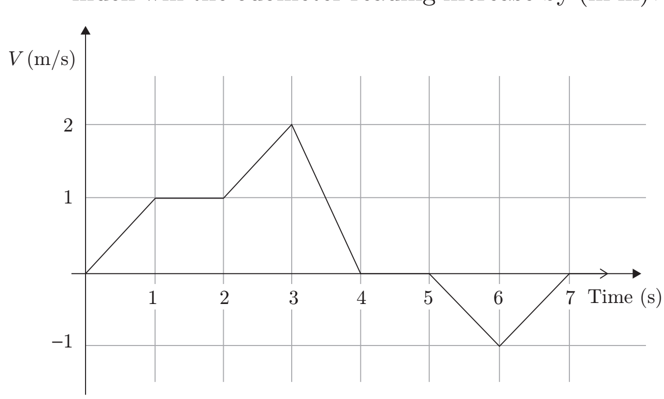
For A to be minimum,

$$\frac{dA}{dx} = 0$$
$$\Rightarrow \frac{2x}{16} + 4\left(\frac{340}{6} - \frac{x}{6}\right)x - \frac{1}{6} = 0$$
$$\Rightarrow \frac{x}{8} + \frac{x}{9} - \frac{340}{9} = 0$$
$$\Rightarrow \frac{x}{4} = 40$$

Since $\dfrac{d^2A}{dx^2} > 0$ for any value of x, A has a minima at $\dfrac{x}{4} = 40$.

Ans. (b)

312. The velocity V of a vehicle along a straight line is measured in m/s and plotted as shown with respect to time in seconds. At the end of the 7 seconds, how much will the odometer reading increase by (in m)?



(a) 0
(b) 3
(c) 4
(d) 5

(GATE 2016, 2 Marks)

Solution: Odometer reading = Overall area

$$= \left(\frac{1}{2}\right)\times1\times1 + 1\times1 + 1\times1 + \left(\frac{1}{2}\right)\times1\times1 + \left(\frac{1}{2}\right)\times1\times2 + \left(\frac{1}{2}\right)\times2\times1$$
$$= \frac{1}{2} + 1 + 1 + \frac{1}{2} + 1 + 1 = 5$$

Ans. (d)

313. The overwhelming number of people infected with rabies in India has been flagged by the World Health Organization as a source of concern. It is estimated that inoculating 70% of pets and stray dogs against rabies can lead to a significant reduction in the number of people infected with rabies.

Which of the following can be logically inferred from the above sentences?

(a) The number of people in India infected with rabies is high.
(b) The number of people in other parts of the world who are infected with rabies is low.
(c) Rabies can be eradicated in India by vaccinating 70% of stray dogs.
(d) Stray dogs are the main source of rabies worldwide.

(GATE 2016, 2 Marks)

Solution: The information given in the passage does not help us conclude whether stray dogs are the main source of rabies globally. Thus option (d) cannot be inferred.
It is said that 70% of both pets and stray dogs need to be inoculated to reduce the number of people infected with rabies. Thus option (c) also cannot be inferred.

No information about the number of people infected in other parts of the world has been given. Thus option (b) is also eliminated.

The passage states that WHO is concerned because of the huge number of rabies patients in India. This means that the number is quite high otherwise the WHO would not have been concerned. Thus option (a) is the correct inference.

Ans. (a)

314. A flat is shared by four first year undergraduate students. They agreed to allow the oldest of them to enjoy some extra space in the flat. Manu is two months older than Sravan, who is three months younger than Trideep. Pavan is one month older than Sravan. Who should occupy the extra space in the flat?

(a) Manu
(b) Sravan
(c) Trideep
(d) Pavan

(GATE 2016, 2 Marks)

Solution: Manu is two months older than Sravan, who is three months younger than Trideep, that means

$$\text{Trideep} > \text{Manu} > \text{Sravan}$$
$$\text{1 month} \quad \text{2 month}$$

Pavan is one month older than Sravan

So,

$$\text{Trideep} > \text{Manu} > \text{Pavan} > \text{Sravan}$$
$$\text{1 month} \quad \text{1 month} \quad \text{1 month}$$

That means Trideep is the oldest among then and hence occupy the extra space in the flat.

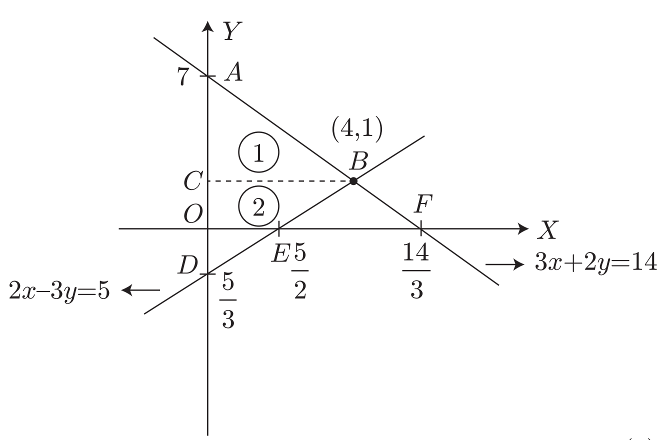
Ans. (c)

315. Find the area bounded by the lines $3x+2y=14$, $2x-3y=5$ in the first quadrant.

(a) 14.95
(b) 15.25
(c) 15.70
(d) 20.35

(GATE 2016, 2 Marks)

Solution:



$$3x+2y=14 \qquad (1)$$

$$2x-3y=5 \qquad (2)$$

Solving Eqs. (1) and (2), we get the points of intersection as $B(4, 1)$.

Now find the coordinates of the point of intersection of each line with the coordinate axes.

Therefore, the required area

$$A = A_1 + A_2$$

$$\Rightarrow A = \left(\frac{1}{2}\right)\times 6\times 4 + \left(\frac{1}{2}\right)\left[4+\left(\frac{5}{2}\right)\right]\times 1$$

[Area of Δ ABC + Area of trapezium $BCOE$]

$$\Rightarrow A = 15.25$$

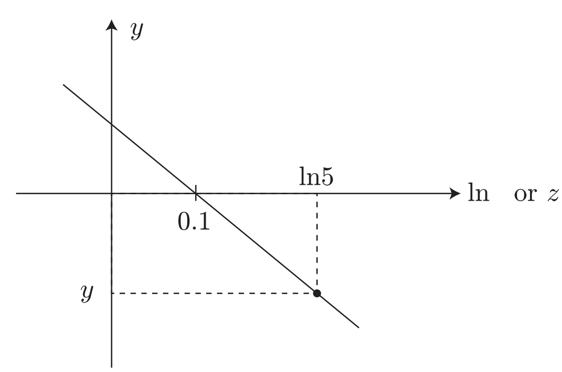
Ans. (b)

316. A straight line is fit to a data set ln (x, y). This line intercepts the abscissa at ln $x=0.1$ and has a slope of -0.02. What is the value of y at $x=5$ from the fit?

(a) -0.030
(b) -0.014
(c) 0.014
(d) 0.030

(GATE 2016, 2 Marks)

Solution:



Since the figure will be straight line having intercept 0.1 at abscissa and slope$=-0.2$

Hence, equation of line

$$z = \left(\frac{1}{m}\right)y + c$$

$$\Rightarrow z = -\left(\frac{1}{0.02}\right)y + 0.1$$

$$\Rightarrow z = \ln x = -\left(\frac{1}{0.02}\right) + 0.1$$

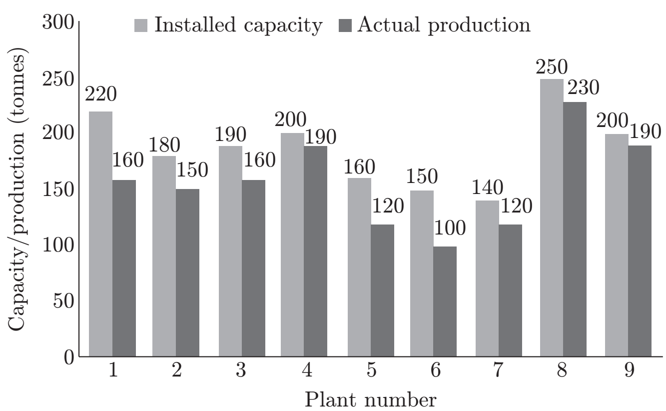
Now at $x=5$,

$$z = \ln 5 = -(1/0.02) + 0.1$$

$$\Rightarrow y = -0.030$$

Ans. (a)

317. The following graph represents the installed capacity for cement production (in tonnes) and the actual production (in tonnes) of nine cement plants of a cement company. Capacity utilization of a plant is defined as ratio of actual production of cement to installed capacity. A plant with installed capacity of at least 200 tonnes is called a large plant and a plant with lesser capacity is called a small plant. The difference between total production of large plants and small plants, in tonnes is ________.



(GATE 2016, 2 Marks)

Solution: Plants 1, 4, 8 and 9 are large plants.

Total actual production of large plants = 160 + 190 + 230 + 190 = 770 tonnes = A

Total actual production of small plants = 150 + 160 + 120 + 100 + 120 = 650 tonnes = B

Therefore,

$$A - B = 120 \text{ tonnes}$$

Ans. 120

318. A poll of students appearing for masters in engineering indicated that 60% of the students believed that mechanical engineering is a profession unsuitable for women. A research study on women with masters or higher degrees in mechanical engineering found that 99% of such women were successful in their professions.

Which of the following can be logically inferred from the above paragraph?

(a) Many students have misconceptions regarding various engineering disciplines.

(b) Men with advanced degrees in mechanical engineering believe women are well suited to be mechanical engineers.

(c) Mechanical engineering is a profession well suited for women with masters or higher degrees in mechanical engineering.

(d) The number of women pursuing higher degrees in mechanical engineering is small.

(GATE 2016, 2 Marks)

Solution: Option (a) can be logically inferred from the given paragraph.

Ans. (a)

319. Sourya committee had proposed the establishment of Sourya Institutes of Technology (SITs) in line with Indian Institutes of Technology (IITs) to cater to the technological and industrial needs of a developing country.

Which of the following can be logically inferred from the above sentence?

Based on the proposal,

(i) In the initial years, SIT students will get degrees from IIT.
(ii) SITs will have a distinct national objective.
(iii) SIT like institutions can only be established in consultation with IIT.
(iv) SITs will serve technological needs of a developing country.

(a) (iii) and (iv) only

(b) (i) and (iv) only

(c) (ii) and (iv) only

(d) (ii) and (iii) only

(GATE 2016, 2 Marks)

Solution: Option (c) can be logically inferred from the given sentence.

Ans. (c)

320. Shaquille O' Neal is a 60% career free throw shooter, meaning that he successfully makes 60 free throws out of 100 attempts on average. What is the probability that he will successfully make exactly 6 free throws in 10 attempts?

(a) 0.2508 (b) 02816

(c) 0.2934 (d) 0.6000

(GATE 2016, 2 Marks)

Solution: Probability of successful free throw = 0.6
Probability of unsuccessful free throw = 0.4

Any 6 throws out of 10 may be successful, so choose 6 out of 10 as ${}^{10}C_6$. Therefore,

$$P = {}^{10}C_6\,(0.6)^6 \times (0.4)^4 = 0.2508$$

Ans. (a)

321. The numeral in the units position of $211^{870} + 146^{127} \times 3^{424}$ is ________.

(GATE 2016, 2 Marks)

Solution: Considering the given expression

$$211^{870} + 146^{127} \times 3^{424}$$

For unit place, we will find the unit place of each of the number (i.e. 211^{870}, 146^{127} and 3^{424}) given in the above expression.

Unit place of $211^{870} = (1)^{870} = 1$

Unit place of $146^{127} = (6)^{127} = 6$

Unit place of 3^{424}
$$\begin{aligned} &= (3)^{424} = ((3)^5)^{84} \times (3)^4 \\ &= (3)^{84} \times (3)^4 \\ &= ((3)^5)^{16} \times (3)^4 \times (3)^4 \\ &= (3)^{16} \times (3)^8 \\ &= ((3)^5)^3 \times (3)^9 = (3)^{12} \\ &= ((3)^5)^2 \times (3)^2 \\ &= (3)^4 = 1 \end{aligned}$$

Hence, overall the unit place of the expression is

$$1 + 6 \times 1 = 1 + 6 = 7$$

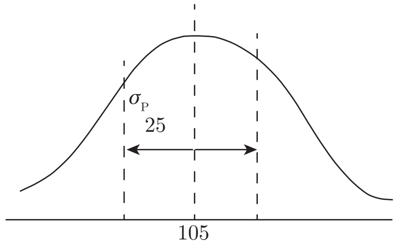
Ans. 7

322. Students taking an exam are divided into two groups, P and Q such that each group has the same number of students. The performance of each of the students in a test was evaluated out of 200 marks. It was observed that the mean of group P was 105, while that of group Q was 85. The standard deviation of group P was 25, while that of group Q was 5. Assuming that the marks were distributed on a normal distribution, which of the following statements will have the highest probability of being TRUE?

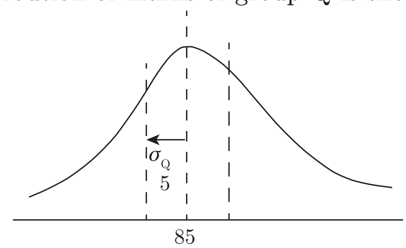
(a) No student in group Q scored less marks than any student in group P.
(b) No student in group P scored less marks than any student in group Q.
(c) Most students of group Q scored marks in a narrower range than students in group P.
(d) The median of the marks of group P is 100.

(GATE 2016, 2 Marks)

Solution: Distribution of marks of group P is shown below:



Distribution of marks of group Q is shown below:



Since $\sigma_Q (= 5) < \sigma_P (= 25)$, most of the students of group Q scored marks in a narrower range than the students of group P.

Ans. (c)

323. A smart city integrates all modes of transport, uses clean energy and promotes sustainable use of resources. It also uses technology to ensure safety and security of the city, something which critics argue, will lead to a surveillance state.

Which of the following can be logically inferred from the above paragraph?

(i) All smart cities encourage the formation of surveillance states.
(ii) Surveillance is an integral part of a smart city.
(iii) Sustainability and surveillance go hand in hand in a smart city.
(iv) There is a perception that smart cities promote surveillance.

(a) (i) and (iv) only
(b) (ii) and (iii) only
(c) (iv) only
(d) (i) only

(GATE 2016, 2 Marks)

Solution: Sustainability and surveillance go hand in hand in a smart city.

Ans. (c)

324. Find the missing sequence in the letter series.

B, FH, LNP, ________.

(a) SUWY (b) TUVW
(c) TVXZ (d) TWXZ

(GATE 2016, 2 Marks)

Solution:

$$\begin{array}{ccc} B, & FH, & L\ N\ P \\ \downarrow & \downarrow & \downarrow \\ 2 & 6\ 8 & 12\ 14\ 16 \end{array}$$

The next sequence must be

$$\begin{array}{cccc} 20 & 22 & 24 & 26 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ T & V & X & Z \end{array}$$

Ans. (c)

325. The binary operation $\square$ is defined as $a \square b = ab + (a + b)$, where a and b are any two real numbers. The value of the identity element of this operation, defined as the number x such that $a \square x = a$, for any a, is ______.

(a) 0 (b) 1 (c) 2 (d) 10

(GATE 2016, 2 Marks)

Solution: $a \square x = ax + (a + x) = a$ (given)

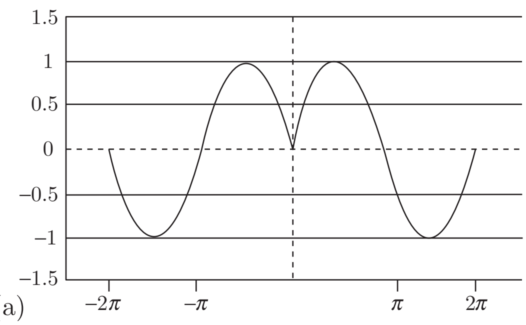
$$\Rightarrow x(a+1) + a = a$$

$$\Rightarrow x(a+1) = 0$$

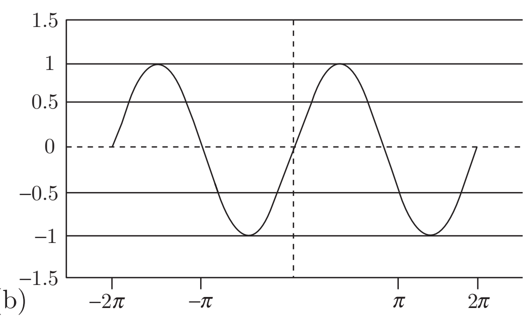
This product has to be zero for any a. It is possible only with $x = 0$.

Ans. (a)

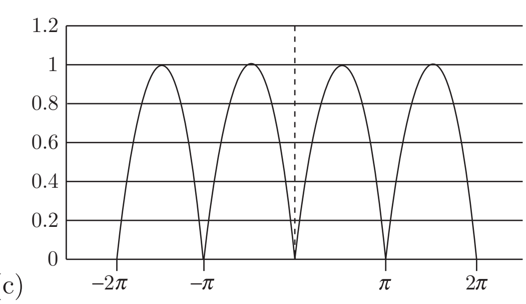
326. Which of the following curves represents the function $y = \ln(|e^{[|\sin(|x|)|]}|)$ for $|x| < 2\pi$? Here, x represents the abscissa and y represents the ordinate.



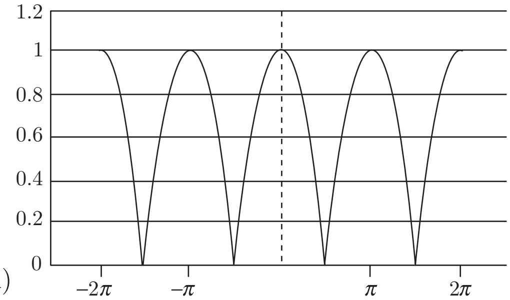
(a)



(b)



(c)



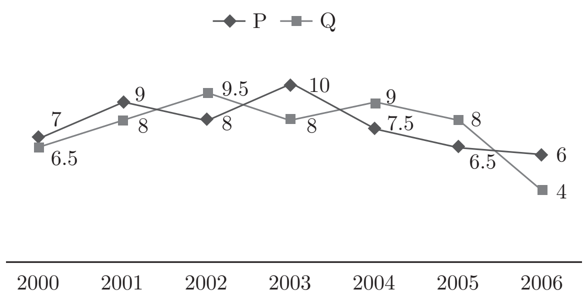
(d)

(GATE 2016, 2 Marks)

Solution: Here option (c) is correct since the given function is a periodic function with period π and a maximum value of 1.

Ans. (c)

327. Two finance companies, P and Q, declared fixed annual rates of interest on the amounts invested with them. The rates of interest offered by these companies may differ from year to year. Year-wise annual rates of interest offered by these companies are shown by the line graph provided below.



If the amounts invested in the companies, P and Q, in 2006 are in the ratio 8:9, then the amounts received after one year as interests from companies P and Q would be in the ratio:

(a) 2:3 (b) 3:4 (c) 6:7 (d) 4:3

(GATE 2016, 2 Marks)

Solution: Amounts invested:

By $P = 8x$ at 6%

By $Q = 9x$ at 4%

$$\text{Interest received by } P \text{ after 1 year} = \frac{6}{100} \times 8x = 0.48x$$

$$\text{Interest received by } Q \text{ after 1 year} = \frac{4}{100} \times 9x = 0.36x$$

Therefore,

$$\text{Ratio} = \frac{0.48x}{0.36x} = \frac{4}{3}$$

Ans. (d)

328. Today, we consider Ashoka as a great ruler because of the copious evidence he left behind in the form of stone carved edicts. Historians tend to correlate greatness of a king at his time with the availability of evidence today.

Which of the following can be logically inferred from the above sentences?

(a) Emperors who do not leave significant sculpted evidence are completely forgotten.
(b) Ashoka produced stone carved edicts to ensure that later historians will respect him.
(c) Statues of kings are a reminder of their greatness.
(d) A king's greatness, as we know him today, is interpreted by historians.

(GATE 2016, 2 Marks)

Solution: Historians tend to correlate greatness of king with the available evidence today. This relates to last option (d).

Ans. (d)

329. Fact 1: Humans are mammals.
Fact 2: Some humans are engineers.
Fact 3: Engineers build houses.

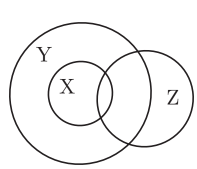
If the above statements are facts, which of the following can be logically inferred?

I. All mammals build houses.
II. Engineers are mammals.
III. Some humans are not engineers.

(a) II only (b) III only
(c) I, II and III (d) I only

(GATE 2016, 2 Marks)

Solution: Draw the Venn diagram, assuming Humans = X, Mammals = Y, Engineers = Z.



Hence,
Statement I is wrong.
Statement II is wrong.
Statement III is right.

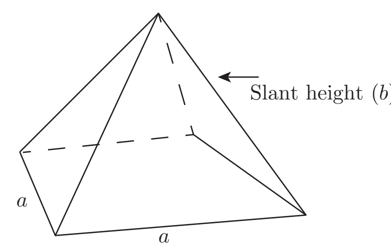
Ans. (b)

330. A square pyramid has a base perimeter x, and the slant height is half of the perimeter. What is the lateral surface area of the pyramid?

(a) x^2 (b) $0.75x^2$
(c) $0.50x^2$ (d) $0.25x^2$

(GATE 2016, 2 Marks)

Solution:



Base perimeter of square pyramid,

$$x = 4a$$

$$a = \frac{x}{4}$$

Slant height,

$$b = \frac{x}{2}$$

Therefore,
Lateral surface area = 4 × Single face area

$$= 4 \times \left(\frac{1}{2} \times \frac{x}{4} \times \frac{x}{2}\right)$$

$$= 4 \times \frac{x^2}{16} = \frac{x^2}{4}$$

$$= 0.25\ x^2$$

Ans. (d)

331. Ananth takes 6 hours and Bharath takes 4 hours to read a book. Both started reading copies of the book at the same time. After how many hours is the number of pages to be read by Ananth, twice that to be read by Bharath? Assume Ananth and Bharath read all the pages with constant pace.

(a) 1 (b) 2 (c) 3 (d) 4

(GATE 2016, 2 Marks)

Solution: Total time taken by Ananth, $t_A = 6$ hours
Total time taken by Bharath, $t_B = 4$ hours
In 1 hour,

Ananth reads $\frac{1}{6}$th of the book.

Bharath reads $\frac{1}{4}$th of the book.

After x hours,

Number of pages to be read by Ananth $= 1 - \frac{x}{6}$

Number of pages to be read by Bharath $= 1 - \frac{x}{4}$

Now,

$$1 - \frac{x}{6} = \left(1 - \frac{x}{4}\right) \times 2$$

$$\Rightarrow \frac{2x}{4} - \frac{x}{6} = 2 - 1$$

$$\Rightarrow \frac{2x}{6} = 1$$

$$\Rightarrow x = 3 \text{ hours}$$

Ans. (c)

332. After Rajendra Chola returned from his voyage to Indonesia, he ________ to visit the temple in Thanjavur.

(a) was wishing (b) is wishing
(c) wished (d) had wished

(GATE 2017, 1 Mark)

Solution: (d) After Rajendra Chola returned from his voyage to Indonesia, he *wished* to visit the temple in Thanjavur.

Ans. (c)

333. Research in the workplace reveal that people work for many reasons _________.

(a) money beside (b) beside money
(c) money besides (d) besides money

(GATE 2017, 1 Mark)

Solution: Research in the workplace reveal that people work for many reasons *besides money.*

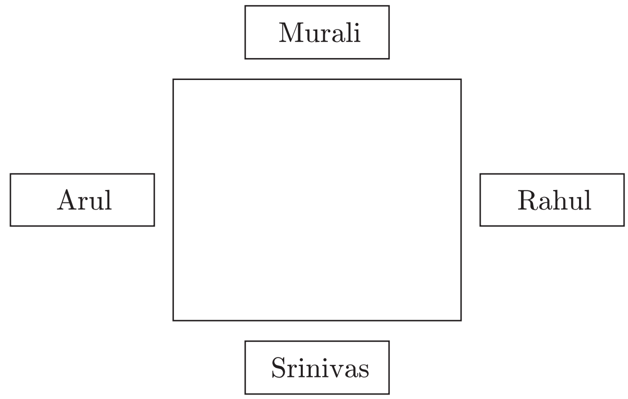
Ans. (d)

334. Rahul, Murali, Srinivas and Arul are seated around a square table. Rahul is sitting to the left of Murali. Srinivas is sitting to the right of Arul. Which of the following pairs are seated opposite each other?

(a) Rahul and Murali (b) Srinivas and Arul
(c) Srinivas and Murali (d) Srinivas and Rahul

(GATE 2017, 1 Mark)

Solution:



From the above figure, we can conclude that Srinivas and Murali are seated opposite each other.

Ans. (c)

335. Find the smallest number y such that the $y \times 162$ is a perfect cube.

(a) 24 (b) 27 (c) 32 (d) 36

(GATE 2017, 1 Mark)

Solution: $y \times 162$ should be a perfect cube.
Now,

$$162 = 2 \times 81 = 2 \times 3^4$$

For this to be a cube, we additionally need 2^2 and 3^2, i.e.

$$4 \times 9 = 36$$

Ans. (d)

336. The probability that a k-digit number does NOT contain the digits 0, 5 or 9 is

(a) 0.3^k (b) 0.6^k (c) 0.7^k (d) 0.9^k

(GATE 2017, 1 Mark)

Solution: Let us first find out the total number of ways.

The first place from the left can be filled in 9 ways.

The second place can be filled in 10 ways.

The third place can be filled in 10 ways and so on till the unit's place.

The total number of ways $= 9 \times 10 \times 10 \times 10 \times 10 \ldots (k-1)$ times $= 9 \times 10^{k-1}$

The favourable number of ways $= 7 \times 7 \times 7 \times 7 \ldots k$ times

Therefore,

$$\text{Required probability} = 7^k/(9 \times 10^{k-1})$$

If we check with the options, the closest such value will be 0.7^k.

Ans. (c)

337. Choose the option with words that are not synonyms.

(a) aversion, dislike (b) luminous, radiant
(c) plunder, loot (d) yielding, resistant

(GATE 2017, 1 Mark)

Solution: The words "yielding" and "resistant" are not synonyms.

Ans. (d)

338. Saturn is ________ to be seen on a clear night with the naked eye.

(a) enough bright
(b) bright enough
(c) as enough bright
(d) bright as enough

(GATE 2017, 1 Mark)

Solution: The correct answer is *bright enough*.

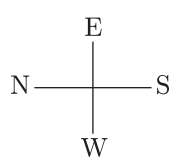
Ans. (b)

339. There are five buildings called V, W, X, Y and Z in a row (not necessarily in that order). V is to the West of W. Z is to the East of X and the West of V. W is to the West of Y. Which is the building in the middle?

(a) V (b) W (c) X (d) Y

(GATE 2017, 1 Mark)

Solution:



The buildings will be as shown:

Y
W
V
Z
X

Therefore, building V is in the middle.

Ans. (a)

340. A test has twenty questions worth 100 marks in total. There are two types of questions. Multiple choice questions are worth 3 marks each and essay questions are worth 11 marks each. How many multiple choice questions does the exam have?

(a) 12 (b) 15 (c) 18 (d) 19

(GATE 2017, 1 Mark)

Solution: Let x be the number of questions of 3 marks. Therefore, the number of questions of 11 marks will be $20 - x$.

Also,

$$3x + 11(20 - x) = 100$$
$$\Rightarrow 3x + 220 - 11x = 100$$
$$\Rightarrow 8x = 120$$
$$\Rightarrow x = 15$$

Ans. (b)

341. There are 3 red socks, 4 green socks and 3 blue socks. You choose 2 socks. The probability that they are of the same colour is

(a) 1/5 (b) 7/20 (c) 1/4 (d) 4/15

Solution:

$$\text{Favourable number of ways} = {}^3C_2 + {}^4C_2 + {}^3C_2 = 3 + 6 + 3 = 12$$

$$\text{Total number of ways} = {}^{10}C_2 = 45$$

Therefore,

$$\text{Required probability} = 12/45 = 4/15$$

Ans. (d)

342. She has a sharp tongue and it can occasionally turn ________.

(a) hurtful (b) left
(c) methodical (d) vital

(GATE 2017, 1 Mark)

Solution: She has a sharp tongue and it can occasionally turn *hurtful*.

Ans. (a)

343. I ________ made arrangements had I ________ informed earlier.

(a) could have, been (b) would have, being
(c) had, have (d) had been, been

(GATE 2017, 1 Mark)

Solution: I *could have* made arrangements had I *been* informed earlier.

Ans. (a)

344. In the summer, water consumption is known to decrease overall by 25%. A Water Board official states that in the summer household consumption decreases by 20%, while other consumption increases by 70%.

Which of the following statements is correct?

(a) The ratio of household to other consumption is 8/17.

(b) The ratio of household to other consumption is 1/17.

(c) The ratio of household to other consumption is 17/8.

(d) There are errors in the official's statement.

(GATE 2017, 1 Mark)

Solution: Let the ratio of household consumption to other consumption be x:y.

Therefore,

$$-20x + 70y = -25(x + y)$$
$$\Rightarrow -20x + 70y = -25x - 25y$$
$$\Rightarrow 5x = -95y$$

which is not possible.

Therefore, option (d) is correct.

Ans. (d)

345. 40% of deaths on city roads may be attributed to drunken driving. The number of degrees needed to represent this as a slice of a pie chart is

(a) 120 (b) 144 (c) 160 (d) 212

(GATE 2017, 1 Mark)

Solution: $100\% = 360°$

$$\Rightarrow 1\% = 3.6°$$

Now,

$$40\% = 40 \times 3.6 = 4 \times 36 = 144°$$

Ans. (b)

346. Some tables are shelves. Some shelves are chairs. All chairs are benches. Which of the following conclusions can be deduced from the preceding sentences?

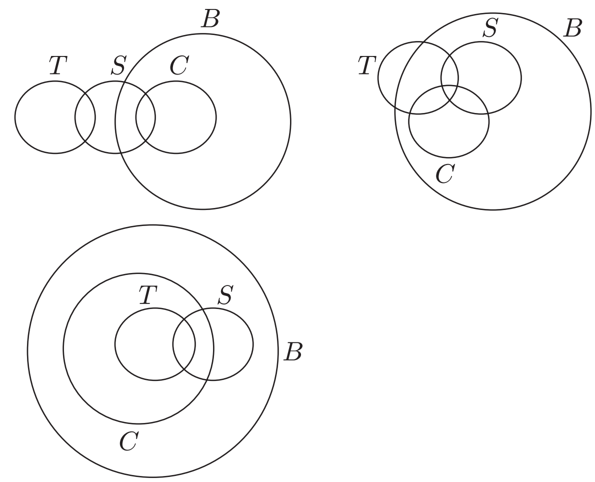
(i) At least one bench is a table

(ii) At least one shelf is a bench

(iii) At least one chair is a table

(iv) All benches are chairs

(a) Only (i) (b) Only (ii)
(c) Only (ii) and (iii) (d) Only (iv)



(GATE 2017, 1 Mark)

Solution: According to the Venn diagram shown above, option (b) is correct.

Ans. (b)

347. The ninth and the tenth of this month are Monday and Tuesday ________.

(a) figuratively (b) retrospectively
(c) respectively (d) rightfully

(GATE 2017, 1 Mark)

Solution: The ninth and the tenth of this month are Monday and Tuesday, *respectively.*

Ans. (c)

348. It is ________ to read this year's textbook ________ the last year's.

(a) easier, than (b) most easy, than
(c) easier, from (d) easiest, from

(GATE 2017, 1 Mark)

Solution: It is *easier* to read this year's textbook *than* the last year's.

Ans. (a)

349. A rule states that in order to drink beer, one must be over 18 years old. In a bar there are 4 people. P is 16 years old, Q is 25 years old, R is drinking milkshake and S is drinking a beer. What must be checked to ensure that the rule is being followed?

(a) Only P's drink
(b) Only P's drink and S's age
(c) Only S's age
(d) Only P's drink, Q's drink and S's age

(GATE 2017, 1 Mark)

Solution: Only P's drink and S's age must be checked to ensure that the rule is being followed.

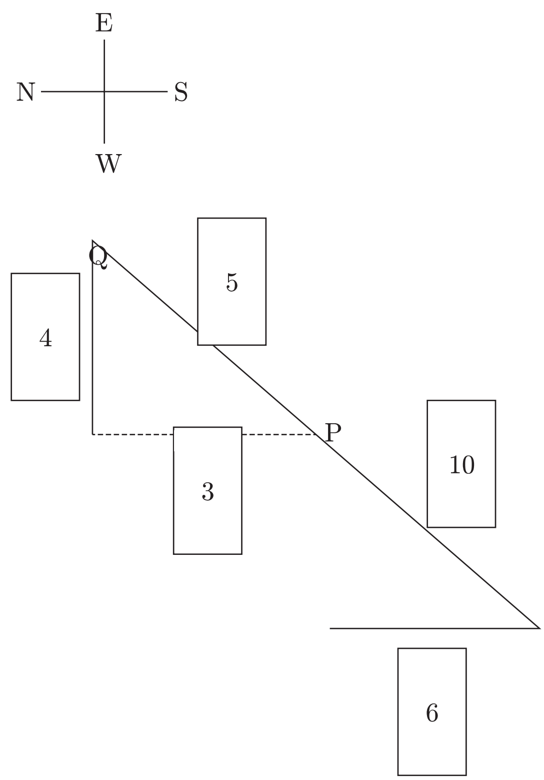
Ans. (b)

350. Fatima starts from point P, goes North for 3 km and then East for 4 km to reach point Q. She then turns to face point P and goes 15 km in that direction. She then goes North for 6 km. How far is she from point P and in which direction should she go to reach point P?

(a) 8 km, East (b) 12 km, North
(c) 6 km, East (d) 10 km, North

(GATE 2017, 1 Mark)

Solution:



Option (a) is correct as Fatima is at the position as shown and therefore needs to go 8 km towards the East direction.

Ans. (a)

351. 500 students are taking one or more courses out of Chemistry, Physics and Mathematics. Registration records indicate course enrolment as follows: Chemistry (329), Physics (186), Mathematics (295), Chemistry and Physics (83), Chemistry and Mathematics (217), and Physics and Mathematics (63). How many students are taking all 3 subjects?

(a) 37 (b) 43 (c) 47 (d) 53

(GATE 2017, 1 Mark)

Solution: We know,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Let x be the number of students who take all the three subjects. Therefore, from the above formula, we have

$$500 = 329 + 186 + 295 - 83 - 217 - 63 - x$$
$$\Rightarrow 500 = 447 + x$$
$$\Rightarrow x = 500 - 447 = 53$$

Ans. (d)

352. He was one of my best _________ and I felt his loss _________.

(a) friend, keenly (b) friends, keen
(c) friend, keener (d) friends, keenly

(GATE 2017, 1 Mark)

Solution: He was one of my best *friends* and I felt his loss *keenly*.

Ans. (d)

353. As the two speakers became increasingly agitated, the debate became _________.

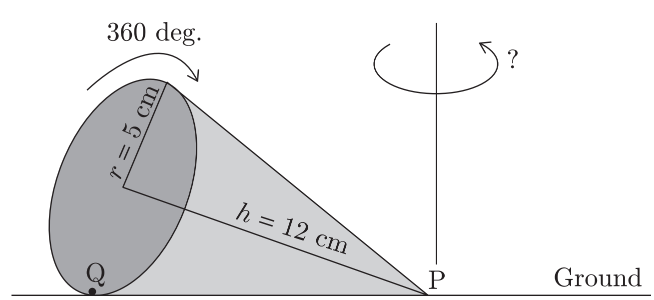
(a) lukewarm (b) poetic
(c) forgiving (d) heated

(GATE 2017, 1 Mark)

Solution: As the two speakers became increasingly agitated, the debate became *heated*.

Ans. (d)

354. A right-angled cone (with base radius 5 cm and height 12 cm), as shown in the figure below is rolled on the ground keeping the point P fixed until the point Q (at the base of the cone, as shown) touches the ground again.

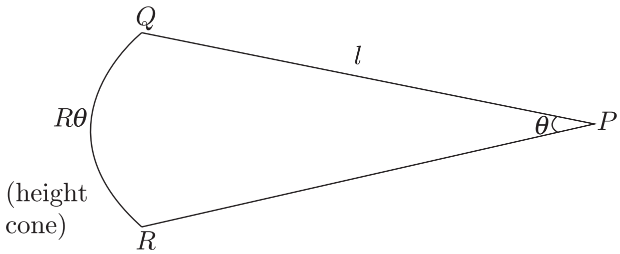


By what angle (in radians) about P does the cone travel?

(a) $\frac{5\pi}{12}$ (b) $\frac{5\pi}{24}$ (c) $\frac{24\pi}{5}$ (d) $\frac{10\pi}{13}$

(GATE 2017, 1 Mark)

Solution: Let l represent the slant height of the cone.



$$l^2 = 5^2 + 12^2 = 25 + 144 = 169$$
$$\Rightarrow l = 13$$

Circumference of base circle = Length of arc QR

$$\Rightarrow 2\pi r = R\theta \ (R = \text{slant height of the cone})$$
$$\Rightarrow 2\pi r = 13\theta$$
$$\Rightarrow 2\pi \times 5 = 13\theta$$
$$\Rightarrow \theta = \frac{10\pi}{13}$$

Ans. (d)

355. In a company with 100 employees, 45 earn Rs. 20,000 per month, 25 earn Rs. 30,000, 20 earn Rs. 40,000, 8 earn Rs. 60,000 and 2 earn Rs. 150,000. The median of the salaries is

(a) Rs. 20,000 (b) Rs. 30,000
(c) Rs. 32,300 (d) Rs. 40,000

(GATE 2017, 1 Mark)

Solution: Arrange all the values in ascending or descending order first.
Now, the number of observations is equal to 100 [even].
Therefore,
Median of these values = Average of the two middlemost observations

$$= \frac{50^{\text{th}}\,\text{Observation} + 51^{\text{th}}\,\text{Observation}}{2}$$
$$= \frac{30{,}000 + 30{,}000}{2} = 30{,}000$$

Ans. (b)

356. P, Q and R talk about S's car collection. P states that S has at least 3 cars. Q believes that S has less than 3 cars. R indicates that to his knowledge, S has at least one car. Only one of P, Q and R is right. The number of cars owned by S is

(a) 0 (b) 1
(c) 3 (d) Cannot be determined

(GATE 2017, 1 Mark)

Solution: It is mentioned that only one of them gave a correct statement. The only possible numerical value for which exactly one of the three statements is true will be 0.

If the number of cars with S is 0, then only the statement given by Q is correct.

Ans. (a)

357. The ways in which this game can be played ________ potentially infinite.

(a) is (b) is being
(c) are (d) are being

(GATE 2017, 1 Mark)

Solution: The ways in which this game can be played *are* potentially infinite.

Ans. (c)

358. If you choose plan P, you will have to ________ plan Q, as these two are mutually ________.

(a) forgo, exclusive (b) forget, inclusive
(c) accept, exhaustive (d) adopt, intrusive

(GATE 2017, 1 Mark)

Solution: Mutually exclusive is an event where only one of two given events can happen. Therefore, option (a) is correct.

Ans. (a)

359. If a and b are integers and $a - b$ is even, which of the following must always be even?

(a) ab (b) $a^2 + b^2 + 1$
(c) $a^2 + b + 1$ (d) $ab - b$

(GATE 2017, 1 Mark)

Solution: If $a - b$ is even either both a and b are even or both a and b are odd.
If a and b are both odd,

- ab will be odd.
- $a^2 + b^2 + 1$ will be odd.
- $a^2 + b + 1$ will be odd.
- But $ab - b$ will always be even.

Ans. (d)

360. A couple has 2 children. The probability that both children are boys if the older one is a boy is

(a) 1/4 (b) 1/3 (c) 1/2 (d) 1

(GATE 2017, 1 Mark)

Solution: A couple has 2 children in 4 ways:

BG, BB, GB, GG

Therefore, the probability that both children are boys if the older one is a boy is 1/2.

Ans. (c)

361. P looks at Q while Q looks at R. P is married, R is not. The number of people in which a married person is looking at an unmarried person is

(a) 0 (b) 1
(c) 2 (d) Cannot be determined

(GATE 2017, 1 Mark)

Solution: In this question, we do not know about the marital status of Q.

(a) Let Q be married.
P looks at Q: Married looking at married: Not considered.
Q looks at R: Married looking at unmarried: Considered.

Therefore, count is 1.

(b) Let Q be unmarried.
P looks at Q: Married looking at unmarried: Considered.
Q looks at R: unmarried looking at unmarried: Not Considered.

Therefore, count is 1.

In both the cases, irrespective of the marital status of Q, the answer will be 1.

Ans. (b)

362. The bacteria in milk are destroyed when it ______________ heated to 80 degree Celsius.

(a) would be (b) will be
(c) is (d) was

(GATE 2017, 1 Mark)

Solution: The bacteria in milk are destroyed when it *is* heated to 80 degree Celsius.

Ans. (c)

363. _________ with someone else's email account is now a very serious offence.

(a) Involving (b) Assisting
(c) Tampering (d) Incubating

(GATE 2017, 1 Mark)

Solution: *Tampering* with someone else's email account is now a very serious offence.

Ans. (c)

364. Consider the following sentences:

All benches are beds. No bed is a bulb. Some bulbs are lamps.

Which of the following can be inferred?

(i) Some beds are lamps.
(ii) Some lamps are beds.

(a) Only (i) (b) Only (ii)
(c) Both (i) and (ii) (d) Neither (i) nor (ii)

(GATE 2017, 1 Mark)

Solution: Neither (i) nor (ii) is correct since some lamps are beds or some beds are lamps is only a possibility and not a definite conclusion.

Ans. (d)

365. If the radius of a right circular cone is increased by 50%, its volume increases by

(a) 75% (b) 100%
(c) 125% (d) 237.5%

(GATE 2017, 1 Mark)

Solution: The formula for the volume of a cone is $\Pi r^2 h$. If the radius increases by 50%, the volume will increase by the successive effect of r since the formula has r^2 in it.

$$\begin{aligned}\% \text{ change in volume} &= 50 + 50 + [(50 \times 50)/100] \\ &= 100 + (2500/100) \\ &= 125\%\end{aligned}$$

Ans. (c)

366. The following sequence of numbers is arranged in increasing order, 1, x, x, x, y, y, 9, 16, 18. Given that the mean and median are equal, and are also equal to twice the mode, the value of y is

(a) 5 (b) 6 (c) 7 (d) 8

(GATE 2017, 1 Mark)

Solution: The given data is 1, x, x, x, y, y, 9, 16, 18.

Median is the middlemost data which is y.
Mean is given by $(1 + 3x + 2y + 9 + 16 + 18)/9$.
Since mean and median are equal, we have

$$\begin{aligned}(1 + 3x + 2y + 9 + 16 + 18)/9 &= y \\ \Rightarrow 3x + 2y + 44 &= 9y \\ \Rightarrow 3x - 7y &= -44 \quad \text{(i)}\end{aligned}$$

Also, since x is the mode and median is twice of mode, we have

$$y = 2x \quad \text{(ii)}$$

Solving Eqs. (i) and (ii), we have

$$\begin{aligned}3x - 14x &= -44 \\ \Rightarrow -11x &= -44 \\ \Rightarrow x &= 4\end{aligned}$$

Therefore, from Eq. (ii)

$$y = 2x = 8$$

Ans. (d)

367. The event would have been successful if you ____________ able to come.

(a) are (b) had been
(c) have been (d) would have been

(GATE 2017, 1 Mark)

Solution: The event would have been successful if you *had been* able to come.

Ans. (b)

368. There was no doubt that their work was thorough.

Which of the words below is closest in meaning to the underlined word above?

(a) pretty (b) complete
(c) sloppy (d) haphazard

(GATE 2017, 1 Mark)

Solution: *Complete* is closest in meaning to the word *thorough*.

Ans. (b)

369. Four cards lie on a table. Each card has a number printed on one side and a colour on the other. The faces visible on the cards are 2, 3, red, and blue.

Proposition: If a card has an even value on one side, then its opposite face is red.
The cards which MUST be turned over to verify the above proposition are

(a) 2, red (b) 2, 3, red
(c) 2, blue (d) 2, red, blue

(GATE 2017, 1 Mark)

Solution: (a) The proposition says that if one side is an even number, then the other side will be red. It does not say anything when the number visible is not even. Also if a side is red, then the number on the opposite side could be an odd number also.

Therefore, to verify the proposition, the two cards must be turned are the one showing blue on its face and the other one showing 2 on its face.

Therefore, option (c) is the correct answer.

Ans. (c)

370. What is the value of x when

$$81\times\left(\frac{16}{25}\right)^{x+2}\div\left(\frac{3}{5}\right)^{2x+4}=144?$$

(a) 1 (b) −1
(c) −2 (d) Cannot be determined

(GATE 2017, 1 Mark)

Solution:

$$81\times\left(\frac{16}{25}\right)^{x+2}\div\left(\frac{3}{5}\right)^{2x+4}=144$$

$$\Rightarrow 3^4\left(\frac{(2^4)^{x+2}}{(5^2)^{x+2}}\right)\div\left(\frac{3}{5}\right)^{2x+4}=144$$

$$\Rightarrow 3^4\left(\frac{2^{4x+8}}{5^{2x+4}}\right)\left(\frac{5^{2x+4}}{3^{2x+4}}\right)=144$$

$$\Rightarrow 3^{4-2x-4}\times 2^{4x+8}=2^4\times 3^2$$

Equating the powers of 3 or the powers of 2, we get $x=-1$.

Ans. (b)

371. Two dice are thrown simultaneously. The probability that the product of the numbers appearing on the top faces of the dice is a perfect square is

(a) 1/9 (b) 2/9 (c) 1/3 (d) 4/9

(GATE 2017, 1 Mark)

Solution: The possible options for the product are {1, 2, 3, 4, 5, 6, 2, 4, 6, 8, 10, 12, 3, 6, 9, 12, 15, 18, 4, 8, 12, 16, 20, 24, 5, 10, 15, 20, 25, 30, 6, 12, 18, 24, 30, 36}.

The perfect squares are {1, 4, 4, 4, 9, 16, 25, 36} which will form the favourable cases.

Therefore,

$$\text{Required probability}=8/36=2/9$$

Ans. (b)

372. "The hold of the nationalist imagination on our colonial past is such that anything inadequately or improperly nationalist is just not history."

Which of the following statements best reflects the author's opinion?

(a) Nationalists are highly imaginative.
(b) History is viewed through the filter or nationalism.
(c) Our colonial past never happened.
(d) Nationalism has to be both adequately and properly imagined.

(GATE 2017, 2 Marks)

Solution: Statement (b) best reflects the author's opinion.

Ans. (b)

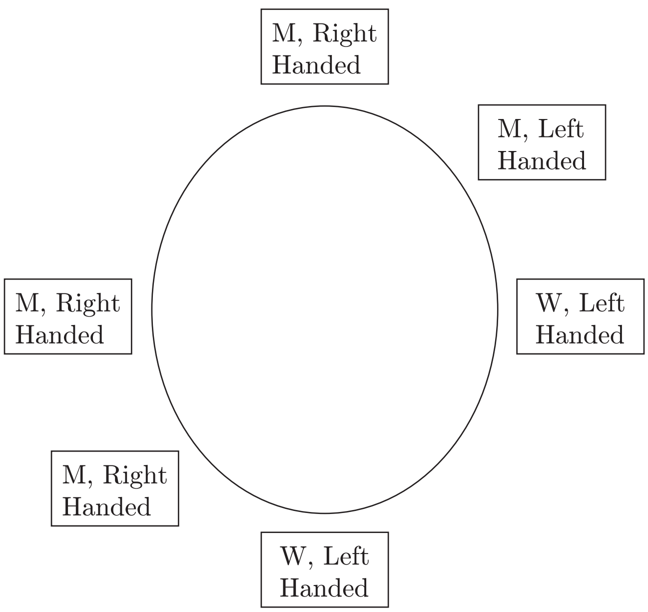
373. Six people are seated around a circular table. There are at least two men and two women. There are at least three right-handed persons. Every woman has a left-handed person to her immediate right.

None of the women are right-handed. The number of women at the table is

(a) 2 (b) 3
(c) 4 (d) Cannot be determined

(GATE 2017, 2 Marks)

Solution: There are 4 men and 2 women. Also, out of the men 3 will be right handed and one man will be left handed.



Ans. (a)

374. The expression $\dfrac{(x+y)-|x-y|}{2}$ is equal to

(a) the maximum of x and y
(b) the minimum of x and y
(c) 1
(d) none of the above

(GATE 2017, 2 Marks)

Solution: The given expression is

$$[(x+y)-|x-y|]/2$$

There are 2 cases:

1. If $x > y$, then $|x-y| = x-y$
   Therefore, $(x+y-x+y)/2 = 2y/2 = y$
2. If $x < y$, then $|x-y| = -(x-y)$
   Therefore, $(x+y+x-y)/2 = 2x/2 = x$

The expression will be equal to the minimum of x or y.

Ans. (b)

375. Arun, Gulab, Neel and Shweta must choose one shirt each from a pile of four shirts coloured red, pink, blue and white, respectively. Arun dislikes the colour red and Shwetha dislikes the colour white. Gulab and Neel like all the colours. In how many different ways can they choose the shirts so that no one has a shirt with a colour he or she dislikes?

(a) 21 (b) 18 (c) 16 (d) 14

(GATE 2017, 2 Marks)

Solution: We will take 2 cases:

1. Arun selects white.
   The other three people can select in 3! ways, i.e. 6 ways.
2. Arun does not select white.
   Arun can select in 2 ways (since red cannot be selected).

   Now, Shweta can select in 2 ways and the remaining two people can select in 2 ways.

Therefore,

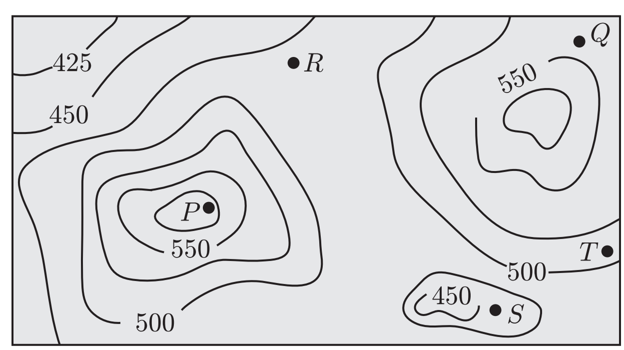
Total number of ways = $2 \times 2 \times 2 = 8$ ways

Hence,

Total of both cases 1 and 2 = $6 + 8 = 14$ ways

Ans. (d)

376. A contour line joins locations having the same height above the mean sea level. The following is a contour plot of a geographical region. Contour lines are shown at 25 m intervals in this plot. If in a flood, the water level rises to 525 m, which of the villages P, Q, R, S, T get submerged?



(a) P, Q (b) P, Q, T (c) R, S, T (d) Q, R, S

(GATE 2017, 2 Marks)

Solution: R, S and T are all below 525 m and will get submerged.

Ans. (c)

377. "We lived in a culture that denied any merit to literary works, considering them important only when they were handmaidens to something seemingly more urgent namely ideology. This was a country where all gestures, even the most private, were interpreted in political terms."

The author's belief that ideology is not as important as literature is revealed by the word:

(a) culture (b) seemingly
(c) urgent (d) political

(GATE 2017, 2 Marks)

Solution: The author's belief that ideology is not as important as literature is revealed by the word *seemingly*.

Ans. (b)

378. There are three boxes. One contains apples, another contains oranges and the last one contains both apples and oranges. All three are known to be incorrectly labeled. If you are permitted to open just one box and then pull out and inspect only one fruit, which box would you open to determine the contents of all three boxes?

(a) The box labelled "apples"
(b) The box labelled "apples and oranges"
(c) The box labelled "oranges"
(d) Cannot be determined

(GATE 2017, 2 Marks)

Solution: It is given that all the three boxes have been incorrectly labelled. we will open the box which is labelled as "apples and oranges".

Let us suppose this box contains apples. Then we are left with two boxes labelled "oranges" and "apples". One of these two boxes will have oranges and the other will have apples and oranges.

Now, the box having the label "oranges" cannot have oranges. Therefore, the box labelled "oranges" will have apples and oranges while the box labelled "apples" will have oranges.

The same will be true if the box having the label "apples and oranges" contains oranges.

Therefore, option (b) is correct.

Ans. (b)

379. X is a 30 digit number starting with the digit 4 followed by the digit 7. Then the number X^3 will have

(a) 90 digits (b) 91 digits
(c) 92 digits (d) 93 digits

(GATE 2017, 2 Marks)

Solution: The number X^3 will have 90 digits.

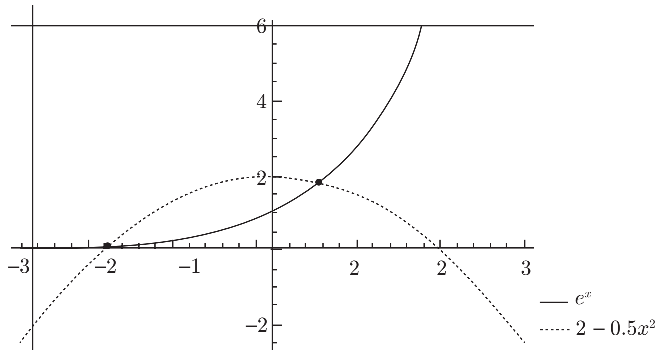
Ans. (a)

380. The number of roots of $e^x + 0.5x^2 - 2 = 0$ in the range [–5, 5] is

(a) 0 (b) 1 (c) 2 (d) 3

(GATE 2017, 2 Marks)

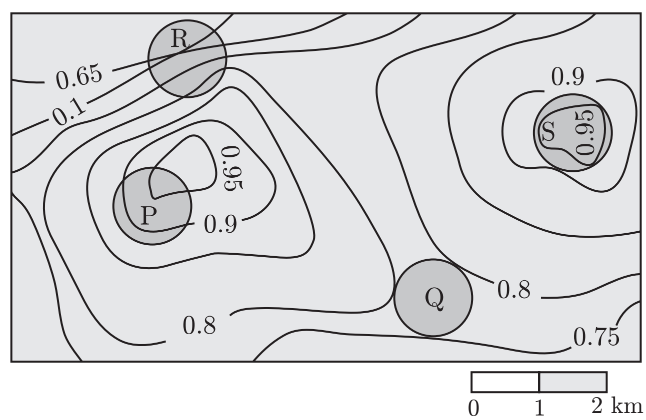
Solution:



As can be clearly seen from the above graph, there will be two solutions in the range of –5 to 5. Therefore, option (c) is correct.

Ans. (c)

381. An air pressure contour line joins locations in a region having the same atmospheric pressure. The following is an air pressure contour plot is a geographical region. Contour lines are shown at 0.05 bar intervals in this plot.



If the possibility of a thunderstorm is given by how fast air pressure rises or drops over a region, which of the following regions is most likely to have a thunderstorm?

(a) P (b) Q (c) R (d) S

(GATE 2017, 2 Marks)

Solution: Air pressure drop is the maximum in the region R as can be seen from the contour lines. Therefore, option (c) is correct.

Ans. (c)

382. "If you are looking for a history of India, or for an account of the rise and fall of the British Raj, or for the reason of the cleaving of the subcontinent into two mutually antagonistic parts and the effects this mutilation will have in the respective sections, and ultimately on Asia, you will not find it in these pages, for though I have spent a lifetime in the country, I lived too near the seat of events, and was too intimately associated with the actors, to get the perspective needed for the impartial recording of these matters."

Here, the word "antagonistic" is closest in meaning to

(a) impartial (b) argumentative
(c) separated (d) hostile

(GATE 2017, 2 Marks)

Solution: Here, the word "antagonistic" is closest in meaning to *hostile*.

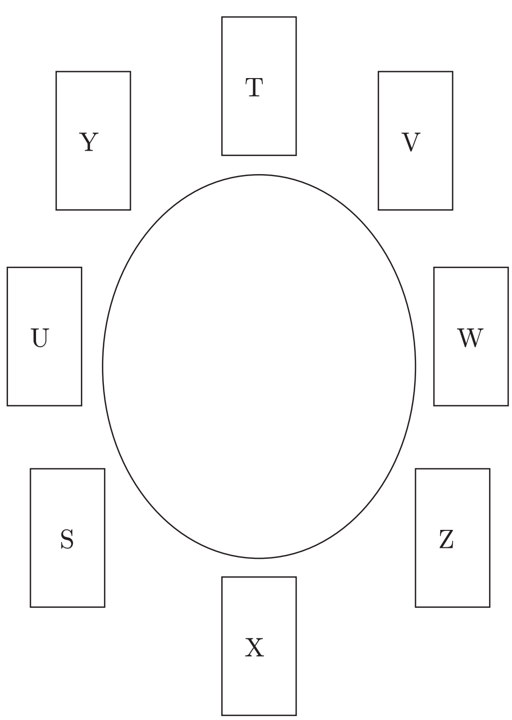
Ans. (d)

383. S, T, U, V, W, X, Y and Z are seated around a circular table. T's neighbours are Y and V, Z is seated third to the left to T and second to the right of s. U's neighbours are S and Y; and T and W are not seated opposite each other. Who is third to the left of V?

(a) X (b) W (c) U (d) T

(GATE 2017, 2 Marks)

Solution:



Therefore, from the above figure, X is the person who is third to the left of V.

Ans. (a)

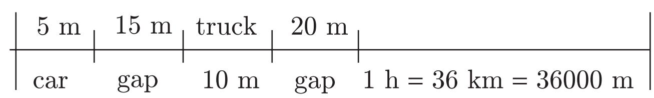
384. Trucks (10 m long) and cars (5 m long) go on a single lane bridge. There must be a gap of at least 20 m after each truck and a gap of at least 15 m after each car. Trucks and cars travel at a speed of 36 km/h. If cars and trucks go alternately, what is the maximum number of vehicles that can use the bridge in one hour?

(a) 1440 (b) 1200 (c) 720 (d) 600

(GATE 2017, 2 Marks)

Solution: Given that speeds of both car and truck = 36 km/hour

In 1 h, they travel = 36 km = 36000 m



Therefore, the maximum number of vehicles that can use the bridge in 1 h

$$= \frac{36000}{50} \times 2 = 720 \times 2 = 1440 \text{ vehicles}$$

Ans. (a)

385. There are 3 Indians and 3 Chinese in a group of 6 people. How many subgroups of this group can we choose so that every subgroup has at least one Indian?

(a) 56 (b) 52 (c) 48 (d) 44

(GATE 2017, 2 Marks)

Solution: Total number of ways of selection

$$\begin{aligned}&= {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6\\&= 6 + 15 + 20 + 15 + 6 + 1\\&= 63 \text{ ways}\end{aligned}$$

Total number of ways of selection such that there is no Indian = ${}^3C_1 + {}^3C_2 + {}^3C_3$

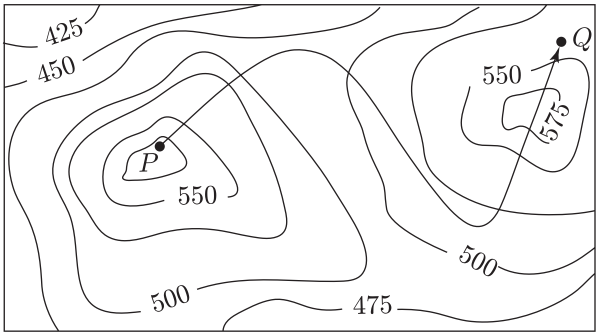
$$= 3 + 3 + 1 = 7 \text{ ways}$$

Therefore,

Total number of ways of selection such that there is at least one Indian = 63 − 7 = 56 ways

Ans. (a)

386. A contour line joins locations having the same height above the mean sea level. The following is a contour plot of a geographical region. Contour lines are shown at 25 m intervals in this plot.



The path from P to Q is best described by

(a) Up-Down-Up-Down (b) Down-Up-Down-Up
(c) Down-Up-Down (d) Up-Down-Up

(GATE 2017, 2 Marks)

Solution: The path from P to Q is best described as follows:

550 – 500 – 575 – 550

Down Up Down

Therefore, option (c) is the correct answer.

Ans. (c)

387. "If you are looking for a history of India, or for an account of the rise and fall of the British Raj or for the reason of the cleaving of the subcontinent into two mutually antagonistic parts and the effects this mutilation will have in the respective sections, and ultimately or Asia, you will not find it in these pages; for though I have spent a lifetime in the country. I lived too near the seat of events, and was too intimately associated with the actors, to get the perspective needed for the impartial recording of these matters."

Which of the following statements best reflects the author's opinion?

(a) An intimate association does not allow for the necessary perspective.
(b) Matters are recorded with an impartial perspective.
(c) An intimate association offers an impartial perspective.
(d) Actors are typically associated with the impartial recording of matters.

(GATE 2017, 2 Marks)

Solution: Option (a) best reflects the author's opinion.

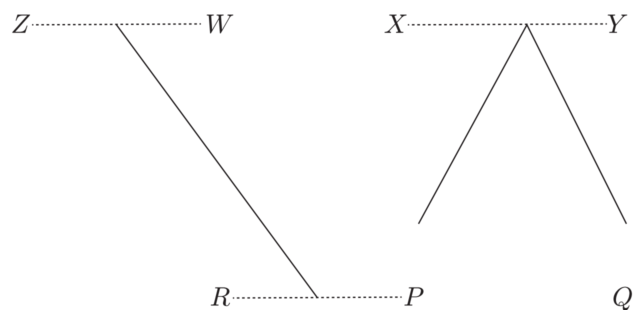
Ans. (a)

388. Each of P, Q, R, S, W, X, Y and Z has been married at most once. X and Y are married and have two children P and Q, Z is the grandfather of the daughter S of P. Further, Z and W are married and are parents of R. Which one of the following must necessarily be FALSE?

(a) X is the mother-in-law of R.
(b) P and R are not married to each other.
(c) P is a son of X and Y.
(d) Q cannot be married to R.

(GATE 2017, 2 Marks)

Solution:



From the figure, P and R are married. So, option b is necessarily false.

Ans. (b)

389. 1200 men and 500 women can build a bridge in 2 weeks, 900 men and 250 women will take 3 weeks to build the same bridge. How many men will be needed to build the bridge in one week?

(a) 3000 (b) 3300 (c) 3600 (d) 3900

(GATE 2017, 2 Marks)

Solution: Let the total work be 42 units.

1200 M + 500 W = 14 days; 3 units per day (2 weeks) (i)

900 M + 250 W = 21 days; 2 units per day (3 weeks) (ii)

From Eq. (i), we have

$$1200\text{ M} + 500\text{ W} = 3\text{ units} \quad \text{(iii)}$$

From Eq. (ii), we have

$$1800\text{ M} + 500\text{ W} = 4\text{ units} \quad \text{(iv)}$$

Solving Eqs. (iii) and (iv), we get

$$600\text{ M} = 1\text{ unit}$$
$$\Rightarrow 1\text{ M} = 1/600\text{ units}$$

Also,

$$500\text{ W} = 1\text{ unit}$$
$$\Rightarrow 1\text{ W} = 1/500\text{ units}$$

Now, the same work has to be completed in 1 week, i.e. 7 days.

If 42 units has to be done in 7 days, 6 units will be done per day.

600 men are able to do 1 unit work per day. Therefore, for 6 units, 3600 men will be required.

Ans. (c)

390. The number of 3-digit numbers such that the digit 1 is never to the immediate right of 2 is

(a) 781 (b) 791 (c) 881 (d) 891

(GATE 2017, 2 Marks)

Solution: The total number of three digit numbers (from 100 to 999) = 900

Let 2 be at the hundred's place and so 1 will be at the ten's place.

Hundred's Place	Ten's Place	Unit's Place
Digit 2	Digit 1	Any of the 10 digits
1 way	1 way	10 ways

Therefore, there are total 10 such numbers.

Let 2 be at the ten's place and so 1 will be at the unit's place.

Hundred's Place	Ten's Place	Unit's Place
Any digit from 1 to 9	Digit 2	Digit 1
9 way	1 way	1 way

Therefore, there are total 9 such numbers.

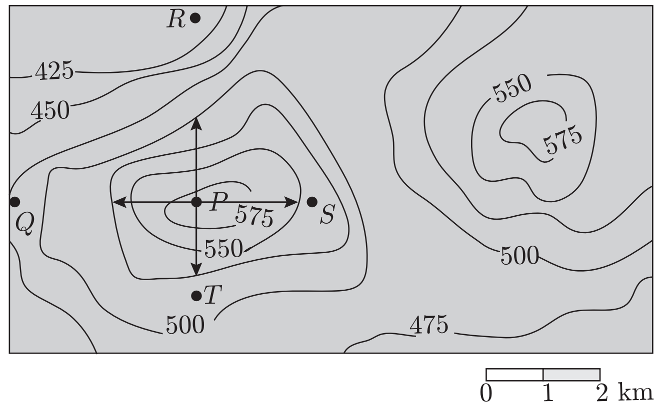
Total 3 digit numbers such that the digit 1 is to the immediate right of digit 2 = 19.

Therefore,

$$\text{Required answer} = 900 - 19 = 881\text{ numbers}$$

Ans. (c)

391. A contour line joins locations having the same height above the mean sea level. The following is a contour plot of a geographical region. Contour lines are shown at 25 m intervals in this plot.



Which of the following is the steepest path leaving from P?

(a) P to Q (b) P to R (c) P to S (d) P to T

(GATE 2017, 2 Marks)

Solution: P to R path is the deepest path from sea level. Therefore, it is the steepest path.

Ans. (b)

392. "Here, throughout the early 1820s, Stuart continued to fight his losing battle to allow his sepoys to wear their caste-marks and their own choice of facial hair on parade, being again reprimanded by the commander-in-chief. His retort that 'A stronger instance than this of European prejudice with relation to this country has never come under my observations' had no effect on his superiors."

According to this paragraph, which of the statements below is most accurate?

(a) Stuart's commander-in-chief was moved by this demonstration of his prejudice.
(b) The Europeans were accommodating of the sepoys' desire to wear their caste-marks.
(c) Stuart's "losing battle" refers to his inability to succeed in enabling sepoys to wear caste-marks.
(d) The commander-in-Chief was exempt from the European prejudice that dictated how the sepoys were to dress.

(GATE 2017, 2 Marks)

Solution: According to the paragraph, option (c) is the most accurate.

Ans. (c)

393. What is the sum of the missing digits in the subtraction problem below?

$$\begin{array}{r} 5\,\_\,\_\,\_\,\_ \\ -4\,8\,\_\,8\,9 \\ \hline 1\,1\,1\,1 \end{array}$$

(a) 8 (b) 10
(c) 11 (d) Cannot be determined

(GATE 2017, 2 Marks)

Solution: There are two possible cases:

(i) Considering

$$50000 - 48889 = 1111$$

Therefore, the sum of the missing digits is

$$0 + 0 + 0 + 0 + 8 = 8$$

(ii) Considering

$$50100 - 48989 = 1111$$

Therefore, the sum of the missing digits is

$$0 + 1 + 0 + 0 + 9 = 10$$

Hence, the correct option is (d) as the answer cannot be determined.

Ans. (d)

394. Let S_1 be the plane figure consisting of the points (x, y) given by the inequalities $|x - 1| \le 2$ and $|y + 2| \le 3$. Let S_2 be the plane figure given by the inequalities $x - y \ge -2$, $y \ge 1$ and $x \le 3$. Let S be the union of S_1 and S_2. The area of S is

(a) 26 (b) 28 (c) 32 (d) 34

(GATE 2017, 2 Marks)

Solution:

For $|x-1| \le 2$

$$|x-1| = x-1, \text{ if } x-1 \ge 0 \text{ or } x \ge 1 \qquad (1)$$

Taking this condition,

$$x - 1 \le 2$$
$$\Rightarrow x \le 3 \qquad (2)$$

Combining Eqs. (1) and (2), we have

$$1 \le x \le 3 \qquad (A)$$

$$|x-1| = -x+1, \quad \text{if } x < 1 \qquad (3)$$

Taking this condition, we have

$$-x + 1 \le 2$$
$$\Rightarrow -x \le 1$$
$$\Rightarrow x \ge -1 \qquad (4)$$

Combining Eqs. (3) and (4), we have

$$-1 \le x \le 1 \qquad (B)$$

Combining Eqs. (A) and (B), we have

$$-1 \le x \le 3 \qquad (C)$$

For $|y+2| \le 3$

$$|y+2| = y+2, \quad \text{if } y+2 \ge 0 \text{ or } y \ge -2 \qquad (5)$$

Taking this condition, we have

$$y + 2 \le 3 \Rightarrow y \le 1 \qquad (6)$$

Combining Eqs. (5) and (6), we have

$$-2 \le y \le 1 \qquad (D)$$

$$|y+2| = -y-2, \quad \text{if } y+2 < 0 \text{ or } y < -2$$

Taking this condition, we have

$$-y - 2 \le 3 \Rightarrow -y \le 5$$
$$\Rightarrow y \ge -5 \qquad (E)$$

Combining Eqs. (D) and (E), we have

$$-5 \le y \le 1 \qquad (F)$$

We will now plot the graph of the union of

$x - y \ge -2$, $y \ge 1$, $x \le 3$, $-1 \le x \le 3$ and $-5 \le y \le 1$

The required area is the area of ΔABC and the area of rectangle ABDE.

Area of ΔABC

$$= \frac{1}{2} \times AB \times BC = \frac{1}{2} \times 4 \text{ units} \times 4 \text{ units} = 8 \text{ sq. units}$$

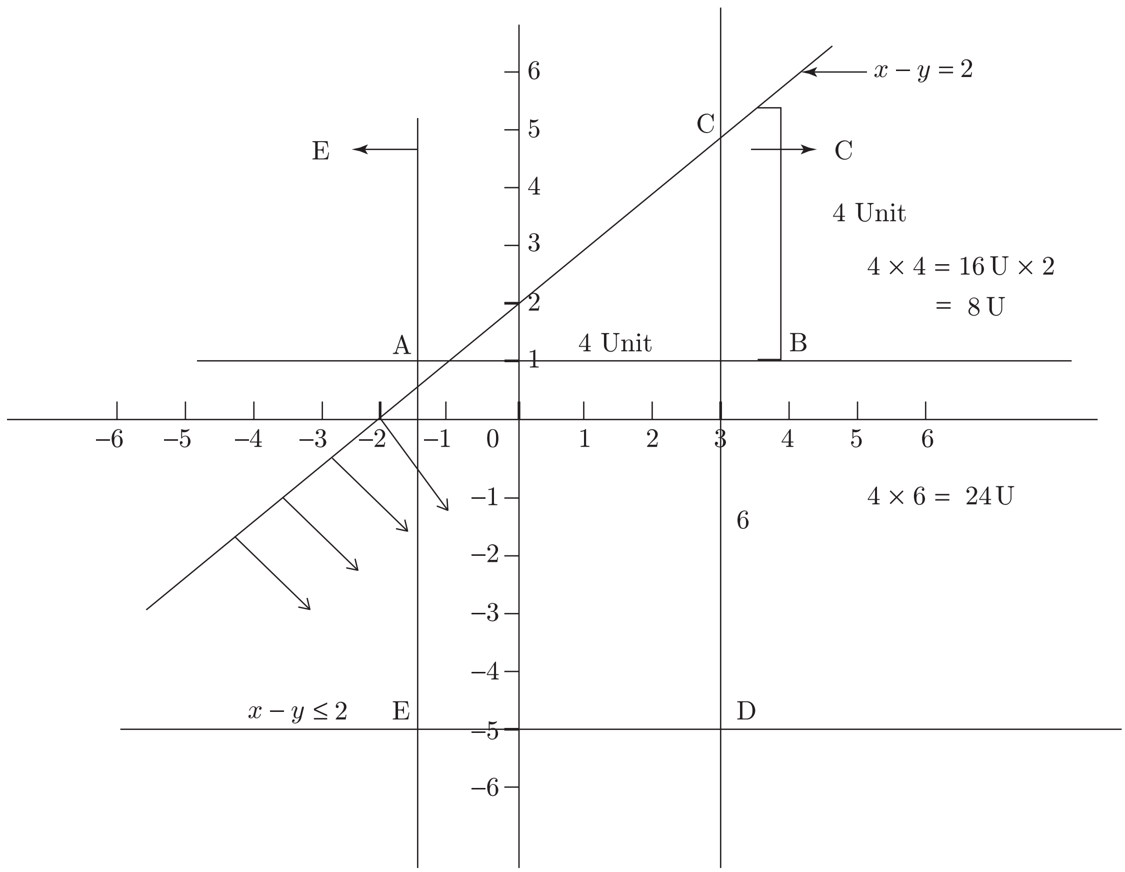
Area of rectangle ABDE

$$= AB \times BD = 4 \text{ units} \times 6 \text{ units} = 24 \text{ sq. units}$$

Therefore,

$$\text{Total area} = 8 \text{ units} + 24 \text{ units} = 32 \text{ sq. units}$$

Ans. (c)



395. Two very famous sportsmen Mark and Steve happened to be brothers, and played for country K. Mark teased James, an opponent from country E. "There is no way you are good enough to play for your country." James replied, "Maybe not, but at least I am the best player in my own family."

Which one of the following can be inferred from this conversation?

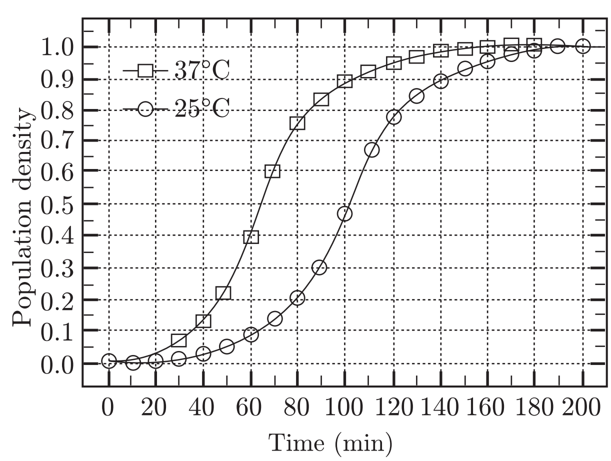
(a) Mark was known to play better than James.
(b) Steve was known to play better than Mark.
(c) James and Steve were good friends.
(d) James played better than Steve.

(GATE 2017, 2 Marks)

Solution: According to the statement given by James, Mark is not the best player in his (Mark's) family. From this statement, we can infer that Steve is a better player than Mark.

Ans. (b)

396. The growth of bacteria (lactobacillus) in milk leads to curd formation. A minimum bacterial population density of 0.8 (in suitable units) is needed to form curd. In the graph below, the population density of lactobacillus in 1 litre of milk is plotted as a function of time, at two different temperatures, 25°C and 37°C.



Consider the following statements based on the data shown above:

(i) The growth in bacterial population stops earlier at 37°C as compared to 25°C.
(ii) The time taken for curd formation at 25°C is twice the time taken at 37°C.

Which one of the following options is correct?

(a) Only (i) (b) Only (ii)
(c) Both (i) and (ii) (d) Neither (i) nor (ii)

(GATE 2017, 2 Marks)

Solution: From the graph, it can be seen that statement (i) is true whereas statement (ii) is not correct.

Ans. (a)

397. "If you are looking for a history of India, or for an account of the rise and fall of the British Raj, or for the reason of the cleaving of the subcontinent into two mutually antagonistic parts and the effects this mutilation will have in the respective sections, and ultimately on Asia, you will not find it in these pages, for though I have spent a lifetime in the country, I lived too near the seat of events, and was too intimately associated with the actors, to get the perspective needed for the impartial recording of these matters."

Which of the following is closest in meaning to "cleaving"?

(a) Deteriorating (b) Arguing
(c) Departing (d) Splitting

(GATE 2017, 2 Marks)

Solution: "Splitting" is closest in meaning to "cleaving."

Ans. (d)

398. X bullocks and Y tractors take 8 days to plough a field. If we halve the number of bullocks and double the number of tractors, it takes 5 days to plough the same filed. How many days will it take X bullocks alone to plough the field?

(a) 30 (b) 35 (c) 40 (d) 45

(GATE 2017, 2 Marks)

Solution: Let the work be 40 units.

$X + Y = 8$ 5 units per day (i)

$\frac{X}{2} + 2Y = 5$ 8 units per day (ii)

Eq. (i) can be written as

$$2X + 2Y = 10 \quad \text{(iii)}$$

Eq. (ii) can be written as

$$\frac{X}{2} + 2Y = 8 \quad \text{(iv)}$$

Solving Eqs. (iii) and (iv), we get

$$3\frac{X}{2} = 2$$
$$\Rightarrow 3X = 4$$
$$\Rightarrow X = 4/3$$

Therefore,

$$\text{Time taken} = 40/X = 40 \times 3/4 = 30 \text{ days}$$

Ans. (a)

399. There are 4 women P, Q, R, S and 5 men V, W, X, Y, Z in a group. We are required to form pairs each consisting of one woman and one man. P is not to be paired with Z, and Y must necessarily be paired with someone. In how many ways can 4 such pairs be formed?

(a) 74 (b) 76 (c) 78 (d) 80

(GATE 2017, 2 Marks)

Solution: There are two possibilities:

(a) If Z is not selected.
The four pairs can be formed in 4! ways, i.e. 24 ways.

(b) If Z is selected.
Z can first form a pair in 3 ways. (P is not paired with Z).
Also, Y has to be definitely selected.

The remaining 2 people from the 3 men available can be chosen in ${}^3C_2$, i.e. 3 ways.
The 3 people selected including Y can now form a pair in 3! ways = 6 ways.
Total number of ways in the second case = 3 ways × 3 ways × 6 ways = 54 ways.
Therefore, from the two possibilities,

$$\text{Total number of ways} = 24 + 54 = 78 \text{ ways}$$

Ans. (c)

400. All people in a certain island are either "Knights" or "Knaves" and each person knows every other person's identity. Knights NEVER lie, and knaves ALWAYS lie.

P says "Both of us are knights". Q says "None of us are knaves".

Which one of the following can be logically inferred from the above?

(a) Both P and Q are knights.
(b) P is a knight; Q is a knave.
(c) Both P and Q are knaves.
(d) The identities of P and Q cannot be determined.

(GATE 2017, 2 Marks)

Solution: There are two possible cases:

1. If P is a knight, then the statement given by P is true as per which both P and Q are knights.

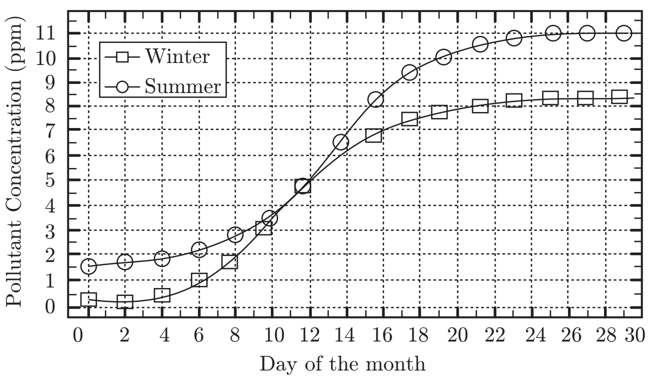
Also, Q says that "none of us are knaves" which is true. This further verifies that Q is also a knight.

2. If P is a knave, then the statement given by P is not true. Also, the statement given by Q is also not true. This leads us to the conclusion that both P and Q are knaves.

Therefore, the correct option is (d).

Ans. (d)

401. In the graph below, the concentration of a particular pollutant in a lake is plotted over (alternate) days of a month in winter (average temperature 10°C) and a month in summer (average temperature 30°C).



Consider the following statements based on the data shown above:

(i) Over the given months, the difference between the maximum and the minimum pollutant concentrations is the same in both winter and summer.

(ii) There are at last four days in the summer month such that the pollutant concentrations on those days are within 1 ppm of the pollutant concentrations on the corresponding days in the winter month.

Which one of the following options is correct?

(a) Only (i) (b) Only (ii)
(c) Both (i) and (ii) (d) Neither (i) nor (ii)

(GATE 2017, 2 Marks)

Solution: The difference between the maximum and the minimum pollutant concentrations in winter = 8 − 0 = 8.

The difference between the maximum and the minimum pollutant concentrations in summer = 10.5 − 1.5 = 9.

Statement (i) is false and statement (ii) is correct from the graph.

Ans. (b)

402. The old concert hall was demolished because of fears that the foundation would be affected by the construction of the new metro line in the area. Modern technology for underground metro construction tried to mitigate the impact of pressurized air pockets created by the excavation of large amounts of soil. But even with these safeguards, it was feared that the soil below the concert hall would not be stable.

From this, one can infer that

(a) the foundations of old buildings create pressurized air pockets underground, which are difficult to handle during metro construction.
(b) metro construction has to be done carefully considering its impact on the foundations of existing buildings.
(c) old buildings in an area form an impossible hurdle to metro construction in that area.
(d) pressurized air can be used to excavate large amounts of soil from underground areas.

(GATE 2017, 2 Marks)

Solution: From the given paragraph, we can infer that metro construction has to be done carefully considering its impact on the foundations of existing buildings.

Ans. (b)

403. Students applying for hostel rooms are allotted rooms in order of seniority. Students already staying in a room will move if they get a room in their preferred list. Preferences of lower ranked applicants are ignored during allocation.

Given the data below, which room will Ajit stay in?

Names	**Student Seniority**	**Current Room**	**Room Preference List**
Amar	1	P	R, S, Q
Akbar	2	None	R, S
Anthony	3	Q	P
Ajit	4	S	Q, P, R

(a) P (b) Q (c) R (d) S

(GATE 2017, 2 Marks)

Solution: As per the preferences given,

Amar will go to room R.
Akbar will go to room S.
Anthony will go to room P.
Ajit will go to room Q.

Ans. (b)

404. The last digit of $(2171)^7 + (2172)^9 + (2173)^{11} + (2174)^{13}$ is

(a) 2 (b) 4 (c) 6 (d) 8

(GATE 2017, 2 Marks)

Solution: Given, $(2171)^7 + (2172)^9 + (2173)^{11} + (2174)^{13}$

The last digit of 2171^7 is 1.
When 9 is divided by 4, the remainder obtained is 1.
Therefore, the last digit of 2172^9 is same as the last digit of 2172^1 which is 2.
When 11 is divided by 4, the remainder obtained is 3.
Therefore, the last digit of 2173^{11} is same as the last digit of 2173^3 which is 7.

When 13 is divided by 4, the remainder obtained is 1.

Therefore, the last digit of 2174^{13} is same as the last digit of 2174^1 which is 4.

Hence,

$$\text{Required last digit} = 1 + 2 + 7 + 4 = 14$$

Therefore, the required last digit is 4.

Ans. (b)

405. Two machines M1 and M2 are able to execute any of four jobs P, Q, R and S. The machines can perform one job on one object at a time. Jobs P, Q, R and S take 30 minutes, 20 minutes, 60 minutes and 15 minutes each respectively. There are 10 objects each requiring exactly 1 job. Job P is to be performed on 2 objects. Job Q on 3 objects, Job R on 1 object and Job S on 4 objects. What is the minimum time needed to complete all the jobs?

(a) 2 hours (b) 2.5 hours
(c) 3 hours (d) 3.5 hours

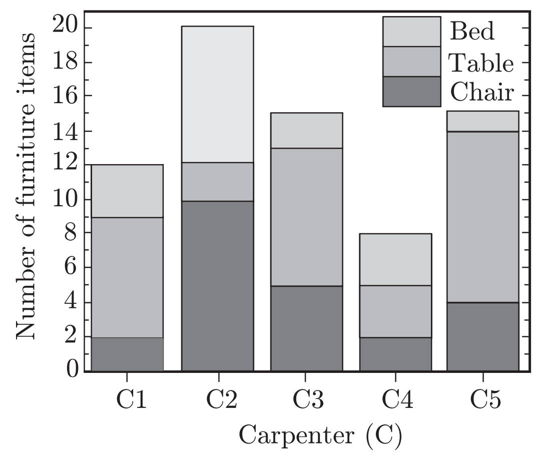
(GATE 2017, 2 Marks)

Solution: M1: P: 1 hour and R: 1 hour
M2: Q: 1 hour and S: 1 hour

Therefore, the minimum time needed to complete all the jobs is 2 hours.

Ans. (a)

406. The bar graph below shows the output of five carpenters over one month, each of whom made different items of furniture: chairs, tables, and beds.



Consider the following statements.

(i) The number of beds made by carpenter C2 is exactly the same as the number of tables made by carpenter C3.
(ii) The total number of chairs made by all carpenters is less than the total number of tables.

Which one of the following is true?

(a) Only (i) (b) Only (ii)
(c) Both (i) and (ii) (d) Neither (i) nor (ii)

(GATE 2017, 2 Marks)

Solution: The number of beds made by carpenter C2 = 8 (12 to 20)
The number of tables made by carpenter C3 = 8 (5 to 13)
Therefore, statement (i) is true.
Now,
Total number of chairs made by all the carpenters = 2 + 10 + 5 + 2 + 4 = 23 chairs
Total number of tables made by all the carpenters = 7 + 2 + 8 + 3 + 10 = 30 tables
Therefore, statement (ii) is also true.

Ans. (c)

407. Bhaichung was observing the pattern of people entering and leaving a car service centre. There was a single window where customers were being served. He saw that people inevitably came out of the centre in the order that they went in. However, the time they spent inside seemed to vary a lot: some people came out in a matter of minutes while for others it took much longer.

From this, what can one conclude?

(a) The centre operates on a first-come-first-served basis, but with variable service times, depending on specific customer needs.
(b) Customers were served in an arbitrary order, since they took varying amounts of time for service completion in the centre.
(c) Since some people came out within a few minutes of entering the centre, the system is likely to operate on a last-come-first-served basis.
(d) Entering the centre early ensured that one would have shorter service times and most people attempted to do this.

(GATE 2017, 2 Marks)

Solution: We can conclude that the centre operates on a first-come-first-served basis, but with variable service times, depending on specific customer needs.

Ans. (a)

408. A map shows the elevations of Darjeeling, Gangtok, Kalimpong, Pelling, and Siliguri. Kalimpong is at a lower elevation than Gangtok. Pelling is at a lower elevation than Gangtok. Pelling is at a higher elevation than Siliguri. Darjeeling is at a higher elevation than Gangtok.

Which of the following statements can be inferred from the paragraph above?

(i) Pelling is at a higher elevation than Kalimpong.
(ii) Kalimpong is at a lower elevation than Darjeeling.
(iii) Kalimpong is at a higher elevation than Siliguri.
(iv) Siliguri is at a lower elevation than Gangtok.

(a) Only (ii) (b) Only (ii) and (iii)
(c) Only (ii) and (iv) (d) Only (iii) and (iv)

(GATE 2017, 2 Marks)

Solution: As per the question, we have

$$\text{Kalimpong} < \text{Gangtok}$$
$$\text{Pelling} < \text{Gangtok}$$
$$\text{Siliguri} < \text{Pelling}$$
$$\text{Gangtok} < \text{Darjeeling}$$

Combining the statements, we have

$$\text{Kalimpong} < \text{Gangtok} < \text{Darjeeling}$$
$$\text{Siliguri} < \text{Pelling} < \text{Gangtok} < \text{Darjeeling}$$

Therefore, only statements (ii) and (iv) are correct.

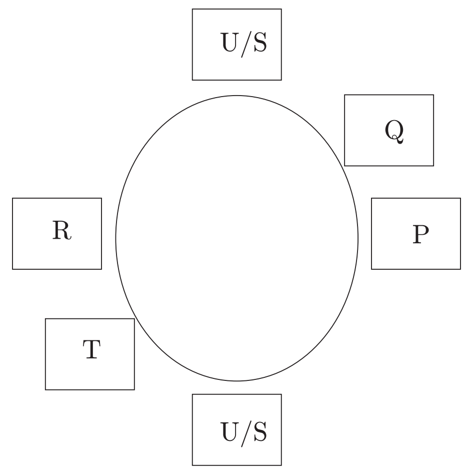
Ans. (c)

409. P, Q, R, S, T and U are seated around a circular table. R is seated two places to the right of Q. P is seated three places to the left of R. S is seated opposite U. If P and U now switch seats, which of the following must necessarily be true?

(a) P is immediately to the right of R.
(b) T is immediately to the left of P.
(c) T is immediately to the left of P or P is immediately to the right of Q.
(d) U is immediately to the right of R or P is immediately to the left of T.

(GATE 2017, 2 Marks)

Solution: Consider the following figure.



If P and U interchange their positions, either T will be immediately to the left of P or P will be immediately to the right of Q. Therefore, option (c) is true.

Ans. (c)

410. Budhan covers a distance of 19 km in 2 hours by cycling one fourth of the time and walking the rest. The next day he cycles (at the same speed as before) for half the time and walks the rest (at the same speed as before) and covers 26 km in 2 hours. The speed in km/h at which Budhan walks is

(a) 1 (b) 4 (c) 5 (d) 6

(GATE 2017, 2 Marks)

Solution: Let the speed for cycling be C km/h and the speed of walking be W km/h.
On the first day, we have

$$(2/4) \times C + (6/4) \times W = 19$$
$$\Rightarrow C/2 + 3W/2 = 19$$
$$\Rightarrow C + 3W = 38 \quad (1)$$

On the second day, we have

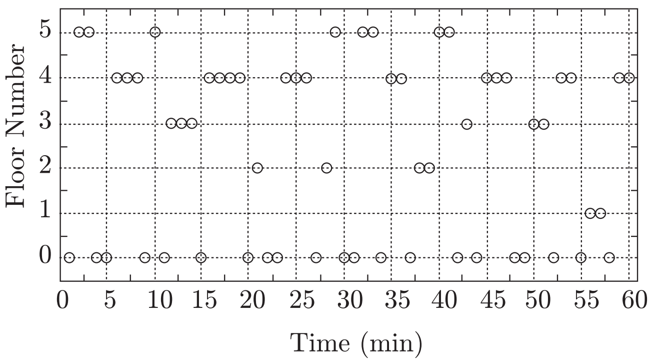
$$(2/2) \times C + (2/2) \times W = 26$$
$$\Rightarrow C + W = 26 \quad (2)$$

Solving Eqs. (1) and (2), we have

$$2W = 12$$
$$W = 6 \text{ km/h}$$

Ans. (d)

411. The points in the graph below represent the halts of a lift for durations of 1 minute, over a period of 1 hour.



Which of the following statements are correct?

(i) The elevator never moves directly from any non-ground floor to another non-ground floor over the one hour period.

(ii) The elevator stays on the fourth floor for the longest duration over the one hour period.

(a) Only (i)
(b) Only (ii)
(c) Both (i) and (ii)
(d) Neither (i) nor (ii)

(GATE 2017, 2 Marks)

Solution: Statement (i) is not correct since the elevator moved from the 2nd floor to the 5th floor during the time period of 25 minutes to 30 minutes.

Statement (ii) is not correct since the elevator has stopped in the ground floor for the maximum time.

Ans. (d)

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The book presents the subjects of Engineering Mathematics and General Aptitude in a systematic, structured and precise manner. It intends to offer GATE aspirants a self-study and do-it-yourself approach by providing comprehensive and step-by-step treatment of each and every aspect of the GATE examination. In addition, the book aims to solve problems faced by aspirants in terms of extensive syllabus coverage of major engineering branches and unavailability of a standard book. The emphasis on fundamental concepts helps in developing the aptitude required for success in GATE. The book has all the three important 'C' qualities, that is Conciseness, Completeness and Correctness.

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